Grafos

Algoritmo de Floyd-Warshall

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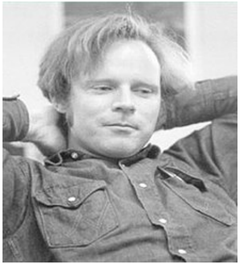
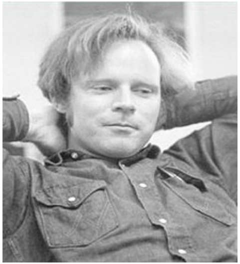
Faculdade UnB Gama



Robert W. Floyd
(1962)



Robert W. Floyd
(1962)

Stephen Warshall
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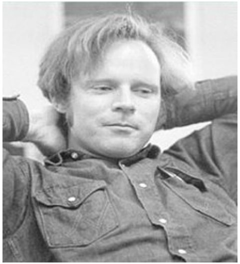

Robert W. Floyd
(1962)

Stephen Warshall
(1962)

Bernard Roy
(1959)

Características do algoritmo de Bellman-Ford

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- $\star$ **Computa o caminho mínimo entre todos os pares de vértices de** $G(V, E)$

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- ⋆ Computa o caminho mínimo entre todos os pares de vértices de $G(V, E)$
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- $\star$ **As distâncias são reduzidas por meio do uso de vértices intermediários**
- $\star$ **Complexidade:** $O(V^3)$

Pseudocódigo

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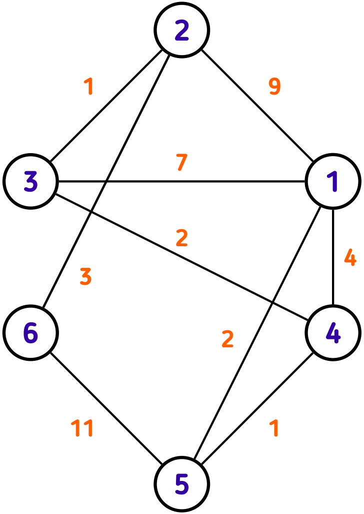
Entrada: um grafo $G(V, E)$
Saída: uma matriz d tal que $d[u][v]$ é a distância mínima em G entre u e v

1. **Faça:**
   (a) $d[u][u] = 0$, **para todos** $u \in V$
   (b) $d[u][v] = w$, **se** $(u, v, w) \in E$
   (c) $d[u][v] = \infty$, **caso contrário**

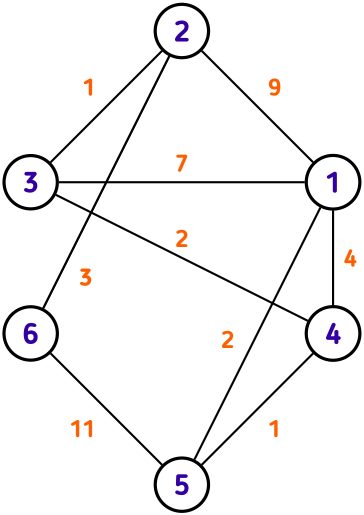
2. **Para cada vértice** k **e todos os pares** $(u, v) \in V^2$, **faça**

$$d[u][v] = \min(d[u][v], d[u][k] + d[k][v])$$

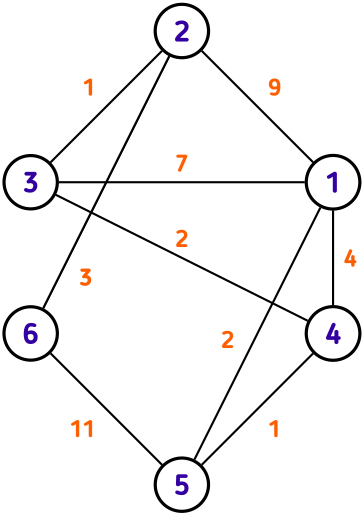
3. **Retorne** d



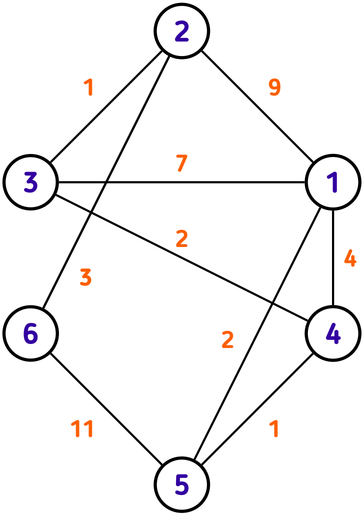
	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						



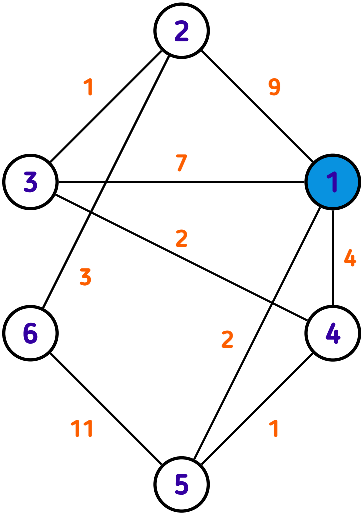
	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0



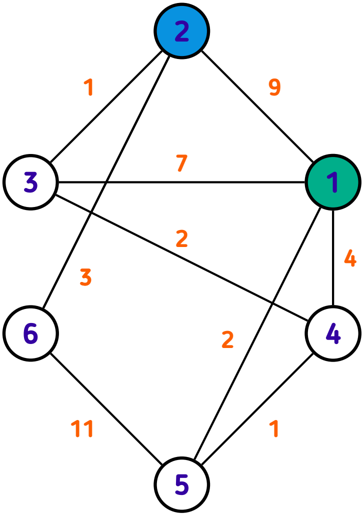
	1	2	3	4	5	6
1	0	9	7	4	2	
2	9	0	1			3
3	7	1	0	2		
4	4		2	0	1	
5	2			1	0	11
6		3			11	0



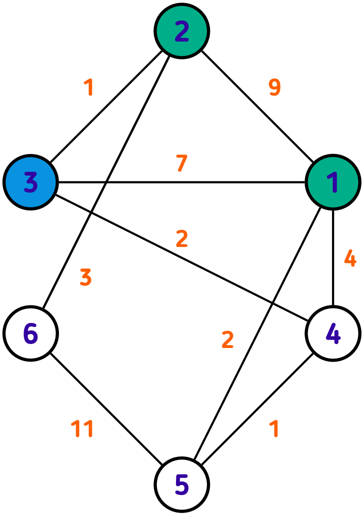
	1	2	3	4	5	6
1	0	9	7	4	2	∞
2	9	0	1	∞	∞	3
3	7	1	0	2	∞	∞
4	4	∞	2	0	1	∞
5	2	∞	∞	1	0	11
6	∞	3	∞	∞	11	0



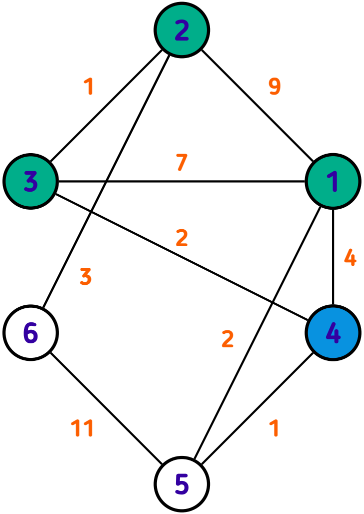
	1	2	3	4	5	6
1	0	9	7	4	2	∞
2	9	0	1	**13**	**11**	3
3	7	1	0	2	**9**	∞
4	4	**13**	2	0	1	∞
5	2	**11**	**9**	1	0	11
6	∞	3	∞	∞	11	0



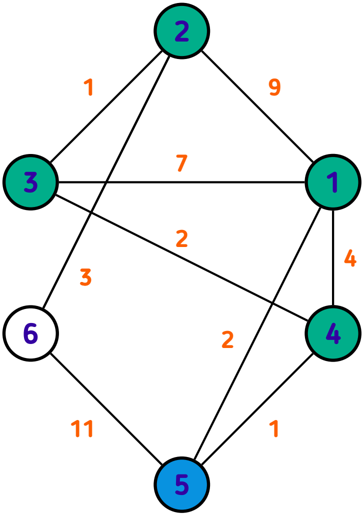
	1	2	3	4	5	6
1	0	9	7	4	2	**12**
2	9	0	1	13	11	3
3	7	1	0	2	9	**4**
4	4	13	2	0	1	**16**
5	2	11	9	1	0	11
6	**12**	3	**4**	**16**	11	0



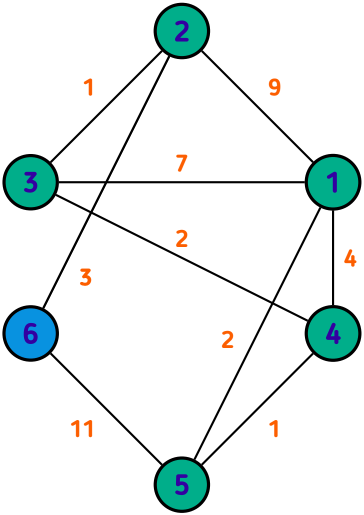
	1	2	3	4	5	6
1	0	**8**	7	4	2	**11**
2	**8**	0	1	**3**	**10**	3
3	7	1	0	2	9	4
4	4	**3**	2	0	1	**6**
5	2	**10**	9	1	0	11
6	**11**	3	4	**6**	11	0



	1	2	3	4	5	6
1	0	**7**	**6**	4	2	**10**
2	**7**	0	1	3	**4**	3
3	**6**	1	0	2	**3**	4
4	4	3	2	0	1	6
5	2	**4**	**3**	1	0	**7**
6	**10**	3	4	6	**7**	0



	1	2	3	4	5	6
1	0	**6**	**5**	**3**	2	**9**
2	**6**	0	1	3	4	3
3	**5**	1	0	2	3	4
4	**3**	3	2	0	1	6
5	2	4	3	1	0	7
6	**9**	3	4	6	7	0



	1	2	3	4	5	6
1	0	6	5	3	2	9
2	6	0	1	3	4	3
3	5	1	0	2	3	4
4	3	3	2	0	1	6
5	2	4	3	1	0	7
6	9	3	4	6	7	0

```cpp
vector<vector<int>> floyd_warshall(int N)
{
    vector<vector<int>> dist(N + 1, vector<int>(N + 1, oo));

    for (int u = 1; u <= N; ++u)
        dist[u][u] = 0;

    for (int u = 1; u <= N; ++u)
        for (auto [v, w] : adj[u])
            dist[u][v] = w;

    for (int k = 1; k <= N; ++k)
        for (int u = 1; u <= N; ++u)
            for (int v = 1; v <= N; ++v)
                if (dist[u][k] < oo and dist[k][v] < oo)
                    dist[u][v] = min(dist[u][v], dist[u][k] + dist[k][v]);

    return dist;
}
```

Identificação de um caminho mínimo

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- O algoritmo de Dijkstra computa as distâncias mínimas, mas não os caminhos mínimos

Identificação de um caminho mínimo

- $\star$ **O algoritmo de Dijkstra computa as distâncias mínimas, mas não os caminhos mínimos**

- $\star$ **Para determinar um caminho mínimo, é preciso definir uma matriz auxiliar** pred, **onde** $\text{pred}[u][v] =$ **antecessor de** v **no caminho mínimo de** u **a** v

Identificação de um caminho mínimo

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$\star$ **Para determinar um caminho mínimo, é preciso definir uma matriz auxiliar** pred, **onde** $\mathsf{pred}[u][v] =$ **antecessor de** v **no caminho mínimo de** u **a** v

$\star$ **No início do algoritmo,**

(a) $\mathsf{pred}[u][u] = u, \forall u \in V$

(b) $\mathsf{pred}[u][v] = u$, **se** $(u, v) \in E$

(c) $\mathsf{pred}[u][v] = \mathsf{undef}$, **caso contrário**

Identificação de um caminho mínimo

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- **Se** k **atualizar** $d[u][v]$**, faça** $\text{pred}[u][v] = \text{pred}[k][v]$

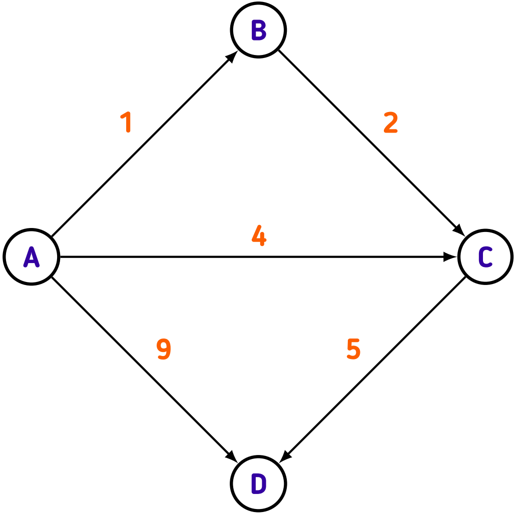
Identificação de um caminho mínimo

- $\star$ **Se k atualizar $d[u][v]$, faça $\textsf{pred}[u][v] = \textsf{pred}[k][v]$**

- $\star$ **A sequência**

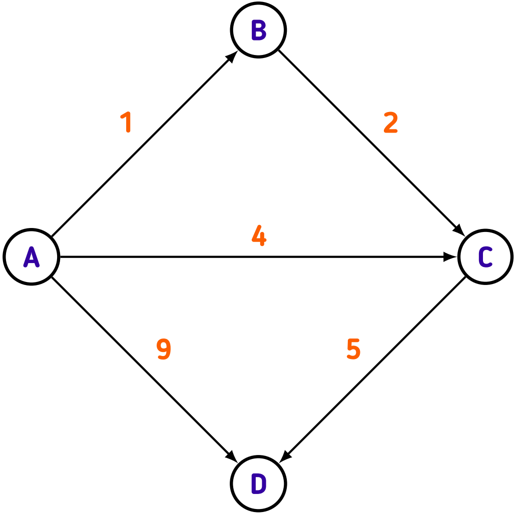
$$p = \{(u, \textsf{pred}^{k-1}[u][v]), \ldots, (\textsf{pred}[\textsf{pred}[u][v]], \textsf{pred}[u][v]), (\textsf{pred}[u][v], v)\}$$

é um caminho mínimo de u a v composto por k arestas e tamanho $d[u][v]$



$$M = \begin{array}{cc} \begin{array}{cccc} \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} \end{array} & \\ \left[\begin{array}{c|c|c|c} & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \end{array}\right] & \begin{array}{c} \mathsf{A} \\ \mathsf{B} \\ \mathsf{C} \\ \mathsf{D} \end{array} \end{array}$$

$$m_{ij} = \mathsf{dist}[i][j]_{\mathsf{pred}[i][j]}$$



$$M = \begin{array}{cc} \begin{array}{cccc} \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} \end{array} & \\ \begin{bmatrix} 0_A & & & \\ & 0_B & & \\ & & 0_C & \\ & & & 0_D \end{bmatrix} & \begin{array}{c} \mathsf{A} \\ \mathsf{B} \\ \mathsf{C} \\ \mathsf{D} \end{array} \end{array}$$

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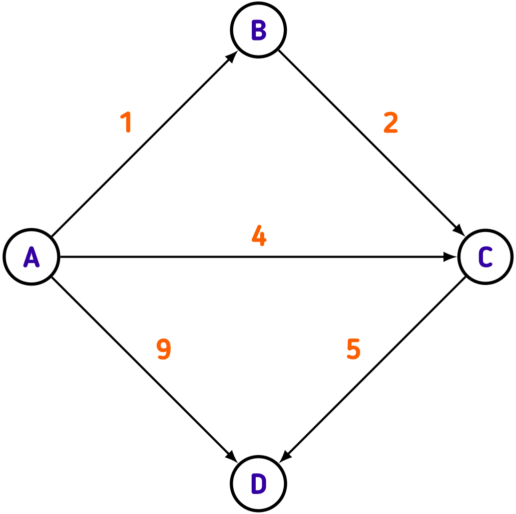
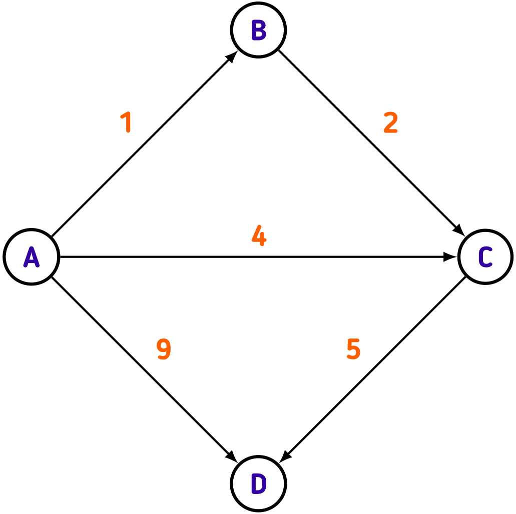


$$M = \begin{array}{cccc|c} A & B & C & D & \\ \hline 0_A & 1_A & 4_A & 9_A & A \\ & 0_B & 2_B & & B \\ & & 0_C & 5_C & C \\ & & & 0_D & D \end{array}$$

$$m_{ij} = \mathsf{dist}[i][j]_{\mathsf{pred}[i][j]}$$



$$M = \begin{array}{|c|c|c|c|c} \multicolumn{1}{c}{\mathsf{A}} & \multicolumn{1}{c}{\mathsf{B}} & \multicolumn{1}{c}{\mathsf{C}} & \multicolumn{1}{c}{\mathsf{D}} & \\ \hline 0_{\mathsf{A}} & 1_{\mathsf{A}} & 4_{\mathsf{A}} & 9_{\mathsf{A}} & \mathsf{A} \\ \hline \infty_{-} & 0_{\mathsf{B}} & 2_{\mathsf{B}} & \infty_{-} & \mathsf{B} \\ \hline \infty_{-} & \infty_{-} & 0_{\mathsf{C}} & 5_{\mathsf{C}} & \mathsf{C} \\ \hline \infty_{-} & \infty_{-} & \infty_{-} & 0_{\mathsf{D}} & \mathsf{D} \\ \hline \end{array}$$

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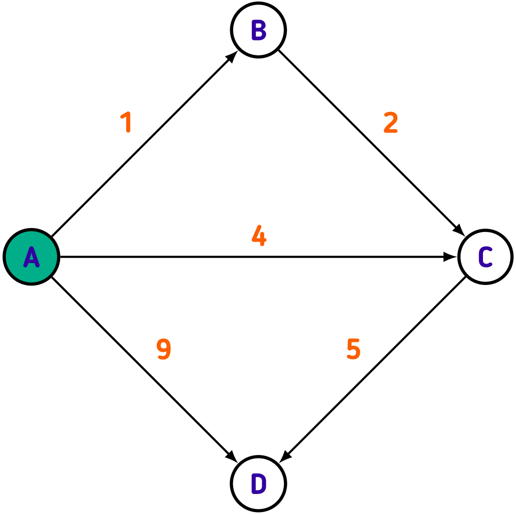

$$M = \begin{array}{cccc|c} \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} & \\ \hline 0_{\mathsf{A}} & 1_{\mathsf{A}} & 4_{\mathsf{A}} & 9_{\mathsf{A}} & \mathsf{A} \\ \infty_{-} & 0_{\mathsf{B}} & 2_{\mathsf{B}} & \infty_{-} & \mathsf{B} \\ \infty_{-} & \infty_{-} & 0_{\mathsf{C}} & 5_{\mathsf{C}} & \mathsf{C} \\ \infty_{-} & \infty_{-} & \infty_{-} & 0_{\mathsf{D}} & \mathsf{D} \end{array}$$

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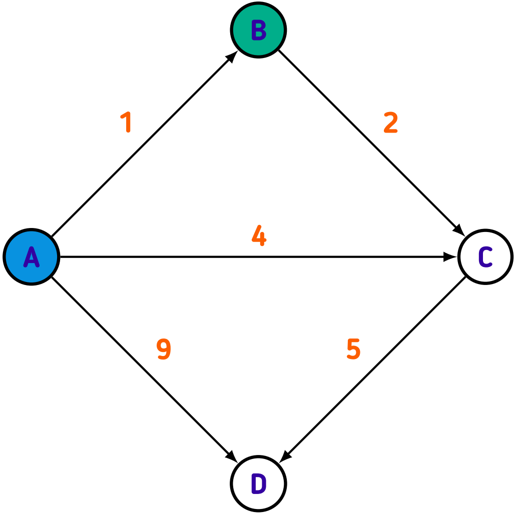
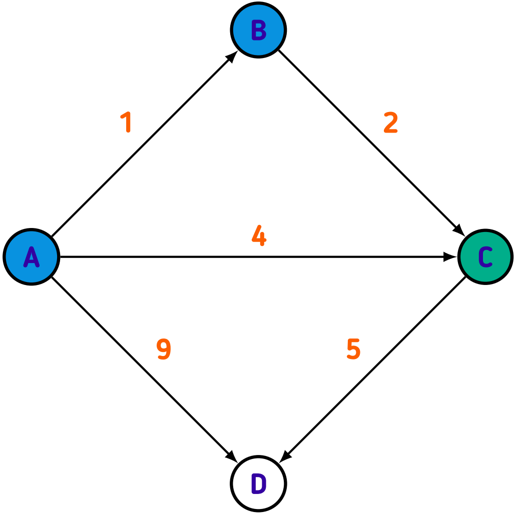


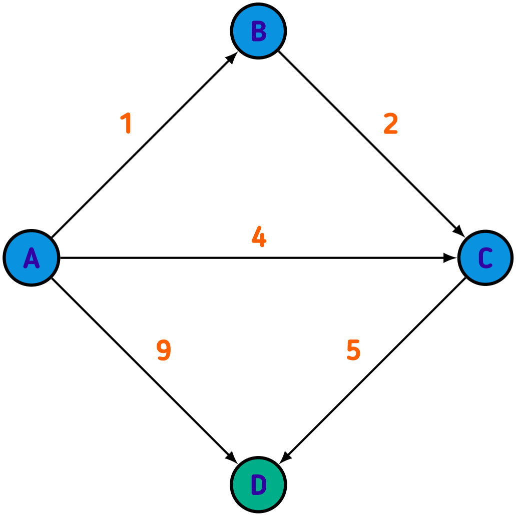
$$M = \begin{array}{c|c|c|c|c}
 & \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} \\
\hline
\mathsf{A} & 0_{\mathsf{A}} & 1_{\mathsf{A}} & \mathbf{3}_{\mathsf{B}} & 9_{\mathsf{A}} \\
\hline
\mathsf{B} & \infty_{-} & 0_{\mathsf{B}} & 2_{\mathsf{B}} & \infty_{-} \\
\hline
\mathsf{C} & \infty_{-} & \infty_{-} & 0_{\mathsf{C}} & 5_{\mathsf{C}} \\
\hline
\mathsf{D} & \infty_{-} & \infty_{-} & \infty_{-} & 0_{\mathsf{D}}
\end{array}$$

$$m_{ij} = \mathsf{dist}[i][j]_{\mathsf{pred}[i][j]}$$



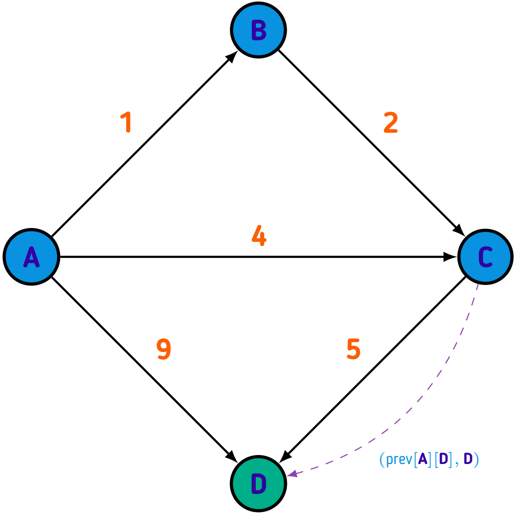
$$M = \begin{array}{cccccc} & \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} & \\ \left[\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \begin{array}{|c|}\hline 0_{\mathsf{A}} \\\hline \infty_{-} \\\hline \infty_{-} \\\hline \infty_{-} \\\hline \end{array} & \begin{array}{c|}\hline 1_{\mathsf{A}} \\\hline 0_{\mathsf{B}} \\\hline \infty_{-} \\\hline \infty_{-} \\\hline \end{array} & \begin{array}{c|}\hline 3_{\mathsf{B}} \\\hline 2_{\mathsf{B}} \\\hline 0_{\mathsf{C}} \\\hline \infty_{-} \\\hline \end{array} & \begin{array}{c|}\hline \mathbf{8}_{\mathsf{C}} \\\hline \mathbf{7}_{\mathsf{C}} \\\hline 5_{\mathsf{C}} \\\hline 0_{\mathsf{D}} \\\hline \end{array} & \begin{array}{c} \mathsf{A} \\ \mathsf{B} \\ \mathsf{C} \\ \mathsf{D} \end{array} \end{array}$$

$$m_{ij} = \mathsf{dist}[i][j]_{\mathsf{pred}[i][j]}$$



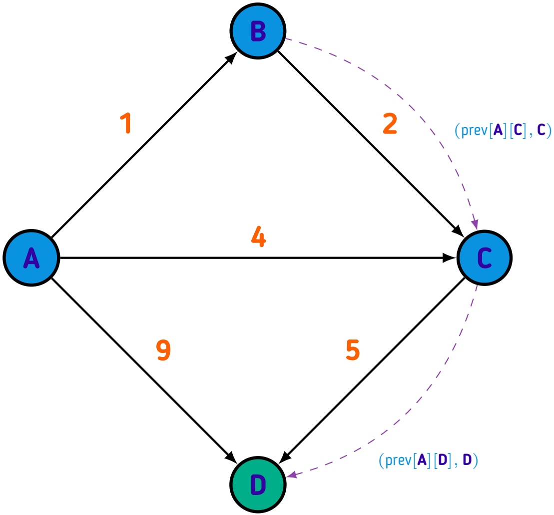
$$
M = \begin{array}{c c}
\begin{array}{cccc} \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} \end{array} & \\
\left[\begin{array}{cccc}
0_{\mathsf{A}} & 1_{\mathsf{A}} & 3_{\mathsf{B}} & 8_{\mathsf{C}} \\
\infty_{-} & 0_{\mathsf{B}} & 2_{\mathsf{B}} & 7_{\mathsf{C}} \\
\infty_{-} & \infty_{-} & 0_{\mathsf{C}} & 5_{\mathsf{C}} \\
\infty_{-} & \infty_{-} & \infty_{-} & 0_{\mathsf{D}}
\end{array}\right] &
\begin{array}{c} \mathsf{A} \\ \mathsf{B} \\ \mathsf{C} \\ \mathsf{D} \end{array}
\end{array}
$$

$$m_{ij} = \mathsf{dist}[i][j]_{\mathsf{pred}[i][j]}$$



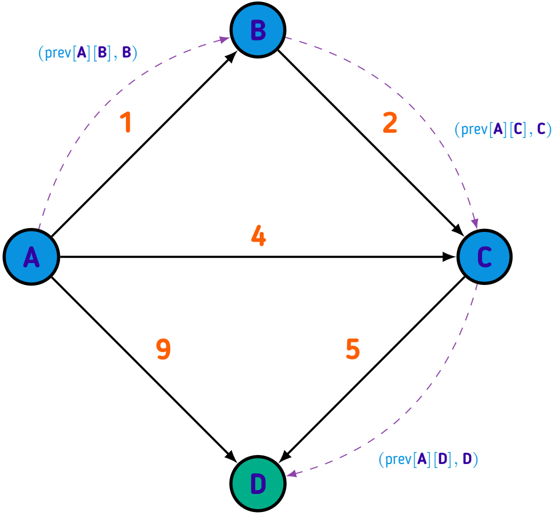
$$M = \begin{array}{cccc|c} \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} & \\ \hline 0_{\mathsf{A}} & 1_{\mathsf{A}} & 3_{\mathsf{B}} & 8_{\mathsf{C}} & \mathsf{A} \\ \infty_{-} & 0_{\mathsf{B}} & 2_{\mathsf{B}} & 7_{\mathsf{C}} & \mathsf{B} \\ \infty_{-} & \infty_{-} & 0_{\mathsf{C}} & 5_{\mathsf{C}} & \mathsf{C} \\ \infty_{-} & \infty_{-} & \infty_{-} & 0_{\mathsf{D}} & \mathsf{D} \end{array}$$

$$m_{ij} = \mathsf{dist}[i][j]_{\mathsf{pred}[i][j]}$$



$$M = \begin{array}{cccc|c} \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} & \\ 0_{\mathsf{A}} & 1_{\mathsf{A}} & 3_{\mathsf{B}} & 8_{\mathsf{C}} & \mathsf{A} \\ \infty_{-} & 0_{\mathsf{B}} & 2_{\mathsf{B}} & 7_{\mathsf{C}} & \mathsf{B} \\ \infty_{-} & \infty_{-} & 0_{\mathsf{C}} & 5_{\mathsf{C}} & \mathsf{C} \\ \infty_{-} & \infty_{-} & \infty_{-} & 0_{\mathsf{D}} & \mathsf{D} \end{array}$$

$$m_{ij} = \mathsf{dist}[i][j]_{\mathsf{pred}[i][j]}$$



$$M = \begin{array}{|c|c|c|c|l} \multicolumn{1}{c}{\mathsf{A}} & \multicolumn{1}{c}{\mathsf{B}} & \multicolumn{1}{c}{\mathsf{C}} & \multicolumn{1}{c}{\mathsf{D}} & \\ \hline 0_{\mathsf{A}} & 1_{\mathsf{A}} & 3_{\mathsf{B}} & 8_{\mathsf{C}} & \mathsf{A} \\ \hline \infty_{-} & 0_{\mathsf{B}} & 2_{\mathsf{B}} & 7_{\mathsf{C}} & \mathsf{B} \\ \hline \infty_{-} & \infty_{-} & 0_{\mathsf{C}} & 5_{\mathsf{C}} & \mathsf{C} \\ \hline \infty_{-} & \infty_{-} & \infty_{-} & 0_{\mathsf{D}} & \mathsf{D} \\ \hline \end{array}$$

$$m_{ij} = \mathsf{dist}[i][j]_{\mathsf{pred}[i][j]}$$

```cpp
pair<vector<vector<int>>, vector<vector<int>>>
floyd_warshall(int N)
{
    vector<vector<int>> dist(N + 1, vector<int>(N + 1, oo));
    vector<vector<int>> pred(N + 1, vector<int>(N + 1, oo));

    for (int u = 1; u <= N; ++u)
    {
        dist[u][u] = 0;
        pred[u][u] = u;
    }

    for (int u = 1; u <= N; ++u)
        for (auto [v, w] : adj[u]) {
            dist[u][v] = w;
            pred[u][v] = u;
        }
```

```
    for (int k = 1; k <= N; ++k)
    {
        for (int u = 1; u <= N; ++u)
        {
            for (int v = 1; v <= N; ++v)
            {
                if (dist[u][k] < oo and dist[k][v] < oo
                    and dist[u][v] > dist[u][k] + dist[k][v]) {
                        dist[u][v] = dist[u][k] + dist[k][v];
                        pred[u][v] = pred[k][v];
                }
            }
        }
    }

    return { dist, pred };
}
```

```cpp
vector<ii> path(int u, int v, const vector<vector<int>>& pred)
{
    vector<ii> p;

    do {
        p.push_back(ii(pred[u][v], v));
        v = pred[u][v];
    } while (v != u);

    reverse(p.begin(), p.end());

    return p;
}
```

Identificação de ciclos negativos

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- O algoritmo de Floyd-Warshall é capaz de detectar ciclos negativos

Identificação de ciclos negativos

- **O algoritmo de Floyd-Warshall é capaz de detectar ciclos negativos**
- **Inicialmente** $\text{dist}[u][u] = 0, \forall u \in V$**, se** G **não tem *autoloops***

Identificação de ciclos negativos

- $\star$ **O algoritmo de Floyd-Warshall é capaz de detectar ciclos negativos**
- $\star$ **Inicialmente** $\textsf{dist}[u][u] = 0, \forall u \in V$**, se** G **não tem** ***autoloops***
- $\star$ **Caso exista um ciclo negativo que passe por** u**, seguir este ciclo de** u **a** u **torna** $\textsf{dist}[u][u] < 0$

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- ⋆ **Assim,** G **terá um ciclo negativo se, ao final do algoritmo,** $\text{dist}[i][i] < 0$ **para algum** $i \in V$

```cpp
bool has_negative_cycle(int N) {
    for (int u = 1; u <= N; ++u)
        for (int v = 1; v <= N; ++v)
            dist[u][v] = u == v ? 0 : oo;

    for (int u = 1; u <= N; ++u)
        for (auto [v, w] : adj[u])
            dist[u][v] = w;

    for (int k = 1; k <= N; ++k)
        for (int u = 1; u <= N; ++u)
            for (int v = 1; v <= N; ++v)
                if (dist[u][k] < oo and dist[k][v] < oo)
                    dist[u][v] = min(dist[u][v], dist[u][k] + dist[k][v]);

    for (int i = 1; i <= N; ++i)
        if (dist[i][i] < 0) return true;

    return false;
}
```

Problemas sugeridos

1. Codeforces Round #179 (Div. 1) – Problem B: Greg and Graph
2. LightOJ – Travel Company
3. OJ 104 – Arbitrage
4. OJ 10171 – Meeting Prof. Miguel...

Referências

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2. HALIM, Felix; HALIM, Steve. *Competitive Programming 3*, 2010.
3. LAAKSONEN, Antti. *Competitive Programmer's Handbook*, 2018.
4. SKIENA, Steven; REVILLA, Miguel. *Programming Challenges*, 2003.