Grafos

Algoritmo de Bellman-Ford

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Proponentes

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Lester Randolph Ford Jr.

(1956)

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Lester Randolph Ford Jr.
(1956)

Richard Ernest Bellman
(1958)

Características do algoritmo de Bellman-Ford

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- $\star$ Não processa, mas identifica ciclos negativos
- $\star$ **Complexidade:** $O(VE)$

Pseudocódigo

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Entrada: **um grafo** $G(V, E)$ **e um vértice** $s \in V$
Saída: **um vetor** d **tal que** $d[u]$ **é a distância mínima em** G **entre** s **e** u

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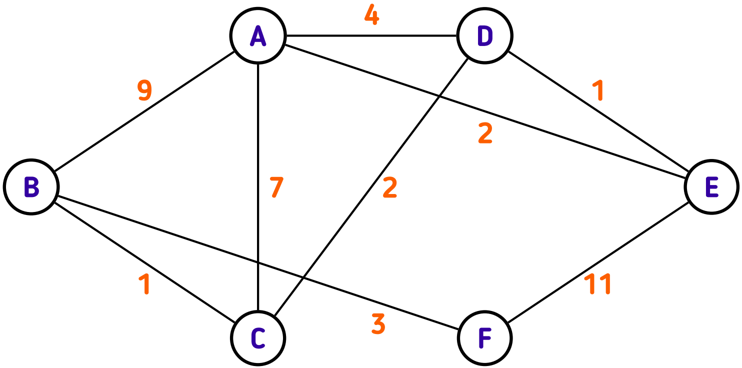
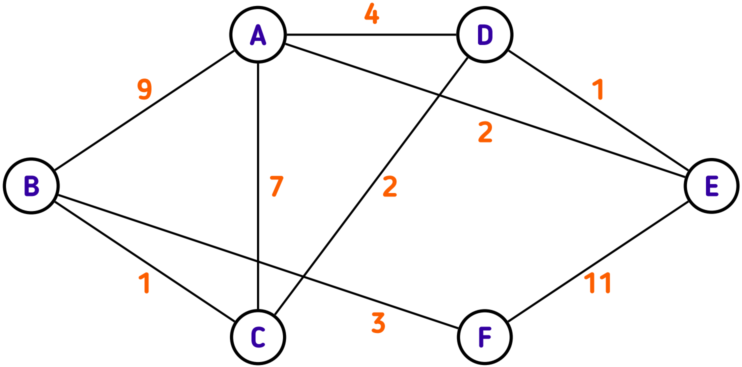
2. **Para cada aresta** $(u, v, w) \in E$, **se** $d[u] + w < d[v]$, **faça** $d[v] = d[u] + w$

3. **Se o vetor** d **foi atualizado ao menos uma vez, volte ao passo** 2.

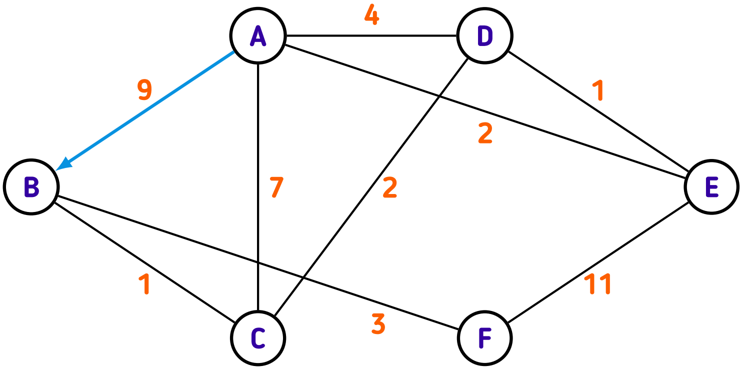
4. **Retorne** d



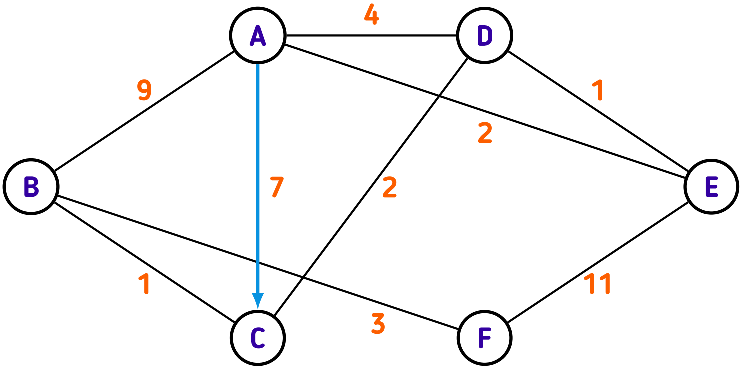
A
D
B
E
C
F
4
9
1
2
7
2
1
11
3



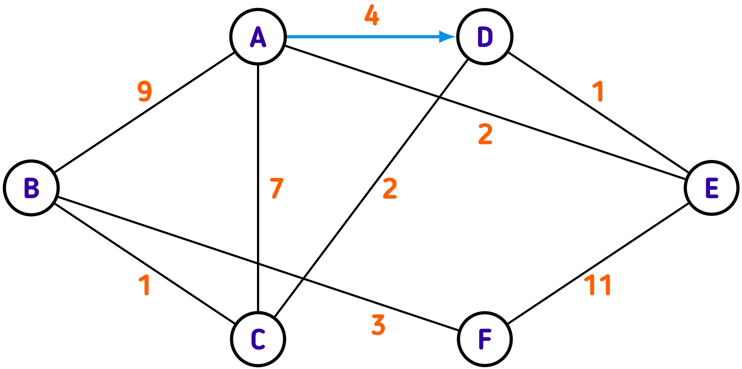
	A	B	C	D	E	F
dist$(u, \mathbf{A})$	0	∞	∞	∞	∞	∞



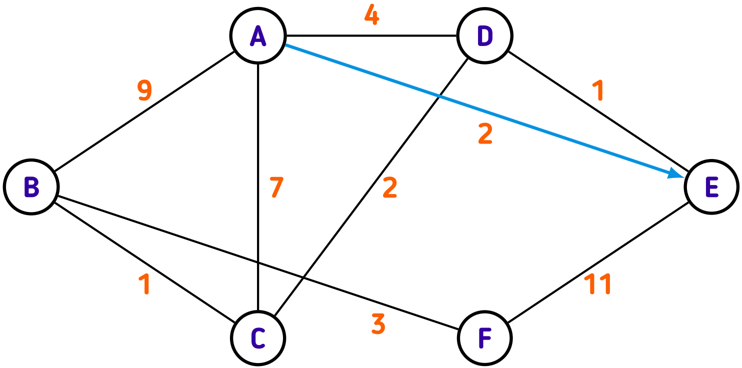
	A	B	C	D	E	F
dist(u, **A**)	0	**9**	∞	∞	∞	∞



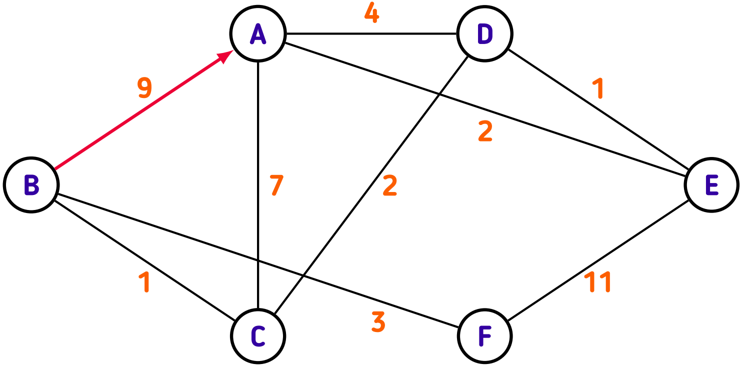
	A	B	C	D	E	F
$\mathrm{dist}(u, \mathbf{A})$	0	9	**7**	∞	∞	∞



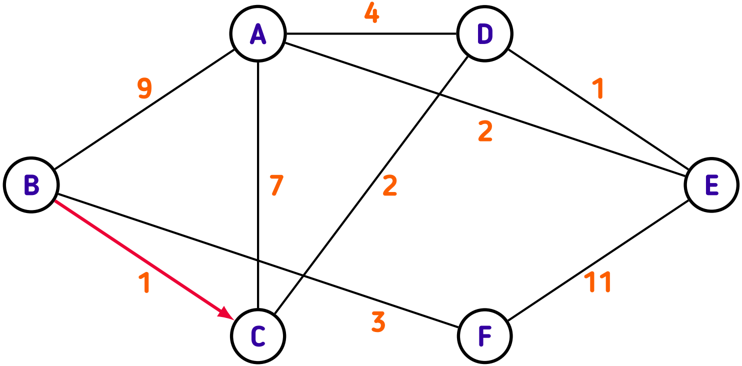
	A	B	C	D	E	F
dist(u, **A**)	0	9	7	**4**	∞	∞



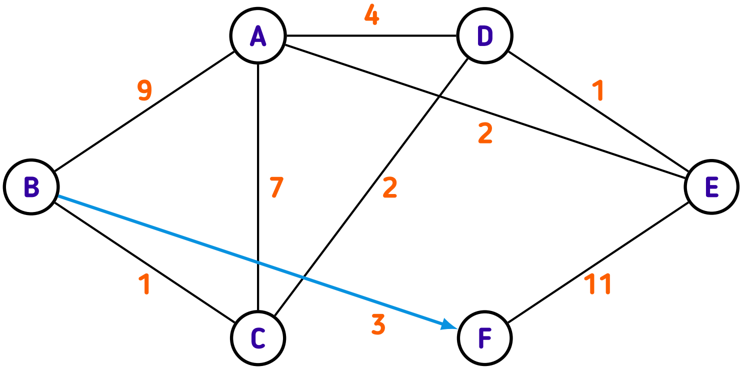
	A	B	C	D	E	F
dist(u, **A**)	0	9	7	4	**2**	∞



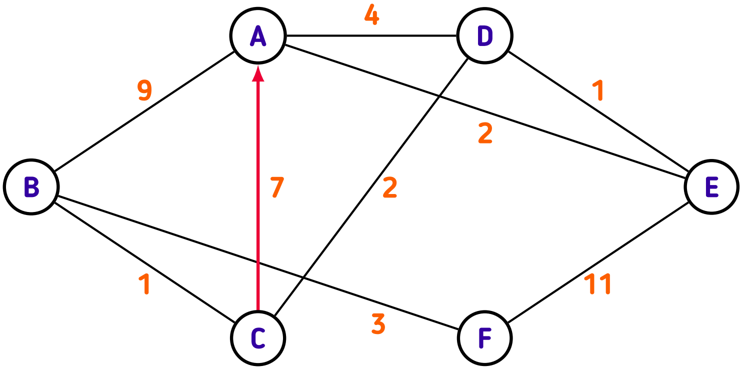
	A	B	C	D	E	F
dist$(u, \mathbf{A})$	0	9	7	4	2	∞



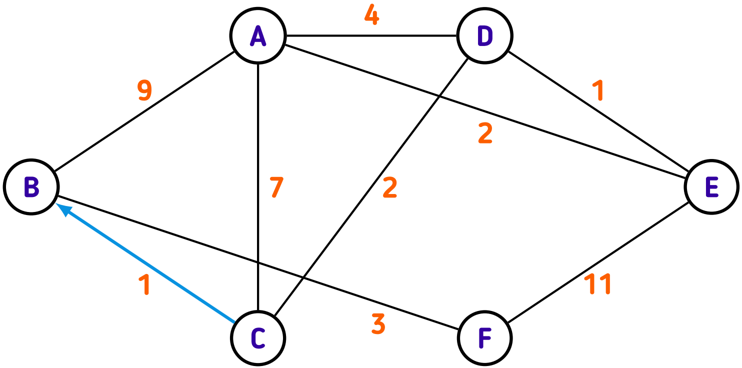
	A	B	C	D	E	F
dist(u, **A**)	0	9	7	4	2	∞



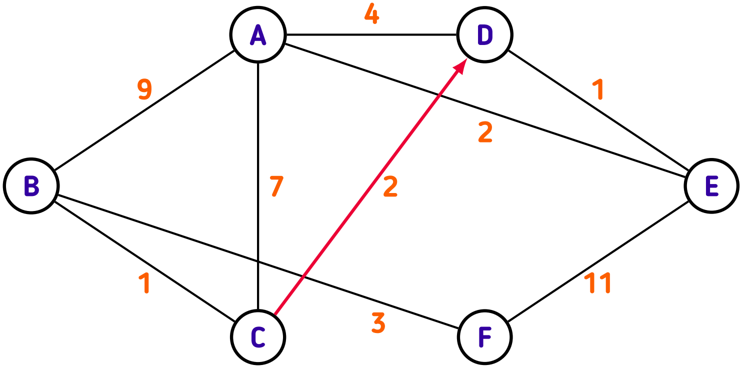
	A	B	C	D	E	F
dist(u, **A**)	0	9	7	4	2	**12**



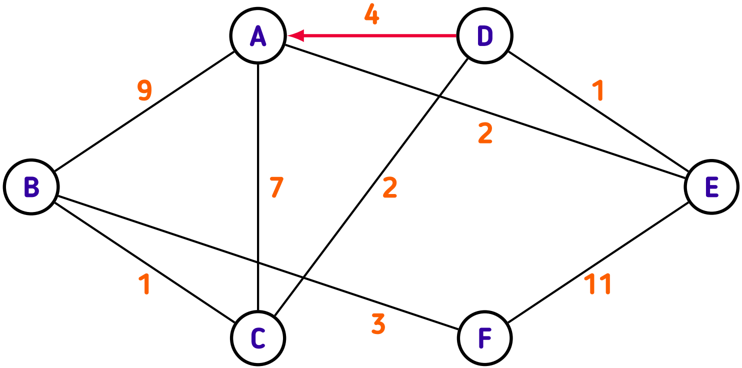
	A	B	C	D	E	F
dist$(u,$ **A**$)$	0	9	7	4	2	12



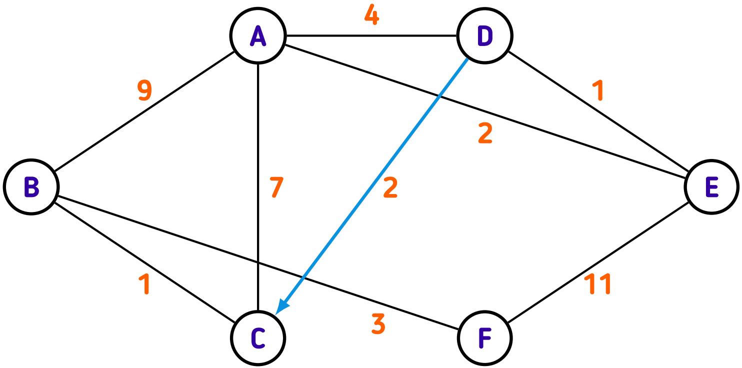
	A	B	C	D	E	F
dist(u, **A**)	0	**8**	7	4	2	12



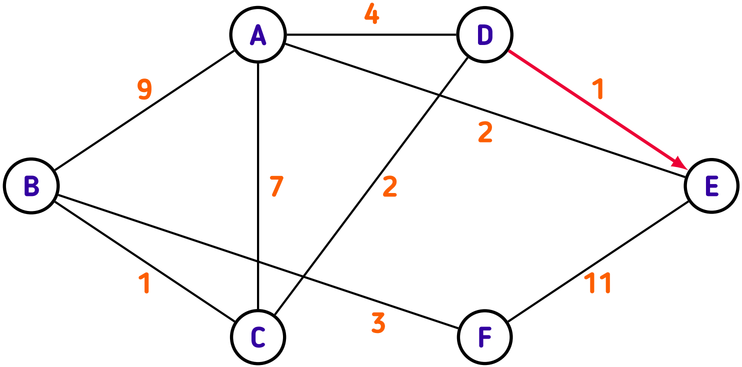
	A	B	C	D	E	F
dist(u, **A**)	0	8	7	4	2	12



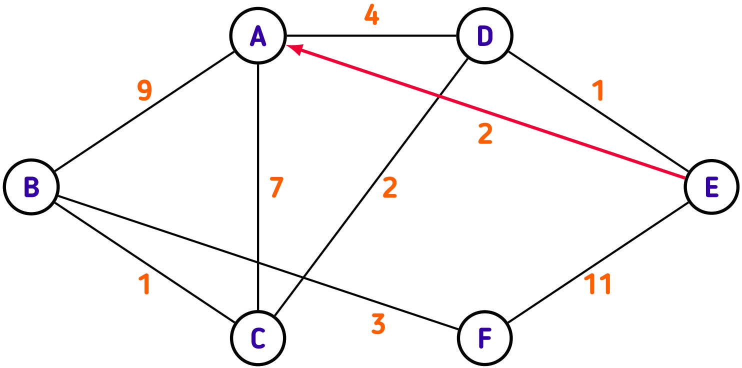
	A	B	C	D	E	F
dist$(u, \textbf{A})$	0	8	7	4	2	12



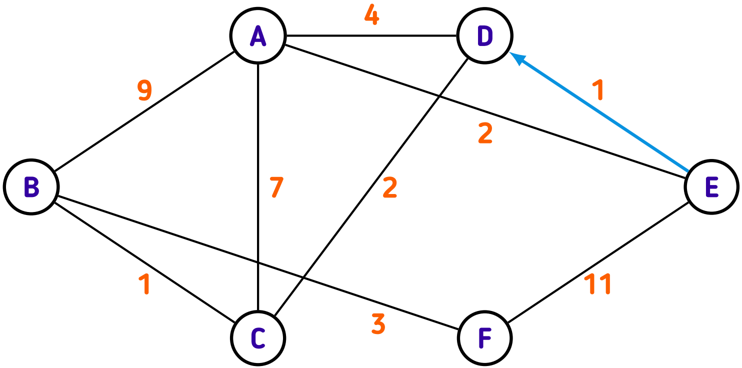
	A	B	C	D	E	F
dist(u, **A**)	0	8	**6**	4	2	12



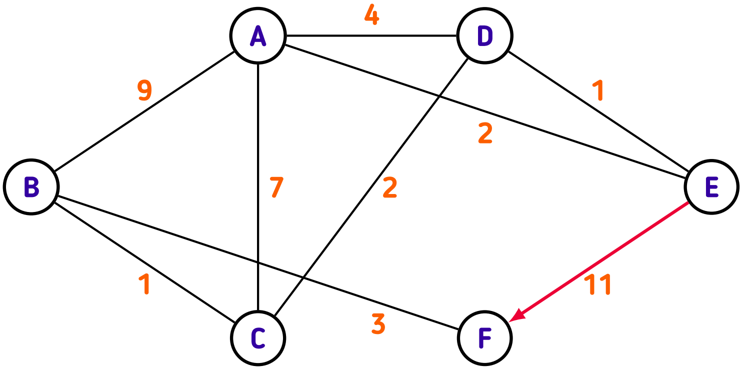
	A	B	C	D	E	F
$\mathsf{dist}(u, \mathbf{A})$	0	8	6	4	2	12



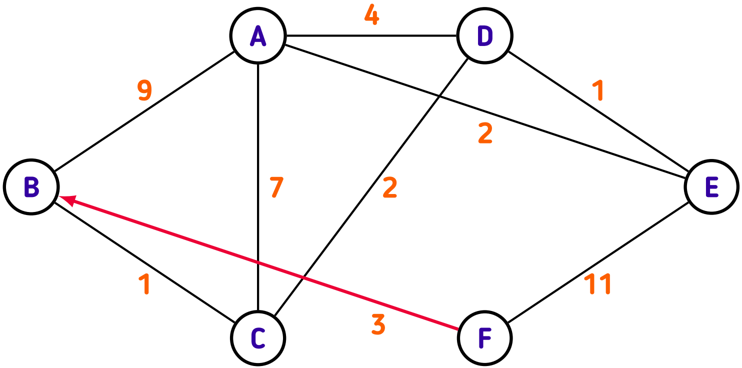
	A	B	C	D	E	F
dist$(u, \textbf{A})$	0	8	6	4	2	12



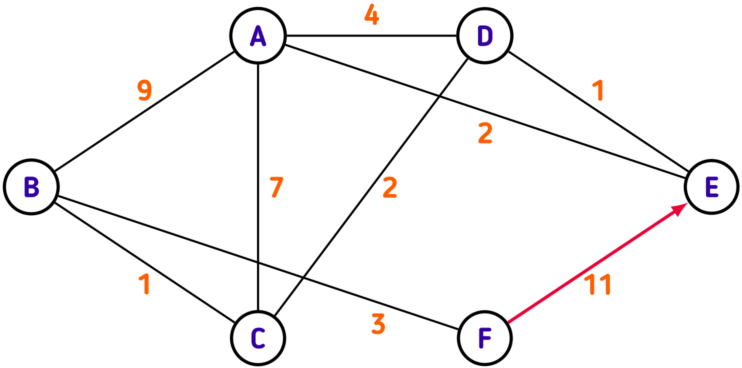
	A	B	C	D	E	F
dist(u, **A**)	0	8	6	**3**	2	12



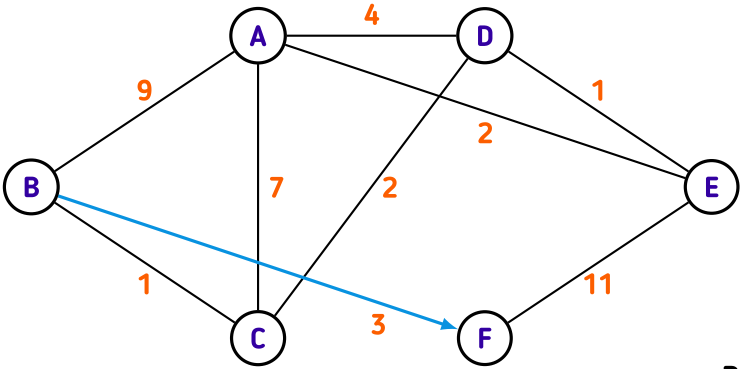
	A	B	C	D	E	F
dist$(u, \mathbf{A})$	0	8	6	3	2	12



	A	B	C	D	E	F
$\text{dist}(u, \textbf{A})$	0	8	6	3	2	12

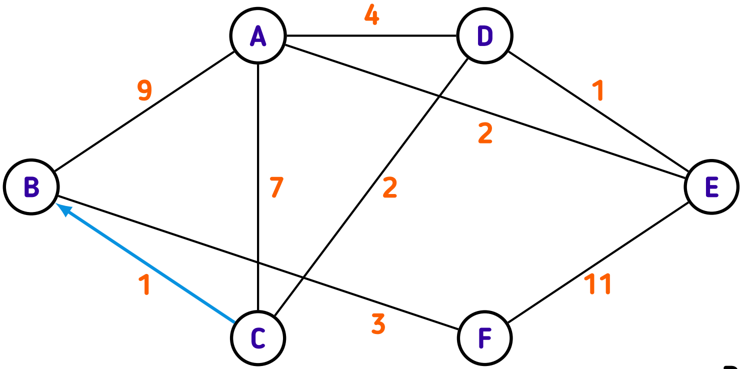


	A	B	C	D	E	F
dist(u, **A**)	0	8	6	3	2	12



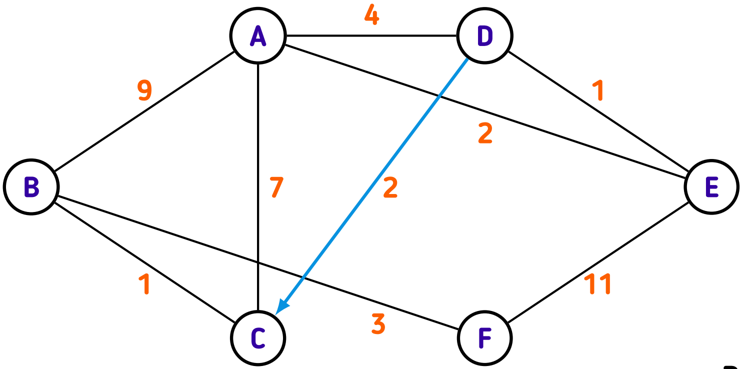
Round #2

	A	B	C	D	E	F
dist$(u, \mathbf{A})$	0	8	6	3	2	**11**



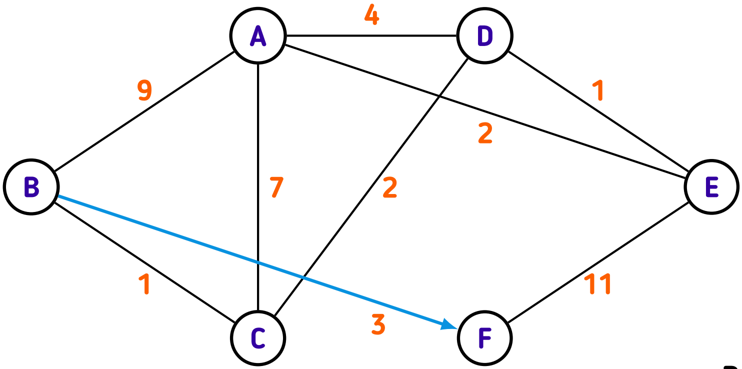
Round #2

	A	B	C	D	E	F
dist(u, **A**)	0	**7**	6	3	2	11



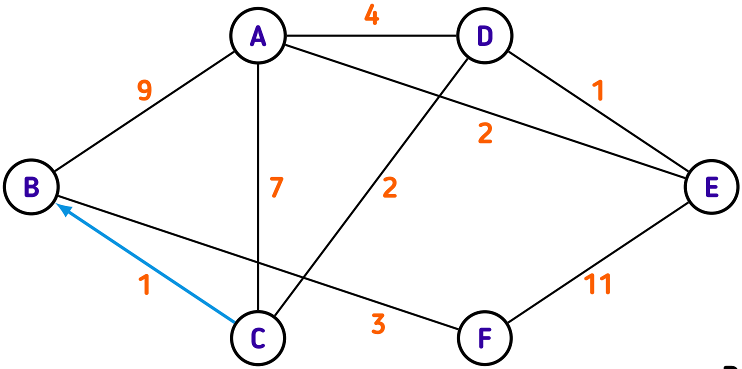
Round #2

	A	B	C	D	E	F
dist(u, **A**)	0	7	**5**	3	2	11



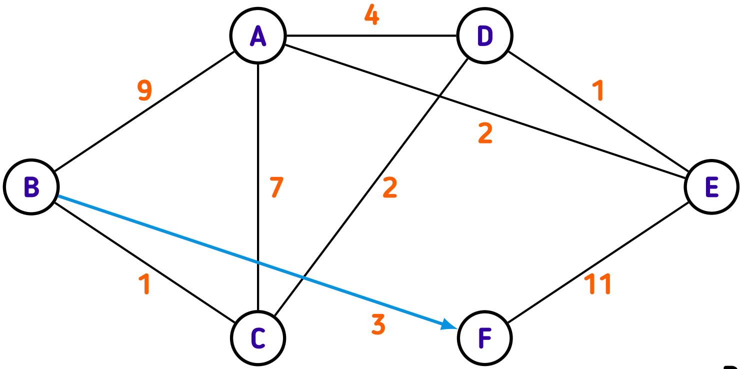
Round #3

	A	B	C	D	E	F
dist(u, **A**)	0	7	5	3	2	**10**



Round #3

	A	B	C	D	E	F
$\text{dist}(u, \textbf{A})$	0	**6**	5	3	2	10



Round #4

	A	B	C	D	E	F
dist(u, **A**)	0	6	5	3	2	**9**

```cpp
vector<int> bellman_ford(int s, int N, const vector<edge>& edges)
{
    const int oo { 1000000010 };

    vector<int> dist(N + 1, oo);
    dist[s] = 0;

    for (int i = 1; i <= N - 1; i++)
        for (auto [u, v, w] : edges)
            dist[v] = min(dist[v], dist[u] + w);

    return dist;
}
```

Identificação do caminho mínimo

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$\star$ **No início do algoritmo,** $\text{pred}[s] = s$ **e** $\text{pred}[u] = \text{undef}$**, se** $u \neq s$

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- ⋆ **Se** (u, v) **atualizar** $d[v]$**, faça** $\text{pred}[v] = u$

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- **Se** (u, v) **atualizar** $d[v]$**, faça** $\mathsf{pred}[v] = u$

- **A sequência**

$$p = \{(s, \mathsf{pred}^{k-1}[u]), \ldots, (\mathsf{pred}[\mathsf{pred}[u]], \mathsf{pred}[u]), (\mathsf{pred}[u], u)\}$$

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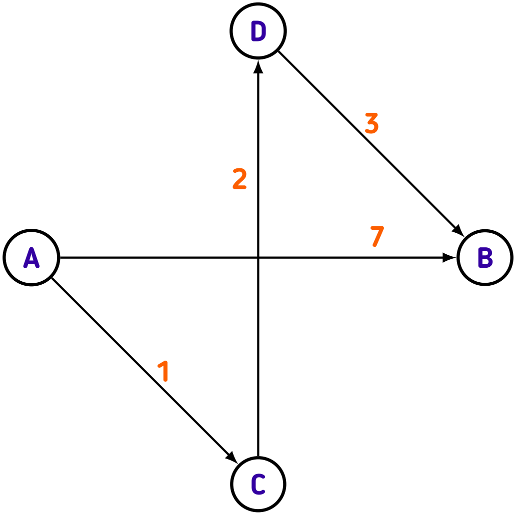
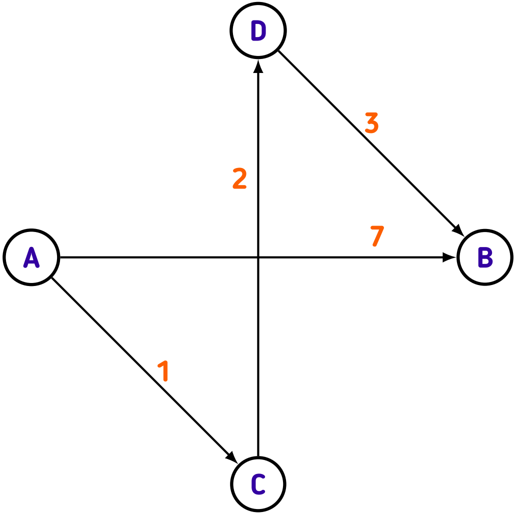
- $\star$ **A sequência**

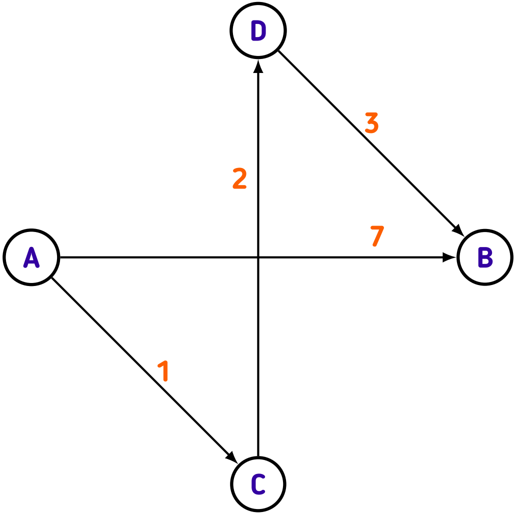
$$p = \{(s, \text{pred}^{k-1}[u]), \ldots, (\text{pred}[\text{pred}[u]], \text{pred}[u]), (\text{pred}[u], u)\}$$

é um caminho mínimo de s **a** u **composto de** k **arestas e tamanho** $d[u]$


D
3
2
7
A
B
1
C

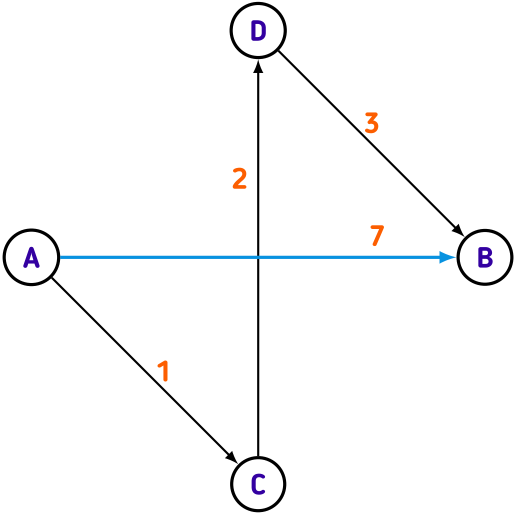


	A	**B**	**C**	**D**
dist(u, **A**)	0	∞	∞	∞



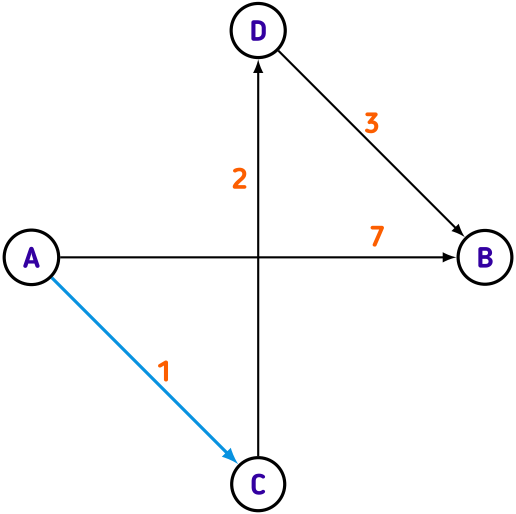
	A	B	C	D
dist$(u, \mathbf{A})$	0	∞	∞	∞

	A	B	C	D
pred(u)	A	-	-	-



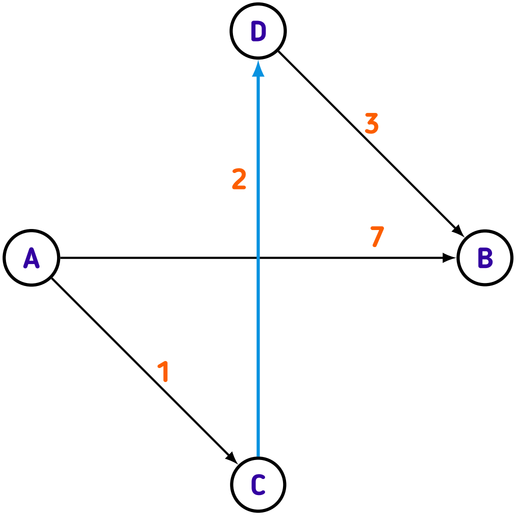
	A	B	C	D
dist$(u, \mathbf{A})$	0	7	∞	∞

	A	B	C	D
pred(u)	A	A	-	-



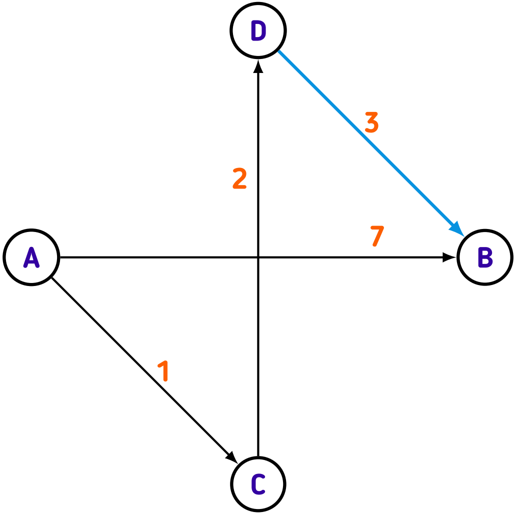
	A	B	C	D
$\text{dist}(u, \textbf{A})$	0	7	**1**	∞

	A	B	C	D
$\text{pred}(u)$	A	A	A	-



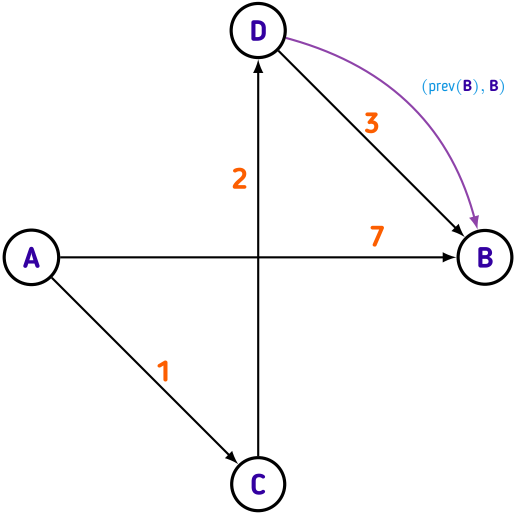
	A	B	C	D
dist$(u, \mathbf{A})$	0	7	1	**3**

	A	B	C	D
pred(u)	A	A	A	C



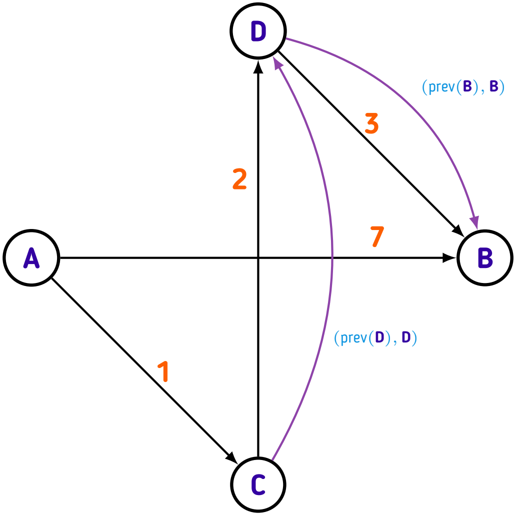
	A	B	C	D
$\mathsf{dist}(u, \mathbf{A})$	0	**6**	1	3

	A	B	C	D
$\mathsf{pred}(u)$	A	D	A	C



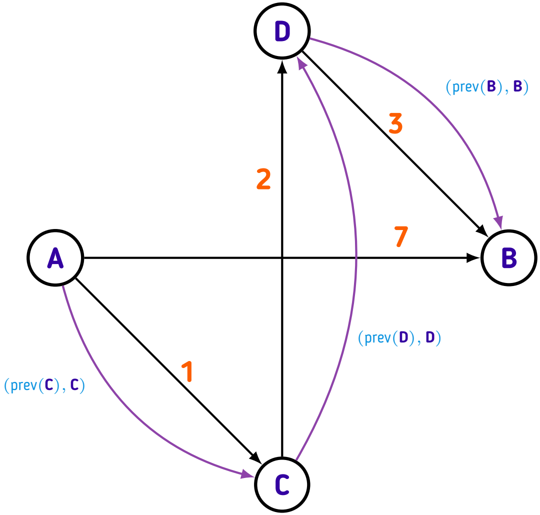
	A	B	C	D
dist$(u, \mathbf{A})$	0	6	1	3

	A	B	C	D
pred(u)	A	D	A	C



	A	B	C	D
dist(u, **A**)	0	6	1	3

	A	B	C	D
pred(u)	A	D	A	C



	A	B	C	D
dist$(u,$ A$)$	0	6	1	3

	A	B	C	D
pred(u)	A	D	A	C

```cpp
pair<vector<int>, vector<int>>
bellman_ford(int s, int N, const vector<edge>& edges)
{
    vector<int> dist(N + 1, oo), pred(N + 1, oo);

    dist[s] = 0;
    pred[s] = s;

    for (int i = 1; i <= N - 1; i++)
        for (auto [u, v, w] : edges)
            if (dist[v] > dist[u] + w) {
                dist[v] = dist[u] + w;
                pred[v] = u;
            }

    return { dist, pred };
}
```

```cpp
vector<ii> path(int s, int u, const vector<int>& pred)
{
    vector<ii> p;
    int v = u;

    do {
        p.push_back(ii(pred[v], v));
        v = pred[v];
    } while (v != s);

    reverse(p.begin(), p.end());

    return p;
}
```

Caminhos mínimos e ciclos

Caminhos mínimos e ciclos

Seja

$$p = \{(a, u_1), (u_1, u_2), \ldots, (v, u_r), \ldots, (u_s, v), \ldots, (u_t, b)\}$$

um caminho de a **a** b **e** $\omega(c)$ **o custo do ciclo** $c = \{(v, u_r), \ldots, (u_s, v)\}$**, isto é**

$$\omega(c) = \sum_{e \in c} w(e)$$

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custo da aresta e

Caminhos mínimos e ciclos

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um caminho de a a b e $\omega(c)$ o custo do ciclo $c = \{(v, u_r), \ldots, (u_s, v)\}$, isto é

$$\omega(c) = \sum_{e \in c} w(e)$$

custo da aresta e

Se p é caminho mínimo de a a b então $\omega(c) = 0$.

Caminhos mínimos e ciclos positivos

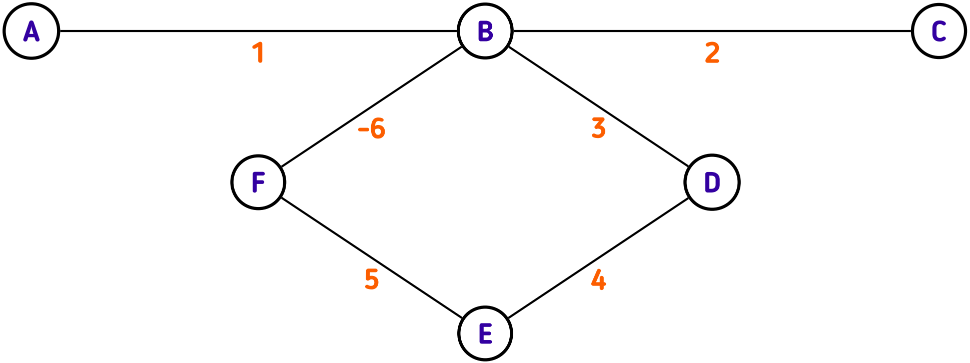
Caminhos mínimos e ciclos positivos

Seja $\omega(c) > 0$ **e**

$$q = \{(a, u_1), (u_1, u_2), \ldots, (u_{r-1}, v), (v, u_{s+1}), \ldots, (u_t, b)\},$$

o caminho resultante da exclusão do ciclo c **de** p**. Então** $\omega(q) < \omega(p)$**, pois**

$$\omega(p) = \sum_{e_i \in p} w(e_i) = \sum_{e_j \in q} w(e_j) + \sum_{e_k \in c} w(e_k) = \omega(q) + \omega(c) > \omega(q)$$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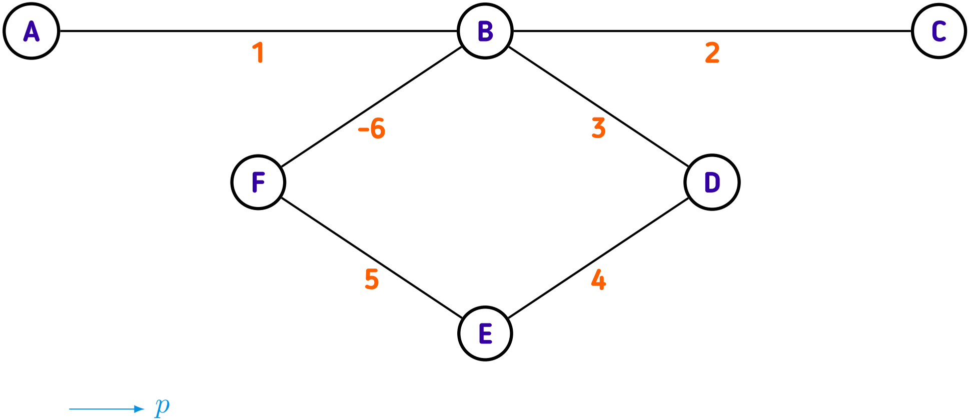
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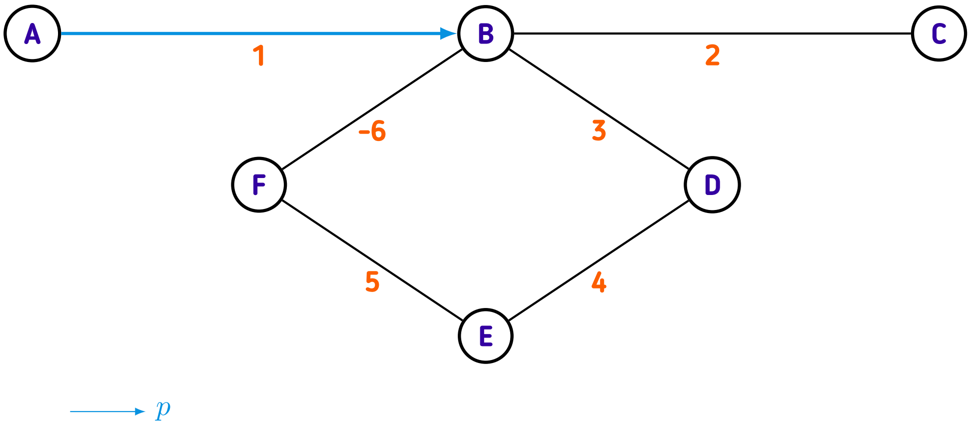
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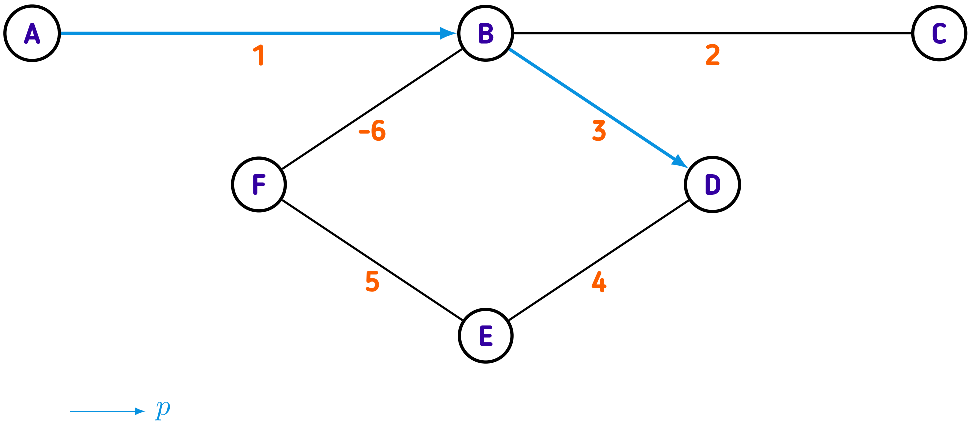
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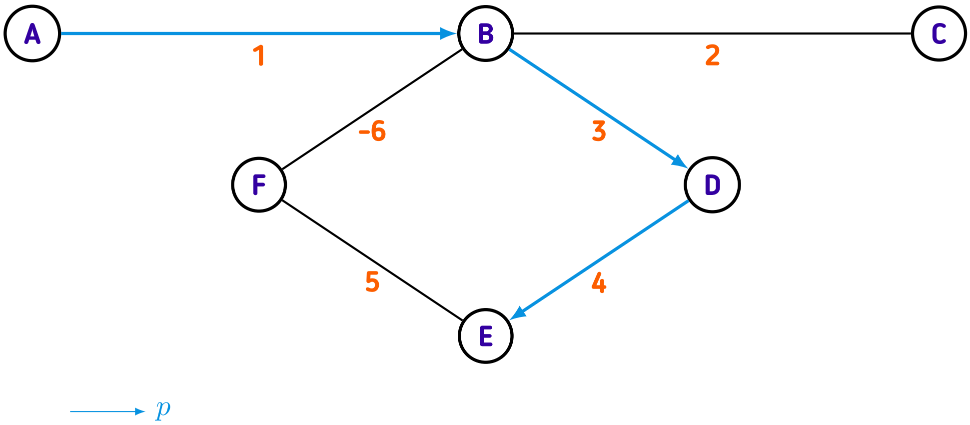

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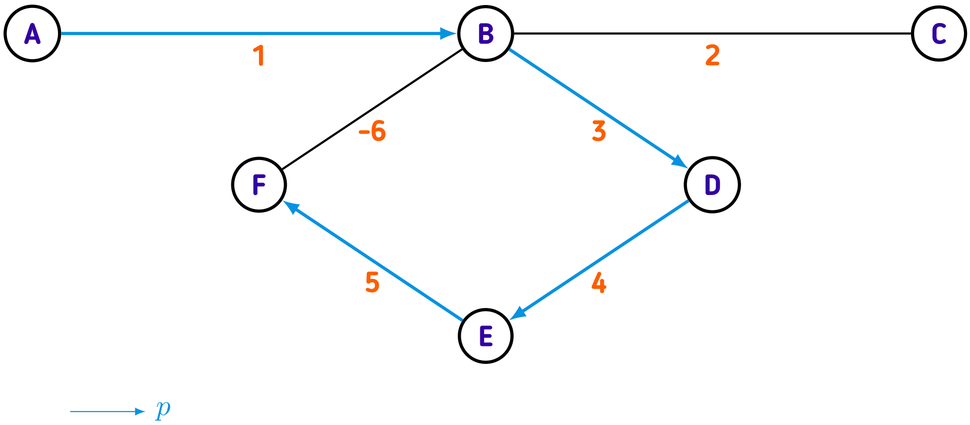
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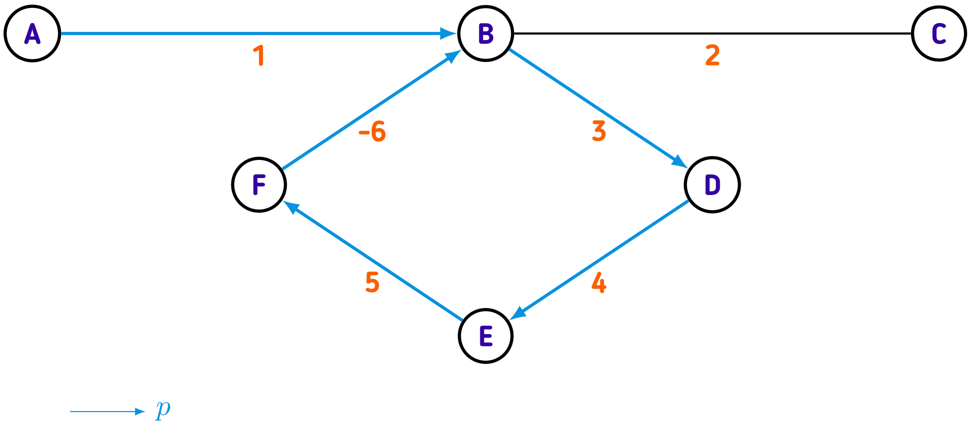
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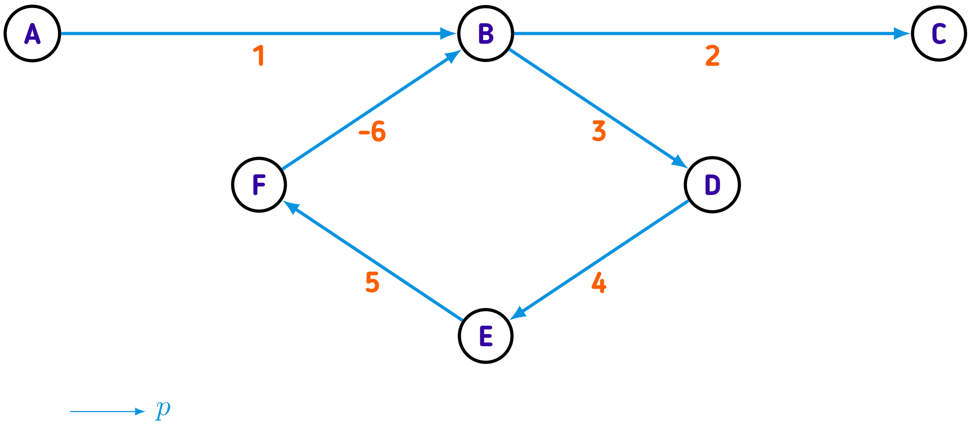
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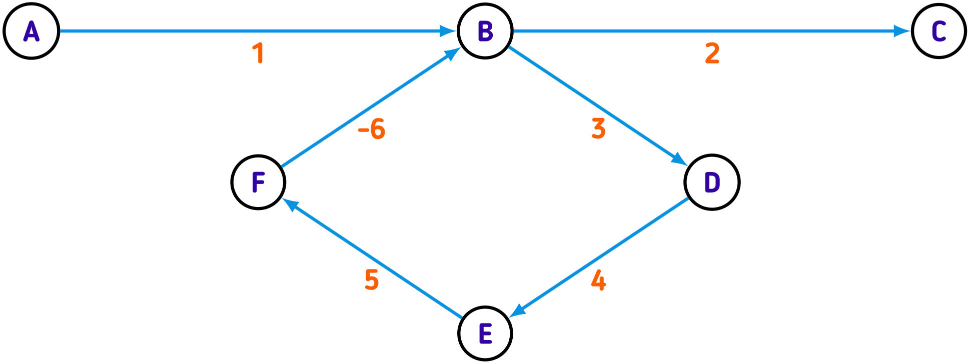
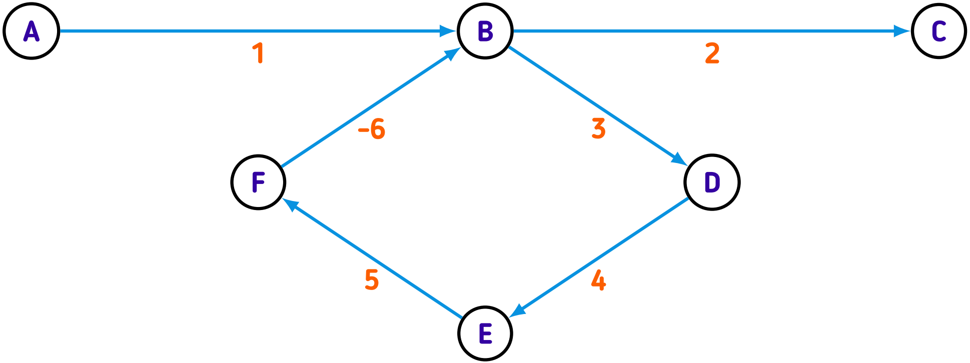
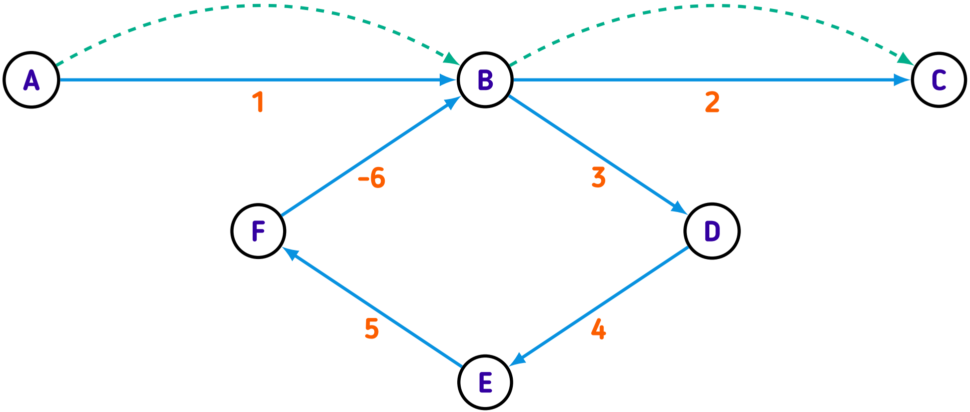


$\longrightarrow p \qquad \omega(p) = 9$



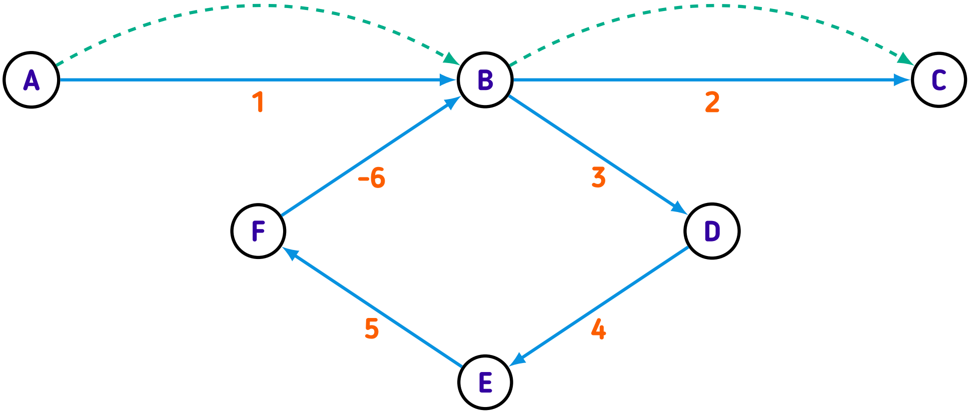
$\longrightarrow p$ $\qquad \omega(p) = 9$

$\dashrightarrow q$



p $\quad\omega(p) = 9$

q



p $\quad \omega(p) = 9$

q $\quad \omega(q) = 3$

Caminhos mínimos e ciclos negativos

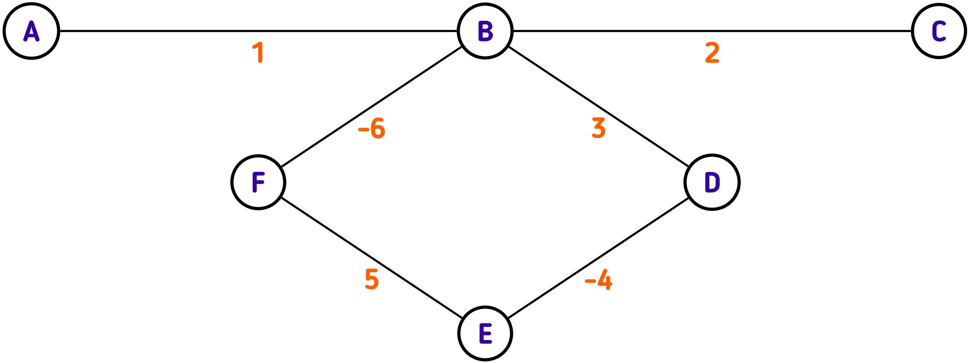
Caminhos mínimos e ciclos negativos

Seja $\omega(c) < 0$ **e**

$$q = \{(a, u_1), (u_1, u_2), \ldots, (v, u_r), \ldots, (u_s, v), (v, u_r), \ldots, (u_s, v), \ldots, (u_t, b)\}$$

o caminho resultante da duplicação do ciclo c **de** p**. Então** $\omega(q) < \omega(p)$**, pois**

$$\omega(q) = \sum_{e_i \in q} w(e_i) = \sum_{e_j \in p} w(e_j) + \sum_{e_k \in c} w(e_k) = \omega(p) + \omega(c) < \omega(p)$$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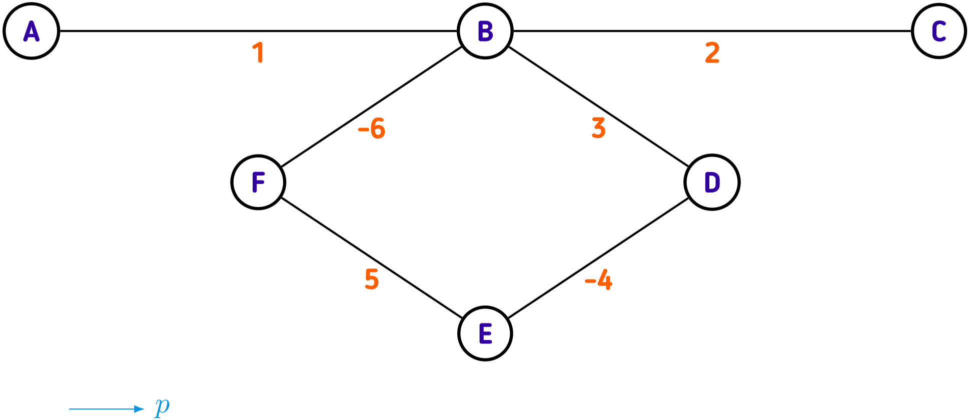
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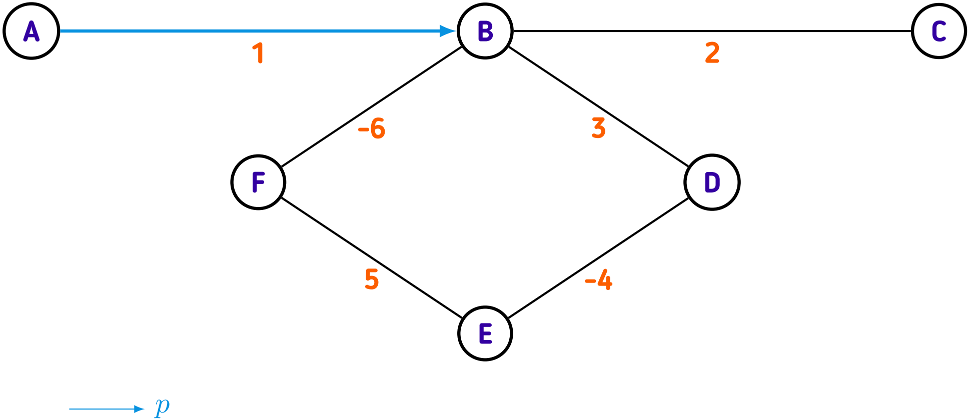
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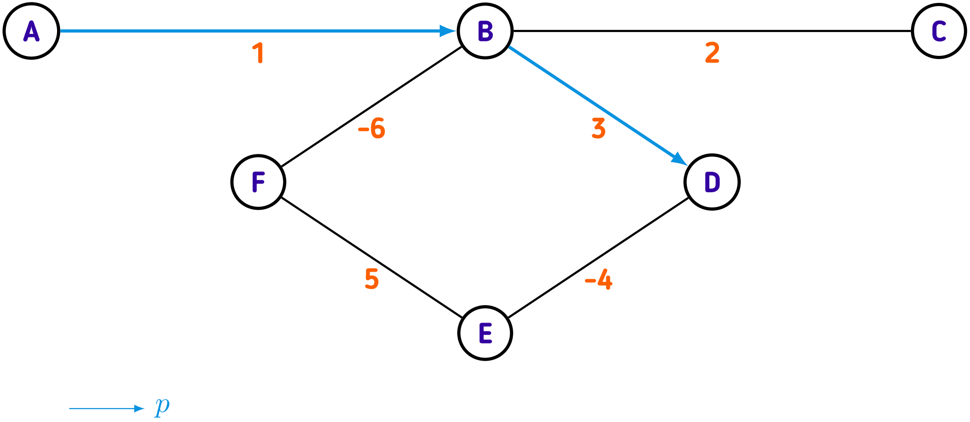
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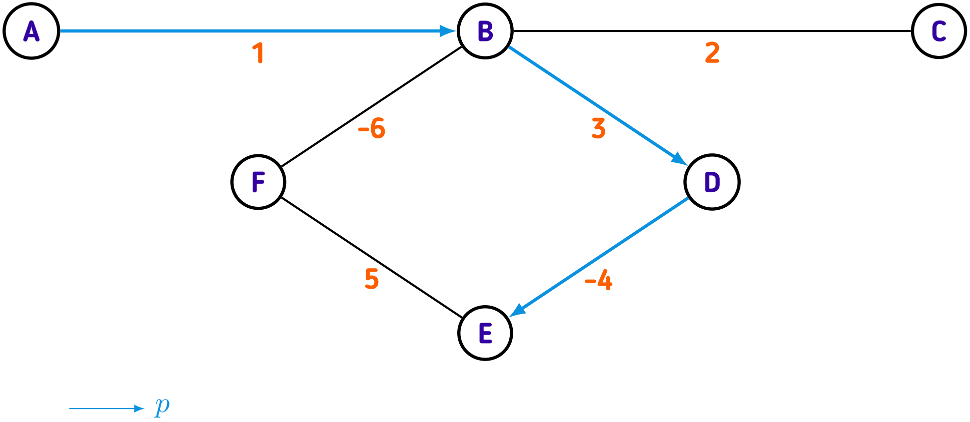
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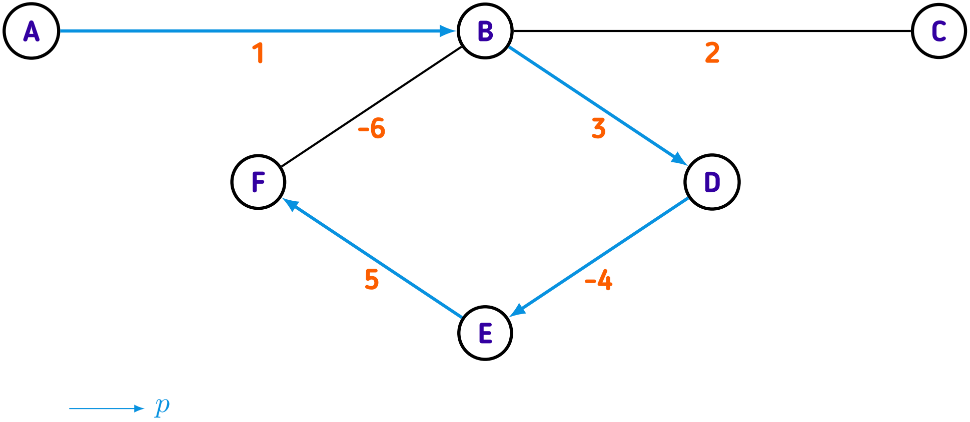
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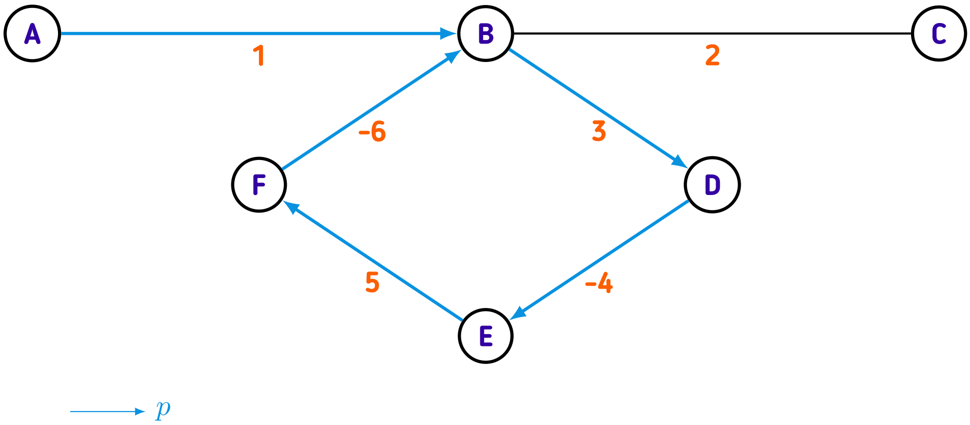
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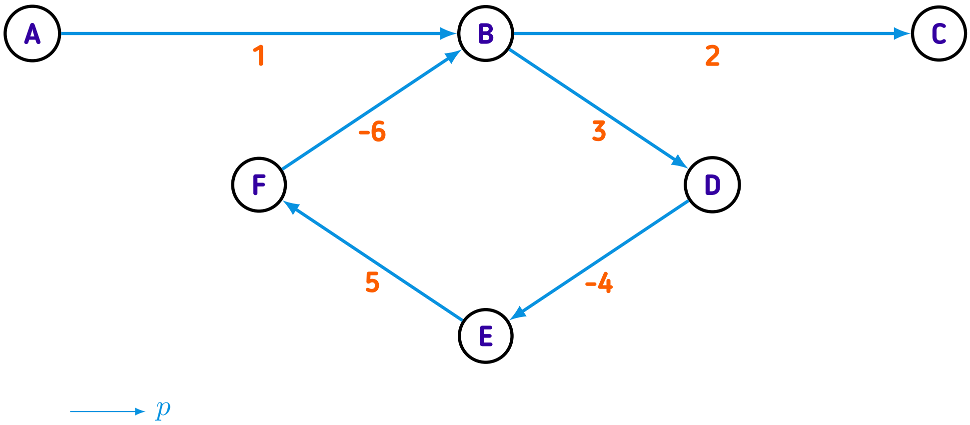
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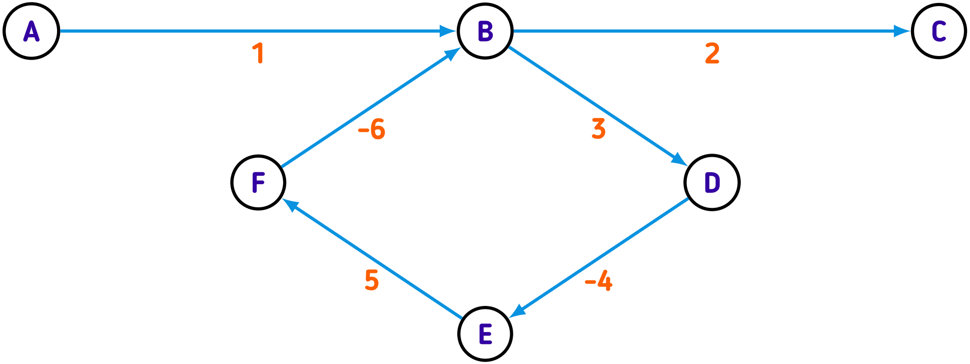
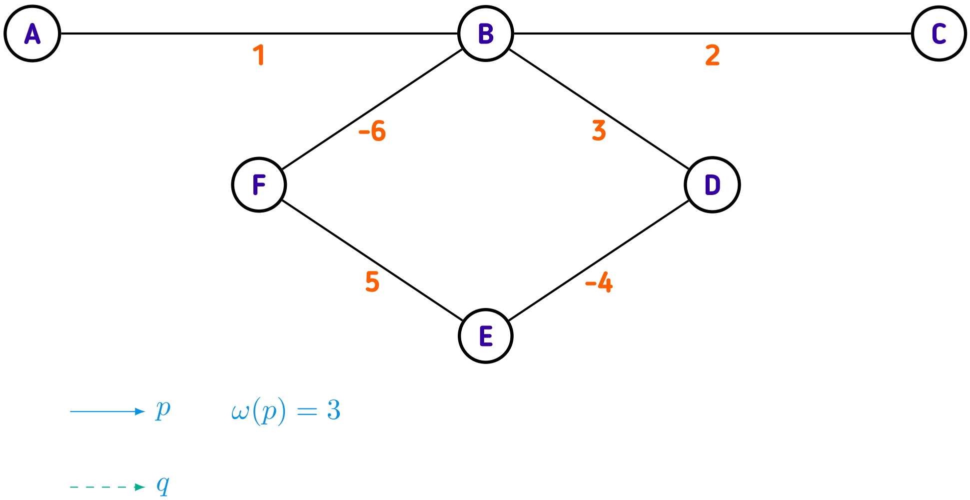


$\longrightarrow p \qquad \omega(p) = 3$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ω(p) = 3
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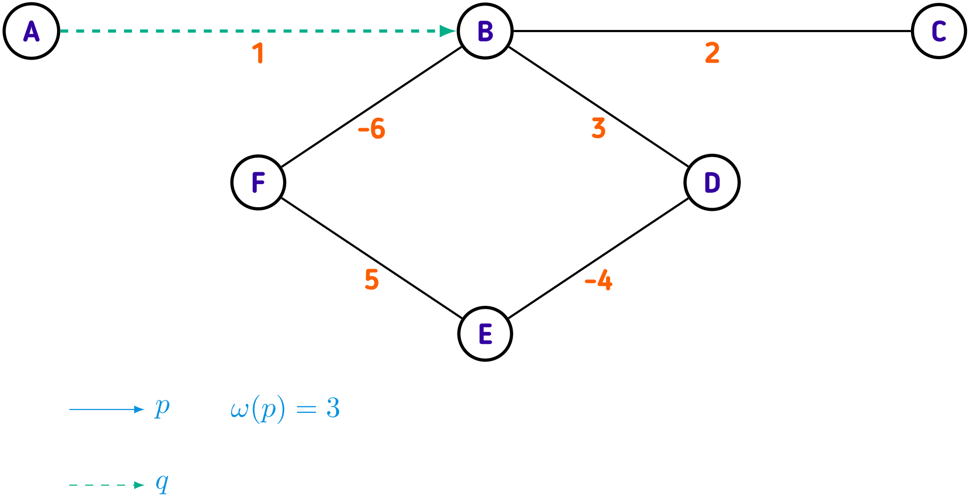
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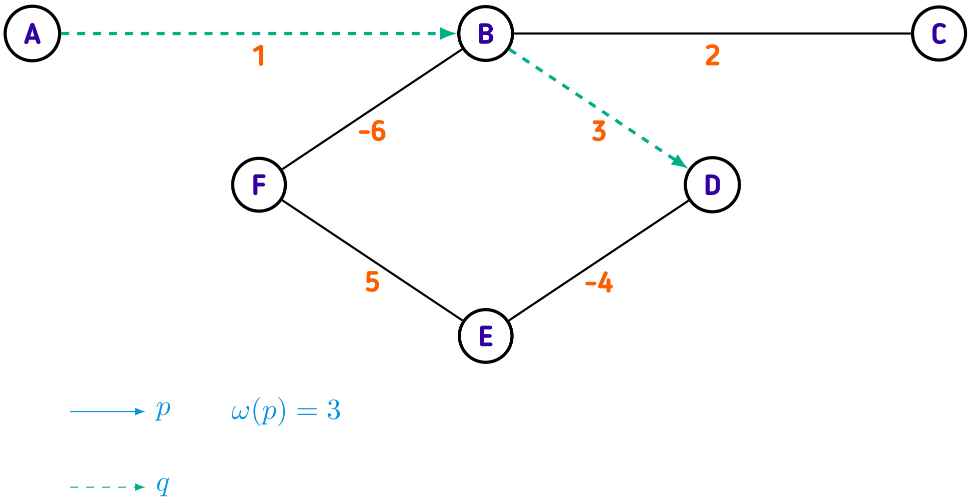
$\omega(p) = 3$
q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$\omega(p) = 3$
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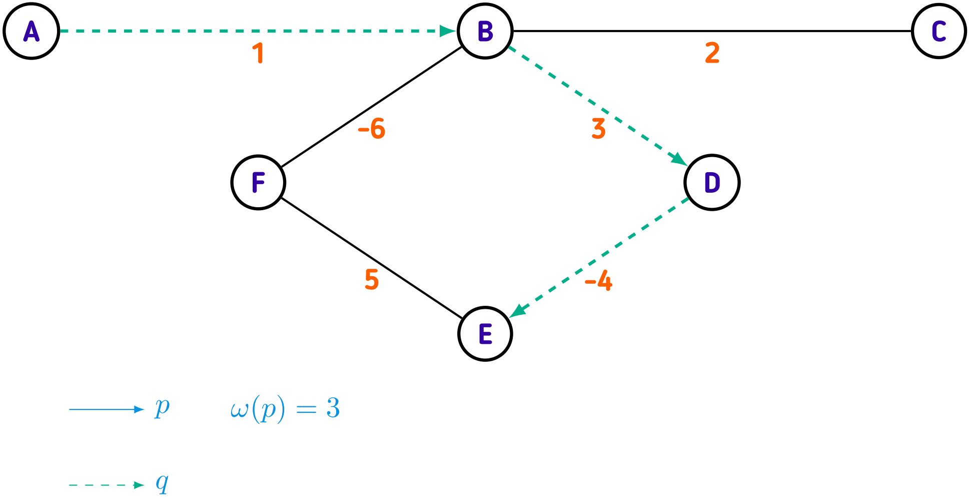
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ω(p) = 3
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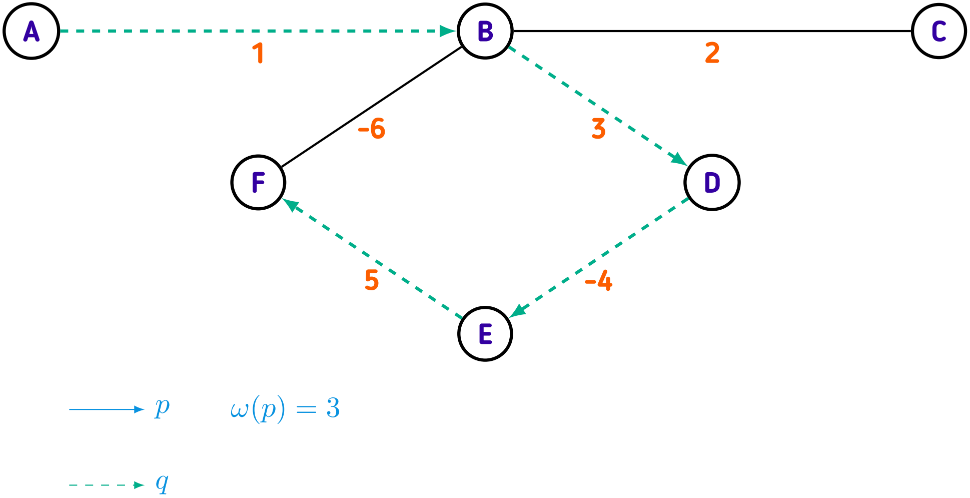
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$\omega(p) = 3$
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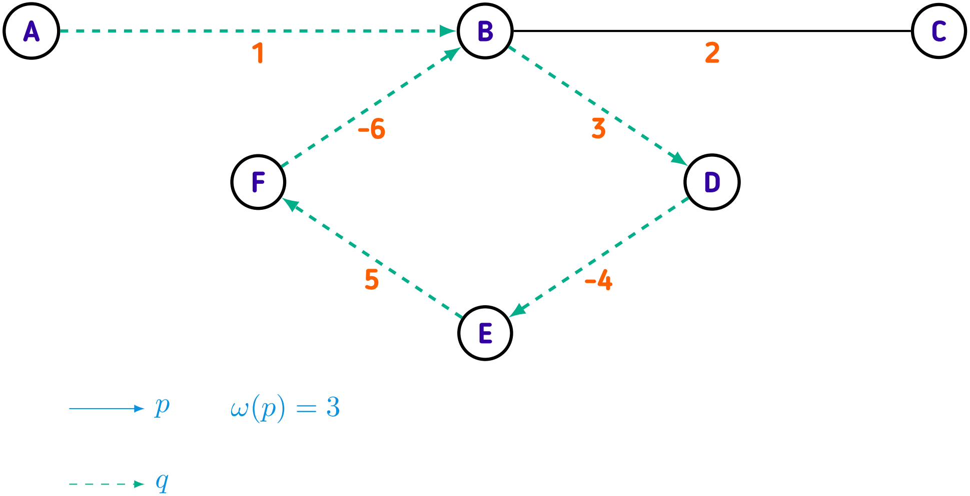
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ω(p) = 3
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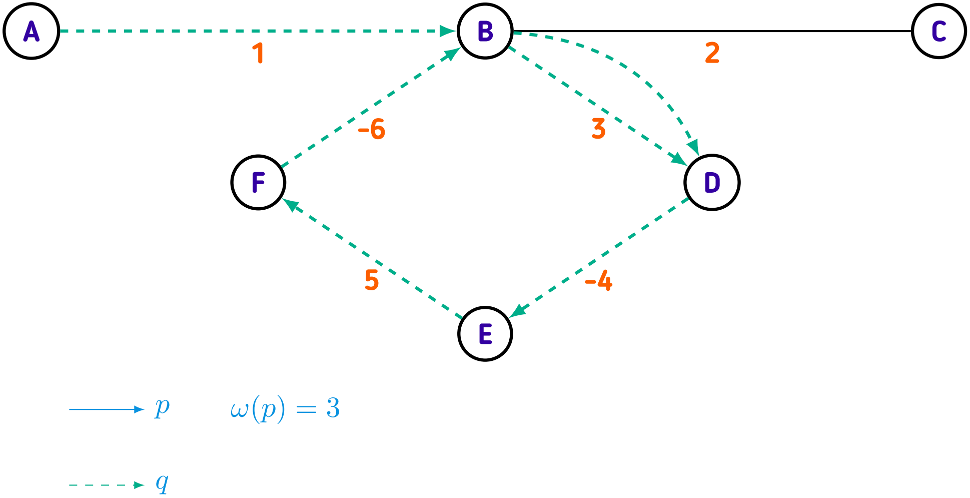
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$\omega(p) = 3$
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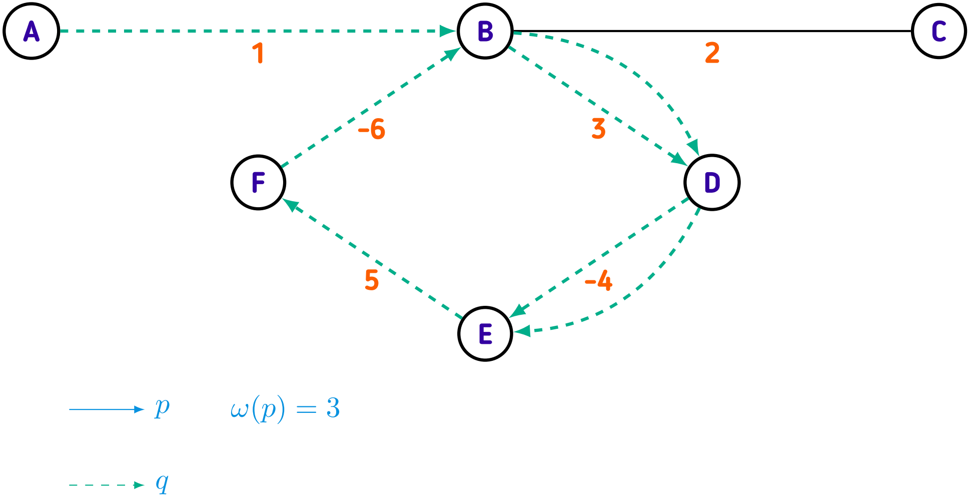
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ω(p) = 3
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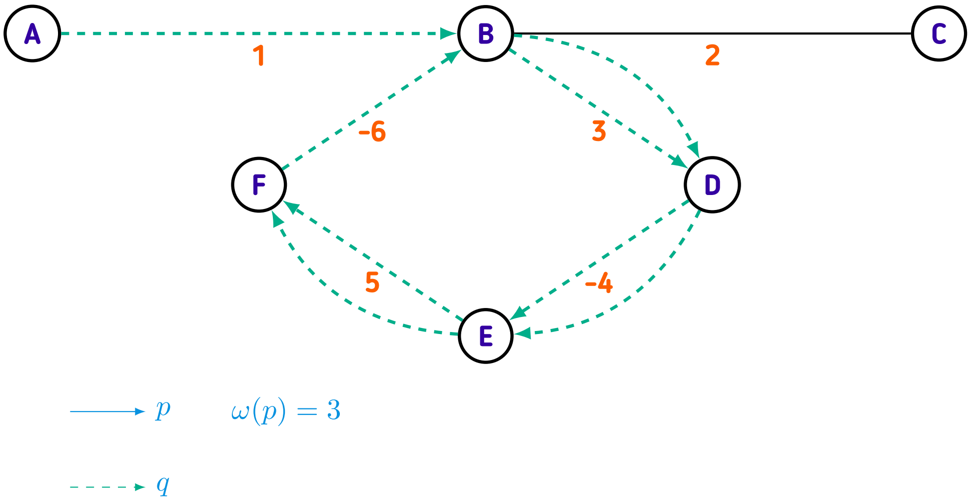
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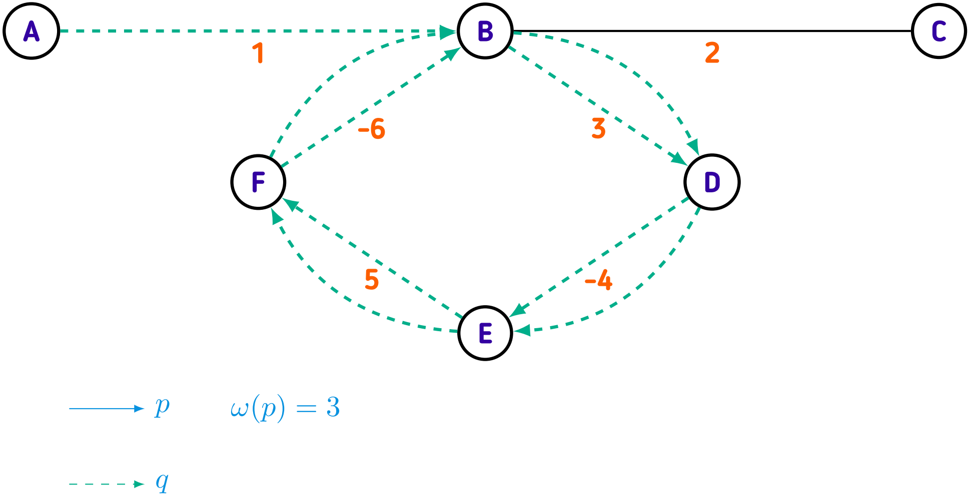
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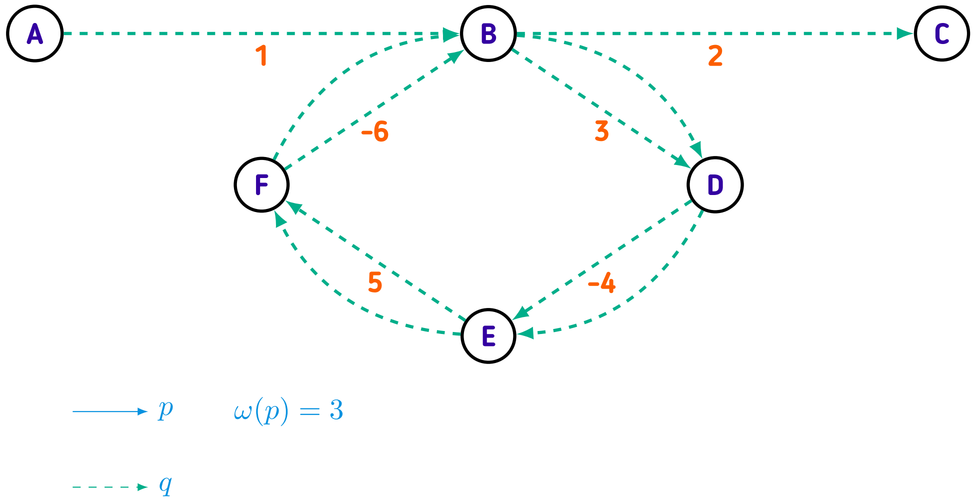
p
ω(p) = 3
q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p
$\omega(p) = 3$
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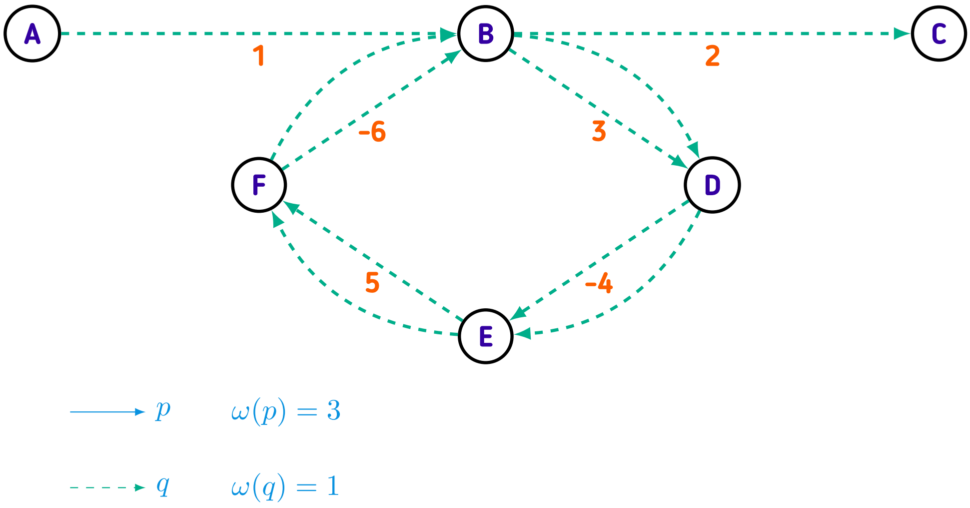
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ω(p) = 3
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ω(q) = 1

Número de rodadas do algoritmo de Bellman-Ford

Número de rodadas do algoritmo de Bellman-Ford

Teorema. Seja $G(V, E)$ um grafo cujos pesos de suas arestas sejam todos não-negativos. Então para qualquer $v \in V$, o caminho mínimo de s a u identificado pelo algoritmo de Bellman-Ford tem, no máximo, $|V| - 1$ arestas.

Detecção de ciclos negativos

Detecção de ciclos negativos

Teorema. Seja $G(V, E)$ um grafo. Se a $|V|$-ésima rodada do algoritmo de Bellman-Ford atualizar o vetor d ao menos uma vez, então G possui pelo menos um ciclo negativo.

```
bool has_negative_cycle(int s, int N, const vector<edge>& edges)
{
    const int oo { 1000000010 };

    vector<int> dist(N + 1, oo);
    dist[s] = 0;

    for (int i = 1; i <= N - 1; i++)
        for (auto [u, v, w] : edges)
            dist[v] = min(dist[v], dist[u] + w);

    for (auto [u, v, w] : edges)
        if (dist[v] > dist[u] + w)
            return true;

    return false;
}
```

Problemas sugeridos

1. AtCoder Beginner Contest 137 - Problem E: Coin Respawn
2. CSES 1673 - High Score
3. OJ 423 - MPI Maelstrom
4. OJ 534 - Frogger

Referências

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2. **LAAKSONEN**, **Antti**. *Competitive Programmer's Handbook*, **2018.**
3. **SKIENA**, **Steven**; **REVILLA**, **Miguel**. *Programming Challenges*, **2003.**
4. **Wikipédia**, *Bellman-Ford algorithm.* **Acesso em 07/07/2021.**
5. **Wikipédia**, *L. R. Ford Jr.* **Acesso em 07/07/2021.**
6. **Wikipédia**, *Richard E. Bellman.* **Acesso em 07/07/2021.**