Introduction to Graphs

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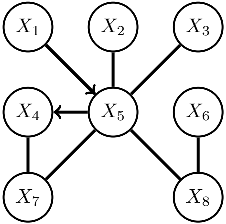
Graphs



Definition

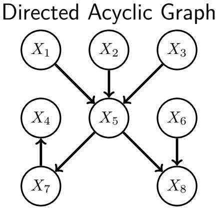
A graph consists of nodes (vertices) and undirected or directed links (edges) between nodes.

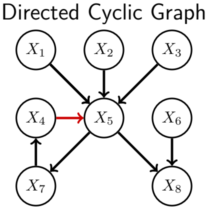
Path

A path from X_i to X_j is a sequence of connected nodes starting at X_i and ending at X_j.

Directed Graphs

All the edges are directed:





DAG

Directed Acyclic Graph: Graph in which by following the direction of the arrows a node will never be visited more than once.

Parents and Children:

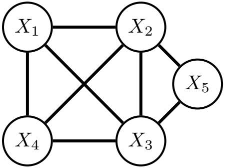
X_i is a parent of X_j if there is a link from X_i to X_j. X_i is a child of X_j if there is a link from X_j to X_i.

Ancestors and Descendants:

The ancestors of a node X_i are the nodes with a directed path ending at X_i. The descendants of X_i are the nodes with a directed path beginning at X_i.

Undirected Graph

All the edges are undirected:



Clique

A clique is a fully connected subset of nodes. (X_1, X_2, X_4) forms a (non-maximal) clique. A non-maximal clique is called cliquo.

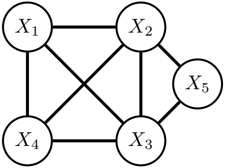
Maximal Clique

Clique which is not a subset of a larger clique. (X_1, X_2, X_3, X_4) and (X_2, X_3, X_5) are both maximal cliques.

Connectivity

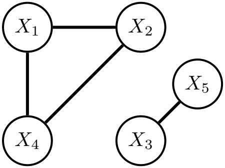
Connected graph

There is a path between every pair of vertices:



Connected components

In a non-connected graph, the connected components are the connected-subgraphs:

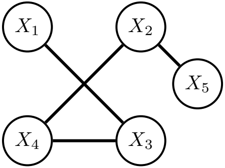


(X_1, X_2, X_4) and (X_3, X_5) are the two connected components.

Connectedness

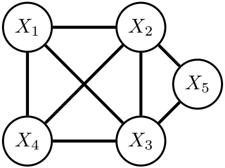
Singly-connected

There is only one path from any node a to another other node b



Multiply-connected

A graph is multiply-connected if it is not singly-connected:

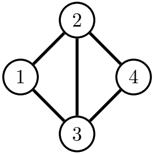


Numerically Encoding Graphs

Edge List

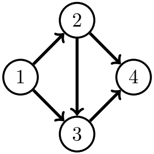
An edge list L is the set of vertex-vertex pairs in the graph.

a)



$$L = \{(1,2)\,,(2,1)\,,(1,3)\,,(3,1)\,,(2,3)\,,(3,2)\,,(2,4)\,,(4,2)\,,(3,4)\,,(4,3)\}$$

b)



$$L = \{(1,2)\,,(1,3)\,,(2,3)\,,(2,4)\,,(3,4)\}$$

Numerically Encoding Graphs

Adjacency Matrix

An adjacency matrix $\mathbf{A}$ is such that $A_{i,j} = 1$ if there is an edge from node i to j, and $A_{i,j} = 0$ otherwise.

a)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

b)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If the nodes are labeled in ancestral order and the graph is directed, the adjacency matrix is called triangular adjacency matrix.

Numerically Encoding Graphs

Clique Matrix

A clique matrix $\mathbf{C}$ contains all maximal cliques, each maximal clique described in one column of the matrix.

a)

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Cliquo Matrix

A cliquo matrix $\mathbf{C}_n$ contains all the n-dimensional maximal cliques, each n-dimensional maximal clique described in one column of the matrix.

a)

$$\mathbf{C}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

In particular, a 2-dimensional cliquo matrix matrix is called incidence matrix.