Sistemas Multimédia

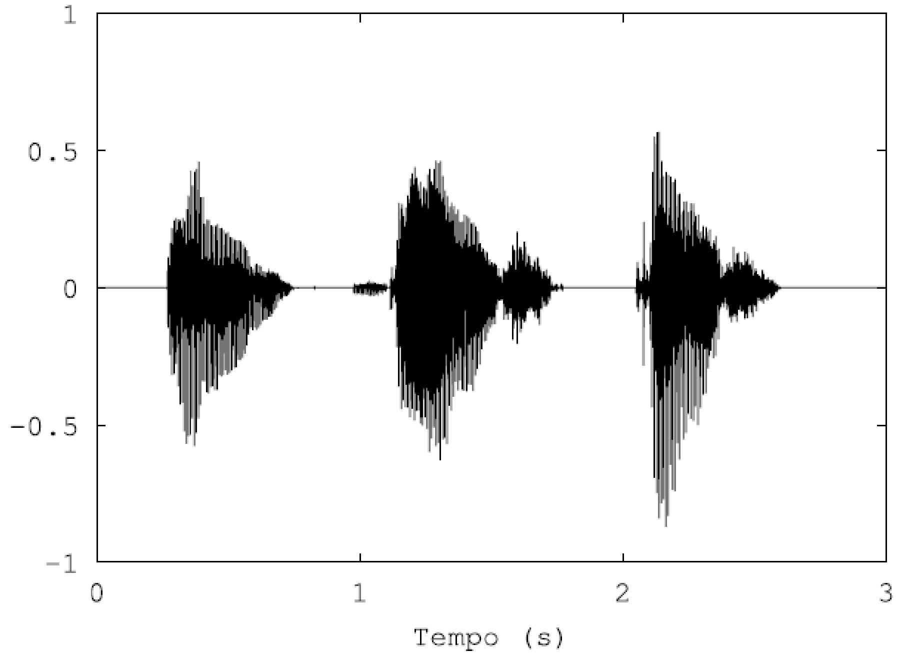
Fundamentals

Departamento de Eletrónica, Telecomunicações e Informática

Universidade de Aveiro – 2024/2025

1. Context

- **Signal:** A signal is a mathematical function that represents information about a certain quantity that varies over time and/or space or other variables of interest.
  - For example, a person's voice can be represented as a time signal that measures changes in air pressure.

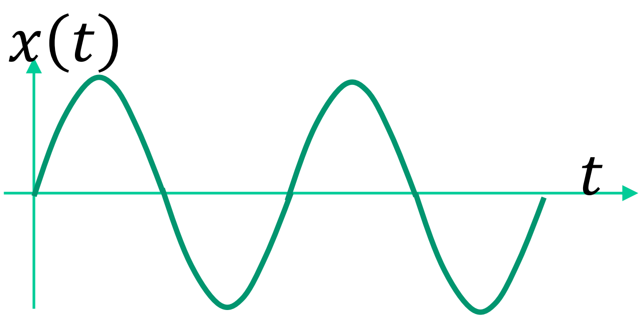


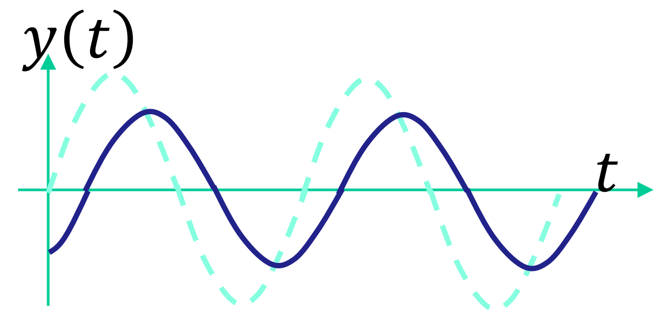
audio signal corresponding to the words "one, two, three"

1. Context

- Signal transformation is done through **systems**
  - **A microphone** (converts audible air pressure vibrations into electrical signals)
- Several systems operating with multimedia signals (sound, image, video, ...) typically preserve the shape of those signals.
- Usually, they are linear systems.
- A property of linear systems is that they respond to a sinusoidal signal with another sinusoidal signal, of the same frequency.

$$x(t) = \sin(\omega_o t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow y(t) = A \cdot \sin(\omega_o t + \phi)$$





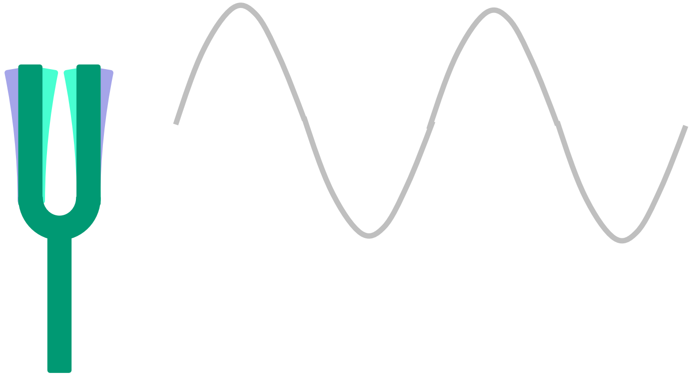
1. Context

- This characteristic allows:
  - Decomposing a given signal into a sum of sinusoids;
  - Analyzing how each sinusoidal component is processed, one-by-one;
  - Determining the result of the original signal processing, by adding the results of each sinusoidal components.
- **This is a very important principle in the analysis and processing of multimedia signals.**

1. Context

- Another observation is the sinusoidal (or quasi-sinusoidal) nature of many oscillatory phenomena.

- For example, a tuning fork (or a string of an instrument, etc.) vibrates in a sinusoidal way (so the sound it produces follows that shape).



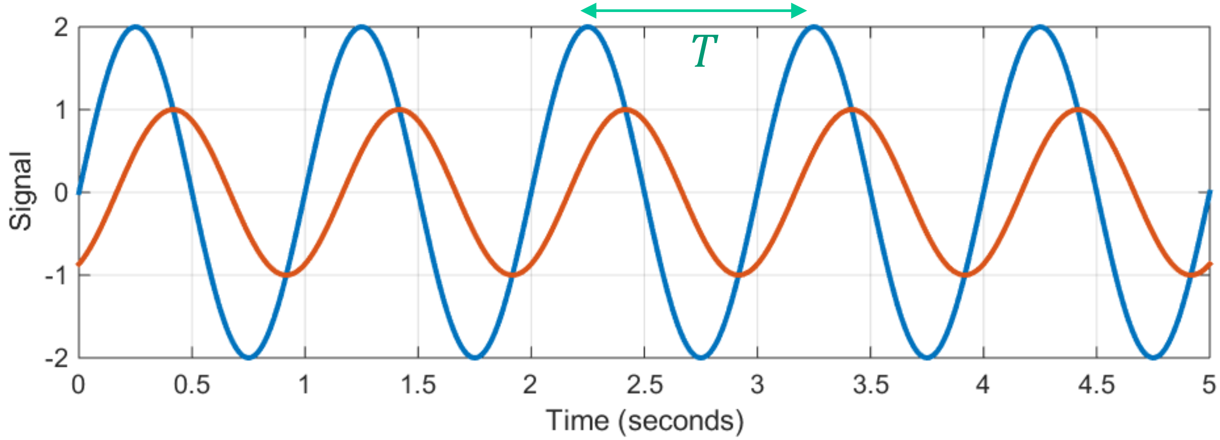
- Thus, the study and composition of sinusoidal signals becomes highly relevant.

2. Sinusoidal Signal

A sinusoidal signal is defined by:

$$x(t) = A\sin(\omega t + \phi)$$

- ω – Frequency, in rad/s.
- f – Frequency, in Hz, where $\omega = 2\pi f$.
- T – Period, in seconds, where $T = 1/f$.
- A – Amplitude.
- ϕ – Phase (with respect to the defined initial moment), in rad.



2. Sinusoidal Signal

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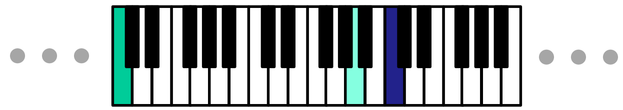
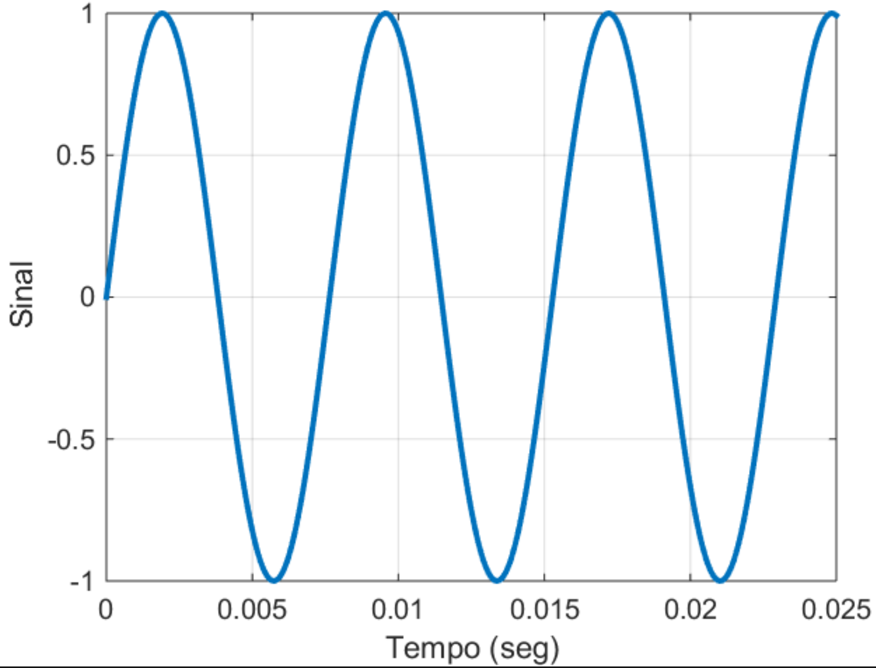
Examples associated with sound:

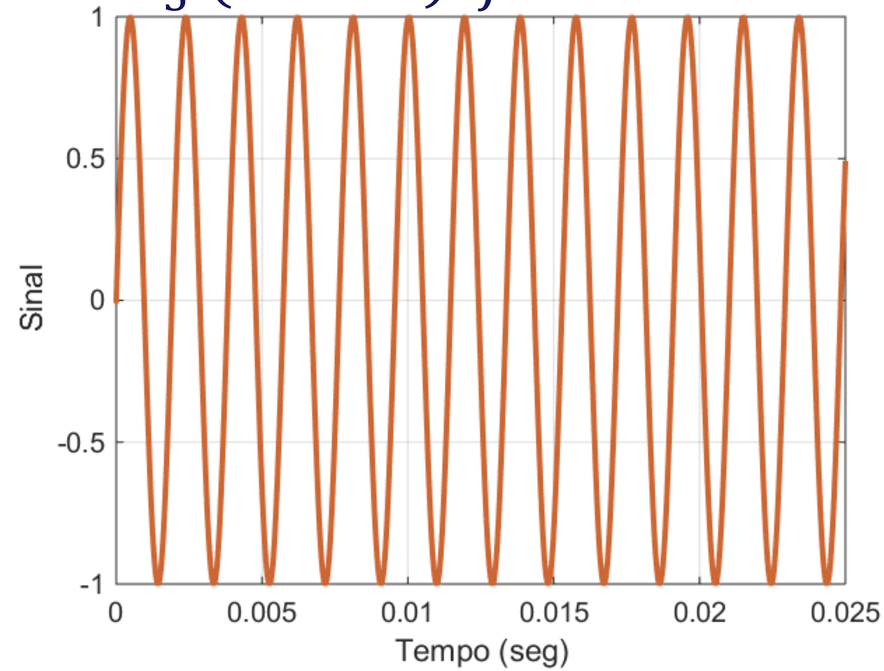


$$f(n) = 2^{\left(\frac{n-49}{12}\right)} 440\ Hz$$

$C_3\ (n = 28): f = 130.8\ Hz$

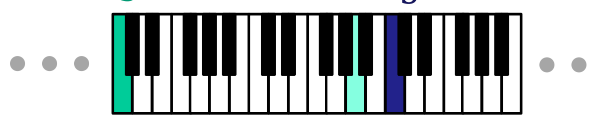


$C_5\ (n = 52): f = 523.3\ Hz$



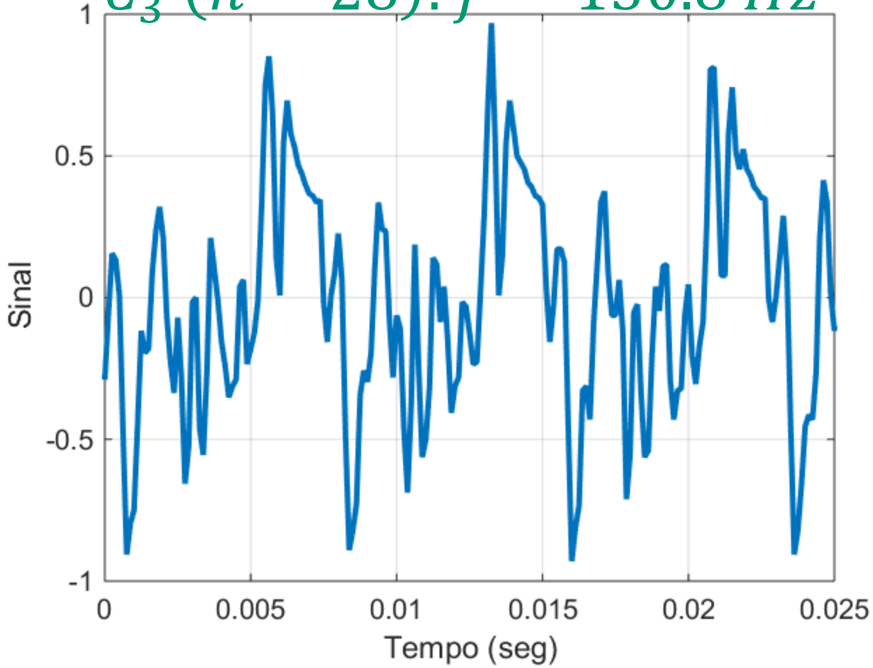
2. Sinusoidal Signal

Examples associated with sound:

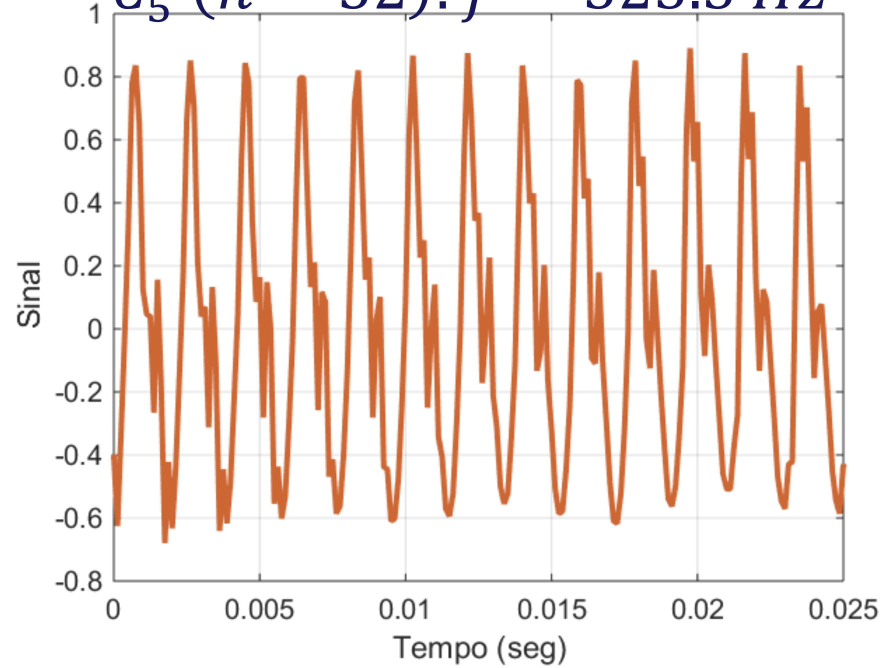


- The same notes generated by a piano.
- Why do they differ from the sinusoidal signal?

C_3 $(n = 28)$: $f = 130.8\ Hz$

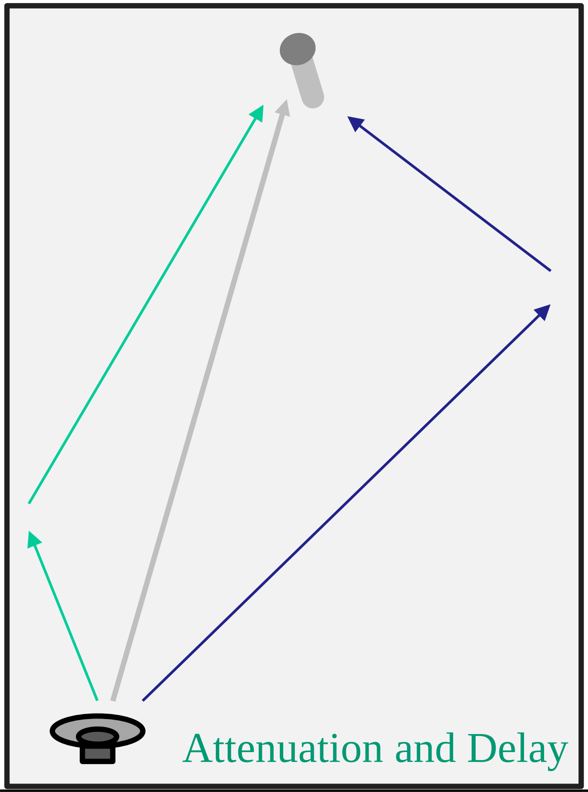


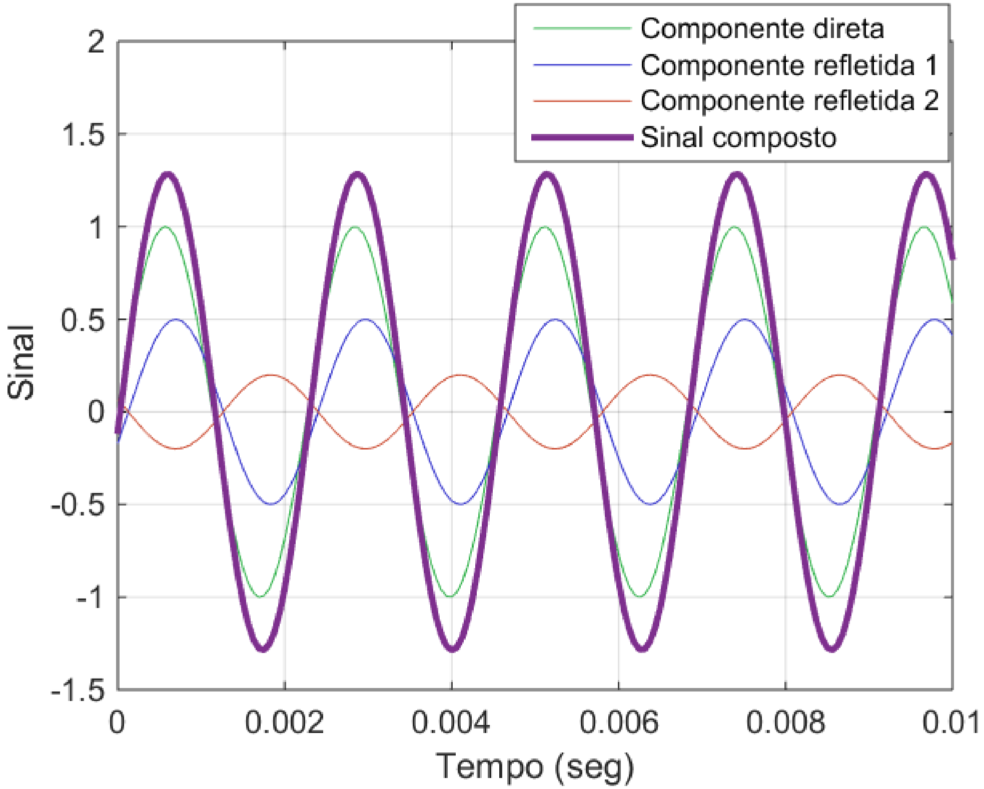
C_5 $(n = 52)$: $f = 523.3\ Hz$



3. Composition of Sinusoidal Signals - Part I

- In real scenarios, the signals contain combinations of multiple sinusoids.
- Example: Sound in a multiple reflection environment:





3. Composition of Sinusoidal Signals - Part I

- The sum of K sinusoidal signals, all of the same frequency f_0, but possibly different A_k amplitudes and ϕ_k phases, results in a sinusoidal signal of the same frequency f_0.

$$y(t) = \sum_{k=1}^{K} A_k \sin(2\pi f_0 t + \phi_k) = A \sin(2\pi f_0 t + \phi)$$

- The amplitude A and phase ϕ of the resulting signal depends on both A_k amplitudes and ϕ_k phases.

- But how can these parameters be determined in a simple and systematized way?

4. Sinusoids and Complex Numbers

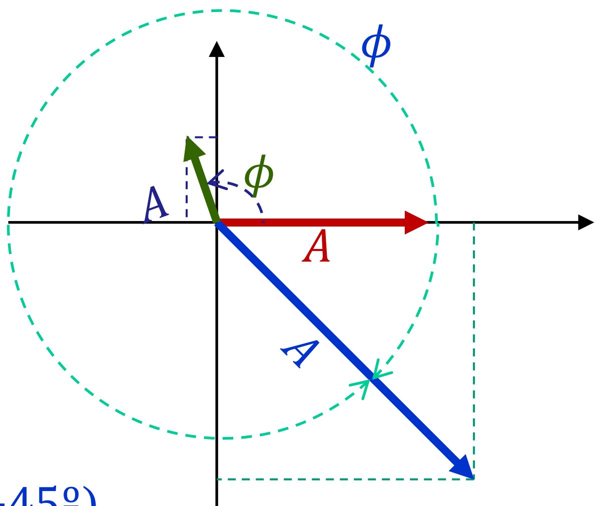
- The representation of sinusoidal signals through complex numbers aims to simplify and systematize the analysis and processing of these signals.

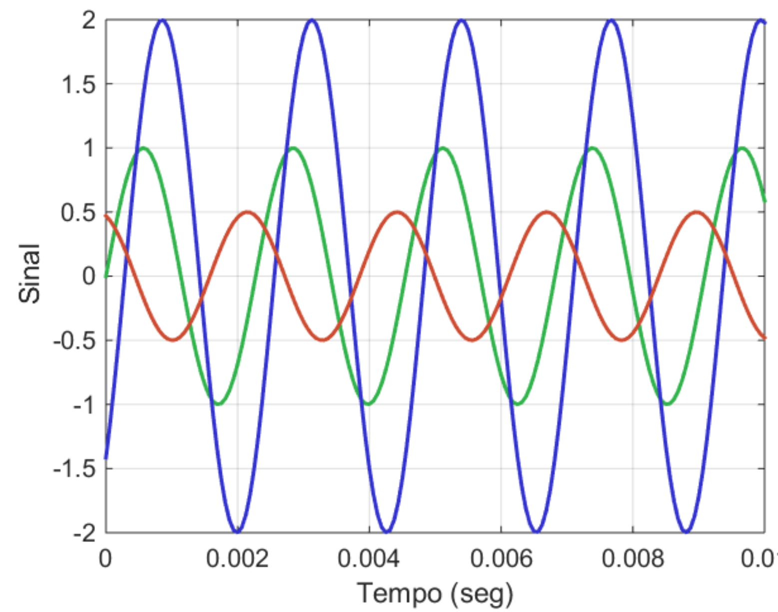
- Being several sinusoids, of the same frequency f_0, characterized by the amplitude and phase parameters, these may be conceptually represented by a vector (termed **Phasor**):

$A = 1$
$\phi = 0º$

$A = 0.5$
$\phi = 110º$

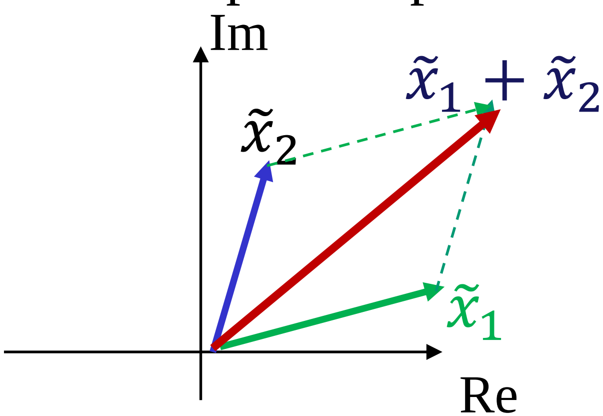
$A = 2$
$\phi = 315º$ (ou $-45º$)

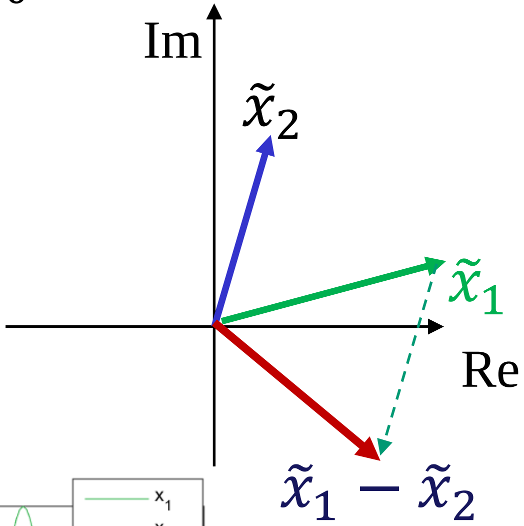


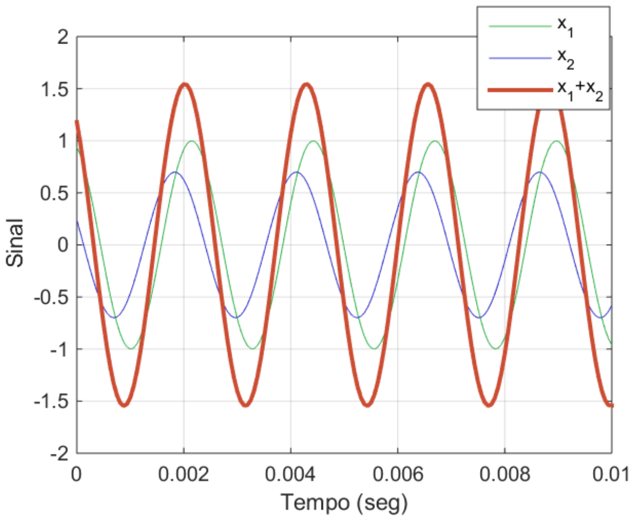


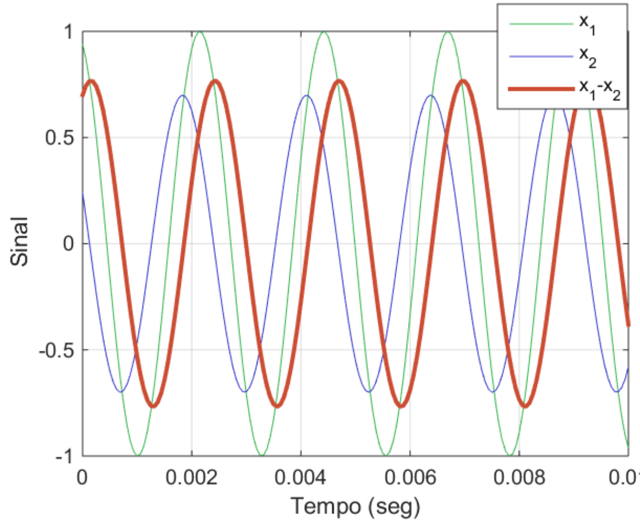
4. Sinusoids and Complex Numbers

- The sum of sinusoids of the same frequency f_0 can be seen as the sum of the respective phasors:









4. Sinusoids and Complex Numbers

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- Recalling some relations regarding complex numbers :

Cartesian Representation

Polar Representation

$$\underbrace{y = a + jb}_{\text{Cartesian Representation}} = \underbrace{Ae^{j\varphi}}_{\text{Polar Representation}}$$

$$A = \sqrt{a^2 + b^2}$$
$$\varphi = \operatorname{atan}(b/a)$$
$$a = A\cos(\varphi)$$
$$b = A\sin(\varphi)$$

- Sum and subtraction :

$$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$

- Multiplication :

$$\left(A_1 e^{j\varphi_1}\right)\left(A_2 e^{j\varphi_2}\right) = A_1 A_2 e^{j(\varphi_1 + \varphi_2)}$$

- Conjugate:

$$(a + jb)^* = a - jb \qquad \left(Ae^{j\varphi}\right)^* = Ae^{-j\varphi}$$

- Recalling some relations regarding complex numbers :

$$\underbrace{y = a + jb}_{\text{Cartesian Representation}} = \underbrace{Ae^{j\varphi}}_{\text{Polar Representation}}$$

- Multiplication by its conjugate:

$$yy^* = \left(Ae^{j\varphi}\right)\left(Ae^{j\varphi}\right)^* = A^2 = a^2 + b^2$$

- Division:

$$\left(A_1e^{j\varphi_1}\right)/\left(A_2e^{j\varphi_2}\right) = (A_1/A_2)e^{j(\varphi_1-\varphi_2)} \qquad \frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{c^2+d^2}$$

- Euler's formula:

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$$

4. Sinusoids and Complex Numbers

- The inverse Euler formulas allow us to write the cosine/sine in terms of complex exponentials

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- These expression can be used to express $\cos(w_0 t + \varphi)$ in terms of two complex numbers

$$A\cos(w_0 t + \varphi) = A\left(\frac{e^{j(w_0 t + \varphi)} + e^{-j(w_0 t + \varphi)}}{2}\right)$$

$$= \frac{1}{2} X e^{j w_0 t} + \frac{1}{2} X^* e^{-j w_0 t}$$

with $X = Ae^{j\varphi}$

4. Sinusoids and Complex Numbers

- From the previous relations the following expression

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos\left(2\pi k f_k + \varphi_k\right)$$

- Can be written as

$$x(t) = X_0 + \sum_{k=1}^{N} \mathrm{Re}\left\{X_k e^{j2\pi f_k t}\right\} \qquad A_0 = X_0 \qquad X_k = A_k e^{j\varphi_k}$$

- By using the Euler formula $x(t)$ can be alternatively represented by

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t} \right\}$$

Called the two-side spectrum as discussed later

- **Note:** As in the case of individual sinusoids, this form follows the fact that the real part of a complex number is equal to one-half the sum of that number and its complex conjugate.

4. Sinusoids and Complex Numbers

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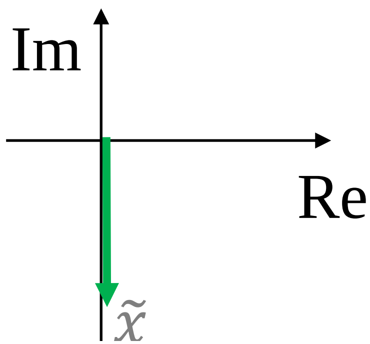
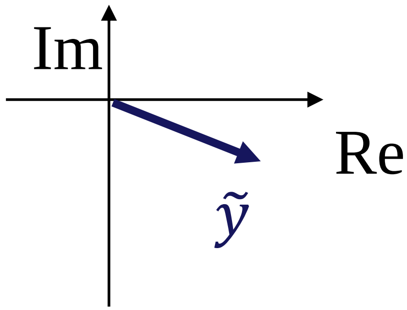
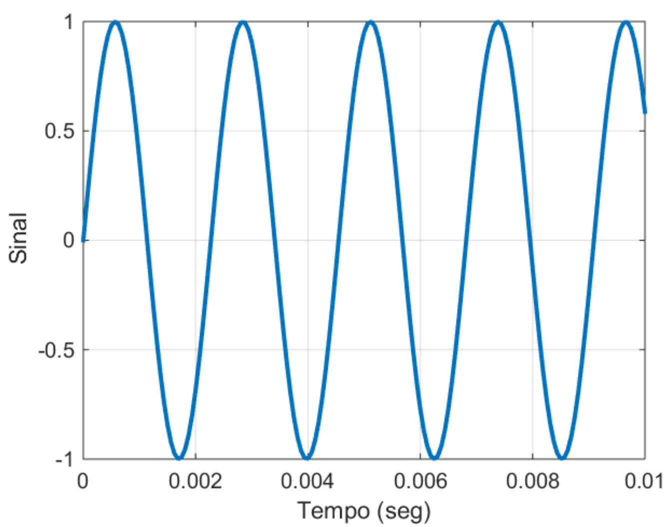
- The sinusoidal response of a linear process can be easily analyzed by the amplitude and phase change that this process imposes:

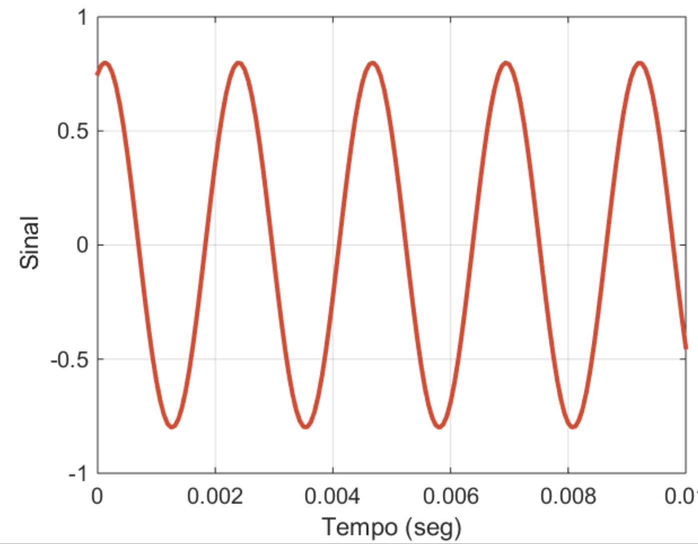




$$\tilde{y} = \left(Ge^{j\delta}\right)\tilde{x} = (GA)e^{j(\varphi+\delta)}$$



$$G = 0.8$$
$$\delta = 70º$$



5. Decomposition in Series of Sinusoids

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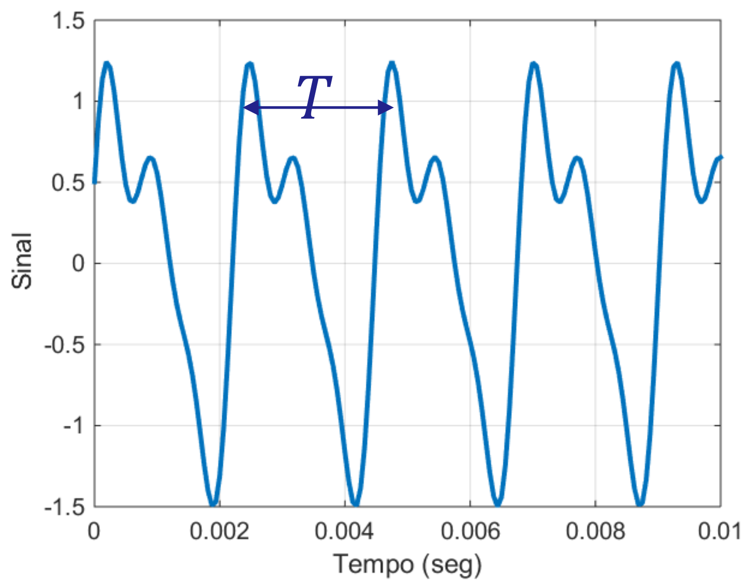
- As a linear process maintains the behavior to each frequency separately, it becomes very suitable to decompose signals into a sum of sinusoidal components.

- Such decomposition is very simple for the case of periodic signals (signals whose shape repeats regularly over time).

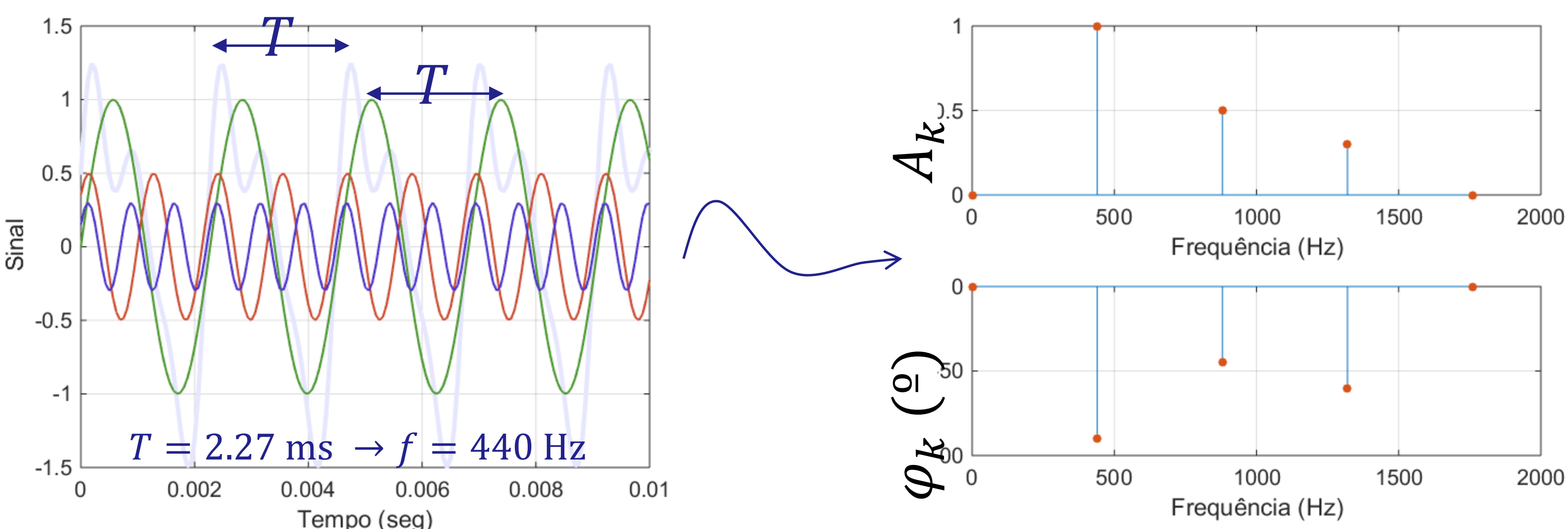


5. Decomposition in Series of Sinusoids

- A periodic signal, of period T, can be described as a sum of sinusoidal signals whose frequencies are multiples of $f = 1/T$.

Harmonic Frequencies

$$x(t) = A_0 + \sum_{k=1}^{K} A_k \cos(2\pi k f t + \varphi_k) = A_0 + \sum_{k=1}^{K} A_k \sin(2\pi k f t + \phi_k)$$



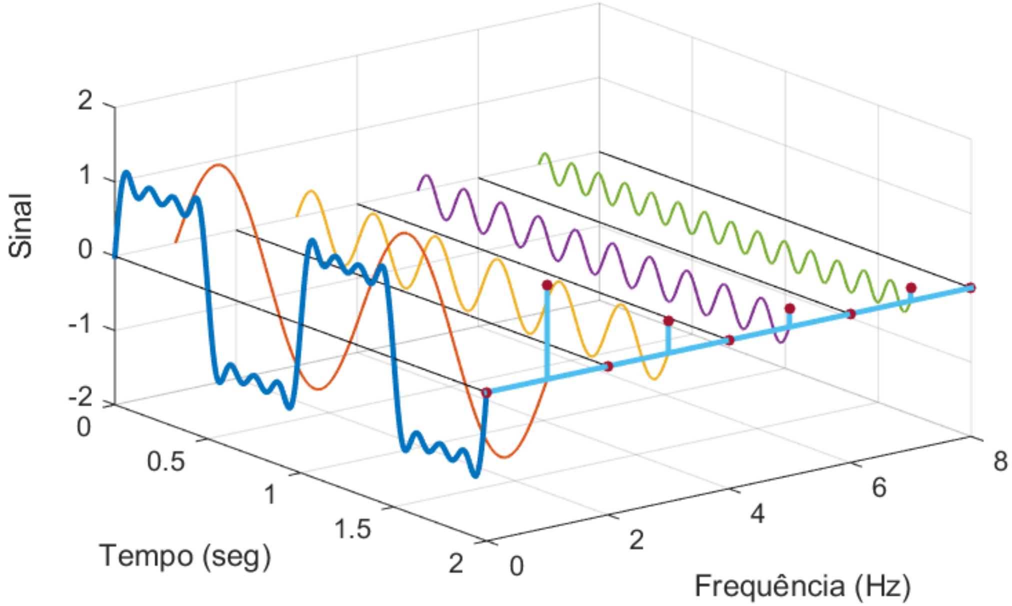
5. Decomposition in Series of Sinusoids

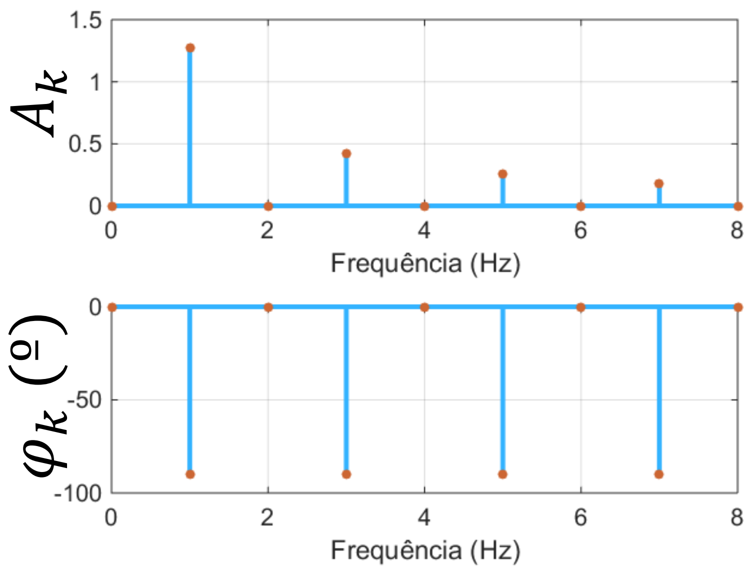
Jean-Baptiste Joseph Fourier
1768 - 1830

- This representation is known as the Fourier Series.
- It can also be described by:

$$x(t) = a_0 + \sum_{k=1}^{K} a_k \cos(2\pi k f t) + \sum_{k=1}^{K} b_k \sin(2\pi k f t)$$

Example (square wave):





6. Composition of Sinusoidal Signals - Part II

- Consider, now, the case in which a signal is composed of a sum of non-harmonically related sinusoids:

$$x(t) = A_0 + \sum_{k=1}^{K} A_k \cos(2\pi f_k t + \varphi_k)$$

- This signal is also a periodic signal if all the frequencies are integer multiples of a common frequency, f_0 (called the fundamental frequency).
- Where the frequency f_k of the kth cosine component is

$f_k = k f_0$ k=1,2.. (harmonic frequencies)

- **Note:**
  - When a function is a sum of two periodic functions, with fundamental periods T_1 and T_2 , all periods of the sum function will be at the same time integer multiples of both T_1 and T_2 , that is

$$T = k_1 T_1 = k_2 T_2 \qquad k_1, k_2 \in \mathbb{Z}$$

  - This is only possible if T_1 / T_2 is a rational number. In this case, the fundamental period will be the least common multiple of T_1 and T_1 , or the fundamental frequency will be the greatest common divisor (GCD) of the two fundamental frequencies.

6. Composition of Sinusoidal Signals - Part II

- The fundamental frequency is the largest f_0 such that $f_k = kf_0$ where k is an integer. Mathematically,

$$f_0 = \gcd(f_1, f_2, \dots, f_K)$$

- If the harmonic frequencies are not integers, it is possible to find a scaling constant γ such that γf_k in an integer for all k. In this case we can use the gcd on the scaled frequencies as follows,

$$f_0 = \frac{1}{\gamma}\gcd(\gamma f_k)$$

- If there is not a greatest common divisor of all frequencies, the composed signal is not periodic.

Example: Two-tone signal

$f_1 = 1\ Hz$ $\quad$ $f_2 = 5/6\ Hz$ $\quad$ $f_o = 1/6\ Hz$

$$x_1(t) = \sin\left(\frac{2\pi}{T_1}t\right) \quad x_2(t) = \sin\left(\frac{2\pi}{T_2}t\right) \quad y(t) = x_1(t) + x_2(t)$$