Category Theory for Programmers

Homework (Session 5)

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Pushout in `C++`

We talked about products and co-products in posets before, where the term supremum and infimum applied. In this more general case applied to `C++` types a diagram of a pushout D is as follows :

$$\begin{tikzcd}
& E & \\
& D & \\
B \arrow[ur] \arrow[uur] & A \arrow[l] \arrow[r] & C \arrow[ul] \arrow[uul]
\end{tikzcd}$$

Any superclass E of B and C is a superclass of D. Which means that D corresponds to a superclass that encompasses all the functionality that the subclasses have in common[1].

Limit of the identity functor

Note ; this is no proof ; you need to show uniqueness of morfisms apex => other objects
can be done by starting from naturality condition of components

The identity functor `Id` maps a category to itself. The initial object is the object for which there's exactly one morphism to every object. When constructing the limit for `Id` any apex will correspond to an initial object (or there wouldn't be a natural transformation from Δ_c to `Id`). If there are many apexes (or initial objects) there will be inversible transformations between them. Indeed, initial objects are unique up to isomorphism, which we knew already.

[1] One has to consider all possible superclasses of B and C. Then D inherits from all of those. Some superclasses may only declare a part of the functionality that's present in both the classes B and C, yet together they declare all of it and nothing more or some wouldn't be superclasses.

Pullback, pushout, ... in `Set`

Starting from the diagram depicted on the previous page and applying analogous reasoning we can see that a pushout D is the smallest superset or the union of B and C. In the opposite category we get :

$$\begin{tikzcd}
& E \arrow[ddl] \arrow[ddr] & \\
& D \arrow[dl] \arrow[dr] & \\
B \arrow[r] & A & C \arrow[l]
\end{tikzcd}$$

Here, the pullback D is the largest subset or the intersection (every other set that's a subset of B and C ought to be a subset of D). A lattice visualises this and the terms supremum and infimum are in order.

As for the initial - and terminal object, they (trivially) correspond to the empty set and the set on which the category is based.

Co-equalizer

Visually :

$$\begin{tikzcd}
& C & \\
B \arrow[ur, "p"] & & A \arrow[ul, "q"'] \arrow[ll, shift right, "f"'] \arrow[ll, shift left, "g"]
\end{tikzcd}$$

Once again q is fully defined by p and one of the morphisms from $A \to B$. We have

$$q = p \cdot f \qquad and \qquad q = p \cdot g \qquad \Rightarrow \qquad p \cdot f = p \cdot g$$

where for every other co-equalizer C' there should be a unique factoriser. Aside from this universal construction, there's a more concrete explanation of what it means in `Set`. There we end up with a partitioning of B that corresponds to an equivalence relationship $f(x) \sim g(x)$. It should be the smallest of such relationships which corresponds to the finest partitioning[2].

Pullback towards terminal object, pushout from initial object

A pullback is a special kind of product because A imposes some additional structure. When you're dealing with A as a terminal object and you're looking for a product, then that structure (morphisms to A from B, C and D) will necessarily be there. In other words, every such product is a pullback towards the terminal object (and vice versa, every such pullback is a product).

In the same vein, when you're dealing with A as an initial object and are looking for a coproduct then there will be unique morphisms from A to B, C and D which necessarily make it a pushout from the initial object.

[2] An example ; let $A = B = \{1, 2, 3\}$, $f = id$, $g(1) = 2$, $g(2) = 1$ and $g(3) = 3$. Then we find a co-equalizer p where $p(1) = p(2) = [1]$ and $p(3) = [3]$. This is a '*better*' co-equalizer than p' with $p'(x) = [1]$ because we find the factoriser h with $h(x) = [1]$ for which $p' = h \cdot p$.