Decisions

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Expected Utility

You are asked if you wish to take a bet on the outcome of tossing a fair coin. If you bet and win, you gain $£100$. If you bet and lose, you lose $£200$. If you don't bet, the cost to you is zero.

$$U(\text{win, bet}) = 100 \quad U(\text{lose, bet}) = -200$$
$$U(\text{win, no bet}) = 0 \quad U(\text{lose, no bet}) = 0$$

Our expected winnings/losses are:

$$\begin{aligned} U(\text{bet}) &= p(\text{win}) \times U(\text{win, bet}) + p(\text{lose}) \times U(\text{lose, bet}) \\ &= 0.5 \times 100 - 0.5 \times 200 = -50 \end{aligned}$$

$$U(\text{no bet}) = 0$$

Based on taking the decision which maximises expected utility, we would therefore be advised not to bet.

Utility of Money

You have $£1,000,000$ in your bank account. You are asked if you would like to participate in a fair coin tossing bet in which, if you win, your bank account will become $£1,000,000,000$. However, if you lose, your bank account will contain only $£1000$. Should you take the bet?

$$U(\text{bet}) = 0.5 \times 1,000,000,000 + 0.5 \times 1000 = 500,000,500$$

$$U(\text{no bet}) = 1,000,000$$

Based on expected utility, we are therefore advised to take the bet.

Utility of Money

- In reality few people who are millionaires are likely to be willing to risk losing almost everything in order to become a billionaire.
- This means that the subjective utility of money is not simply the quantity of money.
- We follow the definitions and concepts of the adopted book!

Decision Trees

Consider the decision problem as to whether or not to go ahead with a fund-raising garden party. If we go ahead with the party and it subsequently rains, then we will lose money (since very few people will show up). If we don't go ahead with the party and it doesn't rain we're free to go and do something else fun.

$$p(Rain = \text{rain}) = 0.6,\ p(Rain = \text{no rain}) = 0.4$$

$$U\,(\text{party}, \text{rain}) = -100, \quad U\,(\text{party}, \text{no rain}) = 500$$

$$U\,(\text{no party}, \text{rain}) = 0, \quad U\,(\text{no party}, \text{no rain}) = 50$$

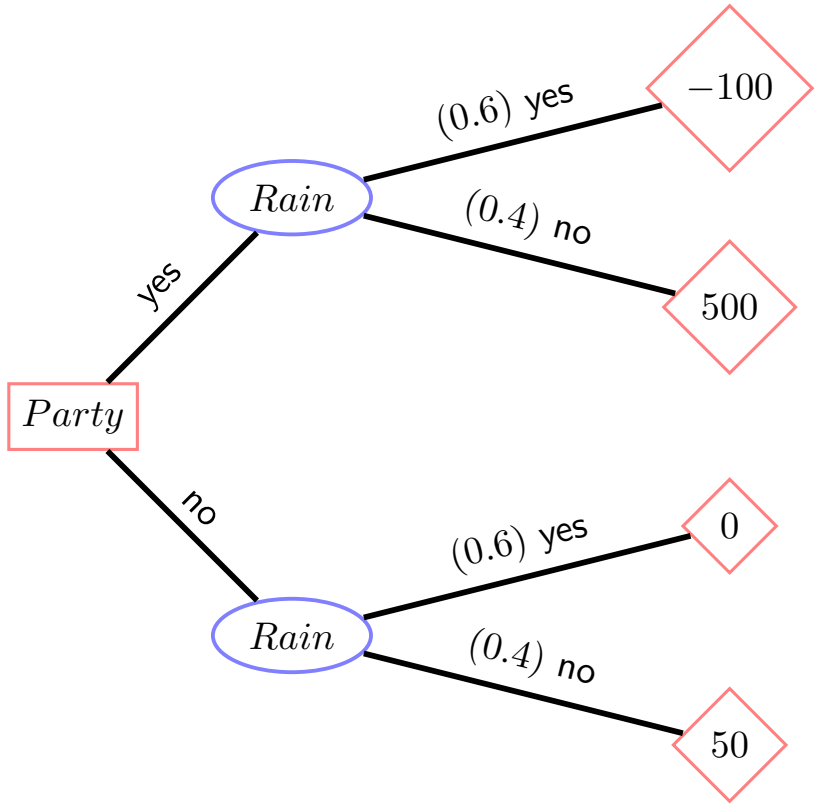
Should we go ahead with the party? Since we don't know what will actually happen to the weather, we compute the expected utility of each decision:

$$U\,(\text{party}) = \sum_{Rain} U(\text{party}, Rain)p(Rain) = -100 \times 0.6 + 500 \times 0.4 = 140$$

$$U\,(\text{no party}) = \sum_{Rain} U(\text{no party}, Rain)p(Rain) = 0 \times 0.6 + 50 \times 0.4 = 20$$

Based on expected utility, we are therefore advised to go ahead with the party.

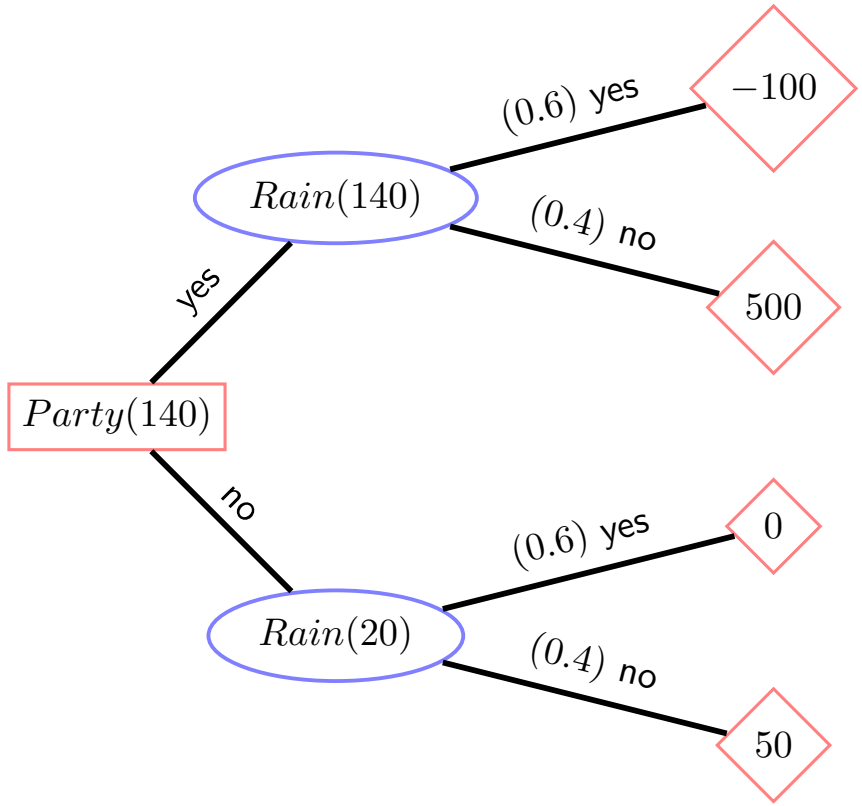
Decision Trees



A decision tree containing chance nodes (denoted with ovals), decision nodes (denoted with squares) and utility nodes (denoted with diamonds). A DT is not a belief network. A DT is an explicit enumeration of the possible choices that can be made, beginning with the leftmost decision node, with probabilities on the links out of 'chance' nodes.

Solving a Decision Tree

Working backwards from the leaves: A chance parent is the expectation of its children. A decision parent is the max of its children.



$$-100 \times 0.6 + 500 \times 0.4 = 140$$

$$0 \times 0.6 + 50 \times 0.4 = 20$$

$$\max(140, 20) = 140$$

Solving a Decision Tree

Matlab

```
>> setup;
>> demoDecParty;
```

The Party-Friend problem

If we decide not to go ahead with the party, we will consider going to visit a friend. In making the decision not to go ahead with the party we have utilities

$$U_{party}\,(\text{no party}, \text{rain}) = 0, \quad U_{party}\,(\text{no party}, \text{no rain}) = 50$$

$$U_{party}\,(\text{party}, \text{rain}) = -100, \quad U_{party}\,(\text{party}, \text{no rain}) = 500$$

$$p(Rain = \text{rain}) = 0.6, \quad p(Rain = \text{no rain}) = 0.4$$

The probability that the friend is in depends on the weather according to

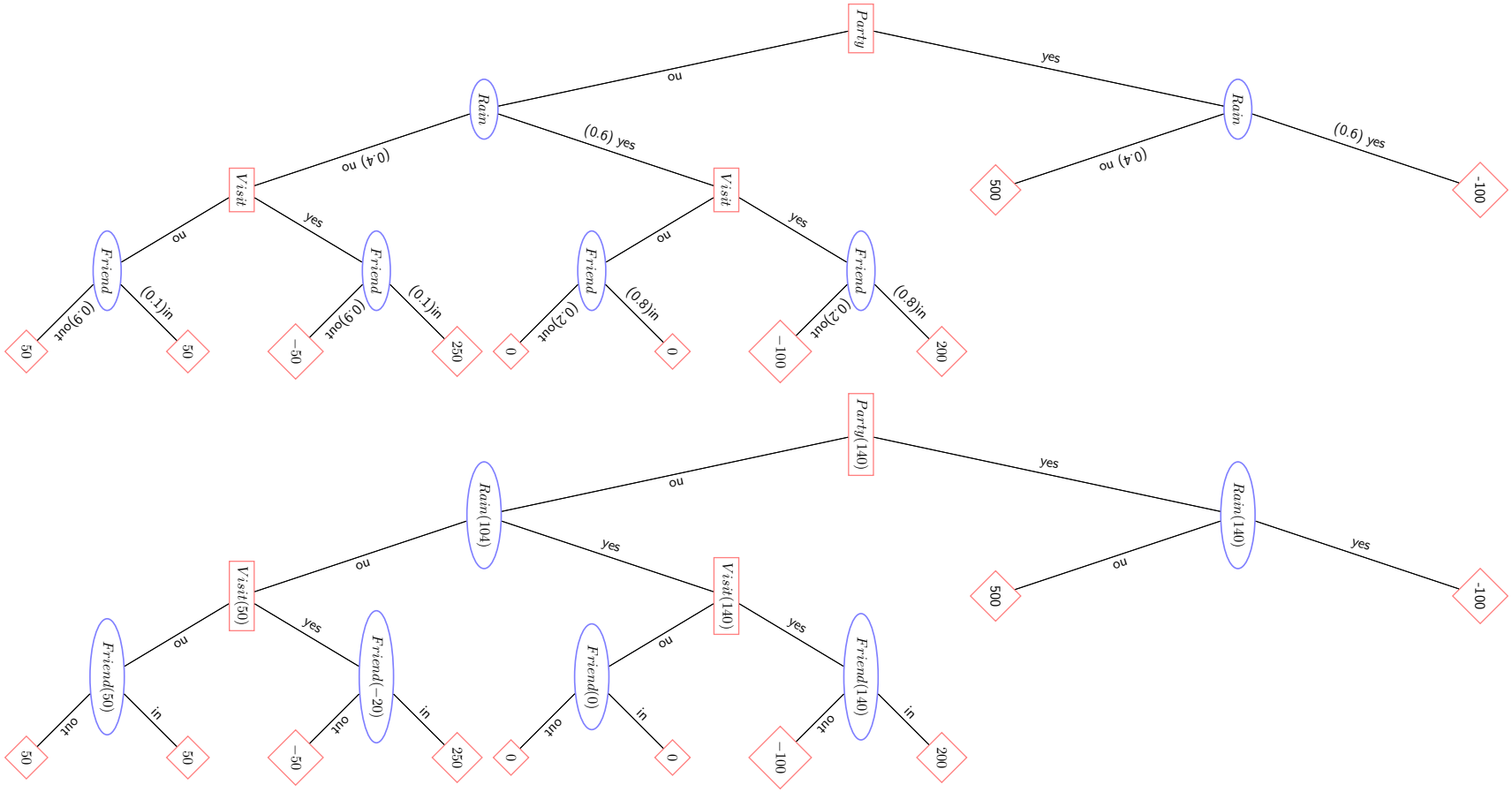
$$p(Friend = \text{in}|\text{rain}) = 0.8, \quad p(Friend = \text{in}|\text{no rain}) = 0.1,$$

The other probabilities are determined by normalization. Additionally, we have

$$U_{visit}\,(\text{friend in}, \text{visit}) = 200, \quad U_{visit}\,(\text{friend out}, \text{visit}) = -100$$

with the remaining utilities zero. The two sets of utilities add up so that the overall utility of any decision sequence is $U_{party} + U_{visit}$.

A bigger DT...



DTs can get very unwieldily even for simple problems. Need a more compact graphical description.

Solving the Party-Friend problem...

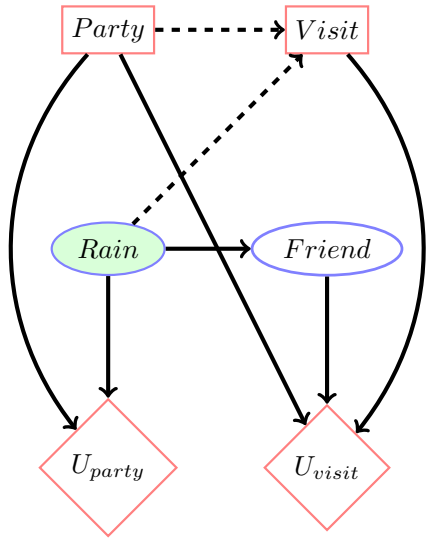
Matlab

```
>> setup;
>> demoDecPartyFriend;
```

Influence Diagrams (ID)

The optimal expected utility for the Party-Friend example is given by summing over un-revealed variables and optimizing over future decisions:

$$\max_{Party} \sum_{Rain} p(Rain) \max_{Visit} \sum_{Friend} p(Friend|Rain)$$
$$\times \left[U_{party}(Party, Rain) + U_{visit}(Visit, Friend)\mathbb{I}\left[Party = \text{no}\right]\right]$$

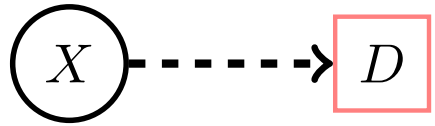


The partial ordering is $Party^* \prec Rain \prec Visit^* \prec Friend$.

An ID looks similar to a DT... However, a DT is analogous to a state-transition diagram. An ID is analogous to a belief network.

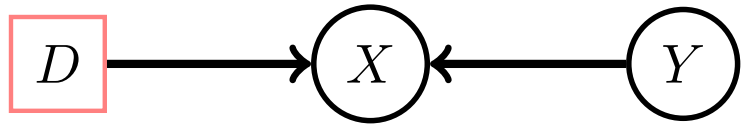
Influence Diagrams

Information Links An information link from a random variable into a decision node



indicates that the state of the variable X will be known before decision D is taken. Information links from another decision node d into D similarly indicate that decision d is known before decision D is taken. We use a dashed link to denote that decision D is not functionally related to its parents.

Random Variables Random variables may depend on the states of parental random variables (as in belief networks), but also decision node states:



As decisions are taken, the states of some random variables will be revealed. To emphasise this we typically shade a node to denote that its state will be revealed during the sequential decision process.

Utilities A utility node is a deterministic function of its parents. The parents can be either random variables or decision nodes.

Partial Ordering

- Begin by writing those variables $\mathcal{X}_0$ whose states are known (evidential variables) before the first decision D_1.
- Then find that set of variables $\mathcal{X}_1$ whose states are revealed before the second decision D_2.
- Subsequently the set of variables $\mathcal{X}_t$ is revealed before decision D_{t+1}.
- The remaining fully-unobserved variables are placed at the end of the ordering:

$$\mathcal{X}_0 \prec D_1 \prec \mathcal{X}_1 \prec D_2, \ldots, \prec \mathcal{X}_{n-1} \prec D_n \prec \mathcal{X}_n$$

with $\mathcal{X}_k$ being the variables revealed between decision D_k and D_{k+1}.

'Partial' refers to the fact that there is no order implied amongst the variables within the set $\mathcal{X}_n$. For notational clarity, we may indicate decision variables with * to reinforce that we maximise over these variables, and sum over the non-starred variables. Where the sets are empty we may omit writing them.

Information Links

No forgetting assumption

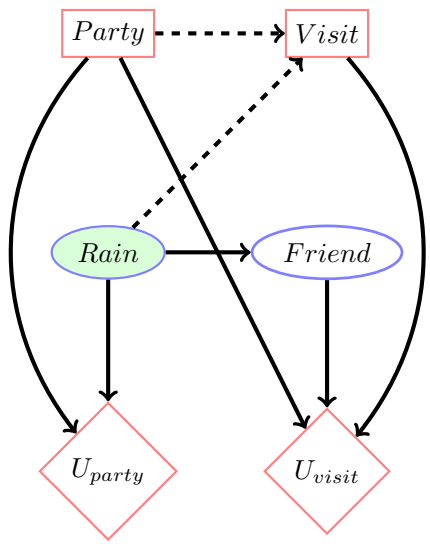
- An information link specifies explicitly which quantities are known before that decision is taken.
- We also implicitly assume that all past decisions and revealed variables are available at the current decision.
- (If we were to include all such information links, IDs would get potentially rather messy.)

Fundamental links

We call an information link fundamental if its removal would alter the partial ordering.

Causal Consistency

- For an Influence Diagram to be consistent a current decision cannot affect the past.
- This means that any random variable descendants of a decision D in the ID must come later in the partial ordering.
- Assuming no-forgetting, this means that for any valid ID there must be a directed path connecting all decisions. This can be a useful check on the consistency of an ID.
- An ID defines a partial ordering of the nodes.



Should I do a PhD?

Consider a decision whether or not to do PhD as part of our education (E). Taking a PhD incurs costs, U_C both in terms of fees, but also in terms of lost income. However, if we have a PhD, we are more likely to win a Nobel Prize (P), which would certainly be likely to boost our Income (I), subsequently benefitting our finances (U_B). The ordering is (excluding empty sets)

$$E^* \prec \{I, P\}$$

$\mathrm{dom}(E) = (\text{do PhD}, \text{no PhD})$, $\mathrm{dom}(I) = (\text{low}, \text{average}, \text{high})$,
$\mathrm{dom}(P) = (\text{prize}, \text{no prize})$.

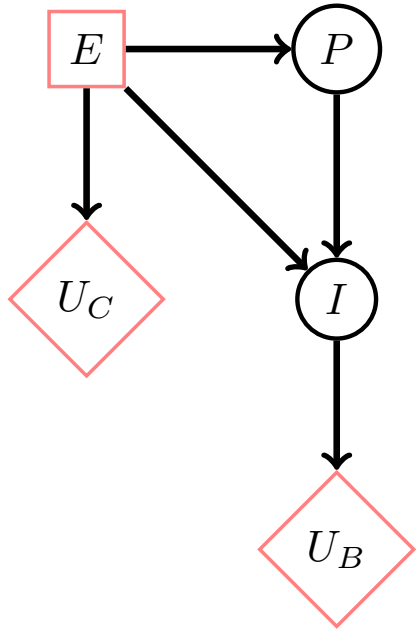
$$p(\text{win Nobel prize}|\text{no PhD}) = 0.0000001 \qquad p(\text{win Nobel prize}|\text{do PhD}) = 0.001$$

$p(\text{low}\|\text{do PhD, no prize}) = 0.1$	$p(\text{average}\|\text{do PhD, no prize}) = 0.5$	$p(\text{high}\|\text{do PhD, no prize}) = 0.4$
$p(\text{low}\|\text{no PhD, no prize}) = 0.2$	$p(\text{average}\|\text{no PhD, no prize}) = 0.6$	$p(\text{high}\|\text{no PhD, no prize}) = 0.2$
$p(\text{low}\|\text{do PhD, prize}) = 0.01$	$p(\text{average}\|\text{do PhD, prize}) = 0.04$	$p(\text{high}\|\text{do PhD, prize}) = 0.95$
$p(\text{low}\|\text{no PhD, prize}) = 0.01$	$p(\text{average}\|\text{no PhD, prize}) = 0.04$	$p(\text{high}\|\text{no PhD, prize}) = 0.95$

The utilities are

$$U_C(\text{do PhD}) = -50000, \quad U_C(\text{no PhD}) = 0,$$
$$U_B(\text{low}) = 100000, \quad U_B(\text{average}) = 200000, \quad U_B(\text{high}) = 500000$$

Should I do a PhD?



The expected utility of Education is

$$U(E) = \sum_{I,P} p(I|E,P)p(P|E)\left[U_C(E) + U_B(I)\right]$$

so that $U(\text{do phd}) = 260174.000$, whilst not taking a PhD is $U(\text{no phd}) = 240000.0244$, making it on average beneficial to do a PhD.

Matlab

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>> setup; demoDecPhD;
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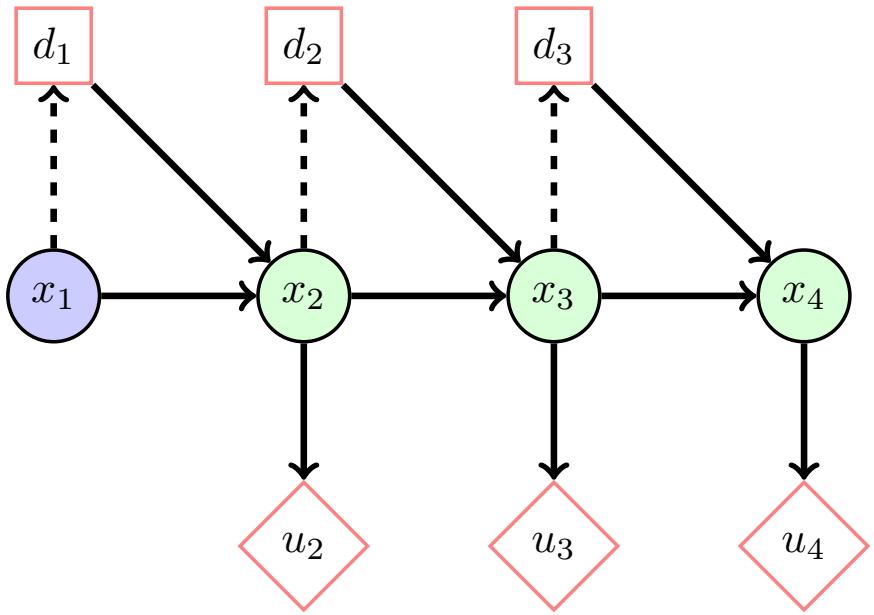
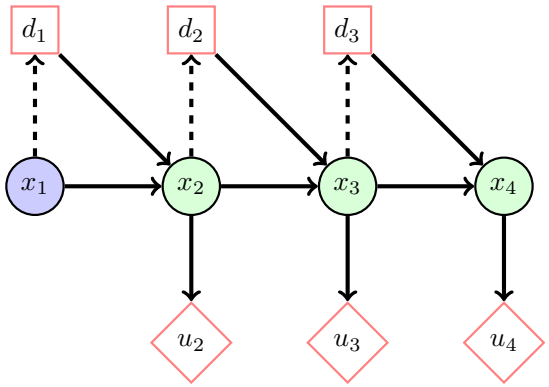
Markov Decision Process (MDP)



Can be used to model planning problems of the form 'how do I get to where I want to be incurring the lowest total cost?'.

Markov Decision Process



We want to make that decision d_1 that will lead to maximal expected total utility

$$U(d_1|x_1) \equiv \sum_{x_2} \max_{d_2} \sum_{x_3} \max_{d_3} \sum_{x_4} \ldots \max_{d_{T-1}} \sum_{x_T} p(x_{2:T}|x_1, d_{1:T-1}) U(x_{1:T})$$

$$U(x_{1:T}) = u(x_2) + u(x_3) + \ldots + u(x_T)$$

Our task is to compute $U(d_1|x_1)$ for each state of d_1 and then choose that state with maximal expected total utility. To carry out the summations and maximisations efficiently, we can use a simple message passing approach.

Markov Decision Process

To decide on how to take the first optimal decision, we need to compute

$$\sum_{x_2} \max_{d_2} \sum_{x_3} \max_{d_3} \sum_{x_4} p(x_4|x_3,d_3)p(x_3|x_2,d_2)p(x_2|x_1,d_1)\left(u(x_2)+u(x_3)+u(x_4)\right)$$

Since only $u(x_4)$ depends on x_4 explicitly, we can write

$$\sum_{x_2} \max_{d_2} \sum_{x_3} p(x_3|x_2,d_2)p(x_2|x_1,d_1)\left(u(x_2)+u(x_3)+\max_{d_3}\sum_{x_4} p(x_4|x_3,d_3)u(x_4)\right)$$

Defining a message and corresponding value

$$u_{3\leftarrow 4}(x_3) \equiv \max_{d_3}\sum_{x_4} p(x_4|x_3,d_3)u(x_4), \qquad v(x_3) \equiv u(x_3)+u_{3\leftarrow 4}(x_3)$$

we can write

$$U(d_1|x_1) = \sum_{x_2}\max_{d_2}\sum_{x_3} p(x_3|x_2,d_2)p(x_2|x_1,d_1)\left(u(x_2)+v(x_3)\right)$$

In a similar manner, only the last term depends on x_3 and hence

$$U(d_1|x_1) = \sum_{x_2} p(x_2|x_1,d_1)\left(u(x_2)+\max_{d_2}\sum_{x_3} p(x_3|x_2,d_2)v(x_3)\right)$$

Markov Decision Process

Defining similarly the value

$$v(x_2) \equiv u(x_2) + \max_{d_2} \sum_{x_3} p(x_3|x_2, d_2) v(x_3)$$

then

$$U(d_1|x_1) = \sum_{x_2} p(x_2|x_1, d_1) v(x_2)$$

Given $U(d_1|x_1)$ above, we can then find the optimal decision d_1 by

$$d_1^*(x_1) = \operatorname*{argmax}_{d_1} U(d_1|x_1)$$

Bellman's Equation

In an MDP we can define utility messages recursively as

$$u_{t-1\leftarrow t}(x_{t-1}) \equiv \max_{d_{t-1}} \sum_{x_t} p(x_t|x_{t-1}, d_{t-1}) \left[u(x_t) + u_{t\leftarrow t+1}(x_t)\right]$$

The value

It is more common to define the value of being in state x_t as

$$v_t(x_t) \equiv u(x_t) + u_{t\leftarrow t+1}(x_t), \qquad v_T(x_T) = u(x_T)$$

and write then the equivalent recursion

$$v_{t-1}(x_{t-1}) = u(x_{t-1}) + \max_{d_{t-1}} \sum_{x_t} p(x_t|x_{t-1}, d_{t-1}) v_t(x_t)$$

Backtracking

The optimal decision d_t^* is then given by

$$d_t^* = \operatorname*{argmax}_{d_t} \sum_{x_{t+1}} p(x_{t+1}|x_t, d_t) v_{t+1}(x_{t+1})$$

Temporally Unbounded MDPs

The infinite T case would appear to be ill-defined since the sum of utilities

$$u(x_1) + u(x_2) + \ldots + u(x_T)$$

will in general be unbounded. If we let $u^* = \max_s u(s)$ be the largest value of the utility and consider the sum of modified utilities for a chosen discount factor $0 < \gamma < 1$

$$\sum_{t=1}^{T} \gamma^t u(x_t) \leq u^* \sum_{t=1}^{T} \gamma^t = \gamma u^* \frac{1-\gamma^T}{1-\gamma}$$

In the limit $T \to \infty$ this means that the summed modified utility $\gamma^t u(x_t)$ is finite:

$$\lim_{T\to\infty} \sum_{t=1}^{T} \gamma^t u(x_t) \leq u^* \frac{\gamma}{1-\gamma}$$

Temporally Unbounded MDPs

The only modification required to our previous discussion is to include a factor γ in the message definition. Assuming that we are at convergence, we define a value $v(x_t = \mathsf{s})$ dependent only on the state s, and not the time.
This means we replace the time-dependent Bellman's value recursion with the time-independent equation

$$v(\mathsf{s}) \equiv u(\mathsf{s}) + \gamma \max_{\mathsf{d}} \sum_{\mathsf{s}'} p(x_t = \mathsf{s}' | x_{t-1} = \mathsf{s}, d_{t-1} = \mathsf{d}) v(\mathsf{s}')$$

We then need to solve for the value $v(\mathsf{s})$ for all states s. The optimal decision policy when one is in state $x_t = \mathsf{s}$ is then given by

$$\mathsf{d}^*(\mathsf{s}) = \operatorname*{argmax}_{\mathsf{d}} \sum_{\mathsf{s}'} p(x_{t+1} = \mathsf{s}' | x_t = \mathsf{s}, d_t = \mathsf{d}) v(\mathsf{s}')$$

For a deterministic transition p (*i.e.* for each decision d, only one state s′ is available), this means that the best decision is the one that takes us to the accessible state with highest value.

Value Iteration

In the $T = \infty$ case, we need to actually know the value to take a decision. This means we need to solve Bellman's $T = \infty$ equations first.

- A naive procedure is to iterate the value recursion until convergence, assuming some initial guess for the values (say uniform)

$$v^{new}(\mathsf{s}) \equiv u(\mathsf{s}) + \gamma \max_{\mathsf{d}} \sum_{\mathsf{s}'} p(x_t = \mathsf{s}' | x_{t-1} = \mathsf{s}, d_{t-1} = \mathsf{d}) v^{old}(\mathsf{s}')$$

- One can show that this value iteration procedure is guaranteed to converge to a unique optimum.
- The convergence rate depends somewhat on the discount γ – the smaller γ is, the faster is the convergence.

Policy Iteration

First assume we know the optimal decision $\mathsf{d}^*(\mathsf{s})$ for any state s. Then

$$v(\mathsf{s}) = u(\mathsf{s}) + \gamma \sum_{\mathsf{s}'} p(x_t = \mathsf{s}' | x_{t-1} = \mathsf{s}, \mathsf{d}^*(\mathsf{s})) v(\mathsf{s}')$$

The maximisation over d has disappeared since we have assumed we already know the optimal decision for each state s. For fixed $\mathsf{d}^*(\mathsf{s})$, the above is now linear in the value. Defining the value $\mathbf{v}$ and utility $\mathbf{u}$ vectors and transition matrix $\mathbf{P}$,

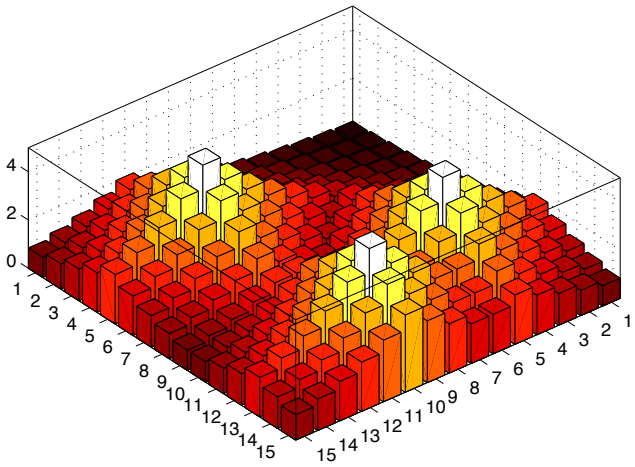
$$[\mathbf{v}]_{\mathsf{s}} = v(\mathsf{s}), \qquad [\mathbf{u}]_{\mathsf{s}} = u(\mathsf{s}), \qquad [\mathbf{P}]_{\mathsf{s}',\mathsf{s}} = p(\mathsf{s}' | \mathsf{s}, \mathsf{d}^*(\mathsf{s}))$$

in matrix notation:

$$\mathbf{v} = \mathbf{u} + \gamma \mathbf{P}^{\mathsf{T}} \mathbf{v} \Leftrightarrow \left(\mathbf{I} - \gamma \mathbf{P}^{\mathsf{T}}\right) \mathbf{v} = \mathbf{u} \Leftrightarrow \mathbf{v} = \left(\mathbf{I} - \gamma \mathbf{P}^{\mathsf{T}}\right)^{-1} \mathbf{u}$$

These linear equations are readily solved with Gaussian Elimination. Using this, the optimal policy is recomputed. The two steps of solving for the value, and recomputing the policy are iterated until convergence.

Solving an MDP



Value Iteration on a set of 225 states, corresponding to a 15×15 two dimensional grid. Deterministic transitions to neighbours on the grid, $\{\text{stay}, \text{left}, \text{right}, \text{up}, \text{down}\}$. There are three goal states, each with utility 1 – all other states have utility 0. Plotted is the value $v(\text{s})$ for $\gamma = 0.9$ after 30 updates of Value Iteration, where the states index a point on the $x - y$ grid. The optimal decision for any state on the grid is to go to the neighbouring state with highest value.

Matlab

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>> setup; demoMDP;
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Reinforcement Learning

A form of learning

- This is an MDP in which the transition and possibly reward function (utility) is not a priori known.
- As one interacts with the environment, one may begin to build an understanding of these and then define a policy accordingly.

Exploration Exploitation dilemma

- But wait! If I don't collect enough information, why should I start to act as if I know the transition and reward perfectly? This is the 'exploration-exploitation' dilemma.
- There are nice ways to address this using Bayesian methods.

Further Topics

More complex structures

- Can solve more general IDs using a form of Junction Tree algorithm.
- Many real world problems are structured MDPs. This creates a huge state space (Games, Scheduling, Landing airplanes, Life)

POMDPs

- In a Partially Observable (POMDP) not all the states of the MDP are observed.
- This creates a more complex problem for which no efficient message passing algorithms are available.
- Still an active research area.

Game Theory

In a game, the transition depends also on the opponents strategy. Still a form of MDP but complicated by having to learn your own policy in relation to an opponents policy.