Mathematical modeling and numerics of opinion consensus and polarisation dynamics

Bruno Fernández Carballo

Supervisor: Gissell Estrada-Rodriguez

Universitat Politècnica de Catalunya

FME

February 19, 2026

Overview

Motivation

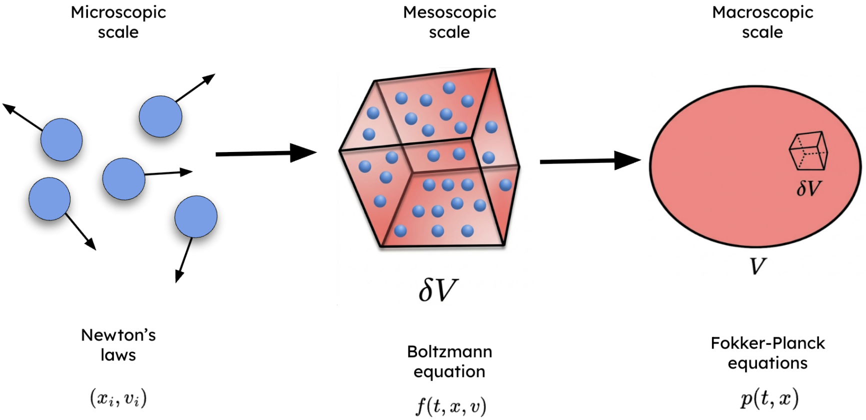


Figure: Microscopic, mesoscopic and macroscopic scales.

Microscopic Description of Gases

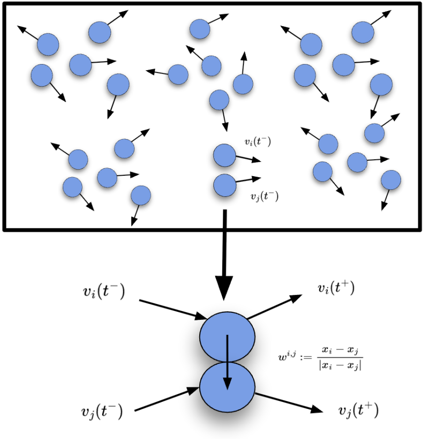


Figure: Hard-spheres description.

- We consider a system of N hard-spheres of diameter ϵ in the phase space $\mathbb{T}^{dN} \times \mathbb{R}^{dN}$. We denote $x_i \in \mathbb{T}^d$ the position and $v_i \in \mathbb{R}^d$ the velocity of each particle for $i = 1, \dots, N$.
- The equations of motion are:

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = 0$$

provided $|x_i(t) - x_j(t)| > \epsilon$ for $1 \leq i \neq j \leq N$.

Microscopic Description of Gases

- We assume that interactions are modeled as binary elastic collisions:

$$v_i(t^-) + v_j(t^-) = v_i(t^+) + v_j(t^+), \qquad \text{(conservation of momentum)}$$

$$|v_i(t^-)|^2 + |v_j(t^-)|^2 = |v_i(t^+)|^2 + |v_j(t^+)|^2. \qquad \text{(cons. of kinetic energy)}$$

- Particle velocities after a collision are

$$\begin{aligned} v_i(t^+) &= v_i(t^-) - w^{i,j} \cdot (v_i(t^-) - v_j(t^-)) w^{i,j} \\ v_j(t^+) &= v_j(t^-) + w^{i,j} \cdot (v_i(t^-) - v_j(t^-)) w^{i,j}. \end{aligned}$$

- Note that the interaction domain can be written as

$$(x_1, v_1, \dots, x_N, v_N) \in \mathcal{D}_N^\epsilon \times \mathbb{R}^{dN} := \Omega,$$

where $\mathcal{D}_N^\epsilon := \{(x_1, \dots, x_N) \in \mathbb{T}^{dN}, \forall i \neq j, |x_i - x_j| > \epsilon\}$.

Liouville equation

- **Idea:** From microscopic to mesoscopic description.
- **Notation:** We define $Z_N := (X_N, V_N)$ with $X_N = (x_1, \dots, x_N)$ and $V_N = (v_1, \dots, v_N)$, i.e. the positions and velocities of all the N particles.
- We introduce the N-particle distribution function $f_N(t, \cdot) \in L^1(\mathbb{T}^{dN} \times \mathbb{R}^{dN})$ which satisfies

$$\int_{\mathbb{T}^{dN} \times \mathbb{R}^{dN}} f_N(t, Z_N)\, dZ_N = 1.$$

Liouville equation

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N = 0, \quad \forall Z_N \in \Omega,$$

for $f_N(t, Z_N)$ probability density function + Boundary Conditions.

Boltzmann equation

- **Objective:** study asymptotically the behavior of the first marginal of f_N, defined as
$$f_N^{(1)}(t,z_1) := \int_{\Omega\setminus\{z_1\}} f_N(t,Z_N)dz_2\dots dz_N,$$
where $z_i = (x_i, v_i)$.
- Integrating Liouville equation in $\Omega \setminus \{z_1\}$, we obtain
$$\partial_t f_N^{(1)} + v_1 \cdot \nabla_{x_1} f_N^{(1)} = \alpha \left(C_{1,2} f_N^{(2)} \right),$$
where $f_N^{(2)}$ is the second marginal and $\alpha := N\epsilon^{d-1}$.
- $C_{1,2}$ is the collision operator that acts of $f_N^{(2)}$, and is given by
$$C_{1,2} f_N^{(2)} = \int_{\mathbb{R}^d} \int_{\mathbb{S}_+^{d-1}} \Big(f_N^{(2)}(t, x_1, v_1', x_1 + \epsilon w, v_2') \\ - f_N^{(2)}(t, x_1, v_1, x_1 - \epsilon w, v_2) \Big) ((v_2 - v_1)\cdot w)_+ \, dw \, dv_2,$$
where $w = \frac{x_2 - x_1}{|x_2 - x_1|}$, $\mathbb{S}^{d-1} = \{w \in \mathbb{R}^d : \|w\| = 1\}$ is the unit sphere and $\mathbb{S}_+^{d-1} := \mathbb{S}_+^{d-1}(v_i - v_j) = \{w \in \mathbb{S}^{d-1} : w \cdot (v_i - v_j) > 0\}$.

Boltzmann equation

Molecular chaos assumption

$$f_N^{(2)}(t, z_1, z_2) \approx f_N^{(1)}(t, z_1) f_N^{(1)}(t, z_2)$$

- **Boltzmann-Grad limit:** $N \to \infty, \epsilon \to 0$ such that $\alpha = N\epsilon^{d-1}$ remains finite.

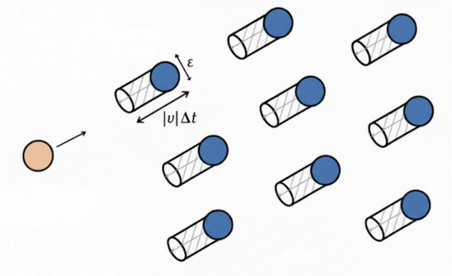


Figure: Collisional volume given by the collision cylinder.

Boltzmann equation

Boltzmann equation for hard-spheres

When $f := \lim_{N\to\infty} f_N^{(1)}$ exists, then f satisfies

$$\partial_t f + v \cdot \nabla_x f = \alpha \mathcal{Q}(f, f),$$

where $\mathcal{Q}$ is the collision operator given by

$$\begin{aligned} \mathcal{Q}[f, f](v) :=& \int_{\mathbb{S}_+^{d-1}\times\mathbb{R}^d} [f(v')f(v_1') - f(v)f(v_1)]((v - v_1)\cdot w)_+ dwdv_1 \\ =&\mathcal{Q}^+(f, f) - \mathcal{Q}^-(f, f), \end{aligned}$$

where v', v_1' and v, v_1 are the post- and pre-collision velocities, respectively. Note that $\mathcal{Q}^+(f, f)$ is the gain term and $\mathcal{Q}^-(f, f)$ is the loss term.

Microscopic description of Opinions

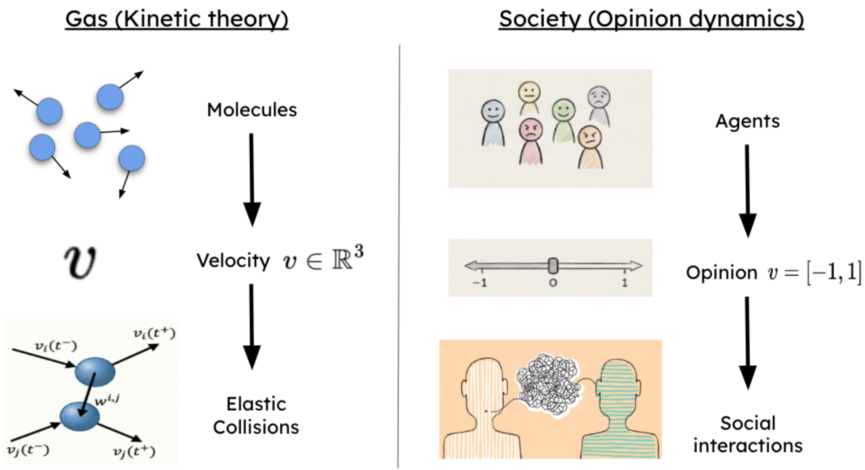


Figure: Analogy between kinetic theory and opinion dynamics.

Microscopic description of Opinions

- Let $v \in [-1, 1] := \mathcal{V}$ an agent opinion.
- Opinions cannot cross boundaries.
- We consider the binary interaction between two agents with initial opinions v and v_* as follows:

$$\begin{aligned} v' &= v - \gamma P(|\cdot|)(v - v_*) + \eta D(|v|), \\ v'_* &= v_* - \gamma P(|\cdot|)(v_* - v) + \eta_* D(|v_*|), \end{aligned}$$

where v' and v'_* are the post-interaction opinions.

Microscopic description of Opinions

- Let $v \in [-1, 1] := \mathcal{V}$ an agent opinion.
- Opinions cannot cross boundaries.
- We consider the binary interaction between two agents with initial opinions v and v_* as follows:

$$v' = v - \gamma P(|\cdot|)(v - v_*) + \eta D(|v|),$$
$$v'_* = v_* - \gamma P(|\cdot|)(v_* - v) + \eta_* D(|v_*|),$$

  where v' and v'_* are the post-interaction opinions.

- $\gamma \in (0, 1/2)$ is a constant parameter which represents the compromise propensity.
- η and η_* are two random variables with the same distribution Θ with zero mean and variance σ^2, and take values in a bounded set $\mathcal{B} \subseteq \mathcal{V}$. They model the random uncertainty in human interactions.

Microscopic description of Opinions

- Let $v \in [-1, 1] := \mathcal{V}$ an agent opinion.
- Opinions cannot cross boundaries.
- We consider the binary interaction between two agents with initial opinions v and v_* as follows:

$$
\begin{aligned}
v' &= v - \gamma P(|\cdot|)(v - v_*) + \eta D(|v|), \\
v_*' &= v_* - \gamma P(|\cdot|)(v_* - v) + \eta_* D(|v_*|),
\end{aligned}
$$

  where v' and v_*' are the post-interaction opinions.

- $P(|\cdot|)$ and $D(|\cdot|)$ are functions that describe the local relevance of compromise and diffusion for a given opinion, respectively.
- An example for P is $P(|v - v_*|) = \mathbb{1}_{|v - v_*| \le r} = \begin{cases} 1 & \text{if } |v - v_*| \le r, \\ 0 & \text{if } |v - v_*| > r. \end{cases}$
- An example for D is $D(|v|) = (1 - |v|^2)^\nu, \quad \nu \ge 0.$

Boltzmann-type equation for Opinion dynamics

- Let $f(t, v)$ denote the distribution of opinion $v \in \mathcal{V}$ at time $t \geq 0$.
- Denote v', v'_* and v, v_* the post- and pre-collisional velocities/opinions of two interacting particles/agents.
- For the opinion model, the strong form can be written as

$$\partial_t f(t, v') = \int_{\mathcal{B}^2} \int_{\mathcal{V}} \left(\frac{1}{J} B(v, v_*) f(v) f(v_*) - B(v', v'_*) f(v') f(v'_*) \right) dv'_* d\eta d\eta_*,$$

where $B_{(v,v_*)\to(v',v'_*)} = \Theta(\eta)\Theta(\eta_*)\chi(|v'| \leq 1)\chi(|v'_*| \leq 1)$ and Θ is a distribution function.

- Recall the "classic" Boltzmann equation in the homogeneous case:

$$\partial_t f(t, v) = \int_{\mathbb{S}_+^{d-1}} \int_{\mathbb{R}^d} \left(f(v') f(v'_*) - f(v) f(v_*) \right) B(|v - v_*|, \omega) dv'_* dw,$$

where $B(|v - v_*|, \omega) = ((v - v_*) \cdot w)_+$.

Numerics - Preliminaries

- **Idea:** Use Monte Carlo methods.
- The Monte Carlo method, based on the Law of Large numbers, encompasses a broad class of computational algorithms that rely on repeated random sampling.
- In the specific context of kinetic energy we use Direct simulation Monte Carlo (DSMC) methods, which simulate the gas itself instead to solve the mathematical expression.
- Deterministic numerical methods, such as finite difference and finite volume, are computationally expensive due to the high-dimensionality of the phase space.
- For a d-dimensional problem, if we discretize each dimension with M points, we need M^{2d} points in total.

Numerics - Preliminaries

- Recall that

$$\mathcal{Q}(f,f)(v) = \underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{S}_+^{d-1}} B(|v-v_*|,\omega) f(v')f(v_*')\,d\omega dv_*}_{\mathcal{Q}^+(f,f)}$$
$$- \underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{S}_+^{d-1}} B(|v-v_*|,\omega) f(v)f(v_*)\,d\omega dv_*}_{\mathcal{Q}^-(f,f)}\,.$$

- Then, we can write the loss term $\mathcal{Q}^-(f,f)$ as

$$\mathcal{Q}^-(f,f)(v) = f(v)\left(\int_{\mathbb{R}^d}\int_{\mathbb{S}_+^{d-1}} B(|v-v_*|,\omega)\, f(v_*) d\omega\, dv_*\right).$$

- We consider Maxwellian molecules ($B = cte.$) to simplify the scheme, that allows to write $P(f,f) = \mathcal{Q}^+(f,f)$ and $\mu f = \mathcal{Q}^-(f,f)$.

Numerics - Preliminaries

- Note that $\frac{\partial f}{\partial t} = \frac{1}{\varepsilon}[P(f,f) - \mu f] = \frac{1}{\varepsilon}(\mathcal{Q}^+ - \mathcal{Q}^-)$, where $\varepsilon := \frac{1}{\alpha}$ is the Knudsen number.
- Consider a time interval $[0, t_{max}]$, discretized in n_{TOT} intervals of size Δt, and let us denote $f^n(v)$ an approximation of $f(n\Delta t, v)$.
- Using a probabilistic interpretation of the Forward Euler time discretization, we get:

$$f^{n+1} = \left(1 - \frac{\mu \Delta t}{\varepsilon}\right) f^n + \frac{\mu \Delta t}{\varepsilon} \frac{P(f^n, f^n)}{\mu}.$$

- A first algorithm determined whether each particle collided or not, by using the previous probabilistic interpretation.
- A conservative version was introduced by computing the expected number of collisions, $N_c = \text{Round}(N\mu\Delta t/2\varepsilon)$.

- We are going to test the conservative algorithm for the case of the Kac equation.
- The Kac equation can be written

$$\frac{\partial f}{\partial t} = \frac{1}{\varepsilon}\int_0^{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2\pi}[f(t,v')f(t,v'_*) - f(t,v)f(t,v_*)]\,dv_*\,d\theta,$$

where $v' = v\cos\theta - v_*\sin\theta, \quad v'_* = v\sin\theta + v_*\cos\theta.$

- $\theta = 2\pi Rand$, where $Rand \sim U(0,1)$.
- The analytical solution is given by

$$f(t,v) = \frac{1}{2}\left[\frac{3}{2}(1-C(t))\sqrt{C(t)} + (3C(t)-1)C(t)^{3/2}v^2\right]e^{-C(t)v^2},$$

where $C(t) = \left[3 - 2e^{-\sqrt{\pi}t/16}\right]^{-1}.$

- The asymptotic solution is $f_\infty(v) = \frac{1}{2\sqrt{3}}e^{-\frac{v^2}{3}}$.

The Kac equation

- At the end of the simulation we get $\{v_j(t)\}_{j=1}^{N}$.
- Let $\{V_l\}_{l=0}^{N_g}$ be a uniform velocity grid, covering the domain of interest $\{V_l = V_{\min} + l\Delta V, l = 0, \dots, N_g\}$.
- The reconstructed numerical solution is $f_{num}(t,v) = \frac{1}{N}\sum_{j=1}^{N} W_H(v - v_j(t))$, where

$$W_H(x) = \frac{1}{H} W\left(\frac{x}{H}\right), \qquad W(x) = \begin{cases} 3/4 - |x|^2 & \text{if } |x| \le 0.5, \\ (|x| - 3/2)^2/2 & \text{if } 0.5 < |x| \le 1.5, \\ 0 & \text{otherwise.} \end{cases}$$

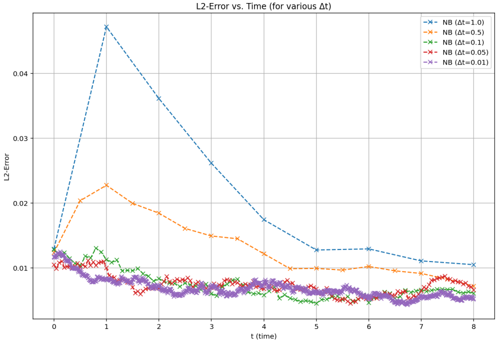
- We set the parameters $H = 0.2$ and $\Delta V = 0.0625$.
- For the numerical computations, we evaluate f_{num} at the specific grid points $v = V_l$.

The Kac equation

- The L^2 error at a given time t is

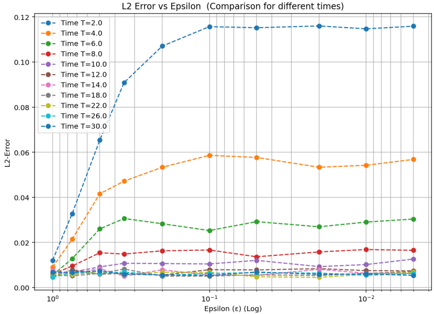
$$E_{L^2}(t) = \left(\int_{v_{min}}^{v_{max}} \left| f_{num}(t,v) - f(t,v) \right|^2 dv \right)^{1/2}.$$

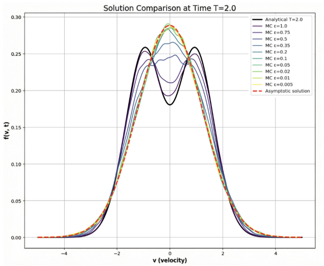
- In the plot we represent the L^2 error between the numerical solution and the analytical solution for $t \in [0,8]$ varying the time step Δt.

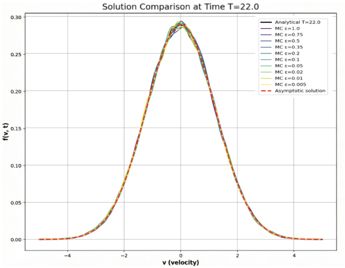


The Kac equation

- In the left graphic we represent the L^2 error between the numerical solution and the analytical solution in different times T varying ε.
- In the right plots the temporal evolution of f for various ε values is shown.





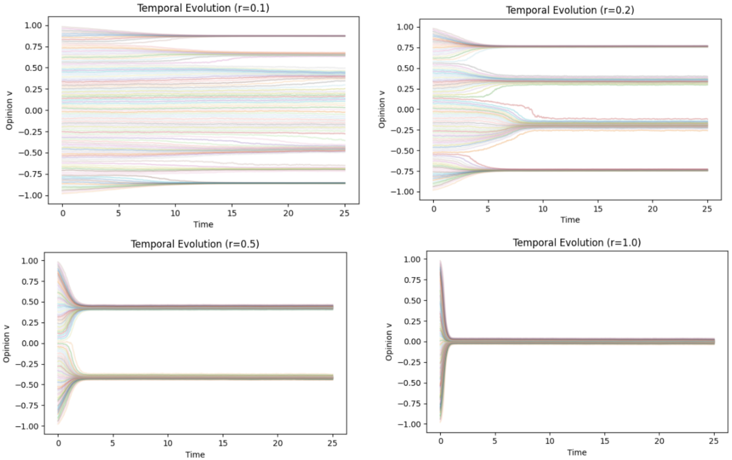


Numerics of Opinion dynamics

- We use the conservative scheme to simulate Opinion dynamics.
- Let $P(v - v_*) = \mathbb{1}_{|v-v_*|\leq r}$ and $D(v) = (1 - v^2)^{\nu}, \quad \nu \geq 0.$
- Set $N = 10^4$, $\gamma = 0.25$, $\Delta t = 0.05$, $\nu = 2$.
- We want to analyze how the parameters r and σ affect the society.
- Recall that r is the confidence radius and σ models the noise inherent in human interactions.

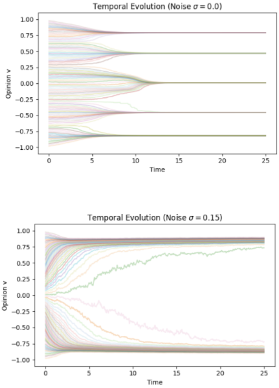
Numerics of Opinion dynamics

- We fix the noise $\sigma = 0.01$ and vary r.
- Let $P(v - v_*) = \mathbb{1}_{|v-v_*|\leq r}$ and $D(v) = (1 - v^2)^\nu, \quad \nu \geq 0.$
- Set $N = 10^4$, $\gamma = 0.25$, $\Delta t = 0.05$, $\nu = 2$.

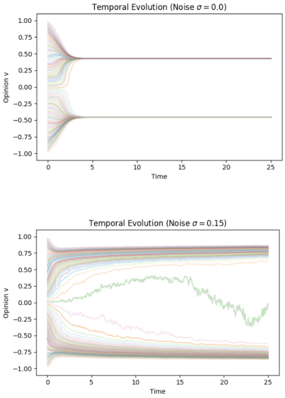


Numerics of Opinion dynamics

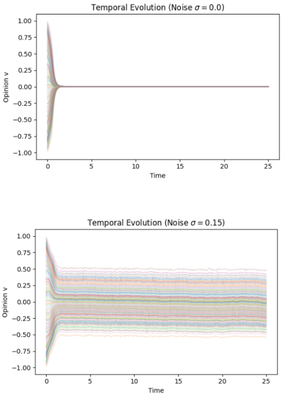
- We fix some values for r, and vary the noise σ.
  - $r = 0.15$



  - $r = 0.4$

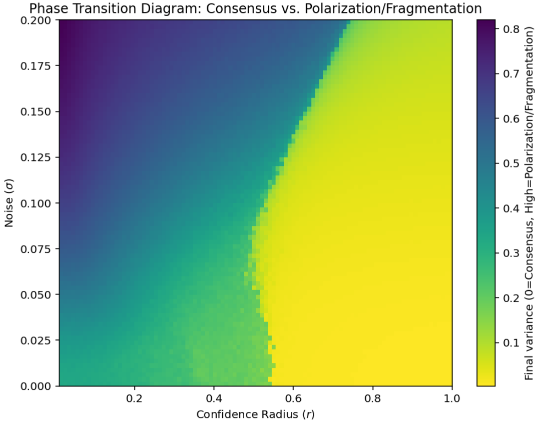


  - $r = 0.8$



Numerics of Opinion dynamics

- At the end of the simulation, we compute the sample variance between all the final agents' opinions $S^2 = \frac{\sum_{i=1}^{N} |v_i - \bar{v}|}{N-1}$, where $\bar{v}$ is the sample mean.



Summary

Main results

- Hard-spheres framework adapted to opinion dynamics.
- Validated an algorithm to simulate the Boltzmann equation.
- Analyzed how the confidence radius r and the noise σ drive the society to consensus, polarization and fragmentation.

Summary

Main results

- Hard-spheres framework adapted to opinion dynamics.
- Validated an algorithm to simulate the Boltzmann equation.
- Analyzed how the confidence radius r and the noise σ drive the society to consensus, polarization and fragmentation.

References

Pareschi, Lorenzo and Russo, Giovanni. "An introduction to Monte Carlo method for the Boltzmann equation".
In *ESAIM: Proceedings (Vol. 10, pp. 35-75). EDP Sciences, 2001.*

Toscani, Giuseppe. "Kinetic models of opinion formation".
In *Commun. Math. Sci. 4.1 (2006): 481-496.*

Bodineau, T., Gallagher, I., Saint-Raymond, L., Simonella, S. "One-sided convergence in the Boltzmann–Grad limit".
In Annales de la Faculté des sciences de Toulouse: Mathématiques (Vol. 27, No. 5, pp. 985-1022).2018.

Nanbu algorithm for a Maxwellian Gas

Algorithm 1 Nanbu for a Maxwellian Gas

Sample initial particle velocities $\{v_i^0, i = 1, \ldots, N\} \sim f_0(\mathbf{v})$
for $n = 0$ to $n_{TOT} - 1$ **do**
    for $i = 1$ to N **do**
        draw a uniform random number $\xi \sim U(0, 1)$
        if $\xi < 1 - \mu \Delta t / \varepsilon$ **then**
            no collision: $v_i^{n+1} = v_i^n$
        else
            select random index $j \in \{1, \ldots, N\} \setminus \{i\}$
            compute v_i' by performing the collision between particle i and j
            update: $v_i^{n+1} = v_i'$
        end if
    end for
end for

Nanbu-Babovsky algorithm for a Maxwellian Gas

Algorithm 2 Nanbu-Babovsky (NB) for a Maxwellian Gas

1: Sample initial particle velocities $\{v_i^0, i = 1, \ldots, N\} \sim f_0(\mathbf{v})$
2: **for** $n = 0$ to $n_{TOT} - 1$ **do**
3: set $N_c = \text{Round}(N\mu\Delta t/2\varepsilon)$
4: select N_c pairs (i, j) uniformly among all possible pairs and:
5: compute the collision between i and j, then compute v_i', v_j'
6: set $v_i^{n+1} = v_i', v_j^{n+1} = v_j'$
7: set $v_k^{n+1} = v_k^n$ for the particles that have not been selected
8: **end for**

- $Round(\cdot)$ is a rounding stochastic function.

Lanford's Theorem

Theorem

Consider a system of N particles interacting as hard-spheres of diameter ϵ. Let $f_0 : \mathbb{T}^d \times \mathbb{R}^d \mapsto \mathbb{R}^+$ be a continuous probability density such that

$$f_0(x, v)e^{\frac{\beta}{2}|v|^2} \leq e^{-\mu},$$

for some $\beta > 0$, $\mu \in \mathbb{R}$. Assuming that these N hard-spheres are initially distributed according to f_0 and independent (molecular chaos), there exists a positive time depending only on β and μ ($T^* = T^*(\beta, \mu)$) such that, in the Boltzmann-Grad limit, the distribution function of the particles converges uniformly on $[0, \frac{T^*}{\alpha}] \times \mathbb{T}^d \times \mathbb{R}^d$ to the solution of the Boltzmann equation.