Grafos

Algoritmo de Floyd-Warshall

Prof. Edson Alves

Faculdade UnB Gama

Proponentes

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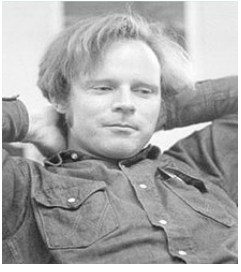
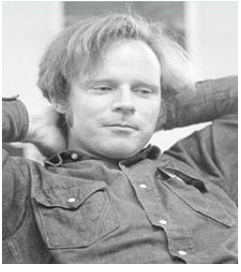


Robert W. Floyd
(1962)

Proponentes



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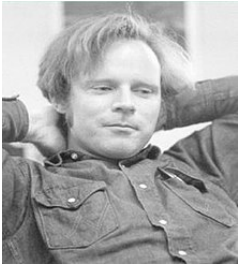

Robert W. Floyd
(1962)

Bernard Roy
(1959)

Características do algoritmo de Bellman-Ford

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- ⋆ **Complexidade:** $O(V^3)$

Pseudocódigo

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Saída: uma matriz d tal que $d[u][v]$ é a distância mínima em G entre u e v

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1. **Faça:**
   (a) $d[u][u] = 0$, **para todos** $u \in V$
   (b) $d[u][v] = w$, **se** $(u, v, w) \in E$
   (c) $d[u][v] = \infty$, **caso contrário**

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2. **Para cada vértice** k **e todos os pares** $(u, v) \in V^2$**, faça**

$$d[u][v] = \mathsf{min}(d[u][v], d[u][k] + d[k][v])$$

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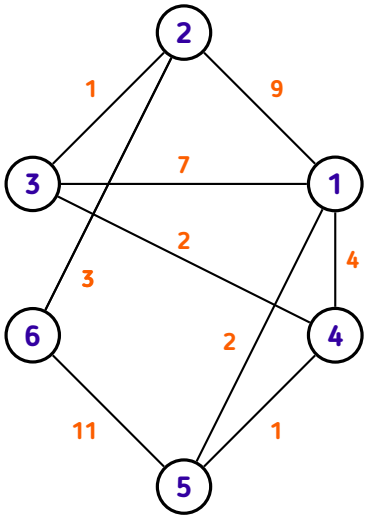
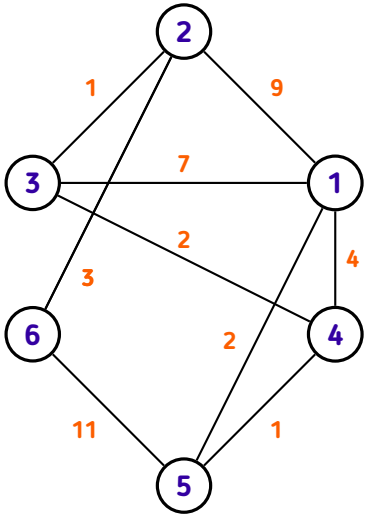
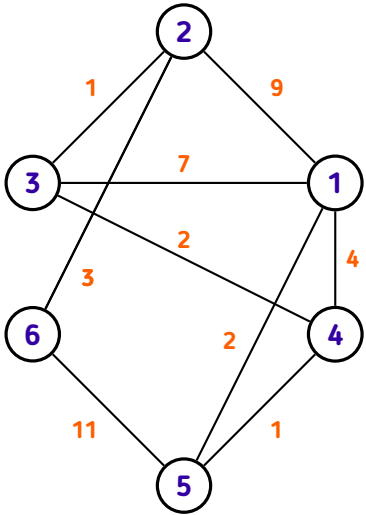
3. **Retorne** d



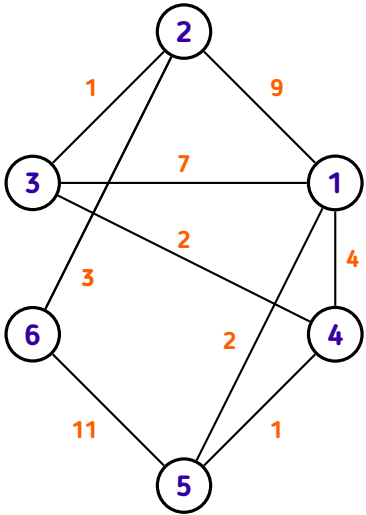
	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						



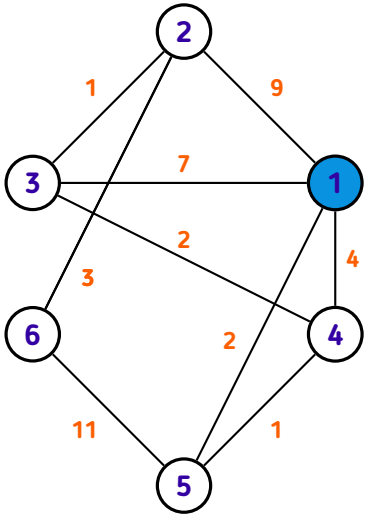
	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0



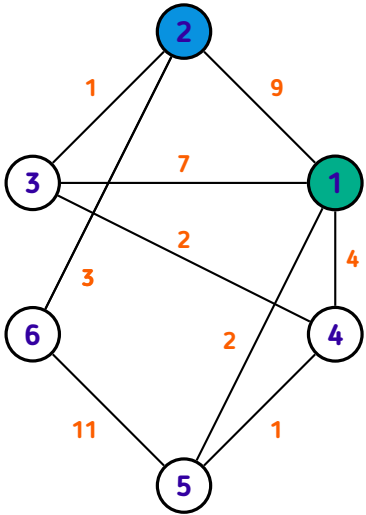
	1	2	3	4	5	6
1	0	9	7	4	2	
2	9	0	1			3
3	7	1	0	2		
4	4		2	0	1	
5	2			1	0	11
6		3			11	0



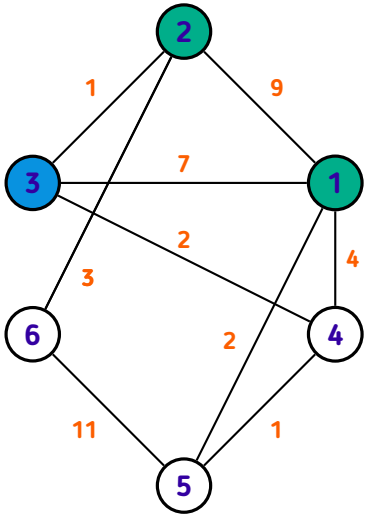
	1	2	3	4	5	6
1	0	9	7	4	2	∞
2	9	0	1	∞	∞	3
3	7	1	0	2	∞	∞
4	4	∞	2	0	1	∞
5	2	∞	∞	1	0	11
6	∞	3	∞	∞	11	0



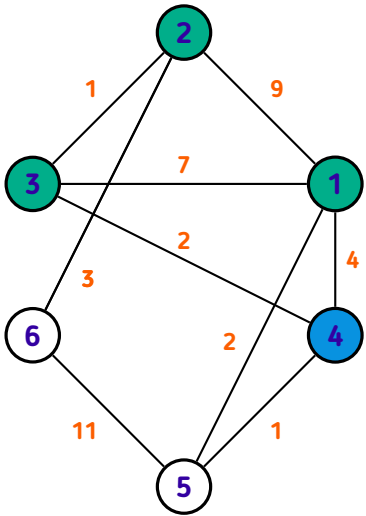
	1	2	3	4	5	6
1	0	9	7	4	2	∞
2	9	0	1	**13**	**11**	3
3	7	1	0	2	**9**	∞
4	4	**13**	2	0	1	∞
5	2	**11**	**9**	1	0	11
6	∞	3	∞	∞	11	0



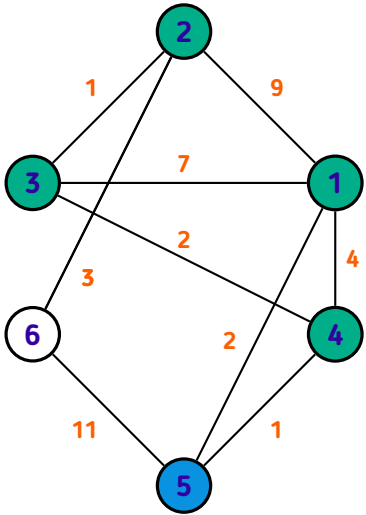
	1	2	3	4	5	6
1	0	9	7	4	2	**12**
2	9	0	1	13	11	3
3	7	1	0	2	9	**4**
4	4	13	2	0	1	**16**
5	2	11	9	1	0	11
6	**12**	3	**4**	**16**	11	0



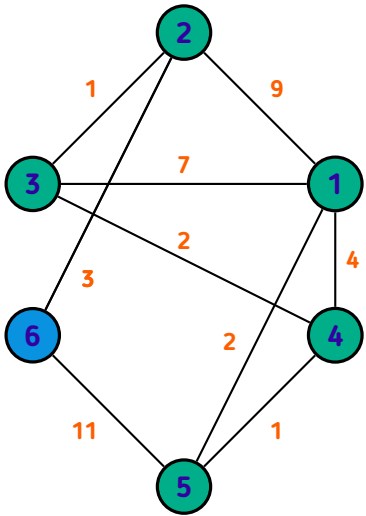
	1	2	3	4	5	6
1	0	**8**	7	4	2	**11**
2	**8**	0	1	**3**	**10**	3
3	7	1	0	2	9	4
4	4	**3**	2	0	1	**6**
5	2	**10**	9	1	0	11
6	**11**	3	4	**6**	11	0



	1	2	3	4	5	6
1	0	**7**	**6**	4	2	**10**
2	**7**	0	1	3	**4**	3
3	**6**	1	0	2	**3**	4
4	4	3	2	0	1	6
5	2	**4**	**3**	1	0	**7**
6	**10**	3	4	6	**7**	0



	1	2	3	4	5	6
1	0	**6**	**5**	**3**	2	**9**
2	**6**	0	1	3	4	3
3	**5**	1	0	2	3	4
4	**3**	3	2	0	1	6
5	2	4	3	1	0	7
6	**9**	3	4	6	7	0



	1	2	3	4	5	6
1	0	6	5	3	2	9
2	6	0	1	3	4	3
3	5	1	0	2	3	4
4	3	3	2	0	1	6
5	2	4	3	1	0	7
6	9	3	4	6	7	0

```cpp
vector<vector<int>> floyd_warshall(int N)
{
    vector<vector<int>> dist(N + 1, vector<int>(N + 1, oo));

    for (int u = 1; u <= N; ++u)
        dist[u][u] = 0;

    for (int u = 1; u <= N; ++u)
        for (auto [v, w] : adj[u])
            dist[u][v] = w;

    for (int k = 1; k <= N; ++k)
        for (int u = 1; u <= N; ++u)
            for (int v = 1; v <= N; ++v)
                if (dist[u][k] < oo and dist[k][v] < oo)
                    dist[u][v] = min(dist[u][v], dist[u][k] + dist[k][v]);

    return dist;
}
```

Identificação de um caminho mínimo

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- **O algoritmo de Dijkstra computa as distâncias mínimas, mas não os caminhos mínimos**

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⋆ **Para determinar um caminho mínimo, é preciso definir uma matriz auxiliar** pred**, onde** $\text{pred}[u][v] =$ **antecessor de** v **no caminho mínimo de** u **a** v

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⋆ **No início do algoritmo,**

(a) $\text{pred}[u][u] = u, \forall u \in V$

(b) $\text{pred}[u][v] = u$**, se** $(u, v) \in E$

(c) $\text{pred}[u][v] = \text{undef}$**, caso contrário**

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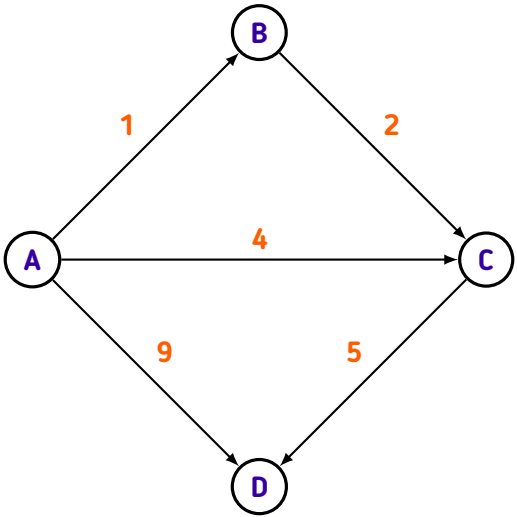
- ⋆ **Se** k **atualizar** $d[u][v]$**, faça** $\mathsf{pred}[u][v] = \mathsf{pred}[k][v]$

Identificação de um caminho mínimo

- **Se** k **atualizar** $d[u][v]$**, faça** $\mathsf{pred}[u][v] = \mathsf{pred}[k][v]$
- **A sequência**

$$p = \{(u, \mathsf{pred}^{k-1}[u][v]), \ldots, (\mathsf{pred}[\mathsf{pred}[u][v]], \mathsf{pred}[u][v]), (\mathsf{pred}[u][v], v)\}$$

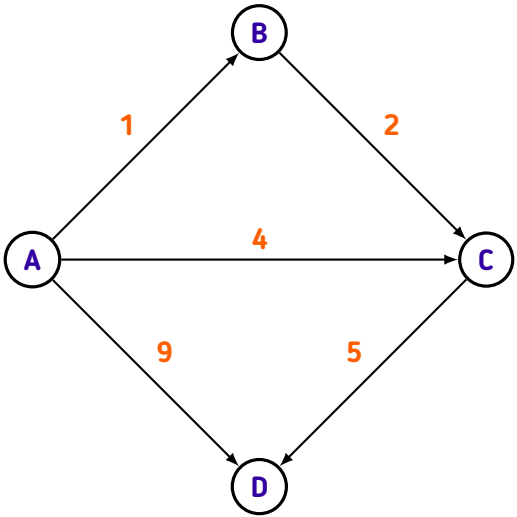
é um caminho mínimo de u **a** v **composto por** k **arestas e tamanho** $d[u][v]$



$M =$

	A	B	C	D
A				
B				
C				
D				

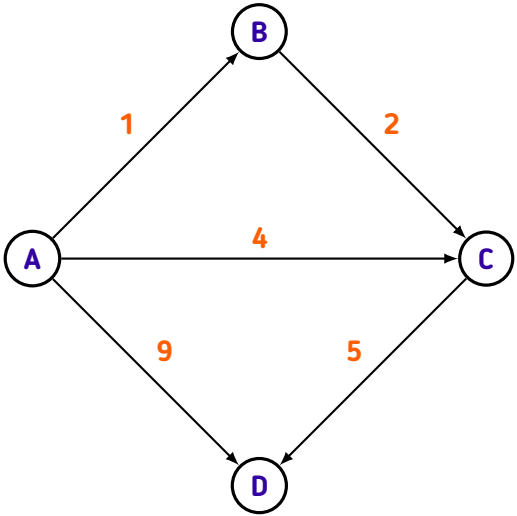
$$m_{ij} = \text{dist}[i][j]_{\text{pred}[i][j]}$$



$M =$

	A	B	C	D	
	0_A				A
		0_B			B
			0_C		C
				0_D	D

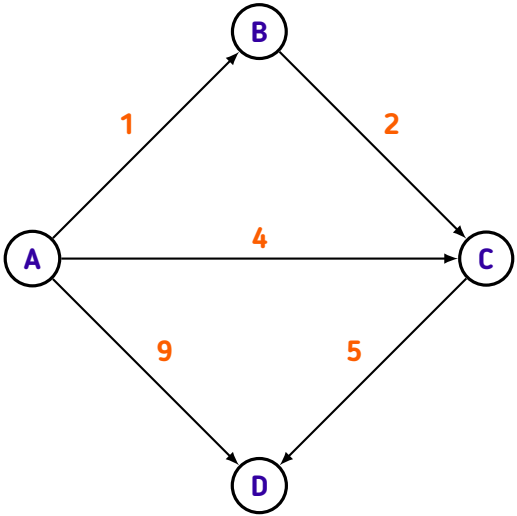
$$m_{ij} = \text{dist}[i][j]_{\text{pred}[i][j]}$$



$M =$

	A	B	C	D	
	0_A	1_A	4_A	9_A	A
		0_B	2_B		B
			0_C	5_C	C
				0_D	D

$$m_{ij} = \text{dist}[i][j]_{\text{pred}[i][j]}$$



$M =$

	A	B	C	D	
	0_A	1_A	4_A	9_A	A
	∞_-	0_B	2_B	∞_-	B
	∞_-	∞_-	0_C	5_C	C
	∞_-	∞_-	∞_-	0_D	D

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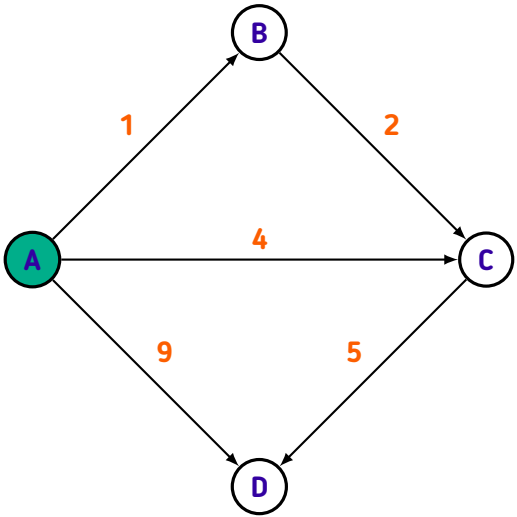

$M =$

	A	B	C	D	
	0_A	1_A	4_A	9_A	A
	∞-	0_B	2_B	∞-	B
	∞-	∞-	0_C	5_C	C
	∞-	∞-	∞-	0_D	D

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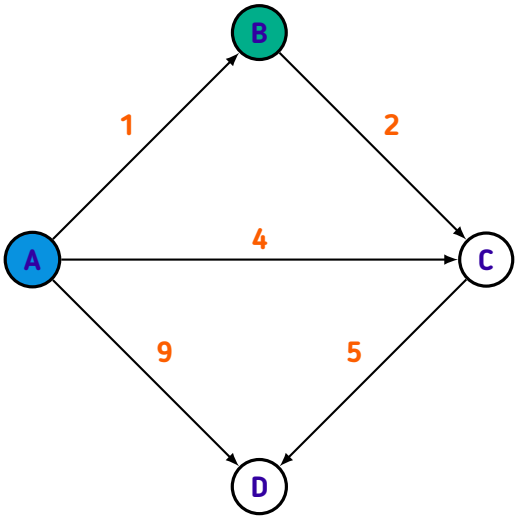

$M =$

	A	B	C	D	
	0_A	1_A	3_B	9_A	A
	∞-	0_B	2_B	∞-	B
	∞-	∞-	0_C	5_C	C
	∞-	∞-	∞-	0_D	D

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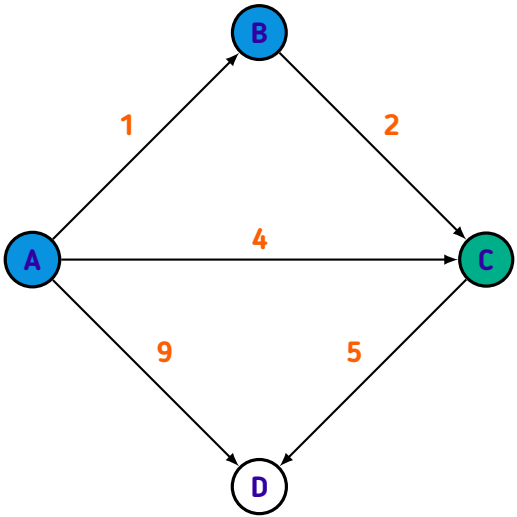
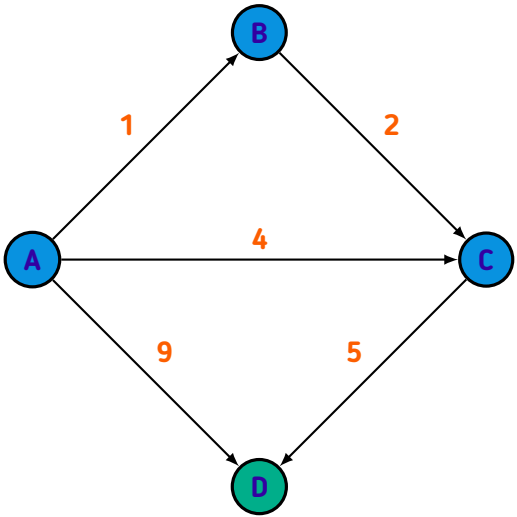


$$M = \begin{array}{c|cccc} & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \\ \hline \mathbf{A} & 0_{\mathbf{A}} & 1_{\mathbf{A}} & 3_{\mathbf{B}} & \mathbf{8_C} \\ \mathbf{B} & \infty_{\text{-}} & 0_{\mathbf{B}} & 2_{\mathbf{B}} & \mathbf{7_C} \\ \mathbf{C} & \infty_{\text{-}} & \infty_{\text{-}} & 0_{\mathbf{C}} & 5_{\mathbf{C}} \\ \mathbf{D} & \infty_{\text{-}} & \infty_{\text{-}} & \infty_{\text{-}} & 0_{\mathbf{D}} \end{array}$$

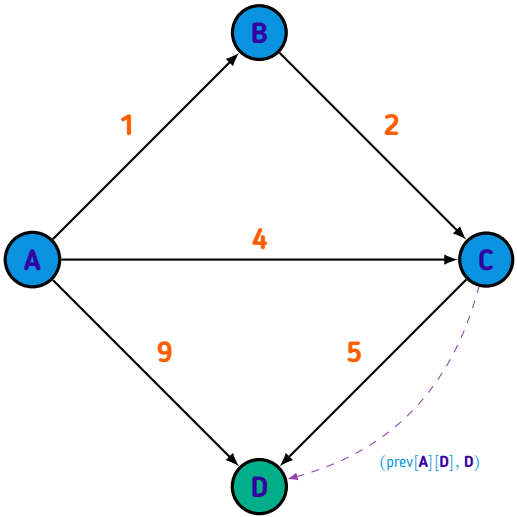
$$m_{ij} = \text{dist}[i][j]_{\text{pred}[i][j]}$$



$M =$

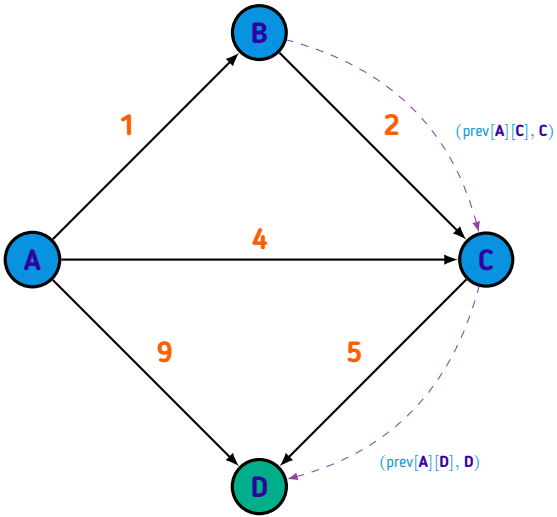
	A	B	C	D	
	0_A	1_A	3_B	8_C	A
	∞_-	0_B	2_B	7_C	B
	∞_-	∞_-	0_C	5_C	C
	∞_-	∞_-	∞_-	0_D	D

$$m_{ij} = \text{dist}[i][j]_{\text{pred}[i][j]}$$



$$M = \begin{array}{c|cccc} & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \\ \hline \mathbf{A} & 0_{\mathbf{A}} & 1_{\mathbf{A}} & 3_{\mathbf{B}} & 8_{\mathbf{C}} \\ \mathbf{B} & \infty_{\text{-}} & 0_{\mathbf{B}} & 2_{\mathbf{B}} & 7_{\mathbf{C}} \\ \mathbf{C} & \infty_{\text{-}} & \infty_{\text{-}} & 0_{\mathbf{C}} & 5_{\mathbf{C}} \\ \mathbf{D} & \infty_{\text{-}} & \infty_{\text{-}} & \infty_{\text{-}} & 0_{\mathbf{D}} \end{array}$$

$$m_{ij} = \text{dist}[i][j]_{\text{pred}[i][j]}$$



$M =$

A	B	C	D	
0_A	1_A	3_B	8_C	A
∞_-	0_B	2_B	7_C	B
∞_-	∞_-	0_C	5_C	C
∞_-	∞_-	∞_-	0_D	D

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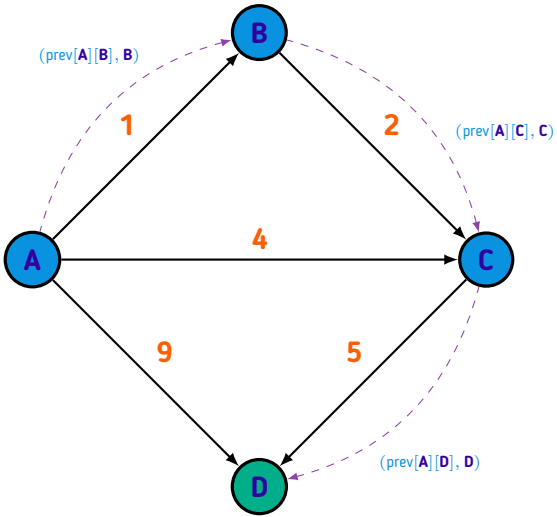

$M =$

A	B	C	D	
0_A	1_A	3_B	8_C	A
∞-	0_B	2_B	7_C	B
∞-	∞-	0_C	5_C	C
∞-	∞-	∞-	0_D	D

$$m_{ij} = \text{dist}[i][j]_{\text{pred}[i][j]}$$

```
pair<vector<vector<int>>, vector<vector<int>>>
floyd_warshall(int N)
{
    vector<vector<int>> dist(N + 1, vector<int>(N + 1, oo));
    vector<vector<int>> pred(N + 1, vector<int>(N + 1, oo));

    for (int u = 1; u <= N; ++u)
    {
        dist[u][u] = 0;
        pred[u][u] = u;
    }

    for (int u = 1; u <= N; ++u)
        for (auto [v, w] : adj[u]) {
            dist[u][v] = w;
            pred[u][v] = u;
        }
```

```
    for (int k = 1; k <= N; ++k)
    {
        for (int u = 1; u <= N; ++u)
        {
            for (int v = 1; v <= N; ++v)
            {
                if (dist[u][k] < oo and dist[k][v] < oo
                    and dist[u][v] > dist[u][k] + dist[k][v]) {
                        dist[u][v] = dist[u][k] + dist[k][v];
                        pred[u][v] = pred[k][v];
                }
            }
        }
    }

    return { dist, pred };
}
```

```
vector<ii> path(int u, int v, const vector<vector<int>>& pred)
{
    vector<ii> p;

    do {
        p.push_back(ii(pred[u][v], v));
        v = pred[u][v];
    } while (v != u);

    reverse(p.begin(), p.end());

    return p;
}
```

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⋆ **Assim,** G **terá um ciclo negativo se, ao final do algoritmo,** $\text{dist}[i][i] < 0$ **para algum** $i \in V$

```
bool has_negative_cycle(int N) {
    for (int u = 1; u <= N; ++u)
        for (int v = 1; v <= N; ++v)
            dist[u][v] = u == v ? 0 : oo;

    for (int u = 1; u <= N; ++u)
        for (auto [v, w] : adj[u])
            dist[u][v] = w;

    for (int k = 1; k <= N; ++k)
        for (int u = 1; u <= N; ++u)
            for (int v = 1; v <= N; ++v)
                if (dist[u][k] < oo and dist[k][v] < oo)
                    dist[u][v] = min(dist[u][v], dist[u][k] + dist[k][v]);

    for (int i = 1; i <= N; ++i)
        if (dist[i][i] < 0) return true;

    return false;
}
```

Problemas sugeridos

1. **Codeforces Round #179 (Div. 1) – Problem B: Greg and Graph**
2. **LightOJ – Travel Company**
3. **OJ 104 – Arbitrage**
4. **OJ 10171 – Meeting Prof. Miguel...**

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