BRUNO BORGES DE SOUZA

LISTA 2

EXERCÍCIO 8.11

• Considere $I = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right) dx$

Usando a transformação $Z = \Sigma^{-1/2}(x-\mu)$ mostrar que

$$I = \sqrt{\det(2\pi\Sigma)}$$

• Primeiro vamos mostrar que $I_0 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$ (Exercício 8.4).

Se a multiplicarmos pela integral $\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy$, temos:

$$I_0^2 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2} dx\, dy$$

$$I_0^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dx\, dy$$

Ao mudarmos para coordenadas polares, observando que

$-\frac{1}{2}(x^2+y^2) = -\frac{r^2}{2}$,

$$I_0^2 = \int_{-0}^{2\pi}\int_{-0}^{\infty} e^{-r^2/2}\, r\, dr\, d\theta$$

Podemos utilizar o Teorema de Fubini, integrando em r e depois em θ, chegando a:

$u = -r^2/2$

$du = -r\,dr \Rightarrow \int_0^{\infty} e^{-r^2/2} r\, d\cancel{r} = -\int_0^{\infty} e^{u} \cancel{r} \frac{du}{\cancel{r}} = -e^{u}\Big|_0^{\infty} = -e^{-r^2/2}\Big|_0^{\infty} = -(0-1) = 1$

$dr = -\frac{du}{r}$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} r\, dr\, d\theta = \int_0^{2\pi} d\theta = 2\pi$$

Como $I_0^2 = 2\pi \therefore I_0 = \sqrt{2\pi}$

· Agora, podemos diagonalizar $\Sigma = EAE^T$, com $E^TE = I$ e $A = diag(\lambda_1, \dots, \lambda_D)$

Se fizermos $Z = \Sigma^{-1/2}(x-\mu)$, temos:

$$Z = EA^{-1/2}E^T(x-\mu)$$

e $$Z^TZ = (x-\mu)^T E A^{-1/2} \underbrace{E^TE}_{I} A^{-1/2} E^T (x-\mu) = (x-\mu)^T \underbrace{E A^{-1} E^T}_{\Sigma^{-1}} (x-\mu)$$

e daí $$\exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right) = \exp\left(-\frac{1}{2}Z^TZ\right) = e^{-\frac{1}{2}\sum_i \lambda_i \gamma_i^2}.$$

O termo $\sum_i \lambda_i \gamma_i^2$, onde λ_i são os valores da diagonal e os γ_i colunas da matriz $E(x-\mu)$, vem da multiplição:

$$Z^TZ = [\gamma_1 \dots \gamma_D] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_D \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_D \end{bmatrix}$$

· Prosseguindo:

$$1 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}Z^TZ} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_i \lambda_i \gamma_i^2} d\gamma = \int_{-\infty}^{\infty} \prod_{i=1}^{D} e^{-\frac{1}{2}\lambda_i \gamma_i^2} d\gamma$$

$$= \prod_{i=1}^{D} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda_i \gamma_i^2} d\gamma_i$$

Usando o fato de que $\int_{-\infty}^{\infty} e^{-1/2\, x^2} dx = \sqrt{2\pi}$:

$$\prod_{i=1}^{D} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda_i \gamma_i^2} d\gamma_i = \prod_{i=1}^{D} (2\pi \lambda_i)^{-\frac{1}{2}} = \sqrt{(2\pi)^D \prod_{i=1}^{D} \lambda_i}$$

· Lembrando que os valores da diagonal de ~~(Σ)~~ A são os autovalores de Σ, e o produto dos autovalores de Σ é o determinante de Σ. Então $\prod_{i=1}^{D} \lambda_i = det(\Sigma)$ e daí:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_i \lambda_i \gamma_i^2} d\gamma = \sqrt{(2\pi)^D |\Sigma|} \quad \text{ou} \quad \sqrt{det(2\pi\Sigma)}$$