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On the Black-Scholes European Option Pricing Model Robustness and Generality

Hellinton Hatsuo Takada[a] and José de Oliveira Siqueira[b]

[a] *Banco Itaú S.A.*
[b] *University of São Paulo*

Abstract. The common presentation of the widely known and accepted Black-Scholes European option pricing model explicitly imposes some restrictions such as the geometric Brownian motion assumption for the underlying stock price. In this paper, these usual restrictions are relaxed using maximum entropy principle of information theory, Pearson's distribution system, market frictionless and risk-neutrality theories to the calculation of a unique risk-neutral probability measure calibrated with market parameters.

Keywords: Black-Scholes European Option Pricing Model, Maximum-Entropy Principle, Information Theory, Risk-Neutral Probability, Pearson's Distribution System.
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I. INTRODUCTION

The usual approaches for Black-Scholes modeling of European option explicitly require a geometric Brownian motion assumption for the underlying stock price, market completeness, dynamically completeness and efficiency. The necessity of all these restrictions is questionable when using information theory elements. In this paper, these restrictions are relaxed and a risk-neutral probability measure, an implied volatility estimator and a pricing formula for European options are presented.

The concept of entropy is usually introduced to provide a quantitative measure of the uncertainty or missing information associated with each probability distribution. Additionally, the Maximum Entropy Principle (MEP) is commonly applied to obtain a unique distribution that is considered the least prejudiced distribution compatible with the given constraints or known information.

Some families of distributions, called systems of distributions, have been constructed which are intended to provide approximations to as wide a variety of observed distributions as it is possible. Pearson Distribution System (PDS) provides one solution by allowing putting the boundary condition into the constraints of the entropy maximization problem. Combining the market frictionless and risk-neutrality theories with MEP and PDS, it is possible to develop a probability distribution without pre-assuming a stochastic process for the underlying state variable.

This paper is organized as follows: the relationship between market expectation and the entropy concept is given in section II; the determination of a unique risk-neutral probability measure using MEP and PDS is done in section III; in section IV, the pricing of the European option is discussed. Finally, conclusions are presented in section V.

II. MARKET EXPECTATION AND ENTROPY CONCEPT

Entropy is a measure of the average uncertainty in a random variable. It is necessary to assume that there are N investors in the market to build a relationship between the entropy concept from information theory and market beliefs associated with asset prices or rate of return in finance. The no arbitrage principle assumes that all investors prefer more to less. Additionally, since there is no information to give a higher weight to some particular investor's point of view the assumption that each investor's beliefs are equally probable is applicable. The market belief by definition is the arithmetic average of all investors' beliefs [1] [2]. Suppose there are m states associated with the future asset price S_T. Investor k believes that state j occurs with probability p_{kj}, where $k = 1,...,N$ and $j = 1,...,m$. Obviously, the investor belief is supposed to be related to a unique state. Consequently, the market belief for state j is expressed as

$$p_j = \frac{\left(\sum_{k=1}^{N} p_{kj}\right)}{N}, j = 1,...,m. \quad (1)$$

The entropy for a market belief is given by:

$$H(S_\tau) = \frac{1}{N} \ln \frac{N!}{(Np_1)!(Np_2!)...(Np_m!)}. \quad (2)$$

A formal proof to establish a relationship between the market beliefs and entropy is presented in [3]. Considering a large number of investors, the entropy depends just on the different states:

$$H(S_\tau) = -\sum_{j=1}^{m} p_j \ln p_j . \quad (3)$$

Describing the beliefs using a probability density function $f_{S_\tau}(s_\tau)$ the entropy definition changes to the following [4]:

$$H(S_\tau) = -\int_P f_{S_\tau}(s_\tau) \ln\left[f_{S_\tau}(s_\tau)\right] ds_\tau, s_\tau \in P. \quad (4)$$

III. RISK-NEUTRAL PROBABILITY MEASURE

The estimation and specification of the risk-neutral probability measure is a central theme in finance. Using the entropic approach, in this section, and considering the maximum market ignorance principle it is presented a robust and parsimonious alternative method, based on the MEP of the Information Theory and on PDS, to the calculation of a unique uncertainty-price risk-neutral probability measure. Before presenting the obtained results, the following definitions are specified:

Definition 1. *Market Frictionless*: There are no transaction costs, no taxes, no restrictions on short sales and no difference between borrowing and lending rates.

Definition 2. *Risk-Neutrality:* Trades takes place continuously without arbitrage.

The unique risk-neutral probability distribution obtained is summarized in the following theorem:

Theorem 1: The informational set $I = \{\xi; r; s_0; \sigma^2\}$ is given, where $\xi = [a; b]$ is a real interval such that $0 < a < b$, r is the riskless interest rate, s_0 is the current price of the underlying asset and σ^2 is an upper bound for the market belief of the implied volatility. I corresponds time to expiration or maturity date τ. Additionally, current European option prices for the underlying asset considered are $\{c_i\}$ for strikes $\{k_{\tau,i}\}$, $i = 1, \ldots, m$. The unique maximum entropy risk-neutral probability density function of the random variable representing the uncertainty price for a risk-neutral and frictionless market is a beta probability density function:

$$q_{S_\tau}(s_\tau) = \frac{1}{B(\alpha,\beta)} \frac{\left[u(s_\tau - a) - u(s_\tau - b)\right](s_\tau - a)^{\alpha-1}(b - s_\tau)^{\beta-1}}{(b-a)^{\alpha+\beta-1}}. \tag{5}$$

The beta probability function parameters α and β are the solutions of the following system of equations:

$$\beta = \frac{\alpha\left[b - s_0 \exp(r\tau)\right]}{\left[s_0 \exp(r\tau) - a\right]} \tag{6}$$

$$\begin{aligned}\sum_{i=1}^{m}\left[(b-k_{\tau,i})B_{\frac{b-k_{\tau,i}}{b-a}}(\beta,1+\alpha) - (k_{\tau,i}-a)B_{\frac{b-k_{\tau,i}}{b-a}}(\beta+1,\alpha)\right] &= \\ = B(\alpha,\beta)\sum_{i=1}^{m} c_i \exp(r\tau)\end{aligned} \tag{7}$$

Finally, the beta, incomplete beta and unit step functions are the standard ones with the following respective definitions:

$$B(\alpha,\beta)=\int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy\,, \tag{8}$$

$$B_z(\alpha,\beta)=\int_0^z y^{\alpha-1}(1-y)^{\beta-1}dy, z\in[0;1]\,, \tag{9}$$

$$u(\eta)=\begin{cases}1,\eta\geq 0\\ 0,\eta<0\end{cases}. \tag{10}$$

Proof of Theorem 1: The unique maximum entropy risk-neutral probability density function of the random variable representing the uncertainty price for a risk-neutral and frictionless market is the solution of the following programming problem considering the maximum market ignorance principle:

$$q_S(s_\tau)=\arg\max_{f_{S_\tau}(s_\tau)}-\int_\xi f_{S_\tau}(s_\tau)\ln\left[f_{S_\tau}(s_\tau)\right]ds_\tau$$

s.t.:

$$(i)\int_\xi f_{S_\tau}(s_\tau)ds_\tau=1 \text{ and } f_{S_\tau}(s_\tau)>0,\forall s_\tau\in\xi$$

$$(ii)\int_\xi s_\tau f_{S_\tau}(s_\tau)ds_\tau=s_0\exp(r\tau)$$

$$(iii)\int_\xi s_\tau^2 f_{S_\tau}(s_\tau)ds_\tau\leq\sigma^2+\left[s_0\exp(r\tau)\right]^2$$

$$(iv)\int_\xi \ln(s_\tau-a)f_{S_\tau}(s_\tau)ds_\tau=E_{f_{S_\tau}}\left[\ln(S_\tau-a)\right]$$

$$(v)\int_\xi \ln(b-s_\tau)f_{S_\tau}(s_\tau)ds_\tau=E_{f_{S_\tau}}\left[\ln(b-S_\tau)\right]$$

$$(vi)\int_\xi \max\{s_\tau-k_{\tau,i},0\}f_{S_\tau}(s_\tau)ds_\tau=c_i\exp(r\tau),i=1,...,m$$

Explanation of the restrictions imposed: (*i*) guarantees a probability density function, (*ii*) represents the arbitrage-free assumption, (*iii*) indicates the upper bound on the market belief for the implied volatility, (*iv*) and (*v*) are domain restrictions equally adopted by [1] [2] and, finally, (*vi*) is an additional market calibration to take into account current option prices.

The corresponding Lagrangian for the programming problem is given by:

$$
\begin{aligned}
\ell\left(f_{S_\tau}(s_\tau)\right) = &-\int_\xi f_{S_\tau}(s_\tau)\ln\left[f_{S_\tau}(s_\tau)\right]ds_\tau + \lambda_0\left[\int_\xi f_{S_\tau}(s_\tau)\,ds_\tau - 1\right] \\
&+\lambda_1\left[\int_\xi s_\tau f_{S_\tau}(s_\tau)\,ds_\tau - s_0\exp(r\tau)\right] \\
&+\lambda_2\left[\int_\xi \ln(s_\tau - a) f_{S_\tau}(s_\tau)\,ds_\tau - E_{f_{S_\tau}}\left[\ln(S_\tau - a)\right]\right] \\
&+\lambda_3\left[\int_\xi \ln(b - s_\tau) f_{S_\tau}(s_\tau)\,ds_\tau - E_{f_{S_\tau}}\left[\ln(b - S_\tau)\right]\right] \\
&+\sum_{i=1}^{m}\lambda_4^i\left[\int_\xi \max\left\{s_\tau - k_{\tau,i}, 0\right\} f_{S_\tau}(s_\tau)\,ds_\tau - c_i\exp(r\tau)\right]
\end{aligned}
.
$$

Optimizing the functional and using the F.O.C.:

$$
\begin{aligned}
\ln q_{S_\tau}(s_\tau) = &-1 + \hat{\lambda}_0 + \hat{\lambda}_1 s_\tau + \\
&+\hat{\lambda}_2\ln(s_\tau - a) + \hat{\lambda}_3\ln(b - s_\tau) + \sum_{i=1}^{m}\hat{\lambda}_4^i\max\left\{s_\tau - k_{\tau,i}, 0\right\}
\end{aligned}
.
$$

Differentiating by $q_{S_\tau}(s_\tau)$:

$$
\frac{1}{q_{S_\tau}(s_\tau)}\frac{dq_{S_\tau}(s_\tau)}{ds_\tau} = \hat{\lambda}_1 + \frac{1}{s_\tau - a}\hat{\lambda}_2 - \frac{1}{b - s_\tau}\hat{\lambda}_3 + \sum_{i=1}^{m} I_{k_{\tau,i}}(s_\tau)\hat{\lambda}_4^i,
$$

where $I_{k_{\tau,i}}(s_\tau) = \begin{cases} 0, & k_{\tau,i} \geq s_\tau \\ 1, & k_{\tau,i} < s_\tau \end{cases}$ is an indicator function.

Multimodal distributions are not physically possible for the case in study. The following restriction must be respected to avoid getting multimodal distributions:

$$
\hat{\lambda}_1 + \sum_{i=1}^{m} I_{k_{\tau,i}}(s_\tau)\hat{\lambda}_4^i = 0
$$

and, consequently, the following PDS is obtained:

$$
\frac{1}{q_{S_\tau}(s_\tau)}\frac{dq_{S_\tau}(s_\tau)}{ds_\tau} = \frac{\left(-b\hat{\lambda}_2 - a\hat{\lambda}_3\right) + \left(\hat{\lambda}_2 + \hat{\lambda}_3\right)s_\tau}{ab - s_\tau(a+b) + s_\tau^2}
$$

Performing a variable substitution:

$$\hat{\lambda}_2 = \alpha - 1$$
$$\hat{\lambda}_3 = \beta - 1$$

and imposing restriction (*i*), the relationship follows:

$$(b-a)^{\alpha+\beta-1} \exp\left(\hat{\lambda}_0 - \sum_{i=1}^{m} I_{k_{\tau,i}}(s_\tau) \hat{\lambda}_4^i k_{\tau,i} - 1 \right) \stackrel{\Delta}{=} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{1}{B(\alpha,\beta)}.$$

Consequently, the risk-neutral probability density function from Eq. 5 is obtained. From the fact the probability density function is beta and using (*ii*), it follows Eq. 6. Moreover, combining the conditions (*vi*) the following expression is obtained:

$$\sum_{i=1}^{m} \left[(b-k_{\tau,i}) \int_0^{b-k_{\tau,i}} y^{\beta-1}(b-a-y)^{\alpha-1}\, dy - \int_0^{b-k_{\tau,i}} y^{\beta}(b-a-y)^{\alpha-1}\, dy \right] =$$
$$= B(\alpha,\beta)(b-a)^{\alpha+\beta-1} \exp(r\tau) \sum_{i=1}^{m} c_i.$$

Simplifying the last expression, it provides Eq. 7.

□

To assure model consistency, restriction (*iii*) must be verified after evaluating α and β for the beta probability density function. Since the expectation and variance of a beta-distributed random variable are well known, the following inequality must be respected for model validity purposes:

$$\frac{\alpha\beta(b-a)^2}{(\alpha+\beta)^2(\alpha+\beta+1)} - \frac{(\beta a + \alpha b)^2}{(\alpha+\beta)^2} \leq \sigma^2. \tag{11}$$

[1] and [2] also obtained different beta distributions independently. However, the restrictions imposed in these works were different and our results consider more market parameters for calibration enabling a better approximation to the reality.

From the same arguments used to obtain Eq. 11, an estimator for the implied volatility is presented in the next corollary, which is straightforward to prove:

Corollary 2: An estimator for the implied volatility of the underlying asset of a European option under the same assumptions of Theorem 1 is:

$$\sigma_{S_\tau} = \frac{1}{\alpha+\beta}\sqrt{\frac{\alpha\beta}{\alpha+\beta+1}(b-a)^2 - (\alpha b+\beta a)^2}\,. \tag{12}$$

IV. EUROPEAN OPTION PRICING

In this section, it is derived an option-pricing formula. The result is introduced in the next corollary for a European call, since the expression for a European put is analogous:

Corollary 3: The exact pricing formula for a European call for a specific strike under the same assumptions of Theorem 1 is:

$$c_0(k_\tau) = \frac{\exp(-r\tau)}{B(\alpha,\beta)(b_\tau - a_\tau)^{\alpha+\beta-1}} \cdot \left[\int_{k_\tau}^{b_\tau} (s_\tau - k_\tau)(s_\tau - a_\tau)^{\alpha-1}(b_\tau - s_\tau)^{\beta-1}\, ds_\tau\right] \tag{13}$$

Proof of Corollary 3: The demonstration is straightforward if considering the following well-known equality:

$$c_0(k_\tau) = \exp(-r\tau)\,\mathrm{E}_{q_{S_\tau}}\left[\max\{S_\tau - k_\tau; 0\}\right]$$

and substituting the expectation by its definition.

□

V. CONCLUSION

The usual application approaches of the widely known and accepted Black-Scholes model for European option pricing explicitly impose some restrictions such as the geometric Brownian motion assumption for the underlying stock price, market completeness, dynamically completeness and efficiency. In this paper, these restrictions were relaxed and a risk-neutral probability measure, an implied volatility estimator and a pricing formula were presented.

The future is uncertain but, despite this fact, there is some information useful today to restrict the possible outcomes. More than one distribution can fit the available information. The MEP and PDS gives some hint as how to proceed and they are the base of the approach presented in this paper to introduce a Black-Scholes model imposing just market frictionless and risk-neutrality.

Clearly, the approach adopted estimates a distribution using Newtonian calculus simplifying the mathematical demonstrations while Ito calculus is usually adopted for these purposes. A beta distribution obtained in this work describes the random variable representing the underlying asset price. Different beta distributions were also obtained independently by [1] and [2]. However, the restrictions imposed in these works were different and our results consider more market parameters for calibration enabling a better approximation to the reality.

Future works will include extensive simulations using market data to compare the obtained formulas with usually encountered ones in the related literature.

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