On Execution Strategies and Minimum Discrimination Information Principle

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Outline

Electronic Trading

In 1960-70s, systems supporting electronic trading began to appear. Consequently, there was no more need to have a person trading on an exchange floor. In the mid 1990s, several large stock exchanges in USA and Europe were trading a considerable proportion of their volume electronically. Since then, the stock exchanges from regions such as Asia or Latin America also started the migration to electronic trading. Clearly, the equity markets were the fastest in the adoption of electronic trading in the world. On the other hand, the bond markets have been the slowest to enter in the electronic trading era. For a historical review of the electronic trading and the evolution of exchanges see [1].

Electronic Trading

Obviously, the electronic trading has enabled the automation of trading strategies [2]. The increase of investments in high-tech trading environments caused the proliferation of automated trading strategies and the called high-frequency trading (HFT). HFT relies on the high speed and frequency of the trades to obtain gains. However, it is important to notice that only part of the trading strategies is HFT[1]. Trading strategies are investment strategies and there are several possible classifications to group them. Execution strategies, the object of study of this paper, represent a group of trading strategies and they are not necessarily HFT. Actually, an execution strategy is an algorithm that receives an order that must be completely executed with some constraints and its performance is always measured against a predefined benchmark.

[1] For examples of trading strategies considered HFT see [3].

Execution Strategies

In practice, execution strategies are part of the service provided by traders and brokers to their clients[2]. Usually, the clients are portfolio managers from large asset management firms, pension funds, family offices, and so on. The portfolio managers need to execute huge amount of orders in several different markets according to some constraints such as predetermined execution time intervals and low market impact to avoid adverse price distortions. Clearly, the automation of execution strategies is suitable for such a scenario and keeps the operational risks very low. Actually, the best execution is an important concern when talking about trading regulation [4] and the use of automated algorithms makes the execution process easy to audit. In the literature, there are some approaches for the design of execution strategies from *ad hoc* procedures to optimization frameworks [5].

[2] It is important to notice that traders usually have internal clients while brokers have external clients.

Concepts

An order is generated by an investment portfolio manager according to some investment strategy. Then, the order is delivered to a trader or broker to be executed in a stock exchange using an execution strategy. Clearly, an order is the input of an execution strategy and its definition is given in the following.

Definition (Order)

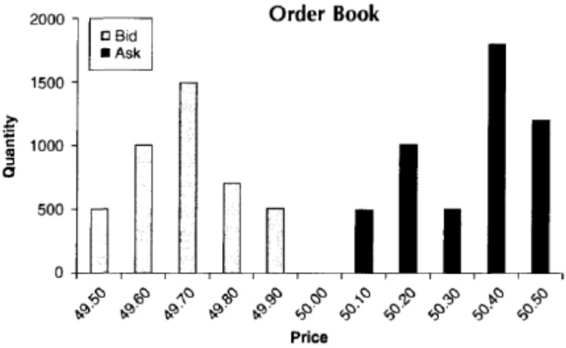
An order is represented by a pair (V, T) such that $V \in \Re_{\neq 0}$ represents an amount of contracts to be executed and $T = [t_s, t_e]$, $t_s < t_e$, $t_s \wedge t_e \in \Re_+$ represents a time interval.

Given an order (V, T), the trader or broker are usually not allowed to change the amount to be executed V or exceed the time constraint T. Additionally, it is important to notice that (V, T) represents a buy or a sell order. We consider $V > 0$ when buying and $V < 0$ when selling.

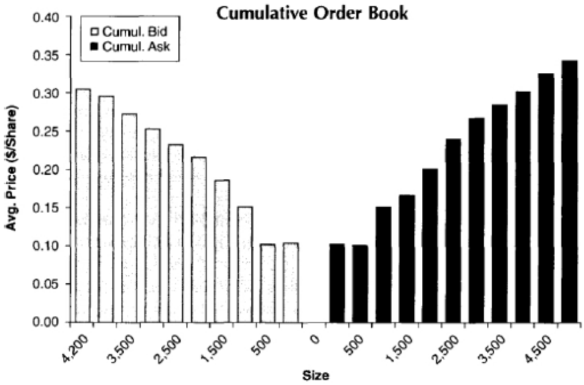
Concepts

	Limit Order Book				
	BID		ASK		
	Qty	Price	Price	Qty	
	100	34.00	38.00	100	
	200	34.00	39.00	600	→ Market Maker
Market Maker ←	600	33.00	39.00	100	
	100	32.00	40.00	200	
	300	32.00	42.00	300	
	100	30.00	42.00	200	
	100	28.00	43.00	100	
			44.00	100	
			45.00	200	

Concepts


Order Book
Bid
Ask
Quantity
2000
1500
1000
500
0
49.50
49.60
49.70
49.80
49.90
50.00
50.10
50.20
50.30
50.40
50.50
Price

Concepts


Cumulative Order Book
Cumul. Bid
Cumul. Ask
Avg. Price ($/Share)
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
4,200
3,500
2,500
1,500
500
0
500
1,500
2,500
3,500
4,500
Size

Concepts

Definition (Execution Strategy)

Given an order (V, T), an execution strategy S is an execution density function $\nu_S : T \to [0,1]$ such that

$$\nu_S(t) \geq 0, \ \forall t \in T \tag{1}$$

and

$$\int_T \nu_S(t)\, dt = 1.^{a} \tag{2}$$

The execution density function ν_S represents the way the amount V is executed over the time interval T in terms of percentage[b]

[a]The equation (2) is called execution completeness since the execution strategy aims to execute all the amount V.

[b]In this paper, we consider the amount of contracts V infinitely divisible. In practice, there is always a minimum lot size.

Concepts

According to the previous definition, $\nu_S(t)V$ represents the amount executed by the execution strategy S at the time instant $t \in T$. After executing the orders, the *ex post* mean price of S is given by

$$\mu_S = \int_T \nu_S\,(t)\,p\,(t)\,dt, \tag{3}$$

where $p(t), t \in T$ represents the *ex post* prices assumed continuous over the execution interval T. The quantity μ_S is important because represents a measure of performance of the execution strategy and it is independent of the amount executed V. Additionally, $\mu_S V$ represents the financial value that will impact the investment portfolio. When buying ($V > 0$), $\mu_S V$ represents a cash outflow. When selling ($V < 0$), $\mu_S V$ represents a cash inflow.

Best Price

Naively, the idea is to obtain a lower value of μ_S when buying or a higher value of μ_S when selling. The performance of execution strategies are always measured against an *ex post* benchmark. In this context, we define the best price (BP) metric and the BP execution strategy.

Definition (BP Metric)

Given (V, T), the BP metric μ_{BP} is given by

$$\mu_{BP} = \min_t \operatorname{sgn}(V) p\left(t\right), \tag{4}$$

where sgn is the signum function.

Definition (BP Execution Strategy)

The execution strategy that aims to have as its *ex post* mean price the BP metric μ_{BP} is the BP execution strategy ν_{BP}.

Best Price

The difficulty in determining ν_{BP} is that the price over T is a stochastic process $P\left(t\right), t \in T$. Using the known *ex post* continuous $p\left(t\right), t \in T$ and assuming the existence of $t_{BP} = \arg\min_t \operatorname{sgn}(V) p\left(t\right)$, we have $\mu_{BP} = p(t_{BP})$. Since

$$\nu_{BP} = \arg\min_{\nu_S} \operatorname{sgn}(V) \int_T \nu_S\left(t\right) p\left(t\right) dt, \tag{5}$$

subject to (1) and (2), we have

$$\nu_{BP}\left(t\right) = \delta(t - t_{BP}), \tag{6}$$

where δ is the Dirac delta function.

Best Price

It is evident that the implementation of BP execution strategies in the real world is very difficult since we do not have p *a priori*. However, even knowing the price trajectory p *a priori* the solution (6) is unfeasible. The lack of liquidity causes problems when V is large and the execution becomes impossible or causes adverse price distortion. It is necessary to take into account the market liquidity over T. In other words, the market liquidity is the source of contracts from where the executions are made possible.

Opposite to the idea of market liquidity is the idea of execution impact cost. We define execution impact cost in the following slide...

Execution Impact Cost

Definition (Execution Impact Cost)

Given (V, T), the execution impact cost of an execution strategy S with its execution density function ν_S is represented by

$$\phi_S(\nu_S, \ell) = D\left(\nu_S \| \, \ell\right) = \int_T \nu_S\left(t\right) \ln\left(\frac{\nu_S\left(t\right)}{\ell\left(t\right)}\right) dt, \tag{7}$$

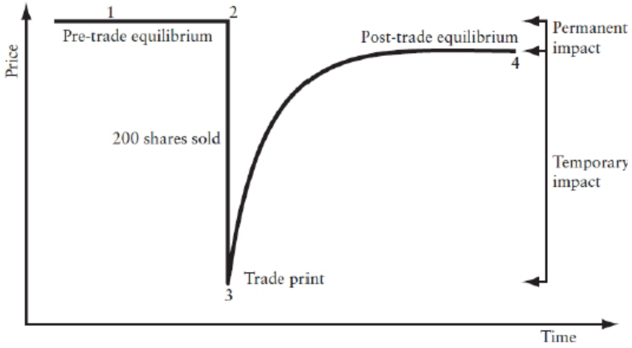
where $D(.\|.)$ is the Kullback-Leibler divergence and $\ell : T \to [0, 1]$ is the market liquidity density function such that

$$\ell\left(t\right) \geq 0, \;\forall t \in T \tag{8}$$

and

$$\int_T \ell\left(t\right) dt = 1. \tag{9}$$

Concepts



Price
1
2
Pre-trade equilibrium
Post-trade equilibrium
4
Permanent
impact
200 shares sold
Temporary
impact
Trade print
3
Time

Execution Impact Cost

In the context of execution strategies, the natural idea is to minimize the execution impact cost ϕ_S. Basically, the minimization of ϕ_S is an application of the minimum discrimination information principle. For example, considering the most uninformative density function for the market liquidity (i.e. $\ell \propto 1$ - a uniform market liquidity density function), it is obvious that (6) does not minimize the divergence (7). Actually, the solution (6) maximizes the divergence (7) and it represents one of the worst execution strategies from this point of view.

Time Weighted Average Price

It was discussed the difficulties to implement BP strategies and the fact that they represent the worst case from the execution impact cost point of view. On the other hand, the time weighted average price (TWAP) metric μ_{TWAP} is an achievable alternative benchmark used in practice.

Definition (TWAP Metric)

The TWAP metric μ_{TWAP} is given by

$$\mu_{TWAP} = \frac{1}{\tau} \int_T p(t)\, dt, \tag{10}$$

where τ is the amount of time for the execution ($\tau = \int_T dt = t_e - t_s$).

Time Weighted Average Price

It is very important to differentiate the TWAP metric from the TWAP execution strategy. The first is a benchmark μ_{TWAP} used to analyze the performance of an execution strategy while the second is an execution strategy described by the execution density function ν_{TWAP}.

Definition (TWAP Execution Strategy)

The execution strategy that aims to have as its *ex post* mean price the TWAP metric μ_{TWAP} is the TWAP execution strategy ν_{TWAP}.

The following theorem obtains ν_{TWAP} using the minimum discrimination information principle...

Time Weighted Average Price

Theorem

Given (V, T), the TWAP execution strategy is the minimum discrimination information execution strategy and $\nu_{TWAP}(t), t \in T$ is the uniform execution density function under no other information.

Proof.

Under no other information, $\ell \propto 1$ over the execution interval T and it is possible to write

$$\ell(t) = \frac{1}{\tau}, \forall t \in T, \tag{11}$$

where $\tau = \int_T dt$.

Applying the minimum discrimination information principle on the execution impact cost (...)

Time Weighted Average Price

Proof.

(...)

$$\nu_{S^*} = \arg\min_{\nu_S} D\left(\nu_S \| \, \ell\right) \tag{12}$$

subject to (1) and (2), it results using properties of Kullback-Leibler divergence (see [7]) that

$$\nu_{S^*}\left(t\right) = \frac{1}{\tau}, \forall t \in T. \tag{13}$$

The *ex post* mean price of the execution strategy S^* is given by

$$\mu_{S^*} = \int_T \nu_{S^*}\left(t\right) p\left(t\right) dt = \frac{1}{\tau}\int_T p\left(t\right) dt. \tag{14}$$

(...)

Time Weighted Average Price

Proof.

(...) Obviously, $\mu_{S^*} = \mu_{TWAP}$ and, consequently, S^* is the TWAP execution strategy. Finally,

$$\nu_{TWAP}(t) = \frac{1}{\tau}, \forall t \in T. \tag{15}$$

□

Despite the behavior of the stochastic process of the prices $P(t)$, it is important to notice that the TWAP execution strategy (15) always produces the TWAP metric μ_{TWAP} as its mean price. Consequently, TWAP metric is easily achieved. However, TWAP is useful only when there is no other information to be used in the execution process.

Volume Weighted Average Price

TWAP metric considers the price of a trade with a smaller volume as important as the price of a trade with a larger volume! Obviously, the metric would be more fair if considering prices weighted by the respective executed volumes. Then, it is important to define volume weighted average price (VWAP) metric.

Volume Weighted Average Price

Definition (VWAP Metric)

The VWAP metric μ_{VWAP} is given by

$$\mu_{VWAP} = \int_T \vartheta(t)\, p(t)\, dt, \tag{16}$$

where $\vartheta : T \to [0, 1]$ is the executed volume density function such that

$$\vartheta(t) \geq 0,\ \forall t \in T \tag{17}$$

and

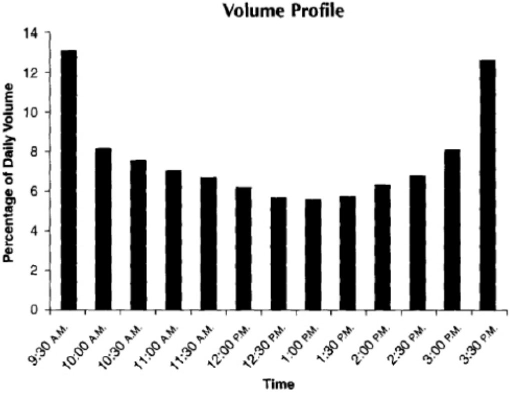
$$\int_T \vartheta(t)\, dt = 1. \tag{18}$$

Volume Weighted Average Price

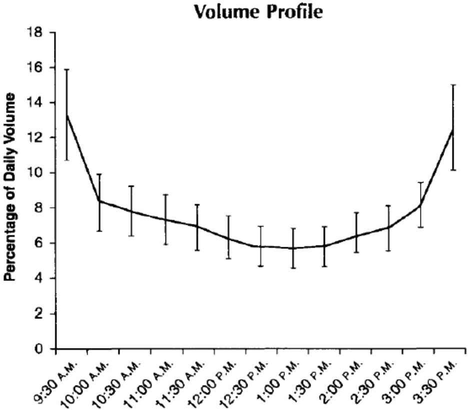
The executed volume density function ϑ is an execution density function representing all the executed trades in a market and not the executed trades by only one execution strategy as in the execution density function definition. It is important to state that ϑ is known only *a posteriori*. It is intuitive that the market liquidity $\ell(t)$, the source of contracts to execution strategies, can be proxied by $\vartheta(t)$.

For equities, it is known that the total executed trades $U(t), t \in T_d$ of a market over the trading day T_d has a pattern. Usually, the pattern has a U-shape and practitioners obtain this pattern using historical executed volumes of the last 21 days [5]. The U-shape implies higher volume of executed trades during the beginning and at the end of the day. Additionally, it is possible to take $\vartheta \propto U$.

Volume Weighted Average Price



Volume Weighted Average Price



Volume Weighted Average Price

Definition (VWAP Execution Strategy)

The execution strategy that aims to have as its *ex post* mean price the VWAP metric μ_{VWAP} is the VWAP execution strategy ν_{VWAP}.

The TWAP metric is very naive because the related optimum execution strategy, the TWAP execution strategy, distributes uniformly the volume V over the execution interval T and does not take into account any other information from the market. The following theorem obtains ν_{VWAP} using the minimum discrimination information principle...

Volume Weighted Average Price

Theorem

The VWAP execution strategy is the minimum discrimination information execution strategy and the $\nu_{VWAP}(t)$ is the executed volume density function $\vartheta(t)$ when market liquidity density function $\ell(t)$ is represented by $\vartheta(t)$ over the execution interval T.

Proof.

Since $\ell(t) = \vartheta(t)$ over the execution interval T and applying the minimum discrimination information principle on the execution impact cost

$$\nu_{S^*} = \arg\min_{\nu_S} D\left(\nu_S \| \vartheta\right) \tag{19}$$

subject to (1) and (2), it results using properties of Kullback-Leibler divergence (see [7]) that (...)

Volume Weighted Average Price

Proof.

(...)

$$\nu_{S^*}(t) = \vartheta(t), \forall t \in T. \tag{20}$$

The *ex post* mean price of the execution strategy S^* is given by

$$\mu_{S^*} = \int_T \nu_{S^*}(t)\, p(t)\, dt = \int_T \vartheta(t) p(t)\, dt. \tag{21}$$

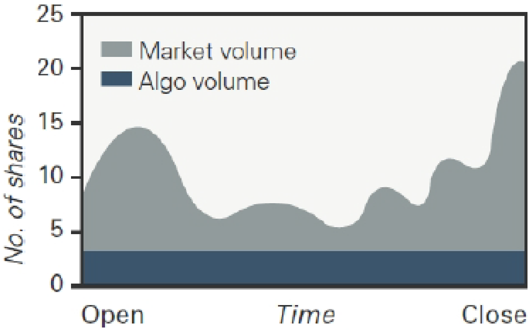
Obviously, $\mu_{S^*} = \mu_{VWAP}$ and, consequently, S^* is the VWAP execution strategy. Finally,

$$\nu_{VWAP}(t) = \vartheta(t), \forall t \in T. \tag{22}$$

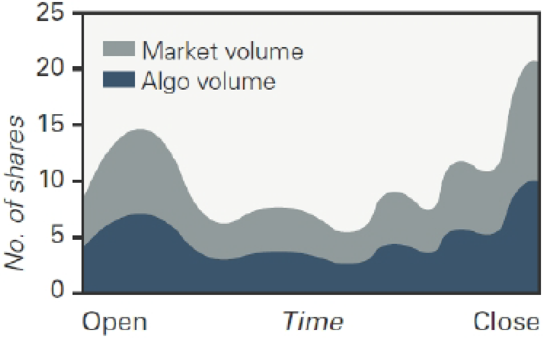
□

Clearly, μ_{VWAP} is more difficult to achieve than μ_{TWAP} because ϑ is not known *a priori*!

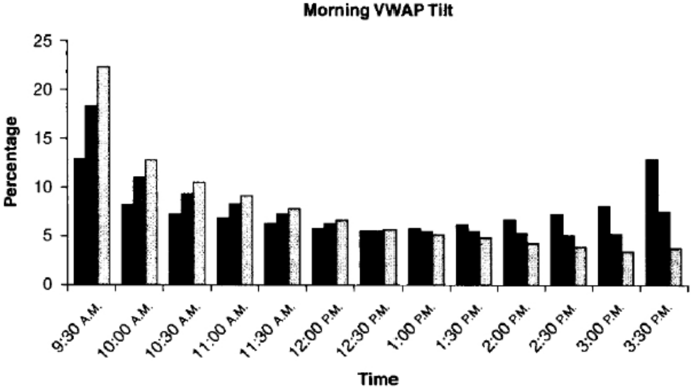
Illustration



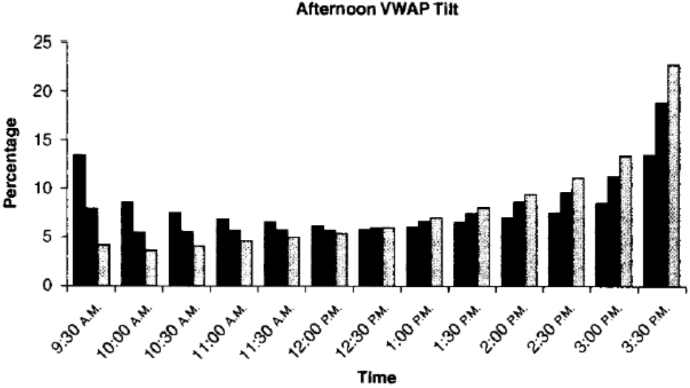
Illustration



VWAP Tilt


Morning VWAP Tilt
25
20
15
10
5
0
Percentage
9:30 A.M.
10:00 A.M.
10:30 A.M.
11:00 A.M.
11:30 A.M.
12:00 P.M.
12:30 P.M.
1:00 P.M.
1:30 P.M.
2:00 P.M.
2:30 P.M.
3:00 P.M.
3:30 P.M.
Time

VWAP Tilt


Afternoon VWAP Tilt
Percentage
25
20
15
10
5
0
9:30 A.M.
10:00 A.M.
10:30 A.M.
11:00 A.M.
11:30 A.M.
12:00 P.M.
12:30 P.M.
1:00 P.M.
1:30 P.M.
2:00 P.M.
2:30 P.M.
3:00 P.M.
3:30 P.M.
Time

VWAP Tilt

Traders and brokers like to be able to change the behavior of VWAP execution strategy according to some beliefs they have or some constraints from their clients. The VWAP tilt algorithms represent execution strategies derived from the pure VWAP execution strategy. Given an order (V, T), a common constraint is to fix the percentage $\gamma, 0 \leq \gamma \leq 1$ of V to be executed at the beginning or end of T.

Generically, we have the following problem

$$\nu_{VWAP,\gamma} = \arg\min_{\nu_S} D\left(\nu_S \| \vartheta\right), \tag{23}$$

subject to (1), (2), $\int_{t_s}^{t_\gamma} \nu_S(t)\, dt = \gamma$ and $\int_{t_\gamma}^{t_e} \nu_S(t)\, dt = 1 - \gamma$, where $0 \leq \gamma \leq 1$ and $t_\gamma \in T = [t_s, t_e]$.

VWAP Tilt

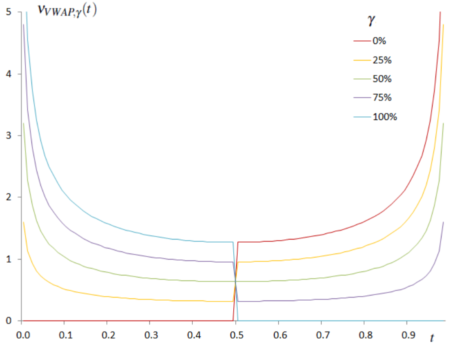
Solving analytically the functional and a system of non-linear equations, an optimal solution to the optimization problem is given by

$$\nu_{VWAP,\gamma}(t) = \begin{cases} \frac{\gamma\vartheta(t)}{\int_{t_s}^{t_\gamma}\vartheta(x)dx}, & \forall t \in [t_s, t_\gamma] \\ \frac{(1-\gamma)\vartheta(t)}{\int_{t_\gamma}^{t_e}\vartheta(x)dx}, & \forall t \in]t_\gamma, t_e] \end{cases} . \quad (24)$$

In the literature, $\vartheta(t), t \in T$ has been approximated by a beta density function Beta(α, β) [9].

VWAP Tilt

We illustrate some VWAP tilt execution strategy densities obtained for different values of γ considering $\alpha = \beta = 0.5$, $T = [0, 1]$ and $t_\gamma = 0.5$. Obviously, $\gamma = 50\%$ represents the VWAP execution strategy. VWAP γ-tilted execution strategy for some values of γ $(0\%, 25\%, 50\%, 75\%, 100\%)$:



Conclusions

- We introduce for the first time information theory to the study of execution strategies.
- We propose an execution impact cost function based on Kullback-Leibler divergence.
- We derive the well-known TWAP and VWAP execution strategies using the minimum discrimination information principle.

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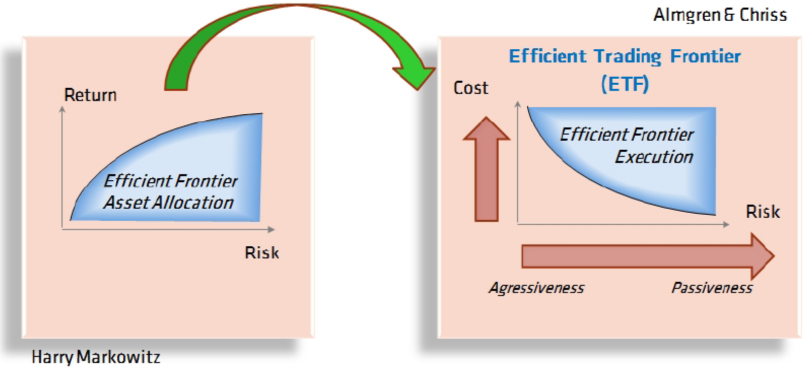
Almgren & Chriss
Efficient Trading Frontier
(ETF)
Return
Efficient Frontier
Asset Allocation
Risk
Harry Markowitz
Cost
Efficient Frontier
Execution
Risk
Agressiveness
Passiveness