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Selection of Optimal Investment Portfolios with Cardinality Constraints

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Abstract—**We consider the problem of selecting an optimal portfolio within the standard mean-variance framework extended to include constraints of practical interest, such as limits on the number of assets that can be included in the portfolio and on the minimum and maximum investments per asset and/or groups of assets. The introduction of these realistic constraints transforms the selection of the optimal portfolio into a mixed integer quadratic programming problem. This optimization problem, which we prove to be NP-hard, is difficult to solve, even approximately, by standard optimization techniques. A hybrid strategy that makes use of genetic algorithms and quadratic programming is designed to provide an accurate and efficient solution to the problem.**

I. Introduction

THE selection of optimal portfolios is a problem of great interest in the area of quantitative finance [1-9]. A rational investor creates a portfolio by investing in different assets whose behavior is uncertain. She is concerned not only with the expected return of the portfolio but also with the uncertainty associated with her investment. This uncertainty or portfolio risk can be reduced by diversification, i.e. by investing in a number of different portfolios whose returns are not fully correlated. This naturally leads to the question of how the capital should be allocated between the different assets available in the market in order to guarantee the maximum expected return with the lowest risk possible. In mathematical terms, the portfolio should be Pareto optimal. That is, for a given level of risk exposure there should not be any other portfolio with a higher expected return; conversely, for a given expected return there should not be any portfolio whose expected risk is lower. In the standard Markowitz model [1], the variance of the portfolio returns is chosen as the measure of risk. The set of Pareto optimal combinations of expected return and variance of return is known as the efficient frontier. In real markets investments are subject to a number of trading restrictions. Furthermore, the profile of the investor and her preferences may lead to the use of further constraints in the allocation of capital among the assets that make up the portfolio. The types of constraints considered are cardinality constraints (invest in a fixed number of different assets), floor and ceiling constraints (set the minimum and maximum investment in a given asset) the amount concentration constraints (set the limits for the fraction of capital that is invested in a group of assets), and so on. These constraints make the portfolio selection a difficult problem with both combinatorial and continuous optimization aspects. In order to address the computational challenge of constructing the efficient frontier in the Markowitz model with realistic constraints we propose a hybrid algorithm that combines evolutionary techniques with quadratic programming. A genetic algorithm (GA) is used to address the purely combinatorial problem of selecting the optimal subset from the available assets. For a given subset, selected by the GA, the problem of determining the optimal amount of capital that should be invested in each asset is solved by quadratic programming. A number of alternative chromosome representations for the genetic algorithm are explored in our work. A simple binary representation requires the use of either penalty functions or repair mechanisms to maintain the cardinality constraints. A set representation, used in combination with a especially designed recombination operator (Random Assorting Recombination: RAR) is a much more appropriate encoding for this problem. Finally, computational results are reported for each approach on a set of benchmark test problems publicly available from the OR-Library [14-15]. A comparison of these results with those obtained with general purpose heuristic optimization methods, shows that the hybrid strategy proposed leads to more accurate efficient frontiers with a reasonable computational effort. In particular, the strategy that uses a set representation for the chromosomes and the RAR (w=1) recombination operator yields the best results.

II. Optimal Portfolio Selection

A. Markowitz Mean-variance Model

In 1952 H. Markowitz [1] published the article “Portfolio selection”, where a quantitative framework for portfolio optimization is established. The problem to solve is how a rational investor should deal with the uncertainty associated with investing in assets whose future evolution is unknown. The market prices of assets can be modeled as random variables characterized by an average expected return and a variance. If the evolution of the asset were deterministic (zero variance) the solution to the portfolio selection problem would simply be to invest in the asset whose expected return is the largest. In the presence of uncertainty, and as a result of the reduction of the uncertainty arising

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from diversification, the deterministic solution need not be optimal. Instead, the rational investor may prefer to settle for a lower expected return with lower uncertainty (i.e. lower risk).

Consider the market prices of the set of N assets available for investments at instant t

$$\{S_i(t)\}_{i=1}^{N} \tag{1}$$

The value of a portfolio composed of $c_i(t)$ units of the *ith* asset, whose market price is $S_i(t)$, is

$$P(t)=\sum_{i=1}^{N} c_i(t)\, S_i(t) \tag{2}$$

The return of the portfolio in the interval $[t,t+\Delta t]$ is

$$\begin{aligned} r_P(t) &= \frac{P(t+\Delta t)-P(t)}{P(t)} = \sum_{i=1}^{N} w_i\, r_i(t) = \mathbf{w}^t \cdot \mathbf{r}(t); \\ r_i(t) &= \frac{S_i(t+\Delta t)-S_i(t)}{S_i(t)}; \\ w_i &= \frac{c_i(t)S_i(t)}{\sum_{m=1}^{n} c_m(t)\, S_m(t)}; \qquad \sum_{i=1}^{N} w_i = 1; \quad 0 \le w_i \le 1 \end{aligned} \tag{3}$$

where

$$\mathbf{r}(t)^t = \left(r_1(t)\ \ r_2(t) \ldots r_N(t)\right) \tag{4}$$

is the (transposed) vector of asset returns, and

$$\mathbf{w}^t = \left(w_1\ w_2\ \ldots\ w_N\right) \tag{5}$$

is the (transposed) vector of investment weights, which is assumed to be held constant (by dynamic portfolio management) during the interval $[t,\ t+\Delta t]$. Hence, the portfolio return in this time interval, $r_P(t)$ is a convex combination of the asset returns.

Since the future evolution of the market prices of the assets is subject to uncertainties, we model the asset returns as random variables, which, assuming stationarity, can be characterized by a constant N-dimensional vector of average returns $<\mathbf{r}>$, and a constant (N×N) covariance matrix $\mathbf{\Sigma}$. The diagonal elements of this covariance matrix are the variances of the returns of the individual assets, and the off-diagonal elements are the covariances between the returns of different assets. In the mean-variance model, these are the only parameters needed as inputs for the portfolio selection. It is usually assumed that the statistical properties of the series of returns will be similar in the near future to those in the recent past. With this hypothesis, their values can be estimated from recent historic data, possibly with exponential smoothing to compensate for weak non-stationarities in the time series.

In terms of these quantities the expected return and variance of the portfolio are

$$\begin{aligned} &< r_P > = \mathbf{w}^t \cdot <\mathbf{r}>, \\ &Var(r_P) = \mathbf{w}^t \cdot \mathbf{\Sigma} \cdot \mathbf{w}. \end{aligned} \tag{6}$$

The goal of portfolio selection is to maximize the expected return while minimizing the investment risk, which in this model is measured by the variance in the returns of the portfolio. Since these are opposing goals the problem is a multiobjective optimization problem and has two equivalent formulations that are dual of each other.

In the first formulation the investor fixes a value for the expected return, R^*, and tries to find the portfolio that minimizes the risk

$$\begin{aligned} &Min \quad \mathbf{w}^t \cdot \mathbf{\Sigma} \cdot \mathbf{w}, \\ &\quad s.t. \quad \mathbf{w}^t \cdot <\mathbf{r}> = R^* \\ &\qquad 0 \le w_i \le 1, \qquad \sum_{i=1}^{N} w_i = 1 \end{aligned} \tag{7}$$

Equivalently the investor selects the level of risk she is ready to accept $(\sigma^*)^2$, and maximizes the expected return.

$$\begin{aligned} &Max \quad \mathbf{w}^t \cdot <\mathbf{r}>, \\ &\quad s.t. \quad \mathbf{w}^t \cdot \mathbf{\Sigma} \cdot \mathbf{w} = \left(\sigma^*\right)^2 \\ &\qquad 0 \le w_i \le 1, \qquad \sum_{i=1}^{N} w_i = 1 \end{aligned} \tag{8}$$

We focus on the of the first problem, which is a quadratic optimization problem that can be solved by a standard numerical algorithm. The set of points in the space of risk-returns that are solutions to both (7) and (8) is the efficient frontier.

B. Restrictions in Portfolio Selection Problems

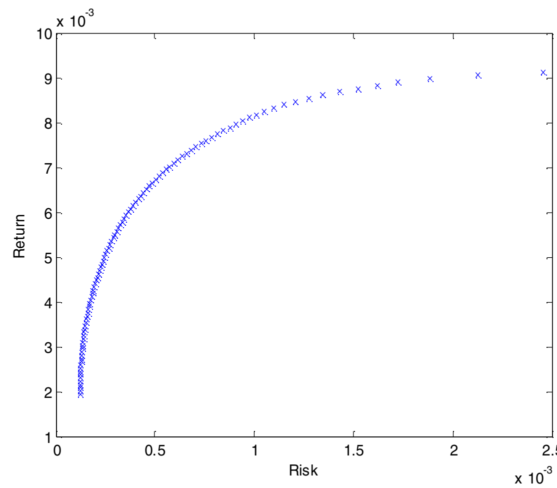

Fig. 1. Efficient frontier: Set of points that correspond to the expected return, and risk (variance of return) of portfolios that are Pareto optimal.

In real-world portfolio selection problems there exist restrictions beyond the ones that are a mathematical consequence of the classical mean-variance framework. Additional restrictions may be introduced to control the amount of diversification, to reflect existence of fees and limits for trading, or they may simply reflect the profile, preferences and goals of the investor. In this work we assume that the investor wants to create a new portfolio (i.e. there are no restrictions associated with the rebalancing of the investments in a portfolio, such as turnover constraints), that the weights are continuous (i.e. the amounts invested in any given asset can be arbitrary) and that there are no transaction costs.

The first type of restrictions considered are lower and upper bounds in the percent of the portfolio which may be invested in any single asset

$$a_i \leq w_i \leq b_i, \qquad i = 1,2,\ldots N \tag{9}$$

or in a group of assets (capital concentration restrictions)

$$l_m \leq \sum_{i=1}^{N} A_{mi} w_i \leq u_m, \qquad m = 1,2,\ldots M \tag{10}$$

where $\mathbf{A}$ is an (M×N) matrix whose rows are the coefficients of the linear combination in the different linear inequalities. The last restriction limits the percentage of the portfolio that is invested in assets in a given sector. All these restrictions are linear and do not increase the difficulty of the optimization problem, which still can be solved by quadratic programming.

The second type of restrictions are cardinality constraints. In markets with a large number of products available for investing, it may be sensible to set a maximum on the number of different assets in the portfolio. This restriction can be expressed as

$$\left|\{i \in \{1,2,\ldots,M\} : w_i \neq 0\}\right| \leq c, \tag{11}$$

i.e. the number of assets whose percentage in the portfolio is nonzero should not be larger than a value *c,* fixed by the investor. With these restrictions, the selection of the optimal portfolio is a mixed integer quadratic programming problem that is difficult to solve by standard optimization techniques.

In summary, with this restrictions the constrained optimization problem is

$$\begin{aligned} Min \quad & \mathbf{w}^t \cdot \mathbf{\Sigma} \cdot \mathbf{w}, \\ s.t. \quad & \mathbf{w}^t \cdot \langle \mathbf{r} \rangle = R^* \\ & \sum_{i=1}^{N} w_i = 1 \\ & \sum_{i=1}^{N} z_i \leq c, \qquad z_i \in \{0,1\} \\ & a_i z_i \leq w_i \leq b_i z_i,\ a_i > 0,\ b_i > 0, \quad i = 1,2,\ldots,N \\ & l_m \leq \sum_{i=1}^{N} A_{mi} w_i \leq u_m, \qquad m = 1,2,\ldots,M \end{aligned} \tag{12}$$

where the indicator variables $\{z_i,\ i = 1, 2,..., N\}$ have been introduced to indicate the inclusion ($z_i = 1$) or exclusion ($z_i = 0$) of the ith asset in the portfolio. These variables will be the genes of the chromosomes used in the binary representation of the problem.

Without loss of generality, the inequality in the cardinality restriction can be replaced with an equality restriction

$$\sum_{i=1}^{N} z_i = c, \qquad z_i \in \{0,1\}. \tag{13}$$

In this problem, we search the minimal risk portfolio that includes exactly *c* different assets.

Using matrix notation, the cardinality constrained optimization problem is

$$\begin{aligned} Min \quad & \mathbf{w}^t \cdot \mathbf{\Sigma} \cdot \mathbf{w}, \\ s.t. \quad & \mathbf{w}^t \cdot \langle \mathbf{r} \rangle = R^* \\ & \mathbf{w}^t \cdot \mathbf{1} = 1 \\ & \mathbf{z}^t \cdot \mathbf{1} = c \\ & \mathbf{a}^{[\mathbf{z}]} \leq \mathbf{w}^{[\mathbf{z}]} \leq \mathbf{b}^{[\mathbf{z}]},\ \mathbf{a}^{[\mathbf{z}]} > \mathbf{0},\ \mathbf{b}^{[\mathbf{z}]} > \mathbf{0}, \\ & \mathbf{l} \leq \sum_{i=1}^{N} \mathbf{A} \cdot \mathbf{w} \leq \mathbf{u}. \end{aligned} \tag{14}$$

The quantity $\mathbf{z}^t = (z_1\ z_2\ \ldots\ z_N)$ is the (transposed) vector of indicators. The components of the vector **0 (1)** are all 0's (all 1's) and the vector has the appropriate dimensions. Quantities with superscript [**z**] are constructed by extracting from the subscripted vectors or matrices the elements whose indexes correspond to nonzero values for the corresponding component of **z**. Inequalities should be understood as element-wise operations.

Note that for fixed values of the indicator variables (i.e., once the subset of assets that are included in the portfolio has been selected), the problem is simply a quadratic optimization problem, with a quadratic form as the cost function, and linear constraints.

Obviously, the difficult problem is the selection of the *c* assets that should be included in the optimal portfolio. This is a combinatorial optimization problem that is proved to be NP-hard in the Appendix.

III. Previous Work

The asset selection problem can be exactly solved for problems with a small number of assets by exhaustive search. This strategy becomes unfeasible for universes of more than approximately 30 assets. In particular, in one of the problems considered in this investigation, the NIKKEI index, the universe includes 225 different assets. The number of possible combinations of 10 different assets is close to 10^{16}. For each of these combinations a quadratic problem needs to be solved. With a typical desktop computer, the computation would take of the order of millions of years.

Branch and Bound techniques are used in Refs. [3],[7] to address the complexity of the search. However, the time needed to identify the solution still grows exponentially with the size of the universe of assets. In [3] the solution proposed is to use a distributed environment, and in [7] the number of combinations that are explored is limited, which means that the algorithm is not guaranteed to find a global optimum.

Many of the investigations on the portfolio selection problem give up the goal of finding the global optimum and attempt to find near-optimal solutions. Different strategies used include Simulated Annealing [5-6],[9], Tabu Search [7] and Genetic Algorithms [2-3],[5],[7]. Most of these methods perform a search in the space of real-valued weights. Their main difficulty is the generation of feasible portfolios that satisfy the restrictions. Alternatively, candidate solutions that violate some or all of the restrictions are allowed, but put at a disadvantage by the use of penalty functions.

The use of hybrid algorithms in portfolio selection, or in the related index tracking problem, that use quadratic programming in combination with other general optimization methods (genetic algorithms) or specific heuristics have also been investigated in the literature [6-7],[12]. These methods have the advantage of handling separating the problem of selecting the optimal subset of assets and the quadratic optimization problem that results once this subset is selected. The strategy we propose in this article is of this type. We expect to improve the results of previous work by using an encoding scheme in combination with mutation and recombination operators especially adapted to the problem..

IV. A Hybrid Strategy for Portfolio Selection

The asset selection problem can be reformulated as a minimization problem in the space of the indicator variables for the function F(**z**), defined as the solution of a quadratic optimization problem

$$\begin{aligned} F(\mathbf{z}) = Min & \left[\left(\mathbf{w}^{[\mathbf{z}]}\right)^t \cdot \mathbf{\Sigma}_{\mathbf{z}} \cdot \mathbf{w}^{[\mathbf{z}]} \right] \\ & s.t. \quad \left(\mathbf{w}^{[\mathbf{z}]}\right)^t \cdot < \mathbf{r}^{[\mathbf{z}]} > = R^* \\ & \left(\mathbf{w}^{[\mathbf{z}]}\right)^t \cdot \mathbf{1} = 1 \\ & \mathbf{a}^{[\mathbf{z}]} \leq \mathbf{w}^{[\mathbf{z}]} \leq \mathbf{b}^{[\mathbf{z}]}, \mathbf{a}^{[\mathbf{z}]} > \mathbf{0}, \mathbf{b}^{[\mathbf{z}]} > \mathbf{0}, \\ & \mathbf{l} \leq \sum_{i=1}^{N} \mathbf{A}^{[\mathbf{z}]} \cdot \mathbf{w}^{[\mathbf{z}]} \leq \mathbf{u}. \end{aligned} \tag{15}$$

with the cardinality restriction

$$\sum_{i=1}^{N} z_i = c, \qquad z_i \in \{0,1\}.$$

The value of F(**z**) can be calculated by using a standard quadratic solver.

To address the combinatorial optimization problem we propose a genetic algorithm where the chromosomes are a collection of discrete-valued genes that encode the information contained in the vector of indicators **z**. The fitness function that is used in the selection of individuals in the population is

$$fitness(\mathbf{z}) = -F(\mathbf{z}). \tag{16}$$

In the implementations of the genetic algorithm proposed, the preferred strategy to enforce the cardinality constraint is to start the algorithm with a population whose individuals satisfy the constraint and to use appropriately defined mutation and recombination operators that preserve cardinality. In case that it is not possible to design such operators, we use either penalty terms in the fitness function or repair the chromosomes that do not satisfy the constraint.

In what follows we explore two different encoding schemes for the chromosomes and introduce the corresponding mutation and crossover operators.

A. Binary Encoding

In binary encoding a chromosome is a bitstring whose length is equal to the number of assets that are available for investment, N. The value of *ith* component of **z** is stored in the *ith* position on the chromosome. For instance, in a problem with N = 5 and cardinality constraint c = 3, the individual

1	2	3	4	5
0	1	1	0	1

(17)

represents the minimum risk portfolio with investments in the products labeled 2,3 and 5. As a mutation operator we use a permutation of two bits. For instance, from individual (17), a mutation that interchanges the bits in positions 1 and 3 generates the individual

1	2	3	4	5
1	1	0	0	1

(18)

which corresponds to a portfolio with investments in assets 1, 2 and 5.

For the crossover operation, and to avoided any biases due to the original labeling of the assets, we use uniform crossover. This operation does not preserve the cardinality of the parent chromosomes. In order to enforce this constraint penalty terms can be used that lower the fitness of individuals violating the cardinality constraint. We have considered the use of death penalty, which assigns individuals with the incorrect cardinality the worst possible value for the cost function, and penalty functions (linear, quadratic, logarithmic) where the amount of penalization depends on the amount by which the constraint is violated. Chromosome repair methods, where the value of the genes is modified until the cardinality constraint is satisfied have also been employed. In particular we use either random repair, where the genes to modify are selected at random and a heuristic repair. This heuristic is based in the proposal made in [6] where the portfolio is optimized without the cardinality constraint. Then, the bit-values of the genes that correspond to assets with the lowest weights in the unconstrained optimal portfolio are set to zero.

B. Subset Encoding

In subset encoding a chromosome is a set of labels that correspond to the assets that are included in the portfolio. For instance, the individual represented by {1,2,5} corresponds to the minimum risk portfolio with investments only in the products labeled 1, 2 and 5. The mutation operator is the same as the one used in binary encoding. For the crossover operation two different operators are employed: Random Respectful Recombination (R3) and Random Assorting Recombination (RAR) [11].

In R3 a child chromosome has the same cardinality as the parents. It is composed by all the assets present in both parents and a random selection of the assets present in only one of the parents. For instance, if the two parents selected are **P1** ≡ {1,2,5}, and **P2** ≡ {1,3,4}, we construct the intersection set **A** ≡ P1 ∩ P2 = {1}, and the difference set **D** ≡ P1 Δ P2 = {2,3,4,5}. The children are selected at random from the following candidates:

Children: {1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5}, {1,4,5}

Note that the parent chromosomes are also in the list of candidate children. The results in the portfolio selection problem confirm that this operator tends to overexploit the information existing in the parents and does not introduce sufficient variability to allow the population to escape from optima that are local.

To apply Random Assorting Recombination (RAR) we select a positive value for the integer parameter w and carry out the following steps:

1. <u>Create auxiliary sets</u> (A,B,C,D,E)
   The initial composition of these sets is
   - A : Assets present in both parents.
   - B : Assets not present in any of the parents.
   - C ≡ D : Assets present in only one parent.
   - E : Empty set.
2. <u>Build set G</u>:
   - w copies of elements from A and B.
   - 1 copy of elements from C and D.
3. <u>Build the child chromosome</u>

   3.1 Repeat

   Extract an element from G (without replacement)
   - If the element comes from A or C, and is not an element of E, include it in the child chromosome.
   - If the element comes from B or D, include it in set E.

   Until chromosome is complete or bag G is empty.

   3.2 If chromosome is not complete include assets not yet included selected at random.

One of the interesting properties of the RAR operator is that, due to step 3.2, it can generate implicit mutations in the recombination operation.

V. Experimental Results

In order to test the hybrid strategy proposed and the different choices of encoding, mutation and crossover operators we have carried experiments in a series of problems from the OR-library [14-15]. This repository contains historical estimates of the expected return vector and covariance of returns matrix for groups of assets using weekly samples. The assets considered in each problem are those included in different stock market indices: Hang Seng (HK, 31 assets), DAX (DE 85 assets), FTSE (GB, 89 assets), S&P (US, 110 assets), Nikkei (JP, 225 assets).

The experiments reported involve computing the optimal portfolio for 100 points on the efficient frontier. An example of the efficient frontier consisting of Pareto optimal portfolios that can be constructed by investment on the different assets of the Nikkei index can be seen in Fig. 2.

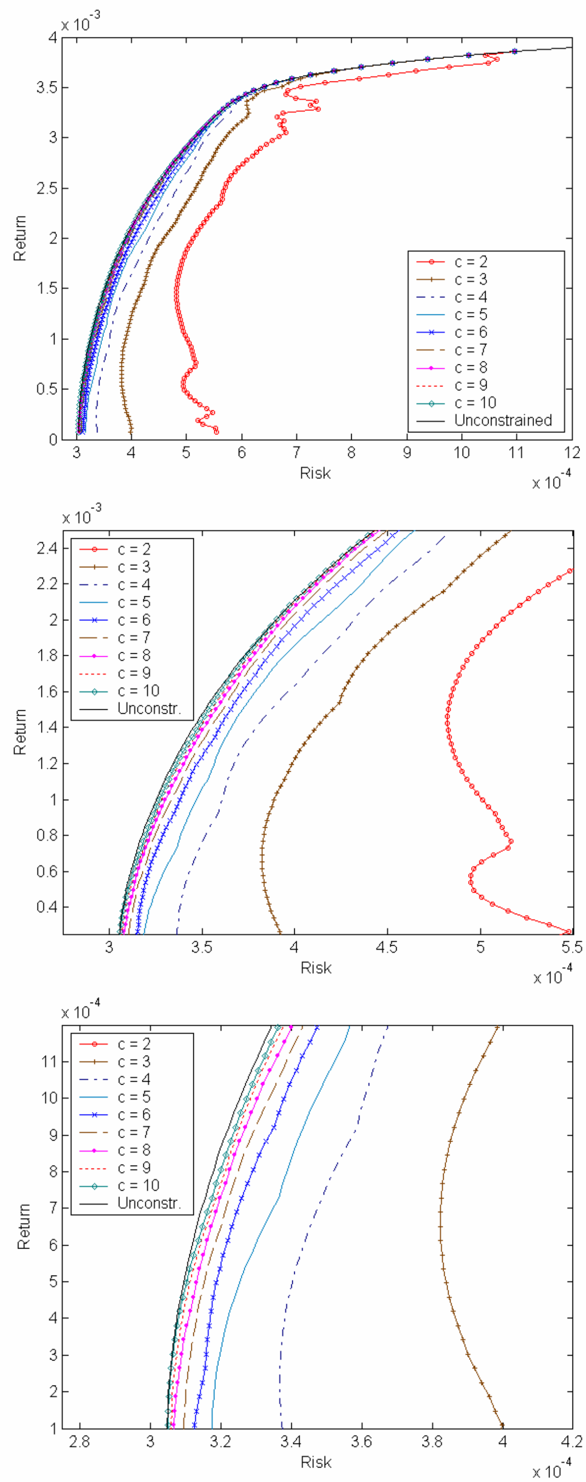

Fig. 2. Efficient frontiers for the Nikkei index problem with different cardinality constraints (c=2,3,...,10) at different levels of detail.

To assess the performance of the different implementations of the hybrid algorithm we perform 30 executions per point in order to compute statistics of the success rate of the algorithm (i.e., percentage of runs that reach the best result). The points in the efficient frontier correspond to equally spaced values of the expected Return. Besides cardinality restrictions, we also use lower bound constraints (minimum investment percentage in each of the

TABLE I
RESULTS FOR BINARY ENCODING

Strategy	Index	D (%) [best result]	Success rate	Time (s)
Death Penalty	Hang Seng	0.00324393	0.98	390.2
	DAX	2.70352541	0.86	456.0
	FTSE	1.94071104	0.93	953.0
	S&P	4.76045939	0.91	932.6
	Nikkei	0.27886285	0.82	1009.5
Linear Penalty	Hang Seng	**0.00321150**	1.00	353.9
	DAX	2.62436368	0.86	420.1
	FTSE	**1.92150019**	0.93	886.3
	S&P	4.72453770	0.93	862.2
	Nikkei	0.26253413	0.84	927.6
Quadratic Penalty	Hang Seng	0.00852899	0.98	295.8
	DAX	2.62819331	0.87	509.3
	FTSE	1.93368663	0.93	853.0
	S&P	4.73613891	0.92	874.5
	Nikkei	0.26531752	0.83	943.3
Log penalty	Hang Seng	0.00609327	0.98	295.7
	DAX	2.60767150	0.87	402.3
	FTSE	1.93009183	0.93	885.7
	S&P	4.73001827	0.92	917.6
	Nikkei	0.26443788	0.83	967.5
Random repair	Hang Seng	**0.00321150**	1.00	417.3
	DAX	2.59586957	0.89	610.2
	FTSE	1.92600177	0.97	1084.3
	S&P	4.70413615	0.96	1091.2
	Nikkei	0.25047296	0.86	1149.2
Heuristic repair	Hang Seng	**0.00321150**	1.00	415.5
	DAX	2.55216320	0.95	560.3
	FTSE	1.92522299	0.96	1216.9
	S&P	4.69979026	0.97	1205.4
	Nikkei	0.23133926	0.91	1436.7

Results for the efficient frontier with binary encoding. Values in boldface correspond to the optimal results achieved by any of the methods investigated.

TABLE II
RESULTS FOR SUBSET ENCODING

	Index	D (%) [best result]	Success rate	Time (s)
R3	Hang Seng	0.02019022	0.95	371.6
	DAX	4.33637863	0.34	337.7
	FTSE	2.37012069	0.40	633.0
	S&P	5.75089932	0.36	608.2
	Nikkei	2.01302119	0.12	638.5
RAR (w=1)	Hang Seng	**0.00321150**	1.00	535.3
	DAX	2.53620543	0.99	598.9
	FTSE	1.92991152	0.94	1257.9
	S&P	**4.69506892**	0.99	1163.7
	Nikkei	**0.20197748**	1.00	1601.2
RAR (w=2)	Hang Seng	**0.00321150**	1.00	453.8
	DAX	2.53187099	0.99	510.9
	FTSE	1.93036965	0.93	1076.9
	S&P	4.69702029	0.99	1035.6
	Nikkei	**0.20197748**	1.00	1471.8
RAR (w=3)	Hang Seng	**0.00321150**	1.00	508.8
	DAX	**2.53180074**	0.99	552.7
	FTSE	1.93255857	0.92	1079.7
	S&P	4.70596057	0.98	1083.2
	Nikkei	**0.20197748**	1.00	1465.8

Results for the efficient frontier with subset encoding. Values in boldface correspond to the optimal results achieved by any of the methods investigated.

TABLE III
COMPARISON WITH TABU SEARCH [7]

Index	D (%) [best result, this work]	D (%) [best result, Tabu Search]	Improvement (%)
Hang Seng	**0.00321150**	0.00344745	7.34
DAX	**2.53180074**	2.53845	0.26
FTSE	**1.92150019**	1.92711	0.29
S&P	**4.69506892**	4.69426	0.01
Nikkei	**0.20197748**	0.204786	1.39

assets is 1 %, i.e. $\{a_i = 0.01, b_i = 1, i=1,2,...,N\}$).

The errors reported are average relative distance between the cardinality constrained (c = 10) and the unconstrained efficient frontier. That is, for a given value of the expected return of the portfolio, R^*_n, we calculate the risk value of the minimum risk portfolio in the constrained problem $\sigma_c(R^*_n)$ and the corresponding value for the unconstrained problem $\sigma(R^*_n)$, and from these

$$D = \frac{1}{100}\sum_{n=1}^{100}\frac{\sigma_c(R_n^*) - \sigma(R_n^*)}{\sigma(R_n^*)} \quad (19)$$

The values reported in the third column of Tables I and II are the best result for D in the 30 executions. In the fourth column we report the overall percentage of runs in which the algorithm succeeded in finding the best portfolio (success rate). The fifth column is the time used to compute the whole efficient frontier in a Pentium IV 3.2GHz.

Optimizations are performed using a steady state genetic algorithm with binary tournament selection. The population consists of 100 individuals. Mutation and crossover probabilities are $p_m = 0.01$, $p_x = 1.00$, respectively. To favor diversity and to prevent early convergence repeated phenotypes are not allowed. No elitism is used.

The results show that the in binary encoding the repair strategies seem to be better than the correction strategies based on adding a penalty term to the fitness function. From the two repair strategies used, the heuristic one seems to be better suited to the problem. The subset encoding gives generally better results than the binary encoding, especially when used in combination with the RAR operator. The best overall performance is given buy the subset encoding with the RAR (w=1) crossover operator.

Finally, Table III presents a comparison of the proposed method with the results of Tabu Search reported in [7]. The results displayed in this table show that the hybrid technique proposed leads to small but consistent and significant improvements over Tabu Search for all the problems. A detailed comparison with other investigations has proven difficult because of insufficient information or difficulties to reproduce the empirical setup.

VI. CONCLUSION

We have designed a hybrid strategy that combines evolutionary techniques with quadratic programming. This hybrid algorithm is efficient and gives a near-optimal solution to the problem of selecting the optimal investment portfolio with cardinality constraints. In particular it outperforms Tabu Search in several benchmark problems. It would be interesting to use this hybrid strategy in combination with other optimization tools, such as Simulated Annealing, and to compare the results. Further research needs to be made to consider other realistic constraints, and to apply these techniques to the related index tracking problem.

APPENDIX

In this appendix we prove that finding the subset of assets that should be included in the optimal portfolio is an NP-hard problem. The asset selection problem is related to the *Subset Sum* problem, which is NP-complete [16]. The *Subset Sum* problem consists in extracting from a given set of integers S = {s_1, s_2,..., s_N} a subset of elements whose sum is equal to zero.

Assume that an algorithm ***A*** solves the asset selection problem in polynomial time exists. Then, the *Subset Sum* problem would also be solvable in polynomial time by solving a collection of N asset selection problems with cardinality constraints c = 1,2,..., N. In each of these problems we assume equal expected returns for the assets

$$r_i = R, \quad i = 1,2,\ldots,N, \tag{A.1}$$

an expected return for the portfolio $R^*=R$, a covariance matrix of the form

$$\Sigma_{ij} = s_i s_j, \quad i,j = 1,2,\ldots,N, \tag{A.2}$$

and restrictions on the weights

$$b_i = a_i = \frac{1}{c}, \quad c > 0 \quad i = 1,2,\ldots,N. \tag{A.3}$$

Under these conditions, the asset selection problem becomes

$$\begin{aligned} Min \quad & \left(\frac{1}{c} \sum_{i=1}^{N} z_i s_i \right)^2, \\ s.t. \quad & \sum_{i=1}^{N} z_i = c \qquad z_i \in \{0,1\} \end{aligned} \tag{A4}$$

If for some value of c the minimum found is zero, the values of the indicator variables {z_i , i = 1, 2,..., N} corresponding to that minimum are the solution of the *Subset Sum* problem. If for all values of c the minimum found is larger than zero, then the particular instance of the *Subset Sum* problem has no solution.

Unless NP = P, no algorithm with the properties of ***A*** exists. Hence, the asset selection problem is at least as hard as any NP-complete problem. These types of problems are NP-hard, and unless NP=P, no polynomial time algorithm that computes a solution to any of this problems exists.

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