Technische Universität Wien

194.027 Hybrid Classic-Quantum Systems

Lecturer: Vincenzo De Maio

# Assignment 1: Basics of Quantum Computing

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Due Date: November 6, 2023

Submitted: November 1, 2023

# Contents

# 1 Question 1

First of all, let's define $v_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. In order for $v_1$, $v_2$ and $v_3$ to form an orthonormal basis for $R^3$, there are a few conditions that need to be met:

- Each of the three vectors must be mutually orthogonal to guarantee linear independence and their suitability as a basis for $R^3$. This can be achieved by ensuring their dot products equal zero, a condition already met by the provided vectors $v_1$ and $v_2$.
- The vectors should be unit vectors with a magnitude of 1, a criterion already met by $v_1$ and $v_2$, and $v_3$ should adhere to the same principle. We can simplify calculations by using the dot product, as it results in the square of the vector's magnitude when multiplied to itself, which should also be 1 in this scenario.

With this in mind, we can reach the following system of equations:

$$\begin{cases} \langle v_1 | v_3 \rangle = 0 \\ \langle v_2 | v_3 \rangle = 0 \\ \langle v_3 | v_3 \rangle = 1 \end{cases} \Leftrightarrow \begin{cases} \frac{2}{3}x - \frac{1}{3}y - \frac{2}{3}z = 0 \\ -\frac{\sqrt{2}}{2}x - -\frac{\sqrt{2}}{2}z = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases} \Leftrightarrow \begin{cases} \frac{2}{3}x - \frac{1}{3}y + \frac{2}{3}x = 0 \\ z = -x \\ \rule{2cm}{0.4pt} \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} y = 4x \\ \rule{2cm}{0.4pt} \\ x^2 + 16x^2 + x^2 = 1 \end{cases} \Leftrightarrow \begin{cases} \rule{1cm}{0.4pt} \\ \rule{1cm}{0.4pt} \\ x^2 = \frac{1}{18} \end{cases} \Leftrightarrow \begin{cases} y = \frac{4}{3\sqrt{2}} \\ z = -\frac{1}{3\sqrt{2}} \\ x = \frac{1}{3\sqrt{2}} \end{cases} \vee \begin{cases} y = -\frac{4}{3\sqrt{2}} \\ z = \frac{1}{3\sqrt{2}} \\ x = -\frac{1}{3\sqrt{2}} \end{cases} \tag{1}$$

If we pick $x = \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}$, we get the following result:

$$v_3 = \begin{bmatrix} \frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{6} \end{bmatrix}$$

# 2 Question 2

## 2.1 Question 2.1

First of all, let's define $|v\rangle = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ as the generic eigenvector of $A$. We should consider the following definitions about eigenvalues and eigenvectors:

$$A\,|v\rangle = \alpha\,|v\rangle \Leftrightarrow (A - \alpha I)\,|v\rangle = 0 \tag{2}$$

$$|A - \alpha I| = 0 \tag{3}$$

Using equation 3, we are able to find the eigenvalues of $A$:

$$\begin{vmatrix} -\alpha & 0 & -1 \\ 0 & 1-\alpha & 0 \\ -1 & 0 & -\alpha \end{vmatrix} = 0 \Leftrightarrow -\alpha \begin{vmatrix} 1-\alpha & 0 \\ 0 & -\alpha \end{vmatrix} - \begin{vmatrix} 0 & 1-\alpha \\ -1 & 0 \end{vmatrix} = 0 \Leftrightarrow \tag{4}$$
$$\Leftrightarrow -\alpha^3 + \alpha^2 + \alpha - 1 = 0 \Leftrightarrow \alpha = 1 \vee \alpha = -1$$

The eigenvalues of $A$ are 1 and -1. In order to find the respective eigenvectors, we can use equation 2. For $\alpha = 1$:

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow x = -z \tag{5}$$

If we say that $x = t_1$ and $y = t_2$, we can represent $|v\rangle$ as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{6}$$

For this reason, **the eigenvalue 1 corresponds to 2 eigenvectors:** $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ **and** $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$**.**

As for $\alpha = -1$, the process is similar:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} x = z \\ y = 0 \end{cases} \tag{7}$$

If we say that $x = t_1$, we can represent $|v\rangle$ as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

For this reason, **the eigenvalue -1 corresponds to the eigenvector:** $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$**.**

## 2.2 Question 2.2

In order for the set of eigenvectors to form an orthonormal basis, there are a few conditions that need to be met:

- Each of the three vectors must be mutually orthogonal to guarantee linear independence and their suitability as a basis. This can be achieved by ensuring their dot products equal zero.
- The vectors should be unit vectors with a magnitude of 1. We can simplify calculations by using the dot product, as it results in the square of the vector's magnitude when multiplied to itself, which should also be 1 in this scenario.

For the set of 3 vectors, we can choose any linear dependent vector of the ones calculated in 2.1. This way, we can define the vectors like below, where $a$, $b$ and $c$ are real numbers.

$$|v1\rangle = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} \quad |v2\rangle = b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} \quad |v3\rangle = c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ c \end{bmatrix} \tag{9}$$

We can then verify the first condition:

$$\begin{cases} \langle v_1, v_2 \rangle = a \times 0 + 0 \times b - a \times 0 = 0 \\ \langle v_1, v_3 \rangle = a \times c - a \times c = 0 \\ \langle v_2, v_3 \rangle = 0 \times c + b \times 0 + 0 \times c = 0 \end{cases} \tag{10}$$

All possible vectors are linearly independent, so they can form a basis. However, as for the second condition, only a subset of those can form an orthonormal basis:

$$\begin{cases} \langle v_1, v_1 \rangle = 1 \\ \langle v_2, v_2 \rangle = 1 \\ \langle v_3, v_3 \rangle = 1 \end{cases} \Leftrightarrow \begin{cases} a^2 + (-a)^2 = 1 \\ b^2 = 1 \\ c^2 + c^2 = 1 \end{cases} \Leftrightarrow \begin{cases} a = \frac{\sqrt{2}}{2} \vee a = -\frac{\sqrt{2}}{2} \\ b = 1 \vee b = -1 \\ c = \frac{\sqrt{2}}{2} \vee c = -\frac{\sqrt{2}}{2} \end{cases} \tag{11}$$

All the vectors that satisfy those conditions can form an orthonormal basis. For example, the vectors $|v_1\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$, $|v_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $|v_3\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$.

## 2.3 Question 2.3

Let's start by picking up the normalized eigenvectors calculated in 2.2:

$$|v_1\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \quad |v_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad |v_3\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \tag{12}$$

Since they form an orthonormal basis, we can use them to create a unitary matrix, $U = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$. We can prove this matrix is unitary by verifying that its transposed conjugate is equal to its inverse: $U^{-1} = U^\dagger$.

$$U^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \quad U^\dagger = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{13}$$

## 2.4 Question 2.4

A way to show that $U$ satisfies the relation would be to calculate it and verify the result. Nevertheless, we can prove that the relation holds for any unitary matrix formed from normalized eigenvectors of the matrix $A$.

Let's start by defining $U = \begin{bmatrix} |v_1\rangle & |v_2\rangle & |v_3\rangle \end{bmatrix}$. From this definition, we can also say that $U^\dagger = \begin{bmatrix} \langle v_1| \\ \langle v_2| \\ \langle v_3| \end{bmatrix}$, since the conjugate transpose is the same as converting a *ket* to a *bra*. Then, if we multiply $A$ to this expression, we will get $U^\dagger A = \begin{bmatrix} \langle v_1| A \\ \langle v_2| A \\ \langle v_3| A \end{bmatrix}$. After multiplying $U$, we get the matrix:

$$U^\dagger A U = \begin{bmatrix} \langle v_1| A |v_1\rangle & \langle v_1| A |v_2\rangle & \langle v_1| A |v_3\rangle \\ \langle v_2| A |v_1\rangle & \langle v_2| A |v_2\rangle & \langle v_2| A |v_3\rangle \\ \langle v_3| A |v_1\rangle & \langle v_3| A |v_2\rangle & \langle v_3| A |v_3\rangle \end{bmatrix} \tag{14}$$

Since $v_1$, $v_2$ and $v_3$ are eigenvectors of $A$, we can use the equality $A|v\rangle = \alpha|v\rangle$. Furthermore, $\alpha_i$ is a constant, so it can be moved to the start of the matrix multiplication:

$$U^\dagger A U = \begin{bmatrix} \langle v_1| \alpha_1 |v_1\rangle & \langle v_1| \alpha_1 |v_2\rangle & \langle v_1| \alpha_1 |v_3\rangle \\ \langle v_2| \alpha_2 |v_1\rangle & \langle v_2| \alpha_2 |v_2\rangle & \langle v_2| \alpha_2 |v_3\rangle \\ \langle v_3| \alpha_3 |v_1\rangle & \langle v_3| \alpha_3 |v_2\rangle & \langle v_3| \alpha_3 |v_3\rangle \end{bmatrix} = \begin{bmatrix} \alpha_1 \langle v_1|v_1\rangle & \alpha_1 \langle v_1|v_2\rangle & \alpha_1 \langle v_1|v_3\rangle \\ \alpha_2 \langle v_2|v_1\rangle & \alpha_2 \langle v_2|v_2\rangle & \alpha_2 \langle v_2|v_3\rangle \\ \alpha_3 \langle v_3|v_1\rangle & \alpha_3 \langle v_3|v_2\rangle & \alpha_3 \langle v_3|v_3\rangle \end{bmatrix} \tag{15}$$

Since the vectors used are normalized, their dot products are equal to 1. On the other hand, the dot products between the different vectors are always equal to 0 since they are all orthogonal, hence:

$$U^\dagger A U = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \tag{16}$$

# 3 Question 3

Since the state is represented in the $\{|0\rangle, |1\rangle\}$ basis, we can retrieve the probabilities of retrieving either outcome upon measurement in the same basis by calculating the square of the respective coefficients:

$$P(|0\rangle) = (\frac{1}{\sqrt{3}})^2 = \frac{1}{3} \quad P(|1\rangle) = (\frac{\sqrt{2}}{\sqrt{3}})^2 = \frac{2}{3} \tag{17}$$

To measure the same state in the $\{|+\rangle, |-\rangle\}$ basis, we can first transform it into that basis and then use the same method to calculate the probabilities. We can do that by applying the following calculations:

$$\begin{cases} |0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \\ |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}} \end{cases} \tag{18}$$

$$|\psi\rangle = \frac{1}{\sqrt{3}}(\frac{|+\rangle + |-\rangle}{\sqrt{2}}) + \frac{\sqrt{2}}{\sqrt{3}}(\frac{|+\rangle - |-\rangle}{\sqrt{2}}) = \frac{1+\sqrt{2}}{\sqrt{6}}|+\rangle + \frac{1-\sqrt{2}}{\sqrt{6}}|-\rangle \tag{19}$$

$$\begin{aligned} P(|+\rangle) &= \frac{1+\sqrt{2}}{\sqrt{6}}^2 = \frac{3+2\sqrt{2}}{\sqrt{6}} \approx 97.1\% \\ P(|-\rangle) &= \frac{1-\sqrt{2}}{\sqrt{6}}^2 = \frac{3-2\sqrt{2}}{\sqrt{6}} \approx 2.9\% \end{aligned} \tag{20}$$

## 3.1 Bonus Question

The matrix representation for the unitary operation $U$ can be found by writing the equation in matrix notation and solving the equation system for $U$. Let's also assume that $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$U\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\sqrt{3}}\\\dfrac{\sqrt{2}}{\sqrt{3}}\end{bmatrix}\Leftrightarrow\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\sqrt{3}}\\\dfrac{\sqrt{2}}{\sqrt{3}}\end{bmatrix}\Leftrightarrow\begin{cases}a=\dfrac{1}{\sqrt{3}}\\c=\dfrac{\sqrt{2}}{\sqrt{3}}\end{cases}\tag{21}$$

Assuming that $b=0$ and $d=0$, we get the matrix $U=\begin{bmatrix}\frac{1}{\sqrt{3}} & 0\\\frac{\sqrt{2}}{\sqrt{3}} & 0\end{bmatrix}$.

# 4 Question 4

The circuit to generate the Bell state is displayed in Figure 1.

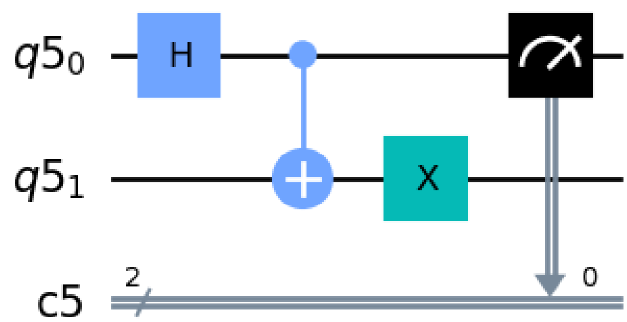


Figure 1: Circuit used to generate the Bell state

In an intuitive way, the state $|\Psi^+\rangle$ is entangled because we gain knowledge about both qubits by measuring just one of them. For instance, when we measure the more significant qubit, we automatically determine the value of the less significant qubit as its complement (e.g., measuring 1 implies $|10\rangle$, and measuring 0 implies $|01\rangle$). This principle also works with the other qubit. By definition, this state is entangled because it cannot be represented as a tensor product of the individual basis states, making it inseparable.

The state $|\psi\rangle$ lacks entanglement as it comprises only a single qubit, rendering it a superposition. Quantum entanglement is a characteristic of two or more qubits, enabling the expression of correlations that defy classical systems' capabilities.

# 5 Question 5

The solution for the quantum circuit is to use a qubit and apply a Hadamard gate, resulting in 50% chance to land in either 0 or 1 after the measurement. The 0 and 1 will represent the different faces of the coin. The circuit in Figure 2 displays the solution used.

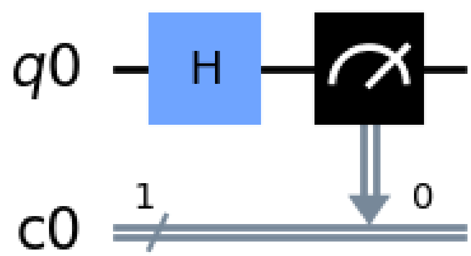


Figure 2: Circuit used for the coin flip

The width of the circuit is 1 since we only use one qubit. The depth of the circuit is also 1 since there's only one gate applied (Hadamard) before measurement.

By running the circuit on a perfect simulator with 10,000 shots, we got the probability distribution shown in Figure 3.

On the other hand, by running it on a noisy simulator (in this case, FakeManilaV2), we got a probability distribution that looks very different from the perfect one. In a real (noisy) quantum machine, the coin flip wouldn't be as perfect as one wished. We can see this in Figure 4.

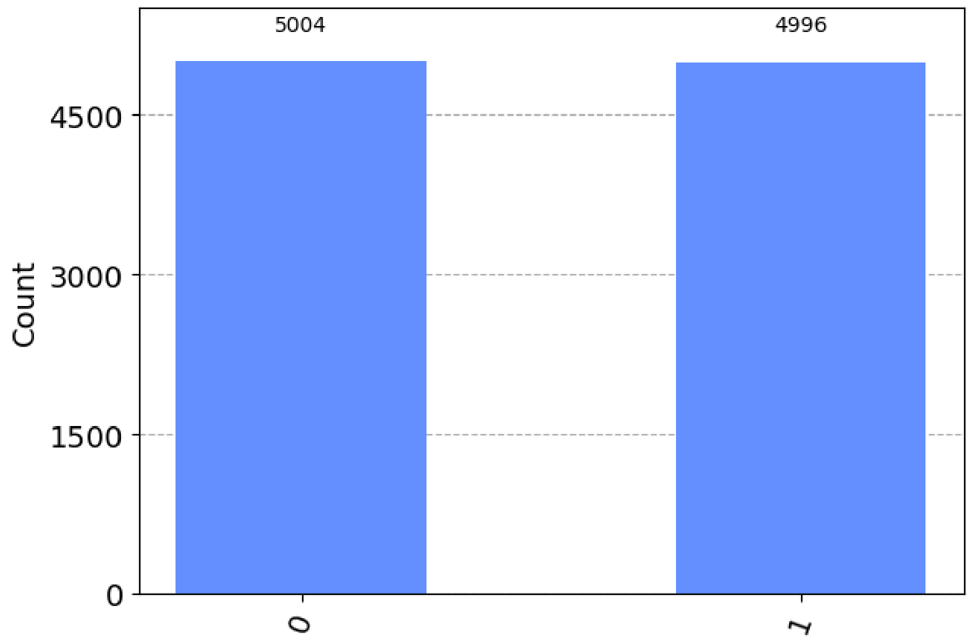


Figure 3: Probability distribution for the perfect simulator of a coin flip

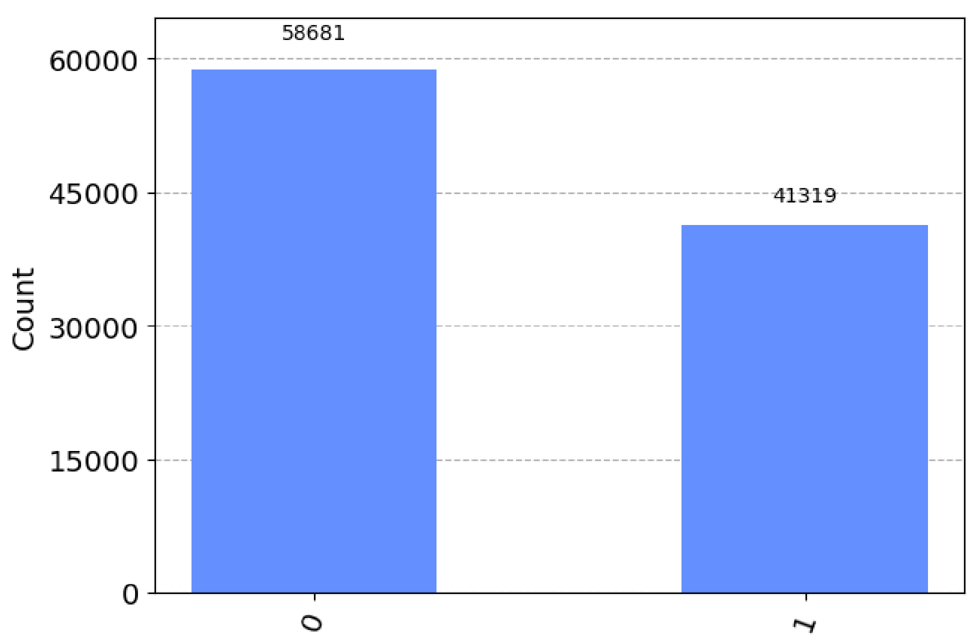


Figure 4: Probability distribution for the noisy simulator of a coin flip

# 6 Question 6

The solution for the quantum circuit is to adapt the Bell state mentioned in question 4 and apply an extra *CNOT* gate to a third qubit, as shown in Figure 5. This way, we'll have 50% chance to measure $|000\rangle$ and 50% chance to measure $|111\rangle$.

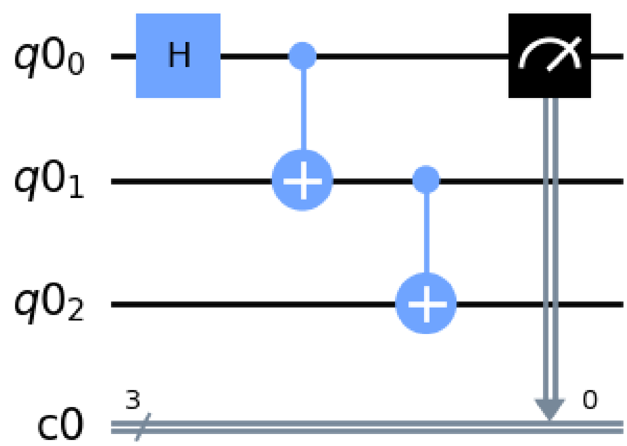


Figure 5: Circuit used to generate the state

This state is entangled for the same reasons as question 4. In sum, it is entangled because we learn about all qubits by measuring just one of them (although it would be enough to have a weaker correlation). For instance, when we measure the more significant qubit, we automatically determine the values of the others (e.g., measuring 1 implies $|111\rangle$, and measuring 0 implies $|000\rangle$). By definition, this state is entangled because it cannot be represented as a tensor product of the individual basis states, making it inseparable.

The width of the circuit is 3 since we use three qubits. The depth of the circuit is also 3 since we apply three gates before measuring the first qubit, and no parallelization is possible. The second *CNOT* can be parallel with the measurement gate, but we do not consider the latter in this calculation.

By running the circuit on a perfect simulator with 10,000 shots, we got the probability distribution shown in Figure 6.

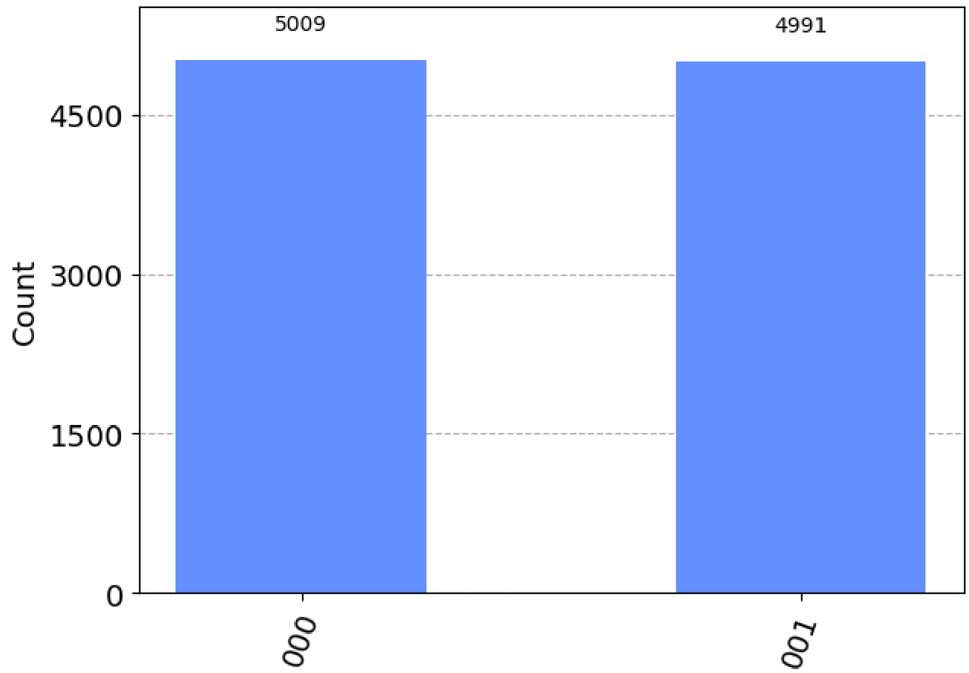


Figure 6: Probability distribution for the perfect simulator

On the other hand, by running it on a noisy simulator (in this case, FakeManilaV2), we got a probability distribution that looks very different from the perfect one. In a real (noisy) quantum machine, we would not get a balanced 50/50 distribution, as shown in Figure 7.

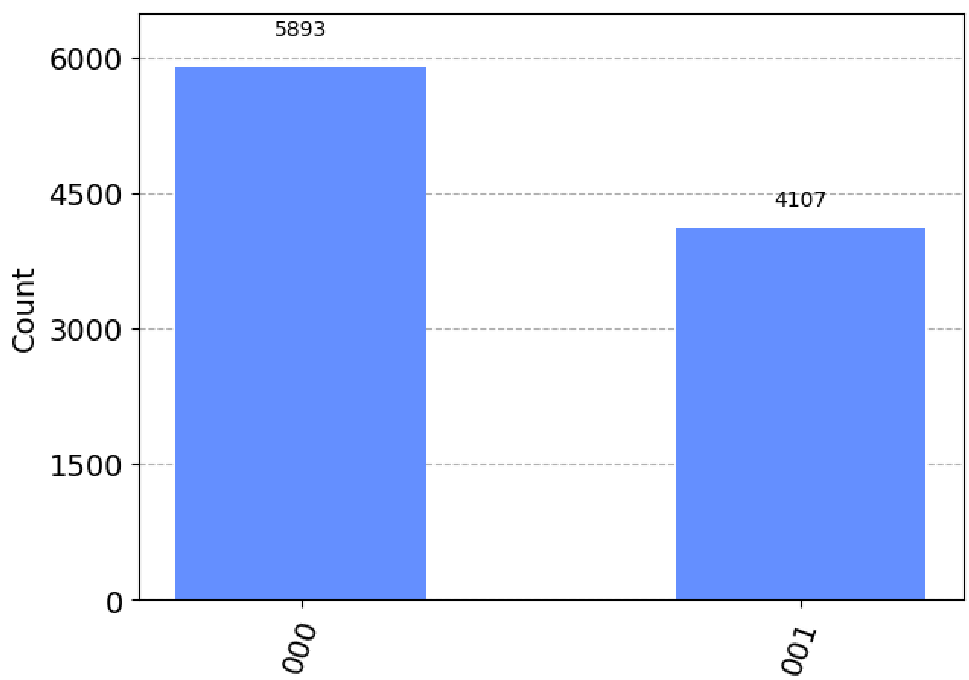


Figure 7: Probability distribution for the noisy simulator

# 7 How to Run the Source Code

The source code for questions 4, 5, and 6 is entirely written in Jupyter Notebooks. Within the notebooks, you'll discover the code for creating and executing the circuits presented in this report, along with graphical plot outputs.

In addition to having Jupyter and Python3 installed, it's important to verify the presence of the following libraries: *qiskit, qiskit[visualization], pdflatex* (optional but useful to generate Latex). To execute the code, you can follow the standard Jupyter Notebook procedure. Just open the notebook with your integrated development environment (IDE) and run the cells in the sequence they are presented.