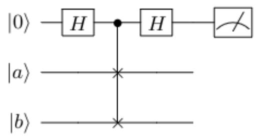

FIG. 17: Quantum circuit of *SwapTest* quantum subroutine.

The steps of the subroutine are presented below.

Initialize
Initialize two states $|a\rangle$ and $|b\rangle$ as well as a control qubit $|0\rangle$ resulting in the state:

$$|\psi_0\rangle = |0, a, b\rangle \ . \tag{124}$$

The states $|a\rangle$ and $|b\rangle$ consist of n qubits each.

Apply Hadamard gate
Apply Hadamard gate on the control qubit resulting in a superposition:

$$|\psi_1\rangle = (H \otimes I^{\otimes n} \otimes I^{\otimes n})\,|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0, a, b\rangle + |1, a, b\rangle)\,. \tag{125}$$

Apply *SWAP* gate
Apply controlled *SWAP* gate on $|a\rangle$ and $|b\rangle$ states which swaps a and b providing that the control qubit is in state $|1\rangle$. As a result:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0, a, b\rangle + |1, b, a\rangle)\,. \tag{126}$$

Apply Hadamard gate
Apply second Hadamard gate on the control qubit resulting in the state:

$$|\psi_3\rangle = \frac{1}{2}\,|0\rangle\,(|a, b\rangle + |b, a\rangle) + \frac{1}{2}\,|1\rangle\,(|a, b\rangle - |b, a\rangle)\,. \tag{127}$$

Measurement
Measure the control qubit. The probability of measuring control qubit being in state $|0\rangle$ is given by:

$$\begin{aligned} P(|0\rangle) &= |\frac{1}{2}\,\langle 0|0\rangle\,(|a, b\rangle + |b, a\rangle) + \frac{1}{2}\,\langle 0|1\rangle\,(|a, b\rangle - |b, a\rangle)|^2 \\ &= \frac{1}{4}|(|a, b\rangle + |b, a\rangle)|^2 \\ &= \frac{1}{4}(\langle b|b\rangle\,\langle a|a\rangle + \langle b|a\rangle\,\langle a|b\rangle + \langle a|b\rangle\,\langle b|a\rangle + \langle a|a\rangle\,\langle b|b\rangle) \\ &= \frac{1}{2} + \frac{1}{2}|\langle a|b\rangle|^2 \end{aligned} \tag{128}$$

Thus, we successfully linked an overlap $\langle a|b\rangle$ with measurement probability of the control qubit in the final quantum state. The probability $P(|0\rangle) = 0.5$ means that the states $|a\rangle$ and $|b\rangle$ are orthogonal, whereas the probability $P(|0\rangle) = 1$ indicates that the states are identical. The subroutine should be repeated several times to obtain a good estimator of probability. The advantage of using swap test is that the states $|a\rangle$ and $|b\rangle$ can be unknown before procedure and simple measurement is performed on the control qubit which has two eigenstates. The time complexity is negligible as the procedure does not depend on a number of qubits representing the input states, however we may pay attention to the preparation time of identical copies of $|a\rangle$ and $|b\rangle$.

With subroutine *SwapTest* presented we can move on to the algorithm retrieving the Euclidean distance $|a-b|^2$ between two real valued vectors a and b. The algorithm was described by Lloyd, Mohseni and Rebentrost [22].

Representation of classical data as quantum states
The classical information in vector a is encoded as [23, 24]:

$$|a|^{-1}a \rightarrow |a\rangle = \sum_{i=1}^{N} |a|^{-1} a_i \, |i\rangle \ . \tag{129}$$

Norm of quantum state is normalized with this definition $\langle a|a\rangle = |a|^{-2}a^2 = 1$, which leads to the correct definition of a quantum state. The N dimensional training vector can be translated into $n = \log_2 N$ qubits. As an example the vector with eight features can be stored in three qubits containing $2*2*2 = 8$ entries to express vector components a_i as probability amplitudes.

Initialize
Initialize two quantum states:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0, a\rangle + |1, b\rangle)\,, \tag{130}$$

$$|\phi\rangle = \frac{1}{\sqrt{Z}}(|a|\,|0\rangle + |b|\,|1\rangle) \tag{131}$$

with $Z = |a|^2 + |b|^2$.

Use quantum subroutine *SwapTest*
Evaluate an overlap $\langle\phi|\psi\rangle$ using subroutine *SwapTest*.

Calculate distance
Calculate Euclidean distance by noting that:

$$|a-b|^2 = 2Z|\,\langle\phi|\psi\rangle\,|^2\,. \tag{132}$$

This holds true, because:

$$\langle\phi|\psi\rangle = \frac{1}{\sqrt{2Z}}(|a|\,|a\rangle - |b|\,|b\rangle) \tag{133}$$

and using the fact how the classical information has been prepared in Eq. (129) the inner product could be expressed as:

$$\langle\phi|\psi\rangle = \frac{1}{\sqrt{2Z}}(a-b) \tag{134}$$

so that $|\,\langle\phi|\psi\rangle\,|^2 = \frac{1}{2Z}|a-b|^2$. Moreover, using this algorithm it is also easy to calculate the inner product between two vectors noticing that:

$$a^T b = \frac{1}{2}(|a|^2 + |b|^2 - |a-b|^2)\,. \tag{135}$$

To summarize, in *DistCalc* subroutine we prepare two states, apply subroutine *SwapTest* and repeat that procedure to obtain an acceptable estimate of probability. Providing that the states are already prepared the subroutine *SwapTest* does not depend on the size of a feature vector. The preparation of states in subroutine *DistCalc* is proved to has $\mathcal{O}(\log N)$ time complexity [22], which makes sense as the classical information is encoded in $n = log_2 N$ qubits and we expect the time complexity to be proportional. The classical algorithms require $\mathcal{O}(N)$ to calculate Eucidean distance between two vectors, thus there is an exponential speed-up. The measurement during swap test is causing the deconherence of input states, thus the quantum memory should contain multiple copies of input states, however it does not change time complexity for large N, as the number of states preparation will be always much smaller than N.