Additional topics

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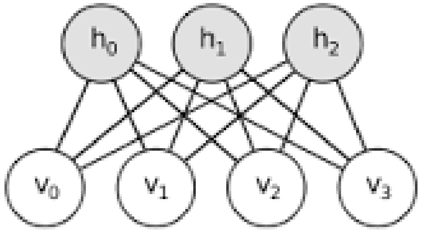
Lecture 10

Contents - additional topics

1. Restricted kernel machines and deep learning
2. Kernel spectral clustering
3. LS-SVM for recurrent networks and optimal control problems

Restricted kernel machines

Restricted Boltzmann Machines (RBM)



- Markov random field, bipartite graph, stochastic binary units
  Layer of visible units v and layer of hidden units h
  No hidden-to-hidden connections
- Energy:

$$E(v, h; \theta) = -v^T W h - c^T v - a^T h \quad \text{with} \quad \theta = \{W, c, a\}$$

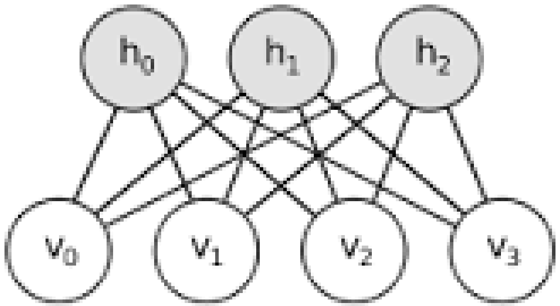
Joint distribution:

$$P(v, h; \theta) = \frac{1}{Z(\theta)} \exp(-E(v, h; \theta))$$

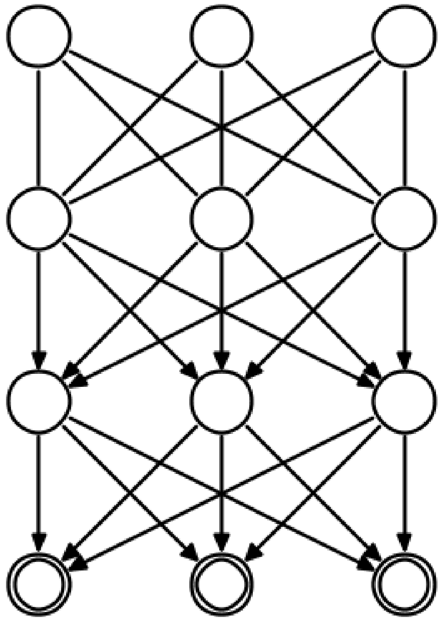
with partition function $Z(\theta) = \sum_v \sum_h \exp(-E(v, h; \theta))$

[Hinton, Osindero, Teh, Neural Computation 2006]

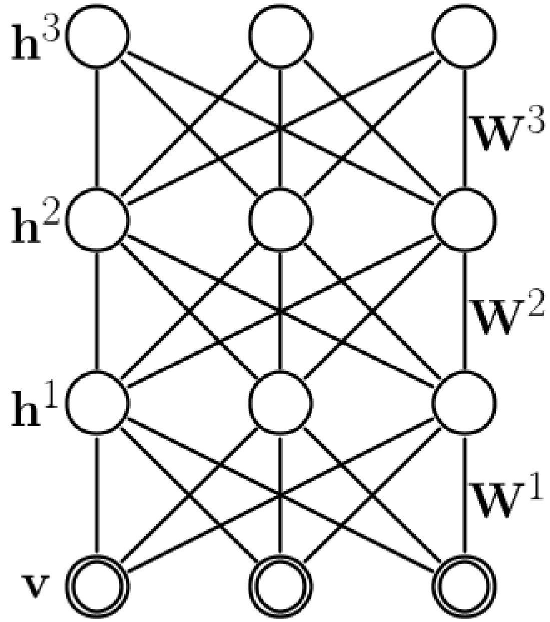
RBM and deep learning



Deep Belief Network



Deep Boltzmann Machine



$p(v, h)$

$p(v, h^1, h^2, h^3, ...)$

[Hinton et al., 2006; Salakhutdinov, 2015]

Energy function

- RBM:

$$E = -v^T W h$$

- Deep Boltzmann machine (two layers):

$$E = -v^T W^1 h^1 - {h^1}^T W^2 h^2$$

- Deep Boltzmann machine (three layers):

$$E = -v^T W^1 h^1 - {h^1}^T W^2 h^2 - {h^2}^T W^3 h^3$$

Restricted Kernel Machines (RKM)

- Kernel machine interpretations in terms of **visible and hidden units** (similar to Restricted Boltzmann Machines (**RBM**))

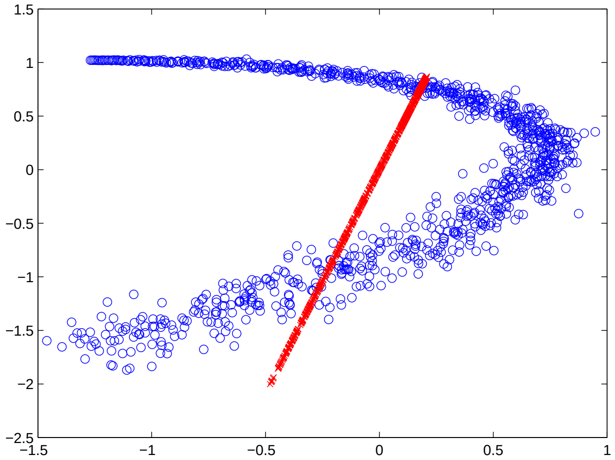
- Restricted Kernel Machine (**RKM**) representations for
  - LS-SVM regression/classification
  - Kernel PCA
  - Matrix SVD
  - Parzen-type models
  - other

- Based on principle of **conjugate feature duality** (with hidden features corresponding to dual variables)

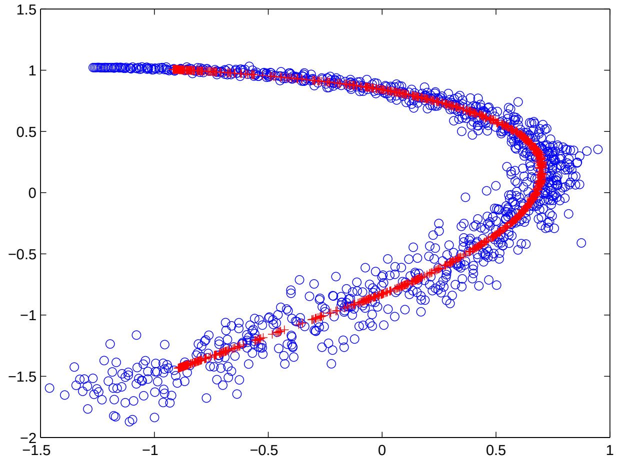
- Deep Restricted Kernel Machines (Deep RKM)

[Suykens, Neural Computation, 2017]

Kernel principal component analysis (KPCA)



linear PCA



kernel PCA (RBF kernel)

Kernel PCA [Schölkopf et al., 1998]:
take eigenvalue decomposition of the kernel matrix

$$\begin{bmatrix} K(x_1, x_1) & \dots & K(x_1, x_N) \\ \vdots & & \vdots \\ K(x_N, x_1) & \dots & K(x_N, x_N) \end{bmatrix}$$

(applications in dimensionality reduction and denoising)

Kernel PCA: classical LS-SVM approach

- Primal problem: [Suykens et al., 2002]: **model-based** approach

$$\min_{w,b,e} \frac{1}{2}w^T w - \frac{1}{2}\gamma \sum_{i=1}^{N} e_i^2 \;\; \text{s.t.} \;\; e_i = w^T \varphi(x_i) + b, \; i = 1, ..., N.$$

- Dual problem corresponds to kernel PCA

$$\Omega^{(c)}\alpha = \lambda\alpha \;\; \text{with} \;\; \lambda = 1/\gamma$$

with $\Omega_{ij}^{(c)} = (\varphi(x_i) - \hat{\mu}_\varphi)^T(\varphi(x_j) - \hat{\mu}_\varphi)$ the *centered kernel matrix* and $\hat{\mu}_\varphi = (1/N)\sum_{i=1}^{N} \varphi(x_i)$.

- Interpretation:
  1. pool of candidate components (objective function equals zero)
  2. select relevant components

From KPCA to RKM representation (1)

Model:

$$e = W^T \varphi(x)$$

objective J
$=$ regularization term $\mathrm{Tr}(W^T W)$
- $(\frac{1}{\lambda})$ variance term $\sum_i e_i^T e_i$

$\downarrow$ use property $e^T h \leq \frac{1}{2\lambda} e^T e + \frac{\lambda}{2} h^T h$

RKM representation:

$$e = \sum_j h_j K(x_j, x)$$

obtain $J \leq \overline{J}(h_i, W)$
solution from stationary points of $\overline{J}$:
$\frac{\partial \overline{J}}{\partial h_i} = 0$, $\frac{\partial \overline{J}}{\partial W} = 0$

From KPCA to RKM representation (2)

- Objective

$$
\begin{aligned}
J &= \frac{\eta}{2}\mathrm{Tr}(W^T W) - \frac{1}{2\lambda}\sum_{i=1}^{N} e_i^T e_i \;\; \text{s.t. } e_i = W^T \varphi(x_i), \forall i \\
&\leq -\sum_{i=1}^{N} e_i^T h_i + \frac{\lambda}{2}\sum_{i=1}^{N} h_i^T h_i + \frac{\eta}{2}\mathrm{Tr}(W^T W) \;\; \text{s.t. } e_i = W^T \varphi(x_i), \forall i \\
&= -\sum_{i=1}^{N} \varphi(x_i)^T W h_i + \frac{\lambda}{2}\sum_{i=1}^{N} h_i^T h_i + \frac{\eta}{2}\mathrm{Tr}(W^T W) \triangleq \overline{J}
\end{aligned}
$$

- Stationary points of $\overline{J}(h_i, W)$:

$$
\begin{cases}
\dfrac{\partial \overline{J}}{\partial h_i} = 0 \;\; \Rightarrow \;\; W^T \varphi(x_i) = \lambda h_i, \;\; \forall i \\
\dfrac{\partial \overline{J}}{\partial W} = 0 \;\; \Rightarrow \;\; W = \dfrac{1}{\eta}\sum_i \varphi(x_i) h_i^T
\end{cases}
$$

From KPCA to RKM representation (3)

- Elimination of W gives the eigenvalue decomposition:

$$\frac{1}{\eta} K H^T = H^T \Lambda$$

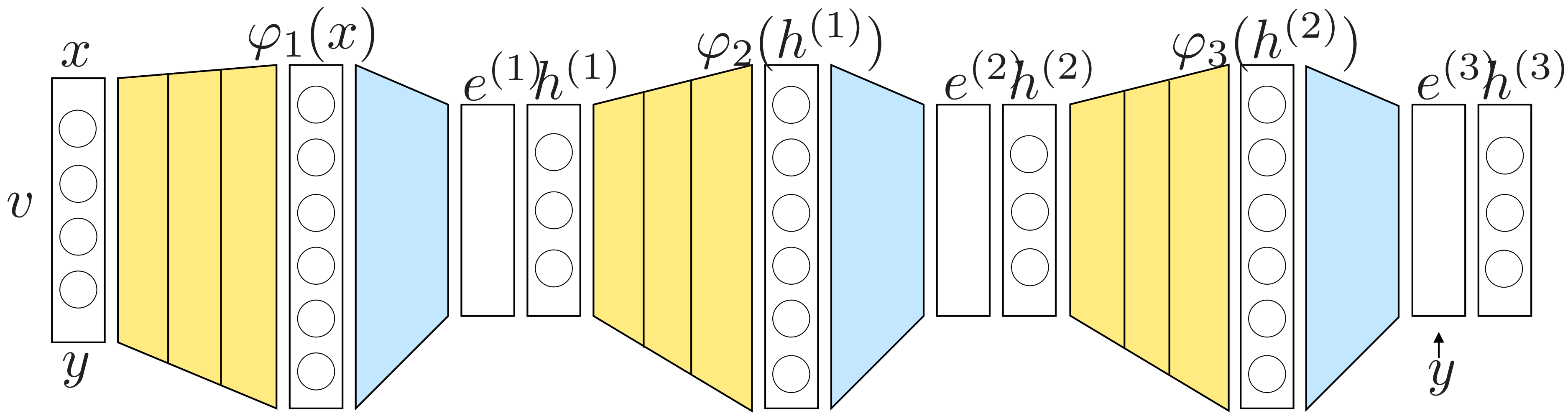
where $H = [h_1 ... h_N] \in \mathbb{R}^{s \times N}$ and $\Lambda = \mathrm{diag}\{\lambda_1, ..., \lambda_s\}$ with $s \leq N$

- Primal and dual model representations

$$\mathcal{M} \begin{array}{l} \nearrow \quad (P)_{\mathrm{RKM}}: \quad \hat{e} = W^T \varphi(x) \\ \\ \searrow \quad (D)_{\mathrm{RKM}}: \quad \hat{e} = \frac{1}{\eta} \sum_j h_j K(x_j, x). \end{array}$$

Deep Restricted Kernel Machines

Deep RKM: example



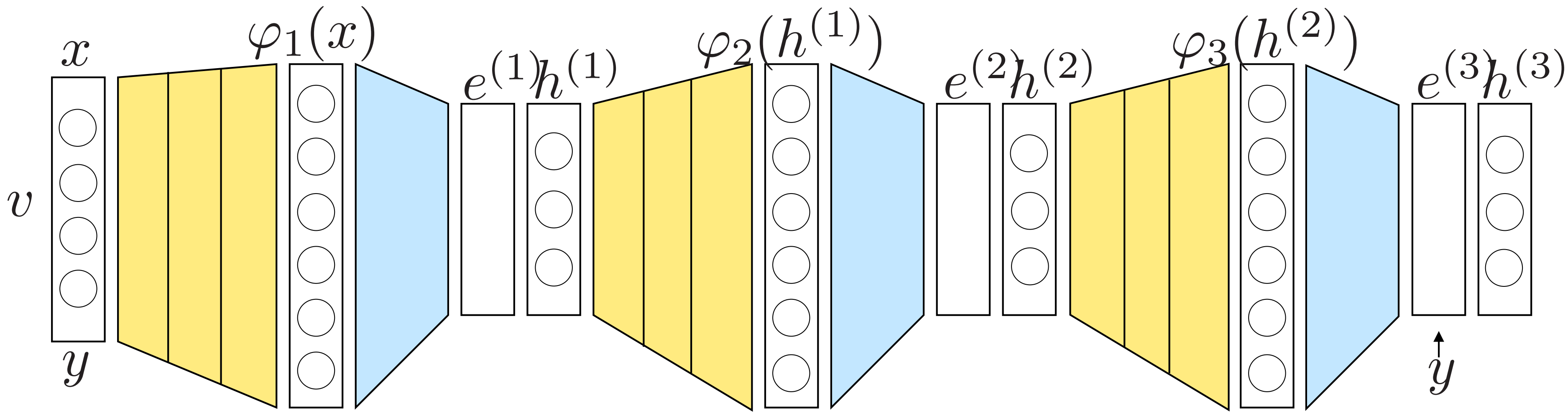
Deep RKM: KPCA + KPCA + LSSVM [Suykens, 2017]

Coupling of RKMs by taking sum of the objectives

$$J_{\text{deep}} = \overline{J}_1 + \overline{J}_2 + \underline{J}_3$$

Multiple *levels* and multiple *layers* per level.

in more detail ...



$$
\begin{aligned}
J_{\text{deep}} \;\; = \;\; & -\sum_{i=1}^{N} \varphi_1(x_i)^T {\color{red}W_1} h_i^{(1)} + \frac{\lambda_1}{2}\sum_{i=1}^{N} {h_i^{(1)}}^T h_i^{(1)} + \frac{\eta_1}{2}\mathrm{Tr}(W_1^T W_1) \\
& -\sum_{i=1}^{N} \varphi_2(h_i^{(1)})^T {\color{red}W_2} h_i^{(2)} + \frac{\lambda_2}{2}\sum_{i=1}^{N} {h_i^{(2)}}^T h_i^{(2)} + \frac{\eta_2}{2}\mathrm{Tr}(W_2^T W_2) \\
& +\sum_{i=1}^{N} (y_i^T - \varphi_3(h_i^{(2)})^T {\color{red}W_3} - b^T) h_i^{(3)} - \frac{\lambda_3}{2}\sum_{i=1}^{N} {h_i^{(3)}}^T h_i^{(3)} + \frac{\eta_3}{2}\mathrm{Tr}(W_3^T W_3)
\end{aligned}
$$

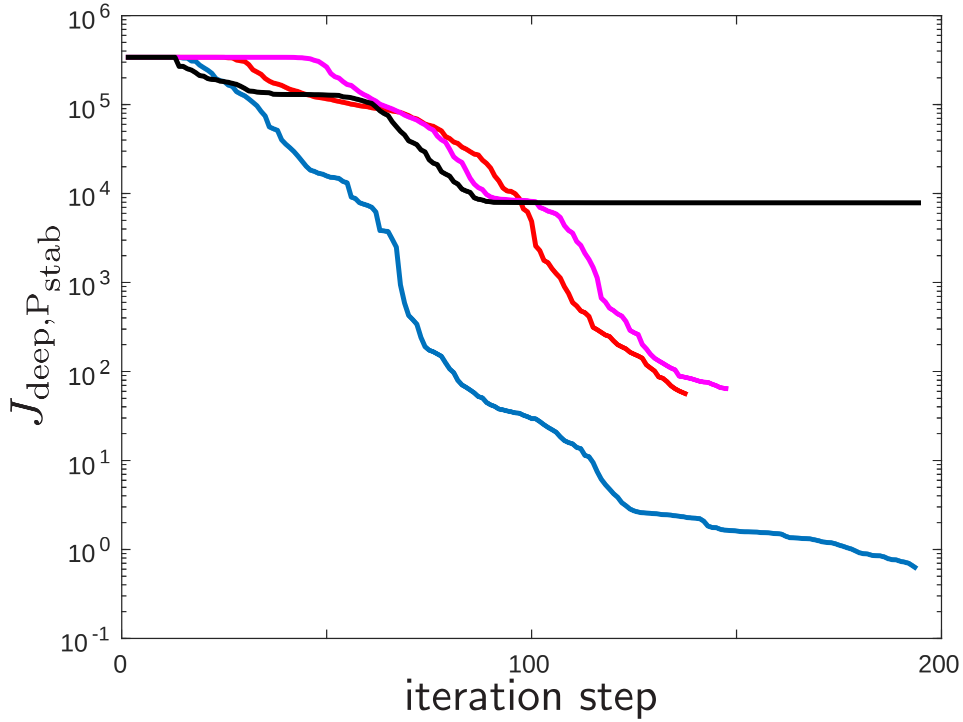
Primal and dual model representations

$$\mathcal{M} \begin{array}{l} \nearrow \quad (P)_{\mathrm{DeepRKM}}: \begin{array}{l} \hat{e}^{(1)} = W_1^T \varphi_1(x) \\ \hat{e}^{(2)} = W_2^T \varphi_2(\Lambda_1^{-1} \hat{e}^{(1)}) \\ \hat{y} = W_3^T \varphi_3(\Lambda_2^{-1} \hat{e}^{(2)}) + b \end{array} \\ \\ \searrow \quad (D)_{\mathrm{DeepRKM}}: \begin{array}{l} \hat{e}^{(1)} = \frac{1}{\eta_1} \sum_j h_j^{(1)} K_1(x_j, x) \\ \hat{e}^{(2)} = \frac{1}{\eta_2} \sum_j h_j^{(2)} K_2(h_j^{(1)}, \Lambda_1^{-1} \hat{e}^{(1)}) \\ \hat{y} = \frac{1}{\eta_3} \sum_j h_j^{(3)} K_3(h_j^{(2)}, \Lambda_2^{-1} \hat{e}^{(2)}) + b \end{array} \end{array}$$

The framework can be used for training deep feedforward neural networks and deep kernel machines [Suykens, 2017].

(Other approaches: e.g. kernels for deep learning [Cho & Saul, 2009], mathematics of the neural response [Smale et al., 2010], deep gaussian processes [Damianou & Lawrence, 2013], convolutional kernel networks [Mairal et al., 2014], multi-layer support vector machines [Wiering & Schomaker, 2014])

Training process



Objective function (logarithmic scale) during training on the ion data set:

- black color: level 3 objective only
- $J_{\rm deep}$ for $c_{\rm stab} = 1, 10, 100$ (blue, red, magenta color) in stabilization term

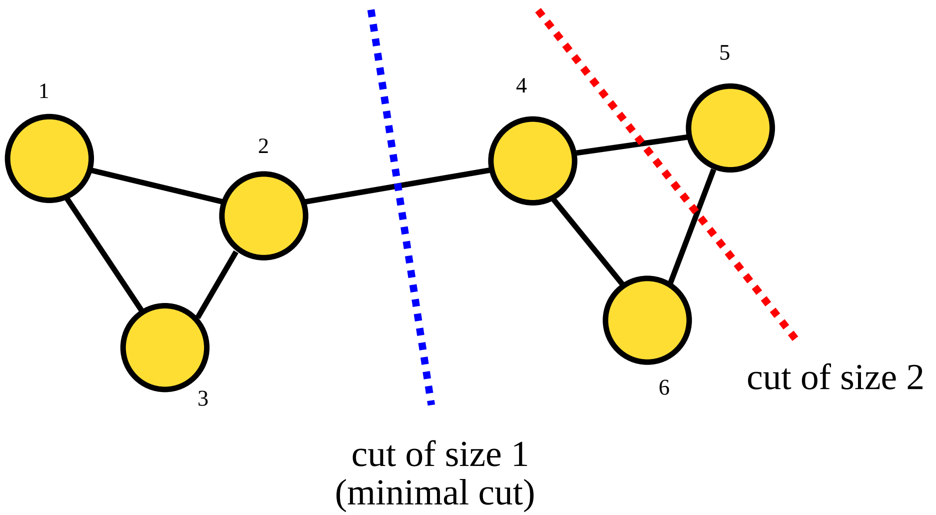
Kernel spectral clustering

Spectral graph clustering

Minimal cut: given the graph $\mathcal{G} = (V, E)$, find clusters $\mathcal{A}_1, \mathcal{A}_2$

$$\min_{q_i \in \{-1,+1\}} \frac{1}{2} \sum_{i,j} w_{ij} (q_i - q_j)^2$$

with cluster membership indicator q_i ($q_i = 1$ if $i \in \mathcal{A}_1$, $q_i = -1$ if $i \in \mathcal{A}_2$) and $W = [w_{ij}]$ the weighted adjacency matrix.



Spectral graph clustering

- Relaxation to **Min-cut** spectral clustering problem

$$\min_{\tilde{q}^T \tilde{q}=1} \ \tilde{q}^T L \tilde{q}$$

with $L = D - W$ the unnormalized graph Laplacian, degree matrix $D = \mathrm{diag}(d_1, ..., d_N)$, $d_i = \sum_j w_{ij}$, giving

$$L\tilde{q} = \lambda \tilde{q}.$$

Cluster member indicators: $\hat{q}_i = \mathrm{sign}(\tilde{q}_i - \theta)$ with threshold θ.

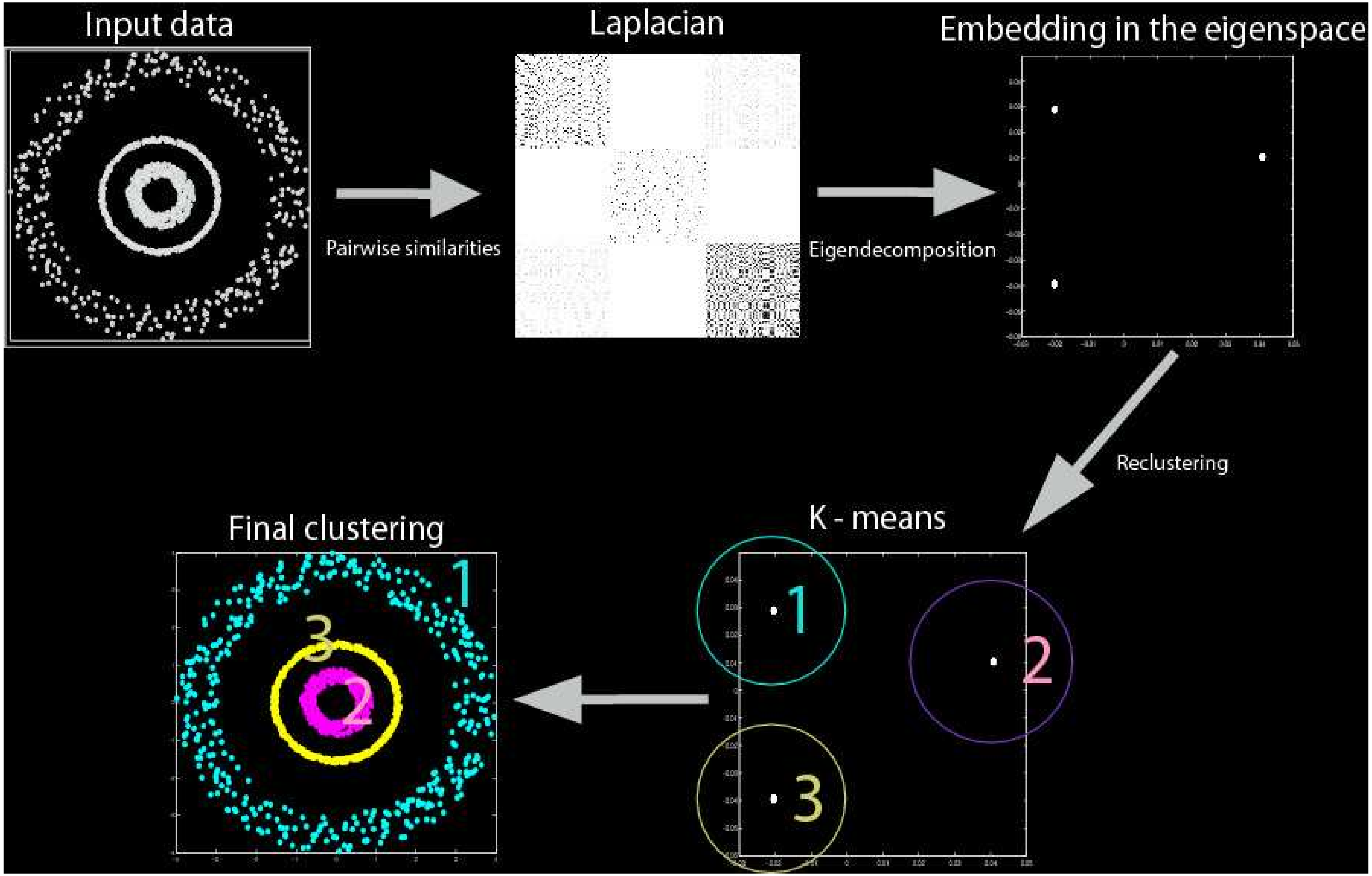
- **Normalized cut**

$$L\tilde{q} = \lambda D \tilde{q}$$

[Fiedler, 1973; Shi & Malik, 2000; Ng et al. 2002; Chung, 1997; von Luxburg, 2007]

- **Discrete version** to continuous problem (Laplace operator)
[Belkin & Niyogi, 2003; von Luxburg et al., 2008; Smale & Zhou, 2007]

Spectral clustering + K-means



Kernel spectral clustering: case of two clusters

- **Underlying model** (primal representation):

$$\hat{e}_* = w^T \varphi(x_*) + b$$

with $\hat{q}_* = \mathrm{sign}[\hat{e}_*]$ the estimated cluster indicator at any $x_* \in \mathbb{R}^d$.

- **Primal problem:** training on given data $\{x_i\}_{i=1}^N$

$$\begin{array}{ll} \displaystyle\min_{w,b,e} & -\frac{1}{2} w^T w + \gamma \frac{1}{2} \sum_{i=1}^{N} v_i \, e_i^2 \\ \text{subject to} & e_i = w^T \varphi(x_i) + b, \quad i = 1, ..., N \end{array}$$

with **positive weights** v_i (will be related to inverse degree matrix).

[Alzate & Suykens, IEEE-PAMI, 2010]

Lagrangian and conditions for optimality

- Lagrangian:

$$\mathcal{L}(w,b,e;\alpha) = -\frac{1}{2}w^T w + \gamma \frac{1}{2}\sum_{i=1}^{N} v_i e_i^2 - \sum_{i=1}^{N} \alpha_i (e_i - w^T \varphi(x_i) - b)$$

- Conditions for optimality:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial w} = 0 & \Rightarrow \quad w = \sum_i \alpha_i \varphi(x_i) \\ \frac{\partial \mathcal{L}}{\partial b} = 0 & \Rightarrow \quad \sum_i \alpha_i = 0 \\ \frac{\partial \mathcal{L}}{\partial e_i} = 0 & \Rightarrow \quad \alpha_i = \gamma v_i e_i, \;\; i = 1, ..., N \\ \frac{\partial \mathcal{L}}{\partial \alpha_i} = 0 & \Rightarrow \quad e_i = w^T \varphi(x_i) + b, \;\; i = 1, ..., N \end{cases}$$

- Eliminate w, b, e, write solution in α.

Kernel-based model representation

- **Dual problem:**

$$V M_V \Omega \alpha = \lambda \alpha$$

with

$\lambda = 1/\gamma$
$M_V = I_N - \frac{1}{1_N^T V 1_N} 1_N 1_N^T V$: weighted centering matrix
$\Omega = [\Omega_{ij}]$: kernel matrix with $\Omega_{ij} = \varphi(x_i)^T \varphi(x_j) = K(x_i, x_j)$

- **Dual model representation:**

$$\hat{e}_* = \sum_{i=1}^{N} \alpha_i K(x_i, x_*) + b$$

with $K(x_i, x_*) = \varphi(x_i)^T \varphi(x_*)$.

Choice of weights v_i

- Take $V = D^{-1}$ where $D = \text{diag}\{d_1, ..., d_N\}$ and $d_i = \sum_{j=1}^{N} \Omega_{ij}$
- This gives the **generalized eigenvalue problem**:

$$M_D \Omega \alpha = \lambda D \alpha$$

  with $M_D = I_N - \frac{1}{1_N^T D^{-1} 1_N} 1_N 1_N^T D^{-1}$

  This is a modified version of random walks spectral clustering.

- Note that $\text{sign}[e_i] = \text{sign}[\alpha_i]$ if $\gamma v_i > 0$ (on training data)

  ... but $\text{sign}[e_*]$ applies beyond training data

Kernel spectral clustering: more clusters

- Case of k clusters: additional sets of constraints

$$
\begin{array}{ll}
\min\limits_{w^{(l)},e^{(l)},b_l} & -\frac{1}{2}\sum_{l=1}^{k-1} w^{(l)^T} w^{(l)} + \frac{1}{2}\sum_{l=1}^{k-1} \gamma_l e^{(l)^T} D^{-1} e^{(l)} \\
\text{subject to} & e^{(1)} = \Phi_{N\times n_h} w^{(1)} + b_1 1_N \\
& e^{(2)} = \Phi_{N\times n_h} w^{(2)} + b_2 1_N \\
& \vdots \\
& e^{(k-1)} = \Phi_{N\times n_h} w^{(k-1)} + b_{k-1} 1_N
\end{array}
$$

where $e^{(l)} = [e_1^{(l)}; ...; e_N^{(l)}]$ and $\Phi_{N\times n_h} = [\varphi(x_1)^T; ...; \varphi(x_N)^T] \in \mathbb{R}^{N\times n_h}$.

- **Dual problem**: $M_D \Omega \alpha^{(l)} = \lambda D \alpha^{(l)}$, $l = 1, ..., k-1$.

[Alzate & Suykens, IEEE-PAMI, 2010]

Primal and dual model representations

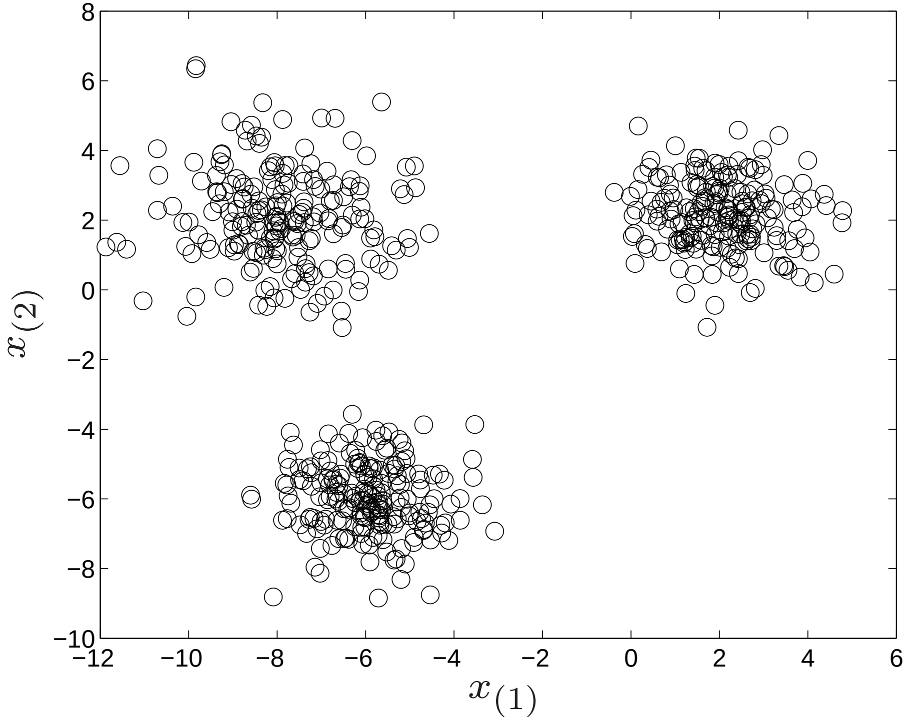
k clusters
$k-1$ sets of constraints (index $l = 1, ..., k-1$)

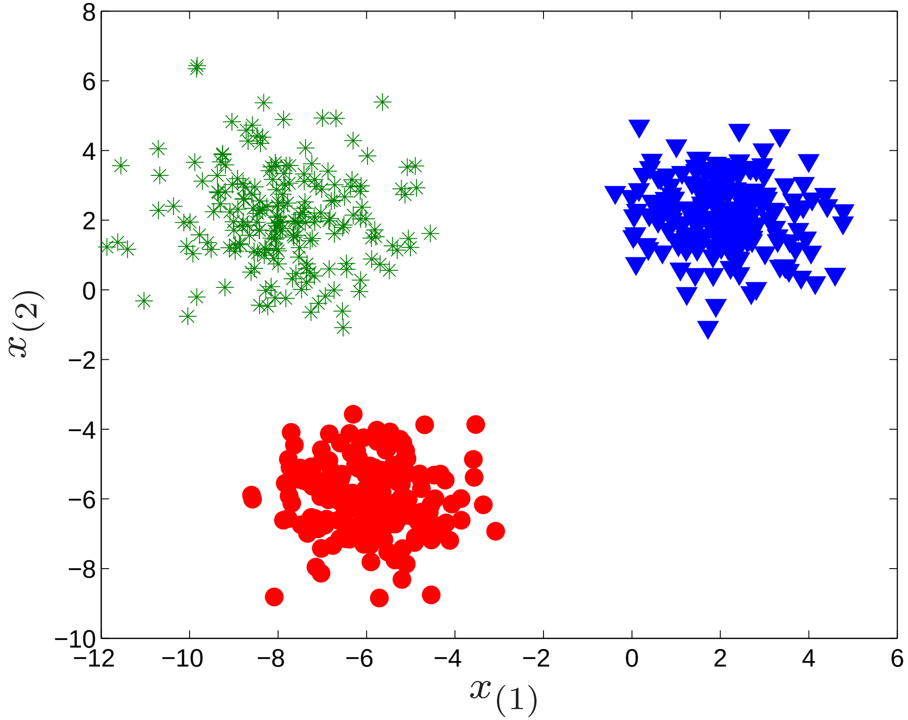
$$
\mathcal{M} \begin{array}{l} \nearrow \quad (P): \quad \mathrm{sign}[\hat{e}_*^{(l)}] = \mathrm{sign}[{w^{(l)}}^T \varphi(x_*) + b_l] \\ \\ \searrow \quad (D): \quad \mathrm{sign}[\hat{e}_*^{(l)}] = \mathrm{sign}[\sum_j \alpha_j^{(l)} K(x_*, x_j) + b_l] \end{array}
$$

- *Note:* additional sets of constraints also in multi-class and vector-valued output LS-SVMs [Suykens et al., 1999]

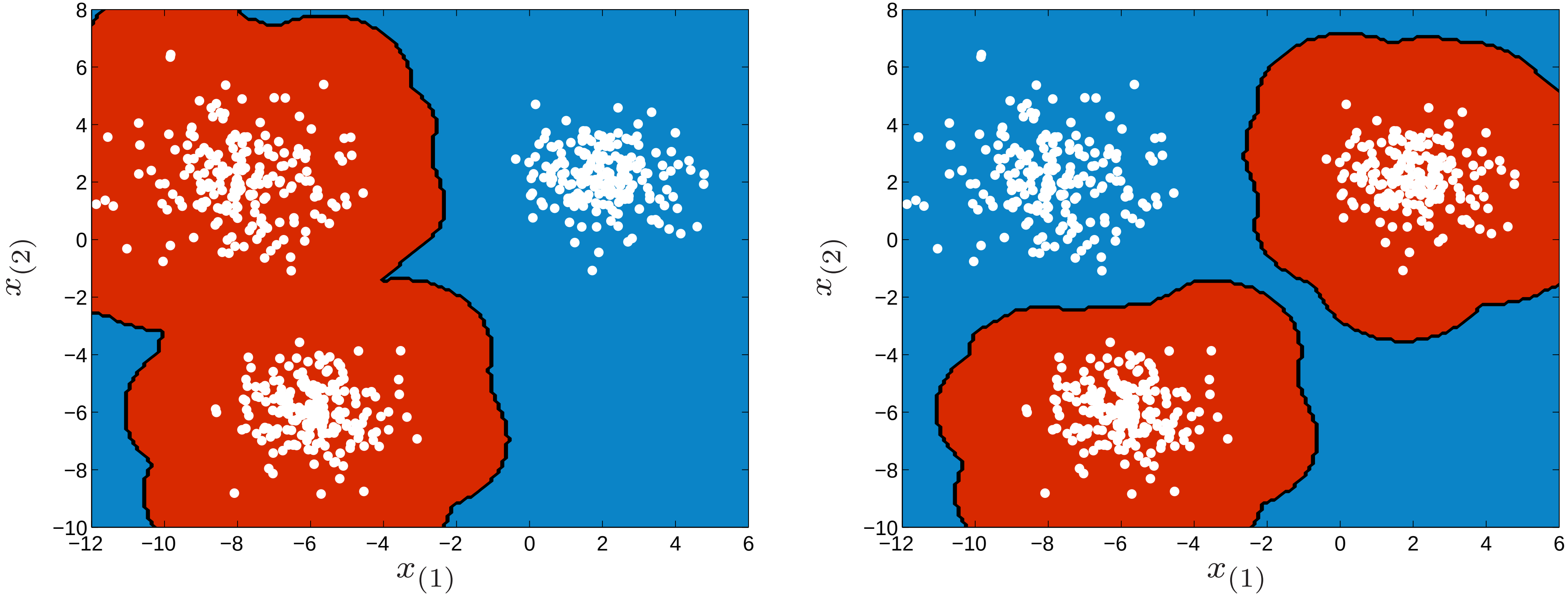
- *Advantages:* out-of-sample extensions, model selection procedures, large scale methods

Out-of-sample extension and coding





Out-of-sample extension and coding



Piecewise constant eigenvectors and extension (1)

- **Definition.** [Meila & Shi, 2001] Vector α is called *piecewise constant* relative to a partition $(\mathcal{A}_1, ..., \mathcal{A}_k)$ iff $\alpha_i = \alpha_j \ \forall x_i, x_j \in \mathcal{A}_p, p = 1, ..., k$.

- **Proposition.** [Alzate & Suykens, 2010] Assume

  (i) a training set $\mathcal{D} = \{x_i\}_{i=1}^N$ and validation set $\mathcal{D}^v = \{x_m^v\}_{m=1}^{N_v}$ i.i.d. sampled from the same underlying distribution;
  (ii) a set of k clusters $\{\mathcal{A}_1, ..., \mathcal{A}_k\}$ with $k > 2$;
  (iii) an isotropic kernel function such that $K(x, z) = 0$ when x and z belong to different clusters;
  (iv) the eigenvectors $\alpha^{(l)}$ for $l = 1, ..., k-1$ are piecewise constant.

  Then validation set points belonging to the same cluster are *collinear* in the $k-1$ dimensional subspace spanned by the columns of $E^v \in \mathbb{R}^{N_v \times (k-1)}$ where $E_{ml}^v = e_m^{(l)} = \sum_{i=1}^N \alpha_i^{(l)} K(x_i, x_m^v) + b_l$.

Piecewise constant eigenvectors and extension (2)

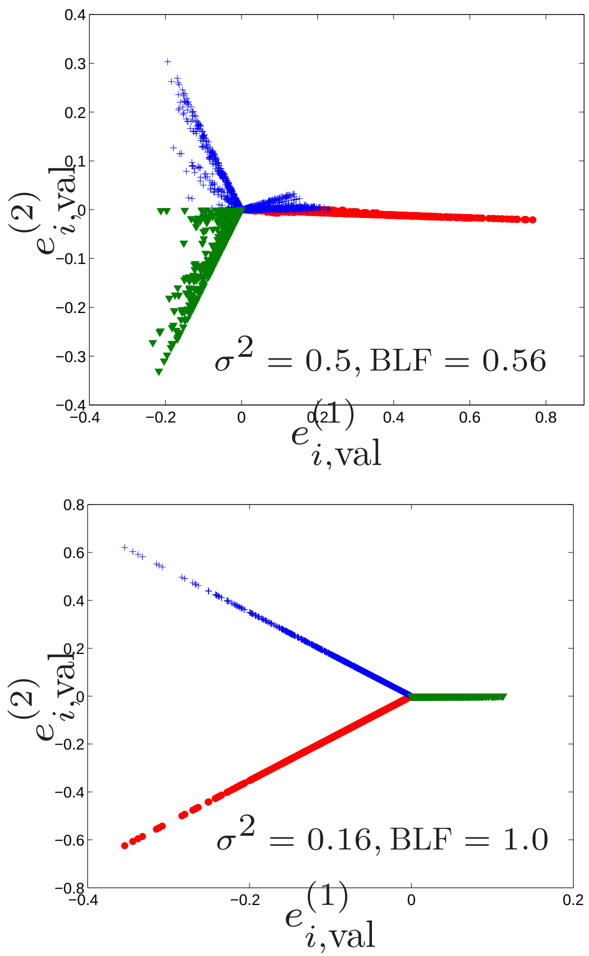
- **Key aspect of the proof:** for $x_* \in \mathcal{A}_p$ one has

$$\begin{aligned} e_*^{(l)} &= \sum_{i=1}^N \alpha_i^{(l)} K(x_i, x_*) + b^{(l)} \\ &= c_p^{(l)} \sum_{i \in \mathcal{A}_p} K(x_i, x_*) + \sum_{i \notin \mathcal{A}_p}^N \alpha_i^{(l)} K(x_i, x_*) + b^{(l)} \\ &= c_p^{(l)} \sum_{i \in \mathcal{A}_p} K(x_i, x_*) + b^{(l)} \end{aligned}$$

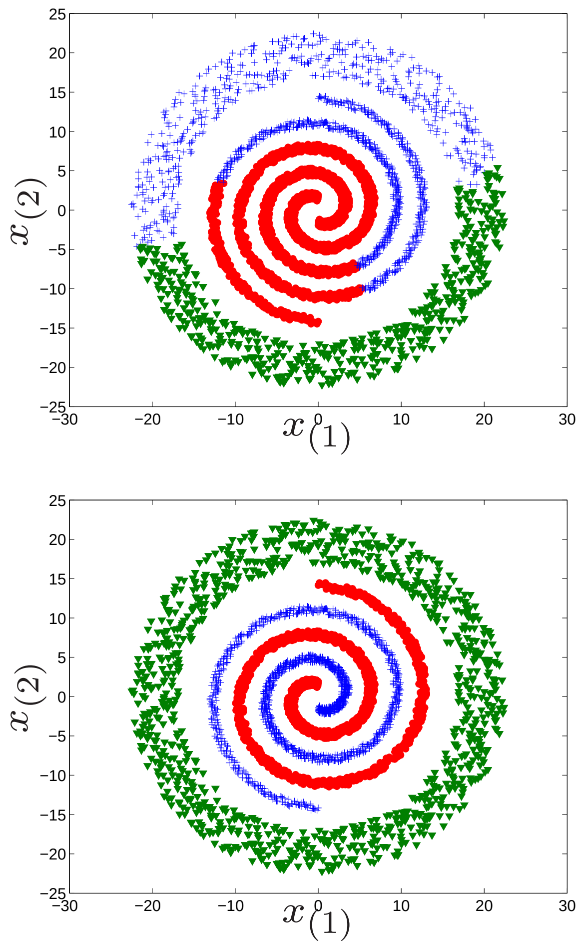
- **Model selection** to determine kernel parameters and k:
  Looking for line structures in the space $(e_i^{(1)}, e_i^{(2)}, ..., e_i^{(k-1)})$,
  evaluated on validation data (aiming for good generalization)

- **Choice kernel:**
  Gaussian RBF kernel
  χ^2-kernel for images

Model selection (looking for lines): toy problem 1

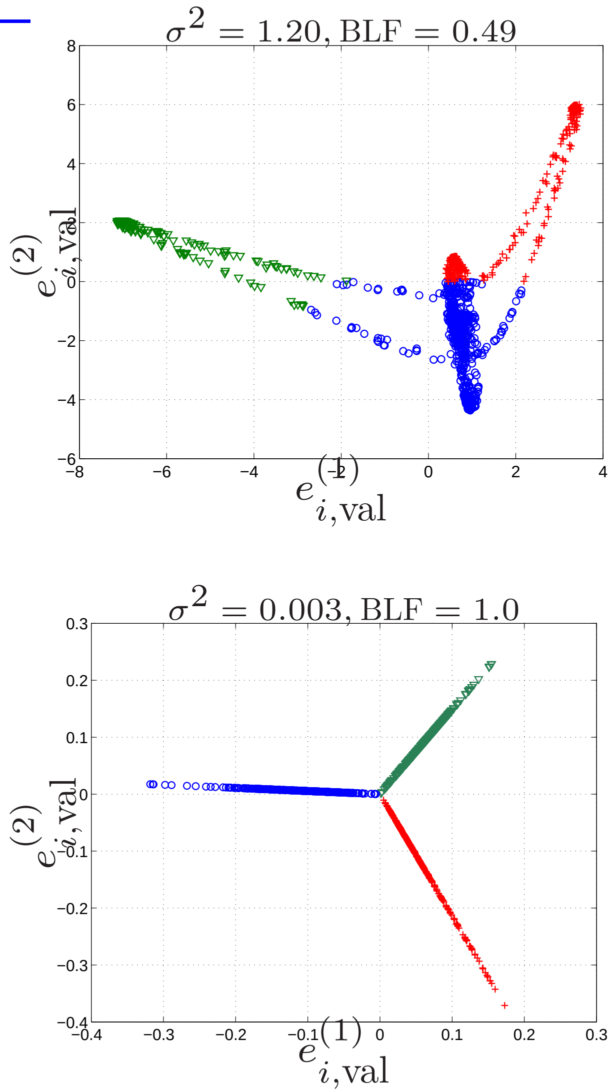


validation set

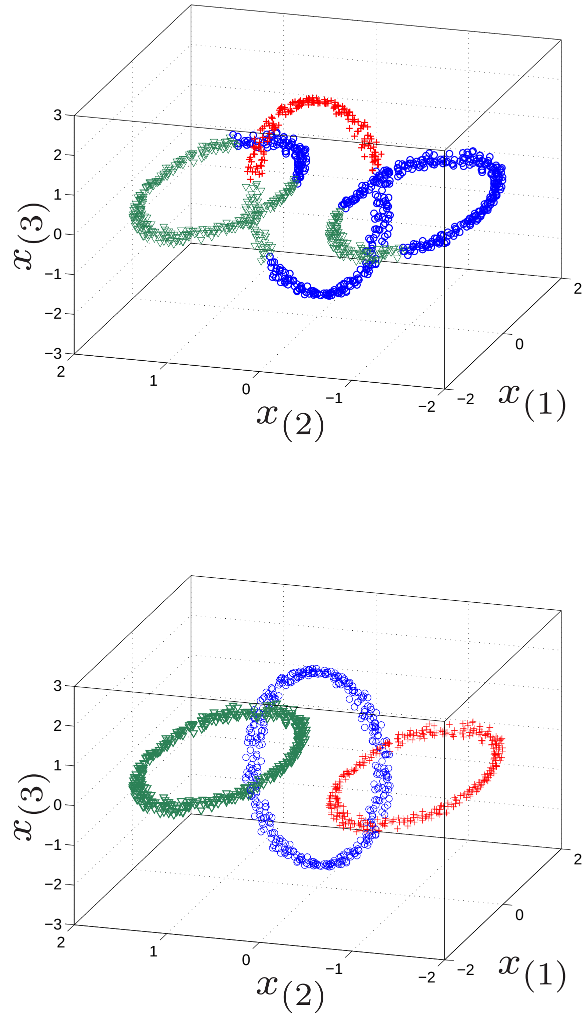


train + validation + test data

Model selection (looking for lines): toy problem 2

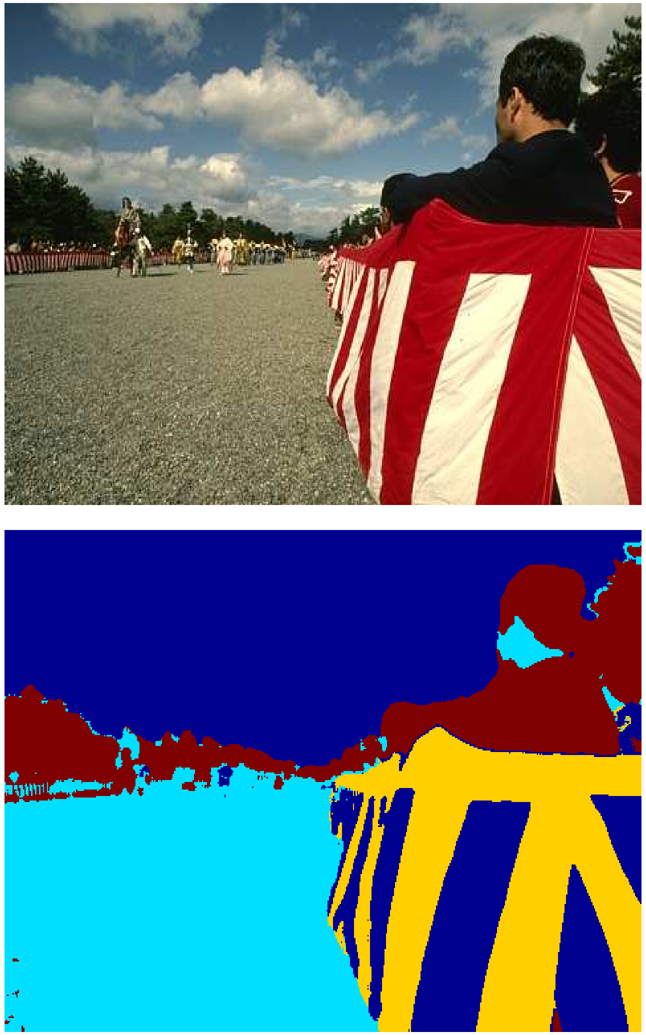


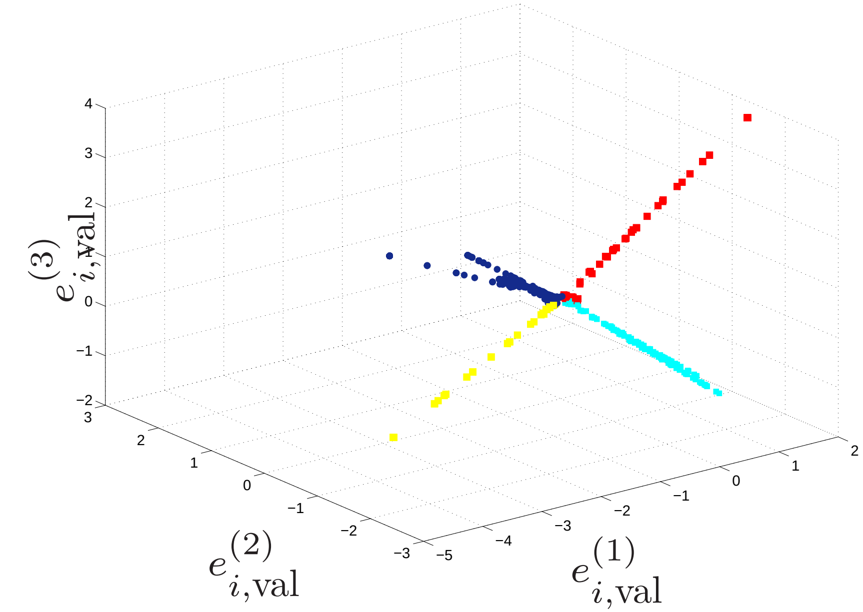
validation set



train + validation + test data

Example: image segmentation (looking for lines)





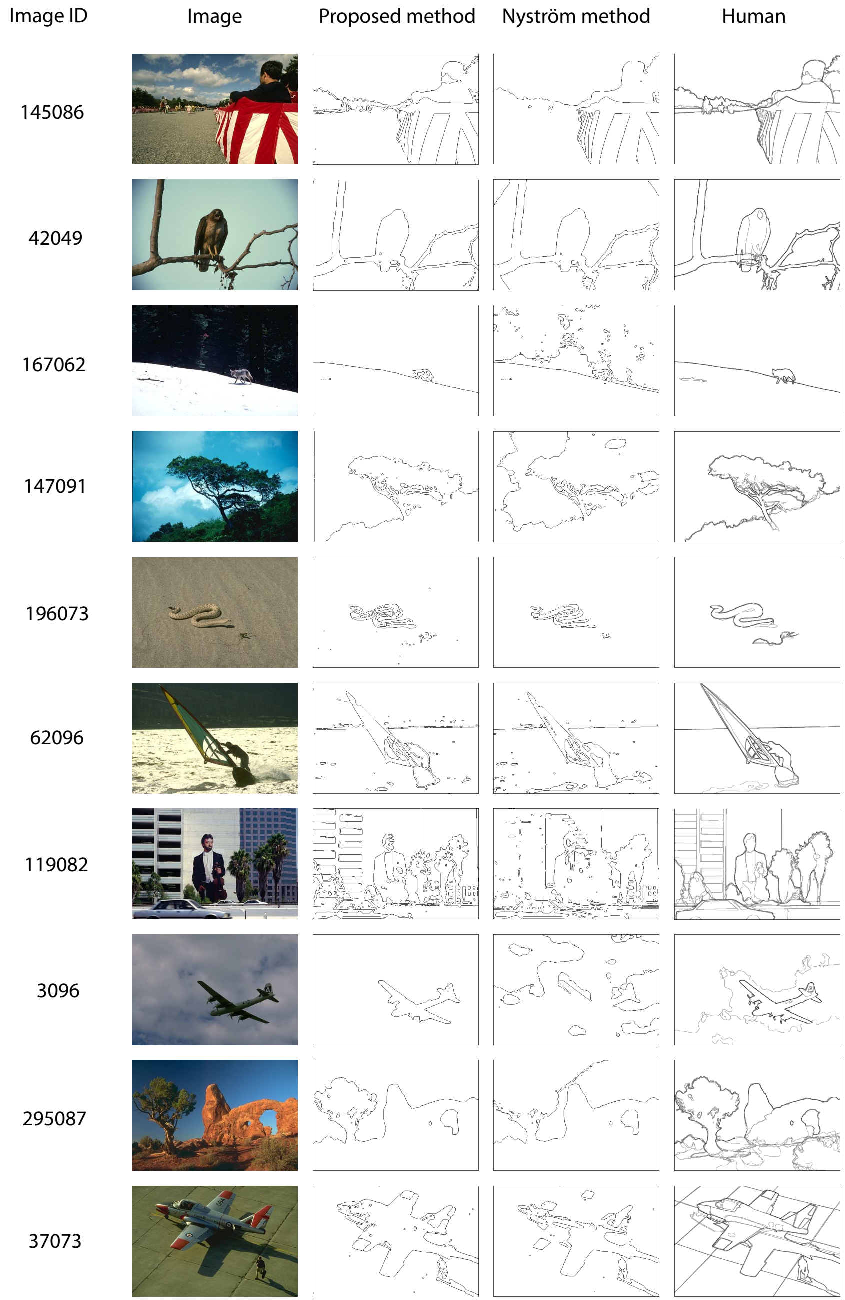
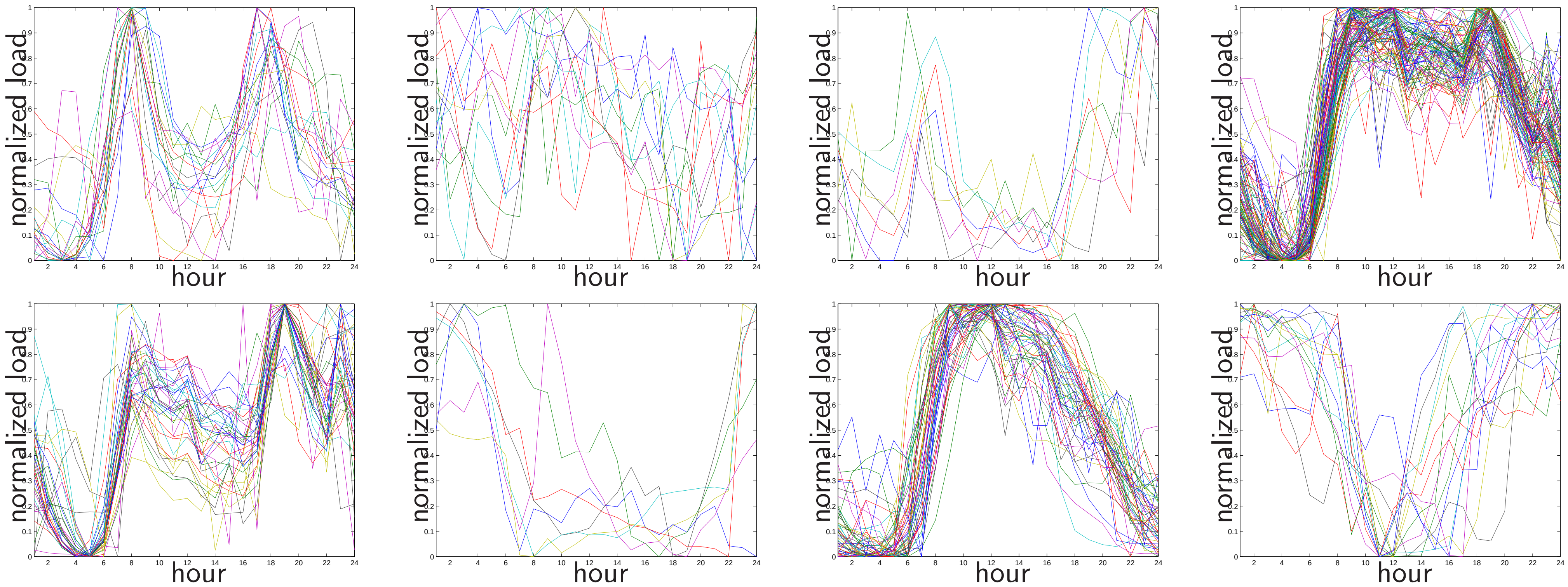


Image ID
Image
Proposed method
Nyström method
Human
145086
42049
167062
147091
196073
62096
119082
3096
295087
37073

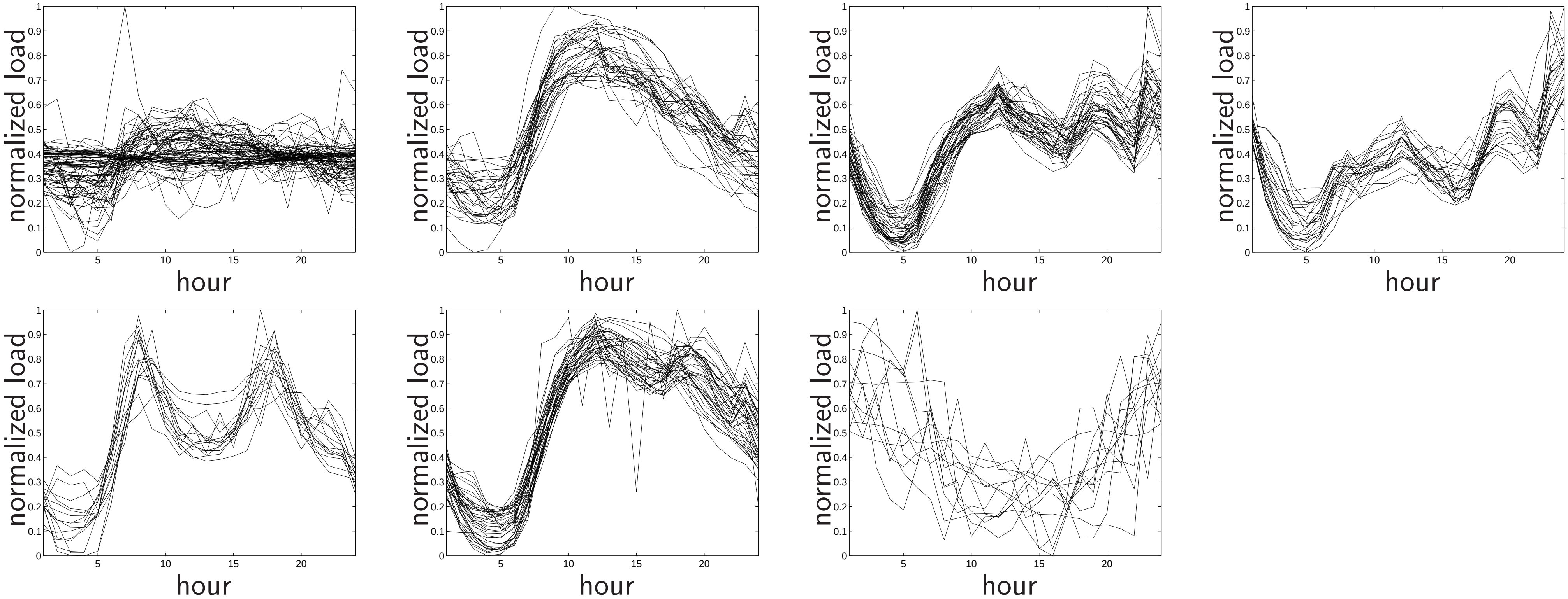
Example: power grid - identifying customer profiles (1)

Power load: 245 substations, hourly data (5 years), $d = 43.824$
Periodic AR modelling: dimensionality reduction $43.824 \rightarrow 24$
k-means clustering applied after dimensionality reduction

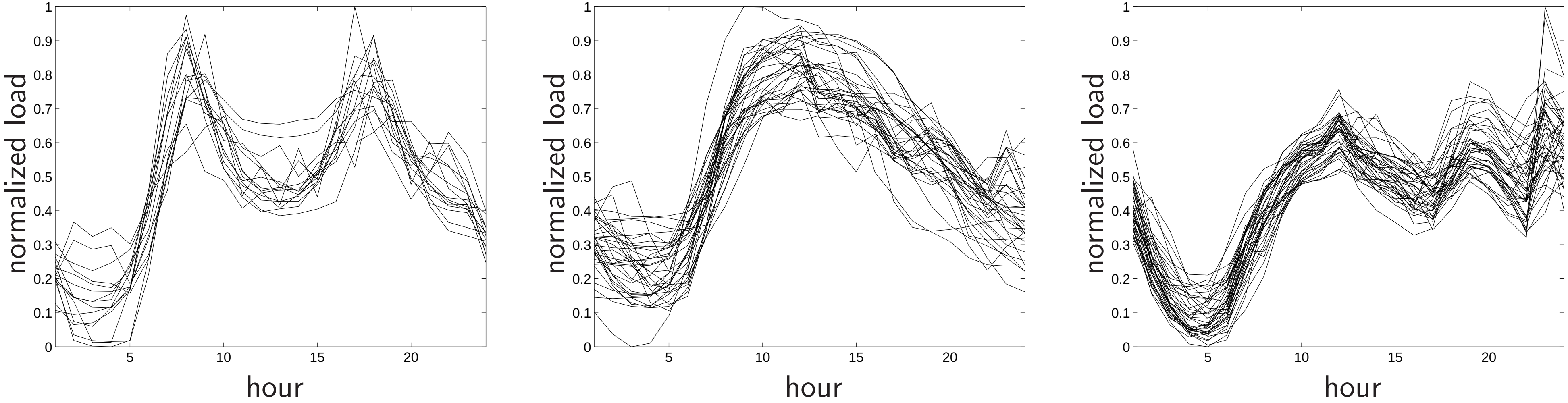


Example: power grid - identifying customer profiles (2)

Application of kernel spectral clustering, directly on $d = 43.824$
Model selection on kernel parameter and number of clusters
[Alzate, Espinoza, De Moor, Suykens, 2009]



Example: power grid - identifying customer profiles (3)



Electricity load: 245 substations in Belgian grid (1/2 train, 1/2 validation)
$x_i \in \mathbb{R}^{43.824}$: spectral clustering on **high dimensional data** (5 years)

3 of 7 detected clusters:
- *1: Residential profile: morning and evening peaks*
- *2: Business profile: peaked around noon*
- *3: Industrial profile: increasing morning, oscillating afternoon and evening*

Kernel spectral clustering: adding prior knowledge

- Pair of points $x_\dagger, x_\ddagger$: $c = 1$ must-link, $c = -1$ cannot-link
- Primal problem [Alzate & Suykens, IJCNN 2009]

$$\begin{array}{ll} \displaystyle\min_{w^{(l)},e^{(l)},b_l} & \displaystyle -\frac{1}{2}\sum_{l=1}^{k-1} {w^{(l)}}^T w^{(l)} + \frac{1}{2}\sum_{l=1}^{k-1} \gamma_l {e^{(l)}}^T D^{-1} e^{(l)} \\ \text{subject to} & e^{(1)} = \Phi_{N\times n_h} w^{(1)} + b_1 1_N \\ & \vdots \\ & e^{(k-1)} = \Phi_{N\times n_h} w^{(k-1)} + b_{k-1} 1_N \end{array}$$

-

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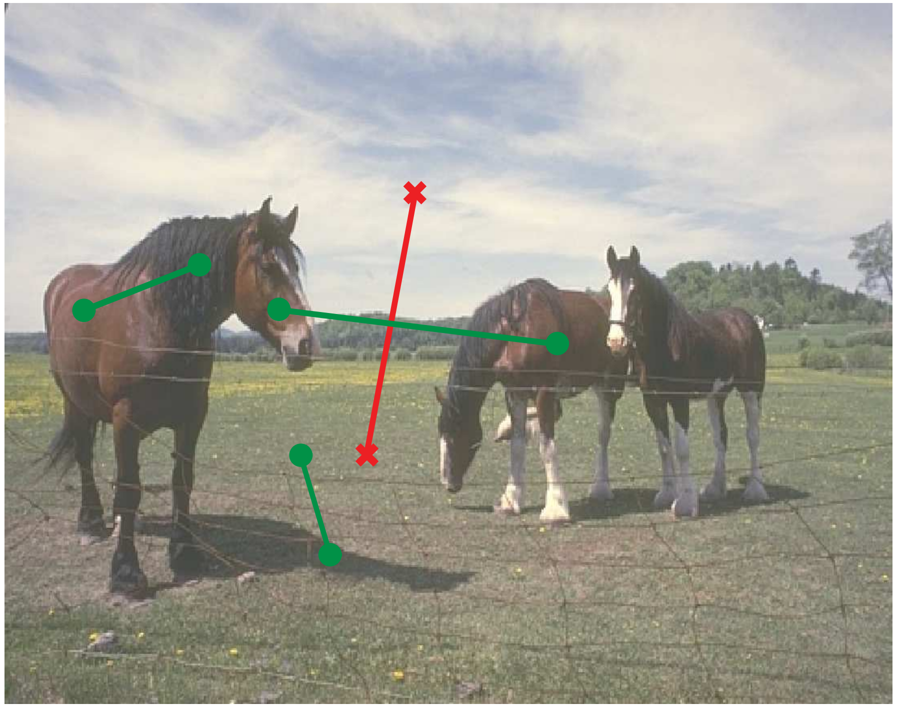
- Dual problem: yields rank-one downdate of the kernel matrix

Adding prior knowledge

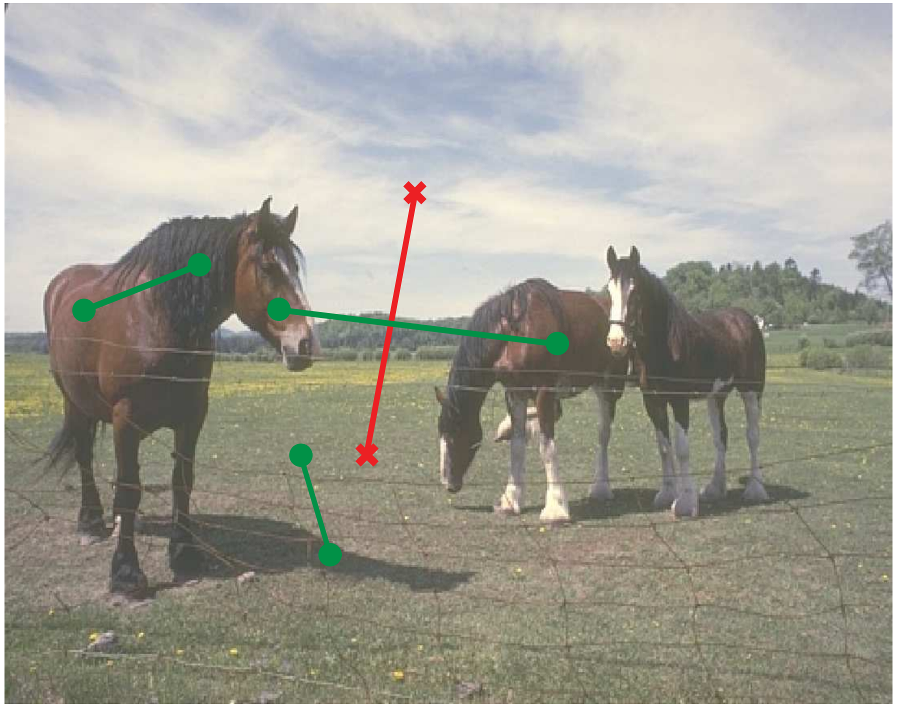
original image



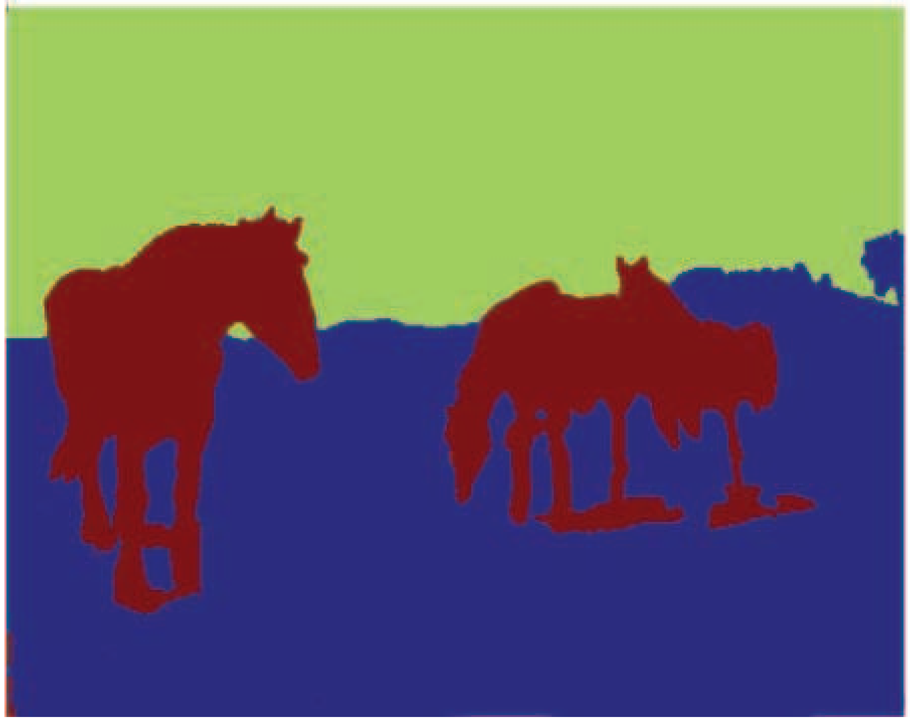
without constraints

Adding prior knowledge

original image



with constraints



LS-SVM in recurrent networks and optimal control

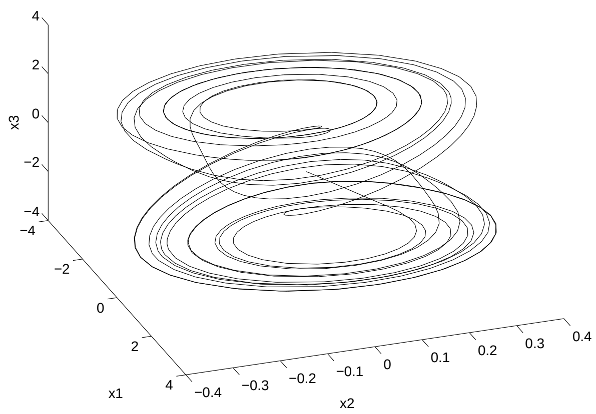
Recurrent least squares support vector machines

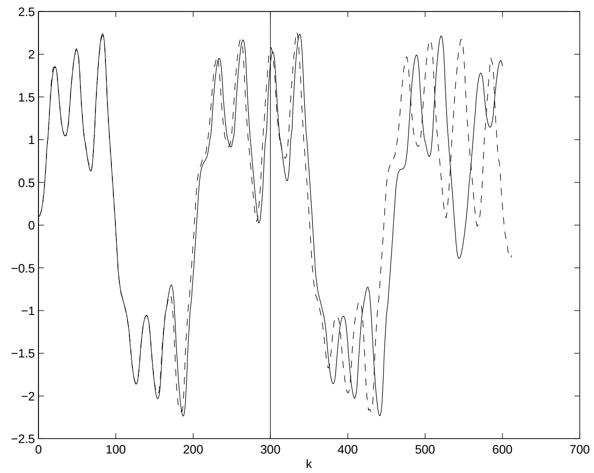
- Nonlinear AR model structure (feedforward):
  $\hat{y}_k = f(y_{k-1}, y_{k-2}, ..., y_{k-p})$
  Nonlinear OE model structure (recurrent):
  $\hat{y}_k = f(\hat{y}_{k-1}, \hat{y}_{k-2}, ..., \hat{y}_{k-p})$

- Recurrent LS-SVM (leads to set of nonlinear equations with kernel trick)

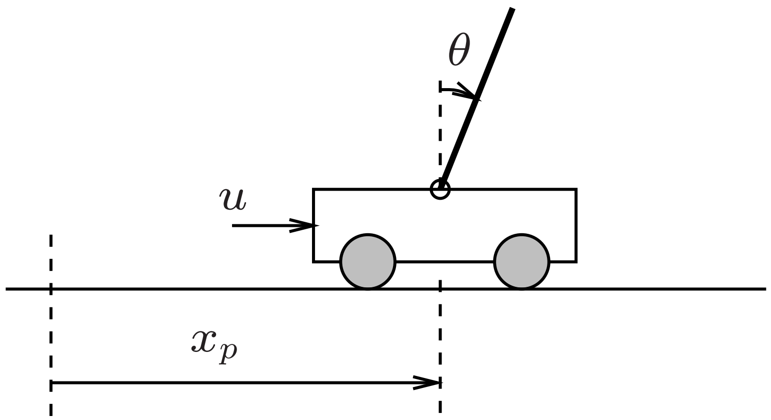
$$\min_{w,b,e} \frac{1}{2} w^T w + \gamma \frac{1}{2} \sum_k e_k^2 \;\; \text{s.t.} \;\; y_k - e_k = w^T \varphi(y_{k-1|k-p} - e_{k-1|k-p}) + b, \;\; \forall k$$

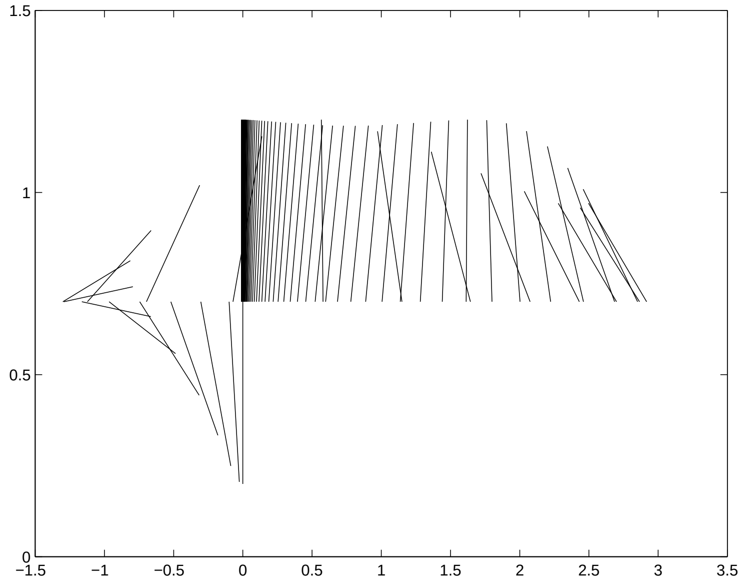
- Example: chaotic time-series prediction





Nonlinear control





min Control objective + LS−SVM objective

subject to $\begin{cases} \text{System dynamics (time k = 1, 2, ... , N)} \\ \text{LS−SVM controller (time k = 1, 2, ... , N)} \end{cases}$

Merging optimal control and support vector machine objectives:

Approximate solutions to optimal control problems [Suykens et al., NN 2001]

N-stage optimal control problem (1)

- Optimal control problem:

$$\min \mathcal{J}_N(x_k, u_k) = \rho(x_{N+1}) + \sum_{k=1}^{N} h(x_k, u_k)$$

subject to the **system dynamics**

$$x_{k+1} = f(x_k, u_k), \quad k = 1, ..., N \ (x_1 \text{ given})$$

where

$x_k \in \mathbb{R}^n$	state vector
$u_k \in \mathbb{R}$	input
$\rho(\cdot)$, $h(\cdot, \cdot)$	positive definite functions

N-stage optimal control problem (2)

- Lagrangian:

$$\mathcal{L}_N(x_k, u_k; \lambda_k) = \mathcal{J}_N(x_k, u_k) + \sum_{k=1}^{N} \lambda_k^T [x_{k+1} - f(x_k, u_k)]$$

with Lagrange multipliers $\lambda_k \in \mathbb{R}^n$.

- Conditions for optimality:

$$\left\{ \begin{array}{llll} \frac{\partial \mathcal{L}_N}{\partial x_k} & = & \frac{\partial h}{\partial x_k} + \lambda_{k-1} - (\frac{\partial f}{\partial x_k})^T \lambda_k = 0, \quad k = 2, ..., N & \text{(adjoint equation)} \\ \frac{\partial \mathcal{L}_N}{\partial x_{N+1}} & = & \frac{\partial \rho}{\partial x_{N+1}} + \lambda_N = 0 & \text{(adjoint final condition)} \\ \frac{\partial \mathcal{L}_N}{\partial u_k} & = & \frac{\partial h}{\partial u_k} - \lambda_k^T \frac{\partial f}{\partial u_k} = 0, \quad k = 1, ..., N & \text{(variational condition)} \\ \frac{\partial \mathcal{L}_N}{\partial \lambda_k} & = & x_{k+1} - f(x_k, u_k) = 0, \quad k = 1, ..., N & \text{(system dynamics)} \end{array} \right.$$

Optimal control using LS-SVM (1)

- **Optimal control problem:**

$$\min \mathcal{J}(x_k, u_k, w, e_k) = \mathcal{J}_N(x_k, u_k) + \frac{1}{2} w^T w + \gamma \frac{1}{2} \sum_{k=1}^{N} e_k^2$$

subject to the **system dynamics**

$$x_{k+1} = f(x_k, u_k), \quad k = 1, ..., N \ (x_1 \text{ given})$$

and the **LS-SVM based control law**

$$u_k = w^T \varphi(x_k) + e_k, \quad k = 1, ..., N$$

- Actual control signal applied to the plant: $w^T \varphi(x_k)$

Optimal control using LS-SVM (2)

- Lagrangian: $\mathcal{L}(x_k, u_k, w, e_k; \lambda_k, \alpha_k) = \mathcal{J}_N(x_k, u_k) + \frac{1}{2}w^T w + \gamma \frac{1}{2} \sum_{k=1}^N e_k^2 + \sum_{k=1}^N \lambda_k^T [x_{k+1} - f(x_k, u_k)] + \sum_{k=1}^N \alpha_k [u_k - w^T \varphi(x_k) - e_k]$

- Conditions for optimality:

$$
\left\{
\begin{array}{lcll}
\frac{\partial \mathcal{L}}{\partial x_k} & = & \frac{\partial h}{\partial x_k} + \lambda_{k-1} - (\frac{\partial f}{\partial x_k})^T \lambda_k - & \\
 & & \alpha_k \frac{\partial}{\partial x_k}[w^T \varphi(x_k)] = 0, \quad k = 2, ..., N & \text{(adjoint equation)} \\
\frac{\partial \mathcal{L}}{\partial x_{N+1}} & = & \frac{\partial \rho}{\partial x_{N+1}} + \lambda_N = 0 & \text{(adjoint final condition)} \\
\frac{\partial \mathcal{L}}{\partial u_k} & = & \frac{\partial h}{\partial u_k} - \lambda_k^T \frac{\partial f}{\partial u_k} + \alpha_k = 0, \quad k = 1, ..., N & \text{(variational condition)} \\
\frac{\partial \mathcal{L}}{\partial w} & = & w - \sum_{k=1}^N \alpha_k \varphi(x_k) = 0 & \text{(support vectors)} \\
\frac{\partial \mathcal{L}}{\partial e_k} & = & \gamma e_k - \alpha_k = 0 \quad k = 1, ..., N & \text{(support values)} \\
\frac{\partial \mathcal{L}}{\partial \lambda_k} & = & x_{k+1} - f(x_k, u_k) = 0, \quad k = 1, ..., N & \text{(system dynamics)} \\
\frac{\partial \mathcal{L}}{\partial \alpha_k} & = & u_k - w^T \varphi(x_k) - e_k = 0, \quad k = 1, ..., N & \text{(LSSVM control)}
\end{array}
\right.
$$

Eliminating w, e gives $u_k = \sum_{l=1}^N \alpha_l K(x_l, x_k)$.