Mølmer's virtual cavity method

Bruno O. Goes

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These notes are devoted to an important tool to perform the simulations of driven systems: Mølmer's virtual cavity method [1, 5]. It is based on the general SLH (**S**cattering matrix, co**L**lapse operators and **H**amiltonian) framework [2], that is built upon the input-output formalism [4] introduced by Collet and Gardiner and on its further development of cascaded systems [3].

1 SLH framework in a nutshell

The SLH framework [2] ascribes a triple $G = (S, L, H)$ for each of the systems involved in the physical problem. By drawing a network connecting the components (or modules) and using the correct prescribed composition rules one obtains a final triple G_{Total} that fully describes the dynamics of the system and all its components.

For the sake of analogy, we can say it is similar to the circuit laws of electrical components. To obtain the net resistance, capacitance, tension, etc. in a circuit it is possible to start from the bottom and use the Maxwell's equations or one simply draws a diagram with all the components correctly connected and then obtain the desired answer using the proper laws. The SLG framework has the same philosophy.It is grounded on the solidly stablished input-output formalism [4] and cascaded quantum systems theory [3]. Hence, it is valid when these are valid, namely the Markovian limit of light-matter interaction where there is no back-action on the source, and where there is dispersionless propagation of the field between components.

The advantage of such a framework is two-folded: it is modular, each component is a node of a "circuit" described by its own triple, and it is straightforward to apply. The workflow goes as follows:

1. **Modules description:** Each quantum system, or module (component) is associated with a triple of operator $G = (S, L, H)$ where S is the scattering matrix describing the transformation between what enter (the input) and what goes out (the output) of it, L stands for the collapse operator of the system, and H is the Hamiltonian describing the system.

2. **Draw a quantum input-output network:** By routing the output of one module to the input of another one creates a network.

3. **Network reduction:** the algebraic rules of the SLH framework are used in order to eliminate all the internal connection and describe everything in the network by a single triple G.

4. **Derivation of the equations of motion.**

5. **Simulate the network** with the most appropriated computational method.

The algebraic rules are the following [2].

1. **Series product or cascaded rule.** Let $G_1 = (\mathbf{S}_1, \mathbf{L}_1, H_1)$ and $G_2 = (\mathbf{S}_2, \mathbf{L}_2, H_2)$ be the triples describing two localized quantum systems with the same number of inputs and outputs. The output of G_1 is routed to be the input of G_2, meaning the systems are in series. The series product, denoted as $G_2 \triangleleft G_1$ is given by,

$$\begin{aligned} G_T &= (\mathbf{S}_T, \mathbf{L}_T, H_T) = G_2 \triangleleft G_1, \\ G_2 \triangleleft G_1 &= (\mathbf{S}_2, \mathbf{L}_2, H_2) \triangleleft (\mathbf{S}_1, \mathbf{L}_1, H_1) \\ &= \left(\mathbf{S}_2\mathbf{S}_1, \mathbf{L}_2 + \mathbf{S}_2\mathbf{L}_1, H_1 + H_2 - \frac{i}{2}(\mathbf{L}_2^\dagger \mathbf{S}_2 \mathbf{L}_1 - \text{h.c.}) \right) \end{aligned}$$

   where h.c. stands for hermitian conjugate. Importantly, $G_2 \triangleleft G_1 \neq G_1 \triangleleft G_2$, implying a directionality. Indeed, if G_2 describes a beam-splitter and G_1 described a cavity we have two completely different physical situations with we route the field of a beam-splitter into a cavity or if we do the other way around.

2. **Concatenation product with direct coupling.** Now we consider the tuples $G_1 = (\mathbf{S}_1, \mathbf{L}_1, H_1 + H_{int})$ and $G_2 = (\mathbf{S}_2, \mathbf{L}_2, H_2)$ where H_{int} is a possible internal interaction between the two systems. This coupling is ambiguously located, we adopt the convention that it is added in the first tuple. The systems are in parallel with each other. The concatenation product of G_1 and G_2, denoted as $G_2 \boxplus G_1$, is given by,

$$\begin{aligned} G_T &= G_2 \boxplus G_1, \\ G_2 \boxplus G_1 &= (\mathbf{S}_2, \mathbf{L}_2, H_2) \boxplus (\mathbf{S}_1, \mathbf{L}_1, H_1) \\ &= \left(\begin{bmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{bmatrix}, H_1 + H_2 + H_{int} \right). \end{aligned}$$

   For further rules and discussion on the method we refer the reader to Ref. [2].

2 Mølmer's virtual cavity method

We consider an incident pulse with a temporal envelope $u(t)$ and arbitrary quantum state that interacts with strength $\sqrt{\gamma}$ with an arbitrary quantum system observable c, see Fig. 1. The system is described by a Hamiltonian H_s. For the sake of concreteness, we can consider it to be a qubit inside a resonator. Mølmer's theory [1,5] is simple yet elegant: the situation of arbitrary traveling pulses is modeled by *virtual cavities* with *complex, time-dependent mirror couplings*, $g_u(t)$ and $g_v(t)$, tuned in such a way that they exactly eject mode $u(t)$ and absorb mode $v(t)$. The traveling incident and output pulses are stored in two virtual cavities. This allows the derivation of a master equation where the input and output pulses are treated as *single oscillator modes* that both couple to the local system in a *cascaded manner*. Hence, with the previous discussion of the SLH framework it suffices two write down the correct tuple for the system and then couple it to the two virtual cavities using the series product. The heart of the method lies in the fact that any wave packet can be emitted as the output from or absorbed as the input to—a *virtual one-sided cavity* with time-dependent complex coupling $\bar{g}_u(t)$ to its input continuum fields. This time dependent coupling is tunned in such a way that the real temporal form $u(t)$ is leaked out by the cavity. The form of

the coupling is [5],

$$g_u(t) = \frac{\bar{u}(t)}{\sqrt{1 - \int_0^t dt' |u(t')|^2}}$$

The source is described by the annihilation operator a, obeying the bosonic algebra $[a, a^\dagger] = \mathbb{I}$ and we consider the target system to be a qubit with lowering operator σ. The source may be driven by an input field a_{in}.

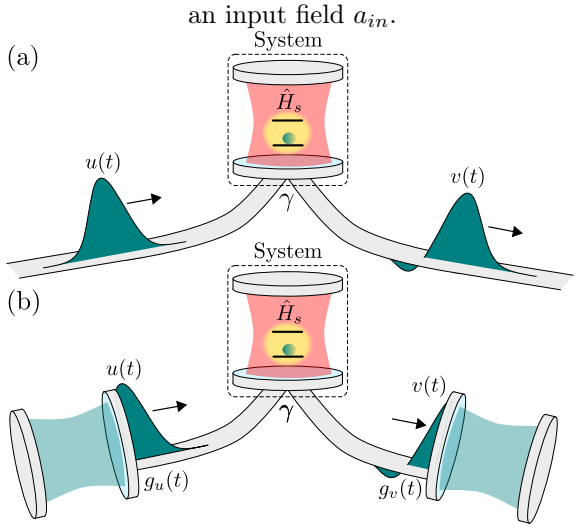

Figure 1: **Mølmer's virtual cavity sketch.** Figure taken from Ref. [5]. a) An incident pulse with a temporal shape $u(t)$ and arbitrary quantum state content interacts with strength $\sqrt{\gamma}$ with an arbitrary quantum system observable. The system is described by a Hamiltonian H_S, and it is depicted here as a qubit inside a resonator. The method provides the full quantum state of light occupying any reflected temporal pulse $v(t)$. b) The physical scenario of arbitrary traveling pulses in (a) is modelled by virtual cavities with complex, time dependent mirror-couplings, $g_u(t)$ and $g_v(t)$, finely tuned so that they eject and absorb precisely the modes $u(t)$ and $v(t)$.

3 SLH framework for the spin-photon interface

Let's consider the scenario of a for level system (4LS) interacting with two orthogonal polarization modes presented in Fig. 2, as in Ref. [7], to apply the formalism.

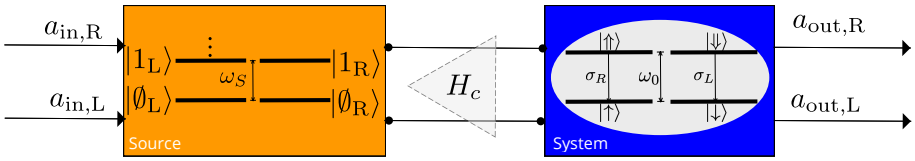

Figure 2: **SLH quantum network diagram for source-4LS cascaded interaction mediated by two orthogonal polarization modes.**

The light source can be modeled as a degenerate two-mode cavity a_R, a_L that is leaking R and

$L-$polarized light. This is described by the triple,

$$G_{\text{source}} = \left(\mathbb{I}, \begin{bmatrix} \sqrt{\eta\kappa}a_R \\ \sqrt{\eta\kappa}a_L \\ \sqrt{(1-\eta)\kappa}a_R \\ \sqrt{(1-\eta)\kappa}a_L \end{bmatrix}, \omega_s(a_R^\dagger a_R + a_L^\dagger a_L) \right).$$

The spin-photon interface not subjected to any magnetic field is described as a degenerate four-level system (4LS) composed of two independent two-level transition with circular polarized selection rules. The triple describing this system is,

$$G_{\text{4LS}} = \left(\mathbb{I}, \begin{bmatrix} \sqrt{\gamma}\sigma_R \\ \sqrt{\gamma}\sigma_L \\ 0 \\ 0 \end{bmatrix}, \omega_0(\sigma_R^\dagger \sigma_R + \sigma_L^\dagger \sigma_L) \right).$$

This is a cascaded system, so that using Rule (1) we obtain,

$$G_{\text{Tot}} = G_{\text{4LS}} \triangleleft G_{\text{source}} = \left(\mathbb{I}, \begin{bmatrix} \sqrt{\gamma}\sigma_R + \sqrt{\eta\kappa}a_R \\ \sqrt{\gamma}\sigma_L + \sqrt{\eta\kappa}a_L \\ \sqrt{(1-\eta)\kappa}a_R \\ \sqrt{(1-\eta)\kappa}a_L \end{bmatrix}, H_T \right)$$

where,

$$H_T = \omega_0(\sigma_R^\dagger \sigma_R + \sigma_L^\dagger \sigma_L) + \omega_s(a_R^\dagger a_R + a_L^\dagger a_L) - \frac{i}{2}\sqrt{\eta\kappa\gamma}\left(\sigma_R^\dagger a_R - a_R^\dagger \sigma_R + \sigma_L^\dagger a_L - a_L^\dagger \sigma_L\right). \quad (1)$$

For each polarization $j = R, L$ the source-4LS input output relations are,

$$b_{\text{in},j} - b_{\text{out},j} = \sqrt{\eta_{\text{losses}}}\,(\sqrt{\kappa\gamma}a_j + \sqrt{\gamma}\sigma_j)$$

where the efficiency factor $\sqrt{\eta_{\text{losses}}}$ accounts for possible additional losses due to imperfect transmission or detection on the system and is inserted *ad hoc*. Ideally $\eta_{\text{losses}} = 1$.

Assuming that the input fields on the source are in vacuum state $b_{\text{in},j} = |\emptyset\rangle$, the total cascaded system evolves under a Lindblad master equation,

$$\partial_t \rho = \mathscr{L}[\rho],$$

where the Liouviallian super-operator $\mathscr{L}$ is composed of three terms,

$$\mathscr{L}[\rho] = \mathscr{U}[\rho] + \mathscr{D}_0[\rho] + \mathscr{D}_d[\rho]$$

with

$$\mathscr{U}[\rho] = -i[H_T, \rho] \quad (2)$$

accounting for coherent dynamics,

$$\mathscr{D}_0[\rho] = \sum_{j=R,L} D[\sqrt{\gamma}\sigma_j, \rho],$$

describing the decay of the 4LS, $\mathscr{D}_0$ is composed of the jth element of the vector $\mathbf{L}_{\text{Tot}}$, and

$$\mathscr{D}_d[\rho] = D[\sqrt{\gamma^\star}(\sigma_R^\dagger \sigma_R + \sigma_L^\dagger \sigma_L), \rho] + D[\sqrt{\kappa^\star}(a_R^\dagger a_R + a_L^\dagger a_L), \rho]$$

where,

$$D[L, \bullet] = \left(L \bullet L^\dagger - \frac{1}{2} \left\{ L^\dagger L, \bullet \right\} \right)$$

is the dissipator. The dissipation channels with dephasing rates $\gamma^\star$ and $\kappa^\star$ are introduced on a phenomenological ground to account for possible additional experimental losses introduced due to the interaction with the solid state environment, such as Overhauser field or spectral wandering. By assuming that the dephasing acts with the same intensity in both polarization, i.e. it acts in a correlated manner, it does not introduce any depolarization or spin decoherence, only temporal decoherence plays a role. The addition of pure dephasing on the cavity $\kappa^\star$ is equivalent to the physical situation of having one mirror of the cavity that jitters on the timescale $\tau_{\text{jit}} = 1/\gamma^\star$. The solution of the master equation is easily obtained via vectorization method.

4 Simulating quantum dynamics with Melt!

Melt! is a useful Mathematica library specialized for the dynamics of quantum systems [6]. It has various quantum information, quantum thermodynamics and plot styling functions loaded that aids us to study quantum system in a straightforward and human friendly environment.

The workflow to use Melt! is as follows:

1. **Load the necessary operators:** in our case bosonic operators with "LoadBosonicOperators[dimFockspace]" where dimFockspace is the truncated dimension of the Fock space (in principle infinite). It loads the quadrature, creation and annihilation operators and also the thermal and coherent states. We also need the "LoadPauliMatrices[]" that loads pre-defined Pauli matrices

2. **Construct the Hilbert space:** using the proper tensor produce "kron[operator1,operator2,...]".

3. **Define the physical system:** Hamiltonian, collapse and Liouvillian "Liouvillian[H,list of collapse operators]" construct the vectorized form of the Liouvillian;

4. **Have fun!:** solve the dynamics numerically (or analytically) using standard linear algebra. For high dimensional systems it may be quite cumbersome if not impossible to obtain an analytical formula, but for 2 level systems or so it is quite easy.

References

[1] Victor Rueskov Christiansen, Alexander Holm Kiilerich, and Klaus Mølmer. Interaction of quantum systems with single pulses of quantized radiation. *arXiv:2203.07477 [quant-ph]*, March 2022. arXiv: 2203.07477.

[2] Joshua Combes, Joseph Kerckhoff, and Mohan Sarovar. The SLH framework for modeling quantum input-output networks. *Advances in Physics: X*, 2(3):784–888, May 2017.

[3] C. W. Gardiner. Driving a quantum system with the output field from another driven quantum system. *Physical Review Letters*, 70(15):2269–2272, April 1993.

[4] C. W. Gardiner and M. J. Collett. Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation. *Physical Review A*, 31(6):3761–3774, June 1985.

[5] Alexander Holm Kiilerich and Klaus Mølmer. Input-Output Theory with Quantum Pulses. *Physical Review Letters*, 123(12):123604, September 2019.

[6] Gabriel T. Landi. Melt! a mathematica quantum library. May 2023.

[7] Maria Maffei, Bruno O. Goes, Stephen C. Wein, Andrew N. Jordan, Loïc Lanco, and Alexia Auffèves. Energy-efficient entanglement generation and readout in a spin-photon interface, May 2022. arXiv:2205.09623 [quant-ph].