Master thesis

Algant programm

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Algebraic unknotting number

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Notation

- $\mathbb{Z}$, the set of integers.
- $\mathbb{Q}$, the set of rational numbers.
- $\mathbb{Z}[t^{\pm 1}]$ and Λ, the set of Laurent polynomials with coefficients in $\mathbb{Z}$.
- $\mathbb{Q}(t)$ and Ω, the fraction field of Λ.
- $\overline{f}$, denotes $f(t^{-1})$ being f an element of $\mathbb{Q}(t)$.
- K, an oriented knot or a number field.
- lk, the linking number.
- $m(K)$, the Nakanishi index of the knot K.
- $V(K)$, the Seifert matrix of the knot K.
- $X(K)$, exterior of the knot K.
- $\mathbb{S}^n$, the $n-$dimensional sphere.
- D^n, the $n-$dimensional disk.
- $\mathcal{O}_L$, the ring of integers of the number field L.
- U_L, the unit group of the number field L.
- C_L, the class group of the number field L.

Contents

Introduction

In 2011 Maciej Borodzik and Stefan Friedl were able to compute the algebraic unknotting number for almost all knots up to 12 crossings. Just 19 knots were resistant to their computations.
This algorithm was made in the following way:

1. Computing an upper bound of the algebraic unknotting number applying a sequence of random algebraic moves to the Seifert matrix and checking whether the resulting matrix is S-equivalent to the 0×0-matrix.

2. Computes different lower bounds such as the Nakanishi index, the Levine–Tristram signature and the Lickorish obstruction.

The upper bound given obtained in (1) matches some of the lower bounds obtained in (2) for all knots up to 12 crossings except for the following knots:

$$\begin{array}{lllllll} 12a_{0050} & 12a_{0141} & 12a_{0364} & 12a_{0649} & 12a_{0728} & 12a_{0791} & 12a_{0901} \\ 12a_{1049} & 12a_{1054} & 12a_{1064} & 12a_{1138} & 12a_{1141} & 12a_{1234} & 12a_{1236} \\ 12a_{1264} & 12n_{0200} & 12n_{0260} & 12n_{0657} & 12n_{0864}. & & \end{array}$$

In 2013 Maciej Borodzik and Stefan Friedl were able to find an algebraic invariant, $n(K)$, that is equivalent to the algebraic unknotting number. However, in general, this invariant is very hard to compute.

The goal of this thesis is to compute the algebraic unknotting number of the remaining 12 crossing knots.

In order to do it, we made an algorithm that, for a given knot K, allows us to give an upper bound of the Nakanishi index. In the case that the upper bound is 1, it computes the Blanchfield form of K and also checks whether $n(K) = 1$. This algorithm will allow us to compute the Blanchfield form for all the knots of the list above except for the knots $12a_{1054}, 12a_{1141}, 12a_{1264}, 12n_{0260}, 12n_{0657}$ and $12a_{1049}$. For these cases the algorithm shows us that the upper bound of the Nakanishi index is 2. Therefore, in order to prove that the Nakanishi index of these knots is 2 we will introduce some obstructions that will allow us to prove it for some cases. For the other ones we will introduce another technique that allows us to prove that the nakanishi index is 1, and then using the same algorithm as before compute $n(K)$.

The thesis is structured in four chapters and an appendix and it's divided in two main parts.

The first part is composed by the first and second chapters. In it, we introduce all the basic classical invariants and concepts that will allow us to prove part of the main result. The second part is composed by the third and fourth chapters and the appendix. In it, we will give some algorithms that allow us to compute these invariants for some of the knots and then deduce the algebraic unknotting number using the first part.

In the first chapter, we give a quick overview of some concepts from knot theory. We are not going to describe the concepts in detail but only define the language and introduce a few basic notions, such as the *classical invariants*, the definition of the algebraic unknotting number and the algebraic Gordian distance. Moreover, we prove an obstruction of the algebraic unknotting number and one of the most basic lower bounds of the algebraic unknotting number given by the Nakanishi index. Finally we define the Dehn surgery that will be used in the proof of the main theorem.

In the second chapter, we introduce important concepts such as the twisted homology, the Blanchfield form and the $n(K)$ invariant. Then, we state the main theorem and deduce the $n(K) = 1$ version. After, we prove the inequality $n(K) \leq u_a(K)$ and finally we prove that the 8_{10} knot has algebraic unknotting number 1.

In the third chapter, we explain the algorithm made to compute the Nakanishi index ($m(K)$) of a knot and the Blanchfield form in the case that $m(K) = 1$. Also we give a list of the Blanchfield forms and Alexander polynomials of the knots for which Knotorius [1] was not able to determinate the algebraic unknotting number. Then, we introduce the row and column class invariants and the pullback diagram method and, using the results obtained together with some computations, we determine the Nakanishi index for the knots $12n_{0657}, 12a_{1054}$ and $12a_{1141}$. Moreover, nearly by accident, we will deduce using the upper bound given by *Knotinfo* [2] and the lower bound of the Nakanishi index of $12n_{0657}$, that the unknotting number of $12n_{0657}$ is 2.

In the fourth chapter, we explain the algorithm made to compute the invariant $n(K)$ and $n_p(K)$ in the cases were $m(K) = 1$. Then, for the cases that we were not able to compute the nakanishi index, we give a method that allows us to find a generator and the Blanchfield form. Finally, we give the table of the results obtained using the algorithms explained in these notes for the knots for which *Knotorius* [1] was not able to determine the algebraic unknotting number.

The results given in the last section of chapter fourth, should not be understood as a complete truth, but more as a motivation to implement the methods given throughout the thesis in knot softwares. The main reason for it is that we do not give a complete proof of it since a lot of the computations (such as class group, unit group, gcd,..) are computed by *sage* or a similar algebraic computational system. To put in other words, we give a computer-assisted proof. In some cases, it would be easy to give a rigorous proof for these computations, but for others would take too long to even think about it. So, if the reader has enough trust in the computational system and/or has a weak definition of mathematical proof, one could say that the work of Maciej Borodzik and Stefan Friedl via *Knotorius* together with the humble calculations of this thesis achieve to compute the algebraic unknotting number for all the knots up to 12 crossings.

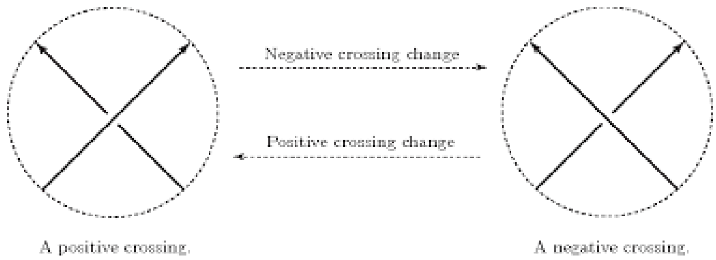
Chapter 1

Algebraic unknotting number

In this chapter we will introduce the unknotting number and the algebraic unknotting number. Then we will give some lower bounds for these invariants that were used in *Knotorius* [1] to compute the algebraic unknotting number. In the last part of this chapter we will introduce the algebraic unknotting operation, used to find upper bounds for the algebraic unknotting number, the Gordian distance, and prove the most easy non-trivial lower bound for the algebraic unknotting number given by the Nakanishi index. Finally, we will see how to obtain a crossing change from a surgery. This technique will be used in chapter 2 to prove the main theorem.

1.1 Basic definitions

In this chapter $K \subseteq \mathbb{S}^3$ will denote an oriented knot. A **crossing change** is one of the two local moves on a knot diagram that are given in the following figure.



The **unknotting number** of a knot K, $u(K)$, is defined to be the minimal number of crossing changes necessary to turn K into the unknot. The unknotting number is one of the most elementary invariants of a knot, but also one of the most intractable. Whereas upper bounds can be found easily using diagrams, it is much harder to find non-trivial lower bounds.

Example 1.1.1. The knots given in Figure 1,

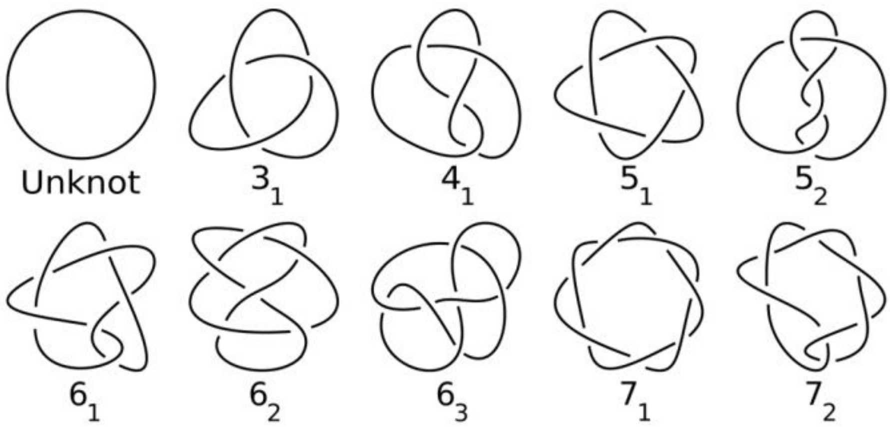


Figure 1.

have the following unknotting numbers

0	1	1	2	1
1	1	1	3	1

.

From the diagrams it is easy to find explicit crossing changes to check that the given numbers are upper bounds. But how can someone prove that these are the actual unknotting numbers?

In these notes we will give some results for an invariant closely related to $u(K)$, namely the **algebraic unknotting number** $u_a(K)$. This invariant is defined as the minimal number of crossing changes necessary to turn K into a knot with trivial Alexander polynomial; though the original definition of the algebraic unknotting number given by Murakami [3] is expressed in terms of "algebraic unknotting moves" on Seifert matrices, by [4] and [5] both definitions are equivalent.

Since the Alexander polynomial of the unknot is trivial, we deduce that $u(K) \geq u_a(K)$. But in general it is not an equality. For example, for any non-trivial knot K with trivial Alexander polynomial, we have $u(K) \geq 1$ and $u_a(K) = 0$.

1.2 Review of classical invariants

Let F be a Seifert surface of a knot K (i.e, a closed surface in $\mathbb{S}^3$ such that $\partial F = K$) and let $v_1, ..., v_n$ be the collection of embedded simple closed curves on F which represents a basis for $H_1(F;\mathbb{Z})$. The corresponding Seifert matrix V is defined as the matrix with $(i,j)-$entry given by $lk(v_i, v_j^+)$, where v_j^+ denotes the positive push-off of v_j and lk denotes the linking number. Clearly, the Seifert matrix depends on different choices, so it's not an invariant. However, the $S-$equivalence of the Seifert matrix is well known to be an invariant of K (see [6] chapter 6). By abuse of notation we will denote $V = V_K$ as a representative of the $S-$equivalence class. A **classical invariant** of a knot is an invariant which is determined by V_K.

Given a knot $K \subseteq \mathbb{S}^3$ and $U(K)$ a tubular neighberhood, we denote by $X(K) = \overline{\mathbb{S}^3 \setminus U(K)}$ the exterior of K and we denote by $\Sigma(K)$ its branched cover. We now give several well-known examples of classical invariants. Some of them will play an important role in these notes (in the following, matrix V is a $2n \times 2n$ Seifert matrix of K):

- The **Alexander module** $H_1(X(K);\mathbb{Z}[t^{\pm 1}])$, see the following chapter for a precise definition.

- The **Alexander polynomial** is defined as

$$\Delta_K(t) = t^{-n}\det(tV - V^T) \in \mathbb{Z}[t^{\pm 1}]$$

  Note that $\Delta_K(t)$ is well-defined with no indeterminacy, and $\Delta_K(1) = 1$. One can prove that the order of the Alexander module is the Alexander polynomial (see [6] for details), using the fact that $tV - V^T$ is a presentation matrix of the Alexander module.

- The isometry type of the **linking pairing**

$$l(K) : H_1(\Sigma(K);\mathbb{Z}) \times H_1(\Sigma(K);\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z},$$

  which is isometric to the pairing

$$\mathbb{Z}^{2n}/(V+V^T)\mathbb{Z}^{2n} \times \mathbb{Z}^{2n}/(V+V^T)\mathbb{Z}^{2n} \to \mathbb{Q}/\mathbb{Z}.$$

- The **Blanchfield pairing**

$$\lambda(K) : H_1(X(K),\mathbb{Z}[t^{\pm 1}]) \times H_1(X(K),\mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}],$$

which is a Hermitian non-singular pairing on the Alexander module and is isometric to the following Hermitian non-singular pairing

$$\begin{aligned}\mathbb{Z}[t^{\pm1}]^{2g}/A\mathbb{Z}[t^{\pm1}]^{2g} \times \mathbb{Z}[t^{\pm1}]^{2g}/A\mathbb{Z}[t^{\pm1}]^{2g} &\to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}] \\ (a,b) &\mapsto \overline{a}^T(t-1)A^{-1}b,\end{aligned}$$

In this chapter, we will use this last expression of the Blanchfield form. However, in the following chapter we will explain the Blanchfield form in more detail.

- The **Nakanishi index** $m(K)$, defined as the minimal number of generators of the Alexander module. Observe that if $m(K) = 1$, then $H_1(X(K); \mathbb{Z}[t^{\pm1}]) \cong \mathbb{Z}[t^{\pm1}]/\Delta_K$.
- Given $z \in \mathbb{S}^1$ the **Levine-Tristram signature**, defined as
$$\sigma_z(K) = \mathrm{sign}(V(1-z) + V^T(1-z^{-1})).$$
- Given $z \in \mathbb{C} \setminus \{0,1\}$, the **nullity**, defined as
$$\eta_z(K) := \mathrm{null}(V(1-z) + V^T(1-z^{-1})).$$
Moreover, by convention $\eta_1(K)$ is 0.
- Let K be a tame knot in $\mathbb{S}^3 = \partial D^4$. Then it is known that K always bounds a properly and topologically locally flatly embedded compact connected orientable surface F in D^4. The **topological 4-ball genus** is defined as $g_4^{top}(K)$ of K to be the minimum genus of such embedded surfaces bounded by K.

Observation 1.2.1. By [7] two knots have S-equivalent Seifert matrices if and only if the corresponding Blanchfield forms are isometric.

Definition 1.2.2. We say that two knots K_1 and K_2 are **algebraically equivalent** if V_{K_1} and V_{K_2} are S-equivalent. In particular, we say that a knot K is algebraically trivial if $[V_K] = [\emptyset]$, i.e, by the above observation, $\Delta_K = 1$.

1.3 Classical bounds for the unknotting number

We will now summarise some of the classical lower bounds on the unknotting number that were used to compute the algebraic unknotting number for the knots up to 12 crossings.

One of the first non-trivial lower bound of the algebraic unknotting number is the inequality

$$u_a(K) \geq m(K).$$

It has been known since Murasugi's work that the Levine-Tristram signature gives rise to lower bounds on the unknotting number. In particular, the following inequality holds

$$u_a(K) \geq \frac{1}{2}(\max\{\eta_z(K) + \sigma_z(K) : z \in \mathbb{S}^1\} + \max\{\eta_z(K) - \sigma_z(K) : z \in \mathbb{S}^1\}).$$

Thanks to Saeki [5], the topological 4−ball genus $g_4^{top}(K)$ is a lower bound for the algebraic unknotting number $u_a(K)$. Livingston [8] introduced a more computable classical invariant, $\rho(K)$, which gives a lower bound on $g_4^{top}(K)$.

We will now recall several classical obstructions on $u_a(K)$ for knots K with "small" algebraic unknotting number. If K can be unknotted using a single $\epsilon-$crossing change (with $\epsilon \in \{\pm 1\}$), then by the work of Lickorish [9] there exists a generator h of $H_1(\Sigma(K); \mathbb{Z})$ such that

$$l(h,h) = \frac{-2\epsilon}{\det(K)} \in \mathbb{Q}/\mathbb{Z}.$$

Finally, the Stoimenow obstruction [10] states that if $|\sigma(K)| = 4$ and $\det(K)$ is a square and has no divisors of the form $4r+3$, then $u(K) > 2$.

In chapter 2 we will give another algebraic invariant, $n(K)$, that subsumes all these lower bounds and obstructions.

1.4 Algebraic unknotting operation

Let K be a knot and V a Seifert matrix of K, we say that W is obtained from an **algebraic unknotting operation** on V if

$$W = \begin{pmatrix} \epsilon & \delta & 0 \\ 0 & x & M \\ 0 & N^T & V \end{pmatrix}$$

for $\epsilon, \delta = \pm 1$ and $x \in \mathbb{Z}$, where M and N are row vectors.

Let V be a Seifert matrix, and let $[V]$ be its $S-$equivalence class. The **algebraic Gordian distance** $d_G^a([V],[V'])$ between two S-equivalence classes of Sefiert matrices is the minimal number of algebraic unknotting operations needed to turn a Seifert matrix in $[V]$ into a Seifert matrix in $[V']$. In the following lemma we will prove that, for a given knot K with Seifert surface V, we have that $u_a(K) \geq d_G^a([V],[\emptyset])$.

Lemma 1.4.1. *Let K be a knot and K' a knot obtained by applying to K a $\epsilon-$crossing change then the Seifert matrix of K' is obtained from an algebraic unknoting operation on the Seifert matrix of K.*

Proof. We construct Seifert surfaces F and F' of K and K' as shown in the figure. We choose a basis $\mathcal{B}$ for $H_1(F;\mathbb{Z})$ and let $V(K)$ be the Seifert matrix of K defined by $\mathcal{B}$. We add generators α and β as shown in the figure so that $\{\alpha, \beta\}$ is a basis for $H_1(F';\mathbb{Z})$. Then, computing the linking number of each one of the elements of the basis $H_1(F';\mathbb{Z})$, we obtain that the Seifert matrix $V(K')$ of K' defined on this basis has the form

$$V(K') = \begin{pmatrix} \epsilon & \delta & 0 \\ 0 & x & M \\ 0 & N^T & V \end{pmatrix},$$

as we wanted to prove.

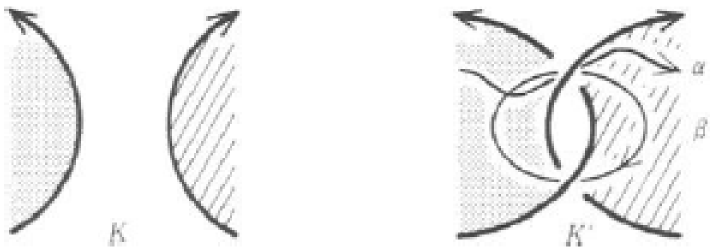


Figure 2.

□

Moreover, one can prove that for a given matrix M obtained from an algebraic unknotting operation on a Seifert matrix of a knot K there exists a knot K' obtained from K after an ϵ-crossing change with Seifert matrix M(see [3] for details). In particular, for a knot K with Seifert matrix V we have that $u_a(K) = d^a_G([V], [\emptyset])$.

Now we have the tools to prove that $u_a(K) \geq m(K)$.

Proposition 1.4.2. *For a knot K we have that $u_a(K) \geq m(K)$.*

Proof. We prove this theorem by induction on the algebraic unknotting number. If $u_a(K) = 0$, we have that K is a knot with trivial Alexander polynomial. Since $tV - V^T$ is the presentation matrix of the Alexander module, we would have that this matrix is invertible in $\mathbb{Z}[t^{\pm 1}]$. Therefore, we would deduce that the Alexander module is trivial, in particular $m(K) = 0$. Assume that the inequality is true for any knot K with $u(K) < n$. Let K' be a knot with $u(K') = n$. Let K be a knot with $u(K) = n - 1$ obtained from K' by an $\epsilon-$crossing change. Then, by 1.4.1 we have that

$$tV(K') - V(K')^T = \begin{pmatrix} \epsilon(t-1) & \delta t & 0 \\ -\delta & a(t-1) & (t-1)N^T \\ 0 & (t-1)N & tV(K) - V(K)^T \end{pmatrix}.$$

Multiplying $tV(K') - V(K')^T$ on the left by an invertible matrix over $\mathbb{Z}[t^{\pm 1}]$, we can transform $tV(K') - V(K')^T$ into a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & x & * \\ 0 & * & tV(K) - V(K)^T \end{pmatrix}.$$

Since $m(K) \leq n-1$, we can transform $tV(K) - V(K)^T$ into an $(n-1) \times (n-1)$ presentation matrix. Hence $tV(K') - V(K')^T$ can be transformed into an $n \times n$ presentation matrix. It follows that $m(K') \leq n$. $\square$

Theorem 1.4.3. *Let K and K' be two knots with V and V' Seifert matrices. If the algebraic Gordian distance $d_G^a([V],[V']) = 1$, then there exists $a \in H_1(X(K); \mathbb{Z}[t^{\pm 1}])$ and $a' \in H_1(X(K'); \mathbb{Z}[t^{\pm 1}])$ such that $Bl_K(a,a) \equiv \pm \frac{\Delta_{K'}}{\Delta_K}$ (mod $\mathbb{Z}[t^{\pm 1}]$) and $Bl_{K'}(a',a') \equiv \pm \frac{\Delta_K}{\Delta_{K'}}$ (mod $\mathbb{Z}[t^{\pm 1}]$).*

Proof. If $[V]$ and $[V']$ have algebraic Gordian distance one, there exists $W \in [V]$ and $W' \in [V']$ such that W can be obtained from W' by an algebraic unknotting operation. By definition, the algebraic unknotting operation assigns W' to $\begin{pmatrix} \epsilon & 0 & 0 \\ 1 & x & M \\ 0 & N^T & W' \end{pmatrix}$ for $\epsilon = \pm 1$. Therefore, we have

$$tW - W^T = \begin{pmatrix} \epsilon(t-1) & -1 & 0 \\ t & x(t-1) & tM - N \\ 0 & tN^T - M^T & tW' - W'^T \end{pmatrix}$$

Let a_1 be the first element of the basis for $H_1(X(K); \mathbb{Z}[t^{\pm 1}])$ so that the Blanchfield pairing $Bl_K(a_1, a_1)$ is the $(1,1)-$entry of the matrix $(t-1)(tW - W^T)^{-1}$ modulo $\mathbb{Z}[t^{\pm 1}]$. The inverse of a non-singular matrix M is equal to $\frac{\text{adj}M}{\text{det}M}$, where adj$M$ is the adjugate matrix.

$$\begin{aligned}
Bl_K(a_1,a_1) &\equiv (t-1)\frac{[\mathrm{adj}(tW-W^T)]_{1,1}}{\det(tW-W^T)} \\
&\equiv (t-1)\frac{\det\begin{pmatrix} x(t-1) & tM-N \\ tN^T-M^T & tW'-W'^T \end{pmatrix}}{\det(tW-W^T)}
\end{aligned}$$

The Alexander polynomial are given by

$$\begin{aligned}
\Delta_K &= t^{-g}\det(tW-W^T) \\
\Delta_{K'} &= t^{1-g}\det(tW'-W'^T)
\end{aligned}$$

where $2g$ is the size of W. The determinant

$$\det(tW-W^T) = \epsilon(t-1)\det\begin{pmatrix} x(t-1) & tM-N \\ tN^T-M^T & tW'-W'^T \end{pmatrix} + t\det(tW'-W'^T).$$

Putting it all togheter, we have

$$Bl_K(a_1,a_1) \equiv \frac{\epsilon(\Delta_{K'}-\Delta_K)}{\Delta_K} \equiv \epsilon\frac{\Delta_{K'}}{\Delta_K} \pmod{\mathbb{Z}[t^{\pm 1}]}$$

The equation for $Bl_{K'}$ can be obtained analogously. □

Theorem 1.4.4 (Fogel, Murakami)**.** *If K is a knot and K' is a knot with trivial alexander polynomial, then the converse of 1.4.3 is also true.*

Proof. See [4] for details. □

We will discuss briefly the converse of 1.4.3 in the appendix and in the following chapter we will see a generalization of this last result.

1.5 Dehn surgery

In this section we will describe the operation of Dehn surgery on knots. This operation will allow us to obtain the crossing changes after a surgery operation.

Let K be a knot in $\mathbb{S}^3$ (or a 3-manifold M). Let T be the boundary of a regular neighbourhood of K. There are two special homology classes of curves on T, namely the **meridian** μ and the **longitude** λ.

The meridian is represented by a simple curve in T that bounds a disc in $U(K) = D^2 \times \mathbb{S}^1$, for example $(\partial D^2) \times 1 \subseteq \partial(D^2 \times \mathbb{S}^1) \subseteq \mathbb{S}^1 \times \mathbb{S}^1$.

The longitude λ is a curve that intersects the meridian transversely in exactly one point and bounds a surface in $\mathbb{S}^3 \setminus U(K)$. In the general case, we simply take any closed curve that intersects μ transversely in one point. In terms of these coordinates, surgeries at K are parametrised by elements of $\mathbb{Z}^2$, or alternatively, by the slope in $\mathbb{Q} \cup \{\infty\}$. To sew in the solid torus $D^2 \times \mathbb{S}^1$, we can first attach the disc $D = D^2 \times \{1\}$, and then its complement, which is a ball. The map is determined, up to isotopy, by the homology class of ∂D. The disc D can be attached to any simple closed non-separating curve on T to give an attaching map for the solid torus. Such curves are of the form $p\mu + q\lambda$, with $(p, q) = 1$. The homology class of ∂D is canonical up to sign. As $(p, q) = 1$, and the surgery determined by (p, q) is the same as that one determined by $(-p, -q)$, we can parametrise the Dehn surgeries by $p/q \in \mathbb{Q} \cup \{\infty\}$. This surgery is described as the p/q-surgery on the knot K.

Example 1.5.1. Let U be an unknot in $\mathbb{S}^3$. For a given tubular neighbourhood W of U, we know that $X(U) = \overline{\mathbb{S}^3 \setminus W}$ is the solid torus. Thus, the result of a surgery is just the result of gluing together two solid tori. More specifically, a p/q-surgery gives $L(p, q)$. In particular, the result of $\pm 1-$surgery on $K \subseteq \mathbb{S}^3$ is homeomorphic to $\mathbb{S}^3$. We write this space as $\mathbb{S}^3_{\pm}(U)$.

Definition 1.5.2. Let $K \subseteq \mathbb{S}^3$ be a knot. We define the 0-**framed surgery** of K to be the 3-manifold obtanied appliying a 0-surgery on K. We denote this manifold by $M(K)$.

Now let K be a knot and U be the unknot such that $\mathrm{lk}(K, U) = 0$ and U bounds a disk D that intersects K exactly in two points. Then, we already know that the result of $\pm-$surgery on U is homoeomorphic to $\mathbb{S}^3$. But the image of K on $\mathbb{S}^3_{\pm}(U)$ is obtained by K after applying a $\mp$-crossing change at the points of K intersected by D.

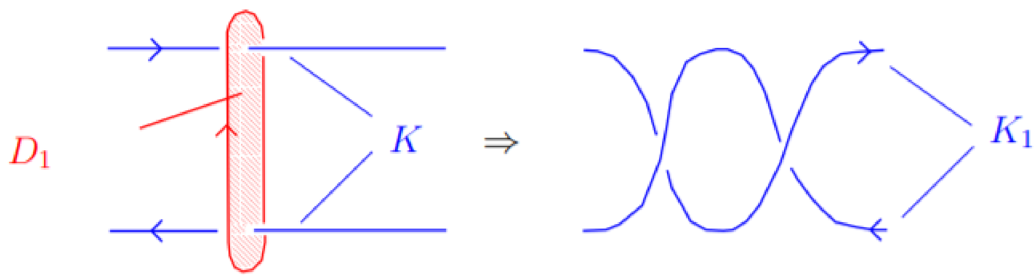


Figure 3: Positive surgery on K. [11]

Chapter 2

The Blanchfield form and the $n(K)$ invariant

In this chapter, we will explain in detail important concepts. Such as the twisted homology, the Blanchfield form and the $n(K)$ invariant. Then, we will state the main theorem and prove part of it. As a corollary we will obtain the $n(K) = 1$ version. And finally, using the previous results, we will prove that the 8_{10} knot has algebraic unknotting number equal to 1. In order to understand the proof one should be familiarised with twisted homology and spectral sequences. However, in order to understand the statements, it is enough to be familiarised with algebraic topology and basic knot theory.

2.1 Homologies of complexes over $\mathbb{Z}[t^{\pm 1}]$

Let C_* be any chain of finitely generated free $\mathbb{Z}[t^{\pm 1}]-$modules and let M be any $\mathbb{Z}[t^{\pm 1}]-$module. We can consider the corresponding homology and cohomology modules:

$$H_*(C; M) := H_*(C \otimes_{\mathbb{Z}[t^{\pm 1}]} M), \text{ and}$$
$$H^*(C; M) := H_*(\mathrm{Hom}_{\mathbb{Z}[t^{\pm 1}]}(C_*, M)).$$

By [12] (theorem 2.3) there is a spectral sequence $E^r_{p,q}$ with

$$E^2_{p,q} = \mathrm{Ext}^p_{\mathbb{Z}[t^{\pm 1}]}(H_q(C), M),$$

which converges to $H^*(C, M)$. This spectral sequence is called the Universal Coefficient Spectral Sequence, or UCSS for abrevation. We note that for any two $\mathbb{Z}[t^{\pm1}]-$modules H and M, the module $\mathrm{Ext}^0_{\mathbb{Z}[t^{\pm1}]}(H,M)$ is canonically isomorphic to $\mathrm{Hom}_{\mathbb{Z}[t^{\pm1}]}(H,M)$.

Also, since $\mathbb{Z}[t^{\pm1}]$ has cohomological dimension 2, we have that

$$\mathrm{Ext}^p_{\mathbb{Z}[t^{\pm1}]}(H,M)=0$$

for any $p>2$. Finally, note that considering $\mathbb{Z}$ as a $\mathbb{Z}[t^{\pm1}]-$module with trivial $t-$action, then $\mathbb{Z}$ admits a resolution of length 1, in particular

$$\mathrm{Ext}^p_{\mathbb{Z}[t^{\pm1}]}(\mathbb{Z},N)=0$$

for any $p>1$.

2.2 Twisted homology, cohomology groups and Poincaré duality

Let X be a topological space and let $\phi:\pi_1(X)\to\langle t\rangle$ be an epimorphism into the infinite cyclic group generated by t. We denote by $\pi:\tilde{X}\to X$ the corresponding infinite cyclic covering of X. Given a subspace $Y\subseteq X$ we write $\tilde{Y}:=\pi^{-1}(Y)$.

The deck transformation induces a canonical $\mathbb{Z}[t^{\pm1}]-$action on $C_*(\tilde{X},\tilde{Y};\mathbb{Z})$ and, thus, we can view $C_*(\tilde{X},\tilde{Y};\mathbb{Z})$ as a $\mathbb{Z}[t^{\pm1}]-$module.

Now let M be a module over $\mathbb{Z}[t^{\pm1}]$. We can consider $H_*(X,Y;M)$ and $H^*(X,Y;M)$ to be the homology and the cohomology of the chain complexes $C_*(\tilde{X},\tilde{Y};\mathbb{Z})\otimes_{\mathbb{Z}[t^{\pm1}]}M$. In this text, we will be interested in the cases $M=\mathbb{Z}[t^{\pm1}]$ and $M=\mathbb{Z}_{(p)}[t^{\pm1}]$, for some p positive integer. This last one will not be important through this chapter but could be usefull for future computations.

Now, let $K\subseteq\mathbb{S}^3$ be an oriented knot, we denote by $\phi:\pi_1(X(K))\to\langle t\rangle$ the epimorphism given by sending the oriented meridian to t. For different choices of ϕ, the resulting modules $H_*(X,Y;\mathbb{Z}[t^{\pm1}])$ and $H^*(X,Y;\mathbb{Z}[t^{\pm1}])$ will be anti-isomorphic, i.e. multiplication by t in one module corresponds to multiplication by t^{-1} in the other module. Since this does not affect any of the arguments we will usually not record the choice of ϕ in our notation.

Theorem 2.2.1 (Poincaré duality for twisted homology)**.** *Let X be an orientable n-manifold and W a union of components of ∂X. Then, for any $\mathbb{Z}[t^{\pm1}]$-module M, the Poincaré duality [13] (chapter 2) defines an isomorphism of $\mathbb{Z}[t^{\pm1}]$-modules*

$$H_i(X,W;M)\cong\overline{H^{n-i}(X,\partial X\setminus W;M)}.$$

In particular, for $W = \emptyset$, we get the canonical isomorphism

$$H_i(X; M) \cong \overline{H^{n-i}(X, \partial X; M)}.$$

Here, given a $\mathbb{Z}[t^{\pm 1}]-$module N, we denote by $\overline{N}$ to be the same abelian group as N but with involuted $\mathbb{Z}[t^{\pm 1}]-$action, i.e, multiplication by t on $\overline{N}$ corresponds to multiplication by t^{-1} on N.

2.3 The homological definition of the Blanchfield form

Let $K \subseteq \mathbb{S}^3$ be a knot. We consider the following sequence of maps:

$$\begin{aligned}\Phi : H_1(X(K), \mathbb{Z}[t^{\pm 1}]) &\to H_1(X(K), \partial X(K), \mathbb{Z}[t^{\pm 1}]) \\ &\to \overline{H^2(X(K), \mathbb{Z}[t^{\pm 1}])} \xleftarrow{\cong} \overline{H^1(X(K), \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}])} \\ &\to \overline{\mathrm{Hom}_{\mathbb{Z}[t^{\pm 1}]}(H_1(X(K), \mathbb{Z}[t^{\pm 1}]), \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}])}.\end{aligned}$$

Here the first map is the inclusion induced map, the second map comes from Poincaré's duality isomorphism, the third map comes from the long exact sequence in cohomology corresponding to the coefficients

$$0 \to \mathbb{Z}[t^{\pm 1}] \to \mathbb{Q}(t) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \to 0,$$

and the last map is the evaluation map. All these maps are isomorphisms, and hence define a non-singular form

$$\begin{aligned}\mathrm{Bl}(K) : H_1(X(K); \mathbb{Z}[t^{\pm 1}]) \times H_1(X(K), \mathbb{Z}[t^{\pm 1}]) &\to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ (a, b) &\mapsto \Phi(a)(b)\end{aligned}$$

called the **Blanchfield form** of K (over $\mathbb{Z}[t^{\pm 1}]$). This form is well-known to be Hermitian, in particular $\mathrm{Bl}(K)(a_1, a_2) = \overline{\mathrm{Bl}(K)(a_2, a_1)}$ and $\mathrm{Bl}(K)(\mu_1 a_1, \mu_2 a_2) = \overline{\mu_1}\mathrm{Bl}(K)(a_1, a_2)\mu_2$ for $\mu_i \in \mathbb{Z}[t^{\pm 1}]$, $a_i \in H_1(X(K); \mathbb{Z}[t^{\pm 1}])$.

Observe that we can define the Blanchfield form over $\mathbb{Z}_{(p)}[t^{\pm 1}]$ analogously, obtaning

$$\begin{aligned}\mathrm{Bl}(K, \mathbb{Z}_{(p)}[t^{\pm 1}]) : H_1(X(K); \mathbb{Z}_{(p)}[t^{\pm 1}]) \times H_1(X(K), \mathbb{Z}_{(p)}[t^{\pm 1}]) &\to \mathbb{Q}(t)/\mathbb{Z}_{(p)}[t^{\pm 1}] \\ (a, b) &\mapsto \Phi(a)(b).\end{aligned}$$

Lemma 2.3.1. *$Bl(K, \mathbb{Z}_{(p)}[t^{\pm 1}])$ is isometric to $Bl(K) \otimes id_{\mathbb{Z}_{(p)}[t^{\pm 1}]}$.*

Proof. Follows from the definitions and from the fact that $\mathbb{Z}_{(p)}[t^{\pm 1}]$ is flat over $\mathbb{Z}[t^{\pm 1}]$. □

Lemma 2.3.2. *Let V be the Seifert surface of K and $A = tV - V^T$. Then, the Blanchfield form over $\mathbb{Z}[t^{\pm 1}]$ is isometric to the following form*

$$\begin{aligned} \mathbb{Z}[t^{\pm 1}]^{2g}/A\mathbb{Z}[t^{\pm 1}]^{2g} \times \mathbb{Z}[t^{\pm 1}]^{2g}/A\mathbb{Z}[t^{\pm 1}]^{2g} &\to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ (a,b) &\mapsto \overline{a}^T(t-1)A^{-1}b. \end{aligned}$$

In particular, $Bl(K, \mathbb{Z}_{(p)}[t^{\pm 1}])$ is isometric to $\lambda(A) \otimes id_{\mathbb{Z}_{(p)}[t^{\pm 1}]}$.

Proof. See [14], section 8. □

Definition 2.3.3 (Invariants $n(K)$ and $n_p(K)$)**.** Given an Hermitian $n \times n-$matrix A over $\mathbb{Z}[t^{\pm 1}]$ (resp. over $\mathbb{Z}_{(p)}[t^{\pm 1}]$ for p a positive integer) with $\det A \neq 0$, we denote by $\lambda(A)$ the pairing

$$\begin{aligned} \lambda(A) : \mathbb{Z}[t^{\pm 1}]^n/A\mathbb{Z}[t^{\pm 1}]^n \times \mathbb{Z}[t^{\pm 1}]^n/A\mathbb{Z}[t^{\pm 1}]^n &\to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ (a,b) &\mapsto \overline{a}^T A^{-1} b, \end{aligned}$$

(resp. $\lambda(A) \otimes_{\mathbb{Z}[t^{\pm 1}} \mathbb{Z}_{(p)}[t^{\pm 1}]$) where a and b represent column vectors in $\mathbb{Z}[t^{\pm 1}]^n$. Note that $\lambda(A)$ is a non-singular Hermitian pairing.

Let K be a knot. We define $n(K)$ (resp. $n_p(K)$) to be the minimal size of a Hermitian matrix A over $\mathbb{Z}[t^{\pm 1}]$ (resp. $\mathbb{Z}_{(p)}[t^{\pm 1}]$) such that

- $\lambda(A) \cong \mathrm{Bl}(K)$ (resp. $Bl(K, \mathbb{Z}_{(p)}[t^{\pm 1}])$).
- The matrix $A(1)$ is congruent over $\mathbb{Z}$ (resp. $\mathbb{Z}_{(p)}$) to a diagonal matrix which has $\pm 1's$ (resp. $\pm p^n$ for some $n \in \mathbb{Z}$) on the diagonal.

Observation 2.3.4.
- Clearly we have that $n_p(K) \leq n(K)$.
- $n(K) = 0$ if and only if the Alexander polynomial of K is trivial.
- $n(K)$ subsumes the Nakanishi index.

From the following lemma we will obtain that $n(K)$ is actually defined.

Lemma 2.3.5. *For any knot K we have that $n(K) \leq deg\ \Delta_K(t) + 1$.*

Proof. Follows from lemma 2.4.1 and the fact that every indefinite, odd symmetric bilinear pairing over $\mathbb{Z}$ is diagonalizable. □

Now we are able to state the main theorem.

Theorem 2.3.6. *Let $K \subseteq \mathbb{S}^3$ be a knot, then*

$$n(K) = u_a(K).$$

In the next section we will prove that $n(K) \leq u_a(K)$ following the arguments given by [15]. The proof of the other inequality can be seen in [11].

Clearly, the invariant $n(K)$ is very difficult to compute in general, however, the case where $n(K)$ equals to 1 has an easier description.

Corollary 2.3.7. *Let $K \subseteq \mathbb{S}^3$ be a knot with Nakanishi index 1 and the associated Blanchfield form is isometric to $Bl(K) : \mathbb{Z}[t^{\pm 1}]/\Delta_K \times \mathbb{Z}[t^{\pm 1}]/\Delta_K \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$ defined as $(a, b) \mapsto \overline{a}\frac{q(t)}{\Delta_K(t)}b$, for $q(t)$ a symmetric element of $\mathbb{Z}[t^{\pm 1}]$. Then the following are equivalent:*

- $u_a(K) = 1$.
- *There exists $f(t)$ an element of $\mathbb{Z}[t^{\pm 1}]$ such that*

$$f(t)f(t^{-1})q(t) \equiv \pm 1 \pmod{\Delta_K \mathbb{Z}[t^{\pm 1}]}.$$

Proof. Since $m(K) = 1$, we have that $H_1(X(K), \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z}[t^{\pm 1}]/\Delta_K$. Observe that the automorphisms of a cyclic module are given by the multiplication of an element of the module. So $n(K) = 1$ if and only if there exists $\phi : \mathbb{Z}[t^{\pm 1}]/\Delta_K \to \mathbb{Z}[t^{\pm 1}]/\Delta_K$ an automorphism given by $x \mapsto f(t)x$ for $f(t) \in (\mathbb{Z}[t^{\pm 1}]/\Delta_K(t))^\times$ such that $Bl(K)(\phi(a_1)), \phi(a_2)) = \pm\overline{a_1}\frac{1}{\Delta_K}a_2$ for all $a_1, a_2 \in \mathbb{Z}[t^{\pm 1}]/\Delta_K$. In particular, we have that

$$\frac{\overline{f(t)}q(t)f(t)}{\Delta_K(t)} \equiv \pm\frac{1}{\Delta_K} \pmod{\mathbb{Z}[t^{\pm 1}]},$$

or equivalently $f(t)\overline{f(t)}q(t) \equiv \pm 1 \pmod{\Delta_K \mathbb{Z}[t^{\pm 1}]}$. The statement follows from 2.3.6. $\square$

For some knots, we are not able to determine whether the Nakanishi index is 1 or 2. In these cases, it could be useful to extend the ring $\mathbb{Z}[t^{\pm 1}]$ to $\mathbb{Z}_{(p)}[t^{\pm 1}]$ for a positive integer p such that the matrix $A = tV - V^T$ is $\mathbb{Z}_{(p)}[t^{\pm 1}]$-equivalent to a 1×1-matrix so that we can use the next observation. In chapter 4 we will discuss this particular situation.

Observation 2.3.8. If the module $H_1(X(K); \mathbb{Z}_{(p)}[t^{\pm 1}])$ has one generator, then we can describe the Blanchfield form over $\mathbb{Z}_{(p)}[t^{\pm 1}]$

$$\mathrm{Bl}(K, \mathbb{Z}_{(p)}[t^{\pm 1}]) : \mathbb{Z}_{(p)}[t^{\pm 1}]/\Delta_K \times \mathbb{Z}_{(p)}[t^{\pm 1}]/\Delta_K \to \mathbb{Q}(t)/\mathbb{Z}_{(p)}[t^{\pm 1}]$$

as $(a, b) \mapsto \overline{a}\frac{q(t)}{\Delta_K(t)}b$. Then similarly to 2.3.7, the following are equivalent:

- $n_p(K) = 1$.
- There exists f an element of $\mathbb{Z}_{(p)}[t^{\pm 1}]$ and $n \in \mathbb{Z}$ such that

$$f(t)f(t^{-1})q \equiv \pm p^n \pmod{\Delta_K \mathbb{Z}_{(p)}[t^{\pm 1}]}.$$

In particular, if the equation does not a have solution we obtain that $u_a(K) \geq 2$.

2.4 Proof $n(K) \leq u_a(K)$

In order to prove the inequality, we will show that, for a given knot K that can be turned into a knot with trivial Alexander polynomial using u_+ positive and u_- negative crossing changes, the $0-$framed surgery cobounds a $4-$manifold with certain properties. We will then show that a matrix representing the equivariant intersection pairing on such $4-$manifold gives, in fact, a presentation matrix for the Blanchfield pairing of K. Before that, we will give a description of the Blanchfield form that will be used in the proof of the theorem.

Let V be any matrix of size $2k$ which is $S-$equivalent to a Sefiert matrix for K. Note that $V - V^T$ is antisymetric and it satisfies $\det(V - V^T) = (-1)^k$. It is well known that, possibly after replacing V by PVP^T for an appropriate P (see [6] for details) , the following equality holds:

$$V - V^T = \begin{pmatrix} 0 & \mathrm{Id}_k \\ -\mathrm{Id}_k & 0 \end{pmatrix}.$$

We now define $A_K(t)$ to be the matrix

$$\begin{pmatrix} (1-t^{-1})^{-1}\mathrm{Id}_k & 0 \\ 0 & \mathrm{Id}_k \end{pmatrix} V \begin{pmatrix} \mathrm{Id}_k & 0 \\ 0 & (1-t)\mathrm{Id}_k \end{pmatrix} +$$
$$+ \begin{pmatrix} \mathrm{Id}_k & 0 \\ 0 & (1-t^{-1})\mathrm{Id}_k \end{pmatrix} V^T \begin{pmatrix} (1-t)^{-1}\mathrm{Id}_k & 0 \\ 0 & \mathrm{Id}_k \end{pmatrix}.$$

Note that $A_K(t)$ is an Hermitian matrix defined over $\mathbb{Z}[t^{\pm 1}]$ and that $\det A_K(1) = (-1)^k$. Also, we have

$$\begin{pmatrix} (1-t^{-1})\mathrm{Id}_k & 0 \\ 0 & \mathrm{Id}_k \end{pmatrix} A_K(t) \begin{pmatrix} (1-t)\mathrm{Id}_k & 0 \\ 0 & \mathrm{Id}_K \end{pmatrix} = (1-t)V + (1-t^{-1})V^T.$$

Lemma 2.4.1. *Let K be a knot and $A_K(t)$ as above. Then $\lambda(A_K(t)) \cong \lambda(K)$.*

Proof. We write $\Lambda_0 = \mathbb{Z}[t^{\pm 1}, (1-t)^{-1}]$, $\Lambda = \mathbb{Z}[t^{\pm 1}]$ and $\Omega = \mathbb{Q}(t)$. We let

$$P := \begin{pmatrix} t^{-1}\mathrm{Id}_k & 0 \\ 0 & (t-1)^{-1}\mathrm{Id}_K \end{pmatrix}.$$

We consider now the following diagram:

$$\begin{tikzcd}
\Lambda^{2k}/A_K(t)\Lambda^{2k} \times \Lambda^{2k}/A_K(t)\Lambda^{2k} \arrow[r, "A_K(t)^{-1}"] \arrow[d] & \Omega/\Lambda \arrow[d] \\
\Lambda_0^{2k}/A_K(t)\Lambda_0^{2k} \times \Lambda_0^{2k}/A_K(t)\Lambda_0^{2k} \arrow[r, "A_K(t)^{-1}"] \arrow[d, "(v,w)\mapsto(Pv,Pw)"] & \Omega/\Lambda_0 \arrow[d] \\
\Lambda_0^{2k}/PA_K(t)\Lambda_0^{2k} \times \Lambda_0^{2k}/PA_K(t)\Lambda_0^{2k} \arrow[r, "(PA_K(t)\overline{P}^T)^{-1}"] \arrow[d, "="] & \Omega/\Lambda_0 \arrow[d] \\
\Lambda_0^{2k}/(tV-V^T)\Lambda_0^{2k} \times \Lambda_0^{2k}/(tV-V^T)\Lambda_0^{2k} \arrow[r, "(t-1)(tV-V^T)^{-1}"] & \Omega/\Lambda_0 \\
\Lambda^{2k}/(tV-V^T)\Lambda^{2k} \times \Lambda^{2k}/(tV-V^T)\Lambda^{2k} \arrow[r, "(t-1)(tV-V^T)^{-1}"] \arrow[u] & \Omega/\Lambda \arrow[u]
\end{tikzcd}$$

Here, the top vertical maps and the bottom vertical maps are induced by the inclusion $\Lambda \to \Lambda_0$. Now we will prove that these vertical maps are isomorphisms. We have that the determinant of $tV - V^T$ and $A_K(t)$ are equal, up to sign, to the Alexander polynomial $\Delta_K(t)$, therefore we have that $\pm\Delta_K(t)c = 0$ for all $c \in \Lambda^{2k}/A_K(t)\Lambda^{2k}$ and $c \in \Lambda^{2k}/(tV-V^T)\Lambda^{2k}$. Since $\Delta_K(1) = \pm 1$, we have that

$$\mp\Delta_K(t) \pm 1 = q(t)(t-1)$$

for some $q(t) \in \mathbb{Z}[t^{\pm 1}]$. Therefore we deduce that

$$q(t)(t-1)c = \pm c,$$

so $(t-1)$ is an isomorphism of the modules $\Lambda^{2k}/A_K(t)\Lambda^{2k}$ and $\Lambda^{2k}/(tV - V^T)\Lambda^{2k}$. It follows that the aforementioned vertical maps are isomorphisms of Λ-modules. For the third vertical map we use the fact that

$$PA_K(t)\overline{P}^T = (t-1)^{-1}(tV - V^T)$$

and that

$$PA_K(t)\Lambda_0^{2k} = PA_K(t)\overline{P}^T\Lambda_0^{2k} = (t-1)^{-1}(tV - V^T)\Lambda_0^{2k} = (tV - V^T)\Lambda_0^{2k}.$$

Since all vertical maps on the left side in the above commutative diagram are isomorphisms, we deduce that $\lambda(A_K(t)) \cong \lambda(K)$. □

Given a knot $K \subseteq \mathbb{S}^3$ we denote by $M(K)$ the $0-$framed surgery on K. Furthermore, given a topological 4-manifold W with boundary M, we consider the following sequence of maps

$$H_2(W;\mathbb{Z}) \xrightarrow{i} H_2(W,M;\mathbb{Z}) \xrightarrow{\mathrm{PD}} H^2(W,\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathbb{Z}}(H_2(W;\mathbb{Z}),\mathbb{Z}),$$

where i denotes the inclusion induced map, PD denotes the Poincaré duality isomorphism and ev denotes the evaluation map. This defines a pairing

$$H_2(W;\mathbb{Z}) \times H_2(W;\mathbb{Z}) \to \mathbb{Z},$$

called **ordinary intersection pairing** of W, which is well-known to be symmetric. Now we give a technical lemma that we will use implicitly several times.

Lemma 2.4.2. *Suppose the following holds:*

- *M is connected,*
- *$H_1(M;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is an isomorphism,*
- *$H_1(W;\mathbb{Z})$ is torsion-free,*

then the ordinary intersection pairing is non-singular.

Proof. We have that PD is an isomorphism by Poincaré's duality theorem and since $H_1(W;\mathbb{Z})$ is torsion-free, thanks to the Universal Coefficient Theorem, we have that ev : $H^2(W;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_2(W;\mathbb{Z}),\mathbb{Z})$ is an isomorphism. The assumption that $H_1(M;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is an isomorphism implies by Poincaré's duality theorem that $H^2(M;\mathbb{Z}) \to H^3(W,M;\mathbb{Z})$ is an isomorphism. From the Universal Coefficient Theorem it also follows that $\mathrm{Hom}_{\mathbb{Z}}(H_2(M;\mathbb{Z}),\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_3(W,M;\mathbb{Z}),\mathbb{Z})$ is an isomorphism. But $H_2(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z})$ and $H_3(W,M;\mathbb{Z}) \cong H^1(W;\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(W;\mathbb{Z}),\mathbb{Z})$ are torsion-free. Thus, it follows that $H_3(W,M;\mathbb{Z}) \to H_2(M;\mathbb{Z})$ is an isomorphism as well. It now follows that the ordinary intersection pairing is non-singular. □

We now consider a topological connected 4-manifold W with boundary M such that $\pi_1(W) \cong \mathbb{Z}$. We then consider the following sequence of maps

$$H_2(W;\mathbb{Z}[t^{\pm 1}]) \xrightarrow{i} H_2(W,M;\mathbb{Z}[t^{\pm 1}]) \xrightarrow{\text{PD}} \overline{H^2(W;\mathbb{Z}[t^{\pm 1}])} \xrightarrow{\text{ev}} \overline{\text{Hom}_{\mathbb{Z}[t^{\pm 1}]}(H_2(W;\mathbb{Z}[t^{\pm 1}]),\mathbb{Z}[t^{\pm 1}])}, \tag{2.4.1}$$

where the first map is again the inclusion map, the second map is Poincaré's duality isomorphism and the third map is the evaluation map. The composition map defines a pairing

$$H_2(W;\mathbb{Z}[t^{\pm 1}]) \times H_2(W;\mathbb{Z}[t^{\pm 1}]) \to \mathbb{Z}[t^{\pm 1}],$$

which is well-known to be Hermitian. We refer to this pairing as the **twisted intersection pairing** on W.

Definition 2.4.3. Let K be a knot and $M(K)$ the 0-framed surgery on K. We shall say that a 4-manifold W **tamely cobounds** $M(K)$ if the following conditions are satisfied:

- $\partial W = M(K)$,
- the inclusion map $H_1(M(K);\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is an isomorphism,
- $\pi_1(W) \cong \mathbb{Z}$.

Furthermore, if the intersection form on $H_2(W;\mathbb{Z})$ is diagonalizable, we say that W **strictly cobounds** $M(K)$.

The proof of the inequality has two key ingredients that we will expose in form of theorems.

Theorem 2.4.4. *Let K be a knot. Suppose there exists a topological 4-manifold W, which tamely cobounds $M(K)$. Then $H_2(W;\mathbb{Z}[t^{\pm 1}])$ is free of rank $b_2(W) := rk_{\mathbb{Z}} H_2(W;\mathbb{Z})$. Furthermore, if B is an integral matrix representing the ordinary intersection pairing of W, then there exists a basis $\mathcal{B}$ for $H_2(W;\mathbb{Z}[t^{\pm 1}])$ such that the matrix $A(t)$ representing the twisted intersection pairing with respect to $\mathcal{B}$ has the following two properties:*

- *$\lambda(A(t)) \cong \lambda(K)$.*
- *$A(1) = B$.*

We will divide the proof of this theorem in different lemmas. From now on K will denote a knot, W a connected 4−manifold that tamely cobounds $M(K)$, M will denote $M(K)$, Λ will denote $\mathbb{Z}[t^{\pm 1}]$ and Ω will denote $\mathbb{Q}(t)$.

Lemma 2.4.5. *The $\Lambda-$module $H_2(W;\Lambda)$ is free of rank $b_2(W)$, i.e, has the same rank as $H_2(W;\mathbb{Z})$.*

Proof. We first show that $H_2(W;\Lambda)$ is free. Note that $H_2(W;\Lambda)$ is a finitely generated $\Lambda-$module since Λ is Noetherian. By [16] (Corollary 3.7) the module $H_2(W;\Lambda)$ is free if and only if $\mathrm{Ext}^i_\Lambda(H_2(W;\Lambda),\Lambda)=0$ for $i=1,2$.

Note that $\pi_1(W)\cong\mathbb{Z}$ implies that $H_1(W;\Lambda)=0$. We also have $H_4(W;\Lambda)=0$. Furthermore, we have an isomorphism $H_0(M;\Lambda)\to H_0(W;\Lambda)$. Thus, from the long exact homology sequence corresponding to the pair (W,M), we conclude that $H_0(W,M;\Lambda)=H_1(W,M;\Lambda)=0$.

Recall that the UCSS given before (see section 2.1) starts with $E^2_{p,q}=\mathrm{Ext}^p_\Lambda(H_q(W;\Lambda),\Lambda)$ and converges to $H^*(W;\Lambda)$. Furthermore the differentials have degree $(1-r,r)$. By the statments above we have $E^2_{p,q}=0$ for $q=1$ and $q=4$. Since Λ has cohomological dimension 2 we also have $E^2_{p,q}=0$ for $p>2$. Finally note that

$$E^2_{2,0}=\mathrm{Ext}^2_\Lambda(H_0(W;\Lambda),\Lambda)=\mathrm{Ext}^2_\Lambda(\Lambda/(t-1)\Lambda,\Lambda)=0.$$

It now follows from the UCSS that we have a monomorphism

$$E^2_{1,2}=\mathrm{Ext}^1_\Lambda(H_2(W;\Lambda),\Lambda)\to H^3(W;\Lambda).$$

But $H^3(W;\Lambda)\cong\overline{H_1(W,M;\Lambda)}=0$. Similarly, it follows from the UCSS that we have another monomorphism

$$E^2_{2,2}=\mathrm{Ext}^2_\Lambda(H_2(W,\Lambda),\Lambda)\to H^4(W;\Lambda).$$

But $H^4(W;\Lambda)\cong\overline{H_0(W,M;\Lambda)}=0$. This concludes the proof of the claim that $H_2(W;\Lambda)$ is a free $\mathbb{Z}[t^{\pm1}]-$module. Now will prove that $H_2(W;\Lambda)$ is a free Λ-module of rank $s:=b_2(W)$. Since Ω is flat over Λ, it suffices to show that $\dim_\Omega(H_2(W;\Omega))=s$. It is clear that $H_i(W;\Omega)=0$ for $i=0,1,4$. Furthermore, $H_3(W;\Omega)\cong\overline{H^1(W,M;\Omega)}$, but since Ω is field, we have by the Universal Coefficient Theorem that the last term is isomorphic to $\overline{H_1(W,M;\Omega)}$ which is zero. Thus, we can calculate

$$\dim_\Omega(H_2(W;\Omega))=\sum_{i=0}^4(-1)^i\dim_\Omega(H_i(W;\Omega))=\chi(W).$$

Now note that $b_0(W)=b_1(W)=1$ and $b_4(W)=0$. Also, since we assume that $H_1(M;\mathbb{Z})\to H_1(W;\mathbb{Z})$ is an isomorphism, we have that $H^3(W;\mathbb{Z})\cong H_1(W,M;\mathbb{Z})=0$. Thus, it follows that $b_3(W)=0$ and we see that $\chi(W)=b_2(W)=s$. This concludes the proof of the lemma. □

Now we write $s = b_2(W)$. We pick a basis $\mathcal{B}$ for $H_2(W;\Lambda)$ and denote by $A = A(t)$ the corresponding $s \times s-$matrix representing the twisted intersection pairing. Note that A is a Hermitian $s \times s-$matrix. By the argument given on [17] we see that the matrix $A(1)$ represents the ordinary intersection pairing on $H_2(W;\mathbb{Z})$. In particular there exists an integral matrix P such that $PA(1)P^T = B$. After acting on the basis $\mathcal{B}$ by the matrix P we can assume, without loss of generality, that $A(1) = B$. The following lemma concludes the proof of the theorem 2.4.4.

Lemma 2.4.6. *The pairing $\lambda(A)$ is isometric to $\lambda(K)$.*

In order to prove this lemma we first want to prove the following claim.
Claim. The following:

$$0 \to H_2(W;\Lambda) \to H_2(W,M;\Lambda) \to H_1(M;\Lambda) \to 0$$

is a short exact sequence.

In order to prove this claim we first consider the following exact sequence

$$H_2(W;\Lambda) \to H_2(W,M;\Lambda) \to H_1(M;\Lambda) \to H_1(W;\Lambda) \to ...$$

Recall that $H_1(W;\Lambda) = 0$. Also, note that

$$H_2(M;\Lambda) \otimes_\Lambda \Omega \cong H_2(M;\Omega) \cong \overline{H^1(M;\Omega)} \cong \overline{\mathrm{Hom}_\Omega(H_1(M;\Omega),\Omega)} = 0$$

due to $H_1(M;\Omega) \cong H_1(M;\Lambda) \otimes_\Lambda \Omega = 0$ (using that $\Delta_K(t)H_1(M;\Lambda) = 0$, we deduce that $H_1(M;\Lambda)$ is torsion). In particular, we have that $H_2(M;\Lambda)$ is torsion and the map $H_2(M;\Lambda) \to H_2(W;\Lambda)$ is trivial since $H_2(W;\Lambda)$ is a free module. This concludes the proof of the claim.

Now we define the Blanchfield pairing on $H_1(M;\Lambda)$ and the intersection pairing on $H_2(W,M;\Lambda)$. First of all, similar to what we did in the beginning of this section, we can consider the following sequence of isomorphisms:

$$H_1(M;\Lambda) \overset{PD}{\to} \overline{H^2(M;\Lambda)} \overset{\cong}{\leftarrow} \overline{H^1(M;\Omega/\Lambda)} \overset{\mathrm{ev}}{\to} \mathrm{Hom}_\Lambda(H_1(M;\Lambda),\Omega/\Lambda).$$

This defines a Hermitian non-singular pairing

$$H_1(M;\Lambda) \times H_1(M;\Lambda) \to \Omega/\Lambda.$$

It is well-known that the natural map $H_1(X(K);\Lambda) \to H_1(M;\Lambda)$ is an isomorphism, and it follows immediately that the Blanchfield pairing on $X(K)$ is isometric to the pairing on M.

Secondly, we can consider the following sequence of maps

$$\begin{aligned} H_2(W,M;\Lambda) &\overset{PD}{\to} \overline{H^2(W;\Lambda)} \to \overline{H^2(W;\Omega)} \cong \overline{H^2(W,M;\Omega)} \\ &\overset{\mathrm{ev}}{\to} \overline{\mathrm{Hom}(H^2(W,M;\Lambda),\Omega)}. \end{aligned}$$

The isomorphism is deduced from the fact that $H_1(M;\Lambda)$ is $\Lambda-$torsion. Therefore, from the long exact sequence (W,M), we have that the inclusion induced map $H^2(W,M;\Omega) \to H^2(W;\Omega)$ is an isomorphism. The other maps are given by Poincaré's duality theorem, the inclusion of rings and the evaluation homomorphism. This sequence of maps defines an Hermitian pairing

$$H_2(W,M;\Lambda) \times H_2(W,M;\Lambda) \to \Omega.$$

Claim. The intersection pairing on W, the intersection pairing on $H_2(W,M;\Lambda)$ and the Blanchfield pairing on M fit into the following commutative diagram, where the left vertical maps form a short exact sequence.

$$\begin{array}{ccc} H_2(W;\Lambda) \times H_2(W;\Lambda) & \longrightarrow & \Lambda \\ \downarrow & & \downarrow \\ H_2(W,M;\Lambda) \times H_2(W,M;\Lambda) & \longrightarrow & \Omega \\ \downarrow & & \downarrow \\ H_1(M;\Lambda) \times H_1(M;\Lambda) & \longrightarrow & \Omega/\Lambda \end{array} \tag{2.4.2}$$

Proof. In the previous claim we already showed that the left vertical maps form a short exact sequence. Now consider the following diagram

$$\begin{array}{ccc} H_2(W;\Lambda) \times H_2(W;\Lambda) & \longrightarrow & \Lambda \\ \downarrow & & \downarrow \\ H_2(W,\Omega) \times H_2(W,\Omega) & \longrightarrow & \Omega \\ \downarrow & & \downarrow \\ H_2(W,M;\Omega) \times H_2(W,M;\Omega) & \longrightarrow & \Omega \\ \uparrow & & \uparrow \\ H_2(W,M;\Lambda) \times H_2(W,M;\Lambda) & \longrightarrow & \Omega. \end{array}$$

The pairings on Ω-homology are defined analogously to the corresponding pairings on $\Lambda-$homology, and the vertical maps are the obvious maps. It follows easily from the definitions that this is a commutative diagram. Since the image of $H_2(W;\Lambda) \to H_2(W,M;\Omega)$ lies in the image of $H_2(W,M;\Lambda) \to H_2(W,M;\Omega)$ it follows that the top square in the diagram of the claim commutes.

Now, consider the following diagram

$$\begin{tikzcd}
H_2(W,M;\Lambda) \arrow[r] \arrow[d] & H_1(M;\Lambda) \arrow[d] \\
\overline{H^2(W;\Lambda)} \arrow[d] & \overline{H^2(M;\Lambda)} \\
\mathrm{Hom}(H_2(W,M;\Lambda),\Omega) \arrow[d] & \overline{H^1(M;\Omega/\Lambda)} \arrow[u] \arrow[d] \\
\overline{\mathrm{Hom}(H_2(W,M;\Lambda),\Omega/\Lambda)} & \overline{\mathrm{Hom}(H_1(M;\Lambda),\Omega/\Lambda)} \arrow[l]
\end{tikzcd}$$

where the left middle vertical map is part of the definition of the intersection pairing on $H_2(W,M;\Lambda)$. Moreover, the horizontal maps are the maps induced by long exact sequence corresponding to the pair (W,M). By [18] (section 6) this diagram commutes. This implies that the lower square in the claim also commutes. $\square$

Claim. The evaluation map

$$H^2(W;\Lambda) \stackrel{\mathrm{ev}}{\to} \mathrm{Hom}_\Lambda(H_2(W;\Lambda),\Lambda)$$

is an isomorphism.

Proof. In order to prove the claim we have to study the UCSS corresponding to $H^2(W;\Lambda)$. Note that $\mathrm{Ext}^1_\Lambda(H_0(W;\Lambda)) = \Lambda/(t-1)\Lambda$ is $\Lambda-$torsion, hence the differential

$$d_2 : E^2_{1,0} = \mathrm{Ext}^1_\Lambda(H_0(W;\Lambda),\Lambda) \to E^2_{0,2} = \mathrm{Ext}^0_\Lambda(H_2(W;\Lambda),\Lambda)$$

is zero since $\mathrm{Ext}^0_\Lambda(H_2(W;\Lambda),\Lambda) = \mathrm{Hom}_\Lambda(H_2(W;\Lambda),\Lambda)$ is $\Lambda-$torsion free. It follows (using the earlier discussion) that the UCSS for $H^2(W;\Lambda)$ gives rise to the desired isomorphism

$$H^2(W;\Lambda) \stackrel{\cong}{\to} \mathrm{Ext}^0_\Lambda(H_2(W;\Lambda),\Lambda) = \mathrm{Hom}_\Lambda(H_2(W;\Lambda),\Lambda). \tag{2.4.3}$$

$\square$

Recall that we picked a basis $\mathcal{B}$ for $H_2(W;\Lambda)$ and that we denote by $A = A(t)$ the corresponding matrix representing the twisted intersection pairing on $H_2(W;\Lambda)$. Note that by Poincaré's duality theorem and by the above claim we have two isomorphisms

$$H_2(W,M;\Lambda) \stackrel{PD}{\to} \overline{H^2(W;\Lambda)} \stackrel{ev}{\to} \overline{\mathrm{Hom}_\Lambda(H_2(W;\Lambda),\Lambda)}. \tag{2.4.4}$$

Now, we endow $H_2(W,M;\Lambda)$ with the basis $\mathcal{C}$ which is dual to $\mathcal{B}$. It follows easily from (2.4.1) and (2.4.4) that the inclusion induced map $H_2(W;\Lambda) \to H_2(W,M;\Lambda)$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ is given by A.

We now rewrite the diagram 2.4.2 in terms of our bases, obtaining the following diagram

$$\begin{tikzcd}
\Lambda^s \times \Lambda^s \arrow[r, "(v,w)\mapsto \overline{v}^T A w"] \arrow[d, "(v,w)\mapsto(Av,Aw)"] & \Lambda \arrow[d] \\
\Lambda^s \times \Lambda^s \arrow[r, "(v,w)\mapsto \overline{v}^T A^{-1} w"] \arrow[d] & \Omega \arrow[d] \\
H_1(M;\Lambda) \times H_1(M;\Lambda) \arrow[r] & \Omega/\Lambda.
\end{tikzcd}$$

Now the statement of lemma 2.4.6 follows from this diagram and the fact that the left vertical maps form a short exact sequence. This concludes the proof of theorem 2.4.4.□

Theorem 2.4.7. *Let K be a knot such that u_+ positive crossing changes and u_- negative crossing changes turn K into a knot J with trivial Alexander polynomial. Then there exists an oriented topological 4-manifold W which strictly cobounds $M(K)$. Moreover, the intersection pairing on $H_2(W;\mathbb{Z})$ is represented by a diagonal matrix of size $u_- + u_+$ such that u_+ entries are equal to -1 and u_- entries are equal to $+1$.*

Proof. We first recall the following interpretation of a crossing change. Let $K \subseteq \mathbb{S}^3$ be a knot and suppose we perform an $\epsilon-$crossing change along a crossing. We denote by $D \subseteq \mathbb{S}^3$ an embedded disk which intersects K in precisely two points with opposite orientations, one point on each strand involved in the crossing change. If we perform an $\epsilon-$surgery on the curve $c := \partial D$, then the resulting $3-$manifold Σ is diffeomorphic to $\mathbb{S}^3$ and $K \subseteq \Sigma$ is the result of performing an $\epsilon-$crossing change.

For the rest of the proof, we will use the following notation: let $c_1, ..., c_s$ be simple closed curves which form the unlink in $\mathbb{S}^3$ and let $\epsilon_1, ..., \epsilon_s \in \{1,-1\}$.

Then, we can denote by $\Sigma(c_1, ..., c_s, \epsilon_1, ..., \epsilon_s)$ the result of performing ϵ_i-surgery along the c_i for $i = 1, ..., s$. Note that this 3−manifold is isomorphic to the standard 3−sphere.

Let K be a knot such that u_+ positive crossing changes and u_- negative crossing changes turn K into a knot J with trivial Alexander polynomial. We write $s = u_+ + u_-$, $n_i = -1$ for $i = 1, ..., u_+$, and $n_i = 1$ for $i = u_+ + 1, ..., s$. By the above discussion there exists simple closed curves $c_1, ..., c_s$ in $X(J)$ with the following properties:

- $c_1, .., c_s$ are the unlink in $\mathbb{S}^3$,
- the linking numbers $lk(c_i, K)$ are zero,
- the image of J in $\Sigma(c_1, ..., c_s, n_1, ..., n_s)$ is the knot K.

Note that the curves lie in $\mathbb{S}^3 \setminus V(J)$ and therefore we can view them as lying in $M(J)$. The manifold $M(K)$ is then the result of n_i-surgeries on $c_i \subseteq M(J)$ for $i = 1, ..., s$.

Since J is a knot with trivial Alexander polynomial, it follows from Freedman's theorem [19] that J is topologically slice, in fact; there exists a locally flat slice disc $D \subseteq B^4$ for J such that $\pi_1(B^4 \setminus D) \cong \mathbb{Z}$. We now write $X := B^4 \setminus V(D)$. Then X is an oriented topological 4-manifold X with the following properties

1. $\partial X = M(J)$ as oriented manifolds,
2. $\pi_1(X) \cong \mathbb{Z}$,
3. $H_1(M(J); \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is an isomorphism,
4. $H_2(X; \mathbb{Z}) = 0$.

Let W be the 4−manifold which is obtained by adding 2−handles along $c_1, ..., c_s \subseteq M(J)$ with framings $n_1, ..., n_s$ to X. Then $\partial W \cong M(K)$ as oriented manifolds. From now on we will write $M := M(K)$. Since the curves $c_1, ..., c_s$ are nullhomologus, the map $H_1(M; \mathbb{Z}) \to H_1(W; \mathbb{Z})$ is an isomorphism and $\pi_1(W) \cong \mathbb{Z}$. Hence, the following is left to prove:

Claim. The ordinary intersection pairing on W is represented by a diagonal matrix of size s with u_+ diagonal entries equal to -1 and u_- diagonal entries equal to 1.

Recall that the curves $c_1, ..., c_s$ form the unlink in $\mathbb{S}^3$ and that the linking numbers $\mathrm{lk}(c_i, J)$ are zero. Therefore the curves $c_1, ..., c_s$ are also nullhomologus in $M(J)$. Thus we can find disjoint surfaces in $M(J) \times [0,1]$ such that $\partial F_i = c_i \times \{1\}$. By adding the cores of the 2−handles attached to c_i, we obtain closed surfaces $C_1, ..., C_s$ in W. It is clear that $C_i \cdot C_j = 0$ for $i \neq j$ and $C_i \cdot C_i = n_i$.

Using the Mayer-Vietoris sequence we will see that the surfaces $C_1, ..., C_s$ present a basis for $H_2(W; \mathbb{Z})$. Write $W := X \cup H$, where $H \cong \cup_{i=1}^s B^2 \times B^2$ is the set of 2−handles attached to $c_1, ..., c_s$. Then write $Y := X \cap H$, so that

$$Y = \cup_{i=1}^s N(c_i) \cong \cup_{i=1}^s (S^1 \times D^2).$$

We have from the Mayer-Vietoris sequence

$$... \to H_2(X) \oplus H_2(H) \to H_2(W) \to H_1(Y) \to H_1(X) \oplus H_1(H) \to H_1(W) \to 0.$$

Now, since $H_1(Y)$ is generated by all the $\mathbb{S}^1-$factors i.e. the longitudes $c_1, ..., c_s$, and $H_1(H) = H_2(H) = H_2(X) = 0$, the sequence becomes

$$0 \to H_2(W) \to \langle c_1, ..., c_s \rangle \xrightarrow{i_*} H_1(X) \to H_1(W) \to 0.$$

In our case, we have that $\mathbb{S}^3 \setminus X$ induces an isomorphism $H_1(\mathbb{S}^3 \setminus K) \to H_1(X)$ by the properties of X. Since i_* is induced by inclusion and the longitudes $c_1, ..., c_s$ are nullhomologus in $\mathbb{S}^3 \setminus K$, these are nullhomologus in X. Therefore, i_* is the zero map. Then we have that $H_2(W) \cong H_1(Y) = \langle c_1, ..., c_s \rangle$.

In particular, the intersection matrix on W with respect to this basis is given by $C_i \cdot C_j$, i.e, it is a diagonal matrix such that u_+ diagonal entries are equal to -1 and u_- diagonal entries are equal to $+1$. This concludes the proof of the theorem. □

Remark 2.4.8. *Rereading the proof, one can observe that we actually proved that*

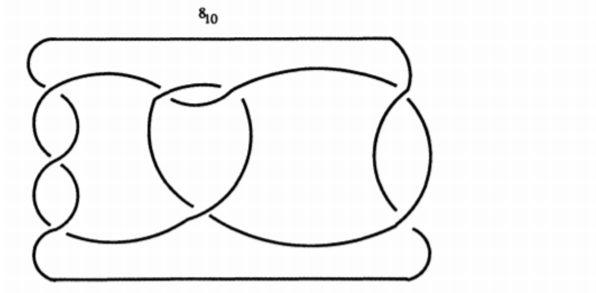
$$u_a(K) \geq \{dim_\Omega(H_2(W; \Lambda) \otimes \Omega) : W \text{ tamely cobounds } M(K)\}.$$

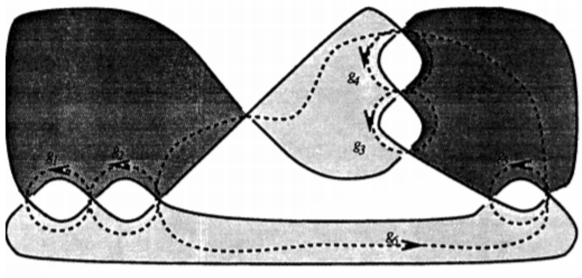
Reinterpretating [11], the last inequality is an equality.

2.5 Knot 8_{10}

In this section we will compute the invariant $n(K)$ for $K = 8_{10}$. This example is interesting because from the lower bounds introduced at the beginning of these notes gives the trivial lower bound $u_a(K) \geq 1$. But, using a naive approach it's very hard to find a crossing change that makes K a knot with trivial Alexander polynomial. On the other hand, we will see that $n(K) = 1$ and therefore, we will obtain that $u_a(8_{10}) = 1$.

In the following figures we show a picture of the knot 8_{10} and a figure of this knot that allows us to compute the Seifert matrix in a more obvious way.





A Seifert matrix can be read from the last figure as

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{pmatrix}$$

Using the algorithm of chapter 3 that allows us to reduce the matrix $tV - V^T$ to a small square matrix, we can reduce it to a 1×1 matrix with coefficient $\Delta(t) = t^3 - 3t^2 + 6t - 7 + 6t^{-1} - 3t^{-2} + t^{-3}$. So we have that the Alexander module $H_1(X(8_{10}); \mathbb{Z}[t^{\pm 1}])$ is cyclic. Now, using the algorithm explained in the following chapter, we obtain that the Blanchfield form of K is given by

$$\begin{aligned} Bl(8_{10}) : \mathbb{Z}[t^{\pm 1}]/\Delta \times \mathbb{Z}[t^{\pm 1}]/\Delta &\to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ (a, b) &\mapsto \overline{a}\frac{z^4 + z^2 + 1}{\Delta(t)} b \end{aligned}$$

where $z = t^{1/2} + t^{-1/2}$. Observe that, for $f(t) = t^3 - t^2 + 2t - 1$, we have that $f(t)f(t^{-1})q/\Delta \equiv 1/\Delta \pmod{\mathbb{Z}[t^{\pm 1}]}$. Using the corollary 2.3.7 we obtain that the algebraic unknotting number of 8_{10} is 1.

Chapter 3

The Nakanishi index: Different approaches to compute it

In this chapter, we will explain the algorithm made to compute the Nakanishi index of a knot and, in the case where $m(K) = 1$, computes the Blanchfield form. Then we will give a list with the Blanchfield forms and Alexander polynomials of the knots for which Knotorius [1] was not able to determine the algebraic unknotting number. Then we will introduce the row/column class invariants and the pullback diagram method. Finally, using these concepts and after doing some computations, we determine the Nakanishi index for the knots $12n_{0657}$, $12a_{1054}$ and $12a_{1141}$, and we will prove that the unknotting number of $12n_{0657}$ is 2.

3.1 Preliminaries

Definition 3.1.1. Let $K \subseteq \mathbb{S}^3$ be a knot and $H_1(X(K);\mathbb{Z}[t^{\pm 1}])$ the Alexander module of K. We define the Nakanishi index of K as

$$m(K) := \min\{n : \text{there exists } n \text{ elements that generate } H_1(X(K);\mathbb{Z}[t^{\pm 1}])\}.$$

Or more generally, for a given $\mathbb{Z}[t^{\pm 1}]-$module N we define the Nakanishi index of K with coefficients in N as

$$m_N(K) := \min\{n : \text{there exists } n \text{ elements that generate } H_1(X(K);\mathbb{Z}[t^{\pm 1}])\otimes_{\mathbb{Z}[t^{\pm 1}]} N\}.$$

Observation 3.1.2.

- Given V the Seifert matrix of the knot K, we know that $A = tV - V^T$ is a presentation matrix of $H_1(X(K);\mathbb{Z}[t^{\pm 1}])$ and therefore we have that $m(K) \leq 2g$.

- For a given $\mathbb{Z}[t^{\pm1}]-$module N, we have that $\cdot \otimes_{\mathbb{Z}[t^{\pm1}]} N$ is a right exact functor. Therefore, we deduce that $m_N(K) \leq m(K)$.

Remark 3.1.3. *The reader can already have the intuition that this algebraic invariant may be, in general, really hard to compute, and the main reason is that the ring $\mathbb{Z}[t^{\pm1}]$ is not a principal domain. However, in this chapter we will give different approaches that will allow us to compute this invariant in many cases or, at least, give good bounds.*

Let V be a $n \times n$ Seifert matrix of K and $A = tV - V^T$. Then we have the following exact sequence

$$0 \to \mathbb{Z}[t^{\pm1}]^n \xrightarrow{A} \mathbb{Z}[t^{\pm1}]^n \to H_1(X(K); \mathbb{Z}[t^{\pm1}]) \to 0.$$

It is known that two presentation matrices A and B of $H_1(X(K); \mathbb{Z}[t^{\pm1}])$ differ by a sequence of matrix moves of the following forms and their inverses [6](Theorem 6.1):

1. Permutation of rows and/or columns.
2. Replacing A for $\begin{pmatrix} A & 0 \\ 0 & u \end{pmatrix}$ with $u \in R^*$.
3. Addition of an extra column of zeros to the matrix A.
4. Addition of a scalar multiple of a row (or column) to another row (or column).

If two matrices A and B over a ring R differ by a sequence of these moves, they are called $R-$**equivalent**. So an equivalent definition of the Nakanishi index would be

$m(K) = \min\{n :$ there exists a $n \times n$ presentation matrix of the Alexander module$\}$.

We develop an algorithm that computes an upper bound of the Nakanishi index, finding a matrix that is $\mathbb{Z}[t^{\pm1}]$-equivalent to the original matrix $tV - V^T$ and giving us the change of basis that converts one to the other (this will be usefull in order to compute the Blanchfield form).

3.2 Algorithm for Nakanishi index

In this section, we will explain how does the algorithm used to reduce the presentation matrix of the Alexander modules works.

Synopsis

- Input: V, Seifert matrix of the knot K and N a natural number.
- Output: Gives a presentation matrix of the knot K of "small" dimension and an upper bound of the Nakanishi index.

The algorithm works as follows. Let N be a natural number and V a Seifert matrix of a knot K. The natural number measures how hard the algorithm is trying to compute a minimal presentation of $A = tV - V^T$.

1. Computes the presentation matrix $A = tV - V^T = (a_{ij})_{1 \leq i,j \leq 2g}$.
2. Defines $n =$ dimension of A and the variables $\text{effort1}, \text{effort2} = 0$.
3. For each i, j checks if $a_{i,j}$ is an element of $\mathbb{Z}[t^{\pm 1}]^{\times}$.
   - If some $a_{i,j}$ is an invertible element; it swaps it to the position $1, 1$ of the matrix and does gauss reductions by columns and rows obtaining a matrix of the form
   $$\begin{pmatrix} a_{i,j} & 0 \\ 0 & A' \end{pmatrix}$$
   Changes A by A' and n by $n-1$ and goes back to 3.
   - If for all i, j we have that $a_{i,j}$ is not invertible then the algorithm jumps to the next step.
4. For each $i, j, k \in \{1, ..., n\}$ we compute $(a_{i,k}, a_{j,k})$ and $(a_{k,i}, a_{k,j})$
   - If for some i, j, k we have that $(a_{i,k}, a_{j,k}) = (1)$, computes $\alpha, \beta \in \mathbb{Z}[t^{\pm 1}]$ such that $\alpha a_{i,k} + \beta a_{j,k} = 1$. Then, if necessary, it swaps rows and columns of A in order to have the elements $a_{i,k}$ and $a_{j,k}$ in the positions $1, 1$ and $2, 1$ and replaces A by this new matrix. Then we do the following operation:
   $$\left(\begin{array}{cc|c} \alpha & \beta & \multirow{2}{*}{0} \\ -a_{j,k} & a_{i,k} & \\ \hline \multicolumn{2}{c|}{0} & Id \end{array}\right) A = \left(\begin{array}{cc|c} 1 & * & \multirow{2}{*}{B} \\ 0 & * & \\ \hline \multicolumn{2}{c|}{C} & A' \end{array}\right)$$

and replaces A by the matrix on the right side of the equality. Observe that the determinant of the first matrix is 1. Analogously, if $(a_{k,i}, a_{k,j}) = (1)$ for some k, j, i it applies gauss reduction by columns and rows. After this we have a matrix of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$$

It replaces n by $n-1$, A by A' and goes back to step 3.

- If this first condition does not hold, the algorithm jumps to the next step.

5. If effort1 $< N$, then it constructs a random invertible matrix P with coefficients in $\mathbb{Z}[t^{\pm 1}]$, multiplies A by P, goes back to step 3 and effort1 $\leftarrow$ effort1 $+ 1$. Otherwise, it goes to the next step.

6. If effort2 $< N$, then replaces n by $n+1$, replaces A by $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ and effort2 $\leftarrow$ effort2 $+ 1$, then it jumps back to step 5. Otherwise the algorithm stops.

In each one of the steps, if $n = 1$ the algorithm stops.

As a result of the algorithm we will obtain a matrix of the form $B := \mathrm{Id} \oplus A'$. Moreover, if in each one of the steps one keeps the change of basis made, we can obtain matrices C_1 and C_2 such that $C_1 A C_2 = B$, with C_1, C_2 invertible matrices over $\mathbb{Z}[t^{\pm 1}]$.

Now, let K be a knot with Seifert matrix V and $A = tV - V^T$ a presentation matrix of the Alexander module. Suppose that using this algorithm we can find invertible matrices C_1, C_2 over $\mathbb{Z}[t^{\pm 1}]$ such that

$$C_1 A C_2 = \begin{pmatrix} \mathrm{Id}_{2g-1} & 0 \\ 0 & \Delta_K(t) \end{pmatrix}.$$

We already know that the Blanchfield form is isometric to the pairing

$$\begin{aligned} \lambda(A) : \mathbb{Z}[t^{\pm 1}]^n / A\mathbb{Z}[t^{\pm 1}]^n \times \mathbb{Z}[t^{\pm 1}]^n / A\mathbb{Z}[t^{\pm 1}]^n &\to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ (a, b) &\mapsto \overline{a}^T (t-1) A^{-1} b, \end{aligned}$$

write $D = \begin{pmatrix} \mathrm{Id}_{2g-1} & 0 \\ 0 & \Delta_K(t) \end{pmatrix}$. Then we have that

$$\begin{aligned} \mathbb{Z}[t^{\pm 1}]^n / A\mathbb{Z}[t^{\pm 1}]^n &= \mathbb{Z}[t^{\pm 1}]^n / C_1^{-1} D C_2^{-1} \mathbb{Z}[t^{\pm 1}]^n \\ &= \mathbb{Z}[t^{\pm 1}]^n / C_1^{-1} D \mathbb{Z}[t^{\pm 1}]^n \\ &\cong \mathbb{Z}[t^{\pm 1}]^n / D\mathbb{Z}[t^{\pm 1}]^n \cong \mathbb{Z}[t^{\pm 1}] / \Delta_K(t) \mathbb{Z}[t^{\pm 1}], \end{aligned}$$

where the first isomorphism is given by C_1 and the second isomorphism is given by the projection in the last coordinate. Putting it all together, we obtain that $\lambda(A)$ is isometric to

$$\begin{aligned} \mathbb{Z}[t^{\pm 1}]/\Delta_K(t) \times \mathbb{Z}[t^{\pm 1}]/\Delta(t) &\to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ (a,b) &\mapsto \overline{a}\frac{q(t)}{\Delta_K(t)}b \end{aligned}$$

where $\frac{q(t)}{\Delta_K(t)}$ is the (n,n) component of the matrix $\overline{C_1^{-1}}(t-1)A^{-1}C_1^{-1}$. Observe that usually, each time we apply the algorithm we will obtain different q because the matrices C_1 produced by the algorithm are not unique. Even though the algorithm can give different presentations of the Blanchfield form, all of them are isometric.

In the next page we will apply this algorithm to the Seifert matrix (obtained in the webpage Knotinfo [2]) of the listed of knots in [15] with unknown algebraic unknotting number and Nakanishi index equal 1. Also observe that the Alexander polynomials in the following list don't correspond to the definition in chapter 1. However, one can easily obtain the Alexander polynomial multiplying by a suitable unit of $\mathbb{Z}[t^{\pm 1}]$.

Knots	q	Alexander polynomial $\Delta_K(t)$
$12a_{0050}$	$t^8-9t^7+28t^6-50t^5+60t^4-50t^3+28t^2-9t+1$	$1-8t+20t^2-30t^3+33t^4-30t^5+20t^6-8t^7+t^8$
$12a_{0141}$	$t^{14}-8t^{13}+24t^{12}-24t^{11}-48t^{10}+213t^9-392t^8+468t^7-392t^6+213t^5-48t^4-24t^3+24t^2-8t+1$	$-t^8+7t^7-20t^6+33t^5-39t^4+33t^3-20t^2+7t-1$
$12a_{0364}$	$t^8-9t^7+35t^6-77t^5+100t^4-77t^3+35t^2-9t+1$	$t^8-8t^7+29t^6-63t^5+81t^4-63t^3+29t^2-8t+1$
$12a_{0649}$	$t^8-8t^7+24t^6-42t^5+50t^4-42t^3+24t^2-8t+1$	$t^8-7t^7+17t^6-25t^5+27t^4-25t^3+17t^2-7t+1$
$12a_{0728}$	$2t^8-9t^7+20t^6-30t^5+34t^4-30t^3+20t^2-9t+2$	$-2t^8+8t^7-17t^6+25t^5-29t^4+25t^3-17t^2+8t-2$
$12a_{0791}$	$-3t^6+12t^5-22t^4+26t^3-22t^2+12t-3$	$-3t^6+9t^5-13t^4+13t^3-13t^2+9t-3$
$12a_{0901}$	$t^8-7t^7+20t^6-34t^5+40t^4-34t^3+20t^2-7t+1$	$t^8-9t^7+27t^6-47t^5+55t^4-47t^3+27t^2-9t+1$
$12a_{1064}$	$t^8-5t^7+12t^6-19t^5+22t^4-19t^3+12t^2-5t+1$	$-2t^8+9t^7-21t^6+33t^5-37t^4+33t^3-21t^2+9t-2$
$12a_{1138}$	$-t^6+4t^5-6t^4+6t^3-6t^2+4t-1$	$-3t^6+11t^5-17t^4+17t^3-17t^2+11t-3$
$12a_{1234}$	$t^8-4t^7+9t^6-15t^5+18t^4-15t^3+9t^2-4t+1$	$-2t^8+7t^7-15t^6+25t^5-29t^4+25t^3-15t^2+7t-2$
$12a_{1236}$	$-2t^8+11t^7-30t^6+52t^5-62t^4+52t^3-30t^2+11t-2$	$2t^8-9t^7+21t^6-33t^5+37t^4-33t^3+21t^2-9t+2$
$12n_{0200}$	$2t^3-4t^2+2t$	$2-2t+t^2-2t^3+2t^4$
$12n_{0864}$	$-t^{12}+9t^{11}-40t^{10}+117t^9-241t^8+366t^7-420t^6+366t^5-241t^4+117t^3-40t^2+9t-1$	$t^{10}-5t^9+15t^8-28t^7+33t^6-28t^5+15t^4-5t^3+t^2$

3.3 Obstructions for the Nakanishi index

Suppose that for a given knot K, we obtain that the upper bound of the Nakanishi index is 2. So we are not able to know if the Nakanishi index is 1 or 2. In this section, we will give two different obstructions that will allow us to prove that the Nakanishi index is bigger than 1.

3.3.1 Row/column class

In this section, we define the row and column class invariant of a knot.

Let R be a commutative integral domain, and I, J two ideals of R. We say that I and J are **equivalent** if there exists two non-trivial principal ideals P, Q such that $PI = QJ$. The ideal classes of a ring under this relation form a semi-group with the principal ideal class as identity. If R is an order of a number field we can define this relation to the invertibles ideals, this set forms a group denoted by $\mathrm{Pic}(R)$.

Let A be an $m \times n$ matrix A over a commutative integral domain R. For any integer k consider the kth compound matrix $M^{(k)}$. This matrix is defined as the $\binom{m}{k} \times \binom{n}{k}$ matrix whose entries are the $k \times k$ minor determinants of M, rows and columns written in lexicographic order.

Proposition 3.3.1. *Let M be a matrix over a commutative integral domain R with rank r, then:*

1. $M^{(r)} = 0$ *for k greater than r.*
2. $M^{(r)}$ *contains some nonzero element.*
3. *The rank of $M^{(r)}$ is* 1.

Proof. The first two are deduced by definition. The matrix M is equivalent to a diagonal matrix N over the quotient field of R, and one can check that $M^{(k)}$ is equivalent (the reader should not confuse equivalent with R-equivalent) to $N^{(k)}$. Since the rank of $N^{(r)}$ is 1, the rank of $M^{(r)}$ is 1. □

Definition 3.3.2. Let M be an $m \times n$ matrix of rank r. The ith row (column) ideal ρ_i of M is defined as the ideal generated by the elements of the ith row (column) of $M^{(r)}$

Proposition 3.3.3. *The class of non-trivial ith row (column) ideals does not depend on i.*

Proof. Let ith and kth rows of $M^{(r)}$ rows that contain nonzero elements. Writing $M^{(r)} = (m_{ij})_{i,j}$ and choosing q such that m_{iq} is nonzero, we have for any j

$$\det \begin{pmatrix} m_{iq} & m_{ij} \\ m_{kq} & m_{kj} \end{pmatrix} = 0$$

since $M^{(r)}$ is of rank 1. Thus $m_{iq}m_{kj} = m_{kq}m_{ij}$ and $(m_{iq})\rho_k = (m_{kq})\rho_i$. Then it follows that m_{kq} is nonzero and thus that any two nonzero row ideals are equivalent. The proof is analogous for column ideals. □

Definition 3.3.4. The equivalence class of the nontrivial row (column) ideals is called the **row (column) class** of M.

Proposition 3.3.5. *The row (column) class of a matrix M is invariant under R-equivalences.*

Proof. The row and column class of a matrix remain invariant under the operations $(1)-(5)$. This is clear for $(1)-(4)$; the operation (5) increases the rank of the matrix, but does not change the elements of the row ideals (column ideals) of the matrix. It follows that the row class and column class of the matrix M are invariants under $R-$equivalences. □

Now, let K be a knot with Nakanishi index 1. Therefore we have that the presentation matrix $A = tV - V^T$ is $\mathbb{Z}[t^{\pm 1}]-$equivalent to the 2×2-matrix $\begin{pmatrix} 1 & 0 \\ 0 & \Delta_K(t) \end{pmatrix}$. Therefore we have that $A \pmod{\mathbb{Z}[t^{\pm 1}]/\Delta_K(t)}$ is $\mathbb{Z}[t^{\pm 1}]/\Delta_K(t)$-equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Clearly the column and row class of this matrix over the ring $\mathbb{Z}[t^{\pm 1}]/\Delta_K(t)$ is trivial. Therefore, we deduce the following proposition.

Proposition 3.3.6. *Let K be a knot, A a presentation matrix of the Alexander module and $\Delta_K(t)$ the Alexander polynomial. Then if K has Nakanishi index 1, the column/row class of $A \pmod{\mathbb{Z}[t^{\pm 1}]/\Delta_K(t)}$ is trivial.*

From now on for a given knot K, the row and colummn class of the presentation matrix of A will mean the column and row class of

$$A \pmod{\mathbb{Z}[t^{\pm 1}]/\Delta_K(t)}.$$

3.3.2 Pull-back diagram method

In this section, $\Lambda = \mathbb{Z}[t^{\pm 1}]$ and M a $\Lambda-$module isomorphic to $\Lambda^n/A\Lambda^n$, for some $n \times n$-matrix A with non zero determinant Δ. Moreover, we suppose that $\Delta = fg$, with $f, g \in \Lambda$ irreducible non-invertible elements. Then we have the following pullback diagram

$$\begin{tikzcd} M/(f)M \cap (g)M \arrow[r] \arrow[d] & M/fM \arrow[d] \\ M/gM \arrow[r] & M/(f,g)M. \end{tikzcd} \tag{3.3.1}$$

Suppose now that M has the following properties:

- $M/fM \cong \Lambda/f$, $M/gM \cong \Lambda/g$,
- $M/(f,g)M \cong \Lambda/(f,g)$,
- and $(f)M \cap (g)M = \{0\}$.

Then, we can rewrite the diagram 3.3.1 in the following way

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \Lambda/f \arrow[d, "\pi"] \\ \Lambda/g \arrow[r, "\phi"] & \Lambda/(f,g). \end{tikzcd}$$

where π_1 is the natural projection and ϕ is given by the composite of the natural projection $\pi_2 : \Lambda/g \to \Lambda/(f,g)$ and an automorphism on $\Lambda/(f,g)$ of $\Lambda-$modules (i.e. a map $\chi : \Lambda/(f,g) \to \Lambda/(f,g)$ given by $\chi(x) = ux$ for $u \in (\Lambda/(f,g))^{\times}$). So we deduce that M is isomorphic to

$$\{(x,y) \in \Lambda/f \times \Lambda/g : \pi_1(x) = u\pi_2(y)\}.$$

On the other hand, we have that

$$\begin{tikzcd} \Lambda/fg \arrow[r] \arrow[d] & \Lambda/f \arrow[d, "\pi_1"] \\ \Lambda/g \arrow[r, "\pi_2"] & \Lambda/(f,g). \end{tikzcd}$$

is a pull-back diagram and therefore we have that

$$\Lambda/fg \cong \{(x,y) \in \Lambda/f \times \Lambda/g : \pi_1(x) = \pi_2(y)\}$$

Clearly if $u \in \pi_1((\Lambda/f)^{\times})\pi_2((\Lambda/g)^{\times})$ then $\Lambda/fg \cong M$ and therefore M is cyclic. In fact, the converse is also true.

Lemma 3.3.7. *In this setting, we have that if $\Lambda/fg \cong M$, then*

$$u \in \pi_1((\Lambda/f)^\times)\pi_2((\Lambda/g)^\times).$$

Proof. See [20] lemma 4.5(v). $\square$

3.4 Examples

3.4.1 Nakanishi index of the knot $12n_{0657}$

In this section we will use *sage* for computing the unit group, the class group for some number fields and the smith form for some matrices.
In [1] the knot module of the knot $K = 12n_{0657}$ is given (changing the sign of the first column) by:

$$A(t) = \begin{pmatrix} -1+4t^2 & 4-4t-5t^3+t^4-t^5 \\ t-3t^2+t^3 & -1+3t^2 \end{pmatrix}$$

i.e. the knot module M is the following $\mathbb{Z}[t^{\pm 1}]$-module

$$\mathbb{Z}[t^{\pm 1}]^2/A(t)\mathbb{Z}[t^{\pm 1}]^2$$

Observation 3.4.1.

- The determinant of $A(t)$ is

  $$\det A(t) = (t^4-3t^3+3t^2-t+1)(t^4-t^3+3t^2-3t+1).$$

  Define $f := t^4-3t^3+3t^2-t+1$ and $g := t^4-t^3+3t^2-3t+1$.

- Let θ be a root of f and τ a root of g, then the fields $K := \mathbb{Q}(\theta)$ and $L := \mathbb{Q}(\tau)$ have class number 1, $\mathcal{O}_K = \mathbb{Z}[\theta]$ and $\mathcal{O}_L = \mathbb{Z}[\tau]$. The second result can be seen computing the discriminant of both polynomials and observing that is 229 that is a prime number. The first result can be seen using the minkowski bound M_K and computing the class of the primes with norm less than M_K.

- $U_K \cong \mathbb{Z}/2 \times \mathbb{Z} = \langle -1 \rangle \times \langle \theta - 1 \rangle$ and $U_L \cong \mathbb{Z}/2 \times \mathbb{Z} = \langle -1 \rangle \times \langle \tau^3 - \tau^2 + 3\tau - 2 \rangle$.

Lemma 3.4.2. *$M/fM \cong \mathbb{Z}[\theta]$.*

Proof. Observe that the $\mathbb{Z}[t^{\pm 1}]-$morphism

$$\begin{aligned} M &\to \mathbb{Z}[\theta]^2/A(\theta)\mathbb{Z}[\theta]^2 \\ (x_1(t), x_2(t)) &\mapsto (x_1(\theta), x_2(\theta)) \end{aligned}$$

induces an isomorphism $M/fM \cong \mathbb{Z}[\theta]^2/A(\theta)\mathbb{Z}[\theta]^2$ of $\mathbb{Z}[t^{\pm 1}]-$modules. We know that $\mathcal{O}_K = \mathbb{Z}[\theta]$ and the class number of K is 1. Therefore we can compute the smith form of the matrix $A(\theta)$. We obtain that

$$\begin{pmatrix} 3 & 4\theta^2 - 4\theta + 8 \\ -\theta^3 + 3\theta^2 - \theta & 4\theta^2 - 1 \end{pmatrix} A(\theta) \begin{pmatrix} 1 & \theta^2 - \theta + 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let C_1 be the first matrix on the left side of the equality, C_2 the third one on the left side of the equality and D the matrix on the right side of the equality. Then we obtain:

$$\begin{aligned} M/fM &\cong \mathbb{Z}[\theta]^2/A(\theta)\mathbb{Z}[\theta]^2 \\ &= \mathbb{Z}[\theta]^2/C_1^{-1}DC_2^{-1}\mathbb{Z}[\theta]^2 \\ &= \mathbb{Z}[\theta]^2/C_1^{-1}D\mathbb{Z}[\theta]^2 \\ &\cong \mathbb{Z}[\theta]^2/D\mathbb{Z}[\theta]^2 \\ &\cong \mathbb{Z}[\theta]. \end{aligned}$$

Where the second isomorphism is given by C_1 multiplication and the third one by the projection in the second coordinate. $\square$

Lemma 3.4.3. *$M/gM \cong \mathbb{Z}[\tau]$.*

Proof. With the same arguments as before we compute the smith form of $A(\tau)$ and we obtain:

$$\begin{pmatrix} 0 & -6\tau^3 + 3\tau^2 - 16\tau + 10 \\ -\tau^3 & -2\tau^3 + \tau^2 - 2\tau + 1 \end{pmatrix} A(\tau) \begin{pmatrix} 3 & -3\tau^3 - 7\tau + 1 \\ -\tau^3 - 3\tau + 3 & -\tau^2 + \tau - 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore we obtain that $M/gM \cong \mathbb{Z}[\tau]$. $\square$

Lemma 3.4.4. *As $\mathbb{Z}[t^{\pm 1}]$-modules we have that $M/(f,g)M \cong \mathbb{Z}[\theta]/(g(\theta))$.*

Proof. Follows easily from lemma 3.4.2. $\square$

Lemma 3.4.5. $fM \cap gM = \{0\}$.

Proof. Observe that

$$144 = (-32t^3 + 64t^2 - 88t + 184)f + (32t^3 - 128t^2 + 152t - 40)g$$

Let $x \in fM \cap gM$, then we have that $(f, g) \subseteq \{p \in \mathbb{Z}[t^{\pm 1}] : px = 0\}$, since $144 \in (f, g)$, we would have that $144x = 0$, but M is a knot module so it does not have $\mathbb{Z}$-torsion, this implies that $x = 0$. $\square$

Thus, we have the following cartesian square,

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M/fM \arrow[d] \\ M/gM \arrow[r] & M/(f,g)M \end{tikzcd}$$

and by the last lemmas we can rewrite this square in the following way

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & \mathbb{Z}[\theta] \arrow[d, "\beta"] \\ \mathbb{Z}[\tau] \arrow[r, "\gamma"] & \mathbb{Z}[\theta]/(g(\theta)) \end{tikzcd}$$

where the maps are defined as follows:

$$\begin{aligned} &\alpha(x) = -\theta^3 + 3\theta^2 - \theta, && \alpha(y) = 4\theta^2 - 1 \\ &\lambda(x) = 3\tau^3 + 7\tau - 1, && \lambda(y) = -\tau^2 + \tau - 2 \\ &\beta = \text{natural projection map.} && \\ &\gamma(\tau) = \overline{\theta}, && \gamma(1) = \overline{\theta^2 + 1}, \end{aligned}$$

where $x = \overline{(1, 0)}$ and $y = \overline{(0, 1)}$. By the discussion made in the last section, we have that the projective $\mathbb{Z}[t^{\pm 1}]/(fg)$-module M is free of rank 1, if and only if, the element $\overline{\theta^2 + 1} \in (\mathbb{Z}[\theta]/(g(\theta)))^\times$ lies in the group

$$\gamma_0(\mathbb{Z}[\tau]^\times)\beta(\mathbb{Z}[\theta]^\times) = \langle\overline{-1}\rangle \times \langle\overline{\theta - 1}\rangle \times \langle\overline{\theta^3 - \theta^2 + 3\theta - 2}\rangle.$$

where γ_0 is defined as $\gamma_0(1) = 1$ and $\gamma(\tau) = \overline{\theta}$. Using *sage* one proves that $(\mathbb{Z}[\theta]/(g(\theta))^\times$ is an abelian group isomorphic to $C_3 \times C_{30}$ therefore in order to prove that $\overline{\theta^2 + 1} \notin \gamma_0(\mathbb{Z}[\tau]^\times)\beta(\mathbb{Z}[\theta]^\times)$, is enough to prove the following claim

$$b \not\equiv \pm(\theta - 1)^i(\theta^3 - \theta^2 + 3\theta - 2)^j \pmod{g(\theta)} \text{ for } i, j \in \{0, ..., 30\}.$$

Using *sage* again we deduce that the claim is true and therefore by the discussion made before we deduce that the Nakanishi index of $12n_{0657}$ is 2.

Corollary 3.4.6. *The algebraic unknotting number and the unknotting number of* $12a_{0657}$ *are* 2.

Proof. By [2] the unknotting number of $12a_{0657}$ is 1 or 2, since $u_a(K) \geq m(K)$, we have that the unknotting number and the algebraic unknotting number of $12a_{0657}$ are 2. □

3.4.2 Nakanishi index of $12a_{1054}$ and $12a_{1141}$

Article [15] gives a 2×2 presentation matrix of the Alexander module of the knot $12a_{1054}$. The matrix is

$$A = \begin{pmatrix} 2t^3 & -1+4t-7t^2+4t^3-t^4 \\ 1-5t+t^2-5t^3+t^4 & 3t \end{pmatrix}$$

and $\det A = t^8 - 9t^7 + 28t^6 - 48t^5 + 55t^4 - 48t^3 + 28t^2 - 9t + 1$.

Proposition 3.4.7. *Let* $f = detA$*,* θ *a root of* f *and* $K = \mathbb{Q}(\theta)$*. Then*

1. $\mathbb{Z}[\theta]^\times = \mathcal{O}_k^\times$.
2. $(2\theta^3, -1+4\theta-7\theta^2+4\theta^3-\theta^4)_{\mathcal{O}_K} = (2\theta^7 - 17\theta^6 + 48\theta^5 - 76\theta^4 + 82\theta^3 - 69\theta^2 + 35\theta - 10)_{\mathcal{O}_K}$.
3. $(2\theta^3, -1+4\theta-7\theta^2+4\theta^3-\theta^4)_{\mathbb{Z}[\theta]}$ *is a non-principal ideal.*

Proof. 1. Using *sage* one can check that the group $\mathcal{O}_K^\times$ is generated by the elements

$$\begin{aligned}
u_0 &= -1 \\
u_1 &= \theta^7 - 9\theta^6 + 28\theta^5 - 48\theta^4 + 55\theta^3 - 48\theta^2 + 28\theta - 8 \\
u_2 &= \theta - 1 \\
u_3 &= 3\theta^7 - 27\theta^6 + 83\theta^5 - 136\theta^4 + 145\theta^3 - 116\theta^2 + 57\theta - 8 \\
u_4 &= \theta^7 - 8\theta^6 + 20\theta^5 - 28\theta^4 + 27\theta^3 - 22\theta^2 + 12\theta - 3
\end{aligned}$$

therefore $U_K^\times = \mathcal{O}_K^\times$.

2. Using *sage* we deduce the equality of ideals. See [21] section 6.5.5.

3. Suppose that $(2\theta^3, -1+4\theta-7\theta^2+4\theta^3-\theta^4)_{\mathbb{Z}[\theta]} = (\lambda)_{\mathbb{Z}[\theta]}$ then we would have that $\lambda = u(2\theta^7 - 17\theta^6 + 48\theta^5 - 76\theta^4 + 82\theta^3 - 69\theta^2 + 35\theta - 10)$ for $u \in \mathcal{O}_K^\times = \mathbb{Z}[\theta]^\times$, and therefore we would have that

$$\frac{2\theta^3}{2\theta^7 - 17\theta^6 + 48\theta^5 - 76\theta^4 + 82\theta^3 - 69\theta^2 + 35\theta - 10} = \frac{u2\theta^3}{\lambda} \in \mathbb{Z}[\theta]$$

but the first term is equal to $1/2\theta^7 - 9/2\theta^6 + 27/2\theta^5 - 20\theta^4 + 20\theta^3 - 27/2\theta^2 + 9/2\theta - 1/2$ and this elements is not in $\mathbb{Z}[\theta]$, so we deduce that the ideal is not principal.

$\square$

Corollary 3.4.8. *The Nakanishi index of the knot* $12a_{1054}$ *is* 2 *and in particular we deduce that the algebraic unknotting number is* 2.

Proof. Suppose that the Nakanishi index of the knot of $12a_{1054}$ is 1, then we would have that the presentation matrix A is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & \Delta(t) \end{pmatrix}$ therefore we would have that the row class would be trivial. But as we proved already the column class of the matrix A is nontrivial, therefore the Nakanishi index is 2. $\square$

Again, article [15] gives a 2×2 presentation matrix of the Alexander module of the knot $12a_{1141}$. The matrix is

$$A = \begin{pmatrix} 1+2t & 2-2t+t^2 \\ 4t-9t^2+6t^3-8t^4+6t^5-t^6 & 1-2t \end{pmatrix}$$

and $\det A = t^8 - 8t^7 + 22t^6 - 34t^5 + 37t^4 - 34t^3 + 22t^2 - 8t + 1$.

Proposition 3.4.9. *Let* $f = \det A$, θ *a root of* f *and* $K = \mathbb{Q}(\theta)$. *Then*

1. $\mathbb{Z}[\theta]^\times = \mathcal{O}_K^\times$.
2. $(1+2\theta, 2-2\theta+\theta^2)_{\mathcal{O}_K} = (-\theta^7 + 15/2\theta^6 - 37/2\theta^5 + 26\theta^4 - 49/2\theta^3 + 21\theta^2 - 23/2\theta + 7/2)_{\mathcal{O}_K}$
3. $I := (1+2\theta, 2-2\theta+\theta^2)_{\mathbb{Z}[\theta]}$ *is a non-principal ideal.*

Proof. The first two are deduced using *sage*, the third one can be deduced from the first two. If I, was principal, then there would exist $\lambda \in \mathbb{Z}[\theta]$ such that $I = (\lambda)$ so we would have that $\lambda = u(-\theta^7 + 15/2\theta^6 - 37/2\theta^5 + 26\theta^4 - 49/2\theta^3 + 21\theta^2 - 23/2\theta + 7/2)$ and therefore we would deduce by (1) that $-\theta^7 + 15/2\theta^6 - 37/2\theta^5 + 26\theta^4 - 49/2\theta^3 + 21\theta^2 - 23/2\theta + 7/2 \in \mathbb{Z}[\theta]$ and this isn't true. Therefore, the ideal I is not principal. $\square$

Corollary 3.4.10. *The Nakanishi index of the knot* $12a_{1141}$ *is* 2 *and in particular we deduce that the algebraic unknoting number is* 2.

Proof. Analogus to the knot $12a_{1054}$. $\square$

Chapter 4

Obstruction $n(K) = 1$

Suppose that K is a knot with Nakanishi index 1, i.e $H_1(X(K), \mathbb{Z}[t^{\pm 1}])$ is cyclic and therefore is isomorphic to $\mathbb{Z}[t^{\pm 1}]/\Delta_K(t)$, where $\Delta_K(t)$ is the Alexander polynomial. Then there exists $q(t) \in \mathbb{Z}[t^{\pm 1}]$ such that the Blanchfield form is given by

$$\begin{aligned} Bl(K) : H_1(X(K), \mathbb{Z}[t^{\pm 1}]) \times H_1(X(K), \mathbb{Z}[t^{\pm 1}]) &\to \mathbb{Q}(t^{\pm 1})/\mathbb{Z}[t^{\pm 1}] \\ (a, b) &\mapsto \overline{a}\frac{q(t)}{\Delta_K(t)}b \end{aligned}$$

If we use the main theorem in this particular case, we deduce that the algebraic unknotting number is 1 if and only if this form is isometric to one of the forms

$$\begin{aligned} \mathbb{Z}[t^{\pm 1}]/\Delta_K(t)\mathbb{Z}[t^{\pm 1}] \times \mathbb{Z}[t^{\pm 1}]/\Delta_K(t)\mathbb{Z}[t^{\pm 1}] &\to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ (a, b) &\mapsto \overline{a}\frac{\pm 1}{\Delta_K(t)}b \end{aligned}$$

and by 2.3.7 we know that the algebraic unknotting number is 1 if and only if there exists $f(t) \in \mathbb{Z}[t^{\pm 1}]$ such that

$$q(t)f(t)f(t^{-1}) \equiv \pm 1 \pmod{\Delta_K}$$

In this chapter, we will give an explicit algorithm to see whether this condition holds for a certain knot.

4.1 The equation $q(t)f(t)f(t^{-1}) \equiv \pm 1 \pmod{p(t)}$

Synopsis

- Input: $p(t)$, an irreducible symmetric element of $\mathbb{Z}[t^{\pm 1}]$ and $q(t)$, a symmetric element of $\mathbb{Z}[t^{\pm 1}]$.
- Output: Tells whether there exists f such that $q(t)f(t)f(t^{-1}) \equiv 1 \pmod{p(t)}$ and computes it.

Let $p(t), q(t)$ be symmetric elements of $\Lambda := \mathbb{Z}[t^{\pm 1}]$, where $p(t)$ is assumed to be irreducible. Suppose we want to know whether there exists an element $f(t)$ of Λ such that

$$q(t)f(t)f(t^{-1}) \equiv \pm 1 \pmod{p(t)}. \tag{4.1.1}$$

Now we will explain an algorithm that checks whether does this $f(t)$ exists.

We first test if $q(t)$ is a unit modulo $p(t)$, i.e, whether there is some $q^*(t) \in \Lambda$ such that $q(t)q^*(t) \equiv 1 \pmod{p(t)}$, and if so, compute such $q^*(t)$. There exists algorithm that can do this task using Grobner bases over $\mathbb{Z}$. In the case that $q(t)$ is not a unit modulo $p(t)$, then we are done, since we know that the equation 4.1.1 does not have a solution. So suppose that $q(t)$ is a unit modulo $p(t)$, and $q^*(t)$ as above has been found. Then, instead of focusing on the equation 4.1.1, we will focus on the equation

$$f(t)f(t^{-1}) \equiv \pm q^*(t) \pmod{p(t)} \tag{4.1.2}$$

Note also that since each element of Λ is a quotient of an element of $\mathbb{Z}[t]$ by a power of t, if 4.1.2 has a soultion with $f(t) \in \Lambda$, then it also has a solution $f(t) \in \mathbb{Z}[t]$.

Write $p(t)$ as $p(t) = t^{-d}P(t)$, where $P(t) \in \mathbb{Z}[t]$ and $P(0) \neq 0$. Since p is symmetric, we have that $\deg P = 2d$ is even. Let $K = \mathbb{Q}(\tau)$ be a number field generated by a root τ of the irreducible polynomial $P(t)$ of degree d (so $p(\tau) = 0$). We identify $R := \Lambda/p\Lambda$ with the subring $\mathbb{Z}[\tau^{\pm}]$ of K.

Since τ^{-1} is also a root of p (due to the symmetry of p) we have that it is an element of K. Then, we have that K has an automorphism σ such that $\sigma(\tau) = \tau^{-1}$. This automorphim gives rise to an endomorphism A of the multiplicative group $K^\times$ of K given by $A(f) = f\sigma(f)$ for $f \in K^\times$. Observe that if we define $L = \mathbb{Q}(\tau + \tau^{-1})$, we have that the morphism of groups A is the restriction of the norm map $\mathrm{Nm}_{K|L} : K^\times \to L^\times$. Clearly, the subring $\mathbb{Z}[\tau^{\pm}]$ of K is invariant under σ, so A restricts to an endomorphism of the group $R^\times$

of units of R. Putting all together we have that the 4.1.2 is equivalent to the equation

$$A(f) = \pm q^*(\tau) \tag{4.1.3}$$

in $R^\times$. Observe that if this equation does not have solution in the larger group $\tilde{R}^\times$ (where $\tilde{R}$ is $R\mathcal{O}_K$ the normalisation of R), then does not have solution in $R^\times$.

Now in order to check whether the equation have solution or not, we should be able to compute $R^\times$ (i.e finding generators and relations). In the examples that we are interested in, will be enough to compute the group $\tilde{R}^\times$ and the endomorphism in this group (except the case $12n_{200}$).

There are two different cases in order to compute $\tilde{R}$.

- If P is monic then we have that $\tilde{R} = \mathcal{O}_K$ is the integral closure of $\mathbb{Z}$ in K. Then we know by Dirichlets theorem, that $\mathcal{O}_K^\times \cong \mu_k \times \mathbb{Z}^{r+s-1}$, where μ_K is the group of roots of unity in K, r is the number of real embeddings of K into $\mathbb{R}$ and s is the number of conjugate pairs of embeddings of K into $\mathbb{C}$. Algebraic computing software like *sage* and *MAGMA* are able to compute this group, giving generators of the torsion group and the free abelian group, these algorithms are summarized in Cohen's book on computational algebraic number theory [21].

- If P isn't monic then we have that $\tilde{R} = \mathcal{O}_K(S_R)$, where $S_R = \{\mathfrak{p} : v_\mathfrak{p}(\tau) \neq 0\}$. Similarly as before we have that $\mathcal{O}_K(S_R)^\times \cong \mu_K \times \mathbb{Z}^{r+s-1+|S_R|}$ and with a similar algorithm, algebraic computing system like *sage* and *MAGMA*, are able to compute it.

Now in order to compute $R^\times$, we use the following well known exact sequence

$$0 \to R^\times \to \tilde{R}^\times \to (\tilde{R}/\mathfrak{f})^\times/(R/\mathfrak{f})^\times,$$

where $\mathfrak{f} = \{a \in \tilde{R} : a\tilde{R} \subseteq R\}$ and it is well known to be non-trivial ideal. Since $\tilde{R}^\times$ and $(\tilde{R}/\mathfrak{f})^\times/(R/\mathfrak{f})^\times$, are computable (see [21] chapter 6) , we can compute $R^\times \cong \mathbb{Z}/d \oplus \mathbb{Z}^{n-1}$.

Moreover, given an element $u \in R^\times$, there exists an algorithm that is able to write u in terms of the generators. Therefore, we can express the automorphism A in terms of these generators, obtaining a $n \times n$ matrix A, n being the number of generators of $R^\times$. Now we can express $q^\star(\tau)$ in terms of these generators, obtaining a vector w with coefficients in $\mathbb{Z}/d \oplus \mathbb{Z}^{n-1}$. Therefore the equation 4.1.3 is equivalent to the following equation

$$Av = w \tag{4.1.4}$$

for some $v \in \mathbb{Z}/d \oplus \mathbb{Z}^{n-1}$. By elementary linear algebra, it is easy to check whether the equation 4.1.4 has solutions. This concludes the algorithm that check if $n(K) = 1$ for knots with Nakanishi index 1.

4.2 Algorithm $n_p(K) = 1$

In some cases, we will be interested to solve the following more generic but similar equation:

$$f(t)f(t^{-1})q(t) \equiv \pm l^n \pmod{p\mathbb{Z}_{(l)}[t^{\pm 1}]} \tag{4.2.1}$$

with $q(t) \in \mathbb{Z}_{(l)}[t^{\pm 1}]$, p in $\mathbb{Z}[t^{\pm 1}]$ irreducible and l a positive integer number, where f is an element of $\mathbb{Z}[t^{\pm 1}]$ and n an integer elements.

Observation 4.2.1. In order to check if the equation 4.2.1 it is enough to check if it has solutions for $n = 0, 1$.

Now to see if the equation 4.2.1 has solutions is the same method as before but we consider $R = \mathcal{O}_K(S_R \cup S_l)$ where S_R is the same set used before and $S_l := \{\mathfrak{p} : v_{\mathfrak{p}}(\tau) \neq 0\}$. And then compute the endomorphism A over $R^\times$ and check if q^* is in the image of A.

4.3 The cases $12n_{260}$, $12a_{1264}$ and $12a_{1049}$

Using the algorithm to reduce the presentation matrix of $12n_{260}$ as much as posible we found that the matrix

$$A = \begin{pmatrix} t^5 - 2t^4 + t^3 - t^2 & -2t^4 + 3t^3 - t^2 \\ 2t^5 - 3t^4 + 3t^3 - t^2 & -4t^4 + 7t^3 - 4t^2 \end{pmatrix}$$

and we are not able to reduce it more with the algorithm.

Lemma 4.3.1. *The column class and row class of A are trivial.*

Proof. The determinant of A is $f := -3t^4 + 5t^3 - 5t^2 + 5t - 3$ and is irreducible. Let θ a root a f and $K = \mathbb{Q}(\theta)$. Using *sage* one is able to compute the picard group of $\mathcal{O}_K$ and one have that $\mathrm{Cl}_K = \langle [\mathfrak{p}_3] \rangle$ of order 2. Therefore, we deduce that the picard group of $\mathcal{O}_K[1/3]$ is trivial. On the other hand, using *sage* one obtain that that $\mathbb{Z}[\theta, \theta^{-1}] = \mathcal{O}_K[1/3]$ and therefore we have that the column and row class are trivial. $\square$

So we are not able to prove that the Nakanishi index of $12n_{260}$ is 2 using the row and column class method (the same happens with $12a_{1264}$ and $12a_{1049}$). Since $\det A =: f$ is irreducible a priori we couldn't use the pullback diagram method to prove that the Nakanishi index is 2.

Observation 4.3.2. Let $f = \det A$ and θ a root of f. Defining $a = \theta + \theta^{-1}$ we have that $\mathbb{Q}(a)$ is a quadratic field and f factors over $\mathbb{Z}[a, 1/3]$ in two polynomials. So now one can try to use the mayer-vietoris sequence to prove that $M \otimes \mathbb{Z}[a, 1/3]$ is not cyclic in order to prove that M is not cyclic. But this fails since if one compute the pull-back diagram method obtain that $M \otimes \mathbb{Z}[a, 1/3]$ is cyclic.

Even though we are not able to compute the nakanishi index with the methods stated before, we can try to compute $n_p(K)$ for p positive integers. However, we try it and we obtained that $n_p(K) = 1$ for each one of them. This fact motivated the results of the following section, that will allow us to prove that the nakanishi index is 1, and moreover will allow us to compute the algebraic unknotting number.

4.4 PID/Dedekind method.

Let K be a knot with Alexander module M and Alexander polynomial Δ_K. If Δ_K is irreducible, we have that for all non trivial element x of the Alexander module holds $\mathrm{Ann}_\Lambda(x) = \mathrm{Ann}_\Lambda(M) = (\Delta_K)$. Therefore, we have that the Λ-module structure on M induces a $R := \Lambda/\Delta_K$-module structure. In this setting we have the following result.

Lemma 4.4.1. *If $R := \Lambda/\Delta_K$ is a principal domain, then M is cyclic module.*

Proof. Since R is a principal domain we have, in particular, that Δ_K is irreducible.

As we already stated the sequnce

$$0 \to \Lambda^n \xrightarrow{A} \Lambda^n \to M \to 0$$

is exact. Since $\cdot \otimes_\Lambda R$ is a right exact functor, we have that the sequence

$$\Lambda^n \otimes_\Lambda R \xrightarrow{A} \Lambda^n \otimes_\Lambda R \to M \otimes_\Lambda R \to 0$$

is exact. By the above discussion we, have that $M \otimes_\Lambda R \cong M$ as Λ-modules. Therefore the last exact sequence can be rewrite as

$$R^n \xrightarrow{\overline{A}} R^n \to M \to 0.$$

Since R is a principal domain, we have that

$$M \cong \mathrm{Torsion}_R(M) \oplus R^m$$

as R-modules, for some $m \geq 1$. Suppose that $\overline{x}$ is an R-torsion element, therefore there exists $f(t) \in \mathbb{Z}[t^{\pm 1}]$ with $f(t) \not\equiv 0 \pmod{\Delta_K}$, (i.e, $f(t)$ coprime to Δ_K over $\mathbb{Q}[t^{\pm 1}]$) such that $f(t)x = 0$. But $\Delta_K x = 0$, then since $f(t)$ and Δ_K are coprime $\mathbb{Q}(t^{\pm 1})$ exists $z \in \mathbb{Z}$ such that $z \in (f(t), \Delta_K)$. Therefore, we deduce that x is a $\mathbb{Z}$-torsion element. Since M is an Alexander module, this implies that $x = 0$, so we have that $\mathrm{Torsion}_R(M) = 0$. On the other hand, $\mathbb{Q}[t^{\pm 1}]$ is flat over $\mathbb{Z}[t^{\pm 1}]$, therefore we have that

$$0 \to \Lambda^n \otimes_\Lambda \mathbb{Q}[t^{\pm 1}] \xrightarrow{A} \Lambda^n \otimes_\Lambda \mathbb{Q}[t^{\pm 1}] \to M \otimes_\Lambda \mathbb{Q}[t^{\pm 1}] \to 0$$

is a short exact sequence. Since $\mathbb{Q}[t^{\pm 1}]$ is a principal domain we have that exists P, Q matrices with coefficient in $\mathbb{Q}[t^{\pm 1}]$ and with determinant in $\mathbb{Q}^\times$, such that PAQ is a diagonal matrix. Since $\det A = \Delta_K$ is irreducible, we deduce that $M \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}[t^{\pm 1}] \cong \mathbb{Q}[t^{\pm 1}]/\Delta_K$. Therefore, we have that $m = 1$, this concluedes the proof of the lemma. $\square$

Observe that the isomorphism $M \cong \mathrm{Torsion}_R \oplus R^m$ is obtained by computing the smith form of $\overline{A}$ over R. Therefore, in the case that R is a principal domain we can compute the Blanchfield form in the following way.

Let $\phi : R \times R \to R$ be the map that makes the following diagram

$$\begin{tikzcd}
M \times M \arrow[r, "\mathrm{BL}_K"] \arrow[d] & \frac{1}{\Delta_k}\Lambda/\Lambda \arrow[d] \\
R \times R \arrow[r] \arrow[d] & \frac{1}{\Delta_K}\Lambda/\Lambda \arrow[d, "\cdot\Delta_K"] \\
R \times R \arrow[r, "\phi"] & R.
\end{tikzcd}$$

commutative. Since all the vertical maps are isomorphisms, we have that ϕ is a Hermitian form, and therefore there exists $q \in R$ symmetric such that $\phi(a, b) = \overline{a}qb$. Using lemma 2.3.2 we can compute q, and by the last commutative diagram we obtain that the Blanchfield form is isometric to the form

$$\begin{aligned} R \times R &\longrightarrow \Omega/\Lambda \\ (a, b) &\longmapsto \overline{a}\frac{q}{\Delta_K}b. \end{aligned}$$

Lemma 4.4.2. *Let $R := \Lambda/\Delta_K$ be a dedekind domain, then $m(K) \leq 2$. In particular, the nakanishi index of K is determinated by the row/column class of K.*

Proof. Analogously, one obtains that M is an $R-$module with no torsion with rank 1. Let L be the fraction field of R. Then we have that $M \to M \otimes_R L \cong L$ is injective. So M is isomorphic to an R-submodule of L i.e M is isomorphic to a fractional ideal. As every fractional ideal in an integral domain is isomorphic to an integral ideal and R is a Dedekind domain, we have that M is isomorphic to an invertible ideal. All ideals in a Dedekind domain are generated by, at most, 2 elements. Therefore, we deduce that $m(K) \leq 2$. □

4.5 Table of results and conclusions

Using the upper bounds found in [1] and using the algorithm that describes $n(K)$ and the lower bounds of the Nakanishi index described before, we obtained the following:

Knots	Nakanishi index	u_a	Method to find u_a
$12a_{0050}$	1	2	$n(K)$
$12a_{0141}$	1	2	$n(K)$
$12a_{0364}$	1	1	$n(K)$
$12a_{0649}$	1	1	$n(K)$
$12a_{0728}$	1	2	$n(K)$
$12a_{0791}$	1	1	$n(K)$
$12a_{0901}$	1	1	$n(K)$
$12a_{1049}$	1	1	$n(K)$, PID
$12a_{1054}$	2	2	Row/Column class
$12a_{1064}$	1	1	$n(K)$
$12a_{1138}$	1	1	$n(K)$
$12a_{1141}$	2	2	Row/Column Class
$12a_{1234}$	1	1	$n(K)$
$12a_{1236}$	1	1	$n(K)$
$12a_{1264}$	1	1	$n(K)$, PID
$12n_{0200}$	1	2	$n(K)$
$12n_{0260}$	1	1	$n(K)$, PID
$12n_{0657}$	2	2 ($u = 2$)	Pullback diagram.
$12n_{0864}$	1	1	$n(K)$

These results should not be understood as a complete truth, since the algorithm writen is not completly rigorous. However, these results should motivate the implementation of all the methods given through the thesis in knot softwares.

Conclusions:

From Knotorius [1] the algebraic unknotting number of the knots up to 11 crossings are completly determined by he maximum absolute value of Levine-Tristram signatures, the Nakanishi index and the Lickorish obstruction. So one could think that these invariants determine the algebraic unknotting number, however the list above shows that there are knots with algebraic unknotting number 2 that are not determined by Levine-Tristram signatures, the Nakanishi index or the Lickorish obstruction.

Observe that the row/columns class invariants, the pullback diagram method and the PID/Dedekind method are obstructions that we can easily compute using an algorithm. In the future, these invariants could be useful to compute the algebraic unknotting number for knots with number of crossings greater than 12.

From the above discussion we obtain, in particular, that for a knot K with Λ/Δ_K a Dedekind domain, we can determine if the algebraic unknotting number of K is 1 algorithmically.

Chapter 5

Appendix

5.1 Pseudo-trivial Alexander polynomials.

Definition 5.1.1. Given an Alexander polynomial Δ (symmetric with $\Delta(1) = 1$), we say that it is **weakly pseudo-trivial,** if for each knot K with $\Delta_K = \Delta$ we have that $u_a(K) = 1$. We say that Δ **strongly pseudo-trivial** if it is weakly pseudo-trivial and all knots with alexander polynomial Δ are algebraically equivalent. We define L the fraction field of $R := \mathbb{Z}[t^{\pm 1}]/\Delta$ and σ the automorphism $\sigma(\tau) = \tau^{-1}$.

Question: Do pseudo-trivial Alexander polynomials exist? Can we determine them?

Proposition 5.1.2. *Let K be a knot with Alexander polynomial Δ_K and R a principal domain. Then,*

- *If $[(R^\sigma)^\times : \pm Nm_{L|L^\sigma} R^\times] = 1$, then Δ_K is weakly pseudo-trivial.*
- *If $[(R^\sigma)^\times : Nm_{L|L^\sigma} R^\times] = 1$, then Δ_K is strongly pseudo-trivial.*

Proof. The proof is deduced from last chapter and the fact that two knots are algebraically equivalent if and only if the corresponding Blanchfield form are equivalent. □

Example 5.1.3. The knot 4_1 is strongly pseudo-trivial.

The following proposition will show that the converse of 1.4.3 is true in more cases.

Proposition 5.1.4. *Let K be a knot and J be a strongly pseudo-trivial knot. Then, if $Bl_K(x,x) \equiv \pm\frac{\Delta_J}{\Delta_K} \pmod{\mathbb{Z}[t^{\pm 1}]}$ for some x in $H_1(X(K);\mathbb{Z}[t^{\pm 1}])$, we have that $d_G^a(K,J) \leq 1$.*

Proof. Essentially we will follow the arguments in [11]. Let x be an element of $H_1(X(K);\mathbb{Z}[t^{\pm 1}])$ such that $Bl_K(x,x) \equiv \epsilon\frac{\Delta_J}{\Delta_K} \pmod{\mathbb{Z}[t^{\pm 1}]}$ with $\epsilon = \pm 1$. By [11] (lemma 5.5) there exists disk D with the following properties:

- D intersects K transversely in two points with opposit signs.
- $\partial D =: c$ represents x.
- If we equipp the curve c with the framing ϵ we obtain that $\mathrm{lk}_t(c,c) = \epsilon\frac{\Delta_J}{\Delta_K}$.

Let J' denote the knot obtanied after applying a $\mp$ crossing change to K in the crossing given by the points $D \cap K$. We denote W to be the 4-manifold obtained by adding a 2-handle along c to $M(K) \times [0,1]$ with framing ϵ. Then, we obtain that $\partial W = -M(K) \cup M(J')$ for some knot J'. By [11] (section 4.3), the manifold W has furthermore the following properties:

- $H_1(W;\mathbb{Z}) = \mathbb{Z}$,
- $H_1(M(K);\mathbb{Z}) \to H_1(W;\mathbb{Z})$ and $H_1((M(J');\mathbb{Z})) \to H_1(W;\mathbb{Z})$ are isomorphisms,
- $H_1(W;\mathbb{Z}[t^{\pm 1}]) \cong H_1(M(K);\mathbb{Z}[t^{\pm 1}])/(c) = 0$.

Furthermore, by [11] (Proposition 4.6), we have that

$$\det(Q_W) \doteq lk_t(c,c) \cdot \Delta_K^2 \doteq \Delta_J \Delta_K$$

It now follows by [11] (Proposition 4.5), that the alexander polynomial of J' is Δ_J. Since Δ_J is strongly pseudo-trivial, we have that J' is algebraically equivalent to J. Therefore, $d_G^a(K,J) \leq 1$. □

5.2 Script $n(K)$

These are the Python scripts used for the calculations.
In this section we will show the calculations made to find the algebraic unknotting number of $12a_{1236}$. For the other cases we compute it analogously. In the following script we compute q^*. Observe that we compute it over $\mathbb{Q}$ since *sage* can't do it over $\mathbb{Z}$. Sometimes (like in this example) we are lucky enough that all the polynomials found have coefficients in $\mathbb{Z}[t,s]$. When were not so lucky we used *MAGMA* with a similar approach.

Remark 5.2.1. *In our case, we don't need to compute q^*, since the Blanchfield form is non-singular and $q \in Nm_{K|L}R^\times$ if and only if $q^* \in Nm_{K|L}R^\times$.*

```
P.<t,s>=PolynomialRing(QQ,2)#12a_1236
z_1=t+s
z_2=t^2+s^2
z_3=t^3+s^3
z_4=t^4+s^4
Al=-s^4*(-2*t^8 + 9*t^7 - 21*t^6 + 33*t^5 - 37*t^4 + 33*t^3 - 21*t^2\
    + 9*t -2)#Alexander polynomial
q=-s^4*(-2*t^8 + 11*t^7 - 30*t^6 + 52*t^5 - 62*t^4 + 52*t^3 - 30*t^2\
    + 11*t -2) #Blanchifield form
I=ideal(Al,q,s*t-1)
I2=ideal(Al,q2,s*t-1)
a=t^0
L=a.lift(I)#inverse blancfhfield form.
S=a.lift(I2)
print(L[1])
d=Al.degree(t)
a=Al.coefficient({t:d,s:0})
2*t^3 + 2*t^2*s + 2*t*s^2 + 2*s^3 - 7*t^2 - 14*t*s - 7*s^2 + 10*t + 10*s + 1
```

In the next script we define the polynomial f that corresponds to $a^{2d-1}P(t/a)$ in the explanation made in chapter 4.

```
#Defines f in order to the find the field.
f=-(-2*t^8 + 9*t^7 - 21*t^6 + 33*t^5 - 37*t^4 + 33*t^3 - 21*t^2 + 9*\
   t -2)
a=2
g=P.hom([t/a,a*s])
f=a^(8-1)*g(f)
print(f)
I=ideal(s*t-1)
h=I.reduce(f)
t^8 - 9*t^7 + 42*t^6 - 132*t^5 + 296*t^4 - 528*t^3 + 672*t^2 - 576*t + 256
```

In the next script we define the number field $\mathbb{Q}(\tau)$, and we compute the generators $\tilde{R}^*$.

```
reset()
a=2 #rewrite
R.<x>=ZZ[]
f=x^8 - 9*x^7 + 42*x^6 - 132*x^5 + 296*x^4 - 528*x^3 + 672*x^2 - \
   576*x +256#rewrite
l=f.discriminant()
K.<z>=NumberField([f])
O=K.ring_of_integers()
print(O.gens()) #We deduce that R is dedekind.
print((1/K.discriminant()).factor())
#q=#rewrite
t=z/a
s=1/t
O=K.maximal_order()
S=K.ideal(a).prime_factors()
U=UnitGroup(K,S=tuple(S))
gen=U.gens()
gensv=U.gens_values()
L=[gen[i].multiplicative_order() for i in range(0,len(gen))] #\
   construction of the abelian group that represent the unit group.
L2=[]
for x in L:
    if x==+Infinity:
        L2.append(0)
    else:
        L2.append(x)
len(gen)#number of generators
```

```
(1, 1/2*z^7 + 1/2*z, 1/4*z^7 + 1/4*z^2, 7/8*z^7 + 1/8*z^3, 1/16*z^7 + 1/16*z^4, 31/32*z^7
+ 1/32*z^5, 49/64*z^7 + 1/64*z^6, z^7)
2^42
9
```

In the next script we check whether $q^*(\tau)$ is in $\tilde{R}^\times$. It is important to say that the function **in** does not work properly, but in general is easy to check by hand if $U.log(\pm q_0)$ is in H.

```
#write abcdefg... until number of generators.
A.<a,b,c,d,e,f,g,h,i>=AbelianGroup(L2)
#ABC=[a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,w,x,y,z]
g=A.gens()
unit=U.gens_values()
print(unit[0])
unit2=A.gens()
G=K.automorphisms()
print(G[1](t)*t==1) #check if is the endomorphism that we want.
ID=G[0]
tau=G[1]
U.log(unit[0])
tau(unit[1])
L=G[0]*G[1]
```

```
L3=[U.log(G[0](x)*G[1](x)) for x in U.gens_values()]
L3
#Define qstar i -qstar
q_0= 2*t^3 + 2*t^2*s + 2*t*s^2 + 2*s^3 - 7*t^2 - 14*t*s - 7*s^2 + \
   10*t + 10*s+ 1#rewrite
print(G[1](q_0)==q_0)
q_02=-q_0
q_02
U.log(q_0)
L4=[A(vector(L3[i])) for i in range(0,len(L3))]
print(L4)
H=A.subgroup(L4)
H
A(vector(U.log(q_02)))
```

```
-1
True
(1, 0, 0, 0, 0, 0, 0, 0, 0)
1/32*z^6 - 7/32*z^5 + 3/4*z^4 - 7/4*z^3 + 11/4*z^2 - 4*z + 3
[(0, 0, 0, 0, 0, 0, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0, 0, 0), (0, 0, 2, 0, 0, 0, 0, 0, 0),
(1, 1, 1, 0, 0, 0, 0, 0, 0), (0, -1, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1, 0, 1, 0),
(0, 0, 0, 0, 0, 0, 1, 0, 1), (0, 0, 0, 0, 0, 1, 0, 1, 0), (0, 0, 0, 0, 0, 0, 1, 0, 1)]
True
1/32*z^6 - 7/32*z^5 + 3/4*z^4 - 7/4*z^3 + 11/4*z^2 - 4*z + 1
(1, 0, 1, 0, 0, 0, 0, 0, 0)
[1, b^2, c^2, a*b*c, b^-1, f*h, g*i, f*h, g*i]
Multiplicative Abelian subgroup isomorphic to Z x Z x Z x Z x Z x Z x Z x Z generated by
{b^2, c^2, a*b*c, b^-1, f*h, g*i, f*h, g*i}
c
```

In the case that we do these computations and obtain that $U.log(q_0)$ is in H we check have to check if $\tilde{R} = R$. We can do that computing the discriminant or computing the generators of $\mathcal{O}_K$. In this case we obtain that $R = \tilde{R}$.

5.3 Script $12n_{0657}$

The following script will show the computations made to deduce that the Nakanishi index of $12n_{0657}$ is 2. To show the first lemmas we essentially used the command **.smith_form()**.

```
R.<t>=ZZ[]
M1,M2=matrix([[-(1-4*t^2),4-4*t-5*t^3+t^4-t^5],[-(-t+3*t^2-t^3)\
    ,-1+3*t^2]]),matrix([[-(1-4*t^2),4-4*t-5*t^3+t^4-t^5],[-(-t+3*t\
    ^2-t^3),-1+3*t^2]])
h=M1.det()
print(h.factor())
M1
```

```
(t^4 - 3*t^3 + 3*t^2 - t + 1) * (t^4 - t^3 + 3*t^2 - 3*t + 1)
[              4*t^2 - 1 -t^5 + t^4 - 5*t^3 - 4*t + 4]
[          t^3 - 3*t^2 + t                 3*t^2 - 1]
```

```
R.<t>=ZZ[]
f=t^4 - 3*t^3 + 3*t^2 - t + 1
g=t^4 - t^3 + 3*t^2 - 3*t + 1
L1.<w1>=NumberField(f,'w1')
L2.<w2>=NumberField(g,'w2')
O1=NumberField(f,'w1').ring_of_integers()
O2=NumberField(g,'w2').ring_of_integers()
m1=matrix(O1,M1)
m2=matrix(O2,M2)
UK1=UnitGroup(L1)
UK2=UnitGroup(L2)
UK1.gens()
UK2.gens()
d1,u1,v1=m1.smith_form()
print(u1*m1*v1==d1)
d2,u2,v2=m2.smith_form()
print(u1^(-1))
print(v1)
print(u2)
print(v2)
b= 14*w1^3 - 17*w1^2 - 2*w1 - 5
J=O1.ideal(w1^4 - w1^3 + 3*w1^2 - 3*w1 + 1)
print(J.reduce(b))
```

```
for i in range(0,31):
    for j in range(0,31):
        if J.reduce(b)==J.reduce((w1-1)^i*(w1^3 -w1^2 + 3*w1 - 2)^j)\
    or J.reduce(-b)==J.reduce((w1-1)^i*(w1^3 -w1^2 + 3*w1 - 2)^j):
            i,j
J.idealstar()
(u0, u1)
(u0, u1)
True
[       4*w1^2 - 1 -4*w1^2 + 4*w1 - 8]
[w1^3 - 3*w1^2 + w1                 3]
[          1 w1^2 - w1 + 2]
[          0             1]
[                        0 -6*w2^3 + 3*w2^2 - 16*w2 + 10]
[                    -w2^3     -2*w2^3 + w2^2 - 2*w2 + 1]
[               3 -3*w2^3 - 7*w2 + 1]
[ -w2^3 - 3*w2 + 3     -w2^2 + w2 - 2]
w1^2 + 1
Multiplicative Abelian group isomorphic to C30 x C3
```

b is the component $(2,2)$ for the matrix $u1^{-1}u2$ seeing both matrix over $L1$ (changing $w2$ for $w1$).

5.4 Nakanishi index

The following scripts will show how we compute the Nakanishi index and the Blanchfield form.

```
def xxgcd(a,b):
    """return (g, x, y) such that a*x + b*y = g = gcd(a, b) but not \
   exactly"""
    x0, x1, y0, y1 = 0, 1, 1, 0
    nose=False
    if(a==0 or b==0):
        return False
    else:
        if (a%b==0 or b%a==0):
            return False
    while ( a!=0 and nose==False):
        if(b.leading_coefficient()%a.leading_coefficient()!=0):
            return False
        q, b, a = b // a, a, b % a
        y0, y1 = y1, y0 - q * y1
        x0, x1 = x1, x0 - q * x1
        nose=invertibilitat(a)
    if(a!=0):
        q, b, a = b // a, a, b % a
        y0, y1 = y1, y0 - q * y1
        x0, x1 = x1, x0 - q * x1
    return b, x0, y0
def invertibilitat(a):
    "checks if the element a is invertible."
    if(a==1 or a==-1 or a==t^(a.degree()) or a==-t^(a.degree())):
        return True
    else:
        return False
def troba1t(B,R,l,n):
    "find invertible elements of the matrix B and applies a change \
   of matrix to bring it in the position (1,1)"
    Aux1=identity_matrix(R,n)
    Aux2=identity_matrix(R,n)
```

```
    for i in range(l,n):
        for j in range(l,n):
            if (B[i][j]==1 or B[i][j]==-1 or B[i][j]==t^(B[i][j].\
degree()) or B[i][j]==-t^(B[i][j].degree())):
                B.swap_columns(l,j)
                Aux1=elementary_matrix(R,n,row1=l,row2=j)
                B.swap_rows(l,i)
                Aux2=elementary_matrix(R,n,row1=l,row2=i)
                return 0,Aux2,Aux1
    return 1,Aux1,Aux2
def eliminacio(B,l,n):
    "does gaussian elimination"
    Aux1=identity_matrix(R,n)
    Aux2=identity_matrix(R,n)
    H=block_matrix([[identity_matrix(R,l+1),0],[0,B[l,l]*\
identity_matrix(R,n-l-1)]])
    Aux1=H*identity_matrix(R,n)
    Aux2=identity_matrix(R,n)
    B=H*B
    for i in range(l+1,n):
        Aux1=elementary_matrix(R,n,row1=i,row2=l,scale=-B[i,l]//B[l,\
l])*Aux1
    for i in range(l+1,n):
        for j in range(l+1,n):
            B[i,j]=B[i,j]=B[i,j]-B[l,j]*(B[i,l]//B[l,l])
    for i in range(l+1,n):
        if(B[l,i]%B[l,l]!=0):
            B=B*elementary_matrix(R,n,row1=i,scale=B[l,l])
            Aux2=Aux2*elementary_matrix(R,n,row1=i,scale=B[l,l])
        Aux2=Aux2*elementary_matrix(R,n,row1=i,row2=l,scale=-B[l,i\
]//B[l,l]).transpose()
    for i in range(l+1,n):
        B[i,l],B[l,i]=0,0
    return B,Aux1,Aux2
def luckcol(B,R,l,n):
    "finds if there is two elements in the same col with xxgcd=1"
    Aux1=identity_matrix(R,n)
    Aux2=identity_matrix(R,n)
    for k in range(l,n):
        for i in range(l,n-1):
            for j in range(i+1,n):
                if(xxgcd(B[i][k],B[j][k])!=False):
                    g,s,q=xxgcd(B[i][k],B[j][k])
                    if invertibilitat(g):
                        B.swap_rows(l,i)
                        Aux1=elementary_matrix(R,n,row1=l,row2=i)*\
Aux1
```

```
                        B.swap_rows(l+1,j)
                        Aux1=elementary_matrix(R,n,row1=l+1,row2=j)*\
Aux1
                        B.swap_columns(l,k)
                        Aux2=Aux2*elementary_matrix(R,n,row1=l,row2=\
k)
                        H=identity_matrix(R,n)
                        H[l,l]=s
                        H[l,l+1]=q
                        H[l+1,l]=-B[l+1,l]
                        H[l+1,l+1]=B[l,l]
                        Aux1=H*Aux1
                        B=H*B

                        return True,B, Aux1, Aux2
    return False,B,Aux1,Aux2
def luckfil(B,R,l,n):
    "finds if there is two elements in the same col with xxgcd=1"
    Aux1=identity_matrix(R,n)
    Aux2=identity_matrix(R,n)
    for k in range(l,n):
        for i in range(l,n-1):
            for j in range(i+1,n):
                if(xxgcd(B[k][i],B[k][j])!=False):
                    g,s,q=xxgcd(B[k][i],B[k][j])
                    if invertibilitat(g):
                        B.swap_columns(l,i)
                        Aux2=Aux2*elementary_matrix(R,n,row1=l,row2=\
i)
                        B.swap_columns(l+1,j)
                        Aux2=Aux2*elementary_matrix(R,n,row1=l+1,\
row2=j)
                        B.swap_rows(l,k)
                        Aux1=elementary_matrix(R,n,row1=l,row2=k)*\
Aux1
                        H=identity_matrix(R,n)
                        H[l,l]=s
                        H[l+1,l]=q
                        H[l,l+1]=-B[l,l+1]
                        H[l+1,l+1]=B[l,l]
                        B=B*H
                        Aux2=Aux2*H
                        return True,B,Aux1,Aux2
    return False,B,Aux1,Aux2
def elementarymatrixconstruction(R,l,n):
    "constructs a `pseudorandom invertible matrix with coeffiecients\
    in R"
```

```
    L=identity_matrix(R,n)
    for i in range(0,2):
        C=identity_matrix(R,l)
        A=block_matrix([ [C, 0], [0,elementary_matrix(R, n-l, row1=\
randrange(0,n-l), row2=randrange(0,n-l)) ] ])
        L=A*L
        rnd1=randrange(0,n-l)
        rnd2=randrange(0,n-l)
        if(rnd1!=rnd2):
            A=block_matrix([[C,0],[0,elementary_matrix(R,n-l,row1=\
rnd1,row2=rnd2,scale=R.random_element())]])
            L=A*L
    return L
def canviadebasepersort(B,P,X1,X2,R,l,n):
    Aux=elementarymatrixconstruction(R,l,n)
    B=B*Aux
    X2=X2*Aux
    L=luckcol(B,R,l,n)
    B,X1,X2=L[1],L[2]*X1,X2*L[3]
    if L[0]==True:
        L=eliminacio(B,l,n)
        B=L[0]
        X1=L[1]*X1
        X2=X2*L[2]
        pots=0
        pots2=1
        return B,X1,X2,pots,pots2
    else:
        L=luckfil(B,R,l,n)
        if L[0]==True:
            L=eliminacio(B,l,n)
            B=L[0]
            X1=L[1]*X1
            X2=X2*L[2]
            pots=0
            pots2=1
            return B,X1,X2,pots,pots2
    return B,X1,X2,1,0
n=8#dimension of matrix
l=0
while(n-l!=1):
    R.<t> = ZZ[]
    n=#dimension of matrix
    l=0
    B=matrix(R,)#Seifert matrix
    A,P=(t*B-B.transpose()),(t*B-B.transpose()) #presentation matrix\
    of Alexander module
```

```
 C1=identity_matrix(R,n)
 C2=identity_matrix(R,n)
 esfors=100
 pots=0
 l=0
 esfors2=1
 contado2=0
 while(pots==0 and l<n-1 and contado2<esfors2):
     Aux=elementary_matrix(R, n, row1=randrange(l,n), row2=\
randrange(l,n))
     Aux2=elementary_matrix(R, n, row1=randrange(l,n), row2=\
randrange(l,n))
     A=Aux2*A*Aux
     C2=C2*Aux
     C1=Aux2*C1
     L=trobalt(A,R,l,n)
     pots=L[0]
     C1=L[1]*C1
     C2=C2*L[2]
     if pots==0:
         L=eliminacio(A,l,n)
         A=L[0]
         C1=L[1]*C1
         C2=C2*L[2]

     else:
         L=luckcol(A,R,l,n)
         if L[0]==True:
             A,C1,C2=L[1],L[2]*C1,C2*L[3]
             L=eliminacio(A,l,n)
             A=L[0]
             C1=L[1]*C1
             C2=C2*L[2]
             pots=0
         else:
             L=luckfil(A,R,l,n)
             if L[0]==True:
                 A,C1,C2=L[1],L[2]*C1,C2*L[3]
                 L=eliminacio(A,l,n)
                 A=L[0]
                 C1=L[1]*C1
                 C2=C2*L[2]
                 pots=0
             else:
                 contador=0
                 pots2=0
                 Aux=identity_matrix(R,n)
```

```
                    while(contador<esfors and pots2==0):
                        L=canviadebasepersort(A,P,C1,C2,R,l,n)
                        if L[3]==0:
                            A=L[0]
                            C1=L[1]*C1
                            C2=C2*L[2]
                            pots2=1
                        contador=contador+1
        if pots==0:
            l=l+1
    C1*P*C2==A
    if(C1*P*C2!=A):
        n-1
print(C1*P*C2==A)
s=R.hom([1/t])
S=matrix([[s(C1[i,j]) for i in range(0,n)] for j in range(0,n)])
Bl=(t-1)*P^(-1)
Cvar=S^(-1)
BL=Cvar*Bl*C1^(-1)
f=BL[n-1,n-1]
print(n-1)#nakanishi
#conjugamat(R,C1,n)
#C1
a=f.numerator()
b=f.denominator()
print(f)
a
b
print(A)
print(a%b)
```

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