Tutorial on Asset Allocation Methodologies

Hellinton H. Takada[1,2]

[1]VP Quantitative Research, Itaú Asset Management,
Banco Itaú-Unibanco S.A., Brazil

[2]Department of Applied Mathematics, Institute of Mathematics and Statistics,
University of São Paulo, Brazil

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Outline

Asset Allocation Methodologies

- Conceptually, asset allocation is an investment strategy and consists in determining the proportion of each individual asset, asset class or investment factor in the portfolio.

- Basic asset allocation demo using Matlab.

Goal Functions

Consider a market with $n \in \mathbb{N}_+$ assets, an investor implementing an asset allocation trades quantities of the available assets. The vector of quantities is represented by $\boldsymbol{q} = (q_i)_{n\times 1}$. For instance, the quantities are shares when the asset is an equity and the quantities are contracts when the asset is a derivative. It is also important to point that if the investor buys a quantity of shares of an equity, the trade consumes cash. On the other hand, if the investor sells a quantity of shares of an equity, the trade produces cash. Furthermore, if the investor buys or sells a quantity of contracts of a future, the trade does not consume or produce cash. Obviously, after the trade, a position in future contracts causes financial impacts in the margin balance.

Goal Functions

Since each asset has a financial value, we denote the random vector containing all the financial values at time t as $\boldsymbol{P}_t = (P_t^{(i)})_{n\times 1}$. At the asset allocation decision time t_0, the financial values of the assets are known and they are equal to $\boldsymbol{p}_{t_0}$. Consequently, the financial value of the portfolio at time t_0 is given by

$$v_{t_0}(\boldsymbol{q}) \equiv \boldsymbol{q}'\boldsymbol{p}_{t_0}. \tag{1}$$

Goal Functions

The asset allocation is always done for a time horizon. Considering a time horizon τ, the future financial value of the portfolio at the end of the time horizon is a random variable given by

$$V_{t_0+\tau}(\boldsymbol{q}) \equiv \boldsymbol{q}'\boldsymbol{P}_{t_0+\tau}. \tag{2}$$

Clearly, the final objective of the investor is almost surely related to the random variable $V_{t_0+\tau}(\boldsymbol{q})$.

Goal Functions

Generically, the investor has a set of goal functions Γ. As it will be clarified in the text, Γ is a set of invertible affine functions. Using the *non-satiation principle*, the investor desires the maximization of each goal function. In the following, we list some common goal functions.

- *absolute value*: the investor has the goal of maximizing the financial value of the portfolio $\boldsymbol{q}$ at the end of the time horizon

$$\Gamma_{\text{absolute value}}\left(\boldsymbol{q}\right) \equiv V_{t_0+\tau}\left(\boldsymbol{q}\right) = \boldsymbol{q}'\boldsymbol{P}_{t_0+\tau}; \tag{3}$$

Goal Functions

- *relative value*: the investor has the goal of outperforming the financial value of the portfolio $\boldsymbol{q}$ in relation to a reference portfolio $\boldsymbol{q}_0$ such that $v_{t_0}(\boldsymbol{q}_0) \neq 0$ at the end of the time horizon

$$\Gamma_{\text{relative value}}(\boldsymbol{q}) \equiv V_{t_0+\tau}(\boldsymbol{q}) - \gamma(\boldsymbol{q}) V_{t_0+\tau}(\boldsymbol{q}_0), \quad (4)$$

where γ is given by

$$\gamma(\boldsymbol{q}) \equiv \frac{v_{t_0}(\boldsymbol{q})}{v_{t_0}(\boldsymbol{q}_0)}. \quad (5)$$

Then,

$$\Gamma_{\text{relative value}}(\boldsymbol{q}) = \boldsymbol{q}' \left(\boldsymbol{I}_n - \frac{\boldsymbol{p}_{t_0+\tau} \boldsymbol{q}_0'}{\boldsymbol{q}_0' \boldsymbol{p}_{t_0+\tau}} \right) \boldsymbol{P}_{t_0+\tau}, \quad (6)$$

where $\boldsymbol{I}_n$ is the identity matrix.

Goal Functions

- *profit value*: the investor has the goal of maximizing the gains of the portfolio $\boldsymbol{q}$ at the end of the time horizon

$$\Gamma_{\text{profit value}}\left(\boldsymbol{q}\right) \equiv V_{t_0+\tau}\left(\boldsymbol{q}\right) - v_{t_0}\left(\boldsymbol{q}\right) = \boldsymbol{q}'\left(\boldsymbol{P}_{t_0+\tau} - \boldsymbol{p}_{t_0}\right). \qquad (7)$$

Goal Functions

In practice, the previous goal functions with slightly changes are used in the investment industry. Consequently, it is possible to notice that the goal functions can be written generically as

$$\Gamma_* (\boldsymbol{q}) = \boldsymbol{q}' \boldsymbol{M}, \tag{8}$$

where $\boldsymbol{M}$ is a *market vector* given by

$$\boldsymbol{M} \equiv \boldsymbol{a} + \boldsymbol{b} \boldsymbol{P}_{t_0+\tau}, \tag{9}$$

where $\boldsymbol{a}$ is a constant vector and $\boldsymbol{b}$ is a constant invertible matrix. Clearly, $\boldsymbol{M}$ is a simple invertible affine transformation of the market prices at the end of the time horizon.

Goal Functions

In the following, we identify the market vector M for the goal functions previously presented:

- *absolute value*:

$$\boldsymbol{a} \equiv \boldsymbol{0}_{n\times 1}, \boldsymbol{b} = \boldsymbol{I}_n; \tag{10}$$

- *relative value*:

$$\boldsymbol{a} \equiv \boldsymbol{0}_{n\times 1}, \boldsymbol{b} = \left(\boldsymbol{I}_n - \frac{\boldsymbol{p}_{t_0+\tau}\boldsymbol{q}_0'}{\boldsymbol{q}_0'\boldsymbol{p}_{t_0+\tau}} \right); \tag{11}$$

- *profit value*:

$$\boldsymbol{a} \equiv -\boldsymbol{p}_{t_0}, \boldsymbol{b} = \boldsymbol{I}_n. \tag{12}$$

Goal Functions

Finally, the following two properties are useful to combine different goal functions and their proofs are trivial.

Property

A goal function Γ_* is homogeneous of first degree, i.e.

$$\Gamma_* \left(\lambda \boldsymbol{q} \right) = \lambda \Gamma_* \left(\boldsymbol{q} \right). \tag{13}$$

Property

A goal function Γ_* is additive, i.e.

$$\Gamma_* \left(\boldsymbol{q}_1 + \boldsymbol{q}_2 \right) = \Gamma_* \left(\boldsymbol{q}_1 \right) + \Gamma_* \left(\boldsymbol{q}_2 \right). \tag{14}$$

Market Vector

The market vector M defined in (9) is an invertible affine transformation of the asset prices $P_{t_0+\tau}$ at the investment horizon. Assuming a known probability density function for $P_{t_0+\tau}$ given by $\mathsf{f}_{P_{t_0+\tau}}$; defining a function $\mathbf{g}$ such that $\mathbf{g}\left(P_{t_0+\tau}\right) \equiv a + bP_{t_0+\tau}$; and using the fact that

$$\mathsf{f}_{M}\left(m\right) = \frac{\mathsf{f}_{P_{t_0+\tau}}\left(\mathbf{g}^{-1}\left(m\right)\right)}{\sqrt{\left|\mathsf{J}_{\mathbf{g}}\left(\mathbf{g}^{-1}\left(m\right)\right)\right|^{2}}}, \tag{15}$$

where $\left|\mathsf{J}_{\mathbf{g}}\left(\cdot\right)\right|$ is the determinant of the Jacobian of $\mathbf{g}$.

Market Vector

It follows that the probability density function of the market vector $\boldsymbol{M}$ is given by

$$\mathsf{f}_{\boldsymbol{M}}\left(\boldsymbol{m}\right)=\frac{\mathsf{f}_{\boldsymbol{P}_{t_0+\tau}}\left(\boldsymbol{b}^{-1}\left(\boldsymbol{m}-\boldsymbol{a}\right)\right)}{\sqrt{\left|\boldsymbol{b}\boldsymbol{b}'\right|}}. \tag{16}$$

In terms of characteristic functions, it is also possible to obtain the following relation

$$\phi_{\boldsymbol{M}}\left(\boldsymbol{\omega}\right)=\exp\left(j\boldsymbol{\omega}'\boldsymbol{a}\right)\phi_{\boldsymbol{P}_{t_0+\tau}}\left(\boldsymbol{b}'\boldsymbol{\omega}\right). \tag{17}$$

Market Vector

Example

Consider normally distributed asset prices at the end of the investment horizon such that

$$\boldsymbol{P}_{t_0+\tau} \sim \mathsf{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \tag{18}$$

and the investor's goal function is $\Gamma_{\text{absolute value}}$. Using (16), it follows that

$$\mathsf{f}_{\boldsymbol{M}}\left(\boldsymbol{m}\right) = \mathsf{f}_{\boldsymbol{P}_{t_0+\tau}}\left(\boldsymbol{m}\right) \Rightarrow \boldsymbol{M} \sim \mathsf{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right). \tag{19}$$

Finally,

$$\Gamma_{\text{absolute value}} \sim \mathsf{N}\left(\boldsymbol{q}'\boldsymbol{\mu}, \boldsymbol{q}'\boldsymbol{\Sigma}\boldsymbol{q}\right). \tag{20}$$

Stochastic Dominance

Definition

Strong stochastic domination or *zero-order dominance*. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two allocation quantity vectors and Γ_* is a goal function. The strong stochastic dominance of $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) over $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$) occurs if and only if

$$\Gamma_*(\boldsymbol{q}_1) \geq \Gamma_*(\boldsymbol{q}_2)\,. \tag{21}$$

Stochastic Dominance

An alternative definition for the strong stochastic domination is given in the following:

Definition

Strong stochastic domination or *zero-order dominance* (alternative definition). Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two allocation quantity vectors, Γ_* is a goal function and $\mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)-\Gamma_*(\boldsymbol{q}_2)}$ is the cumulative distribution function of $\Gamma_*(\boldsymbol{q}_1)-\Gamma_*(\boldsymbol{q}_2)$. The strong stochastic dominance of $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) over $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$) occurs if and only if

$$\mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)-\Gamma_*(\boldsymbol{q}_2)}(0) \equiv \mathbb{P}\left\{\Gamma_*(\boldsymbol{q}_1)-\Gamma_*(\boldsymbol{q}_2) \leq 0\right\} = 0. \tag{22}$$

The strong stochastic dominance definition is very restrictive. In practice, the existence of strong stochastic dominance implies in arbitrage opportunities and, usually, such opportunities are rare and temporary. In the following, we define the weak stochastic dominance:

Stochastic Dominance

Definition

Weak stochastic domination or *first-order dominance*. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two allocation quantity vectors, Γ_* is a goal function and F_{Γ_*} is the cumulative distribution function of Γ_*. The weak stochastic dominance of $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) over $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$) occurs if and only if

$$\mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)}(\gamma) \leq \mathsf{F}_{\Gamma_*(\boldsymbol{q}_2)}(\gamma), \forall\gamma \in (-\infty, +\infty) \tag{23}$$

or, equivalently,

$$\Im\left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_1)}\right](\gamma) \leq \Im\left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_2)}\right](\gamma), \forall\gamma \in (-\infty, +\infty), \tag{24}$$

Stochastic Dominance

Definition

... where $\mathsf{f}_{\Gamma_*(\boldsymbol{q}_1)}$ and $\mathsf{f}_{\Gamma_*(\boldsymbol{q}_2)}$ are probability density functions and

$$\Im\left[\mathsf{f}_{\Gamma_*(\boldsymbol{q})}\right](\gamma) \equiv \int_{-\infty}^{\gamma} \mathsf{f}_{\Gamma_*(\boldsymbol{q})}(s)\,\mathsf{d}s = \mathsf{F}_{\Gamma_*(\boldsymbol{q})}(\gamma)\,. \tag{25}$$

Stochastic Dominance

An alternative definition for the weak stochastic domination is given in the following:

Definition

Weak stochastic domination or *first-order dominance* (alternative definition). Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two allocation quantity vectors, Γ_* is a goal function and $\mathsf{Q}_{\Gamma_*}(p) \equiv \inf\left\{\gamma \in \mathbb{R} : p \leq \mathsf{F}_{\Gamma_*(\boldsymbol{q})}(\gamma)\right\}$ is the quantile function of Γ_*. The weak stochastic dominance of $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) over $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$) occurs if and only if

$$\mathsf{Q}_{\Gamma_*(\boldsymbol{q}_1)}(p) \geq \mathsf{Q}_{\Gamma_*(\boldsymbol{q}_2)}(p), \forall p \in (0,1). \tag{26}$$

Stochastic Dominance

The weak stochastic domination is not as restrictive as the strong stochastic domination. If $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) strongly dominates $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$), $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) also weakly dominates $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$). However, the opposite is not always true.

Stochastic Dominance

Example

Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be allocation quantity vectors and Γ_* is a goal function. In addition, $\Gamma_*(\boldsymbol{q}_1) \sim \mathsf{N}(1,1)$, $\Gamma_*(\boldsymbol{q}_2) \sim \mathsf{N}(0,1)$ and $\Gamma_*(\boldsymbol{q}_1) \perp\!\!\!\perp \Gamma_*(\boldsymbol{q}_2)$. It is straightforward to show that

$$\mathsf{F}_{\Gamma_* \boldsymbol{q}_1}(\gamma) \equiv \frac{1}{2}\left[1 + \mathsf{erf}\left(\frac{\gamma - 1}{\sqrt{2}}\right)\right] \leq \frac{1}{2}\left[1 + \mathsf{erf}\left(\frac{\gamma}{\sqrt{2}}\right)\right] \equiv \mathsf{F}_{\Gamma_* \boldsymbol{q}_2}(\gamma), \tag{27}$$

where

$$\mathsf{erf}(x) \equiv \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2} \mathsf{d}t = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} \mathsf{d}t. \tag{28}$$

Then, $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) weakly dominates $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$).

Stochastic Dominance

Example

... Since $\Gamma_*(\boldsymbol{q}_1) \perp\!\!\!\perp \Gamma_*(\boldsymbol{q}_2)$, it follows that

$$\Gamma_*(\boldsymbol{q}_1) - \Gamma_*(\boldsymbol{q}_2) \sim \mathsf{N}(1,2) \Rightarrow \mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)-\Gamma_*(\boldsymbol{q}_2)}(0) = \frac{1}{2}\left[1 + \mathsf{erf}\left(-\frac{1}{2}\right)\right] \neq 0. \tag{29}$$

Then, $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) does not strongly dominates $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$).

Stochastic Dominance

Proposition

For every non-decreasing function $\mathrm{g} : \mathbb{R} \to \mathbb{R}$, $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) weakly dominates $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$) if and only if

$$\int \mathrm{g}(\gamma)\, \mathrm{dF}_{\Gamma_*(\boldsymbol{q}_1)}(\gamma) \geq \int \mathrm{g}(\gamma)\, \mathrm{dF}_{\Gamma_*(\boldsymbol{q}_2)}(\gamma)\,. \tag{30}$$

Stochastic Dominance

Proposition

... **Proof.** Integrating by parts, it follows that

$$\int \mathsf{g}(\gamma)\, d\mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)}(\gamma) = \left[\mathsf{g}(\gamma)\, \mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)}(\gamma)\right]_{-\infty}^{+\infty} - \int \mathsf{g}'(\gamma)\, \mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)}(\gamma)\, d\gamma \tag{31}$$

and

$$\int \mathsf{g}(\gamma)\, d\mathsf{F}_{\Gamma_*(\boldsymbol{q}_2)}(\gamma) = \left[\mathsf{g}(\gamma)\, \mathsf{F}_{\Gamma_*(\boldsymbol{q}_2)}(\gamma)\right]_{-\infty}^{+\infty} - \int \mathsf{g}'(\gamma)\, \mathsf{F}_{\Gamma_*(\boldsymbol{q}_2)}(\gamma)\, d\gamma. \tag{32}$$

Since $\mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)}(-\infty) = \mathsf{F}_{\Gamma_*(\boldsymbol{q}_2)}(-\infty) = 0$ and $\mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)}(+\infty) = \mathsf{F}_{\Gamma_*(\boldsymbol{q}_2)}(+\infty) = 1$, we have

Stochastic Dominance

Proposition

...

$$\int \mathsf{g}(\gamma)\, \mathsf{dF}_{\Gamma_*(\boldsymbol{q}_1)}(\gamma) - \int \mathsf{g}(\gamma)\, \mathsf{dF}_{\Gamma_*(\boldsymbol{q}_2)}(\gamma) = \int \mathsf{g}'(\gamma)\, \mathsf{F}_{\Gamma_*(\boldsymbol{q}_2)}(\gamma)\, \mathsf{d}\gamma - \int \mathsf{g}'(\gamma)\, \mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)}(\gamma)\, \mathsf{d}\gamma = \int \mathsf{g}'(\gamma)\left[\mathsf{F}_{\Gamma_*(\boldsymbol{q}_2)}(\gamma) - \mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)}(\gamma)\right] \mathsf{d}\gamma. \quad (33)$$

Using the fact that $\mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)}(\gamma) \leq \mathsf{F}_{\Gamma_*(\boldsymbol{q}_2)}(\gamma), \forall\gamma \in (-\infty, +\infty)$ and $\mathsf{g}'(\gamma) \geq 0, \forall\gamma \in (-\infty, +\infty)$, we notice that

$$\int \mathsf{g}(\gamma)\, \mathsf{dF}_{\Gamma_*(\boldsymbol{q}_1)}(\gamma) - \int \mathsf{g}(\gamma)\, \mathsf{dF}_{\Gamma_*(\boldsymbol{q}_2)}(\gamma) \geq 0. \quad (34)$$

$\square$

Stochastic Dominance

Proposition

$\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) weakly dominates $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$) implies that $\mathbb{E}\{\Gamma_*(\boldsymbol{q}_1)\} \geq \mathbb{E}\{\Gamma_*(\boldsymbol{q}_2)\}$.

Proof. The proof is trivial using the previous proposition and g $(\gamma) \equiv \gamma$. $\square$

Stochastic Dominance

Again, in practice, even the weak stochastic dominance is difficult to achieve. In the following, we define the second-order dominance:

Definition

Second-order dominance. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two allocation quantity vectors, Γ_* is a goal function, F_{Γ_*} is the cumulative distribution function of Γ_* and f_{Γ_*} is the probability density function of Γ_*. The second-order dominance of $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) over $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$) occurs if and only if

$$\int_{-\infty}^{\gamma} \mathsf{F}_{\Gamma_*(\boldsymbol{q}_1)}(s)\,\mathsf{d}s \leq \int_{-\infty}^{\gamma} \mathsf{F}_{\Gamma_*(\boldsymbol{q}_2)}(s)\,\mathsf{d}s, \forall\gamma \in (-\infty, +\infty) \tag{35}$$

or, equivalently,

$$\Im^2\left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_1)}\right](\gamma) \leq \Im^2\left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_2)}\right](\gamma), \forall\gamma \in (-\infty, +\infty). \tag{36}$$

Stochastic Dominance

Proposition

For every non-decreasing and concave function g : $\mathbb{R} \to \mathbb{R}$, $\Gamma_* (\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) second-order dominates $\Gamma_* (\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$) if and only if

$$\int \mathrm{g}(\gamma)\, \mathrm{dF}_{\Gamma_*(\boldsymbol{q}_1)}(\gamma) \geq \int \mathrm{g}(\gamma)\, \mathrm{dF}_{\Gamma_*(\boldsymbol{q}_2)}(\gamma). \quad (37)$$

Stochastic Dominance

Definition

Third-order dominance. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two allocation quantity vectors, Γ_* is a goal function, F_{Γ_*} is the cumulative distribution function of Γ_* and f_{Γ_*} is the probability density function of Γ_*. The third-order dominance of $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) over $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$) occurs if and only if

$$\Im^3\left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_1)}\right](\gamma) \leq \Im^3\left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_2)}\right](\gamma), \forall \gamma \in (-\infty, +\infty). \quad (38)$$

Stochastic Dominance

Proposition

For every non-decreasing, concave and positively skewed function $g : \mathbb{R} \to \mathbb{R}$, $\Gamma_* (\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) third-order dominates $\Gamma_* (\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$) if and only if

$$\int g(\gamma)\, dF_{\Gamma_*(\boldsymbol{q}_1)}(\gamma) \geq \int g(\gamma)\, dF_{\Gamma_*(\boldsymbol{q}_2)}(\gamma) . \tag{39}$$

Stochastic Dominance

Definition

k-*order dominance*. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two allocation quantity vectors, Γ_* is a goal function, F_{Γ_*} is the cumulative distribution function of Γ_* and f_{Γ_*} is the probability density function of Γ_*. The k-order dominance of $\Gamma_*(\boldsymbol{q}_1)$ (or $\boldsymbol{q}_1$) over $\Gamma_*(\boldsymbol{q}_2)$ (or $\boldsymbol{q}_2$) occurs if and only if

$$\Im^k\left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_1)}\right](\gamma) \leq \Im^k\left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_2)}\right](\gamma), \forall\gamma \in (-\infty, +\infty). \quad (40)$$

Proposition

k-order dominance implies $k+1$-order dominance.

Satisfaction Functions

In this section, we summarize the features of an allocation quantity vector $\boldsymbol{q}$ using a satisfaction function, $\mathcal{S} : \mathbb{R}^n \to \mathbb{R}$. We denote the idea using a map notation:

$$\boldsymbol{q} \mapsto \mathcal{S}\left(\boldsymbol{q}\right). \tag{41}$$

Since it is not unique the way to summarize the information contained in an allocation quantity vector in one number, the definition of the satisfaction function consists of a set of desired (not necessary) features.

Satisfaction Functions

Definition

Money-equivalence of $\mathcal{S}$. The satisfaction function $\mathcal{S}$ is money-equivalent if and only if it is measured in units of money.

Definition

Scale-invariance of $\mathcal{S}$ or *zero degree positive homogeneity of $\mathcal{S}$*. The satisfaction function $\mathcal{S}$ is scale-invariant if and only if

$$\mathcal{S}\left(\lambda \boldsymbol{q}\right) = \mathcal{S}\left(\boldsymbol{q}\right), \forall \lambda \in \mathbb{R}_{+}. \tag{42}$$

Satisfaction Functions

Example

Obviously, the definition 19 contrasts with 20. If

$$\mathcal{S}\left(\boldsymbol{q}\right) \equiv \mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\}, \tag{43}$$

$\mathcal{S}$ is a money-equivalent because $\Gamma_{*}\left(\boldsymbol{q}\right)$ is measured in units of money. On the other hand, if

$$\mathcal{S}\left(\boldsymbol{q}\right) \equiv \frac{\mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\}}{\sqrt{\mathbb{V}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\}}} \tag{44}$$

(in this case, $\mathcal{S}$ is called Sharpe index or Sharpe ratio), $\mathcal{S}$ is scale-invariant and called index of satisfaction. The indexes of satisfaction provide a tool to normalize and evaluate the performance of portfolios with different sizes.

William F. Sharpe

- Winner of the 1990 Nobel Memorial Prize in Economic Sciences.

Satisfaction Functions

Definition

Estimability of $\mathcal{S}$ or *law invariance of* $\mathcal{S}$. The satisfaction function $\mathcal{S}$ is estimable if and only if

$$\boldsymbol{q} \mapsto \Gamma_* (\boldsymbol{q}) \mapsto \left(\mathsf{f}_{\Gamma_*(\boldsymbol{q})}, \mathsf{F}_{\Gamma_*(\boldsymbol{q})}, \phi_{\Gamma_*(\boldsymbol{q})}\right) \mapsto \mathcal{S} (\boldsymbol{q}) . \tag{45}$$

In words, satisfaction function $\mathcal{S}$ is estimable if $\mathcal{S}$ is a functional of any of the equivalent representations of the distribution of the goal function Γ_*.

Satisfaction Functions

Example

The expected value of the goal function is an estimable satisfaction function:

$$\boldsymbol{q} \mapsto \Gamma_* (\boldsymbol{q}) \mapsto \mathsf{f}_{\Gamma_*(\boldsymbol{q})} \mapsto \mathcal{S} (\boldsymbol{q}) \equiv \mathbb{E} \{\Gamma_* (\boldsymbol{q})\} = \int \gamma \mathsf{f}_{\Gamma_*(\boldsymbol{q})} (\gamma) \, \mathsf{d}\gamma. \qquad (46)$$

Satisfaction Functions

The following definition is the minimum requirement that any satisfaction function needs to verify.

Definition

Sensibility of $\mathcal{S}$ or *monotonicity of* $\mathcal{S}$ or *strong stochastic domination consistence of* $\mathcal{S}$. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two allocation quantity vectors and Γ_* is a goal function. The satisfaction function $\mathcal{S}$ is sensible if and only if

$$\Gamma_*\left(\boldsymbol{q}_1\right) \geq \Gamma_*\left(\boldsymbol{q}_2\right) \Rightarrow \mathcal{S}\left(\boldsymbol{q}_1\right) \geq \mathcal{S}\left(\boldsymbol{q}_2\right). \tag{47}$$

The previous definition guarantees the consistence of the satisfaction functions with the strong stochastic domination definition for allocation quantity vectors and goal functions.

Satisfaction Functions

Example

The expected value of the goal function is a sensible satisfaction function:

$$\Gamma_{*}\left(\boldsymbol{q}_{1}\right) \geq \Gamma_{*}\left(\boldsymbol{q}_{2}\right) \Rightarrow \mathcal{S}\left(\boldsymbol{q}_{1}\right) \equiv \mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}_{1}\right)\right\} \geq \mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}_{2}\right)\right\} \equiv \mathcal{S}\left(\boldsymbol{q}_{2}\right). \tag{48}$$

Satisfaction Functions

Definition

Weak stochastic domination consistence of $\mathcal{S}$ or *first-order domination consistence of* $\mathcal{S}$. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two allocation quantity vectors and Γ_* is a goal function. The satisfaction function $\mathcal{S}$ is weak stochastic domination consistent if and only if

$$\Im\left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_1)}\right](\gamma) \leq \Im\left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_2)}\right](\gamma), \forall\gamma \in (-\infty, +\infty) \Rightarrow \mathcal{S}(\boldsymbol{q}_1) \geq \mathcal{S}(\boldsymbol{q}_2). \tag{49}$$

Satisfaction Functions

Definition

k-order stochastic domination consistence of $\mathcal{S}$. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two allocation quantity vectors and Γ_* is a goal function. The satisfaction function $\mathcal{S}$ is k-order stochastic domination consistent if and only if

$$\Im^k \left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_1)}\right](\gamma) \leq \Im^k \left[\mathsf{f}_{\Gamma_*(\boldsymbol{q}_2)}\right](\gamma), \forall \gamma \in (-\infty, +\infty) \Rightarrow \mathcal{S}(\boldsymbol{q}_1) \geq \mathcal{S}(\boldsymbol{q}_2). \tag{50}$$

Satisfaction Functions

The previous definition guarantees the consistence of the satisfaction functions with the k-order dominance definition for allocation quantity vectors and goal functions.

Proposition

If the satisfaction function $\mathcal{S}$ is estimable and sensible, then the satisfaction function $\mathcal{S}$ is weak stochastic domination consistent.

Proposition

k-order dominance consistence of a satisfaction function $\mathcal{S}$ implies in $k-1$-order dominance consistence.

Notice that a higher-order dominance consistence of a satisfaction function implies in a lower-order dominance consistence. On the other hand, a lower-order dominance does not imply in a higher-order dominance.

Satisfaction Functions

Definition

Constancy of $\mathcal{S}$. Let $\boldsymbol{q}$ be an allocation quantity vector and Γ_* is a goal function. The satisfaction function $\mathbb{S}$ has constancy if and only if

$$\Gamma_* (\boldsymbol{q}) \equiv \gamma_{\boldsymbol{q}} \Rightarrow \mathcal{S} (\boldsymbol{q}) = \gamma_{\boldsymbol{q}}, \tag{51}$$

where $\gamma_{\boldsymbol{q}}$ is a constant, i.e. $\Gamma_* (\boldsymbol{q}) \equiv \gamma_{\boldsymbol{q}}$ means that Γ_* is a deterministic function.

Satisfaction Functions

Example

Consider the absolute value (or wealth) as the goal function and that the portfolio is composed of zero-coupon bonds that expire at the investment horizon. In addition, the satisfaction function is the expected value of the absolute value. Since

$$\Gamma_{\text{absolute value}}\left(\boldsymbol{q}\right) \equiv \gamma_{\boldsymbol{q}} \Rightarrow \mathcal{S}\left(\boldsymbol{q}\right) \equiv \mathbb{E}\left\{\Gamma_{\text{absolute value}}\right\} = \gamma_{\boldsymbol{q}}, \tag{52}$$

the satisfaction function $\mathcal{S}$ has constancy.

Satisfaction Functions

Definition

Positive homogeneity of order-one of $\mathcal{S}$. The satisfaction function $\mathcal{S}$ is positive homogeneous of order-one if and only if

$$\mathcal{S}\left(\lambda \boldsymbol{q}\right) = \lambda \mathcal{S}\left(\boldsymbol{q}\right), \forall \lambda \geq 0. \tag{53}$$

Example

Since the goal functions are order-one positive homogeneous, the satisfaction function $\mathcal{S}\left(\boldsymbol{q}\right) \equiv \mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\}$ is positive homogeneous of order-one because

$$\mathcal{S}\left(\lambda \boldsymbol{q}\right) \equiv \mathbb{E}\left\{\Gamma_{*}\left(\lambda \boldsymbol{q}\right)\right\} = \mathbb{E}\left\{\lambda \Gamma_{*}\left(\boldsymbol{q}\right)\right\} = \lambda \mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\} = \lambda \mathcal{S}\left(\boldsymbol{q}\right), \forall \lambda \geq 0. \tag{54}$$

Satisfaction Functions

Proposition

Euler decomposition of satisfaction function $\mathcal{S}$. If the satisfaction function $\mathcal{S}$ is positive homogeneous of order-one, then

$$\mathcal{S}(\boldsymbol{q}) = \sum_{i=1}^{n} q_i \frac{\partial \mathcal{S}(\boldsymbol{q})}{\partial q_i}. \tag{55}$$

The previous proposition implies that the order-one positive homogeneous satisfaction function is the sum of the contributions of each asset. The contribution of each asset is its quantity q_i times its marginal contribution to the satisfaction function $\partial \mathcal{S} / \partial q_i$.

Satisfaction Functions

Proposition

Let $\mathcal{S}$ be an order-one positive homogeneous satisfaction function. The vector of marginal contributions $\partial\mathcal{S}/\partial\boldsymbol{q}$ is order-zero positive homogeneous function or scale-invariant.

Proof. It is straightforward to notice that for each asset i

$$\frac{\partial\mathcal{S}\left(\lambda\boldsymbol{q}\right)}{\partial\left(\lambda q_i\right)} = \frac{\lambda\partial\mathcal{S}\left(\boldsymbol{q}\right)}{\lambda\partial q_i} = \frac{\partial\mathcal{S}\left(\boldsymbol{q}\right)}{\partial q_i}, \forall\lambda \geq 0. \tag{56}$$

$\square$

Satisfaction Functions

Example

Consider the absolute value (or wealth) as the goal function

$$\Gamma_{\text{absolute value}}\left(\boldsymbol{q}\right) \equiv \boldsymbol{q}'\boldsymbol{P}_{t_0+\tau} \tag{57}$$

and the satisfaction function as

$$\mathcal{S}_{\text{absolute value}}\left(\boldsymbol{q}\right) \equiv \mathbb{E}\left\{\Gamma_{\text{absolute value}}\left(\boldsymbol{q}\right)\right\}. \tag{58}$$

The Euler decomposition of the satisfaction function is

$$\mathcal{S}_{\text{absolute value}}\left(\boldsymbol{q}\right) = \sum_{i=1}^{n} q_i \frac{\partial \mathcal{S}_{\text{absolute value}}\left(\boldsymbol{q}\right)}{\partial q_i} = \sum_{i=1}^{n} q_i \mathbb{E}\left\{P_{t_0+\tau,i}\right\}. \tag{59}$$

Satisfaction Functions

Definition

Translation invariance of $\mathcal{S}$. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be allocation quantity vectors and Γ_* is a goal function. The satisfaction function $\mathcal{S}$ is translation invariant if and only if

$$\Gamma_*\left(\boldsymbol{q}_2\right) \equiv \gamma_{q_2}, \gamma_{q_2} \in \mathbb{R} \Rightarrow \mathcal{S}\left(\boldsymbol{q}_1 + \boldsymbol{q}_2\right) = \mathcal{S}\left(\boldsymbol{q}_1\right) + \gamma_{q_2}. \tag{60}$$

Satisfaction Functions

Example

The expected value of the goal function Γ_* is a translation invariant satisfaction function $\mathcal{S}$. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be allocation quantity vectors. Then,

$$\Gamma_*(\boldsymbol{q}_2) \equiv \gamma_{q_2}, \gamma_{q_2} \in \mathbb{R} \Rightarrow \tag{61}$$

$$\mathcal{S}(\boldsymbol{q}_1 + \boldsymbol{q}_2) = \mathbb{E}\{\Gamma_*(\boldsymbol{q}_1 + \boldsymbol{q}_2)\} \tag{62}$$

$$= \mathbb{E}\{\Gamma_*(\boldsymbol{q}_1) + \Gamma_*(\boldsymbol{q}_2)\} \tag{63}$$

$$= \mathbb{E}\{\Gamma_*(\boldsymbol{q}_1)\} + \mathbb{E}\{\Gamma_*(\boldsymbol{q}_2)\} \tag{64}$$

$$= \mathcal{S}(\boldsymbol{q}_1) + \gamma_{q_2}. \tag{65}$$

Satisfaction Functions

Definition

Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be allocation quantity vectors. The satisfaction function $\mathcal{S}$ is

1. *additive* if and only if
$$\mathcal{S}\left(\boldsymbol{q}_1+\boldsymbol{q}_2\right)=\mathcal{S}\left(\boldsymbol{q}_1\right)+\mathcal{S}\left(\boldsymbol{q}_2\right); \tag{66}$$
2. *super-additive* if and only if
$$\mathcal{S}\left(\boldsymbol{q}_1+\boldsymbol{q}_2\right)\geq\mathcal{S}\left(\boldsymbol{q}_1\right)+\mathcal{S}\left(\boldsymbol{q}_2\right); \tag{67}$$
3. *sub-additive* if and only if
$$\mathcal{S}\left(\boldsymbol{q}_1+\boldsymbol{q}_2\right)\leq\mathcal{S}\left(\boldsymbol{q}_1\right)+\mathcal{S}\left(\boldsymbol{q}_2\right). \tag{68}$$

Satisfaction Functions

The super-additive satisfaction function implies that the investor appreciates more the combination of the allocations than the individual allocations. Alternatively, the sub-additive satisfaction function implies that the investor appreciates more the individual allocations that the combination of the allocations.

Example

The expected value of the goal function Γ_* is an additive satisfaction function. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be allocation quantity vectors. Then,

$$\begin{aligned}
\mathcal{S}\left(\boldsymbol{q}_1+\boldsymbol{q}_2\right) &= \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}_1+\boldsymbol{q}_2\right)\right\} && (69)\\
&= \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}_1\right)+\Gamma_*\left(\boldsymbol{q}_2\right)\right\} && (70)\\
&= \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}_1\right)\right\}+\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}_2\right)\right\} && (71)\\
&= \mathcal{S}\left(\boldsymbol{q}_1\right)+\mathcal{S}\left(\boldsymbol{q}_2\right). && (72)
\end{aligned}$$

Satisfaction Functions

Definition

Co-monotonic allocation quantity vectors. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be allocation quantity vectors and Γ_* is a goal function. The vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are co-monotonic if and only if

$$\Gamma_* \left(\boldsymbol{q}_1 \right) = \mathrm{g} \left(\Gamma_* \left(\boldsymbol{q}_2 \right) \right), \tag{73}$$

where $\mathrm{g} : \mathbb{R} \to \mathbb{R}$ is an increasing and deterministic function.

It is important to notice that a combination of co-monotonic allocations does not provide desirable diversification effects. A bad event in one allocation causes a bad event in the other one.

Satisfaction Functions

Definition

Co-monotonicity additiveness $\mathcal{S}$. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be co-monotonic allocation quantity vectors. The satisfaction function is co-monotonic additive if and only if

$$\mathcal{S}\left(\boldsymbol{q}_1+\boldsymbol{q}_2\right)=\mathcal{S}\left(\boldsymbol{q}_1\right)+\mathcal{S}\left(\boldsymbol{q}_2\right). \tag{74}$$

Satisfaction Functions

Example

Consider a stock with price P_t and its call option with strike K that expires at $t_0 + \tau$. The current date is t_0 and the investment horizon is $t_0 + \tau$. In addition, $\boldsymbol{q}_1 \equiv (1\ 0)'$ represents an allocation in one share of the stock and $\boldsymbol{q}_2 \equiv (0\ 1)'$ represents an allocation in one call option over one share of the stock. If the investor has as the goal function the absolute value (or wealth),

$$\Gamma_{\text{absolute value}}\left(\boldsymbol{q}_1\right) = P_{t_0+\tau} \tag{75}$$

and

$$\Gamma_{\text{absolute value}}\left(\boldsymbol{q}_2\right) = \max\left\{P_{t_0+\tau} - K, 0\right\}. \tag{76}$$

Satisfaction Functions

Example

... It is straightforward to notice that

$$\Gamma_{\text{absolute value}}\left(\boldsymbol{q}_2\right) = \max\left\{\Gamma_{\text{absolute value}}\left(\boldsymbol{q}_1\right) - K, 0\right\}. \tag{77}$$

Consequently, $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are co-monotonic allocation quantity vectors. Consider the expected value of the goal function as the satisfaction function. Then,

$$\begin{aligned} \mathcal{S}\left(\boldsymbol{q}_1 + \boldsymbol{q}_2\right) \equiv & \mathbb{E}\left\{\Gamma_{\text{absolute value}}\left(\boldsymbol{q}_1 + \boldsymbol{q}_2\right)\right\} && (78)\\ = & \mathbb{E}\left\{P_{t_0+\tau} + \max\left\{P_{t_0+\tau} - K, 0\right\}\right\} && (79)\\ = & \mathcal{S}\left(\boldsymbol{q}_1\right) + \mathcal{S}\left(\boldsymbol{q}_2\right). && (80) \end{aligned}$$

Consequently, the chosen satisfaction function is co-monotonic additive.

Satisfaction Functions

Definition

Concavity of $\mathcal{S}$. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be allocation quantity vectors. A satisfaction function $\mathcal{S}$ is concave if and only if

$$\mathcal{S}\left(\lambda \boldsymbol{q}_1 + (1-\lambda)\, \boldsymbol{q}_2\right) \geq \lambda \mathcal{S}\left(\boldsymbol{q}_1\right) + (1-\lambda)\, \mathcal{S}\left(\boldsymbol{q}_2\right), \forall \lambda \in [0,1]. \tag{81}$$

Satisfaction Functions

Proposition

A positive homogeneity of order-one and super-additivity of satisfaction function $\mathcal{S}$ implies that the satisfaction function $\mathcal{S}$ is concave.

Proof. The proof is trivial using the definitions of positive homogeneity of order-one of $\mathcal{S}$ and super-additivity of $\mathcal{S}$.

$\square$

Satisfaction Functions

Definition

Convexity of $\mathcal{S}$. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be allocation quantity vectors. A satisfaction function $\mathcal{S}$ is convex if and only if

$$\mathcal{S}\left(\lambda \boldsymbol{q}_1 + (1-\lambda)\,\boldsymbol{q}_2\right) \leq \lambda \mathcal{S}\left(\boldsymbol{q}_1\right) + (1-\lambda)\,\mathcal{S}\left(\boldsymbol{q}_2\right), \forall \lambda \in [0,1]. \tag{82}$$

Satisfaction Functions

Proposition

A positive homogeneity of order-one and sub-additivity of satisfaction function $\mathcal{S}$ implies that the satisfaction function $\mathcal{S}$ is convex.

Proof. The proof is trivial using the definitions of positive homogeneity of order-one of $\mathcal{S}$ and sub-additivity of $\mathcal{S}$.

$\square$

Satisfaction Functions

Definition

Fair allocation quantity vector. An allocation quantity vector $\boldsymbol{q}_f$ is fair relative to the goal function Γ_* if and only if

$$\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}_f\right)\right\} = 0. \tag{83}$$

The fair allocation quantity vector in relation to a goal function is a fair game or fair bet.

Satisfaction Functions

Definition

Risk-free allocation quantity vector. An allocation quantity vector $\boldsymbol{q}_r$ is risk-free relative to the goal function Γ_* if and only if

$$\Gamma_* \left(\boldsymbol{q}_r \right) = \gamma_{\boldsymbol{q}_r}, \gamma_{\boldsymbol{q}_r} \in \mathbb{R}. \tag{84}$$

The risk-free allocation quantity vector $\boldsymbol{q}_r$ in relation to a goal function does not possess any uncertainty.

Satisfaction Functions

Definition

Risk aversion of $\mathcal{S}$. A satisfaction function $\mathcal{S}$ is risk averse if and only if

$$\Gamma_*\left(\boldsymbol{q}_r\right) \equiv \gamma_{\boldsymbol{q}_r}, \gamma_{\boldsymbol{q}_r} \in \mathbb{R}, \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}_f\right)\right\} \equiv 0 \Rightarrow \mathcal{S}\left(\boldsymbol{q}_r\right) \geq \mathcal{S}\left(\boldsymbol{q}_r + \boldsymbol{q}_f\right). \quad (85)$$

From the previous definition, the satisfaction of the risky joint allocation in the risk-free and fair allocation quantity vectors is less than the satisfaction of the deterministic allocation in the fair allocation quantity vector. It is also important to notice that the satisfaction function is risk averse if it rejects non-rewarded risk.

Satisfaction Functions

Definition

Risk premium in terms of $\boldsymbol{q}_r$ and $\boldsymbol{q}_f$ relative to $\mathcal{S}$. Let $\boldsymbol{q}_r$ and $\boldsymbol{q}_f$ be a risk-free and a fair allocation quantity vectors, respectively. The risk premium in terms of $\boldsymbol{q}_r$ and $\boldsymbol{q}_f$ relative to a satisfaction function $\mathcal{S}$ is given by

$$\mathcal{RP}_{\mathcal{S}}\left(\boldsymbol{q}_r, \boldsymbol{q}_f\right) \equiv \mathcal{S}\left(\boldsymbol{q}_r\right) - \mathcal{S}\left(\boldsymbol{q}_r + \boldsymbol{q}_f\right). \tag{86}$$

Satisfaction Functions

Considering that $\boldsymbol{q}$ is an allocation quantity vector, Γ_* is a goal function and there exists $\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}$, we define $\Gamma_f\left(\boldsymbol{q}\right) \equiv \Gamma_*\left(\boldsymbol{q}\right) - \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}$ and $\Gamma_r\left(\boldsymbol{q}\right) \equiv \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}$. Obviously, $\mathbb{E}\left\{\Gamma_f\left(\boldsymbol{q}\right)\right\} = 0$ and, consequently, $\boldsymbol{q}$ is a fair allocation quantity vector relative to the goal function Γ_f. Additionally, $\boldsymbol{q}$ is also a risk-free allocation quantity vector relative to the goal function Γ_r.

Satisfaction Functions

Definition

Risk premium in terms of an allocation quantity vector $\boldsymbol{q}$. The risk premium in terms of an allocation quantity vector $\boldsymbol{q}$ is given by

$$\mathcal{RP}_{\mathcal{S}}\left(\boldsymbol{q}\right) \equiv \mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\} - \mathcal{S}\left(\boldsymbol{q}\right). \tag{87}$$

Using the previous definition of risk premium, it is possible to present the following definition (next slide).

Satisfaction Functions

Definition

Risk aversion, propensity and neutrality of $\mathcal{S}$. A satisfaction function $\mathcal{S}$ is

- *risk averse* if and only if

$$\mathcal{RP}_{\mathcal{S}}\left(\boldsymbol{q}\right) \geq 0, \forall \boldsymbol{q}; \tag{88}$$

- *risk prone* if and only if

$$\mathcal{RP}_{\mathcal{S}}\left(\boldsymbol{q}\right) \leq 0, \forall \boldsymbol{q}; \tag{89}$$

- *risk neutral* if and only if

$$\mathcal{RP}_{\mathcal{S}}\left(\boldsymbol{q}\right) = 0, \forall \boldsymbol{q}. \tag{90}$$

Satisfaction Functions

Example

The satisfaction function $\mathcal{S}(\boldsymbol{q}) \equiv \mathbb{E}\{\Gamma_*(\boldsymbol{q})\}$ is trivially risk neutral because

$$\mathcal{RP}_{\mathcal{S}}(\boldsymbol{q}) = \mathbb{E}\{\Gamma_*(\boldsymbol{q})\} - \mathbb{E}\{\Gamma_*(\boldsymbol{q})\} = 0. \quad (91)$$

Utility Functions

The investor preference relation $\succ$ on a set of outcomes or consequences $\mathcal{O}$ is *complete*, *transitive*, *continuous* and *independent*. The *completeness*, *transitiveness*, *continuity* and *independence* of the preference relation $\succ$ are considered preference axioms.

Definition

Completeness axiom. A preference relation $\succ$ on the set of outcomes $\mathcal{O}$ is *complete* if for any $a, b \in \mathcal{O}$, one of the following statements is valid

$$a \succ b \vee b \succ a \vee a \sim b, \tag{92}$$

where $\sim$ represent the indifference between two outcomes.

Utility Functions

Definition

Transitiveness axiom. A preference relation $\succ$ on the set of outcomes $\mathcal{O}$ is *transitive* if for any $a, b, c \in \mathcal{O}$, the following statement is valid

$$a \succ b \succ c \Rightarrow a \succ c. \tag{93}$$

Definition

Continuity axiom or *Archimedean axiom*. A preference relation $\succ$ on the set of outcomes $\mathcal{O}$ is *continuous* if for any $a, b, c \in \mathcal{O}$ with $a \succ b \succ c$, there exists some $\lambda \in [0, 1]$ such that

$$\lambda a + (1 - \lambda)\, c \sim b. \tag{94}$$

Utility Functions

Definition

Independence axiom or *Substitution axiom*. A preference relation $\succ$ on the set of outcomes $\mathcal{O}$ satisfies *independence* if for all $a, b, c \in \mathcal{O}$ and $\lambda \in [0, 1]$, the following statement is valid

$$a \succ b \Leftrightarrow \lambda a + (1 - \lambda)\, c \succ \lambda b + (1 - \lambda)\, c. \tag{95}$$

Utility Functions

Using only the preference relation with the preference axioms, in the following, we present a definition of utility function.

Definition

Utility function or *Bernoulli utility function*. Given the outcomes $a, b \in \mathcal{O}$ such that $a \succ b$, the function $\mathcal{U} : \mathcal{O} \to \mathbb{R}$ is an utility function if and only if

$$\mathcal{U}\left(a\right) > \mathcal{U}\left(b\right). \tag{96}$$

Obviously, if $a \sim b$, then $\mathcal{U}\left(a\right) = \mathcal{U}\left(b\right)$.

Utility Functions

Form the previous definitions, we have:

Property

Continuity of $\mathcal{U}$. The utility function $\mathcal{U}$ is a continuous function.

Property

Increasing $\mathcal{U}$. The utility function $\mathcal{U}$ is an increasing function.

Utility Functions

Definition

Expected utility function or *Von Neumann-Morgenstern utility function*. Given the utility function $\mathcal{U} : \mathcal{O} \to \mathbb{R}$, the expected utility function is given by

$$\mathbb{E}\left\{\mathcal{U}\left(\Gamma\right)\right\} \equiv \int \mathcal{U}\left(\gamma\right) \mathrm{f}_{\Gamma}\left(\gamma\right) d\gamma, \tag{97}$$

where Γ is a random variable with a probability density function f_{Γ}.

Utility Functions

In the asset allocation context, the set of outcomes $\mathcal{O}$ is achieved through the choices of allocation quantity vectors. Actually, it is possible to use an utility function $\mathcal{U}$ to describe the extend to which the investor enjoys the outcome of a goal function Γ_* in relation to an allocation quantity vector $\boldsymbol{q}$, i.e.

$$\boldsymbol{q} \mapsto \mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}\right)\right)\right\} \equiv \int \mathcal{U}\left(\gamma\right) \mathsf{f}_{\Gamma_*(\boldsymbol{q})}\left(\gamma\right) d\gamma. \tag{98}$$

Considering the form of $\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}\right)\right)\right\}$, the utility function is a satisfaction function. However, the opposite is not necessarily true.

Utility Functions

Example

Consider the following exponential utility function

$$\mathcal{U}\left(\gamma\right) \equiv -e^{-\frac{1}{c}\gamma}, \tag{99}$$

where $c \in R_{+}$ and γ is the absolute value (or wealth) of the investor. In addition, consider the following goal function

$$\Gamma_{\text{absolute value}}\left(\boldsymbol{q}\right) \equiv \boldsymbol{q}'\boldsymbol{P}_{t_0+\tau}. \tag{100}$$

Then, the utility function becomes

$$\mathcal{U}\left(\boldsymbol{q}\right) = -e^{-\frac{1}{c}\boldsymbol{q}'\boldsymbol{P}_{t_0+\tau}}. \tag{101}$$

Utility Functions

Example

... Notice that

$$\begin{aligned} \boldsymbol{q}'\boldsymbol{P}_{t_0+\tau} =& \boldsymbol{q}'\left\{\boldsymbol{P}_{t_0} \odot \left[\left(\boldsymbol{P}_{t_0+\tau} - \boldsymbol{P}_{t_0}\right) \oslash \boldsymbol{P}_{t_0}\right] + \boldsymbol{P}_{t_0}\right\} && (102) \\ =& \boldsymbol{q}'\left\{\boldsymbol{P}_{t_0} \odot \boldsymbol{R}_{t_0+\tau,t_0} + \boldsymbol{P}_{t_0}\right\}, && (103) \end{aligned}$$

where $\odot$ is the Hadamard product, $\oslash$ is the Hadamard division and

$$\boldsymbol{R}_{t_0+\tau,t_0} \equiv \left(\boldsymbol{P}_{t_0+\tau} - \boldsymbol{P}_{t_0}\right) \oslash \boldsymbol{P}_{t_0} \tag{104}$$

is the vector of returns.

Certainty-Equivalent

Since the utility function is usually not measured in terms of money or return, the certainty-equivalent of a utility function is more interpretable. In the following, we define the certainty-equivalent of a utility function.

Definition

Certainty-equivalent. The certainty-equivalent $\mathcal{CE}$ of an allocation quantity vector $\boldsymbol{q}$ relative to a utility function $\mathcal{U}$ and a goal function Γ_* is such that

$$\boldsymbol{q} \mapsto \mathcal{CE}\left(\boldsymbol{q}\right) \equiv \mathcal{U}^{-1}\left(\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}\right)\right)\right\}\right). \tag{105}$$

From the previous definition, the certainty-equivalent $\mathcal{CE}$ is a satisfaction function and it is measured in the same units as the goal function Γ_*. In addition, the utility function is increasing and, consequently, its inverse $\mathcal{U}^{-1}$ is defined and also increasing. Then, there always exists a unique certainty-equivalent.

Certainty-Equivalent

Property

The certainty-equivalent is measured in the same units as the associated goal function.

Proof. The proof is trivial. $\square$

Consequently, if the associated goal function is measured in money units, the certainty-equivalent is a money-equivalent satisfaction function.

Certainty-Equivalent

Property

The certainty-equivalent is an estimable satisfaction function.
Proof. Since

$$q \mapsto \mathcal{CE}(q) \equiv \mathcal{U}^{-1}\left(\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(q\right)\right)\right\}\right), \tag{106}$$

we have

$$q \mapsto \Gamma_*\left(q\right) \mapsto \mathsf{f}_{\Gamma_*(q)} \mapsto \mathcal{CE}. \tag{107}$$

Therefore, the certainty-equivalent is a functional of the probability density function of the investor's goal function. Then, the certainty-equivalent is an estimable satisfaction function.

□

Certainty-Equivalent

Property

The certainty-equivalent is an increasing function of the corresponding expected utility function, i.e.

$$\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_1\right)\right)\right\} \geq \mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_2\right)\right)\right\} \Leftrightarrow \mathcal{CE}\left(\boldsymbol{q}_1\right) \geq \mathcal{CE}\left(\boldsymbol{q}_2\right) \tag{108}$$

with

$$\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_1\right)\right)\right\} = \mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_2\right)\right)\right\} \Leftrightarrow \mathcal{CE}\left(\boldsymbol{q}_1\right) = \mathcal{CE}\left(\boldsymbol{q}_2\right). \tag{109}$$

Proof. The proof is trivial. □

Certainty-Equivalent

Property

Sensibility of $\mathcal{CE}$ or *monotonicity of* $\mathcal{CE}$ or *strong stochastic domination consistence of* $\mathcal{CE}$. The certainty-equivalent is a sensible satisfaction function.

Property

Weak stochastic domination consistence of $\mathcal{CE}$ or *first-order domination consistence of* $\mathcal{CE}$. The certainty-equivalent is weak stochastic domination consistent.

Certainty-Equivalent

Proposition

k-order stochastic domination consistence of $\mathcal{CE}$. The certainty-equivalent is k-order stochastic domination consistent if and only if

$$(-1)^i\,\mathcal{D}^i\mathcal{U} \leq 0, i = 1, 2, \ldots, k, \tag{110}$$

where $\mathcal{D}$ is a differentiation operator.

Certainty-Equivalent

Property

The certainty-equivalent has constancy, i.e.

$$\Gamma_*(\boldsymbol{q}) \equiv \gamma_{\boldsymbol{q}} \Rightarrow \mathcal{CE}(\boldsymbol{q}) = \gamma_{\boldsymbol{q}}, \tag{111}$$

where $\gamma_{\boldsymbol{q}}$ is a constant, i.e. Γ_* is a deterministic function.
Proof. The proof is trivial. $\square$

Property

The certainty-equivalents are not necessarily positive homogeneous of order-one.

Certainty-Equivalent

The following proposition gives a class of order-one positive homogenous certainty-equivalents.

Proposition

The class of power utility functions

$$\mathcal{U}(\gamma) \equiv \gamma^{1-1/c}, c \geq 0, \tag{112}$$

generates a class of order-one positive homogeneous certainty-equivalents.

Certainty-Equivalent

Property

The certainty-equivalents are not necessarily translation invariant.

Proposition

The class of power utility functions

$$\mathcal{U}(\gamma) \equiv \gamma^{1-1/c}, c \geq 0, \tag{113}$$

generates a class of translation invariant certainty-equivalents.

Certainty-Equivalent

The certainty-equivalents are not necessarily super or sub-additive functions. The following proposition gives an utility function that results in an additive certainty-equivalent.

Proposition

The identity utility function implies in an additive certainty-equivalent.
Proof. Using the fact that

$$\mathcal{U}\left(\gamma\right) \equiv \gamma \Leftrightarrow \mathcal{CE}\left(\boldsymbol{q}\right) = \mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\} \tag{114}$$

and that the goal functions are additive, the proof is trivial.

□

Certainty-Equivalent

The certainty-equivalents are not necessarily co-monotonic additive functions. The following proposition gives an utility function that results in a co-monotonic additive certainty-equivalent.

Proposition

The identity utility function implies in an co-monotonic additive certainty-equivalent.

Certainty-Equivalent

Considering the certainty-equivalent as a satisfaction function, using the risk premium definition, it is possible to write

$$\mathcal{RP}_{\mathcal{CE}}\left(\boldsymbol{q}\right) = \mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\} - \mathcal{CE}\left(\boldsymbol{q}\right). \tag{115}$$

Proposition

The certainty-equivalent is a risk averse satisfaction function if and only if the utility function is concave.

Certainty-Equivalent

Proposition

The certainty-equivalent is a risk prone satisfaction function if and only if the utility function is convex.

Proposition

The certainty-equivalent is a risk neutral satisfaction function if and only if the utility function is linear.

Certainty-Equivalent

The risk aversion, propensity and neutrality definitions are global classifications for satisfaction functions. In the following, we present some local definitions of risk aversion, propensity and neutrality.

Definition

Arrow-Pratt absolute risk aversion coefficient. The Arrow-Pratt absolute risk aversion coefficient is given by

$$\mathcal{A}(\gamma) \equiv -\frac{\mathcal{D}^2\mathcal{U}(\gamma)}{\mathcal{D}\mathcal{U}(\gamma)}, \tag{116}$$

where $\mathcal{D}$ is a differentiation operator.

Certainty-Equivalent

Definition

Arrow-Pratt relative risk aversion coefficient. The Arrow-Pratt relative risk aversion coefficient is given by

$$\mathcal{R}(\gamma) \equiv -\gamma \frac{\mathcal{D}^2 \mathcal{U}(\gamma)}{\mathcal{D}\mathcal{U}(\gamma)} = \gamma \mathcal{A}(\gamma), \tag{117}$$

where $\mathcal{D}$ is a differentiation operator.

Certainty-Equivalent

Definition

Local absolute risk aversion, propensity and neutrality of $\mathcal{U}$. An utility function $\mathcal{U}$ is

- *locally absolutely risk averse* at γ if and only if

$$\mathcal{A}(\gamma) > 0; \tag{118}$$

- *locally absolutely risk prone* at γ if and only if

$$\mathcal{A}(\gamma) < 0; \tag{119}$$

- *locally absolutely risk neutral* at γ if and only if

$$\mathcal{A}(\gamma) = 0. \tag{120}$$

Certainty-Equivalent

Definition

Local relative risk aversion, propensity and neutrality of $\mathcal{U}$. An utility function $\mathcal{U}$ is

- *locally relatively risk averse* at γ if and only if

$$\mathcal{R}\left(\gamma\right) > 0; \tag{121}$$

- *locally relatively risk prone* at γ if and only if

$$\mathcal{R}\left(\gamma\right) < 0; \tag{122}$$

- *locally relatively risk neutral* at γ if and only if

$$\mathcal{R}\left(\gamma\right) = 0. \tag{123}$$

Certainty-Equivalent

Example

It is straightforward to obtain the following Maclaurin's approximation in relation to γ_r

$$\mathcal{U}\left(\gamma_f + \gamma_r\right) \approx \mathcal{U}\left(\gamma_f\right) + \gamma_r \mathcal{D}\mathcal{U}\left(\gamma_f\right) + \frac{1}{2}\gamma_r^2 \mathcal{D}^2\mathcal{U}\left(\gamma_f\right). \tag{124}$$

Consider an allocation quantity vector in the risky assets $\boldsymbol{q}_r$, an allocation quantity vector in the risk-free asset $\boldsymbol{q}_f$ and a goal function Γ_*. Then,

$$\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f + \boldsymbol{q}_r\right)\right) = \mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right) + \Gamma_*\left(\boldsymbol{q}_r\right)\right) \tag{125}$$

$$\approx \mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)\right) + \Gamma_*\left(\boldsymbol{q}_r\right)\mathcal{D}\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)\right) + \frac{1}{2}\Gamma_*^2\left(\boldsymbol{q}_r\right)\mathcal{D}^2\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)\right), \tag{126}$$

using the fact that goal functions are additive.

Certainty-Equivalent

Example

... Taking the expectation,

$$\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)+\Gamma_*\left(\boldsymbol{q}_r\right)\right)\right\} \tag{127}$$

$$\approx \mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)\right)+\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}_r\right)\right\}\mathcal{D}\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)\right)+\frac{1}{2}\mathbb{E}\left\{\Gamma_*^2\left(\boldsymbol{q}_r\right)\right\}\mathcal{D}^2\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)\right). \tag{128}$$

Setting $\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}_r\right)\right\} \equiv 0$, it results that

$$\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)+\Gamma_*\left(\boldsymbol{q}_r\right)\right)\right\} \tag{129}$$

$$\approx \mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)\right)+\frac{1}{2}\mathbb{V}\left\{\Gamma_*\left(\boldsymbol{q}_r\right)\right\}\mathcal{D}^2\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)\right). \tag{130}$$

Certainty-Equivalent

Example

... On the other hand,

$$\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)+\Gamma_*\left(\boldsymbol{q}_r\right)\right)\right\}=\mathcal{U}\left(\mathcal{CE}\left(\boldsymbol{q}_f+\boldsymbol{q}_r\right)\right). \tag{131}$$

As it was already presented in equation 115, the risk premium using the certainty-equivalent as a satisfaction function is $\mathcal{RP}_{\mathcal{CE}}\left(\boldsymbol{q}\right)=\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}-\mathcal{CE}\left(\boldsymbol{q}\right)$. Then,

$$\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)+\Gamma_*\left(\boldsymbol{q}_r\right)\right)\right\}=\mathcal{U}\left(\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}_f+\boldsymbol{q}_r\right)\right\}-\mathcal{RP}_{\mathcal{CE}}\left(\boldsymbol{q}_f+\boldsymbol{q}_r\right)\right) \tag{132}$$

$$=\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)-\mathcal{RP}_{\mathcal{CE}}\left(\boldsymbol{q}_f+\boldsymbol{q}_r\right)\right) \tag{133}$$

$$\approx\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}_f\right)\right)-\mathcal{RP}_{\mathcal{CE}}\left(\boldsymbol{q}_f+\boldsymbol{q}_r\right)\mathcal{DU}\left(\Gamma_*\left(\boldsymbol{q}_f\right)\right). \tag{134}$$

Certainty-Equivalent

Example

... Finally,

$$\mathcal{RP}_{\mathcal{CE}}\left(\boldsymbol{q}_f + \boldsymbol{q}_r\right) \approx \frac{\mathbb{V}\left\{\Gamma_*\left(\boldsymbol{q}_r\right)\right\}}{2}\mathcal{A}\left(\Gamma_*\left(\boldsymbol{q}_f\right)\right). \tag{135}$$

Certainty-Equivalent

Proposition

The certainty-equivalent is unaffected by a positive affine transformation of the utility function.

Proof. A positive affine transformation of a utility function $\mathcal{U}$ is

$$\mathcal{U}(\gamma) \mapsto \mathcal{U}^{\star}(\gamma) \equiv a + \mathcal{U}(\gamma)\, b, \tag{136}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}_{+}$. Then,

$$\begin{aligned} \mathcal{CE}(\boldsymbol{q}) \equiv & \mathcal{U}^{\star -1}\left(\mathbb{E}\left\{\mathcal{U}^{\star}\left(\Gamma_{*}(\boldsymbol{q})\right)\right\}\right) & (137)\\ = & \mathcal{U}^{\star -1}\left(\mathbb{E}\left\{a + \mathcal{U}\left(\Gamma_{*}(\boldsymbol{q})\right) b\right\}\right) & (138)\\ = & \mathcal{U}^{\star -1}\left(a + \mathbb{E}\left\{\mathcal{U}\left(\Gamma_{*}(\boldsymbol{q})\right)\right\} b\right) & (139)\\ = & \mathcal{U}^{-1}\left(\mathbb{E}\left\{\mathcal{U}\left(\Gamma_{*}(\boldsymbol{q})\right)\right\}\right). & (140) \end{aligned}$$

□

Certainty-Equivalent

The Arrow-Pratt absolute risk aversion coefficient is an equivalent representation of the utility function. Actually, the following proposition states that a Arrow-Pratt absolute risk aversion coefficient corresponds to a class of utility functions given by the positive affine transformations of the corresponding utility function.

Proposition

The following relation holds between the Arrow-Pratt absolute risk aversion coefficient $\mathcal{A}$ and positive affine transformations of the corresponding utility function $\mathcal{U}$

$$(\mathcal{I} \circ \exp \circ \mathcal{I}) [-\mathcal{A}] = a + b\,\mathcal{U}, \tag{141}$$

where $\mathcal{I}$ is the integration operator, $a \in \mathbb{R}$ and $b \in \mathbb{R}_+$.

Certainty-Equivalent

A way to define parametric forms for the investor's preferences is to specify functional forms for the Arrow-Pratt absolute risk aversion coefficient. Consider the following specification (called *Pearson specification of the utility function*)

$$\mathcal{A}(\gamma) \equiv \frac{\gamma}{a\gamma^2 + b\gamma + c}, \tag{142}$$

where $a, b, c \in \mathbb{R}$. The Pearson specification of the utility function includes as special cases most of the utility functions studied in the financial literature:

- $c = 0$: the *Hyperbolic Absolute Risk Aversion* (HARA) class of utility functions;
- $a = 0, b > 0, c = 0$: the exponential class of utility functions (a special case of HARA class of utility functions)

$$\mathcal{U}(\gamma) \equiv -e^{-\frac{1}{b}\gamma}; \tag{143}$$

Certainty-Equivalent

- $a = -1, b > 0, c = 0$: the quadratic utility functions (a special case of HARA class of utility functions)

$$\mathcal{U}(\gamma) \equiv \gamma - \frac{1}{2b}\gamma^2 \tag{144}$$

with $\gamma \leq b$;

- $a \geq 1, b = 0, c = 0$: the power utility functions (a special case of HARA class of utility functions)

$$\mathcal{U}(\gamma) \equiv \gamma^{1-\frac{1}{a}}; \tag{145}$$

- $a \to 1, b = 0, c = 0$: the logarithmic utility function (a special case of the power utility functions)

$$\mathcal{U}(\gamma) \equiv \ln(\gamma); \tag{146}$$

Certainty-Equivalent

- $a \to \infty, b = 0, c = 0$: the linear or identity utility function (a special case of the power utility functions)

$$\mathcal{U}(\gamma) \equiv \gamma; \tag{147}$$

- $a = 0, b = 0, c > 0$: the error utility function

$$\mathcal{U}(\gamma) \equiv \text{erf}\left(\frac{\gamma}{\sqrt{2c}}\right). \tag{148}$$

The error utility function is not concave or convex. Actually, it is S-shaped and suitable to the called *propect theory* from *behavior finance*.

Certainty-Equivalent

Finally, it is possible to generalize the Pearson specification of the utility functions defining

$$\mathcal{A}(\gamma, \delta) \equiv \mathcal{A}(\gamma - \delta), \tag{149}$$

where δ represents a horizontal shift in the resulting utility functions.

Property

The utility functions from the Hyperbolic Absolute Risk Aversion (HARA) class of utility functions are concave.

Certainty-Equivalent

Definition

Utility basis function $\mathcal{B}_{\mathcal{U}}$. The utility basis function $\mathcal{B}_{\mathcal{U}}$ of a utility function $\mathcal{U}$ is such that

$$\mathcal{U}(\gamma) \equiv \int_{\mathbb{R}} \mathcal{B}_{\mathcal{U}}(\theta, \gamma)\, \mathsf{f}_{\Theta}(\theta)\, d\theta, \tag{150}$$

where the function f_{Θ} is such that

$$\mathsf{f}_{\Theta}(\theta) \geq 0, \forall \theta \in \mathbb{R} \text{ and } \int_{\mathbb{R}} \mathsf{f}_{\Theta}(\theta)\, d\theta = 1. \tag{151}$$

Certainty-Equivalent

Clearly, the expected value of the utility function is given by

$$\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}\right)\right)\right\} = \int_{\mathbb{R}^2} \mathcal{B}_{\mathcal{U}}\left(\theta,\gamma\right) \mathsf{f}_{\Theta}\left(\theta\right) \mathsf{f}_{\Gamma_*\left(\boldsymbol{q}\right)}\left(\gamma\right) d\theta d\gamma \tag{152}$$

$$=\mathbb{E}\left\{\mathcal{B}_{\mathcal{U}}\left(\Theta,\Gamma_*\left(\boldsymbol{q}\right)\right)\right\}, \tag{153}$$

i.e. the expected utility is the expected value of a function of two random variables. The random variable represented by the goal function $\Gamma_*\left(\boldsymbol{q}\right)$ models the market. On the other hand, the other random variable Θ models the investor.

Certainty-Equivalent

Using the market vector $\boldsymbol{M}$ (equation 9) and the fact that $\Gamma_{*}(\boldsymbol{q}) = \boldsymbol{q}'\boldsymbol{M}$ (equation 8), it is possible to write the certainty-equivalent with explicit dependence on allocation as follows

$$\boldsymbol{q} \mapsto \mathcal{CE}(\boldsymbol{q}) \equiv \mathcal{U}^{-1}\left(\mathbb{E}\left\{\mathcal{U}\left(\boldsymbol{q}'\boldsymbol{M}\right)\right\}\right). \tag{154}$$

Certainty-Equivalent

Example

Consider the exponential utility function (equation 143) $\mathcal{U}(\gamma) \equiv -e^{-\frac{1}{b}\gamma}, b > 0$. Then,

$$\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}\right)\right)\right\} = -\mathbb{E}\left\{e^{-\frac{1}{b}\Gamma_*(\boldsymbol{q})}\right\}. \tag{155}$$

A characteristic function ϕ_X of a random variable X is given by

$$\phi_X\left(\omega\right) \equiv \mathbb{E}\left\{e^{i\omega X}\right\}, \tag{156}$$

where $i = \sqrt{-1}$. Consequently,

$$\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}\right)\right)\right\} = -\phi_{\Gamma_*(\boldsymbol{q})}\left(\frac{i}{b}\right). \tag{157}$$

Certainty-Equivalent

Example

... In addition, the certainty-equivalent is given by

$$\mathcal{CE}\left(\boldsymbol{q}\right) \equiv \mathcal{U}^{-1}\left(\mathbb{E}\left\{\mathcal{U}\left(\Gamma_{*}\left(\boldsymbol{q}\right)\right)\right\}\right) = -b\ln\left(\phi_{\Gamma_{*}(\boldsymbol{q})}\left(\frac{i}{b}\right)\right). \tag{158}$$

Using the fact that $\Gamma_{*}\left(\boldsymbol{q}\right) = \boldsymbol{q}'\boldsymbol{M}$ (equation 8), it follows that

$$\mathcal{CE}\left(\boldsymbol{q}\right) = -b\ln\left(\phi_{\boldsymbol{M}}\left(\frac{i}{b}\boldsymbol{q}\right)\right). \tag{159}$$

Consider the following distribution for the market vector

$$\boldsymbol{M} \equiv \boldsymbol{P}_{t_0+\tau} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right). \tag{160}$$

Certainty-Equivalent

Example

... The resulting certainty-equivalent is

$$\mathcal{CE}\left(\boldsymbol{q}\right) = \boldsymbol{q}'\boldsymbol{\mu} - \frac{\boldsymbol{q}'\boldsymbol{\Sigma}\boldsymbol{q}}{2b}. \tag{161}$$

Certainty-Equivalent

Remark

Arrow-Pratt approximation for the certainty-equivalent. The Arrow-Pratt approximation for the certainty-equivalent is given by

$$\mathcal{CE}(\boldsymbol{q}) \approx \mathbb{E}\{\Gamma_*(\boldsymbol{q})\} - \frac{\mathcal{A}(\mathbb{E}\{\Gamma_*(\boldsymbol{q})\})}{2}\mathbb{V}\{\Gamma_*(\boldsymbol{q})\}. \tag{162}$$

Proof. Let $\mathcal{U}$ be a utility function. It is possible to write the following approximation of $\mathcal{U}$ around $\bar{\gamma}$:

$$\mathcal{U}(\gamma) \approx \mathcal{U}(\bar{\gamma}) + \mathcal{D}\mathcal{U}(\bar{\gamma})(\gamma - \bar{\gamma}) + \frac{1}{2}\mathcal{D}^2\mathcal{U}(\bar{\gamma})(\gamma - \bar{\gamma})^2. \tag{163}$$

Certainty-Equivalent

Remark

... Assuming that $\gamma = \Gamma_* (\boldsymbol{q})$ and $\bar{\gamma} = \mathbb{E}\{\Gamma_* (\boldsymbol{q})\}$, we have

$$\begin{aligned} \mathcal{U}(\Gamma_*(\boldsymbol{q})) \approx\; & \mathcal{U}(\mathbb{E}\{\Gamma_*(\boldsymbol{q})\}) && (164)\\ & + \mathcal{D}\mathcal{U}(\mathbb{E}\{\Gamma_*(\boldsymbol{q})\})(\Gamma_*(\boldsymbol{q}) - \mathbb{E}\{\Gamma_*(\boldsymbol{q})\}) && (165)\\ & + \frac{1}{2}\mathcal{D}^2\mathcal{U}(\mathbb{E}\{\Gamma_*(\boldsymbol{q})\})(\Gamma_*(\boldsymbol{q}) - \mathbb{E}\{\Gamma_*(\boldsymbol{q})\})^2 . && (166) \end{aligned}$$

Taking the expectation, we have

$$\begin{aligned} \mathbb{E}\{\mathcal{U}(\Gamma_*(\boldsymbol{q}))\} \approx\; & \mathcal{U}(\mathbb{E}\{\Gamma_*(\boldsymbol{q})\}) && (167)\\ & + \frac{\mathcal{D}^2\mathcal{U}(\mathbb{E}\{\Gamma_*(\boldsymbol{q})\})}{2}\mathbb{E}\left\{(\Gamma_*(\boldsymbol{q}) - \mathbb{E}\{\Gamma_*(\boldsymbol{q})\})^2\right\} && (168)\\ =\; & \mathcal{U}(\mathbb{E}\{\Gamma_*(\boldsymbol{q})\}) + \frac{\mathcal{D}^2\mathcal{U}(\mathbb{E}\{\Gamma_*(\boldsymbol{q})\})}{2}\mathbb{V}\{\Gamma_*(\boldsymbol{q})\} . && (169) \end{aligned}$$

Certainty-Equivalent

Remark

... On the other hand, the following Taylor expansion holds

$$\mathcal{U}^{-1}(z+w) \approx \mathcal{U}^{-1}(z) + \frac{1}{\mathcal{D}\mathcal{U}\left(\mathcal{U}^{-1}(z)\right)} w. \tag{170}$$

Then,

$$\mathcal{U}^{-1}\left(\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*(\boldsymbol{q})\right)\right\}\right) \approx \mathcal{U}^{-1}\left(\mathcal{U}\left(\mathbb{E}\left\{\Gamma_*(\boldsymbol{q})\right\}\right) + \frac{\mathcal{D}^2\mathcal{U}\left(\mathbb{E}\left\{\Gamma_*(\boldsymbol{q})\right\}\right)}{2} \mathbb{V}\left\{\Gamma_*(\boldsymbol{q})\right\}\right) \tag{171}$$

$$\approx \mathcal{U}^{-1}\left(\mathcal{U}\left(\mathbb{E}\left\{\Gamma_*(\boldsymbol{q})\right\}\right)\right) \tag{172}$$

$$+ \frac{\mathcal{D}^2\mathcal{U}\left(\mathbb{E}\left\{\Gamma_*(\boldsymbol{q})\right\}\right)}{2\mathcal{D}\mathcal{U}\left(\mathcal{U}^{-1}\left(\mathcal{U}\left(\mathbb{E}\left\{\Gamma_*(\boldsymbol{q})\right\}\right)\right)\right)} \mathbb{V}\left\{\Gamma_*(\boldsymbol{q})\right\} \tag{173}$$

Certainty-Equivalent

Remark

...

$$\mathcal{U}^{-1}\left(\mathbb{E}\left\{\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}\right)\right)\right\}\right) = \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\} + \frac{\mathcal{D}^2\mathcal{U}\left(\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}\right)}{2\mathcal{D}\mathcal{U}\left(\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}\right)}\mathbb{V}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\} \quad (174)$$

$$= \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\} - \frac{\mathcal{A}\left(\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}\right)}{2}\mathbb{V}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}. \quad (175)$$

□

Certainty-Equivalent

In terms of sensitivity analysis, the second order approximation of the certainty-equivalent is usually considered. It is straightforward to see that

$$\mathcal{CE}\left(\boldsymbol{q}+\delta\boldsymbol{q}\right) \approx \mathcal{CE}\left(\boldsymbol{q}\right) + \delta\boldsymbol{q}'\frac{\partial\mathcal{CE}\left(\boldsymbol{q}\right)}{\partial\boldsymbol{q}} + \frac{1}{2}\delta\boldsymbol{q}'\frac{\partial^2\mathcal{CE}\left(\boldsymbol{q}\right)}{\partial\boldsymbol{q}\partial\boldsymbol{q}'}\delta\boldsymbol{q}. \tag{176}$$

The expressions for the derivatives are given in the following propositions (next slides).

Certainty-Equivalent

Proposition

First-order derivative of the certainty-equivalent. The first-order derivative of the certainty-equivalent is given by

$$\frac{\partial \mathcal{CE}\left(\boldsymbol{q}\right)}{\partial \boldsymbol{q}} = \frac{\mathbb{E}\left\{\mathcal{DU}\left(\boldsymbol{q}'\boldsymbol{M}\right)\boldsymbol{M}\right\}}{\mathcal{DU}\left(\mathcal{CE}\left(\boldsymbol{q}\right)\right)}, \tag{177}$$

where $\mathcal{D}$ is the derivative operator and $\boldsymbol{M}$ is the market vector.

Certainty-Equivalent

Proposition

Second-order derivative of the certainty-equivalent. The second-order derivative of the certainty-equivalent is given by

$$\frac{\partial^2 \mathcal{CE}\left(\boldsymbol{q}\right)}{\partial \boldsymbol{q} \partial \boldsymbol{q}'} = \frac{\mathbb{E}\left\{\mathcal{D}^2 \mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}\right)\right) \boldsymbol{M}\boldsymbol{M}'\right\} - \mathcal{D}^2 \mathcal{U}\left(\mathcal{CE}\left(\boldsymbol{q}\right)\right) \boldsymbol{y}\boldsymbol{y}'}{\mathcal{D}\mathcal{U}\left(\mathcal{CE}\left(\boldsymbol{q}\right)\right)}, \tag{178}$$

where $\mathcal{D}$ is the derivative operator, $\boldsymbol{M}$ is the market vector and

$$\boldsymbol{y} \equiv \mathbb{E}\left\{\frac{\mathcal{D}\mathcal{U}\left(\Gamma_*\left(\boldsymbol{q}\right)\right)}{\mathcal{D}\mathcal{U}\left(\mathcal{CE}\left(\boldsymbol{q}\right)\right)} \boldsymbol{M}\right\}. \tag{179}$$

Certainty-Equivalent

Example

Consider the net gains as the goal function, i.e.

$$\Gamma_* (\boldsymbol{q}) \equiv \boldsymbol{q}' \left(\boldsymbol{P}_{t_0+\tau} - \boldsymbol{p}_{t_0} \right). \tag{180}$$

Obviously, the market vector is

$$\boldsymbol{M} \equiv \boldsymbol{P}_{t_0+\tau} - \boldsymbol{p}_{t_0}. \tag{181}$$

In addition, consider the error function as the utility function

$$\mathcal{U} (\gamma) \equiv \text{erf} \left(\frac{\gamma}{\sqrt{2\eta}} \right), \tag{182}$$

where $\eta > 0$.

Certainty-Equivalent

Example

... The first derivative of the utility function is

$$\mathcal{DU}(\gamma) = \sqrt{\frac{2}{\pi\eta}} e^{-\frac{1}{2\eta}\gamma^2}. \tag{183}$$

Finally, consider that there are n assets and the prices are normally distributed

$$\boldsymbol{P}_{t_0+\tau} \sim \mathsf{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{184}$$

Then, the market vector is also normally distributed

$$\boldsymbol{M} \sim \mathsf{N}(\boldsymbol{\nu}, \boldsymbol{\Sigma}), \boldsymbol{\nu} \equiv \boldsymbol{\mu} - \boldsymbol{p}_{t_0}. \tag{185}$$

Certainty-Equivalent

Example

... It follows that

$$\mathbb{E}\left\{\mathcal{DU}\left(\boldsymbol{q}'\boldsymbol{M}\right)\boldsymbol{M}\right\} = \sqrt{\frac{2}{\pi\eta}}\mathbb{E}\left\{e^{-\frac{1}{2\eta}\left(\boldsymbol{q}'\boldsymbol{M}\right)^2}\boldsymbol{M}\right\} \tag{186}$$

$$= \sqrt{\frac{2}{\pi\eta}}\frac{\left|\boldsymbol{\Sigma}\right|^{-\frac{1}{2}}}{\left(2\pi\right)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\boldsymbol{m}e^{-\frac{1}{2}\boldsymbol{m}'\left(\frac{\boldsymbol{q}\boldsymbol{q}'}{\eta}\right)\boldsymbol{m}}e^{-\frac{1}{2}\left(\boldsymbol{m}-\boldsymbol{\nu}\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{m}-\boldsymbol{\nu}\right)}d\boldsymbol{m}. \tag{187}$$

Defining

$$\boldsymbol{\xi} \equiv \left[\frac{\boldsymbol{q}\boldsymbol{q}'}{\eta} + \boldsymbol{\Sigma}^{-1}\right]^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} \text{ and } \boldsymbol{\Phi} \equiv \left[\frac{\boldsymbol{q}\boldsymbol{q}'}{\eta} + \boldsymbol{\Sigma}^{-1}\right]^{-1}, \tag{188}$$

Certainty-Equivalent

Example

... it follows that

$$\mathbb{E}\left\{\mathcal{DU}\left(\boldsymbol{q}'\boldsymbol{M}\right)\boldsymbol{M}\right\} = \tag{189}$$

$$\sqrt{\frac{2}{\pi\eta}}\frac{|\boldsymbol{\Sigma}|^{-\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\boldsymbol{m}e^{-\frac{1}{2}\left[(\boldsymbol{m}-\boldsymbol{\xi})'\boldsymbol{\Phi}^{-1}(\boldsymbol{m}-\boldsymbol{\xi})+\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}-\boldsymbol{\xi}'\boldsymbol{\Phi}^{-1}\boldsymbol{\xi}\right]}d\boldsymbol{m}. \tag{190}$$

Using the first-order derivative,

$$\frac{\partial\mathcal{CE}\left(\boldsymbol{q}\right)}{\partial\boldsymbol{q}} = \frac{\mathbb{E}\left\{\mathcal{DU}\left(\boldsymbol{q}'\boldsymbol{M}\right)\boldsymbol{M}\right\}}{\mathcal{DU}\left(\mathcal{CE}\left(\boldsymbol{q}\right)\right)} \tag{191}$$

$$= \frac{|\boldsymbol{\Sigma}|^{-\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}}e^{\frac{\eta}{2}\mathcal{CE}(\boldsymbol{q})}\int_{\mathbb{R}^n}\boldsymbol{m}e^{-\frac{1}{2}\left[(\boldsymbol{m}-\boldsymbol{\xi})'\boldsymbol{\Phi}^{-1}(\boldsymbol{m}-\boldsymbol{\xi})+\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}-\boldsymbol{\xi}'\boldsymbol{\Phi}^{-1}\boldsymbol{\xi}\right]}d\boldsymbol{m}. \tag{192}$$

Certainty-Equivalent

Example

... Finally,

$$\frac{\partial \mathcal{CE}\left(\boldsymbol{q}\right)}{\partial \boldsymbol{q}} = c\left(\boldsymbol{q}\right)\left[\frac{\boldsymbol{q}\boldsymbol{q}'}{\eta} + \boldsymbol{\Sigma}^{-1}\right]^{-1} \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \boldsymbol{p}_{t_0}\right), \tag{193}$$

where

$$c\left(\boldsymbol{q}\right) \equiv \frac{\left|\boldsymbol{\Sigma}\right|^{-\frac{1}{2}}}{\left|\frac{\boldsymbol{q}\boldsymbol{q}'}{\eta} + \boldsymbol{\Sigma}^{-1}\right|^{\frac{1}{2}}} e^{\frac{\eta}{2}\mathcal{CE}^2(\boldsymbol{q}) - \frac{1}{2}\left[\boldsymbol{\nu}'\left(\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\left[\frac{\boldsymbol{q}\boldsymbol{q}'}{\eta} + \boldsymbol{\Sigma}^{-1}\right]^{-1}\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\nu}\right]}. \tag{194}$$

Quantiles

A very popular measure of risk is the called Value at Risk (VaR). In the following, we define the VaR measure.

Definition

Value at Risk (VaR). The value at risk is defined as follows

$$\mathsf{VaR}_c(\boldsymbol{q}) \equiv -\mathsf{Q}_{\Gamma_{\text{profit value}}(\boldsymbol{q})}(1-c), \tag{195}$$

where c is the confidence interval, $\boldsymbol{q}$ is the allocation quantity vector, $\Gamma_{\text{profit value}}$ is the profit value goal function and $\mathsf{Q}_{\Gamma_*(\boldsymbol{q})}(p) \equiv \inf\left\{\gamma \in \mathbb{R} : p \leq \mathsf{F}_{\Gamma_*(\boldsymbol{q})}(\gamma)\right\}$ is the quantile function of $\Gamma_*(\boldsymbol{q})$.

Quantiles

As it is possible to notice, the VaR measure is based on the quantile function. Actually, it is possible to define the following general quantile-based satisfaction function:

Definition

General quantile-based satisfaction function. The general quantile-based satisfaction function is defined as follows

$$\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}) \equiv \mathsf{Q}_{\Gamma_*(\boldsymbol{q})}(1-c), \tag{196}$$

where c is the confidence interval, $\boldsymbol{q}$ is the allocation quantity vector and $\mathsf{Q}_{\Gamma_*(\boldsymbol{q})}(p) \equiv \inf\left\{\gamma \in \mathbb{R} : p \leq \mathsf{F}_{\Gamma_*(\boldsymbol{q})}(\gamma)\right\}$ is the quantile function of $\Gamma_*(\boldsymbol{q})$.

Quantiles

In the following, we present some properties of the general quantile-based satisfaction function $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$:

Property

Money-equivalence of $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$. The general quantile-based satisfaction function is a money-equivalent satisfaction function if and only if $\Gamma_*(\boldsymbol{q})$ is a money-equivalent satisfaction function.

Proof. From definition 95, it is trivial to see that $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$ has the same dimension of $\Gamma_*(\boldsymbol{q})$.

$\square$

Quantiles

Property

Estimability of $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$ or *law of invariance of* $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$. The general quantile-based satisfaction function is a estimable satisfaction function.

Proof. From definition 95, $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$ is defined through the following chain rule

$$\boldsymbol{q} \mapsto \Gamma_*(\boldsymbol{q}) \mapsto \mathsf{F}_{\Gamma_*(\boldsymbol{q})}(\gamma) \mapsto \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}) \,. \tag{197}$$

Then, $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$ is estimable.

□

Quantiles

Property

Weak stochastic domination consistence of $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$ or *first-order domination consistence of* $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$. The general quantile-based satisfaction function is weak stochastic domination consistent.
Proof. From definition 95, we have that

$$\mathsf{Q}_{\Gamma_*(\boldsymbol{q}_1)}(1-c) \geq \mathsf{Q}_{\Gamma_*(\boldsymbol{q}_2)}(1-c), \forall c \in (0,1) \Rightarrow \tag{198}$$

$$\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}_1) \geq \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}_2). \tag{199}$$

Consequently, the general quantile-based satisfaction function is weak stochastic domination consistent.

□

Quantiles

Property

Sensibility of $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$ or *Strong stochastic domination consistence of* $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$ or *zero-order domination consistence of* $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$. The general quantile-based satisfaction function is strong stochastic consistent.

Proof. Since $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$ is weak stochastic domination consistent (property 8.3), $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$ is also strong stochastic domination consistent.

$\square$

Quantiles

Property

Constancy of $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$. Let $\boldsymbol{q}$ be an allocation quantity vector and Γ_* is a goal function. The general quantile-based satisfaction function has constancy, i.e.

$$\Gamma_*(\boldsymbol{q}) \equiv \gamma_{\boldsymbol{q}} \Rightarrow \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}) = \gamma_{\boldsymbol{q}}, \tag{200}$$

where $\gamma_{\boldsymbol{q}}$ is a constant, i.e. $\Gamma_*(\boldsymbol{q}) \equiv \gamma_{\boldsymbol{q}}$ means that Γ_* is a deterministic function.

Property

Positive homogeneity of order-one of $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$. The general quantile-based satisfaction function is positive homogeneous of order-one, i.e.

$$\mathcal{Q}_{c,\Gamma_*}(\lambda\boldsymbol{q}) = \lambda\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}), \forall\lambda \geq 0. \tag{201}$$

Quantiles

Property

Translation invariance of $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be allocation quantity vectors and Γ_* is a goal function. The general quantile-based satisfaction function is translation invariant, i.e.

$$\Gamma_*(\boldsymbol{q}_2) \equiv \gamma_{\boldsymbol{q}_2}, \gamma_{\boldsymbol{q}_2} \in \mathbb{R} \Rightarrow \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}_1 + \boldsymbol{q}_2) = \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}_1) + \gamma_{\boldsymbol{q}_2}. \tag{202}$$

Property

Co-monotonicity additiveness of $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$. Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be co-monotonic allocation quantity vectors. The general quantile-based satisfaction function is co-monotonic additive, i.e.

$$\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}_1 + \boldsymbol{q}_2) = \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}_1) + \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}_2). \tag{203}$$

Quantiles

Remarks

The general quantile-based satisfaction functions, $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$, are

- neither concave nor convex functions of the allocation quantity vector;
- neither risk averse, nor risk prone, nor risk neutral functions.

Quantiles

Example

Remember that the goal function is a simple linear function of the allocation quantity vector and the market vector:

$$\Gamma_* (\boldsymbol{q}) \equiv \boldsymbol{q}' \boldsymbol{M}. \tag{204}$$

Consequently, we have

$$\boldsymbol{q} \mapsto \mathcal{Q}_{c,\Gamma_*} (\boldsymbol{q}) \equiv \mathsf{Q}_{\boldsymbol{q}'\boldsymbol{M}} (1 - c) . \tag{205}$$

In addition, consider normally distributed asset prices at the end of the investment horizon such that

$$\boldsymbol{P}_{t_0+\tau} \sim \mathsf{N} (\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{206}$$

Quantiles

Example

... and the profit value goal function

$$\Gamma_{\text{profit value}}\left(\boldsymbol{q}\right) \equiv \boldsymbol{q}'\left(\boldsymbol{P}_{t_0+\tau} - \boldsymbol{p}_{t_0}\right). \tag{207}$$

Consequently,

$$\Gamma_{\text{profit value}}\left(\boldsymbol{q}\right) \sim \mathsf{N}\left(\mu_{\text{profit value}}, \sigma^2_{\text{profit value}}\right), \tag{208}$$

where $\mu_{\text{profit value}} \equiv \boldsymbol{q}'\left(\boldsymbol{\mu} - \boldsymbol{p}_{t_0}\right)$ and $\sigma^2_{\text{profit value}} = \boldsymbol{q}'\boldsymbol{\Sigma}\boldsymbol{q}$.
Finally,

$$\mathcal{Q}_{c,\Gamma_*}\left(\boldsymbol{q}\right) = \mu_{\text{profit value}} + \sigma_{\text{profit value}}\,\mathsf{erf}^{-1}\left(1-2c\right). \tag{209}$$

Quantiles

Result

It is possible to approximate the general quantile-based satisfaction function as follows

$$\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}) \approx A(\boldsymbol{q}) + B(\boldsymbol{q})\, z(1-c) + C(\boldsymbol{q})\, z^2(1-c)\,, \tag{210}$$

where

$$z(p) \equiv \sqrt{2}\text{erf}^{-1}(2p-1) \tag{211}$$

is the quantile of the standard normal distribution,

$$A(\boldsymbol{q}) \equiv \mathbb{E}\{\Gamma_*(\boldsymbol{q})\} - C(\boldsymbol{q})\,, \tag{212}$$

$$B(\boldsymbol{q}) \equiv \sqrt{\mathbb{E}\{\Gamma_*^2(\boldsymbol{q})\} - \mathbb{E}\{\Gamma_*(\boldsymbol{q})\}^2} \tag{213}$$

Quantiles

Result

... and

$$C\left(\boldsymbol{q}\right) \equiv \frac{\mathbb{E}\left\{\Gamma_*^3\left(\boldsymbol{q}\right)\right\} - 3\mathbb{E}\left\{\Gamma_*^2\left(\boldsymbol{q}\right)\right\}\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\} + 2\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}^3}{6\left(\mathbb{E}\left\{\Gamma_*^2\left(\boldsymbol{q}\right)\right\} - \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}^2\right)}. \quad (214)$$

Proof. The approximation is based on Cornish-Fisher expansion. The Cornish-Fisher expansion is an asymptotic expansion used to approximate quantiles of a probability distribution based on its cumulants. Consider a random variable X, the first terms of the Cornish-Fisher expansion are

$$\mathrm{Q}_X \approx \mathbb{E}\left\{X\right\} + \mathbb{S}_d\left\{X\right\}\left[z\left(p\right) + \frac{1}{6}\left(z^2\left(p\right) - 1\right)\mathbb{S}_k\left\{X\right\}\right], \quad (215)$$

Quantiles

Result

... where $\mathbb{S}_d\left\{X\right\}$ is the standard deviation of X

$$\mathbb{S}_d\left\{X\right\} \equiv \sqrt{\mathbb{E}\left\{\left(X-\mathbb{E}\left\{X\right\}\right)^2\right\}} = \sqrt{\mathbb{E}\left\{X^2\right\}-\mathbb{E}^2\left\{X\right\}} \tag{216}$$

and $\mathbb{S}_k\left\{X\right\}$ is the skewness of X

$$\mathbb{S}_k\left\{X\right\} \equiv \mathbb{E}\left\{\left(\frac{X-\mathbb{E}\left\{X\right\}}{\mathbb{S}_d\left\{X\right\}}\right)^3\right\}. \tag{217}$$

Finally, using the fact that $\mathcal{Q}_{c,\Gamma_*}\left(\boldsymbol{q}\right) \equiv \mathsf{Q}_{\Gamma_*(\boldsymbol{q})}\left(1-c\right)$, it is straightforward to obtain the desired approximation.

$\square$

Quantiles

In terms of sensitivity analysis, the second order approximation of the general quantile-based satisfaction function is usually considered. It is straightforward to see that

$$\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}+\delta\boldsymbol{q}) \approx \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}) + \delta\boldsymbol{q}'\frac{\partial\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})}{\partial\boldsymbol{q}} + \frac{1}{2}\delta\boldsymbol{q}'\frac{\partial^2\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})}{\partial\boldsymbol{q}\partial\boldsymbol{q}'}\delta\boldsymbol{q}. \quad (218)$$

The expressions for the derivatives are given in the following propositions.

Quantiles

Proposition

First-order derivative of $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$. The first-order derivative of the general quantile-based satisfaction function is given by

$$\frac{\partial \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \mathbb{E}\left\{\boldsymbol{M} | \boldsymbol{q}'\boldsymbol{M} = \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})\right\}, \tag{219}$$

where $\boldsymbol{M}$ is the market vector.

Quantiles

Proposition

Second-order derivative of $\mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})$. The second-order derivative of the general quantile-based satisfaction function is given by

$$\frac{\partial^2 \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})}{\partial \boldsymbol{q} \partial \boldsymbol{q}'} = - \left. \frac{\partial \ln \mathsf{f}_{\Gamma_*(\boldsymbol{q})}(\gamma)}{\partial \gamma} \right|_{\gamma = \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})} \mathsf{Cov}\left\{ \boldsymbol{M} | \Gamma_*(\boldsymbol{q}) = \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q}) \right\} \tag{220}$$

$$- \left. \frac{\partial \mathsf{Cov}\left\{ \boldsymbol{M} | \Gamma_*(\boldsymbol{q}) = \gamma \right\}}{\partial \gamma} \right|_{\gamma = \mathcal{Q}_{c,\Gamma_*}(\boldsymbol{q})}, \tag{221}$$

where $\mathsf{f}_{\Gamma_*(\boldsymbol{q})}$ is the probability density function of the goal function $\Gamma_*(\boldsymbol{q})$.

Quantiles

Example

In the context of the last example (normal markets), it is straightforward to obtain that

$$\frac{\partial \mathcal{Q}_{c,\Gamma_*}\left(\boldsymbol{q}\right)}{\partial \boldsymbol{q}} = \boldsymbol{\mu} - \boldsymbol{p}_{t_0} + \frac{\boldsymbol{\Sigma q}}{\sqrt{\boldsymbol{q}'\boldsymbol{\Sigma q}}}\sqrt{2}\text{erf}^{-1}\left(1-2c\right) \tag{222}$$

and

$$\frac{\partial^2 \mathcal{Q}_{c,\Gamma_*}\left(\boldsymbol{q}\right)}{\partial \boldsymbol{q}\partial \boldsymbol{q}'} = \boldsymbol{\Sigma}\left(\boldsymbol{I}_n - \frac{\boldsymbol{q}\boldsymbol{q}'\boldsymbol{\Sigma}}{\boldsymbol{q}'\boldsymbol{\Sigma q}}\right)\frac{\sqrt{2}\text{erf}^{-1}\left(1-2c\right)}{\sqrt{\boldsymbol{q}'\boldsymbol{\Sigma q}}}. \tag{223}$$

Previously, the allocation quantity vector $\boldsymbol{q} \equiv (q_i)_{n\times 1}$ was considered as an input for the goal and satisfaction functions. However, the allocation is also a function of the information of the market available until date t_0, i_{t_0}, and the information of the investor's profile, $\mathcal{P}$. In the following, we present a generic definition for the allocation function.

Definition

Allocation function. The allocation function is given by

$$\boldsymbol{q} : [i_{t_0}, \mathcal{P}] \mapsto \mathbb{R}^n, \tag{224}$$

where n is the number of assets, i_{t_0} is the information of the market available until date t_0 and $\mathcal{P}$ is the information of the investor's profile.

There are several possible allocation decision functions. Usually, the allocation function is chosen solving an optimization problem. In the following, we present a generic optimization problem to choose an allocation decision function.

Definition

Optimal allocation function. The optimal allocation function is given by

$$\boldsymbol{q}^* \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}} \left\{\mathcal{S}\left(\boldsymbol{q}\right)\right\}, \tag{225}$$

where $\mathcal{S}$ is the primary satisfaction function and $\mathcal{C}$ is a constraint set. The set of allocation functions that satisfy the constraint set is called feasible set.

Obviously, it is possible to include other satisfaction functions in the optimization problem using the constraint set $\mathcal{C}$ or formulating a single-objective function.

In the remainder of the theory, we consider only *estimable* or *law invariant* satisfaction functions, i.e.

$$\boldsymbol{q} \mapsto \Gamma_* (\boldsymbol{q}) \mapsto \left(\mathsf{f}_{\Gamma_*(\boldsymbol{q})}, \mathsf{F}_{\Gamma_*(\boldsymbol{q})}, \phi_{\Gamma_*(\boldsymbol{q})} \right) \mapsto \mathcal{S} (\boldsymbol{q}) \,. \tag{226}$$

Alternative definitions for *estimable* or *law invariant* satisfaction functions are given in the following.

Definition

Estimability of $\mathcal{S}$ or *law invariance of* $\mathcal{S}$ (alternative definition using central moments). The satisfaction function $\mathcal{S}$ is estimable if and only if

$$\mathcal{S}(\boldsymbol{q}) \equiv \mathcal{H}\left(\mathbb{E}\left\{\Gamma_*(\boldsymbol{q})\right\}, \mathbb{CM}_2\left\{\Gamma_*(\boldsymbol{q})\right\}, \mathbb{CM}_3\left\{\Gamma_*(\boldsymbol{q})\right\}, \ldots\right), \tag{227}$$

where $\mathbb{CM}_k$ is the central moment of order k

$$\mathbb{CM}_k\left\{\Gamma_*(\boldsymbol{q})\right\} \equiv \mathbb{E}\left\{\left(\Gamma_*(\boldsymbol{q}) - \mathbb{E}\left\{\Gamma_*(\boldsymbol{q})\right\}\right)^k\right\}. \tag{228}$$

Definition

Estimability of $\mathcal{S}$ or *law invariance of* $\mathcal{S}$ (alternative definition using raw moments). The satisfaction function $\mathcal{S}$ is estimable if and only if

$$\mathcal{S}(\boldsymbol{q}) \equiv \mathcal{H}\left(\mathbb{E}\left\{\Gamma_*(\boldsymbol{q})\right\}, \mathbb{RM}_2\left\{\Gamma_*(\boldsymbol{q})\right\}, \mathbb{RM}_3\left\{\Gamma_*(\boldsymbol{q})\right\}, \ldots\right), \quad (229)$$

where $\mathbb{RM}_k$ is the raw moment of order k

$$\mathbb{RM}_k\left\{\Gamma_*(\boldsymbol{q})\right\} \equiv \mathbb{E}\left\{\Gamma_*^k(\boldsymbol{q})\right\}. \quad (230)$$

Definition

Iso-satisfaction surface. A iso-satisfaction surface is given by the following relation

$$\mathcal{H}\left(\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}, \mathbb{CM}_2\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}, \mathbb{CM}_3\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}, \ldots\right) = s, \tag{231}$$

where $s \in \mathbb{R}$ is a constant level of satisfaction.

Remark

In practice, the representation of the satisfaction function on the infinite-dimensional space of moments of the distribution of the goal function is very complex. A usual approximation follows

$$\mathcal{S}\left(\boldsymbol{q}\right) \approx \tilde{\mathcal{H}}\left(\mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\}, \mathbb{V}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\}\right). \tag{232}$$

The previous approximation is the *mean-variance approximation*.

Remark

In the mean-variance approximation context, through a Lagrangian formulation to combine the mean and variance, it follows that

$$\tilde{\mathcal{H}}\left(\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\},\mathbb{V}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}\right)=\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}-\lambda\mathbb{V}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}, \tag{233}$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier or coefficient. It is also possible to obtain $\tilde{\mathcal{H}}\left(\mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\},\mathbb{V}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}\right)$ considering the expected value of a quadratic utility function.

The Lagrange multiplier is interpreted as a coefficient of risk aversion when $\lambda \in \mathbb{R}_+$, a coefficient of risk propensity when $\lambda \in \mathbb{R}_-$ and the investor is risk neutral when $\lambda = 0$.

Definition

Mean-variance optimization problem. The mean-variance optimization problem is given by

$$\boldsymbol{q}^*, \lambda^* \equiv \arg \max_{\boldsymbol{q} \in \mathcal{C}, \lambda \in \mathbb{R}} \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\} - \lambda \mathbb{V}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}, \tag{234}$$

where $\mathcal{C}$ is a constraint set.

Remark

It is possible to write the mean-variance optimization problem as follows

$$\boldsymbol{q}^*(v) \equiv \arg\max_{\substack{\boldsymbol{q}\in\mathcal{C} \\ \mathbb{V}\{\Gamma_*(\boldsymbol{q})\}=v}} \mathbb{E}\{\Gamma_*(\boldsymbol{q})\}, \tag{235}$$

where $\mathcal{C}$ is a constraint set and $v \geq 0$. Then,

$$v^* \equiv \arg\max_{v\geq 0} \mathcal{S}(\boldsymbol{q}^*(v)). \tag{236}$$

The optimal solution is $\boldsymbol{q}^*(v^*)$. Obviously, the optimization from the previous definition becomes equivalent to the one presented here when $\mathcal{S}(\boldsymbol{q}) = \mathbb{E}\{\Gamma_*(\boldsymbol{q})\} - \lambda\mathbb{V}\{\Gamma_*(\boldsymbol{q})\}$.

Remark

Alternatively, using the market vector $\boldsymbol{M}$ (remember that $\Gamma_*(\boldsymbol{q}) \equiv \boldsymbol{q}'\boldsymbol{M}$), it is also possible to write the mean-variance optimization problem as follows

$$\boldsymbol{q}_*, \lambda^* \equiv \arg \max_{\boldsymbol{q}\in\mathcal{C},\lambda\in\mathbb{R}} \boldsymbol{q}'\mathbb{E}\{\boldsymbol{M}\} - \lambda\boldsymbol{q}'\text{Cov}\{\boldsymbol{M}\}\boldsymbol{q}. \tag{237}$$

Finally, the market vector is an affine transformation of the market prices $\boldsymbol{M} \equiv \boldsymbol{a} + \boldsymbol{b}\boldsymbol{P}_{t_0+\tau}$. Then,

$$\mathbb{E}\{\boldsymbol{M}\} = \boldsymbol{a} + \boldsymbol{b}\mathbb{E}\{\boldsymbol{P}_{t_0+\tau}\} \tag{238}$$

and

$$\text{Cov}\{\boldsymbol{M}\} = \boldsymbol{b}\,\text{Cov}\{\boldsymbol{P}_{t_0+\tau}\}\,\boldsymbol{b}'. \tag{239}$$

Example

Consider the following goal function

$$\Gamma_{\text{total linear return}}\left(\boldsymbol{q}\right) \equiv \frac{\boldsymbol{q}'\boldsymbol{P}_{t_0+\tau}}{\boldsymbol{q}'\boldsymbol{p}_{t_0}} - 1, \tag{240}$$

where $\boldsymbol{q}'\boldsymbol{p}_{t_0} \neq 0$. In addition, it is possible to write

$$\Gamma_{\text{total linear return}}\left(\boldsymbol{q}\right) = \boldsymbol{w}'\boldsymbol{L}, \tag{241}$$

where

$$\boldsymbol{w} \equiv \frac{\text{diag}\left(\boldsymbol{p}_{t_0}\right)}{\boldsymbol{q}'\boldsymbol{p}_{t_0}}\boldsymbol{q} \tag{242}$$

and

$$\boldsymbol{L} \equiv \boldsymbol{P}_{t_0+\tau} \oslash \boldsymbol{p}_{t_0} - \boldsymbol{1}_{n\times 1}. \tag{243}$$

It is straightforward to see that $\boldsymbol{w}$ represents relative weights (or allocations) and $\boldsymbol{L}$ is a random variable.

Example

... The optimization problem becomes

$$\boldsymbol{w}^{*}(v) \equiv \arg \max_{\boldsymbol{w} \in \mathcal{C}, \boldsymbol{w}' \operatorname{Cov}\{\boldsymbol{L}\} \boldsymbol{w} = v} \boldsymbol{w}' \mathbb{E}\{\boldsymbol{L}\}, \tag{244}$$

where $\mathcal{C}$ is a constraint set and $v \geq 0$.
Finally, it is also important to have:

$$\mathbb{E}\{\boldsymbol{L}\} = \operatorname{diag}(\boldsymbol{p}_{t_0})^{-1} \mathbb{E}\{\boldsymbol{P}_{t_0+\tau}\} - \mathbf{1}_{n \times 1} \tag{245}$$

and

$$\operatorname{Cov}\{\boldsymbol{L}\} = \operatorname{diag}(\boldsymbol{p}_{t_0})^{-1} \operatorname{Cov}\{\boldsymbol{P}_{t_0+\tau}\} \operatorname{diag}(\boldsymbol{p}_{t_0})^{-1}. \tag{246}$$

The mean-variance framework is usually presented in terms of returns and relative weights. However, as it is possible to notice, the approach using returns and relative weights is not as general as the specification using goal functions.

Definition

Mean-variance optimization problem with affine constraints. The mean-variance optimization problem with affine constraints is given by

$$\boldsymbol{q}^* \equiv \arg \max_{\boldsymbol{q}'\boldsymbol{d}=\boldsymbol{c}, \mathbb{V}\{\Gamma_*(\boldsymbol{q})\}=v} \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}, \tag{247}$$

where $\boldsymbol{d}$ is a matrix, $\boldsymbol{c}$ is a vector and $v \geq 0$.

Remark

The budget constraint. The budget constraint is an affine constraint, i.e.

$$\mathcal{C} : \boldsymbol{q}'\boldsymbol{p}_{t_0} = v_{t_0}, \tag{248}$$

where v_{t_0} is the financial value of the portfolio at time t_0.

Proposition

The solutions of the optimization problem from the last definition when $\boldsymbol{c} = c \neq 0$ is a scalar and $\boldsymbol{d}$ is a non-collinear vector with $\mathbb{E}\left\{\boldsymbol{M}\right\}$ are given by

$$\boldsymbol{q}^{*}\left(e\right) = \boldsymbol{q}_{\text{MV}} + \left(e - \mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}_{\text{MV}}\right)\right\}\right) \frac{\boldsymbol{q}_{\text{SR}} - \boldsymbol{q}_{\text{MV}}}{\mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}_{\text{SR}}\right)\right\} - \mathbb{E}\left\{\Gamma_{*}\left\{\boldsymbol{q}_{\text{MV}}\right\}\right\}}, \quad (249)$$

$$e \in \left[\mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}_{\text{MV}}\right)\right\}, \infty\right), \quad (250)$$

where

$$\boldsymbol{q}_{\text{MV}} \equiv \frac{c\,\text{Cov}^{-1}\left\{\boldsymbol{M}\right\}\boldsymbol{d}}{\boldsymbol{d}'\,\text{Cov}^{-1}\left\{\boldsymbol{M}\right\}\boldsymbol{d}} \quad (251)$$

and

$$\boldsymbol{q}_{\text{SR}} \equiv \frac{c\,\text{Cov}^{-1}\left\{\boldsymbol{M}\right\}\mathbb{E}\left\{\boldsymbol{M}\right\}}{\boldsymbol{d}'\,\text{Cov}^{-1}\left\{\boldsymbol{M}\right\}\mathbb{E}\left\{\boldsymbol{M}\right\}}. \quad (252)$$

Proposition

... Consequently, the efficient frontier $\boldsymbol{q}^* (e)$ is a straight semi-line in the n-dimensional space of allocation that lies in the $(n-1)$-dimensional hyperplane determined by the affine constraint. The semi-line goes from $\boldsymbol{q}_{\text{MV}}$ and passes through $\boldsymbol{q}_{\text{SR}}$. The result is known as the *two-fund separation theorem*: a linear combination of two specific portfolios suffices to generate the whole mean-variance efficient frontier.

Remark

In the context of the last proposition, consider the plane defined by

$$(v, e) \equiv \left(\mathbb{V}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}, \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}\right). \tag{253}$$

The feasible set in the (v, e) plane is the region to the right of the following parabola

$$v = \frac{A}{D}e^2 - \frac{2cB}{D}e + \frac{c^2C}{D}, \tag{254}$$

where

$$A \equiv \boldsymbol{d}' \,\mathrm{Cov}^{-1}\left\{\boldsymbol{M}\right\}\boldsymbol{d}, \tag{255}$$

$$B \equiv \boldsymbol{d}' \,\mathrm{Cov}^{-1}\left\{\boldsymbol{M}\right\}\mathbb{E}\left\{\boldsymbol{M}\right\}, \tag{256}$$

$$C \equiv \mathbb{E}\left\{\boldsymbol{M}\right\}' \,\mathrm{Cov}^{-1}\left\{\boldsymbol{M}\right\}\mathbb{E}\left\{\boldsymbol{M}\right\} \tag{257}$$

and

$$D \equiv AC - B^2. \tag{258}$$

Remark

In the context of the last proposition, it is also usual to consider the plane defined by

$$(d, e) \equiv (\mathbb{S}_d \{\Gamma_* (\boldsymbol{q})\}, \mathbb{E} \{\Gamma_* (\boldsymbol{q})\}). \tag{259}$$

Then, the feasible set in the (d, e) plane is the region to the right of the following hyperbola

$$d^2 = \frac{A}{D} e^2 - \frac{2cB}{D} e + \frac{c^2 C}{D}, \tag{260}$$

where the constants are defined as in the last remark.

Remark

In the context of the last proposition, the global minimum variance portfolio is given by

$$\boldsymbol{q}_{\text{MV}} \equiv \frac{c\,\text{Cov}^{-1}\{\boldsymbol{M}\}\,\boldsymbol{d}}{\boldsymbol{d}'\,\text{Cov}^{-1}\{\boldsymbol{M}\}\,\boldsymbol{d}}. \tag{261}$$

Remark

In the context of the last proposition, the global maximum Sharpe ratio portfolio is given by

$$\boldsymbol{q}_{\mathsf{SR}} \equiv \frac{c\,\mathsf{Cov}^{-1}\{\boldsymbol{M}\}\,\mathbb{E}\{\boldsymbol{M}\}}{\boldsymbol{d}'\,\mathsf{Cov}^{-1}\{\boldsymbol{M}\}\,\mathbb{E}\{\boldsymbol{M}\}}. \tag{262}$$

Definition

Mean-variance optimization problem with linear constraints. Then mean-variance optimization problem with linear constraints is given by

$$\boldsymbol{q}^* \equiv \arg \max_{\boldsymbol{q}'\boldsymbol{d}=\boldsymbol{0}, \mathbb{V}\{\Gamma_*(\boldsymbol{q})\}=v} \mathbb{E}\left\{\Gamma_*\left(\boldsymbol{q}\right)\right\}, \tag{263}$$

where $\boldsymbol{d}$ is a matrix, $\boldsymbol{0}$ is a vector of zeros and $v \geq 0$.

Remark

The budget constraint for market-neutral strategies. The budget constraint for market-neutral strategies is given by

$$\mathcal{C} : \boldsymbol{q}'\boldsymbol{p}_{t_0} = 0. \tag{264}$$

Proposition

The solutions of the optimization problem from the last definition when $\boldsymbol{d}$ is a non-collinear vector with $\mathbb{E}\{\boldsymbol{M}\}$ and the vector of zeros is a scalar are given by

$$\boldsymbol{q}^*(e) \equiv e\,\mathrm{Cov}^{-1}\{\boldsymbol{M}\}\left(A\mathbb{E}\{\boldsymbol{M}\} - B\boldsymbol{d}\right), \tag{265}$$

where

$$e \equiv \mathbb{E}\{\Gamma_*(\boldsymbol{q})\}, \tag{266}$$

$$A \equiv \boldsymbol{d}'\,\mathrm{Cov}^{-1}\{\boldsymbol{M}\}\,\boldsymbol{d} \tag{267}$$

and

$$B \equiv \boldsymbol{d}'\,\mathrm{Cov}^{-1}\{\boldsymbol{M}\}\,\mathbb{E}\{\boldsymbol{M}\}. \tag{268}$$

Remark

Considering the plane defined by

$$(d, e) \equiv (\mathbb{S}_d \{\Gamma_* (\boldsymbol{q})\}, \mathbb{E} \{\Gamma_* (\boldsymbol{q})\}), \tag{269}$$

the hyperbola degenerates to

$$e = \pm\sqrt{\frac{D}{A}}d, d \geq 0. \tag{270}$$

It is possible to notice that all the efficient allocations have the same Sharpe ratio which is the highest possible in the feasible set and equal to $\sqrt{D/A}$.

Remark

For estimable satisfaction functions $\mathcal{S}$, it follows that

$$\mathcal{S}\left(\boldsymbol{q}\right) \equiv \mathcal{H}\left(\mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\}, \mathbb{CM}_{2}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\}, \mathbb{CM}_{3}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\}, \ldots\right). \tag{271}$$

Using only the first two moments, it follows the called mean-variance approximation for $\mathcal{S}$:

$$\mathcal{S}\left(\boldsymbol{q}\right) \approx \tilde{\mathcal{H}}\left(\mathbb{E}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\}, \mathbb{V}\left\{\Gamma_{*}\left(\boldsymbol{q}\right)\right\}\right). \tag{272}$$

The mean-variance approximation becomes an exact representation for the satisfaction function in the following cases:

- the satisfaction function is the expected value of a quadratic utility function;
- the market is elliptically distributed.

Remark

In the following, we describe two different mean-variance optimization problems:

1. $$\begin{aligned} &\text{first-step: } \boldsymbol{q}^*(\lambda) \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}} \left\{\mathbb{E}\left\{\Gamma_*(\boldsymbol{q})\right\} - \lambda\mathbb{V}\left\{\Gamma_*(\boldsymbol{q})\right\}\right\} \\ &\text{second-step: } \lambda^* \equiv \arg\max_{\lambda\in\mathbb{R}} \mathcal{S}\left(\boldsymbol{q}^*(\lambda)\right); \end{aligned} \tag{273}$$

2. $$\boldsymbol{q}^* \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}} \mathcal{S}(\boldsymbol{q}), \tag{274}$$

   where $\mathcal{S}(\boldsymbol{q}) \equiv \mathbb{E}\left\{\Gamma_*(\boldsymbol{q})\right\} - \lambda\mathbb{V}\left\{\Gamma_*(\boldsymbol{q})\right\}$ and λ is chosen *a priori* depending on the investor's profile.

It is important to notice that the first mean-variance optimization problem is more general because it considers the possibility that an investor has different values of λ depending on the market conditions.

Remark

Consider the plane $(v, e) \equiv (\mathbb{V}\{\Gamma_*(\boldsymbol{q})\}, \mathbb{E}\{\Gamma_*(\boldsymbol{q})\})$, the following optimization problem

$$\boldsymbol{q}^*(v) \equiv \arg \max_{\boldsymbol{q} \in \mathcal{C}, \mathbb{V}\{\Gamma_*(\boldsymbol{q})\} = v} \mathbb{E}\{\Gamma_*(\boldsymbol{q})\}, \tag{275}$$

where $v \geq 0$, is equivalent to

$$\boldsymbol{q}^*(v) \equiv \arg \max_{\boldsymbol{q} \in \mathcal{C}, \mathbb{V}\{\Gamma_*(\boldsymbol{q})\} \leq v} \mathbb{E}\{\Gamma_*(\boldsymbol{q})\}, \tag{276}$$

if the upper limit of the feasible set increases we shift to right on the horizontal axes.

Remark

Consider the plane $(v, e) \equiv (\mathbb{V}\{\Gamma_*(\boldsymbol{q})\}, \mathbb{E}\{\Gamma_*(\boldsymbol{q})\})$, the following optimization problem

$$\boldsymbol{q}^*(v) \equiv \arg \max_{\boldsymbol{q} \in \mathcal{C}, \mathbb{V}\{\Gamma_*(\boldsymbol{q})\} \leq v} \mathbb{E}\{\Gamma_*(\boldsymbol{q})\}, \tag{277}$$

where $v \geq 0$, has the following *dual formulation*

$$\boldsymbol{q}^*(e) \equiv \arg \min_{\boldsymbol{q} \in \mathcal{C}, \mathbb{E}\{\Gamma_*(\boldsymbol{q})\} \geq e} \mathbb{V}\{\Gamma_*(\boldsymbol{q})\}, \tag{278}$$

where $e \in \mathbb{R}$.

Remark

The following three optimization problems:

1. $$\boldsymbol{q}^*(v) \equiv \arg \max_{\boldsymbol{q} \in \mathcal{C}, \mathbb{V}\{\Gamma_*(\boldsymbol{q})\} = v} \mathbb{E}\{\Gamma_*(\boldsymbol{q})\}, \tag{279}$$
   where $v \geq 0$;
2. $$\boldsymbol{q}^*(v) \equiv \arg \max_{\boldsymbol{q} \in \mathcal{C}, \mathbb{V}\{\Gamma_*(\boldsymbol{q})\} \leq v} \mathbb{E}\{\Gamma_*(\boldsymbol{q})\}, \tag{280}$$
   where $v \geq 0$;
3. $$\boldsymbol{q}^*(e) \equiv \arg \min_{\boldsymbol{q} \in \mathcal{C}, \mathbb{E}\{\Gamma_*(\boldsymbol{q})\} \geq e} \mathbb{V}\{\Gamma_*(\boldsymbol{q})\}, \tag{281}$$
   where $e \in \mathbb{R}$;

are equivalent if the constraints are affine.

Remark

Linear and Compounded Returns. We have already addressed linear returns in example 107. Here, we redefine the random variable of linear returns as follows

$$\boldsymbol{L}_{t,\tau} \equiv \left(L_{t,\tau}^{(i)}\right)_{n\times 1} \tag{282}$$

where

$$L_{t,\tau}^{(i)} \equiv \frac{P_t^{(i)}}{P_{t-\tau}^{(i)}} - 1. \tag{283}$$

Considering the total linear return as the goal function (see example 107), we have the following optimization problem

$$\boldsymbol{w}^*(v) \equiv \arg \max_{\boldsymbol{w}\in\mathcal{C},\boldsymbol{w}'\mathrm{Cov}\left\{\boldsymbol{L}_{t_0+\tau,\tau}\right\}\boldsymbol{w}=v} \boldsymbol{w}'\mathbb{E}\left\{\boldsymbol{L}_{t_0+\tau,\tau}\right\}, \tag{284}$$

where $\mathcal{C}$ is a constraint set, $v \geq 0$, τ is the investment horizon and $\boldsymbol{w}$

Remark

... Instead of using the linear return, it is common to use the compounded return defined as follows:

$$\boldsymbol{C}_{t,\tau} \equiv \left(C_{t,\tau}^{(i)}\right)_{n\times 1} \tag{285}$$

where

$$C_{t,\tau}^{(i)} \equiv \ln\left(\frac{P_t^{(i)}}{P_{t-\tau}^{(i)}}\right). \tag{286}$$

Using the compounded returns, the practitioners adopt the following mean-variance optimization problem:

$$\tilde{\boldsymbol{w}}(v) \equiv \arg \max_{\boldsymbol{w}\in\mathcal{C},\boldsymbol{w}'\mathrm{Cov}\left\{\boldsymbol{C}_{t_0+\tau,\tau}\right\}\boldsymbol{w}=v} \boldsymbol{w}'\mathbb{E}\left\{\boldsymbol{C}_{t_0+\tau,\tau}\right\}, \tag{287}$$

where $\mathcal{C}$ is a constraint set, $v \geq 0$, τ is the investment horizon and $\boldsymbol{w}$ represents relative weights (or allocations).

Remark

... Unfortunately, the previous optimization problem is not, at least theoretically, very well founded.

Remark

The linear and compounded returns equivalence. In the context of remark 9.17, considering a short investment horizon τ and a not-too-volatile market, a first-order Taylor expansion shows that the linear and the compounded returns are approximately the same:

$$L^{(i)}_{t_0+\tau,\tau} \equiv \frac{P^{(i)}_{t_0+\tau}}{P^{(i)}_{t_0}} - 1 \approx \ln\left(\frac{P^{(i)}_{t_0+\tau}}{P^{(i)}_{t_0}}\right) \equiv C^{(i)}_{t_0+\tau,\tau}. \tag{288}$$

Consequently,

$$\boldsymbol{w}^*(v) \equiv \arg \max_{\boldsymbol{w}\in\mathcal{C},\boldsymbol{w}'\mathsf{Cov}\left\{\boldsymbol{L}_{t_0+\tau,\tau}\right\}\boldsymbol{w}=v} \boldsymbol{w}'\mathbb{E}\left\{\boldsymbol{L}_{t_0+\tau,\tau}\right\} \approx \tag{289}$$

$$\arg \max_{\boldsymbol{w}\in\mathcal{C},\boldsymbol{w}'\mathsf{Cov}\left\{\boldsymbol{C}_{t_0+\tau,\tau}\right\}\boldsymbol{w}=v} \boldsymbol{w}'\mathbb{E}\left\{\boldsymbol{C}_{t_0+\tau,\tau}\right\} \equiv \tilde{\boldsymbol{w}}(v). \tag{290}$$

For longer investment horizons and more volatile markets, the approximation is not accurate.

Definition

Total-return asset allocation. For the total-return asset allocation the goal function is the absolute value (or final wealth) at the investment horizon τ

$$\Gamma_{\text{absolute value}}(\boldsymbol{q}) \equiv \boldsymbol{q}'\boldsymbol{P}_{t_0+\tau}. \tag{291}$$

Definition

Benchmark asset allocation. For the benchmark asset allocation the goal function is the overperformance goal function at the investment horizon τ

$$\Gamma_{\text{overperformance}}(\boldsymbol{q}) \equiv \boldsymbol{q}'\boldsymbol{P}_{t_0+\tau} - \gamma\tilde{\boldsymbol{\beta}}'\boldsymbol{P}_{t_0+\tau}, \tag{292}$$

where $\tilde{\beta}$ is the benchmark allocation and γ is a normalization scalar such that

$$\gamma \equiv \frac{\boldsymbol{q}'\boldsymbol{p}_{t_0}}{\tilde{\boldsymbol{\beta}}'\boldsymbol{p}_{t_0}}. \tag{293}$$

Definition

Expected overperformance. The expected overperformance is given by

$$\mathsf{EOP}\left(\boldsymbol{q}\right) \equiv \mathbb{E}\left\{\Gamma_{\mathsf{overperformance}}\left(\boldsymbol{q}\right)\right\}. \tag{294}$$

Definition

Tracking error. The tracking error is given by

$$\mathsf{TE}\left(\boldsymbol{q}\right) \equiv \mathbb{S}_d\left\{\Gamma_{\mathsf{overperformance}}\left(\boldsymbol{q}\right)\right\}. \tag{295}$$

Definition

Information ratio. The information ratio is given by

$$\mathsf{IR}\left(\boldsymbol{q}\right) \equiv \frac{\mathsf{EOP}\left(\boldsymbol{q}\right)}{\mathsf{TE}\left(\boldsymbol{q}\right)}. \tag{296}$$

Proposition

The optimization problem

$$\tilde{\boldsymbol{q}}(v) \equiv \arg \max_{\boldsymbol{q}'\boldsymbol{p}_{t_0}=v_{t_0}, \mathbb{V}\{\Gamma_{\text{absolute value}}(\boldsymbol{q})\}=v} \mathbb{E}\{\Gamma_{\text{absolute value}}(\boldsymbol{q})\}, \tag{297}$$

where $v_{t_0} > 0$ and $v \geq 0$, is called total return optimization problem with budget constraint. The total return efficient frontier is given by

$$\begin{aligned} \tilde{\boldsymbol{q}} = {} & \tilde{\boldsymbol{q}}_{\text{MV}} \\ & + (e - \mathbb{E}\{\Gamma_{\text{absolute value}}(\tilde{\boldsymbol{q}}_{\text{MV}})\}) \\ & \times \frac{\tilde{\boldsymbol{q}}_{\text{SR}} - \tilde{\boldsymbol{q}}_{\text{MV}}}{\mathbb{E}\{\Gamma_{\text{absolute value}}(\tilde{\boldsymbol{q}}_{\text{SR}})\} - \mathbb{E}\{\Gamma_{\text{absolute value}}(\tilde{\boldsymbol{q}}_{\text{MV}})\}}, \end{aligned}$$

where $e \in [\mathbb{E}\{\Gamma_{\text{absolute value}}(\tilde{\boldsymbol{q}}_{\text{MV}})\}, \infty)$,

Proposition

...

$$\tilde{q}_{\mathrm{MV}} \equiv \frac{v_{t_0}\,\mathrm{Cov}^{-1}\left\{\boldsymbol{P}_{t_0+\tau}\right\}\boldsymbol{p}_{t_0}}{\boldsymbol{p}'_{t_0}\,\mathrm{Cov}^{-1}\left\{\boldsymbol{P}_{t_0+\tau}\right\}\boldsymbol{p}_{t_0}} \tag{298}$$

and

$$\tilde{q}_{\mathrm{SR}} \equiv \frac{v_{t_0}\,\mathrm{Cov}^{-1}\left\{\boldsymbol{P}_{t_0+\tau}\right\}\mathbb{E}\left\{\boldsymbol{P}_{t_0+\tau}\right\}}{\boldsymbol{p}'_{t_0}\,\mathrm{Cov}^{-1}\left\{\boldsymbol{P}_{t_0+\tau}\right\}\mathbb{E}\left\{\boldsymbol{P}_{t_0+\tau}\right\}}. \tag{299}$$

Proof. The proof is trivial using the presented results.

Proposition

The optimization problem

$$\hat{\boldsymbol{q}}\left(u\right) \equiv \arg \max_{\boldsymbol{q}'\boldsymbol{p}_{t_0}=v_{t_0},\mathsf{TE}^2(\boldsymbol{q})=u} \mathsf{EOP}\left(\boldsymbol{q}\right), \tag{300}$$

where $v_{t_0} > 0$ and $u \geq 0$, is called benchmark optimization problem with budget constraint.
The following optimization problem is equivalent to the benchmark-relative optimization problem with budget constraint:

$$\hat{\boldsymbol{\rho}}\left(u\right) \equiv \arg \max_{\boldsymbol{\rho}'\boldsymbol{p}_{t_0}=0,\mathbb{V}\{\Gamma_{\text{absolute value}}(\boldsymbol{\rho})\}=u} \mathbb{E}\left\{\Gamma_{\text{absolute value}}\left(\boldsymbol{\rho}\right)\right\}, \tag{301}$$

where $u \geq 0$, $\boldsymbol{\rho} \equiv \boldsymbol{q} - \boldsymbol{\beta}$ and

$$\boldsymbol{\beta} \equiv \frac{v_{t_0}}{\boldsymbol{p}'_{t_0}\tilde{\boldsymbol{\beta}}}\tilde{\boldsymbol{\beta}}. \tag{302}$$

Proposition

... The vector ρ is called relative bets because it represents the difference between the actual allocation q and the normalized benchmark allocation β.

Proof. The proof is trivial.

Proposition

Consider the benchmark-relative optimization problem. The benchmark-relative efficient frontier is given by

$$\hat{\boldsymbol{\rho}} = p \frac{\tilde{\boldsymbol{q}}_{\text{SR}} - \tilde{\boldsymbol{q}}_{\text{MV}}}{\mathbb{E}\left\{\Gamma_{\text{absolute value}}\left(\tilde{\boldsymbol{q}}_{\text{SR}}\right)\right\} - \mathbb{E}\left\{\Gamma_{\text{absolute value}}\left(\tilde{\boldsymbol{q}}_{\text{MV}}\right)\right\}}, \tag{303}$$

where $p \geq 0$,

$$\tilde{\boldsymbol{q}}_{\text{MV}} \equiv \frac{v_{t_0} \operatorname{Cov}^{-1}\left\{\boldsymbol{P}_{t_0+\tau}\right\} \boldsymbol{p}_{t_0}}{\boldsymbol{p}'_{t_0} \operatorname{Cov}^{-1}\left\{\boldsymbol{P}_{t_0+\tau}\right\} \boldsymbol{p}_{t_0}} \tag{304}$$

and

$$\tilde{\boldsymbol{q}}_{\text{SR}} \equiv \frac{v_{t_0} \operatorname{Cov}^{-1}\left\{\boldsymbol{P}_{t_0+\tau}\right\} \mathbb{E}\left\{\boldsymbol{P}_{t_0+\tau}\right\}}{\boldsymbol{p}'_{t_0} \operatorname{Cov}^{-1}\left\{\boldsymbol{P}_{t_0+\tau}\right\} \mathbb{E}\left\{\boldsymbol{P}_{t_0+\tau}\right\}}. \tag{305}$$

Proposition

... Alternatively, it is also possible to express the benchmark-relative efficient frontier as follows:

$$\begin{aligned}\hat{\boldsymbol{q}} = &\ \boldsymbol{\beta} \\ &+ \left(e - \mathbb{E}\left\{\Gamma_{\text{absolute value}}\left(\boldsymbol{\beta}\right)\right\}\right) \\ &\times \frac{\tilde{\boldsymbol{q}}_{\text{SR}} - \tilde{\boldsymbol{q}}_{\text{MV}}}{\mathbb{E}\left\{\Gamma_{\text{absolute value}}\left(\tilde{\boldsymbol{q}}_{\text{SR}}\right)\right\} - \mathbb{E}\left\{\Gamma_{\text{absolute value}}\left(\tilde{\boldsymbol{q}}_{\text{MV}}\right)\right\}},\end{aligned}$$

where $e \in \left[\mathbb{E}\left\{\Gamma_{\text{absolute value}}\left(\boldsymbol{\beta}\right)\right\}, \infty\right)$.
It is interesting to notice that the total return allocations $\tilde{q}$ are benchmark-relative allocations $\hat{q}$, where the benchmark allocation β is the global minimum variance allocation $\tilde{q}_{\text{MV}}$.

Proposition

Consider the plane:

$$(v, e) \equiv \left(\mathbb{V}\left\{\Gamma_{\text{absolute value}}\left(\boldsymbol{q}\right)\right\}, \mathbb{E}\left\{\Gamma_{\text{absolute value}}\left(\boldsymbol{q}\right)\right\}\right). \tag{306}$$

The total return efficient frontier is represented by the upper branch of the following parabola:

$$\tilde{\boldsymbol{q}} : v = \frac{A}{D}e^2 - \frac{2v_{t_0}B}{D}e + \frac{v_{t_0}^2 C}{D}, \tag{307}$$

while the benchmark-relative efficient frontier is represented by the upper branch of the following parabola:

$$\hat{\boldsymbol{q}} : v = \frac{A}{D}e^2 - \frac{2v_{t_0}B}{D}e + \frac{v_{t_0}^2 C}{D} + \delta_{\boldsymbol{\beta}}, \tag{308}$$

Proposition

... where

$$A \equiv \boldsymbol{p}_{t_0}' \mathsf{Cov}^{-1} \left\{ \boldsymbol{P}_{t_0+\tau} \right\} \boldsymbol{p}_{t_0}, \tag{309}$$

$$B \equiv \boldsymbol{p}_{t_0}' \mathsf{Cov}^{-1} \left\{ \boldsymbol{P}_{t_0+\tau} \right\} \mathbb{E} \left\{ \boldsymbol{P}_{t_0+\tau} \right\}, \tag{310}$$

$$C \equiv \mathbb{E} \left\{ \boldsymbol{P}_{t_0+\tau} \right\}' \mathsf{Cov}^{-1} \left\{ \boldsymbol{P}_{t_0+\tau} \right\} \mathbb{E} \left\{ \boldsymbol{P}_{t_0+\tau} \right\}, \tag{311}$$

$$D \equiv AC - B^2 \tag{312}$$

and

$$\delta_{\boldsymbol{\beta}} \equiv \mathbb{V} \left\{ \Gamma_{\text{absolute value}} \left(\boldsymbol{\beta} \right) \right\} - \frac{A}{D} \mathbb{E}^2 \left\{ \Gamma_{\text{absolute value}} \left(\boldsymbol{\beta} \right) \right\} \tag{313}$$

$$+ \frac{2 v_{t_0} B}{D} \mathbb{E} \left\{ \Gamma_{\text{absolute value}} \left(\boldsymbol{\beta} \right) \right\} - \frac{v_{t_0}^2 C}{D} \geq 0. \tag{314}$$

It is important to notice that the benchmark-relative efficient frontier is the total return efficient frontier with a shift equal to $\delta_{\boldsymbol{\beta}}$ to the right.

Proposition

Consider the plane:

$$(u, p) \equiv \left(\mathsf{TE}^2(\boldsymbol{q}), \mathsf{EOP}(\boldsymbol{q})\right). \tag{315}$$

The benchmark-relative efficient frontier is represented by the upper branch of the following parabola:

$$\hat{\boldsymbol{q}} : u = \frac{A}{D} p^2, \tag{316}$$

while the total return efficient frontier is represented by the upper branch of the following parabola:

$$\tilde{\boldsymbol{q}} : u = \frac{A}{D} p^2 + \delta_{\boldsymbol{\beta}}, \tag{317}$$

Proposition

... where

$$A \equiv \boldsymbol{p}_{t_0}' \text{Cov}^{-1}\left\{\boldsymbol{P}_{t_0+\tau}\right\} \boldsymbol{p}_{t_0}, \tag{318}$$

$$B \equiv \boldsymbol{p}_{t_0}' \text{Cov}^{-1}\left\{\boldsymbol{P}_{t_0+\tau}\right\} \mathbb{E}\left\{\boldsymbol{P}_{t_0+\tau}\right\}, \tag{319}$$

$$C \equiv \mathbb{E}\left\{\boldsymbol{P}_{t_0+\tau}\right\}' \text{Cov}^{-1}\left\{\boldsymbol{P}_{t_0+\tau}\right\} \mathbb{E}\left\{\boldsymbol{P}_{t_0+\tau}\right\}, \tag{320}$$

$$D \equiv AC - B^2 \tag{321}$$

and

$$\delta_{\boldsymbol{\beta}} \equiv \mathbb{V}\left\{\Gamma_{\text{absolute value}}\left(\boldsymbol{\beta}\right)\right\} - \frac{A}{D}\mathbb{E}^2\left\{\Gamma_{\text{absolute value}}\left(\boldsymbol{\beta}\right)\right\} \tag{322}$$

$$+\frac{2v_{t_0}B}{D}\mathbb{E}\left\{\Gamma_{\text{absolute value}}\left(\boldsymbol{\beta}\right)\right\} - \frac{v_{t_0}^2 C}{D} \geq 0. \tag{323}$$

Contrary to the previous proposition, in the (u, p) plane, it is important to notice that the total return efficient frontier is the benchmark-relative efficient frontier with a shift equal to $\delta_{\boldsymbol{\beta}}$ to the right.

Definition

Opportunity Cost Function. The opportunity cost function is given by

$$\mathcal{OC}\left(\boldsymbol{q}\right) \equiv \mathcal{S}\left(\boldsymbol{q}^{*}\right) - \mathcal{S}\left(\boldsymbol{q}\right), \boldsymbol{q} \in \mathcal{C}, \tag{324}$$

where $\mathcal{S}$ is the satisfaction function, $\mathcal{C}$ is the constraint set and $\boldsymbol{q}^{*}$ is the optimal allocation function:

$$\boldsymbol{q}^{*} \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}} \left\{\mathcal{S}\left(\boldsymbol{q}\right)\right\}. \tag{325}$$

Definition

Opportunity Cost Function (extended definition). The opportunity cost function is given by

$$\mathcal{OC}(\boldsymbol{q}) \equiv \mathcal{S}(\boldsymbol{q}^*) - \mathcal{S}(\boldsymbol{q}) + \mathcal{C}^+(\boldsymbol{q}), \tag{326}$$

where $\mathcal{S}$ is the satisfaction function, and $\boldsymbol{q}^*$ is the optimal allocation function:

$$\boldsymbol{q}^* \equiv \arg\max_{\boldsymbol{q} \in \mathcal{C}} \{\mathcal{S}(\boldsymbol{q})\}, \tag{327}$$

$\mathcal{C}$ is the constraint set and $\mathcal{C}^+ : \boldsymbol{q} \mapsto \mathbb{R}_+$ is the cost of violating the constraint set $\mathcal{C}$.

Property

$$\mathcal{OC}\left(\boldsymbol{q}\right) \geq 0, \forall \boldsymbol{q} \in \mathcal{C}. \tag{328}$$

Proof. The proof follows from the definition of $\mathcal{OC}$.

Remark

Generically, an additional secondary satisfaction function $\tilde{\mathcal{S}}$ is written in terms of a constraint $\tilde{\mathcal{C}}$. It is natural to write:

$$\tilde{\mathcal{C}} : \tilde{s} - \tilde{\mathcal{S}}\left(\boldsymbol{q}\right) \leq 0, \tag{329}$$

where $\tilde{s}$ is a minimum acceptable value for the satisfaction function $\tilde{\mathcal{S}}$. In addition, it is usual to define the cost of violating the constraint set $\tilde{\mathcal{C}}$ as follows:

$$\mathcal{C}^{+}\left(\boldsymbol{q}\right) \equiv \max\left\{0, \tilde{\boldsymbol{s}} - \tilde{\mathcal{S}}\left(\boldsymbol{q}\right)\right\}. \tag{330}$$

Remark

The market vector $\boldsymbol{M}_{t_0+\tau}$ is a random vector. By definition, the market vector depends on the random vector of prices: $\boldsymbol{M}_{t_0+\tau} \equiv \boldsymbol{a} + \boldsymbol{b}\boldsymbol{P}_{t_0+\tau}$. Usually, the $\boldsymbol{P}_{t_0+\tau}$ is not directly modeled. The modeled random vector $\boldsymbol{X}_{t_0+\tau}$ is called market invariant. In addition, the distribution of the market invariant is described by the vector of parameters $\boldsymbol{\theta}$. Consequently,

$$\boldsymbol{\theta} \mapsto \boldsymbol{X}^{\boldsymbol{\theta}}_{t_0+\tau} \mapsto \boldsymbol{P}^{\boldsymbol{\theta}}_{t_0+\tau} \mapsto \boldsymbol{M}^{\boldsymbol{\theta}}_{t_0+\tau}. \tag{331}$$

The primary satisfaction function is such that

$$\left(\boldsymbol{q}, \boldsymbol{M}^{\boldsymbol{\theta}}_{t_0+\tau}\right) \mapsto \Gamma^{\boldsymbol{\theta}}_{*}\left(\boldsymbol{q}\right) \mapsto \mathcal{S}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right) \tag{332}$$

and any additional secondary satisfaction function $\tilde{\mathcal{S}}^{\boldsymbol{\theta}}_{i}$ considered as a constraint is such that

$$\left(\boldsymbol{q}, \boldsymbol{M}^{\boldsymbol{\theta}}_{t_0+\tau}\right) \mapsto \tilde{\Gamma}^{\boldsymbol{\theta}}_{i}\left(\boldsymbol{q}\right) \mapsto \tilde{\mathcal{S}}^{\boldsymbol{\theta}}_{i}\left(\boldsymbol{q}\right) \mapsto \tilde{\mathcal{C}}^{\boldsymbol{\theta}}_{i}. \tag{333}$$

Remark

... Considering m additional secondary satisfaction functions and the set of non-satisfaction function-related constraints $\tilde{\mathcal{C}}^{\boldsymbol{\theta}}$, the final constraint set is given by

$$\mathcal{C}^{\boldsymbol{\theta}} = \tilde{\mathcal{C}}^{\boldsymbol{\theta}} \cup \left(\bigcup_{i=1}^{m} \tilde{\mathcal{C}}_i^{\boldsymbol{\theta}} \right). \tag{334}$$

Finally, the optimal allocation depends on the market parameters and it is given by

$$\boldsymbol{q}^{*,\boldsymbol{\theta}} \equiv \arg\max_{\boldsymbol{q} \in \mathcal{C}^{\boldsymbol{\theta}}} \left\{ \mathcal{S}^{\boldsymbol{\theta}} \left(\boldsymbol{q} \right) \right\} \tag{335}$$

and the opportunity cost is given by

$$\mathcal{OC}^{\boldsymbol{\theta}} \left(\boldsymbol{q} \right) \equiv \mathcal{S}^{\boldsymbol{\theta}} \left(\boldsymbol{q}^{*,\boldsymbol{\theta}} \right) - \mathcal{S}^{\boldsymbol{\theta}} \left(\boldsymbol{q} \right) + \sum_{i=1}^{m} \mathcal{C}_i^{+,\boldsymbol{\theta}} \left(\boldsymbol{q} \right). \tag{336}$$

Remark

... It is also usual to define

$$C_i^{+,\boldsymbol{\theta}}\left(\boldsymbol{q}\right) \equiv \max\left\{0, \tilde{s}_i - \tilde{\mathcal{S}}_i^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right)\right\}. \tag{337}$$

Example

The linear returns are given by

$$\boldsymbol{L}^{\boldsymbol{\theta}}_{t_0+\tau} \equiv \mathsf{diag}^{-1}\left(\boldsymbol{p}_{t_0}\right)\boldsymbol{P}^{\boldsymbol{\theta}}_{t_0+\tau} - \mathbf{1}_{n\times 1}. \tag{338}$$

We consider the linear returns as our market invariant

$$\boldsymbol{X}^{\boldsymbol{\theta}}_{t_0+\tau} \equiv \boldsymbol{L}^{\boldsymbol{\theta}}_{t_0+\tau}. \tag{339}$$

Assuming

$$\boldsymbol{L}^{\boldsymbol{\theta}}_{t_0+\tau} \sim \mathsf{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right), \tag{340}$$

it follows that

$$\boldsymbol{\theta} = \left\{\boldsymbol{\mu}, \boldsymbol{\Sigma}\right\}. \tag{341}$$

Example

... Consequently,

$$\boldsymbol{P}^{\boldsymbol{\theta}}_{t_0+\tau} \sim \mathsf{N}\left(\boldsymbol{\xi}\left(\boldsymbol{\mu}\right), \boldsymbol{\Phi}\left(\boldsymbol{\Sigma}\right)\right), \tag{342}$$

where

$$\boldsymbol{\xi}\left(\boldsymbol{\mu}\right) \equiv \mathrm{diag}\left(\boldsymbol{p}_{t_0}\right)\left(\mathbf{1}_{n\times 1} + \boldsymbol{\mu}\right) \tag{343}$$

and

$$\boldsymbol{\Phi}\left(\boldsymbol{\Sigma}\right) \equiv \mathrm{diag}\left(\boldsymbol{p}_{t_0}\right)\boldsymbol{\Sigma}\,\mathrm{diag}\left(\boldsymbol{p}_{t_0}\right). \tag{344}$$

Assuming an absolute value goal function,

$$\Gamma^{\boldsymbol{\theta}}_{\text{absolute value}}\left(\boldsymbol{q}\right) \equiv \boldsymbol{q}'\boldsymbol{P}^{\boldsymbol{\theta}}_{t_0+\tau}, \tag{345}$$

Example

... the certainty-equivalent of the exponential utility function is given by

$$\mathcal{CE}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right) = \boldsymbol{q}'\text{diag}\left(\boldsymbol{p}_{t_0}\right)\left(\mathbf{1}_{n\times 1} + \boldsymbol{\mu}\right) - \frac{1}{2b}\boldsymbol{q}'\text{diag}\left(\boldsymbol{p}_{t_0}\right)\boldsymbol{\Sigma}\text{diag}\left(\boldsymbol{p}_{t_0}\right)\boldsymbol{q}, \quad (346)$$

where $b > 0$. Here, we consider the primary satisfaction function given by

$$\mathcal{S}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right) \equiv \mathcal{CE}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right). \quad (347)$$

In addition, assuming the profit value goal function,

$$\Gamma^{\boldsymbol{\theta}}_{\text{profit value}}\left(\boldsymbol{q}\right) \equiv \boldsymbol{q}'\left(\boldsymbol{P}^{\boldsymbol{\theta}}_{t_0+\tau} - \boldsymbol{q}\right), \quad (348)$$

the VaR is given by

$$\text{VaR}^{\boldsymbol{\theta}}_{c}\left(\boldsymbol{q}\right) = -\boldsymbol{\mu}'\text{diag}\left(\boldsymbol{p}_{t_0}\right)\boldsymbol{q} + \sqrt{2\boldsymbol{q}'\text{diag}\left(\boldsymbol{p}_{t_0}\right)\boldsymbol{\Sigma}\,\text{diag}\left(\boldsymbol{p}_{t_0}\right)\boldsymbol{q}}\,\text{erf}^{-1}\left(2c-1\right). \quad (349)$$

Example

... Considering $\text{VaR}_c^{\boldsymbol{\theta}}$ as a secondary goal function, the corresponding constraint is given by

$$\mathcal{C}_2^{\boldsymbol{\theta}} : 0 \geq -\gamma v_{t_0} - \boldsymbol{\mu}' \text{diag}\left(\boldsymbol{p}_{t_0}\right) \boldsymbol{q} + \sqrt{2\boldsymbol{q}' \text{diag}\left(\boldsymbol{p}_{t_0}\right) \boldsymbol{\Sigma}\, \text{diag}\left(\boldsymbol{p}_{t_0}\right) \boldsymbol{q}}\, \text{erf}^{-1}\left(2c - 1\right) \tag{350}$$

where $\gamma, v_{t_0} \in \mathbb{R}_+$.
Solving the optimization problem

$$\boldsymbol{q}^{*,\boldsymbol{\theta}} \equiv \arg\max_{\boldsymbol{q} \in \mathcal{C}^{\boldsymbol{\theta}}} \left\{ \mathcal{S}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right) \right\}, \tag{351}$$

where $\mathcal{C}^{\boldsymbol{\theta}} = \mathcal{C}_1^{\boldsymbol{\theta}} \cup \mathcal{C}_2^{\boldsymbol{\theta}}$, $\mathcal{C}_1^{\boldsymbol{\theta}} : \boldsymbol{q}' \boldsymbol{p}_{t_0} = v_{t_0}$, it follows that

$$\boldsymbol{q}^{*,\boldsymbol{\theta}} = \text{diag}^{-1}\left(\boldsymbol{p}_{t_0}\right) \boldsymbol{\Sigma}^{-1} \left(b\boldsymbol{\mu} + \frac{v_{t_0} - b\mathbf{1}_{n\times1}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{1}_{n\times1}' \boldsymbol{\Sigma}^{-1} \mathbf{1}_{n\times1}} \mathbf{1}_{n\times1} \right) \tag{352}$$

Example

... and

$$\mathcal{S}^{\boldsymbol{\theta}}\left(\boldsymbol{q}^{*,\boldsymbol{\theta}}\right)=\frac{b}{2}\left(C-\frac{B^2}{A}\right)+v_{t_0}\left(1+\frac{B}{A}-\frac{v_{t_0}}{b}\frac{1}{2A}\right), \tag{353}$$

where

$$A\equiv\mathbf{1}_{n\times1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}_{n\times1}, \tag{354}$$

$$B\equiv\mathbf{1}_{n\times1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \tag{355}$$

and

$$C\equiv\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}. \tag{356}$$

Remark

Considering the market invariant $\boldsymbol{X}_t$, the information of the market available until date t_0 is given by

$$i_{t_0} = \{\boldsymbol{x}_{t_0-\tau}, \boldsymbol{x}_{t_0-\tau+1}, \ldots, \boldsymbol{x}_{t_0-1}, \boldsymbol{x}_{t_0}\}\,. \tag{357}$$

Then,

$$i_{t_0} \mapsto \hat{\boldsymbol{\theta}}, \tag{358}$$

where $\hat{\boldsymbol{\theta}}$ is the estimated market vector. Obviously, $\hat{\boldsymbol{\theta}}$ is not necessarily equal to the true market vector $\boldsymbol{\theta}$. Consequently, $\boldsymbol{q}^{*,\boldsymbol{\theta}}$ is not necessarily equal to $\boldsymbol{q}^{*,\hat{\boldsymbol{\theta}}}$.

Definition

Feasible set of market parameters. A feasible set of market parameters $\boldsymbol{\Theta}$ is such that

$$\boldsymbol{\theta} \in \boldsymbol{\Theta}, \tag{359}$$

where $\boldsymbol{\theta}$ is the true market parameter.

Definition

Allocation optimality. An allocation $\boldsymbol{q}$ is optimal if and only if

$$\mathcal{OC}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right) = 0. \tag{360}$$

Definition

Allocation optimality in the presence of uncertainty. An allocation $\boldsymbol{q}$ is optimal in the presence of uncertainty if and only if $\mathcal{OC}^{\tilde{\boldsymbol{\theta}}}\left(\boldsymbol{q}\right)$ is very close to zero $\forall \tilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}$.

Example

Consider the linear return $\boldsymbol{L}_t^{\boldsymbol{\theta}}$ as the market invariant and

$$\boldsymbol{L}_t^{\boldsymbol{\theta}} \sim \mathsf{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right). \tag{361}$$

In addition, consider the correlation matrix

$$\boldsymbol{R}\left(\rho\right) \equiv \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & \ddots & & \vdots \\ \vdots & & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}, \tag{362}$$

the standard deviation vector

$$\boldsymbol{\sigma}\left(\rho\right) \equiv \left(1 + \xi\rho\right)\boldsymbol{v}, \tag{363}$$

Example

... the mean vector

$$\boldsymbol{\mu}(\rho) \equiv p\,\boldsymbol{\sigma}(\rho), \tag{364}$$

$$\rho \in \boldsymbol{\Theta} \equiv [0,1), \tag{365}$$

$\boldsymbol{v}$ is a fixed vector of volatilities and $\xi, p \in \mathbb{R}_{+}$. Consequently,

$$\boldsymbol{L}_t^{\rho} \sim \mathsf{N}(\boldsymbol{\mu}(\rho), \boldsymbol{\Sigma}(\rho)), \tag{366}$$

where $\boldsymbol{\Sigma}(\rho) = \operatorname{diag}(\boldsymbol{\sigma}(\rho))\,\boldsymbol{R}(\rho)\operatorname{diag}(\boldsymbol{\sigma}(\rho))$.

Example

... Finally,

$$\rho \mapsto \mathcal{S}^{\rho}\left(\boldsymbol{q}\right), \rho \in \boldsymbol{\Theta} \equiv [0,1)\,, \tag{367}$$

$$\rho \mapsto \boldsymbol{q}^{*,\rho}, \rho \in \boldsymbol{\Theta} \equiv [0,1)\,, \tag{368}$$

$$\rho \mapsto \mathcal{S}^{\rho}\left(\boldsymbol{q}^{*,\rho}\right), \rho \in \boldsymbol{\Theta} \equiv [0,1)\,, \tag{369}$$

$$\rho \mapsto \mathcal{OC}^{\rho}\left(\boldsymbol{q}\right), \rho \in \boldsymbol{\Theta} \equiv [0,1)\,, \tag{370}$$

and

$$\rho \mapsto \mathcal{C}^{+,\rho}\left(\boldsymbol{q}\right), \rho \in \boldsymbol{\Theta} \equiv [0,1)\,. \tag{371}$$

Definition

Prior allocation function. The prior allocation function is given by

$$\boldsymbol{q}_p : [\mathcal{P}] \mapsto \mathbb{R}^n, \tag{372}$$

where n is the number of assets and $\mathcal{P}$ is the private information of the investor. Notice that $\boldsymbol{q}_p$ does not use the publicly available market information i_{t_0}. Since $i_{t_0} \mapsto \hat{\boldsymbol{\theta}}$, $\boldsymbol{q}_p$ does not use the estimation of the market vector $\hat{\boldsymbol{\theta}}$.

Remark

In the remaining of the text, we are going to denote by $\boldsymbol{\theta}$ any element of $\boldsymbol{\Theta}$ and not only the true market parameters.

Definition

Prior allocation optimality in the presence of uncertainty. A prior allocation $\boldsymbol{q}_p$ is optimal in the presence of uncertainty if and only if $\mathcal{OC}^{\boldsymbol{\theta}}\left(\boldsymbol{q}_p\right)$ is very close to zero $\forall\boldsymbol{\theta}\in\boldsymbol{\Theta}$.

Example

Equally-weighted allocation function. The equally-weighted allocation function is a prior allocation function and is given by

$$\boldsymbol{q}_p \equiv \frac{v_{t_0}}{n}\text{diag}^{-1}\left(\boldsymbol{p}_{t_0}\right)\mathbf{1}_{n\times 1}, \tag{373}$$

where n is the number of assets and v_{t_0} is the initial allocated financial value.

In the context of example 131, the primary satisfaction function is given by

$$\mathcal{S}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}\left(\boldsymbol{q}_p\right) = \mathcal{CE}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}\left(\boldsymbol{q}_p\right) = v_{t_0}\left(1 + \frac{\boldsymbol{\mu}'\mathbf{1}_{n\times 1}}{n}\right) - \frac{v_{t_0}^2}{2b}\frac{\mathbf{1}'_{n\times 1}\boldsymbol{\Sigma}\mathbf{1}_{n\times 1}}{n^2} \tag{374}$$

Example

... and the opportunity cost is given by

$$\mathcal{OC}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}\left(\boldsymbol{q}_p\right) = \mathcal{CE}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}\left(\boldsymbol{q}^{*,\theta}\right) - \mathcal{CE}^{\boldsymbol{\mu},\boldsymbol{\Sigma}}\left(\boldsymbol{q}_p\right) + \mathcal{C}^{+,\boldsymbol{\mu},\boldsymbol{\Sigma}}\left(\boldsymbol{q}_p\right), \tag{375}$$

where

$$\mathcal{C}^{+,\boldsymbol{\mu},\boldsymbol{\Sigma}}\left(\boldsymbol{q}_p\right) = v_{t_0} \max\left\{0, -\gamma - \frac{\mathbf{1}'_{n\times 1}\boldsymbol{\mu}}{n} + \frac{\sqrt{2\mathbf{1}'_{n\times 1}\boldsymbol{\Sigma}\mathbf{1}_{n\times 1}}}{n}\mathrm{erf}^{-1}\left(2c-1\right)\right\}. \tag{376}$$

Definition

Sample-based allocation function. The sample-based allocation function is given by

$$\boldsymbol{q}_s : [i_{t_0}] \mapsto \mathbb{R}^n, \tag{377}$$

where n is the number of assets and i_{t_0} is the publicly available market information until date t_0. Notice that $\boldsymbol{q}_s$ does not use the private information of the investor $\mathcal{P}$.

Remark

The estimated market vector $\hat{\boldsymbol{\theta}}$ is such that

$$i_{t_0} \mapsto \hat{\boldsymbol{\theta}}. \tag{378}$$

Notice that

$$i_{t_0} \equiv \{\boldsymbol{x}_{t_0-\tau}, \boldsymbol{x}_{t_0-\tau+1}, \ldots, \boldsymbol{x}_{t_0-1}, \boldsymbol{x}_{t_0}\} \tag{379}$$

is a particular realization of a sequence of the market invariant random variable $\boldsymbol{X}_t$. Consequently, it is possible to define the publicly available market information until date t_0 as a random variable:

$$I_{t_0}^{\boldsymbol{\theta}} \equiv \{\boldsymbol{X}_{t_0-\tau}, \boldsymbol{X}_{t_0-\tau+1}, \ldots, \boldsymbol{X}_{t_0-1}, \boldsymbol{X}_{t_0}\}. \tag{380}$$

Obviously, the estimator $\hat{\boldsymbol{\theta}}\left(i_{t_0}\right)$ is deterministic while the estimator $\hat{\boldsymbol{\theta}}\left(I_{t_0}\right)$ is a random variable.

Example

Consider the optimal allocation function from equation 352:

$$\boldsymbol{q}^{*,\boldsymbol{\theta}} = \text{diag}^{-1}\left(\boldsymbol{p}_{t_0}\right)\boldsymbol{\Sigma}^{-1}\left(b\boldsymbol{\mu} + \frac{v_{t_0} - b\mathbf{1}_{n\times 1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}_{n\times 1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}_{n\times 1}}\mathbf{1}_{n\times 1}\right), \tag{381}$$

where the unknown market parameters vector is given by

$$\boldsymbol{\theta} = \left\{\boldsymbol{\mu}, \boldsymbol{\Sigma}\right\}. \tag{382}$$

It is possible to obtain $\hat{\boldsymbol{\theta}}\left(i_{t_0}\right) = \left\{\hat{\boldsymbol{\mu}}\left(i_{t_0}\right), \hat{\boldsymbol{\Sigma}}\left(i_{t_0}\right)\right\}$ as follows:

$$\hat{\boldsymbol{\mu}}\left(i_{t_0}\right) = \frac{1}{\tau+1}\sum_{t=1}^{\tau+1}\boldsymbol{l}_{t_0-t+1} \tag{383}$$

Example

... and

$$\hat{\boldsymbol{\Sigma}}\left(i_{t_0}\right)=\frac{1}{\tau+1}\sum_{t=1}^{\tau+1}\left(\boldsymbol{l}_{t_0-t+1}-\hat{\boldsymbol{\mu}}\right)\left(\boldsymbol{l}_{t_0-t+1}-\hat{\boldsymbol{\mu}}\right)'. \tag{384}$$

Then,

$$\boldsymbol{q}^{*,\hat{\boldsymbol{\theta}}\left(i_{t_0}\right)}=\mathrm{diag}^{-1}\left(\boldsymbol{p}_{t_0}\right)\hat{\boldsymbol{\Sigma}}^{-1}\left(i_{t_0}\right)\times \tag{385}$$

$$\left(b\hat{\boldsymbol{\mu}}\left(i_{t_0}\right)+\frac{v_{t_0}-b\mathbf{1}'_{n\times 1}\hat{\boldsymbol{\Sigma}}^{-1}\left(i_{t_0}\right)\hat{\boldsymbol{\mu}}\left(i_{t_0}\right)}{\mathbf{1}'_{n\times 1}\hat{\boldsymbol{\Sigma}}^{-1}\left(i_{t_0}\right)\mathbf{1}_{n\times 1}}\mathbf{1}_{n\times 1}\right). \tag{386}$$

It is possible to model

$$\hat{\boldsymbol{\theta}}\left(I_{t_0}^{\hat{\boldsymbol{\mu}}\left(i_{t_0}\right),\hat{\boldsymbol{\Sigma}}\left(i_{t_0}\right)}\right)=\left\{\hat{\boldsymbol{\mu}}\left(I_{t_0}^{\hat{\boldsymbol{\mu}}\left(i_{t_0}\right),\hat{\boldsymbol{\Sigma}}\left(i_{t_0}\right)}\right),\hat{\boldsymbol{\Sigma}}\left(I_{t_0}^{\hat{\boldsymbol{\mu}}\left(i_{t_0}\right),\hat{\boldsymbol{\Sigma}}\left(i_{t_0}\right)}\right)\right\} \tag{387}$$

as follows:

Example

...

$$\hat{\boldsymbol{\mu}}\left(I_{t_0}^{\hat{\boldsymbol{\mu}}(i_{t_0}),\hat{\boldsymbol{\Sigma}}(i_{t_0})}\right) \sim \mathrm{N}\left(\hat{\boldsymbol{\mu}}\left(i_{t_0}\right), \frac{\hat{\boldsymbol{\Sigma}}\left(i_{t_0}\right)}{\tau+1}\right) \tag{388}$$

and

$$(\tau+1)\,\hat{\boldsymbol{\Sigma}}\left(I_{t_0}^{\hat{\boldsymbol{\mu}}(i_{t_0}),\hat{\boldsymbol{\Sigma}}(i_{t_0})}\right) \sim \mathrm{W}\left(\tau, \hat{\boldsymbol{\Sigma}}\left(i_{t_0}\right)\right), \tag{389}$$

where W is the Wishart distribution with τ degrees of freedom. Increasing the number of observations $\tau \to \infty$, it is possible to show that

$$\hat{\boldsymbol{\mu}}\left(I_{t_0}^{\hat{\boldsymbol{\mu}}(i_{t_0}),\hat{\boldsymbol{\Sigma}}(i_{t_0})}\right) \to \hat{\boldsymbol{\mu}}\left(i_{t_0}\right) \tag{390}$$

and

$$\hat{\boldsymbol{\Sigma}}\left(I_{t_0}^{\hat{\boldsymbol{\mu}}(i_{t_0}),\hat{\boldsymbol{\Sigma}}(i_{t_0})}\right) \to \hat{\boldsymbol{\Sigma}}\left(i_{t_0}\right). \tag{391}$$

Example

... Obviously, the distribution of

$$\boldsymbol{q}^{*,\hat{\boldsymbol{\theta}}\left(I_{t_0}^{\hat{\boldsymbol{\mu}}\left(i_{t_0}\right),\hat{\boldsymbol{\Sigma}}\left(i_{t_0}\right)}\right)} \tag{392}$$

is not known analytically. Consequently, it is obtained through Monte Carlo simulation.

Remark

Generically, the asset allocation optimization problem is given by

$$\boldsymbol{q}^{*,\boldsymbol{\theta}} \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}^{\boldsymbol{\theta}}} \left\{ \mathcal{S}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right) \right\} \tag{393}$$

and the certainty-equivalent satisfaction function is given by

$$\mathcal{S}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right) \equiv \mathcal{CE}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right) \equiv \mathcal{U}^{-1}\left(\mathbb{E}\left\{\mathcal{U}\left(\Gamma_{*}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right)\right)\right\}\right) \tag{394}$$

with

$$\Gamma_{*}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right) \equiv \boldsymbol{q}'\boldsymbol{M}^{\boldsymbol{\theta}}. \tag{395}$$

Remark

... Since $\mathbb{E}\left\{\mathcal{U}\left(\Gamma_{*}^{\boldsymbol{\theta}}\left(\boldsymbol{q}_{1}\right)\right)\right\} > \mathbb{E}\left\{\mathcal{U}\left(\Gamma_{*}^{\boldsymbol{\theta}}\left(\boldsymbol{q}_{2}\right)\right)\right\} \Leftrightarrow \mathcal{CE}^{\boldsymbol{\theta}}\left(\boldsymbol{q}_{1}\right) > \mathcal{CE}^{\boldsymbol{\theta}}\left(\boldsymbol{q}_{2}\right)$ and $\mathbb{E}\left\{\mathcal{U}\left(\Gamma_{*}^{\boldsymbol{\theta}}\left(\boldsymbol{q}_{1}\right)\right)\right\} = \mathbb{E}\left\{\mathcal{U}\left(\Gamma_{*}^{\boldsymbol{\theta}}\left(\boldsymbol{q}_{2}\right)\right)\right\} \Leftrightarrow \mathcal{CE}^{\boldsymbol{\theta}}\left(\boldsymbol{q}_{1}\right) = \mathcal{CE}^{\boldsymbol{\theta}}\left(\boldsymbol{q}_{2}\right)$, the optimization problem becomes

$$\boldsymbol{q}^{*,\boldsymbol{\theta}} \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}^{\boldsymbol{\theta}}}\left\{\mathcal{S}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right)\right\} \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}^{\boldsymbol{\theta}}}\left\{\mathcal{CE}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right)\right\} \tag{396}$$

$$= \arg\max_{\boldsymbol{q}\in\mathcal{C}^{\boldsymbol{\theta}}}\left\{\mathbb{E}\left\{\mathcal{U}\left(\Gamma_{*}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right)\right)\right\}\right\} \tag{397}$$

$$= \arg\max_{\boldsymbol{q}\in\mathcal{C}^{\boldsymbol{\theta}}}\left\{\int \mathcal{U}\left(\boldsymbol{q}'\boldsymbol{m}\right) \mathsf{f}_{\boldsymbol{M}^{\boldsymbol{\theta}}}\left(\boldsymbol{m}|\boldsymbol{\theta}\right) d\boldsymbol{m}\right\}, \tag{398}$$

where $\mathsf{f}_{\boldsymbol{M}^{\boldsymbol{\theta}}}$ is the probability density function of the market vector $\boldsymbol{M}^{\boldsymbol{\theta}}$.

Definition

Bayesian optimization problem with certainty-equivalent satisfaction function. The Bayesian optimization problem with certainty-equivalent satisfaction function is given by

$$\boldsymbol{q}_B^* \left[i_{t_0}, \mathcal{P} \right] \equiv \arg\max_{\boldsymbol{q} \in \mathcal{C}} \left\{ \int \mathbb{E} \left\{ \mathcal{U} \left(\Gamma_*^{\boldsymbol{\theta}} \left(\boldsymbol{q} \right) \right) \right\} \mathsf{f}_{\boldsymbol{\Theta}} \left(\boldsymbol{\theta} | i_{t_0}, \mathcal{P} \right) d\boldsymbol{\theta} \right\}, \quad (399)$$

where $\mathcal{C}$ is the constraints set, i_{t_0} is the publicly available information (e.g. market invariant historical data), $\mathcal{P}$ is the private information (e.g. experience of the investor) and $\mathsf{f}_{\boldsymbol{\Theta}}$ is the posterior probability density function.

Remark

Depending on the context, the notation $\boldsymbol{\Theta}$ represents the random vector of parameters or the domain of the possible values for the random vector of parameters. In the remaining of the text, the meaning of the notation $\boldsymbol{\Theta}$ is clear from the context.

Definition

Market predictive distribution. The market predictive distribution is given by

$$\mathsf{f}_{\boldsymbol{M}}\left(\boldsymbol{m}|i_{t_0},\mathcal{P}\right) \equiv \int \mathsf{f}_{\boldsymbol{M}^{\boldsymbol{\theta}}}\left(\boldsymbol{m}|\boldsymbol{\theta}\right) \mathsf{f}_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}|i_{t_0},\mathcal{P}\right) d\boldsymbol{\theta}. \tag{400}$$

Remark

Using the market predictive distribution, it is possible to write the Bayesian optimization problem with certainty-equivalent satisfaction function as follows:

$$\boldsymbol{q}_B^* \left[i_{t_0}, \mathcal{P} \right] \equiv \underset{\boldsymbol{q} \in \mathcal{C}}{\arg\max} \left\{ \int \mathbb{E} \left\{ \mathcal{U} \left(\Gamma_*^{\boldsymbol{\theta}} \left(\boldsymbol{q} \right) \right) \right\} \mathsf{f}_{\boldsymbol{\Theta}} \left(\boldsymbol{\theta} | i_{t_0}, \mathcal{P} \right) d\boldsymbol{\theta} \right\} \tag{401}$$

$$= \underset{\boldsymbol{q} \in \mathcal{C}}{\arg\max} \left\{ \int \mathcal{U} \left(\boldsymbol{q}' \boldsymbol{m} \right) \int \mathsf{f}_{\boldsymbol{M}^{\boldsymbol{\theta}}} \left(\boldsymbol{m} | \boldsymbol{\theta} \right) \mathsf{f}_{\boldsymbol{\Theta}} \left(\boldsymbol{\theta} | i_{t_0}, \mathcal{P} \right) d\boldsymbol{\theta} d\boldsymbol{m} \right\} \tag{402}$$

$$= \underset{\boldsymbol{q} \in \mathcal{C}}{\arg\max} \left\{ \int \mathcal{U} \left(\boldsymbol{q}' \boldsymbol{m} \right) \mathsf{f}_{\boldsymbol{M}} \left(\boldsymbol{m} | i_{t_0}, \mathcal{P} \right) d\boldsymbol{m} \right\}. \tag{403}$$

Consequently, it is possible to write:

$$\boldsymbol{q}_B^* \left[i_{t_0}, \mathcal{P} \right] = \underset{\boldsymbol{q} \in \mathcal{C}}{\arg\max} \left\{ \mathbb{E} \left\{ \mathcal{U} \left(\Gamma_*^{i_{t_0}, \mathcal{P}} \left(\boldsymbol{q} \right) \right) \right\} \right\}. \tag{404}$$

Example

Considering an exponential utility function, the expected value of the utility function is given by

$$\begin{aligned}
\mathbb{E}\left\{\mathcal{U}\left(\Gamma_{*}^{\boldsymbol{\theta}}(\boldsymbol{q})\right)\right\} =& \mathbb{E}\left\{\mathcal{U}\left(\boldsymbol{q}'\boldsymbol{M}^{\boldsymbol{\theta}}\right)\right\} && (405)\\
=& -\mathbb{E}\left\{e^{-\frac{1}{b}\boldsymbol{q}'\boldsymbol{M}}\right\} && (406)\\
=& -\int e^{-\frac{1}{b}\boldsymbol{q}'\boldsymbol{m}} \mathsf{f}_{\boldsymbol{M}^{\boldsymbol{\theta}}}(\boldsymbol{m}|\boldsymbol{\theta})\, d\boldsymbol{m} && (407)\\
=& -\phi_{\boldsymbol{M}^{\boldsymbol{\theta}}}\left(\frac{i}{b}\boldsymbol{q}\right), && (408)
\end{aligned}$$

where $b > 0$ and $\phi_{\boldsymbol{M}^{\boldsymbol{\theta}}}$ is the characteristic function of $\boldsymbol{M}^{\boldsymbol{\theta}}$.

Example

... Assuming the market is normally distributed,

$$\boldsymbol{M}^{\boldsymbol{\theta}} \sim \mathsf{N}\left(\boldsymbol{\xi}, \boldsymbol{\Phi}\right), \boldsymbol{\theta} \equiv \left\{\boldsymbol{\xi}, \boldsymbol{\Phi}\right\}, \tag{409}$$

it follows that

$$\phi_{\boldsymbol{M}^{\boldsymbol{\theta}}}\left(\boldsymbol{x}\right) = e^{i\boldsymbol{\xi}'\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}'\boldsymbol{\Phi}\boldsymbol{x}}. \tag{410}$$

Consequently, the asset allocation optimization problem is given by

$$\boldsymbol{q}^{*,\{\boldsymbol{\xi},\boldsymbol{\Phi}\}} \equiv \arg \max_{\boldsymbol{q} \in \mathcal{C}^{\{\boldsymbol{\xi},\boldsymbol{\Phi}\}}} \left\{ -e^{-\frac{1}{b}\left(\boldsymbol{\xi}'\boldsymbol{q} - \frac{1}{2b}\boldsymbol{q}'\boldsymbol{\Phi}\boldsymbol{q}\right)} \right\}. \tag{411}$$

Example

... To use the Bayesian approach, we assume that the covariance matrix Φ is known and the posterior distribution of the expected value of the market is

$$\boldsymbol{\xi} \sim \mathsf{N}\left(\boldsymbol{\xi}_1, \frac{\Phi}{T_1}\right), \tag{412}$$

where

$$T_1 \equiv T_0 + T, \tag{413}$$

$$\boldsymbol{\xi}_1 \equiv \frac{1}{T_1}\left(T_0\boldsymbol{\xi}_0 + T\hat{\boldsymbol{\xi}}\right), \tag{414}$$

$$\hat{\boldsymbol{\xi}} \equiv \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{m}_t, \tag{415}$$

T represents the size of the sample used to estimate $\hat{\boldsymbol{\xi}}$, T_0 represents the confidence level on the investor private information $\mathcal{P}$ and $\boldsymbol{\xi}_0$ represents the investor private information on $\mathbb{E}\{\boldsymbol{\xi}\}$.

Example

... Finally, the Bayesian asset allocation optimization problem is given by

$$\boldsymbol{q}_B^* \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}} \left\{ -e^{-\frac{1}{b}\left(\boldsymbol{q}'\boldsymbol{\xi}_1 - \frac{1+T_1}{2bT_1}\boldsymbol{q}'\boldsymbol{\Phi}\boldsymbol{q}\right)} \right\}. \tag{416}$$

Definition

Expected value-based location classical-equivalent estimator of $\boldsymbol{\theta}$. The expected value-based location classical-equivalent estimator of the parameter $\boldsymbol{\theta}$ is given by

$$\hat{\boldsymbol{\theta}}_{CE}\left[i_{t_0},\mathcal{P}\right] \equiv \mathbb{E}_{i_{t_0},\mathcal{P}}\left\{\boldsymbol{\Theta}\right\} \tag{417}$$

$$\equiv \int \boldsymbol{\theta} \mathrm{f}_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}|i_{t_0},\mathcal{P}\right) d\boldsymbol{\theta}. \tag{418}$$

Definition

Mode-based location classical-equivalent estimator of $\boldsymbol{\theta}$. The mode value-based location classical-equivalent estimator of the parameter $\boldsymbol{\theta}$ is given by

$$\hat{\boldsymbol{\theta}}_{CE}\left[i_{t_0},\mathcal{P}\right] \equiv \mathbb{M}_{i_{t_0},\mathcal{P}}\left\{\boldsymbol{\Theta}\right\} \tag{419}$$

$$\equiv \arg\max_{\boldsymbol{\theta}} \mathrm{f}_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}|i_{t_0},\mathcal{P}\right). \tag{420}$$

Definition

Classical-equivalent Bayesian optimization problem. The classical-equivalent Bayesian optimization problem is given by

$$\boldsymbol{q}^*_{CE}\left[i_{t_0},\mathcal{P}\right] \equiv \arg \max_{\boldsymbol{q}\in\mathcal{C}^{\hat{\boldsymbol{\theta}}_{CE}\left[i_{t_0},\mathcal{P}\right]}} \left\{\mathcal{S}^{\hat{\boldsymbol{\theta}}_{CE}\left[i_{t_0},\mathcal{P}\right]}\left(\boldsymbol{q}\right)\right\}. \tag{421}$$

Remark

Considering the market invariant sequence i_{t_0} as a random variable I_{t_0}, the classical-equivalent estimator $\hat{\boldsymbol{\theta}}_{CE}\left[i_{t_0},\mathcal{P}\right]$ becomes the random variable $\hat{\boldsymbol{\theta}}_{CE}\left[I^{\boldsymbol{\theta}}_{t_0},\mathcal{P}\right]$. Then, the classical-equivalent Bayesian allocation $\boldsymbol{q}^*_{CE}\left[i_{t_0},\mathcal{P}\right]$ also becomes a random variable

$$\boldsymbol{q}^*_{CE}\left[I^{\boldsymbol{\theta}}_{t_0},\mathcal{P}\right]. \tag{422}$$

Example

Consider as the market invariant the linear returns

$$\boldsymbol{L}_t \equiv \mathsf{diag}^{-1}(\boldsymbol{P}_{t-1})\,\boldsymbol{P}_t - \mathbf{1}_{n\times 1} \tag{423}$$

and

$$\boldsymbol{L}_t|\boldsymbol{\mu},\boldsymbol{\Sigma} \sim \mathsf{N}(\boldsymbol{\mu},\boldsymbol{\Sigma}). \tag{424}$$

The realized market invariant sequence is summarized as follows:

$$i_{t_0} \equiv \left\{\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\Sigma}};\tau+1\right\}, \tag{425}$$

where $\tau+1$ is the size of the available realized market invariant sequence,

$$\hat{\boldsymbol{\mu}} \equiv \frac{1}{\tau+1}\sum_{i=0}^{\tau}\boldsymbol{l}_{t_0-i} \tag{426}$$

and

Example

...

$$\hat{\boldsymbol{\Sigma}} \equiv \frac{1}{\tau+1} \sum_{i=0}^{\tau} \left(\boldsymbol{l}_{t_0-i} - \hat{\boldsymbol{\mu}}\right) \left(\boldsymbol{l}_{t_0-i} - \hat{\boldsymbol{\mu}}\right)' . \tag{427}$$

On the other hand, considering that

$$\boldsymbol{\mu} | \boldsymbol{\Sigma} \sim \mathsf{N}\left(\boldsymbol{\mu}_0, \frac{\boldsymbol{\Sigma}}{T_0}\right) \tag{428}$$

and

$$\boldsymbol{\Sigma}^{-1} \sim \mathsf{W}\left(\nu_0, \frac{\boldsymbol{\Sigma}_0^{-1}}{\nu_0}\right), \tag{429}$$

the investor's private information is summarized as follows:

$$\mathcal{P} \equiv \left\{\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0; T_0, \nu_0\right\}, \tag{430}$$

Example

... where $\boldsymbol{\mu}_0$ is the investor private information on $\mathbb{E}\{\boldsymbol{\mu}\}$, $\boldsymbol{\Sigma}_0$ is the investor private information on $\mathbb{E}\{\boldsymbol{\Sigma}\}$, T_0 is the confidence of the investor on $\boldsymbol{\mu}_0$ and ν_0 is degrees of freedom of the Wishart distribution representing the confidence of the investor on $\boldsymbol{\Sigma}_0$.
The classical-equivalent estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are

$$\hat{\boldsymbol{\mu}}_{CE}\left[i_{t_0}, \mathcal{P}\right] = \frac{T_0\boldsymbol{\mu}_0 + (\tau + 1)\,\hat{\boldsymbol{\mu}}}{T_0 + \tau + 1} \tag{431}$$

and

$$\hat{\boldsymbol{\Sigma}}_{CE}\left[i_{t_0}, \mathcal{P}\right] = \frac{1}{\nu_0 + \tau + n + 2}\left[\nu_0\boldsymbol{\Sigma}_0 + (\tau + 1)\,\hat{\boldsymbol{\Sigma}} + \frac{(\boldsymbol{\mu}_0 - \hat{\boldsymbol{\mu}})(\boldsymbol{\mu}_0 - \hat{\boldsymbol{\mu}})'}{\frac{1}{\tau+1} + \frac{1}{T_0}}\right], \tag{432}$$

respectively.

Example

... Finally, the classical-equivalent Bayesian allocation is given by

$$\boldsymbol{q}^{*}_{CE}\left[i_{t_0},\mathcal{P}\right] \equiv \arg \max_{\boldsymbol{q}\in\mathcal{C}^{\hat{\boldsymbol{\theta}}_{CE}\left[i_{t_0},\mathcal{P}\right]}} \left\{ \mathcal{S}^{\hat{\boldsymbol{\theta}}_{CE}\left[i_{t_0},\mathcal{P}\right]}\left(\boldsymbol{q}\right) \right\}, \tag{433}$$

where $\hat{\boldsymbol{\theta}}_{CE}\left[i_{t_0},\mathcal{P}\right] \equiv \left\{ \hat{\boldsymbol{\mu}}_{CE}\left[i_{t_0},\mathcal{P}\right], \hat{\boldsymbol{\Sigma}}_{CE}\left[i_{t_0},\mathcal{P}\right] \right\}$. It is important to notice that it is possible to obtain $\boldsymbol{q}^{*}_{CE}\left[I^{\boldsymbol{\mu},\boldsymbol{\Sigma}}_{t_0},\mathcal{P}\right]$ through Monte Carlo simulation.

Remark

Considering a private information v, called market view, it is possible to obtain the market invariant conditional on the market view, $\boldsymbol{X}^v$, with the following probability density function

$$\mathsf{f}_{\boldsymbol{X}|v}\left(\boldsymbol{x}|v\right) = \frac{\mathsf{f}_{\boldsymbol{V}|\mathbf{g}(\boldsymbol{x})}\left(\boldsymbol{v}|\boldsymbol{x}\right)\mathsf{f}_{\boldsymbol{X}}\left(\boldsymbol{x}\right)}{\int \mathsf{f}_{\boldsymbol{V}|\mathbf{g}(\boldsymbol{x})}\left(\boldsymbol{v}|\boldsymbol{x}\right)\mathsf{f}_{\boldsymbol{X}}\left(\boldsymbol{x}\right)d\boldsymbol{x}}, \tag{434}$$

where $\mathbf{g}$ is represents the investor's area of expertise.
It is easy to obtain the previous probability density function. By definition, we have

$$\mathsf{f}_{\boldsymbol{X}|\boldsymbol{v}}\left(\boldsymbol{x}|\boldsymbol{v}\right) \equiv \frac{\mathsf{f}_{\boldsymbol{X},\boldsymbol{V}}\left(\boldsymbol{x},\boldsymbol{v}\right)}{\mathsf{f}_{\boldsymbol{V}}\left(\boldsymbol{v}\right)}, \tag{435}$$

where $\mathsf{f}_{\boldsymbol{X},\boldsymbol{V}}$ is the joint distribution of $\boldsymbol{X}$ and $\boldsymbol{V}$ and

Remark

...

$$\mathsf{f}_{\boldsymbol{V}} \equiv \int \mathsf{f}_{\boldsymbol{X},\boldsymbol{V}}\left(\boldsymbol{x},\boldsymbol{v}\right) d\boldsymbol{x} \tag{436}$$

is the marginal probability density function of $\boldsymbol{V}$. In addition, by definition,

$$\mathsf{f}_{\boldsymbol{X},\boldsymbol{V}}\left(\boldsymbol{x},\boldsymbol{v}\right) \equiv \mathsf{f}_{\boldsymbol{V}|\mathbf{g}(\boldsymbol{x})}\left(\boldsymbol{v}|\boldsymbol{x}\right) \mathsf{f}_{\boldsymbol{X}}\left(\boldsymbol{x}\right). \tag{437}$$

Then,

$$\mathsf{f}_{\boldsymbol{X}|\boldsymbol{v}}\left(\boldsymbol{x}|\boldsymbol{v}\right) \equiv \frac{\mathsf{f}_{\boldsymbol{X},\boldsymbol{V}}\left(\boldsymbol{x},\boldsymbol{v}\right)}{\int \mathsf{f}_{\boldsymbol{X},\boldsymbol{V}}\left(\boldsymbol{x},\boldsymbol{v}\right) d\boldsymbol{x}} \tag{438}$$

$$= \frac{\mathsf{f}_{\boldsymbol{V}|\mathbf{g}(\boldsymbol{x})}\left(\boldsymbol{v}|\boldsymbol{x}\right) \mathsf{f}_{\boldsymbol{X}}\left(\boldsymbol{x}\right)}{\int \mathsf{f}_{\boldsymbol{V}|\mathbf{g}(\boldsymbol{x})}\left(\boldsymbol{v}|\boldsymbol{x}\right) \mathsf{f}_{\boldsymbol{X}}\left(\boldsymbol{x}\right) d\boldsymbol{x}}. \tag{439}$$

Definition

Black-Litterman optimization problem. The Black-Litterman optimization problem is given by

$$\boldsymbol{q}_{BL}^{*}[\boldsymbol{v}] \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}^{\boldsymbol{v}}} \left\{\mathcal{S}^{\boldsymbol{v}}(\boldsymbol{q})\right\}, \tag{440}$$

where $\boldsymbol{v}$ is called market view; $\mathcal{S}^{\boldsymbol{v}}(\boldsymbol{q})$ and $\mathcal{C}^{\boldsymbol{v}}$ are obtained considering the market invariant conditional on the market view, $\boldsymbol{X}^{\boldsymbol{v}}$.

Remarks

Important points:

- in the classical-equivalent approach, the estimates of the market parameters are shrunk toward the investor's prior;
- in the Black-Litterman approach, the market distribution is shrunk toward the investor's prior.

Proposition

Black and Litterman (1990) introduced a particular solution for the optimization problem given some assumptions:

- the market invariant $\boldsymbol{X}$ is considered as follows

$$\boldsymbol{X} \sim \mathsf{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right); \tag{441}$$

- the investor's area of expertise is a linear function of the market invariant

$$\mathbf{g}\left(\boldsymbol{x}\right) \equiv \boldsymbol{\Pi}\boldsymbol{x}, \tag{442}$$

  where $\boldsymbol{\Pi}$ is called pick matrix and selects the linear combination of the market correspondent to a view (the investor does not need to express views on all the market variables and views do not necessarily need to be expressed in absolute terms for each market invariant);

Proposition

...

- the conditional distribution of the investor's views given the outcome of the market is assumed as follows

$$\boldsymbol{V}|\boldsymbol{\Pi x} \sim \mathsf{N}\left(\boldsymbol{\Pi x}, \boldsymbol{\Omega}\right), \tag{443}$$

  where $\boldsymbol{\Omega}$ represents the uncertainty matrix of the investor's views;

- the uncertainty matrix of the investor's views is chosen as follows:

$$\boldsymbol{\Omega} \equiv \left(\frac{1}{c} - 1\right) \boldsymbol{\Pi \Sigma \Pi}', \tag{444}$$

  where $c \in \left]0, 1\right]$; and

- the investor has a private view $\boldsymbol{v}$.

Proposition

... Using the Bayes' rule, the distribution of the market invariant conditional on market view is

$$\boldsymbol{X}|\boldsymbol{v} \sim \mathsf{N}\left(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL}\right), \tag{445}$$

where

$$\boldsymbol{\mu}_{BL}\left(\boldsymbol{v}, \boldsymbol{\Omega}\right) \equiv \boldsymbol{\mu} + \boldsymbol{\Sigma}\boldsymbol{\Pi}'\left(\boldsymbol{\Pi}\boldsymbol{\Sigma}\boldsymbol{\Pi}' + \boldsymbol{\Omega}\right)^{-1}\left(\boldsymbol{v} - \boldsymbol{\Pi}\boldsymbol{\mu}\right) \tag{446}$$

and

$$\boldsymbol{\Sigma}_{BL}\left(\boldsymbol{\Omega}\right) \equiv \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\boldsymbol{\Pi}'\left(\boldsymbol{\Pi}\boldsymbol{\Sigma}\boldsymbol{\Pi}' + \boldsymbol{\Omega}\right)^{-1}\boldsymbol{\Pi}\boldsymbol{\Sigma}. \tag{447}$$

Example

Consider a uni-variate market invariant represented by the linear returns given by

$$X \sim \mathsf{N}\left(\mu, \sigma^2\right). \tag{448}$$

The investor opinion about the market invariant is modeled as follows:

$$V|x \sim \mathsf{N}\left(x, \phi^2\right), \tag{449}$$

where ϕ^2 represents the uncertainty of the investor's view.

Example

... Consequently,

$$X|v \sim \mathsf{N}\left(\tilde{\mu}\left(v,\phi^2\right),\tilde{\sigma}^2\left(\phi^2\right)\right). \tag{450}$$

Taking the expected value of the absolute value goal function as the satisfaction function, the total value available to invest equal to v_{t_0}, and defining

$$w = q/v_{t_0}, \tag{451}$$

the Black-Litterman allocation is given by

$$w^*_{BL} \equiv \arg\max_{w \in \mathcal{C}} \{w\tilde{\mu}\}. \tag{452}$$

Remark

The square of the Mahalanobis distance of the market invariant $\boldsymbol{X}$ from its expected value $\boldsymbol{\mu}$ is given by

$$M^2 \equiv (\boldsymbol{X} - \boldsymbol{\mu})' \, \boldsymbol{\Sigma}^{-1} \, (\boldsymbol{X} - \boldsymbol{\mu}) \,. \tag{453}$$

It is possible to show that

$$M^2 \sim \chi_n^2, \tag{454}$$

i.e. the square of the Mahalanobis distance of the market invariant is distributed as a chi-square with n degrees of freedom.

Remark

... Obviously, since $\boldsymbol{\mu}_{BL}$ is a realization of the random variable $\boldsymbol{X}$, it is possible to calculate the following Mahalanobis distance

$$m^2 \equiv (\boldsymbol{\mu}_{BL} - \boldsymbol{\mu})' \, \boldsymbol{\Sigma}^{-1} \, (\boldsymbol{\mu}_{BL} - \boldsymbol{\mu}) \, . \tag{455}$$

If m^2 is small, the views are not too far from the market model and the consistence of the Black-Litterman expectations with the market model is high.

Remark

... Then, the index of consistence of the Black-Litterman expectations with the market model is defined as follows:

$$C\left(\boldsymbol{v}\right) \equiv \mathsf{P}\left(M^2 \geq m^2\left(\boldsymbol{v}\right)\right) = 1 - \mathsf{F}^{\gamma}_{n,1}, \tag{456}$$

where the dependence of m^2 on $\boldsymbol{v}$ is emphasized and $\mathsf{F}^{\gamma}_{n,1}$ is the gamma cumulative density function (the chi-square distribution is a particular case of the gamma distribution).

Remark

... Finally, the sensitivity of the index of consistence to the views is given by

$$\frac{\partial C\left(\boldsymbol{v}\right)}{\partial \boldsymbol{v}} = -2\mathsf{f}_{n,1}^{\gamma}\left(m^{2}\left(\boldsymbol{v}\right)\right)\left(\boldsymbol{\Pi}\boldsymbol{\Sigma}\boldsymbol{\Pi}' + \boldsymbol{\Omega}\right)^{-1}\boldsymbol{\Pi}\left(\boldsymbol{\mu}_{BL} - \boldsymbol{\mu}\right), \tag{457}$$

where $\mathsf{f}_{n,1}^{\gamma}$ is the probability density function of the chi-square distribution with n degrees of freedom, which is a special case of the gamma probability density function.

In the following, we present the original resampled asset allocation from Michaud (1998), U.S. Patent No. 6,003,018.

Assumptions

The original resampled asset allocation from Michaud (1998) has the following assumptions:

- the problem admits the mean-variance formulation in terms of linear returns and relative weights;
- the market consists of equity-like securities for which the linear returns are market invariants;
- the investment horizon and the estimation interval coincide;
- the investment constraints are such that the dual formulation is correct;
- the constraints do not depend on the unknown market parameters.

Original resampled asset allocation from Michaud (1998). The original resampled asset allocation optimization problem from Michaud (1998) is given by

$$\boldsymbol{w}^{(i)} \equiv \arg\min_{\boldsymbol{w}\in\mathcal{C};\boldsymbol{w}'\boldsymbol{\mu}\geq e^{(i)}} \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}, i = 1, 2, \ldots, I, \tag{458}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the expected values and the covariance matrix of the linear returns of the securities relative to the investment horizon;

$$\left\{e^{(1)}, e^{(2)}, \ldots, e^{(I)}\right\} \tag{459}$$

is the set containing a significant grid of target expected values; and $\mathcal{C}$ is the set of investment constraints. The procedure to obtain the resampled allocations has the following steps:

...

- Estimate the inputs ${}_0\hat{\boldsymbol{\mu}}$ and ${}_0\hat{\boldsymbol{\Sigma}}$ from the analysis of the observed time series of the past linear returns

$$i_T \equiv \{\boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_T\}\,. \tag{460}$$

- Consider i_T a realization of the independent identically distributed market invariant

$$I_T \equiv \{\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots, \boldsymbol{L}_T\} \tag{461}$$

  and assume

$$\boldsymbol{L}_t \sim \text{NIID}\left({}_0\hat{\boldsymbol{\mu}},\ {}_0\hat{\boldsymbol{\Sigma}}\right). \tag{462}$$

- Resample a large number Q of Monte Carlo scenarios

$${}_q i_T \equiv \{{}_q\boldsymbol{l}_1,\ {}_q\boldsymbol{l}_2, \ldots,\ {}_q\boldsymbol{l}_T\}\,, q = 1, 2, \ldots, Q. \tag{463}$$

...

- For each Monte Carlo scenario, estimate the inputs

$$\left\{ {}_q\hat{\boldsymbol{\mu}},\ {}_q\hat{\boldsymbol{\Sigma}} \right\}, q = 1, 2, \ldots, Q. \tag{464}$$

- For each resampled input, obtain the global minimum-variance portfolio

$${}_q\boldsymbol{w}_{MV} \equiv \arg\min_{\boldsymbol{w} \in \mathcal{C}} \boldsymbol{w}'\ {}_q\hat{\boldsymbol{\Sigma}}\boldsymbol{w}, q = 1, 2, \ldots, Q. \tag{465}$$

- For each global minimum-variance portfolio, calculate the respective estimated expected value

$${}_q\underline{e} \equiv\ {}_q\boldsymbol{w}'_{MV}\ {}_q\hat{\boldsymbol{\mu}}, q = 1, 2, \ldots, Q. \tag{466}$$

...

- For each Monte Carlo scenario, obtain the maximum estimated expected value

$$ {}_q\bar{e} \equiv \max\left\{ {}_q\hat{\boldsymbol{\mu}}'\boldsymbol{\delta}^{(1)},\ {}_q\hat{\boldsymbol{\mu}}'\boldsymbol{\delta}^{(2)}, \ldots,\ {}_q\hat{\boldsymbol{\mu}}'\boldsymbol{\delta}^{(n)} \right\}, q = 1, 2, \ldots, Q, \quad (467) $$

  where $\boldsymbol{\delta}^{(j)}$ is a $n \times 1$ vector with 1 in the position indicated by j and zeros in the other positions.
- For each Monte Carlo scenario, obtain the grid of target expected values

$$ \left\{ {}_q e^{(1)},\ {}_q e^{(2)}, \ldots,\ {}_q e^{(I)} \right\}, q = 1, 2, \ldots, Q, \quad (468) $$

  where

...

$$
\begin{aligned}
{}_q e^{(1)} &\equiv {}_q\underline{e} \\
&\vdots \\
{}_q e^{(i)} &\equiv {}_q\underline{e} + \frac{{}_q\overline{e} - {}_q\,\underline{e}}{I-1}(i-1) \\
&\vdots \\
{}_q e^{(I)} &\equiv {}_q\overline{e}.
\end{aligned}
$$

- For each Monte Carlo scenario and target expected values, obtain

$$
{}_q\boldsymbol{w}^{(i)} \equiv \arg\min_{\boldsymbol{w}\in\mathcal{C};\boldsymbol{w}'\,{}_q\hat{\boldsymbol{\mu}}\geq {}_q e^{(i)}} \boldsymbol{w}'\,{}_q\hat{\boldsymbol{\Sigma}}\boldsymbol{w}; q=1,2,\ldots,Q; i=1,2,\ldots,I. \tag{469}
$$

...

- The resampled allocations are defined as follows

$$\boldsymbol{w}_{RS}^{(i)} \equiv \frac{1}{Q} \sum_{q=1}^{Q} {}_{q}\boldsymbol{w}^{(i)}, i = 1, 2, \ldots, I. \tag{470}$$

- The resampled allocation quantities are given by

$$\boldsymbol{q}_{RS}^{(i)} \equiv v_T \text{diag}^{-1} (\boldsymbol{p}_T) \, \boldsymbol{w}_{RS}^{(i)}, i = 1, 2, \ldots, I, \tag{471}$$

where v_T is the initial budget.

Remarks

Unlike the Bayesian and Black-Litterman approaches, where the estimates of the input parameters are smoothed, the resampled approaches average the outputs of a set of optimizations.

Generalized resampled asset allocation. The procedure to obtain the generalized resampled allocations has the following steps:

- Estimate the market parameters ${}_0\hat{\boldsymbol{\theta}} \equiv \hat{\boldsymbol{\theta}}\left[i_T\right]$ from the available time series of the market invariants:

$$i_T \equiv \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_T\}. \tag{472}$$

- Generate a large number J of Monte Carlo realizations:

$${}_j i_T \equiv \{{}_j\boldsymbol{x}_1,\ {}_j\boldsymbol{x}_2, \ldots,\ {}_j\boldsymbol{x}_T\}, j = 1, 2, \ldots, J, \tag{473}$$

  from the following set of random variables

$$I_T^{{}_0\hat{\boldsymbol{\theta}}} \equiv \left\{\boldsymbol{X}_1^{{}_0\hat{\boldsymbol{\theta}}}, \boldsymbol{X}_2^{{}_0\hat{\boldsymbol{\theta}}}, \ldots, \boldsymbol{X}_T^{{}_0\hat{\boldsymbol{\theta}}}\right\}. \tag{474}$$

...

- For each Monte Carlo realization, using the resampled time series, estimate the parameters

$$ {}_j\hat{\boldsymbol{\theta}} \equiv \hat{\boldsymbol{\theta}}\left[{}_j i_T\right], j = 1, 2, \ldots, J. \tag{475} $$

- For each estimated parameters, solve the following optimization problem

$$ {}_j\boldsymbol{q} \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}^{j\hat{\boldsymbol{\theta}}}} \left\{\mathcal{S}^{j\hat{\boldsymbol{\theta}}}\left(\boldsymbol{q}\right)\right\}, j = 1, 2, \ldots, J. \tag{476} $$

- The resampled allocation is obtained as follows:

$$ \boldsymbol{q}_{RS} \equiv \frac{1}{J}\sum_{j=1}^{J} {}_j\boldsymbol{q}. \tag{477} $$

... The resampled asset allocation is summarized in the following chain:

$$i_T \mapsto {}_0\hat{\boldsymbol{\theta}} \mapsto {}_j i_T \mapsto {}_j\hat{\boldsymbol{\theta}} \mapsto {}_j\boldsymbol{q} \mapsto \boldsymbol{q}_{RS}. \tag{478}$$

Using the previously defined notation, it is possible to define the generalized resampled allocation as follows:

Definition

Generalized resampled allocation. The generalized resampled allocation is given by

$$\boldsymbol{q}_{RS}\left[i_T\right] \equiv \mathbb{E}\left\{\boldsymbol{q}\left(\hat{\boldsymbol{\theta}}\left[I_T^{\hat{\boldsymbol{\theta}}[i_T]}\right]\right)\right\}, \tag{479}$$

where $\boldsymbol{q}\left(\hat{\boldsymbol{\theta}}\left[{}_j i_T\right]\right) \equiv {}_j\boldsymbol{q}$.

Example

Consider that

$$\hat{\boldsymbol{\mu}}\left[I_T^{\hat{\boldsymbol{\mu}}[i_T],\hat{\boldsymbol{\Sigma}}[i_T]}\right] \sim \mathsf{N}\left(\hat{\boldsymbol{\mu}}\left[i_T\right], \frac{1}{T}\hat{\boldsymbol{\Sigma}}\left[i_T\right]\right) \tag{480}$$

and

$$T\hat{\boldsymbol{\Sigma}}\left[I_T^{\hat{\boldsymbol{\mu}}[i_T],\hat{\boldsymbol{\Sigma}}[i_T]}\right] \sim \mathsf{W}\left(T-1, \hat{\boldsymbol{\Sigma}}\left[i_T\right]\right), \tag{481}$$

where $\hat{\boldsymbol{\mu}}\left[i_T\right]$ and $\hat{\boldsymbol{\Sigma}}\left[i_T\right]$ are the sample mean and covariance of the linear returns, respectively.

In addition, consider the following optimal allocation function

$$\boldsymbol{q}\left(\boldsymbol{\mu},\boldsymbol{\Sigma}\right) \equiv \left[\mathrm{diag}^{-1}\left(\boldsymbol{p}_T\right)\right]\boldsymbol{\Sigma}^{-1}\left(b\boldsymbol{\mu} + \frac{v_T - b\mathbf{1}'_{n\times 1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}'_{n\times 1}\boldsymbol{\Sigma}^{-1}\mathbf{1}_{n\times 1}}\mathbf{1}_{n\times 1}\right). \tag{482}$$

Example

... Therefore, the allocations obtained sampling the random variable $I_T^{\hat{\boldsymbol{\mu}}[i_T],\hat{\boldsymbol{\Sigma}}[i_T]}$ are given by

$$
\begin{aligned}
{}_j\boldsymbol{q} \equiv & \boldsymbol{q}\left(\hat{\boldsymbol{\mu}}\left[{}_j i_T\right], \hat{\boldsymbol{\Sigma}}\left[{}_j i_T\right]\right) \\
= & \left[\text{diag}^{-1}\left(\boldsymbol{p}_T\right)\right] \hat{\boldsymbol{\Sigma}}^{-1}\left[{}_j i_T\right] \\
& \left(b\hat{\boldsymbol{\mu}}\left[{}_j i_T\right] + \frac{v_T - b\mathbf{1}'_{n\times 1}\hat{\boldsymbol{\Sigma}}^{-1}\left[{}_j i_T\right]\hat{\boldsymbol{\mu}}\left[{}_j i_T\right]}{\mathbf{1}'_{n\times 1}\hat{\boldsymbol{\Sigma}}^{-1}\left[{}_j i_T\right]\mathbf{1}_{n\times 1}}\mathbf{1}_{n\times 1}\right) \\
= & \left[\text{diag}^{-1}\left(\boldsymbol{p}_T\right)\right] \\
& \left(b\hat{\boldsymbol{\Sigma}}^{-1}\left[{}_j i_T\right]\hat{\boldsymbol{\mu}}\left[{}_j i_T\right] + \frac{v_T - b\mathbf{1}'_{n\times 1}\hat{\boldsymbol{\Sigma}}^{-1}\left[{}_j i_T\right]\hat{\boldsymbol{\mu}}\left[{}_j i_T\right]}{\mathbf{1}'_{n\times 1}\hat{\boldsymbol{\Sigma}}^{-1}\left[{}_j i_T\right]\mathbf{1}_{n\times 1}}\hat{\boldsymbol{\Sigma}}^{-1}\left[{}_j i_T\right]\mathbf{1}_{n\times 1}\right).
\end{aligned}
$$

Example

... Finally, the resampled allocation is given by

$$\boldsymbol{q}_{RS}\left[i_T\right] \equiv \mathbb{E}\left\{\boldsymbol{q}\left(\hat{\boldsymbol{\mu}}\left[I_T^{\hat{\boldsymbol{\mu}}[i_T],\hat{\boldsymbol{\Sigma}}[i_T]}\right], \hat{\boldsymbol{\Sigma}}\left[I_T^{\hat{\boldsymbol{\mu}}[i_T],\hat{\boldsymbol{\Sigma}}[i_T]}\right]\right)\right\}. \tag{483}$$

Obviously, it is not possible to calculate analytically the previous expectation. Then, the expectation is obtained through Monte Carlo simulation.

Remark

The opportunity cost and the resampled allocation.

- Consider the set Θ of market parameters broad enough to include the true unknown value. The set Θ is called stress test set of parameters.
- Solve the optimization problem

$$\boldsymbol{q}^{*,\boldsymbol{\theta}} \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}^{\boldsymbol{\theta}}} \left\{ \mathcal{S}^{\boldsymbol{\theta}} \left(\boldsymbol{q} \right) \right\}, \tag{484}$$

  for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.
- Calculate

$$\mathcal{S}^{\boldsymbol{\theta}} \left(\boldsymbol{q}^{*,\boldsymbol{\theta}} \right), \tag{485}$$

  for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Remark

...

- For each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, define

$$I_T^{\boldsymbol{\theta}} \equiv \left\{ \boldsymbol{X}_1^{\boldsymbol{\theta}}, \boldsymbol{X}_2^{\boldsymbol{\theta}}, \ldots, \boldsymbol{X}_T^{\boldsymbol{\theta}} \right\} \tag{486}$$

  and obtain the resampled random allocation $\boldsymbol{q}_{RS}\left[I_T^{\boldsymbol{\theta}}\right]$.
- The opportunity cost is obtained as follows

$$\mathcal{OC}^{\boldsymbol{\theta}}\left(\boldsymbol{q}_{RS}\left[I_T^{\boldsymbol{\theta}}\right]\right) \equiv \tag{487}$$

$$\mathcal{S}^{\boldsymbol{\theta}}\left(\boldsymbol{q}^{*,\boldsymbol{\theta}}\right) - \mathcal{S}^{\boldsymbol{\theta}}\left(\boldsymbol{q}_{RS}\left[I_T^{\boldsymbol{\theta}}\right]\right) + \mathcal{C}^{+,\boldsymbol{\theta}}\left(\boldsymbol{q}_{RS}\left[I_T^{\boldsymbol{\theta}}\right]\right), \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}. \tag{488}$$

Remarks

Important points:

- Intuitively, the resampled asset allocation reduces the sensitivity to the market parameters, giving rise to a less disperse opportunity cost than the sample-based asset allocation.
- The resampled asset allocation can give rise to allocations that violate the investment constraints.
- It is very hard to stress test the performance of the resampled asset allocation due to the excessive computational burden.

Definition

Robust asset allocation. The robust asset allocation is given by

$$\boldsymbol{q}^{*,\boldsymbol{\Theta}} \equiv \arg\min_{\boldsymbol{q}\in\mathcal{C}^{\boldsymbol{\Theta}}} \left\{ \max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left\{ \mathcal{S}^{\boldsymbol{\theta}}\left(\boldsymbol{q}^{*,\boldsymbol{\theta}}\right) - \mathcal{S}^{\boldsymbol{\theta}}\left(\boldsymbol{q}\right) \right\} \right\}, \tag{489}$$

where $\left\{\boldsymbol{q} \in \mathcal{C}^{\boldsymbol{\Theta}}\right\} \equiv \left\{\boldsymbol{q} \in \mathcal{C}^{\boldsymbol{\theta}}, \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}\right\}$.

Remark

The allocation $\boldsymbol{q}^{*,\boldsymbol{\Theta}}$ and its quality depends on the choice for the uncertainty set $\boldsymbol{\Theta}$ of the market parameters. The smaller the range of $\boldsymbol{\Theta}$, the lower the maximum value of the opportunity cost generated by $\boldsymbol{q}^{*,\boldsymbol{\Theta}}$.

Example

Consider as the primary satisfaction function the certainty-equivalent of an exponential utility function, an initial investment budget constraint and a VaR constraint. The robust asset allocation is given by

$$\boldsymbol{q}^{*,\boldsymbol{\Theta}} \equiv \arg\min_{\boldsymbol{q}\in\mathcal{C}^{\boldsymbol{\Theta}}} \left\{ \max_{\{\boldsymbol{\mu},\boldsymbol{\Sigma}\}\in\boldsymbol{\Theta}} \left\{ \mathcal{CE}^{\{\boldsymbol{\mu},\boldsymbol{\Sigma}\}}\left(\boldsymbol{q}^{*,\{\boldsymbol{\mu},\boldsymbol{\Sigma}\}}\right) - \mathcal{CE}^{\{\boldsymbol{\mu},\boldsymbol{\Sigma}\}}\left(\boldsymbol{q}\right) \right\} \right\}, \tag{490}$$

where

$$\mathcal{C}^{\boldsymbol{\Theta}} \equiv \left\{ \boldsymbol{q} : \boldsymbol{q}'\boldsymbol{p}_T = v_T \wedge \mathsf{VaR}^{\{\boldsymbol{\mu},\boldsymbol{\Sigma}\}}\left(\boldsymbol{q}\right) \leq \gamma v_T, \forall \left\{\boldsymbol{\mu},\boldsymbol{\Sigma}\right\} \in \boldsymbol{\Theta} \right\} \tag{491}$$

and $\gamma, v_T \in \mathbb{R}_+$.

Remark

It is possible to use the historical information i_T to estimate the uncertainty set $\hat{\boldsymbol{\Theta}}\left[i_T\right]$. Then, the robust asset allocation becomes

$$\boldsymbol{q}^{*,\hat{\boldsymbol{\Theta}}[i_T]} \equiv \arg \min_{\boldsymbol{q} \in \mathcal{C}^{\hat{\boldsymbol{\Theta}}[i_T]}} \left\{ \max_{\boldsymbol{\theta} \in \hat{\boldsymbol{\Theta}}[i_T]} \left\{ \mathcal{S}^{\boldsymbol{\theta}} \left(\boldsymbol{q}^{*,\boldsymbol{\theta}} \right) - \mathcal{S}^{\boldsymbol{\theta}} \left(\boldsymbol{q} \right) \right\} \right\}, \qquad (492)$$

where $\left\{ \boldsymbol{q} \in \mathcal{C}^{\hat{\boldsymbol{\Theta}}[i_T]} \right\} \equiv \left\{ \boldsymbol{q} \in \mathcal{C}^{\boldsymbol{\theta}}, \forall \boldsymbol{\theta} \in \hat{\boldsymbol{\Theta}}\left[i_T\right] \right\}$.

Example

Consider the linear returns distributed as follows

$$\boldsymbol{L}_t \sim \mathsf{N}\left(\dot{\boldsymbol{\mu}}, \dot{\boldsymbol{\Sigma}}\right), \tag{493}$$

where $\dot{\boldsymbol{\mu}}$ and $\dot{\boldsymbol{\Sigma}}$ are the true expected values and covariance matrix, respectively.

In addition, consider that the covariance matrix $\dot{\boldsymbol{\Sigma}}$ is known and the sample estimator $\hat{\boldsymbol{\mu}}\left[i_T\right]$ for $\dot{\boldsymbol{\mu}}$. Then, it is possible to define the uncertainty set as follows

$$\hat{\boldsymbol{\Theta}}\left[i_T\right] \equiv \left\{\boldsymbol{\mu} : M^2\left(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}\left[i_T\right], \dot{\boldsymbol{\Sigma}}\right) \leq \frac{\mathsf{Q}_{\chi^2_n}(p)}{T}\right\}, p \in (0,1) \tag{494}$$

where

Example

...

$$M^2\left(\boldsymbol{\mu},\hat{\boldsymbol{\mu}}\left[i_T\right],\dot{\boldsymbol{\Sigma}}\right)\equiv\left(\boldsymbol{\mu}-\hat{\boldsymbol{\mu}}\left[i_T\right]\right)'\dot{\boldsymbol{\Sigma}}^{-1}\left(\boldsymbol{\mu}-\hat{\boldsymbol{\mu}}\left[i_T\right]\right) \qquad (495)$$

is the square of the Mahalanobis distance, $\mathrm{Q}_{\chi^2_n}(p)$ is the quantile of the chi-square distribution with n degrees of freedom for a confidence level p and T is the number of observations in the time series i_T. Finally, it is possible to show that

$$\mathbb{P}\left\{\dot{\boldsymbol{\mu}}\in\hat{\boldsymbol{\Theta}}\left[i_T\right]\right\}=p. \qquad (496)$$

Remark

Robust mean-variance asset allocation. A robust version of the mean-variance asset allocation is given by

$$q^{*,\hat{\Theta}} \equiv \arg\max_{q} \left\{ \min_{\mu \in \hat{\Theta}_{\mu}} q'\mu \right\} \tag{497}$$

$$\text{subject to} \left\{ \begin{array}{c} q \in \mathcal{C} \\ \max_{\Sigma \in \hat{\Theta}_{\Sigma}} q'\Sigma q \leq v \end{array} \right. \tag{498}$$

where $\hat{\Theta} \equiv \left\{ \hat{\Theta}_{\mu}, \hat{\Theta}_{\Sigma} \right\}$ is the uncertainty set for the market parameters that are estimated from the available information i_T. It follows some usual uncertainty sets:

Remark

...

- *Elliptical set for expected values, known covariances*:

$$\hat{\boldsymbol{\Theta}}_{\boldsymbol{\mu}} \equiv \left\{ \boldsymbol{\mu} : M^2 \left(\boldsymbol{\mu}, \boldsymbol{m}, \boldsymbol{T} \right) \leq \varsigma^2 \right\} \tag{499}$$

and

$$\hat{\boldsymbol{\Theta}}_{\boldsymbol{\Sigma}} \equiv \hat{\boldsymbol{\Sigma}}, \tag{500}$$

where $\hat{\boldsymbol{\Sigma}}$ is a point estimate of $\boldsymbol{\Sigma}$; $\boldsymbol{m}$ is an n-dimensional vector; $\boldsymbol{T}$ is an $n \times n$ symmetric and positive matrix; M is the Mahalanobis distance; and $\varsigma^2 \equiv \mathrm{Q}_{\chi^2_n}(p)$ is the chi-square distribution with n degrees of freedom for a confidence level $p \in (0, 1)$.

Remark

...

- *Box set for expected values, box set for covariances*:

$$\hat{\boldsymbol{\Theta}}_{\boldsymbol{\mu}} \equiv \left\{\boldsymbol{\mu} : \underline{\boldsymbol{\mu}} \leq \boldsymbol{\mu} \leq \overline{\boldsymbol{\mu}}\right\} \tag{501}$$

and

$$\hat{\boldsymbol{\Theta}}_{\boldsymbol{\Sigma}} \equiv \left\{\boldsymbol{\Sigma} \succ 0 : \underline{\boldsymbol{\Sigma}} \leq \boldsymbol{\Sigma} \leq \overline{\boldsymbol{\Sigma}}\right\}, \tag{502}$$

where $\boldsymbol{\Sigma} \succ 0$ stands for positive definite matrices.

Remark

It is possible to write the robust mean-variance asset allocation problem with elliptical set for expected values and known covariances as follows

$$\boldsymbol{q}^* \equiv \arg\max_{\boldsymbol{q}} \left\{ \boldsymbol{q}'\boldsymbol{m} - \gamma_e \sqrt{\boldsymbol{q}'\boldsymbol{T}\boldsymbol{q}} \right\} \tag{503}$$

$$\text{subject to} \left\{ \begin{array}{c} \boldsymbol{q} \in \mathcal{C} \\ \boldsymbol{q}'\hat{\boldsymbol{\Sigma}}\boldsymbol{q} \leq v \end{array} \right. , \tag{504}$$

where γ_e represents the aversion to estimation risk.
Alternatively, it is also possible to rewrite the previous optimization problem as follows

$$\boldsymbol{q}^* \equiv \arg\max_{\boldsymbol{q}\in\mathcal{C}} \left\{ \boldsymbol{q}'\boldsymbol{m} - \gamma_m \sqrt{\boldsymbol{q}'\hat{\boldsymbol{\Sigma}}\boldsymbol{q}} - \gamma_e \sqrt{\boldsymbol{q}'\boldsymbol{T}\boldsymbol{q}} \right\} \tag{505}$$

where γ_m is the market risk Lagrange multiplier.

Definition

Covariance matrix-based dispersion parameter of posterior distribution $\mathsf{f}_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}|i_T,\mathcal{P}\right)$. The covariance matrix-based dispersion parameter of the posterior distribution $\mathsf{f}_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}|i_T,\mathcal{P}\right)$ is given by

$$\hat{\boldsymbol{S}}\left[i_T,\mathcal{P}\right] \equiv \mathrm{Cov}_{i_T,\mathcal{P}}\left\{\boldsymbol{\Theta}\right\} \tag{506}$$

$$\equiv \int \left(\boldsymbol{\theta}-\mathbb{E}\left\{\boldsymbol{\Theta}\right\}\right)\left(\boldsymbol{\theta}-\mathbb{E}\left\{\boldsymbol{\Theta}\right\}\right)' \mathsf{f}_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}|i_T,\mathcal{P}\right) d\boldsymbol{\theta}. \tag{507}$$

Definition

Modal dispersion-based dispersion parameter of posterior distribution $\mathsf{f}_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}|i_T,\mathcal{P}\right)$. The modal dispersion-based dispersion parameter of the posterior distribution $\mathsf{f}_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}|i_T,\mathcal{P}\right)$ is given by

$$\hat{\boldsymbol{S}}\left[i_T,\mathcal{P}\right] \equiv \mathsf{MD}_{i_T,\mathcal{P}}\left\{\boldsymbol{\Theta}\right\} \tag{508}$$

$$\equiv -\left(\left[\frac{\partial^2 \ln \mathsf{f}_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}|i_T,\mathcal{P}\right)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]_{\boldsymbol{\theta}=\mathbb{M}_{i_T,\mathcal{P}}\left\{\boldsymbol{\Theta}\right\}}\right)^{-1}. \tag{509}$$

Definition

Robust Bayesian asset allocation. The robust Bayesian asset allocation is given by

$$\boldsymbol{q}_{RB}^{*} \equiv \arg \min_{\boldsymbol{q} \in \mathcal{C}^{\hat{\boldsymbol{\Theta}}^{\varsigma}[i_T, \mathcal{P}]}} \left\{ \max_{\boldsymbol{\theta} \in \hat{\boldsymbol{\Theta}}^{\varsigma}[i_T, \mathcal{P}]} \left\{ \mathcal{S}^{\boldsymbol{\theta}} \left(\boldsymbol{q}^{*,\boldsymbol{\theta}} \right) - \mathcal{S}^{\boldsymbol{\theta}} \left(\boldsymbol{q} \right) \right\} \right\}, \tag{510}$$

where

$$\hat{\boldsymbol{\Theta}}^{\varsigma} \left[i_T, \mathcal{P} \right] \equiv \left\{ \boldsymbol{\theta} : \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{CE} \left[i_T, \mathcal{P} \right] \right)' \hat{\boldsymbol{S}}^{-1} \left[i_T, \mathcal{P} \right] \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{CE} \left[i_T, \mathcal{P} \right] \right) \leq \varsigma^2 \right\} \tag{511}$$

and ς is the aversion to the estimation risk.

Remark

The robust Bayesian asset allocation generalizes the prior asset allocation, the sample-based asset allocation and classical-equivalent Bayesian asset allocation.

Remark

Robust Bayesian mean-variance asset allocation. A robust Bayesian version of the mean-variance asset allocation in terms of relative weights and linear returns is given by

$$\boldsymbol{w}^*_{RB} \equiv \arg\max_{\boldsymbol{w}} \left\{ \min_{\boldsymbol{\mu} \in \hat{\boldsymbol{\Theta}}_{\boldsymbol{\mu}}} \boldsymbol{w}'\boldsymbol{\mu} \right\} \tag{512}$$

$$\text{subject to} \begin{cases} \boldsymbol{w} \in \mathcal{C} \\ \max_{\boldsymbol{\Sigma} \in \hat{\boldsymbol{\Theta}}_{\boldsymbol{\Sigma}}} \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} \leq v \end{cases}, \tag{513}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ represent the expected values and the covariance matrix of the linear returns of the assets relative to the investment horizon, $v \in \mathbb{R}_+$, $\hat{\boldsymbol{\Theta}}_{\boldsymbol{\mu}}$ is the uncertainty set for $\boldsymbol{\mu}$ and $\hat{\boldsymbol{\Theta}}_{\boldsymbol{\Sigma}}$ is the uncertainty set for $\boldsymbol{\Sigma}$.

Remark

... The following assumptions are considered to obtain the posterior distributions of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$:

- the market consists of equity-like securities for which the linear returns are market invariants;
- the investment horizon and the estimation interval coincide;
- the linear returns are normally distributed as follows

$$\boldsymbol{L}_t|\boldsymbol{\mu},\boldsymbol{\Sigma} \sim \mathrm{N}\left(\boldsymbol{\mu},\boldsymbol{\Sigma}\right); \tag{514}$$

Remark

...

- the investor's priors are defined as follows:

$$\boldsymbol{\mu}|\boldsymbol{\Sigma} \sim \mathsf{N}\left(\boldsymbol{\mu}_0, \frac{\boldsymbol{\Sigma}}{T_0}\right) \tag{515}$$

  and

$$\boldsymbol{\Sigma}^{-1} \sim \mathsf{W}\left(\nu_0, \frac{\boldsymbol{\Sigma}_0^{-1}}{\nu_0}\right), \tag{516}$$

  where the investor's private information is summarized as

$$\mathcal{P} \equiv \{\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0; T_0, \nu_0\}, \tag{517}$$

  $\{\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0\}$ are the investor's private information on the market parameters and $\{T_0, \nu_0\}$ are the confidence of the investor on these private parameters;

Remark

...

- the market public information is defined as follows

$$i_T \equiv \left\{ \hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}; T \right\}, \tag{518}$$

where T is the length of the linear returns time series,

$$\hat{\boldsymbol{\mu}} \equiv \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{l}_{T-t+1} \tag{519}$$

is the sample mean and

$$\hat{\boldsymbol{\Sigma}} \equiv \frac{1}{T} \sum_{t=1}^{T} \left(\boldsymbol{l}_{T-t+1} - \hat{\boldsymbol{\mu}} \right) \left(\boldsymbol{l}_{T-t+1} - \hat{\boldsymbol{\mu}} \right)' \tag{520}$$

is the sample covariance matrix.

Remark

... Consequently, the uncertainty set for $\boldsymbol{\mu}$ is the location-dispersion ellipsoid of the marginal posterior distribution of $\boldsymbol{\mu}$ is given by

$$\hat{\boldsymbol{\Theta}}_{\boldsymbol{\mu}} \equiv \left\{ \boldsymbol{\mu} : (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}_{CE})' \, \hat{\boldsymbol{S}}_{\boldsymbol{\mu}}^{-1} \, (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}_{CE}) \leq \varsigma_{\boldsymbol{\mu}}^2 \right\} \tag{521}$$

and the uncertainty set for $\boldsymbol{\Sigma}$ is the location-dispersion ellipsoid of the marginal posterior distribution of $\boldsymbol{\Sigma}$ is given by

$$\hat{\boldsymbol{\Theta}}_{\boldsymbol{\Sigma}} \equiv \left\{ \boldsymbol{\Sigma} : \text{vech}' \left(\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}_{CE} \right) \hat{S}_{\boldsymbol{\Sigma}}^{-1} \text{vech} \left(\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}_{CE} \right) \leq \varsigma_{\boldsymbol{\Sigma}}^2 \right\}, \tag{522}$$

Remark

... where $\varsigma_{\boldsymbol{\mu}}$ is the radius of the ellipsoid representing the aversion to estimation risk for $\boldsymbol{\mu}$, $\varsigma_{\boldsymbol{\Sigma}}$ is the radius of the ellipsoid representing the aversion to estimation risk for $\boldsymbol{\Sigma}$, vech is the operator that stacks the columns of a matrix skipping the redundant entries above the diagonal; $\hat{\boldsymbol{\mu}}_{CE}$ is the classical-equivalent estimator of $\boldsymbol{\mu}$; $\hat{\boldsymbol{\Sigma}}_{CE}$ is the classical-equivalent estimator of $\boldsymbol{\Sigma}$; $\hat{\boldsymbol{S}}_{\boldsymbol{\mu}}$ is the scatter matrix for $\boldsymbol{\mu}$; and $\hat{\boldsymbol{S}}_{\boldsymbol{\Sigma}}$ is the scatter matrix for $\boldsymbol{\Sigma}$.

Remark

... Finally, the classical-equivalent estimator of $\boldsymbol{\mu}$ is

$$\hat{\boldsymbol{\mu}}_{CE} = \boldsymbol{\mu}_1, \tag{523}$$

the classical-equivalent estimator of $\boldsymbol{\Sigma}$ is

$$\hat{\boldsymbol{\Sigma}}_{CE} = \frac{\nu_1}{\nu_1 + n + 1}\boldsymbol{\Sigma}_1, \tag{524}$$

the scatter matrix for $\boldsymbol{\mu}$ is

$$\hat{\boldsymbol{S}}_{\boldsymbol{\mu}} = \frac{1}{T_1}\frac{\nu_1}{\nu_1 - 2}\boldsymbol{\Sigma}_1 \tag{525}$$

and the scatter matrix for $\boldsymbol{\Sigma}$ is

$$\hat{\boldsymbol{S}}_{\boldsymbol{\Sigma}} = \frac{2\nu_1^2}{(\nu_1 + n + 1)^3}\left[\boldsymbol{D}_n'\left(\boldsymbol{\Sigma}_1^{-1} \otimes \boldsymbol{\Sigma}_1^{-1}\right)\boldsymbol{D}_n\right]^{-1}, \tag{526}$$

Remark

... where D_n is the duplication matrix defined by $\text{vec}\left(\boldsymbol{\Omega}\right) \equiv D_n \text{vech}\left(\boldsymbol{\Omega}\right))$ with $\boldsymbol{\Omega}$, a symmetric matrix, and vec, the operator that stacks the columns of a matrix; $\otimes$ is the Kronecker product;

$$T_1 \equiv T_0 + T; \tag{527}$$

$$\boldsymbol{\mu}_1 \equiv \frac{1}{T_1}\left(T_0 \boldsymbol{\mu}_0 + T \hat{\boldsymbol{\mu}}\right); \tag{528}$$

$$\nu_1 \equiv \nu_0 + T; \tag{529}$$

and

$$\boldsymbol{\Sigma}_1 \equiv \frac{1}{\nu_1}\left[\nu_0 \boldsymbol{\Sigma}_0 + T\hat{\boldsymbol{\Sigma}} + \frac{\left(\boldsymbol{\mu}_0 - \hat{\boldsymbol{\mu}}\right)\left(\boldsymbol{\mu}_0 - \hat{\boldsymbol{\mu}}\right)'}{\frac{1}{T} + \frac{1}{T_0}}\right]. \tag{530}$$

Remark

It is possible to write the robust Bayesian mean-variance asset allocation problem from the previous remark as follows

$$\boldsymbol{w}^*_{RB} \equiv \arg\max_{\boldsymbol{w}} \left\{ \boldsymbol{w}'\boldsymbol{\mu}_1 - \gamma_{\boldsymbol{\mu}} \sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}_1 \boldsymbol{w}} \right\} \tag{531}$$

$$\text{subject to} \left\{ \begin{array}{c} \boldsymbol{w} \in \mathcal{C} \\ \boldsymbol{w}'\boldsymbol{\Sigma}_1 \boldsymbol{w} \leq \gamma_{\boldsymbol{\Sigma}} \end{array} \right. , \tag{532}$$

where

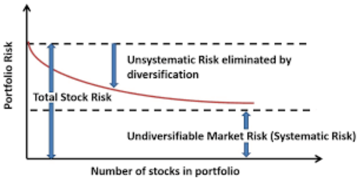
$$\gamma_{\boldsymbol{\mu}} \equiv \sqrt{\frac{\varsigma^2_{\boldsymbol{\mu}}}{T_1} \frac{\nu_1}{\nu_1 - 2}} \tag{533}$$

and

$$\gamma_{\boldsymbol{\Sigma}} \equiv \frac{v}{\frac{\nu_1}{\nu_1+n+1} + \sqrt{\frac{2\nu_1^2 \varsigma^2_{\boldsymbol{\Sigma}}}{(\nu_1+n+1)^3}}}. \tag{534}$$

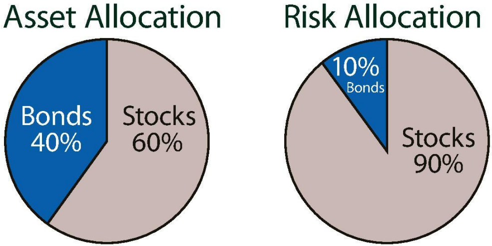
Diversification

- There are some different asset allocation methodologies using entropy measures and divergences to ensure diversification.
- Diversification is a risk management approach to smooth out unsystematic risks from the portfolio.
- Entropy is an accepted measure of diversity. Actually, the greater the level of entropy, the higher the degree of portfolio diversification. The concept of diversification has also slight variations depending on the context.



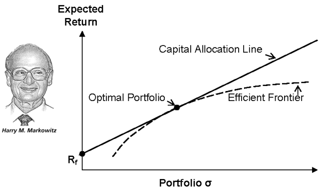
Diversification Approaches

- There are two distinct contexts: the allocation weights-based approaches and the risk-based approaches.



Allocation Weights-Based Approaches

- The idea of diversification is related to the allocation weights over the available asset classes or investment factors. The unrestricted maximum diversification is achieved when the set of allocation weights is given by a uniform distribution.
- The related methodologies are derived from the Markowitz's mean-variance. Mean? Variance?
- Bera and Park (2005) introduced both the maximization of the Shannon entropy and the minimization of the Kullback-Leibler divergence to achieve such diversified allocation weights.



Allocation Weights-Based Approaches

Mean-Variance Asset Allocation

Considering a portfolio of n risky assets, an investor needs to find the allocation weights **w** from the following optimization problem:

$$\max_{\mathbf{w}} \mathbb{E}\,\mathrm{U}\left(\mathbf{w}, \mathbf{R}, \lambda\right) = \max_{\mathbf{w}} \mathbb{E}\left[\mathbf{w}'\mathbf{R}\right] - (\lambda/2)\,\mathrm{Var}\left[\mathbf{w}'\mathbf{R}\right], \tag{535}$$

$$\text{s.t.}: \mathbf{w}'\mathbf{1}_n = 1, \tag{536}$$

where **R** represents the excess returns of the risky assets, $\lambda \geq 0$ is the risk aversion parameter of the investor and $\mathrm{U}\left(\cdot\right)$ a utility function (the expected utility function adopted is exact for elliptical distributions of excess returns).

Allocation Weights-Based Approaches

Entropy Approach for Diversification

Considering a portfolio of n risky assets, an investor needs to find the allocation weights **w** from the following optimization problem to achieve diversification:

$$\max_{\mathbf{w}} - \sum_{i=1}^{n} w_i \log(w_i), \tag{537}$$

$$\text{s.t.}: \mathbf{w} \geq 0, \mathbf{w}'\mathbf{1}_n = 1, \sqrt{\mathbf{w}'\widehat{\Sigma}\mathbf{w}} \leq \sigma_0, \mathbf{w}'\widehat{\mathbf{r}} \geq r_0, \tag{538}$$

where $\widehat{\mathbf{r}}$ is the estimated mean and $\widehat{\Sigma}$ is the estimated covariance of excess returns of the n risky assets. In addition, it is not allowed to short assets and to leverage the portfolio. Finally, σ_0 is the maximum desired risk and r_0 is the minimum desired excess return for the portfolio.

Allocation Weights-Based Approaches

Divergence Approach for Diversification

The Kullback-Leibler divergence $\mathrm{KL}\left(\cdot||\cdot\right)$ is used as the objective function:

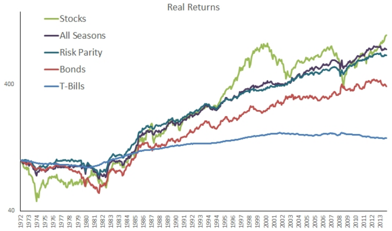
$$\min_{\mathbf{w}} \mathrm{KL}\left(\mathbf{w}||\mathbf{q}\right) = \min_{\mathbf{w}} \sum_{i=1}^{n} w_i \log\left(w_i/q_i\right), \tag{539}$$

$$\text{s.t.} : \mathbf{w} \geq 0, \mathbf{w}'\mathbf{1}_n = 1, \sqrt{\mathbf{w}'\widehat{\Sigma}\mathbf{w}} \leq \sigma_0, \mathbf{w}'\widehat{\mathbf{r}} \geq r_0, \tag{540}$$

where $\mathbf{q}$ is a predefined portfolio. Usually, it is considered $q_i = 1/n$ to ensure portfolio diversification. In addition, it is not allowed to short assets and to leverage the portfolio. Again, σ_0 is the maximum desired risk and r_0 is the minimum desired excess return for the portfolio.

Risk-Based Approaches

- The idea of diversification is related to the risk contribution of each available asset class or investment factor to the total portfolio risk.
- The maximum diversification or the risk parity allocation is achieved when the set of risk contributions is given by a uniform distribution.
- Meucci (2009) introduced the maximization of the Rényi entropy as part of a leverage constrained optimization problem to achieve such diversified risk contributions.



Risk-Based Approaches

Risk Parity

- After the subprime crisis, the risk tolerance of investors decreased and risk-based allocation methodologies have arisen with the idea of combining both risk management and asset allocation. A risk-based allocation seeks for risk diversification and does not use performance forecasts of assets as inputs of the methodologies.
- Risk parity is the most widely used risk-based allocation methodology and has been used by several fund managers. Starting in 1996, one of the pioneers to use risk parity was the *All Weather* hedge fund from Bridgewater.
- The risk parity portfolio has been extensively studied in terms of its properties (Maillard et al., 2010) and compared to other asset allocation heuristics (Chaves et al., 2011).

Risk-Based Approaches

Risk Budgeting

- Risk budgeting is a generalization of risk parity. A theoretical and applied study about risk budgeting techniques is presented by Bruder and Roncalli (2013).
- An example of risk budgeting was investigated by Bruder et al. (2011) where they have studied a portfolio in which the risk contribution of each sovereign bond from a set of countries is done proportional to the gross domestic product (GDP) of the respective country.
- In the literature, there are some proposed optimization frameworks to obtain the risk budgeting and, consequently, risk parity portfolios (Bruder and Roncalli, 2013; Kaya and Lee, 2012).

Risk-Based Approaches

Optimization Frameworks

- The existence and uniqueness (for some cases) of the solution for the optimization problems is proved for particular cases with the imposition of some restrictions such as long-only allocations (Bruder and Roncalli, 2013; Kaya and Lee, 2012), impossibility of leveraging the portfolio (Bruder and Roncalli, 2013) and an upper bound for portfolio volatility (Kaya and Lee, 2012). In this work, these restrictions are relaxed.
- The objective functions used in the optimizations from (Bruder and Roncalli, 2013; Kaya and Lee, 2012) do not possess a strong theoretical basis related to the idea of maximizing risk diversification. However, Meucci (2009) introduced a measure called effective number of bets ENB_α based on Rényi entropy with parameter α. The maximization of the objective function ENB_α brings the idea of maximizing risk diversification.

Risk-Based Approaches

Factors or Assets?

- Deguest et al. (2013), Lohre et al. (2012) and Poddig and Unger (2012) have applied principal component analysis (PCA) to extract uncorrelated factors and analyze the performance of the called factor risk parity (FRP) portfolios.
- Deguest et al. (2013) have developed an optimization framework for FRP portfolios using the concept of ENB_α. Their approach has a leverage restriction and does not include the most general risk budgeting case.

Risk Diversification

The risk-based allocations depend on some measure of each individual asset, asset class or investment factor risk contribution to the total portfolio risk.

Definition 1. (*relative marginal contribution for assets*) Considering n available risky assets, the relative marginal contribution of each asset to the total portfolio risk is given by the following $n \times 1$ vector:

$$\mathbf{p} := \frac{\mathrm{diag}(\mathbf{w})\Sigma\mathbf{w}}{\mathbf{w}'\Sigma\mathbf{w}}, \tag{541}$$

where Σ is the $n \times n$ covariance matrix of risky assets' excess returns, $\mathbf{w}$ is the vector of weights such that $\mathbf{w} \in \Re^{n\times 1}_{\neq \mathbf{0}_{n\times 1}}$ and $\mathrm{diag}(\mathbf{w})$ is a diagonal matrix with $\mathbf{w}$ as its diagonal.

$\mathbf{w} \neq \mathbf{0}_{n\times 1}$ because $\mathbf{w} = \mathbf{0}_{n\times 1}$ represents the absence of allocation in the risky assets and, then, it is not a desired solution. Finally, it is clear that $\sum_{i=1}^{n} p_i = 1$.

Risk Diversification

Meucci (2009) proposed the use of the Rényi entropy with parameter α as a measure of diversification and the measure was called the effective number of bets ENB_α. Instead of using assets or asset classes, ENB_α uses the distribution of the relative marginal risk contribution of each uncorrelated factor as a measure of risk diversification. In practical terms, the uncorrelated factors can be obtained using principal component analysis (PCA). Considering T excess returns of the risky assets represented by the $n \times T$ matrix **r**, the $n \times T$ matrix of uncorrelated factors is given by

$$\mathbf{r}_F = \mathbf{A}'\mathbf{r} \tag{542}$$

and the corresponding covariance matrix is given by

$$\Sigma_F = \mathbf{A}'\Sigma\mathbf{A}, \tag{543}$$

where Σ_F is a positive definite $n \times n$ diagonal matrix and A is an invertible $n \times n$ matrix.

Risk Diversification

We define the relative marginal contribution of each uncorrelated factor to the total portfolio risk analogously to (541).

Definition 2. (*relative marginal contribution for uncorrelated factor*) Considering n available risky assets, the relative marginal contribution of each uncorrelated factor to the total portfolio risk is given by the following $n \times 1$ vector:

$$\mathbf{p}_F := \frac{\mathrm{diag}(\mathbf{w}_F)\Sigma_F \mathbf{w}_F}{\mathbf{w}'_F \Sigma_F \mathbf{w}_F}. \tag{544}$$

where Σ_F is the $n \times n$ diagonal covariance matrix (543), $\mathbf{w}_F$ is the vector of weights such that $\mathbf{w}_F \in \Re^{n\times 1}_{\neq \mathbf{0}_{n\times 1}}$ and $\mathrm{diag}(\mathbf{w}_F)$ is a diagonal matrix with $\mathbf{w}_F$ as its diagonal.

It is important to notice that $\sum_{i=1}^{n} p_{F,i} = 1$ and $p_{F,i} \geq 0, \forall i = 1, \ldots, n$.

Risk Diversification

Using (544), the $\text{ENB}_\alpha(\cdot)$ is defined in the following.

Definition 3. (*effective number of bets*) The effective number of bets of portfolio **w** of n risky assets is given by

$$\mathrm{ENB}_\alpha(\mathbf{w}) := \|\mathbf{p}_F\|_\alpha^{\frac{\alpha}{1-\alpha}}, \alpha \geq 0, \alpha \neq 1, \tag{545}$$

where the $\|\cdot\|_\alpha$ is the α-norm.

Risk Diversification

By definition,

$$\|\mathbf{p}_F\|_\alpha = \left(\sum_{i=1}^{n} p_{F,i}^{\alpha}\right)^{\frac{1}{\alpha}}. \tag{546}$$

Then,

$$\mathrm{ENB}_\alpha(\mathbf{w}) = \left(\sum_{i=1}^{n} p_{F,i}^{\alpha}\right)^{\frac{1}{1-\alpha}}, \alpha \geq 0, \alpha \neq 1. \tag{547}$$

It is important to notice that $\log(\mathrm{ENB}_\alpha(\mathbf{w}))$ is the Rényi entropy $H_\alpha(\mathbf{p}_F)$. In addition, we write ENB_α as a function of $\mathbf{w}$ since $\mathbf{p}_F$ is a function of $\mathbf{w}_F$ (see (544)) and $\mathbf{w}_F$ is a function of $\mathbf{w}$ (see the next property).

Risk Diversification

Property 1. $\mathbf{w}_F = \mathbf{A}^{-1}\mathbf{w}$.

Proof of Property 1. The excess return of the portfolio in terms of assets or uncorrelated factors is the same: $\mathbf{w}'\mathbf{r} = \mathbf{w}_F'\mathbf{r}_F$. Using (542), $\mathbf{w}'\mathbf{r} = \mathbf{w}_F'\mathbf{A}'\mathbf{r} \Rightarrow \mathbf{w} = \mathbf{A}\mathbf{w}_F$. Finally, $\mathbf{w}_F = \mathbf{A}^{-1}\mathbf{w}$.

$\square$

The ENB_α measure achieves its minimum equal to 1 when the portfolio is risk concentrated in only one factor. On the other hand, the ENB_α measure achieves its maximum when the portfolio is totally risk diversified with $p_{F,i} = 1/n, \forall i = 1, \ldots, n$.

Risk Diversification

The following property concerning ENB_α is presented because it will be useful in the following slides...

Property 2. $\mathrm{ENB}_\alpha(\lambda \mathbf{w}) = \mathrm{ENB}_\alpha(\mathbf{w}), \forall \lambda \in \Re_{\neq 0}$.

Proof of Property 2. It is straightforward to state that

$$\mathrm{ENB}_\alpha(\lambda \mathbf{w}) = \mathrm{ENB}_\alpha(\mathbf{p}_F(\mathbf{w}_F(\lambda \mathbf{w}))). \tag{548}$$

Using **Property 1**,

$$\mathrm{ENB}_\alpha(\lambda \mathbf{w}) = \mathrm{ENB}_\alpha(\mathbf{p}_F(\lambda \mathbf{A}^{-1} \mathbf{w})). \tag{549}$$

Risk Diversification

Additionally, using (544), it is trivial to see that

$$\mathbf{p}_F(\lambda \mathbf{w}_F) = \mathbf{p}_F(\mathbf{w}_F), \forall \lambda \in \Re_{\neq 0}. \tag{550}$$

Consequently,

$$\begin{aligned}
\mathrm{ENB}_\alpha(\lambda \mathbf{w}) &= \mathrm{ENB}_\alpha(\mathbf{p}_F(\mathbf{A}^{-1}\mathbf{w})) \\
&= \mathrm{ENB}_\alpha(\mathbf{p}_F(\mathbf{w}_F)) \\
&= \mathrm{ENB}_\alpha(\mathbf{p}_F(\mathbf{w}_F(\mathbf{w}))), \forall \lambda \in \Re_{\neq 0}.
\end{aligned}$$

$$\mathrm{ENB}_\alpha(\lambda \mathbf{w}) = \mathrm{ENB}_\alpha(\mathbf{w}), \forall \lambda \in \Re_{\neq 0}. \tag{551}$$

□

Factor Risk Parity Portfolios

The factor risk parity (FRP) portfolios were defined by Deguest et al. (2013) using the $\mathrm{ENB}_\alpha(\cdot)$ measure. The FRP portfolios were developed under a leverage restriction. Then, we are going to refer to them as restricted FRP (RFRP) portfolios.

Using the $\mathrm{ENB}_\alpha(\cdot)$ measure, the optimization problem to obtain the RFRP portfolios is

$$\max_{\mathbf{w}} \mathrm{ENB}_\alpha(\mathbf{w}), \tag{552}$$

$$\text{s.t. } \mathbf{1}_n'\mathbf{w} = 1, \tag{553}$$

where $\mathbf{1}_n$ is a $n \times 1$ vector of ones. It is important to notice that the restriction in (552) is a non-leverage constraint or budget condition.

Factor Risk Parity Portfolios

Theorem 1. (*RFRP portfolios*) The family of RFRP portfolios is given by

$$\mathbf{w}_{RFRP} = \frac{\mathbf{A}\Sigma_F^{-\frac{1}{2}}\imath_n}{\mathbf{1}_n'\mathbf{A}\Sigma_F^{-\frac{1}{2}}\imath_n}, \tag{554}$$

where $\imath_n := \begin{pmatrix} \pm 1 & \cdots & \pm 1 \end{pmatrix}'$ is a vector of size $n \times 1$ representing all the combinations of ± 1.

Proof of Theorem 1. It was shown by Deguest et al. (2013) that the closed-form expression for the solutions of (552) is given by (554).

$\square$

It is important to notice that (554) implies in 2^{n-1} possible solutions. Additionally, the portfolios with only positive signs coincide with the solutions provided by Maillard et al. (2010).

Factor Risk Parity Portfolios

As we have already mentioned, the non-leverage constraint in (552) is easily relaxed. Our optimization problem to obtain the generalized factor risk parity (GFRP) portfolios is given by

$$\max_{\mathbf{w}} \mathrm{ENB}_{\alpha}(\mathbf{w}). \tag{555}$$

Theorem 2. (*GFRP portfolios*) The family of GFRP portfolios is given by

$$\mathbf{w}_{GFRP} = \lambda \frac{\mathbf{A}\Sigma_F^{-\frac{1}{2}}\imath_n}{\mathbf{1}_n'\mathbf{A}\Sigma_F^{-\frac{1}{2}}\imath_n}, \lambda \in \Re_{\neq 0}. \tag{556}$$

Factor Risk Parity Portfolios

Proof of Theorem 2. It is straightforward to prove (556). Considering a modified version of the optimization problem (552):

$$\mathbf{w}_\lambda = \arg\max_{\mathbf{w}} \mathrm{ENB}_\alpha(\mathbf{w}), \tag{557}$$

$$\text{s.t. } \mathbf{1}_n' \mathbf{w} = \lambda, \lambda \in \Re_{\neq 0}, \tag{558}$$

the solution using **Property 2** is $\mathbf{w}_\lambda = \mathbf{w}_{RFRP}\lambda$. Consequently, it is necessary to solve the problem for each $\lambda \in \Re_{\neq 0}$ to obtain (556).

$\square$

Factor Risk Parity Portfolios

Since the volatility of the portfolio is given by $\sigma_{RFRP} := \sqrt{\mathbf{w}'_{RFRP}\Sigma\mathbf{w}_{RFRP}}$, the volatility of $\mathbf{w}_{GFRP}(\lambda)$ is $\sigma_{GFRP}(\lambda) = |\lambda|\,\sigma_{RFRP}$. It is important to notice that unconstraining the problem, it is possible to obtain risk parity portfolios for any desired volatility or, in other words, risk level. Consequently, our GFRP portfolios are important in terms of asset allocation because it will adapt better to the investor's risk preference than the RFRP portfolios.

Factor Risk Budgeting Portfolios

We generalize the FRP to the factor risk budgeting (FRB). Our optimization problem to obtain the FRB portfolios is given by

$$\min_{\mathbf{w}} D_\alpha \left(\mathbf{p}_F || \mathbf{b}\right), \tag{559}$$

where $\mathbf{b}$ is a $n \times 1$ vector containing the budgets or targets $b_i, i = 1, \ldots, n$ such that $\sum_{i=1}^n b_i = 1$ and $b_i \geq 0$, and $D_\alpha \left(\mathbf{p}_F || \mathbf{b}\right)$ is the Rényi divergence or α-divergence given by

$$D_\alpha \left(\mathbf{p}_F || \mathbf{b}\right) = \frac{1}{\alpha - 1} \log \left(\sum_{i=1}^n \frac{p_{F,i}^\alpha}{b_i^{\alpha-1}} \right), \tag{560}$$

where $0 < \alpha < \infty$ and $\alpha \neq 1$. Considering $\alpha \to 1$, the Rényi divergence becomes the Kullback-Leibler divergence.

Factor Risk Budgeting Portfolios

Obviously, $\mathbf{b}$ works like a probability mass distribution and represents a *prior* to the risk allocation process. Additionally, it is trivial to prove that when $\mathbf{b}$ is uniformly distributed the optimization problem (559) reduces to $\max_{\mathbf{w}} \mathrm{ENB}_\alpha(\mathbf{w})$ (556).

Theorem 3. (*FRB portfolios*) The family of FRB portfolios is given by

$$\mathbf{w}_{FRB} = \lambda \frac{\mathbf{A}\Sigma_F^{-\frac{1}{2}} \imath_n \odot \mathbf{b}^{\odot \frac{1}{2}}}{\mathbf{1}_n' \mathbf{A}\Sigma_F^{-\frac{1}{2}} \imath_n \odot \mathbf{b}^{\odot \frac{1}{2}}}, \lambda \in \Re_{\neq 0}, \tag{561}$$

where $\odot$ is the Hadamard product and $^{\odot}$ is the Hadamard power.

Factor Risk Budgeting Portfolios

Proof of Theorem 3. Since $D_\alpha(\mathbf{p}_F||\mathbf{b})$ is a divergence, $D_\alpha(\mathbf{p}_F||\mathbf{b})$ is minimum and equal to zero when $\mathbf{p}_F = \mathbf{b}$. Consequently,

$$p_{F,k} = b_k \Leftrightarrow \frac{(\sigma_{F,k} w_{F,k})^2}{\mathbf{w}_F' \Sigma_F \mathbf{w}_F} = b_k, \forall k = 1, \ldots, n. \tag{562}$$

Then,

$$w_{F,k} = \pm \frac{\sqrt{\mathbf{w}_F' \Sigma_F \mathbf{w}_F}}{\sigma_{F,k}} \sqrt{b_k}, \forall k = 1, \ldots, n. \tag{563}$$

Using matrix notation, the factor weights are given by:

$$\mathbf{w}_F = \sqrt{\mathbf{w}_F' \Sigma_F \mathbf{w}_F} \Sigma_F^{-\frac{1}{2}} \imath_n \odot \mathbf{b}^{\odot \frac{1}{2}}. \tag{564}$$

Factor Risk Budgeting Portfolios

Considering the relation $\mathbf{w} = \mathbf{A}\mathbf{w}_F$ (see **Property 1**) and the restriction $\mathbf{1}_n'\mathbf{w} = 1$, it is possible to obtain the following expression for leverage-restricted factor weights:

$$\mathbf{w}_{RFRB} = \frac{\mathbf{A}\Sigma_F^{-\frac{1}{2}}\imath_n \odot \mathbf{b}^{\odot\frac{1}{2}}}{\mathbf{1}_n'\mathbf{A}\Sigma_F^{-\frac{1}{2}}\imath_n \odot \mathbf{b}^{\odot\frac{1}{2}}}. \tag{565}$$

Using the same argument from **Proof of Theorem 2**, we obtain (561).

$\square$

Factor Risk Budgeting Portfolios

Considering that $\sigma_{RFRB} := \sqrt{\mathbf{w}'_{RFRB}\Sigma\mathbf{w}_{RFRB}}$, the volatility of $\mathbf{w}_{FRB}(\lambda)$ is given by $\sigma_{FRB}(\lambda) = |\lambda|\, \sigma_{FRB}\,(1)$. It is important to point that the FRB portfolios achieve any desired volatility or, in other words, risk level. Consequently, our FRB portfolios are flexible because it will adapt to the investor's risk preference when varying λ.

Final Remarks

Topics out of the scope of the tutorial:

- Models for Expected Values
- Models for Covariance Matrix
- Models for Transaction Costs
- Asset Liability Management Models
- Multiple-Asset Class Allocation Models
- Multiple-Period Asset Allocation Models

Thank you!

Any questions?