

On execution strategies and minimum discrimination information principle

Hellinton H. Takada

Citation: [AIP Conference Proceedings](#) **1757**, 050003 (2016); doi: 10.1063/1.4959059

View online: <http://dx.doi.org/10.1063/1.4959059>

View Table of Contents: <http://scitation.aip.org/content/aip/proceeding/aipcp/1757?ver=pdfcov>

Published by the [AIP Publishing](#)

Articles you may be interested in

[Minimum-error state discrimination constrained by the no-signaling principle](#)

J. Math. Phys. **51**, 022202 (2010); 10.1063/1.3298647

[Minimum magnetizability principle](#)

J. Chem. Phys. **125**, 056101 (2006); 10.1063/1.2227381

[Learning the Structure of a Bayesian Network: An Application of Information Geometry and the Minimum Description Length Principle](#)

AIP Conf. Proc. **803**, 293 (2005); 10.1063/1.2149807

[New Subharmonic Solution of Quasilinear Elliptic Type Second Order PDEs of the Mathematical Physics with Minimum Principles for Irreversible Processes and Information Theory](#)

AIP Conf. Proc. **735**, 509 (2004); 10.1063/1.1835250

[From Minimum Entropy Production Principle To Minimum Information Loss With Elliptic Type Quasilinear PDEs](#)

AIP Conf. Proc. **707**, 246 (2004); 10.1063/1.1751370

On Execution Strategies and Minimum Discrimination Information Principle

Hellinton H. Takada

*Quantitative Research, Itaú Asset Management, São Paulo, Brazil
Institute of Mathematics and Statistics, University of São Paulo, Brazil*

Abstract. Execution strategies are algorithms used to execute huge amount of orders in stock exchanges. Consequently, they are very important for big players such as institutional investors, asset management firms, pension funds and, so on. In this context, there is a plethora of execution algorithms used by traders and brokers to execute the orders of their clients. In this paper, we define execution strategies as execution density functions. We propose an execution impact cost function based on Kullback-Leibler divergence to be used in the derivation of execution strategies in general. Time weighted average price (TWAP) and volume weighted average price (VWAP) execution strategies are the two fundamental algorithms that originate several other strategies. Formally, we derive the TWAP and VWAP strategies using the minimum discrimination information principle. Additionally, we use the developed theory to obtain some VWAP tilt execution strategies.

Keywords: Information theory, Entropy, Financial markets.

PACS: 89.70.-a, 89.70.Cf, 89.65.Gh.

INTRODUCTION

In 1960-70s, systems supporting electronic trading began to appear. Consequently, there was no more need to have a person trading on an exchange floor. In the mid 1990s, several large stock exchanges in USA and Europe were trading a considerable proportion of their volume electronically. Since then, the stock exchanges from regions such as Asia or Latin America also started the migration to electronic trading. Clearly, the equity markets were the fastest in the adoption of electronic trading in the world. On the other hand, the bond markets have been the slowest to enter in the electronic trading era. For a historical review of the electronic trading and the evolution of exchanges see [1].

Obviously, the electronic trading has enabled the automation of trading strategies [2]. The increase of investments in high-tech trading environments caused the proliferation of automated trading strategies and the called high-frequency trading (HFT). HFT relies on the high speed and frequency of the trades to obtain gains. However, it is important to notice that only part of the trading strategies is HFT¹. Trading strategies are investment strategies and there are several possible classifications to group them. Execution strategies, the object of study of this paper, represent a group of trading strategies and they are not necessarily HFT. Actually, an execution strategy is an algorithm that receives an order that must be completely executed with some constraints and its performance is always measured against a predefined benchmark.

¹ For examples of trading strategies considered HFT see [3].

In practice, execution strategies are part of the service provided by traders and brokers to their clients². Usually, the clients are portfolio managers from large asset management firms, pension funds, family offices, and so on. The portfolio managers need to execute huge amount of orders in several different markets according to some constraints such as predetermined execution time intervals and low market impact to avoid adverse price distortions. Clearly, the automation of execution strategies is suitable for such a scenario and keeps the operational risks very low. Actually, the best execution is an important concern when talking about trading regulation [4] and the use of automated algorithms makes the execution process easy to audit.

There are some approaches for the design of execution strategies from *ad hoc* procedures to optimization frameworks [5]. In this paper, we introduce the use of information theory for the design of execution strategies and define execution strategies as execution density functions. We propose an execution impact cost function based on Kullback-Leibler [6] divergence to be used in the derivation of execution strategies in general. Time weighted average price (TWAP) and volume weighted average price (VWAP) execution strategies are the two fundamental algorithms that originate several other strategies. Formally, we derive the TWAP and VWAP strategies using the minimum discrimination information principle. Additionally, we use the developed theory to obtain some VWAP tilt execution strategies.

EXECUTION STRATEGIES

In this section, we introduce some concepts related to execution strategies necessary to the developments in the remaining of this paper. An order is generated by an investment portfolio manager according to some investment strategy. Then, the order is delivered to a trader or broker to be executed in a stock exchange using an execution strategy. Clearly, an order is the input of an execution strategy and its definition is given in the following.

Definition 1 (Order) *An order is represented by a pair (V, T) such that $V \in \mathfrak{R}_{\neq 0}$ represents an amount of contracts to be executed and $T = [t_s, t_e], t_s < t_e, t_s \wedge t_e \in \mathfrak{R}_+$ represents a time interval.*

Given an order (V, T) , the trader or broker are usually not allowed to change the amount to be executed V or exceed the time constraint T . Obviously, it is possible but not desirable to execute an amount different from V or violate T . For example, the failure during T of the communication to the stock exchange may cause partial or even no execution of the desired amount V . Additionally, it is important to notice that (V, T) represents a buy or a sell order. We consider $V > 0$ when buying and $V < 0$ when selling.

Definition 2 (Execution Strategy) *Given an order (V, T) , an execution strategy S is an execution density function $v_S : T \rightarrow [0, 1]$ such that*

$$v_S(t) \geq 0, \forall t \in T \quad (1)$$

² It is important to notice that traders usually have internal clients while brokers have external clients.

and

$$\int_T v_S(t) dt = 1. \quad (2)$$

The execution density function v_S represents the way the amount V is executed over the time interval T in terms of percentage³.

According to the previous definition, $v_S(t)V$ represents the amount executed by the execution strategy S at the time instant $t \in T$. After executing the orders, the *ex post* mean price of S is given by

$$\mu_S = \int_T v_S(t) p(t) dt, \quad (3)$$

where $p(t), t \in T$ represents the *ex post* prices assumed continuous over the execution interval T . The quantity μ_S is important because represents a measure of performance of the execution strategy and it is independent of the amount executed V . Additionally, $\mu_S V$ represents the financial value that will impact the investment portfolio. When buying ($V > 0$), $\mu_S V$ represents a cash outflow. When selling ($V < 0$), $\mu_S V$ represents a cash inflow.

Naively, the idea is to obtain a lower value of μ_S when buying or a higher value of μ_S when selling. The performance of execution strategies are always measured against an *ex post* benchmark. In this context, we define the best price (BP) metric and the BP execution strategy.

Definition 3 (BP Metric) Given (V, T) , the BP metric μ_{BP} is given by

$$\mu_{BP} = \min_t \text{sgn}(V) p(t), \quad (4)$$

where sgn is the signum function.

Definition 4 (BP Execution Strategy) The execution strategy that aims to have as its *ex post* mean price the BP metric μ_{BP} is the BP execution strategy v_{BP} .

The difficulty in determining v_{BP} is that the price over T is a stochastic process $P(t), t \in T$. Using the known *ex post* continuous $p(t), t \in T$ and assuming the existence of $t_{BP} = \arg \min_t \text{sgn}(V) p(t)$, we have $\mu_{BP} = p(t_{BP})$. Since

$$v_{BP} = \arg \min_{v_S} \text{sgn}(V) \int_T v_S(t) p(t) dt, \quad (5)$$

subject to (1) and (2), we have

$$v_{BP}(t) = \delta(t - t_{BP}), \quad (6)$$

where δ is the Dirac delta function.

³ In this paper, we consider the amount of contracts V infinitely divisible. In practice, there is always a minimum lot size. In addition, the equation (2) is called execution completeness since the execution strategy aims to execute all the amount V .

It is evident that the implementation of BP execution strategies in the real world is very difficult since we do not have *p a priori*. However, even knowing the price trajectory *p a priori* the solution (6) is unfeasible. The lack of liquidity causes problems when V is large and the execution becomes impossible or causes adverse price distortion. It is necessary to take into account the market liquidity over T . In other words, the market liquidity is the source of contracts from where the executions are made possible. Opposite to the idea of market liquidity is the idea of execution impact cost. We define execution impact cost as follows:

Definition 5 (Execution Impact Cost) *Given (V, T) , the execution impact cost of an execution strategy S with its execution density function v_S is represented by*

$$\phi_S(v_S, \ell) = D(v_S \| \ell) = \int_T v_S(t) \ln \left(\frac{v_S(t)}{\ell(t)} \right) dt, \quad (7)$$

where $D(\cdot \| \cdot)$ is the Kullback-Leibler divergence and $\ell : T \rightarrow [0, 1]$ is the market liquidity density function such that

$$\ell(t) \geq 0, \forall t \in T \quad (8)$$

and

$$\int_T \ell(t) dt = 1. \quad (9)$$

In the context of execution strategies, the natural idea is to minimize the execution impact cost ϕ_S . Basically, the minimization of ϕ_S is an application of the minimum discrimination information principle. For example, considering the most uninformative density function for the market liquidity (i.e. $\ell \propto 1$ - a uniform market liquidity density function), it is obvious that (6) does not minimize the divergence (7). Actually, the solution (6) maximizes the divergence (7) and it represents one of the worst execution strategies from this point of view.

TWAP METRIC

In the last section, it was presented that the objective function of a buy (sell) BP strategy is the minimization (maximization) of the execution mean price. Additionally, it was also discussed the difficulties to implement BP strategies and the fact that they represent the worst case from the execution impact cost point of view. On the other hand, the TWAP metric μ_{TWAP} is an achievable alternative benchmark used in practice.

Definition 6 (TWAP Metric) *The TWAP metric μ_{TWAP} is given by*

$$\mu_{TWAP} = \frac{1}{\tau} \int_T p(t) dt, \quad (10)$$

where τ is the amount of time for the execution ($\tau = \int_T dt = t_e - t_s$).

TWAP Execution Strategy from Minimum Discrimination Information Principle

It is very important to differentiate the TWAP metric from the TWAP execution strategy. The first is a benchmark μ_{TWAP} used to analyze the performance of an execution strategy while the second is an execution strategy described by the execution density function v_{TWAP} .

Definition 7 (TWAP Execution Strategy) *The execution strategy that aims to have as its ex post mean price the TWAP metric μ_{TWAP} is the TWAP execution strategy v_{TWAP} .*

The following theorem obtains v_{TWAP} using the minimum discrimination information principle:

Theorem 1 *Given (V, T) , the TWAP execution strategy is the minimum discrimination information execution strategy and $v_{TWAP}(t), t \in T$ is the uniform execution density function under no other information.*

Proof 1 *Under no other information, $\ell \propto 1$ over the execution interval T and it is possible to write*

$$\ell(t) = \frac{1}{\tau}, \forall t \in T, \quad (11)$$

where $\tau = \int_T dt$. Applying the minimum discrimination information principle on the execution impact cost

$$v_{S^*} = \arg \min_{v_S} D(v_S \| \ell) \quad (12)$$

subject to (1) and (2), it results using properties of Kullback-Leibler divergence (see [7]) that

$$v_{S^*}(t) = \frac{1}{\tau}, \forall t \in T. \quad (13)$$

The ex post mean price of the execution strategy S^ is given by*

$$\mu_{S^*} = \int_T v_{S^*}(t) p(t) dt = \frac{1}{\tau} \int_T p(t) dt. \quad (14)$$

Obviously, $\mu_{S^} = \mu_{TWAP}$ and, consequently, S^* is the TWAP execution strategy. Finally,*

$$v_{TWAP}(t) = \frac{1}{\tau}, \forall t \in T. \quad (15)$$

■

Despite the behavior of the stochastic process of the prices $P(t)$, it is important to notice that the TWAP execution strategy (15) always produces the TWAP metric μ_{TWAP} as its mean price. Consequently, TWAP metric is easily achieved. However, TWAP is useful only when there is no other information to be used in the execution process.

VWAP METRIC

Usually, the TWAP metric is not adequate to measure the performance of a broker, trader or execution strategy because it does not take into account the executed volumes over the execution interval. In other words, TWAP metric considers the price of a trade with a smaller volume as important as the price of a trade with a larger volume. Obviously, the metric would be more fair if considering prices weighted by the respective executed volumes. Then, it is important to define VWAP metric.

Definition 8 (VWAP Metric) *The VWAP metric μ_{VWAP} is given by*

$$\mu_{VWAP} = \int_T \vartheta(t) p(t) dt, \quad (16)$$

where $\vartheta : T \rightarrow [0, 1]$ is the executed volume density function such that

$$\vartheta(t) \geq 0, \forall t \in T \quad (17)$$

and

$$\int_T \vartheta(t) dt = 1. \quad (18)$$

The executed volume density function ϑ is an execution density function representing all the executed trades in a market and not the executed trades by only one execution strategy as in the execution density function definition. It is important to state that ϑ is known only *a posteriori*. It is intuitive that the market liquidity $\ell(t)$, the source of contracts to execution strategies, can be proxied by $\vartheta(t)$.

For equities, it is known that the total executed trades $U(t), t \in T_d$ of a market over the trading day T_d has a pattern. Usually, the pattern has a U-shape and practitioners obtain this pattern using historical executed volumes of the last 21 days [5]. The U-shape implies higher volume of executed trades during the beginning and at the end of the day. Additionally, it is possible to take $\vartheta \propto U$.

VWAP Strategy from Minimum Discrimination Information Principle

Again, it is important to differentiate the VWAP metric from the VWAP execution strategy. The first is a benchmark μ_{VWAP} used to analyze the performance of an execution strategy while the second is an execution strategy described by the execution density function v_{VWAP} .

Definition 9 (VWAP Execution Strategy) *The execution strategy that aims to have as its ex post mean price the VWAP metric μ_{VWAP} is the VWAP execution strategy v_{VWAP} .*

The TWAP metric is very naive because the related optimum execution strategy, the TWAP execution strategy, distributes uniformly the volume V over the execution interval T and does not take into account any other information from the market. The following theorem obtains v_{VWAP} using the minimum discrimination information principle:

Theorem 2 *The VWAP execution strategy is the minimum discrimination information execution strategy and the $v_{VWAP}(t)$ is the executed volume density function $\vartheta(t)$ when*

market liquidity density function $\ell(t)$ is represented by $\vartheta(t)$ over the execution interval T .

Proof 2 Since $\ell(t) = \vartheta(t)$ over the execution interval T and applying the minimum discrimination information principle on the execution impact cost

$$v_{S^*} = \arg \min_{v_S} D(v_S \| \vartheta) \quad (19)$$

subject to (1) and (2), it results using properties of Kullback-Leibler divergence (see [7]) that

$$v_{S^*}(t) = \vartheta(t), \forall t \in T. \quad (20)$$

The ex post mean price of the execution strategy S^* is given by

$$\mu_{S^*} = \int_T v_{S^*}(t) p(t) dt = \int_T \vartheta(t) p(t) dt. \quad (21)$$

Obviously, $\mu_{S^*} = \mu_{VWAP}$ and, consequently, S^* is the VWAP execution strategy. Finally,

$$v_{VWAP}(t) = \vartheta(t), \forall t \in T. \quad (22)$$

■

Clearly, μ_{VWAP} is more difficult to achieve than μ_{TWAP} because ϑ is not known *a priori*. The implementation of v_{VWAP} requires some procedure to infer ϑ *ex ante* using, for example, stochastic control. The discussion and development of such a procedure is out of the scope of this paper.

VWAP TILT

The objective of this section is present an application of the developed theory. Traders and brokers like to be able to change the behavior of VWAP execution strategy according to some beliefs they have or some constraints from their clients. The VWAP tilt algorithms represent execution strategies derived from the pure VWAP execution. Given an order (V, T) , a common constraint is to fix the percentage $\gamma, 0 \leq \gamma \leq 1$ of V to be executed at the beginning or end of T . Generically, we have the following problem

$$v_{VWAP, \gamma} = \arg \min_{v_S} D(v_S \| \vartheta), \quad (23)$$

subject to (1), (2), $\int_{t_s}^{t_\gamma} v_S(t) dt = \gamma$ and $\int_{t_\gamma}^{t_e} v_S(t) dt = 1 - \gamma$, where $0 \leq \gamma \leq 1$ and $t_\gamma \in T = [t_s, t_e]$. Solving analytically the functional and a system of non-linear equations, an optimal solution to the optimization problem is given by

$$v_{VWAP, \gamma}(t) = \begin{cases} \frac{\gamma \vartheta(t)}{\int_{t_s}^{t_\gamma} \vartheta(x) dx}, & \forall t \in [t_s, t_\gamma] \\ \frac{(1-\gamma) \vartheta(t)}{\int_{t_\gamma}^{t_e} \vartheta(x) dx}, & \forall t \in]t_\gamma, t_e] \end{cases}. \quad (24)$$

In the literature, $\vartheta(t), t \in T$ has been approximated by a beta density function $\text{Beta}(\alpha, \beta)$ (see [9] for details). In Figure 1, we illustrate some VWAP tilt execution strategy densities obtained for different values of γ considering $\alpha = \beta = 0.5$, $T = [0, 1]$ and $t_\gamma = 0.5$. Obviously, $\gamma = 50\%$ represents the VWAP execution strategy.

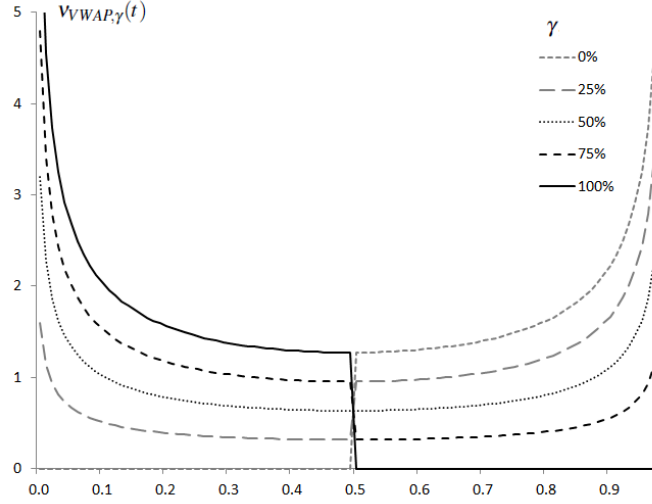


FIGURE 1. VWAP γ -tilted execution strategy for some values of γ (0%, 25%, 50%, 75%, 100%).

CONCLUSIONS

This article introduces for the first time information theory to the study of execution strategies. We propose an execution impact cost function based on Kullback-Leibler divergence and we derive the TWAP and VWAP execution strategies using the minimum discrimination information principle. Additionally, we present an application of the developed theory to derive some VWAP tilt execution strategies.

REFERENCES

1. M. Gorham, and N. Singh, *Electronic Exchanges: The Global Transformation from Pits to Bits*, Elsevier, Inc., Burlington, 2009.
2. K. Kim, *Electronic and Algorithmic Trading Technology: The Complete Guide*, Elsevier Inc., Burlington, 2007.
3. I. Aldridge, *High-Frequency Trading: A Practical Guide to Algorithmic Strategies and Trading Systems*, John Wiley & Sons, Inc., Hoboken, 2013.
4. C. D'Hondt, and J.-R. Giraud, *Transaction Cost Analysis A-Z: A Step towards Best Execution in the Post-MiFID Landscape*, EDHEC Risk and Asset Management Research Centre, Nice, 2008.
5. R. Kissell, and M. Glantz, *Optimal Trading Strategies: Quantitative Approaches for Managing Market Impact and Trading Risk*, AMACOM, Inc., New York, 2003.
6. S. Kullback, and R. A. Leibler, *Ann. Math. Statist.* **22** (1), 79–86 (1951).
7. T. M. Cover, and J. A. Thomas, *Elements of Information Theory*, John Wiley & Sons, Inc., New York, 2006, pp. 409–415.
8. R. Almgren, and N. Chriss, *J. Risk* **3** (2), 5–39 (2000).
9. E. Panas, *Appl. Econ.* **37** (2), 191–199 (2005).