Sistemas Multimédia

Signal representation and processing

Departamento de Eletrónica, Telecomunicações e Informática

Universidade de Aveiro – 2024/2025

Fourier Series

- In the previous classes, we have seen that we can synthesize a variety of periodic waveforms by using a sum of harmonically related sinusoids.

- Can every periodic signal be synthesized as a sum of harmonically related sinusoids?
  - If $K \to \infty$ virtually all periodic waveforms can be synthesized with a sum of cosines and sines.

$$x(t+T) = x(t), \quad \forall t \in \mathbb{R}$$

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{+\infty} a_k \cos\left(2\pi k \frac{t}{T}\right) + \sum_{k=1}^{+\infty} b_k \sin\left(2\pi k \frac{t}{T}\right)$$

Classical Fourier series

- The coefficients of the series can be determined by:

$$a_k = \frac{2}{T}\int_0^T x(t)\cos\left(2\pi k \frac{t}{T}\right)dt \qquad b_k = \frac{2}{T}\int_0^T x(t)\sin\left(2\pi k \frac{t}{T}\right)dt \qquad \forall_{k>0}$$

Fourier Series

$$a_0 = \frac{2}{T}\int_0^T x(t)dt \qquad\qquad b_0 = 0$$

average signal value

- However, it is more practical to use complex exponentials instead of cosines and sines, and thus the previous expression can be rewritten as

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{+j2\pi k\frac{t}{T}}$$

Fourier Synthesis Summation

$$c_k = \frac{1}{T}\int_0^T x(t) e^{-j2\pi k\frac{t}{T}} dt$$

Fourier Analysis Integral

Fourier Series

- It can be shown that

$$a_k = c_k + c_{-k} \qquad b_k = j\left(c_k - c_{-k}\right)$$

- As an example, consider the following signal with period of T and $0 < a < \frac{T}{2}$

$$x(t) = \begin{cases} 1, & if \ \ |t| < a \\ 0, & a \le |t| \le \frac{T}{2} \\ x(t-T), & if \ \ t > \frac{T}{2} \\ x(t+T), & if \ \ t < -\frac{T}{2} \end{cases}$$

- The coefficients c_k are given by

$$c_k = \frac{1}{T}\int_0^T x(t) e^{-j2\pi k \frac{t}{T}} dt$$

$$= \frac{1}{T}\int_{-a}^{a} (1) e^{-j2\pi k \frac{t}{T}} dt$$

$$= \frac{1}{T} \left. \frac{e^{-j2\pi k \frac{t}{T}}}{-j2\pi \frac{k}{T}} \right|_{-a}^{a}$$

$$= \frac{\sin\left(2\pi k \frac{a}{T}\right)}{\pi k}$$

For k=0

$$\lim_{k \to 0} c_k = \frac{2a}{T}$$

$\Rightarrow$ Average value of x(t)

or

$$c_0 = \frac{1}{T}\int_0^T x(t) dt$$

Also known as DC component

Spectrum

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- The previous representation introduced the concept of **Spectrum** of a signal
  - A compact representation of the frequency content, i.e., how the energy of a given signal is spread by the frequencies.

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{+j2\pi k\frac{t}{T}}$$

Function with frequency $f_k = \frac{k}{T}$

C_k represents the complex coefficients in frequency domain (modulus and phase). Concept of “positive” and “negative” frequencies

Recall

$$\cos(2\pi f_0 t) = \frac{e^{+j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

$\Rightarrow$ We can say that the signal $\cos(2\pi f_0 t)$ is composed by two signals, one with a frequency f_0 and other with frequency $-f_0$

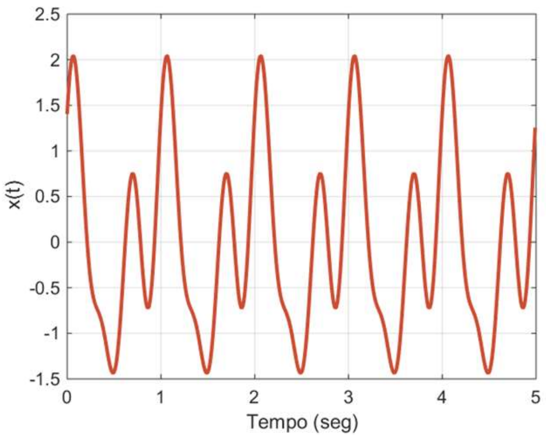
Spectrum

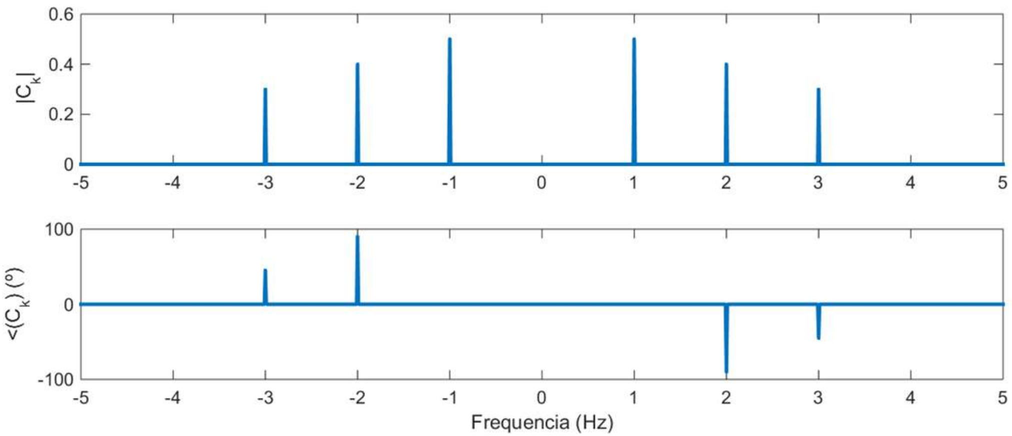
- Example

$$x(t) = \cos(2\pi t) + 0.8\sin(4\pi t) + 0.6\cos(6\pi t - \pi/4)$$

$$x(t) = \frac{1}{2}e^{j2\pi(1)t} + \frac{1}{2}e^{-j2\pi(1)t} + \frac{0.8}{2j}e^{j2\pi(2)t} - \frac{0.8}{2j}e^{-j2\pi(2)t} +$$

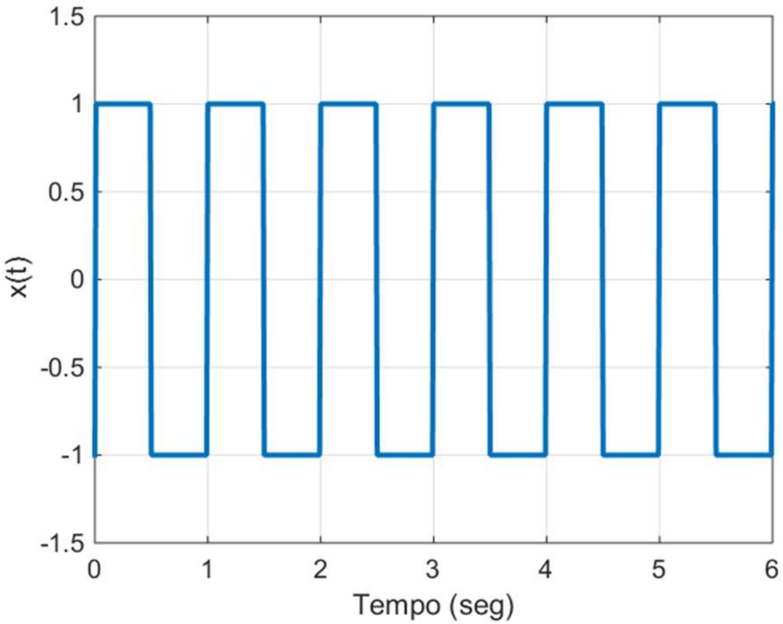
$$\frac{0.6}{2}e^{j2\pi(3)t}\ e^{-j\pi/4} + \frac{0.6}{2}e^{-j2\pi(3)t}\ e^{j\pi/4}$$

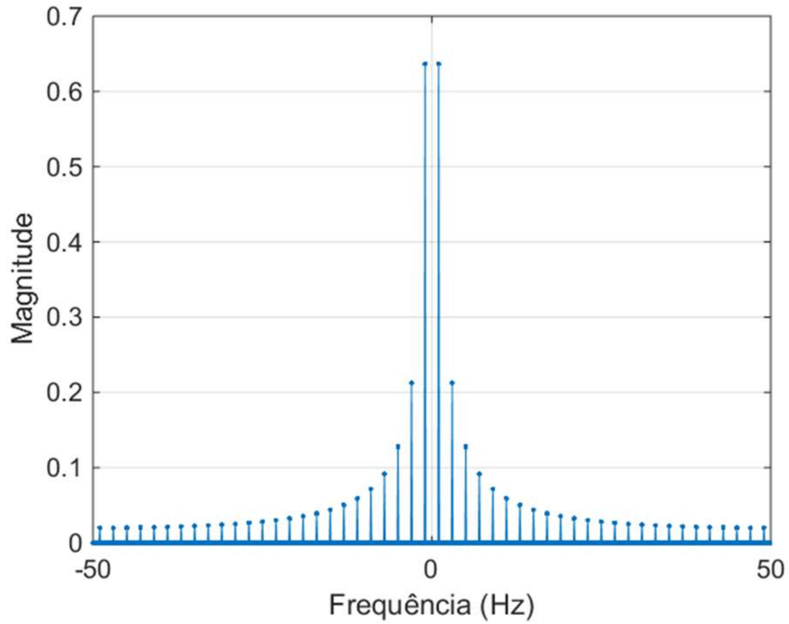




Spectrum

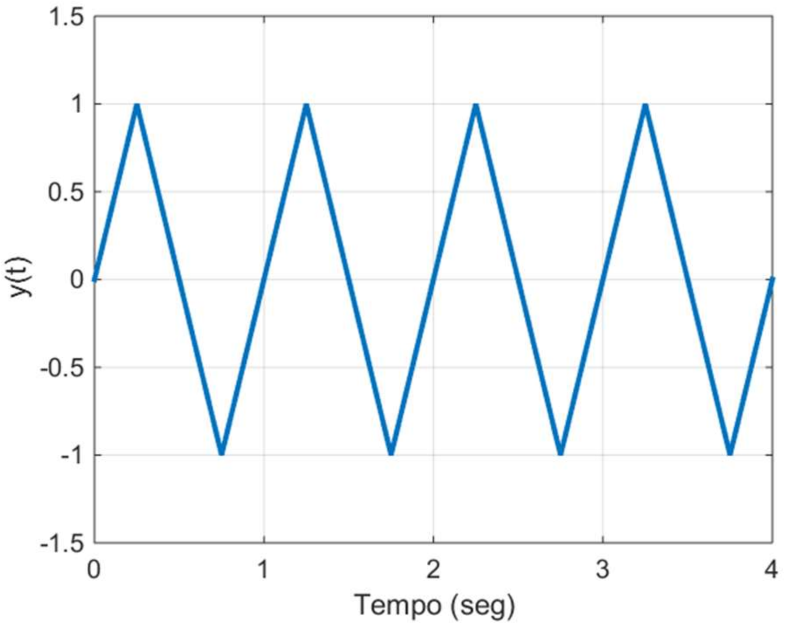
- Square wave:

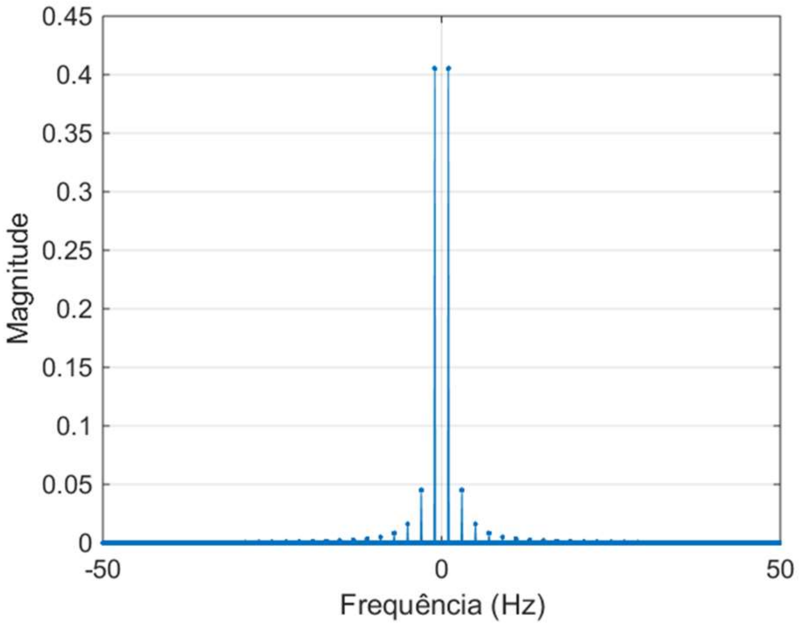




Spectrum

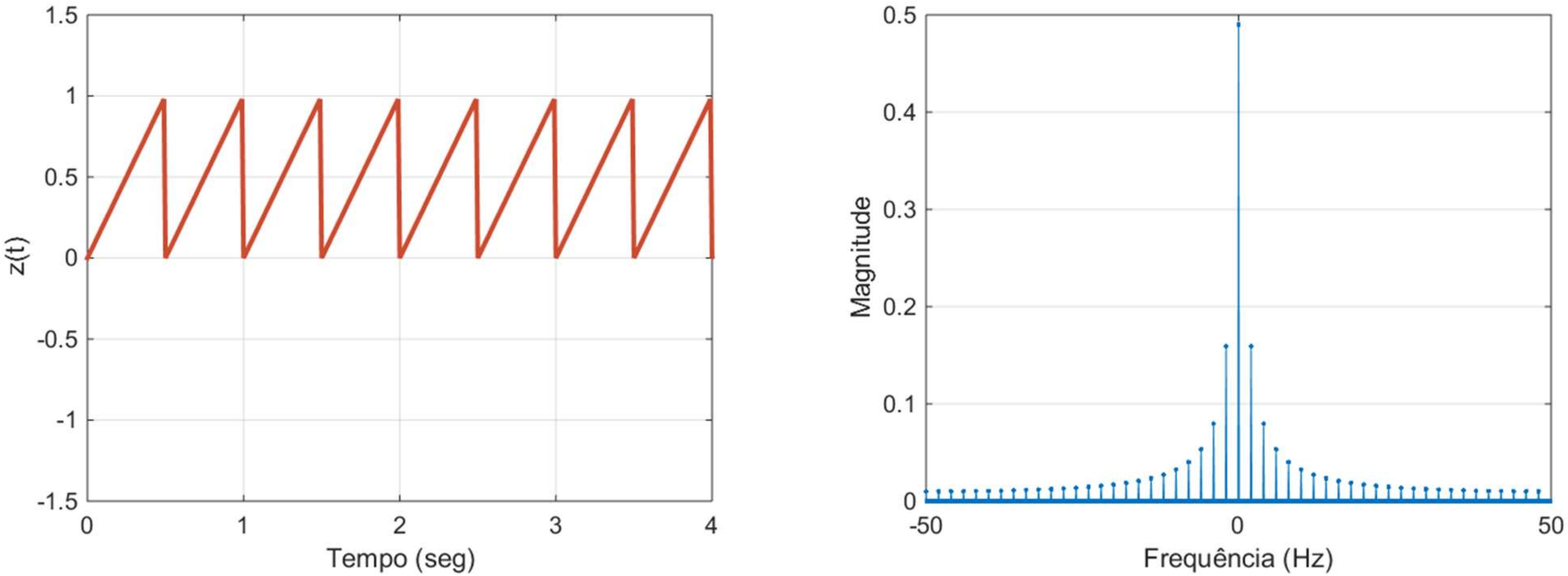
- Triangular wave:





Spectrum

- Sawtooth Wave:



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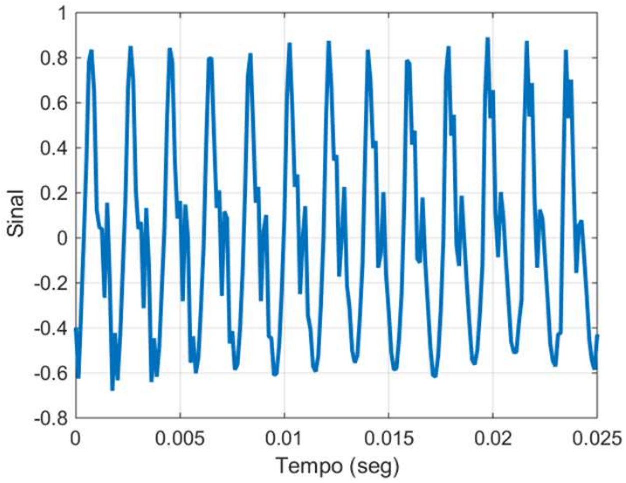
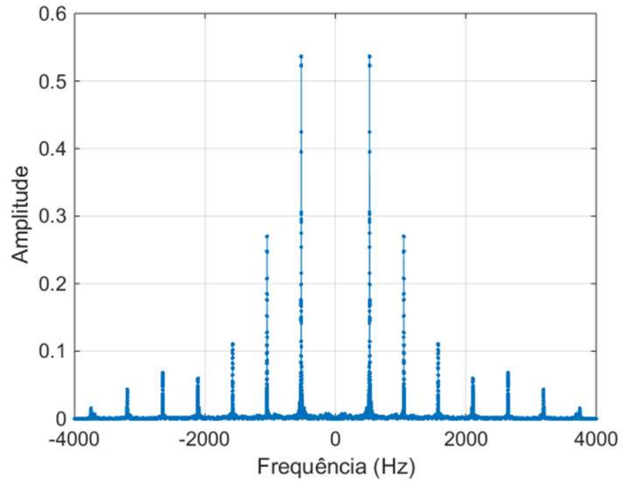
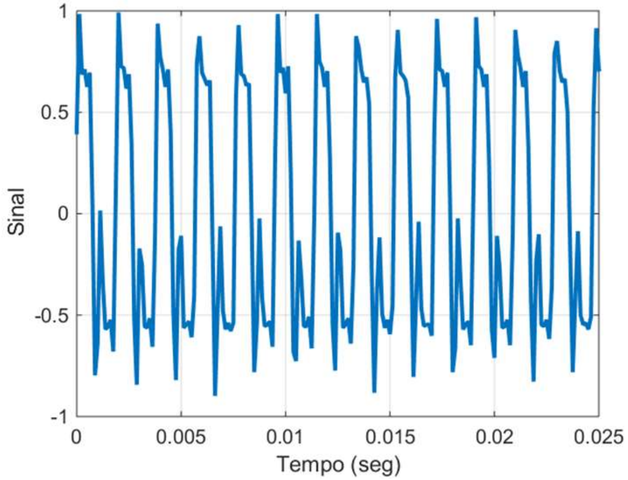
Spectrum

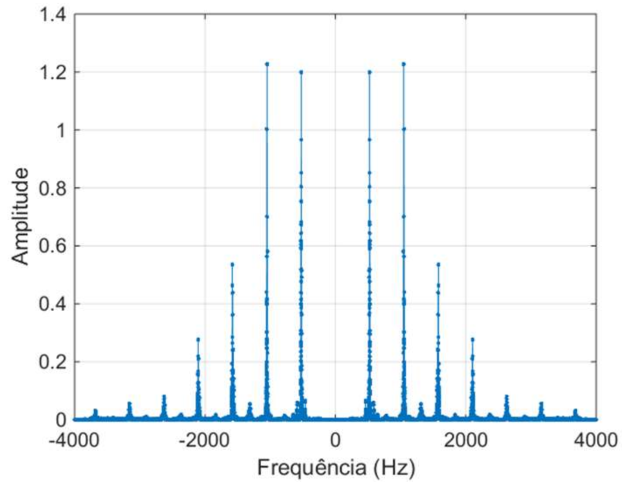
Piano C5 Key:





- Flute C5 Key:





Exercise I

- Consider

$$x(t) = 2\cos(4\pi t) - \sin(6\pi t)$$

1) Compute the fundamental frequency of $x(t)$
2) Expand x(t) in the classical Fourier series
3) Make a graph of $|c_k|$ as a function of frequency

Discrete Fourier Transform

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- In most cases, the signals are discretized in time (they are not continuous signals). So, what happens in the classical Fourier series when the signal $x(t)$, of period *T*, is sampled *N* times in a period?

- In this case, a sampling period of $T_a = T/N$ is used and we can do the following approximation

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_0^T x(t)\, e^{-j2\pi k\frac{t}{T}}\, dt \\
&\approx \frac{1}{T}\sum_{n=0}^{N-1} x(nT_a)\, e^{-j2\pi kn\frac{T_a}{T}}\, T_a \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x(nT_a)\, e^{-j2\pi\frac{kn}{N}} \\
&= \hat{c}_k
\end{aligned}
$$

Note that in the approximation we do $t = nT_a$

dt should be replaced by the step Ta that is given in the values of *t*

Discrete Fourier Transform

- From the $\hat{c}_k$ would be possible to get an approximation of $x(t)$. However, doing the sampling a problem appeared ⇒ this new coefficients are periodic, with a period of *N*.

  - It is easy to see that $\hat{c}_{N+k} = \hat{c}_k \quad \forall k \in \mathbb{N}$, and thus we can only use values of *k* from 0 to *N*-1 (corresponding to 1 period) or even better $|k| < \frac{N}{2}$

- It is however possible to make things more exact. Let's consider that $x(t)$ is sampled getting the *N* samples $x(0), x(T_a), \ldots, x((N-1)T_a)$. Without loss of generality, in the next expressions we will ignore the sampling period.

- Thus, the samples are given by $x(0), x(1), \ldots, x(N-1)$, where the argument is the number of the sample (array index)

Discrete Fourier Transform

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- From the N samples let's compute the equivalent of $\hat{c}_k$ (in the literature it is common to use $X(m)$ instead $\hat{c}_k$, and thus here we will consider $X(m)$)

$$X(m)=\frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j2\pi\frac{mn}{N}} \qquad ,m=0,1,\ldots,N-1$$

Note that X(m) e a periodic function of period N

- We can convert the N samples of x(n) to the N samples of X(m) applying the following linear transformation

$$\underbrace{\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}}_{X} = \frac{1}{N}\underbrace{\begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & \cdots & W_N^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^0 & W_N^{(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}}_{W}\underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{x}$$

with $W_N = e^{-\frac{j2\pi mn}{N}}$

Discrete Fourier Transform

- In matrix notation we have

$$\boldsymbol{X} = \frac{1}{N}\boldsymbol{W}\boldsymbol{x}$$

$$\frac{1}{\sqrt{N}}\boldsymbol{W}$$

Called Fourier Matrix

- The inverse of the matrix is your conjugate for what is easy getting $\boldsymbol{x}$ from $\boldsymbol{X}$, and vice-versa. The following Matlab code lustrate that

```
N =10;                              % value of N
mn =(0:N -1) '*(0:N -1);            % exponents
F=exp (-2i*pi/N*mn )/ sqrt (N);     % Fourier matrix
norm (inv (F)- conj (F))            % inv (F) is almost
                                    % equal to conj (F)
```

Discrete Fourier Transform

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- Less than a scale factor, simply multiply ***X*** by that conjugate, we obtain ***x***. In terms of summation, we get

$$x(n)=\frac{1}{N}\sum_{n=0}^{N-1}X(m)e^{j2\pi\frac{mn}{N}} \qquad ,n=0,1,\ldots,N-1$$

- Note that $x(n)$ is also a periodic signal of period N.

- In the MatLab the DFT ($x(n) \longrightarrow X(m)$) can be computed using the **fft(.)** MatLab function (it does not divide by a factor of N) and the inverse DFT ($X(m) \longrightarrow x(n)$) can be computed using the **ifft(.).**

- How can we associate the samples to the frequencies?

Discrete Fourier Transform

- Notice that we can associate $X(m)$ to $\hat{c}_m$ (here the index k was replaced by m). Thus, $\hat{c}_m$, which is an approximation of c_m , is associated to signal as (remember that $T = NT_a$)

$$e^{j2\pi m\frac{t}{T}} = e^{j2\pi m\frac{t}{NT_a}} = e^{j2\pi\left(\frac{m}{N}f_a\right)t}$$

- So, $X(m)$ is associated to the frequency $\frac{m}{N}fa$. Note that due to the periodicity of $X(m)$, we must interpret the value, m=N as m=0, m=N-1as m=-1, m=N-2 as m=-2 and so on.

- Let's see a simple example to associate the coefficients $X(m)$ to the respective frequencies.

Exercise II

- A periodic signal with a fundamental frequency of 10Hz was sampled 5 times in one period. The $X(m)$ values of your DFT are

m	*X(m)*
0	2
1	3+4j
2	-1
3	-1
4	3-4j

1) Draw the graph of $|X(m)|$ where the xx axis is calibrated in Hz.

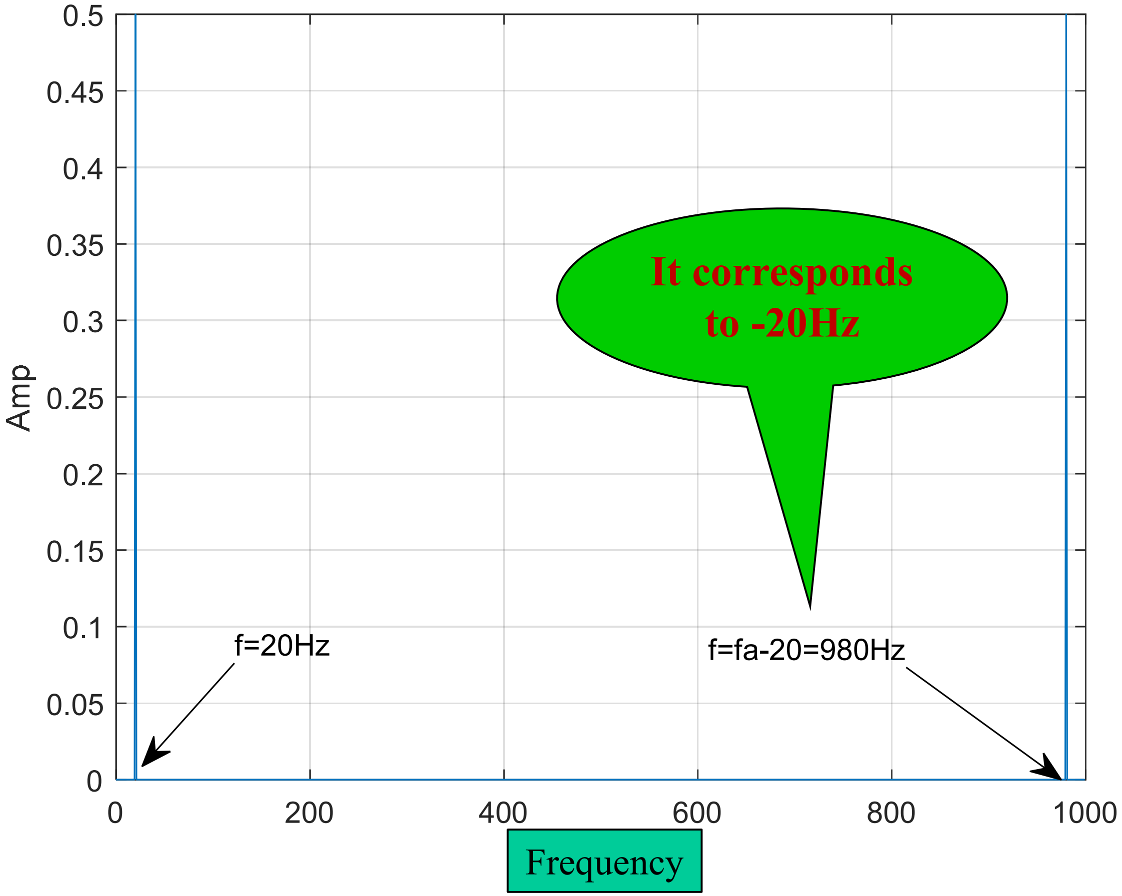
Discrete Fourier Transform

- Example considering $x(t) = \sin(2\pi f_0 t)$

```
N =1000;                % number of samples
Ta =0.001;              % sampling period
fa =1/ Ta;              % sampling frequency
fo =20;                 % signal frequency (Hz)
t =(0:N -1)* Ta;        % time instants
x= sin (2* pi*fo*t);    % signal
X= fft (x)/N;           % DFT
f =(0:N -1)*fa/N;       % DFT frequencies
plot (f,abs(X ));       % graph abs(DFT)
```

Discrete Fourier Transform

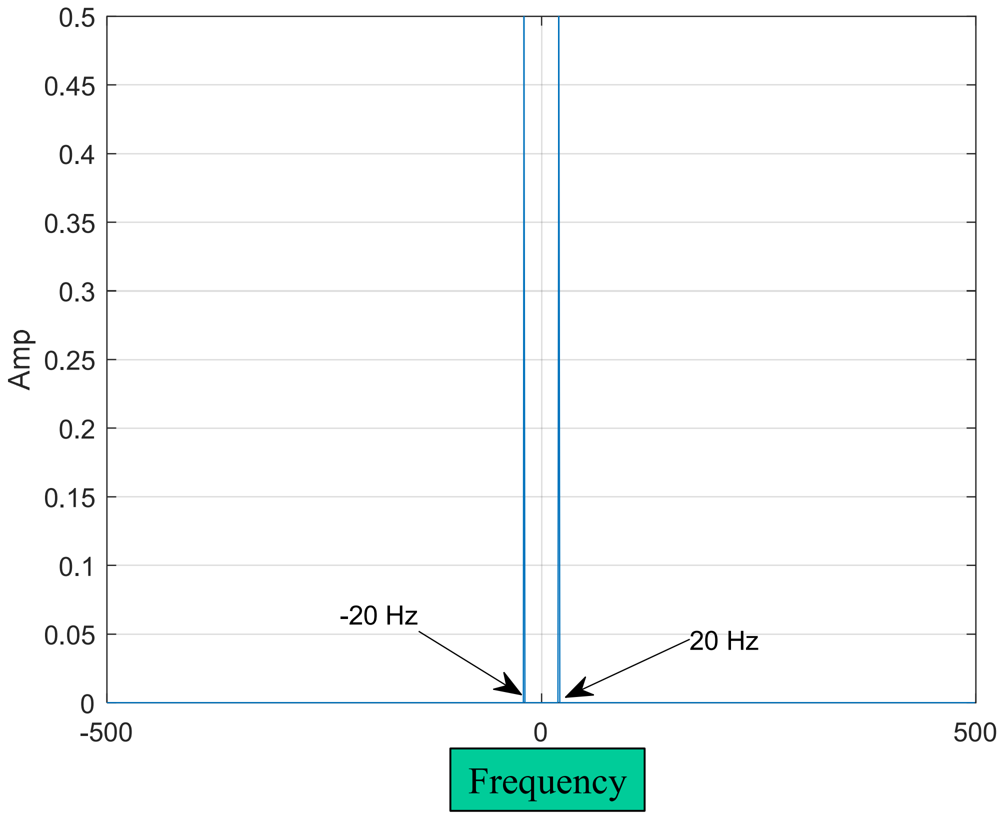
- MatLab Example considering $x(t) = \sin(2\pi(20)t)$



If it was a cosine the graph would be the same (the difference is only in the phase !)

Discrete Fourier Transform

- Example considering $x(t) = \sin(2\pi f_0 t)$



It corresponds to -20Hz

Use **fftshift(X)** to sort from $-f_a/2$ to $+f_a/2$.

f =(0:N -1)*fa/N- fa/2;

Sampling

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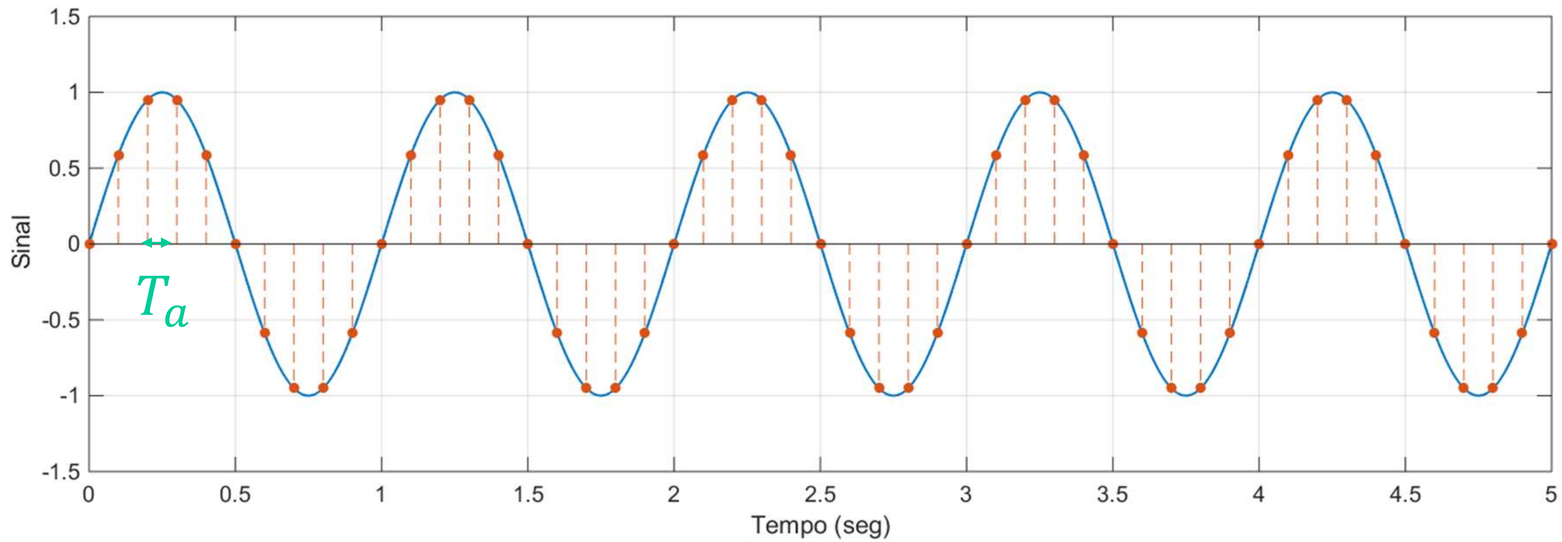
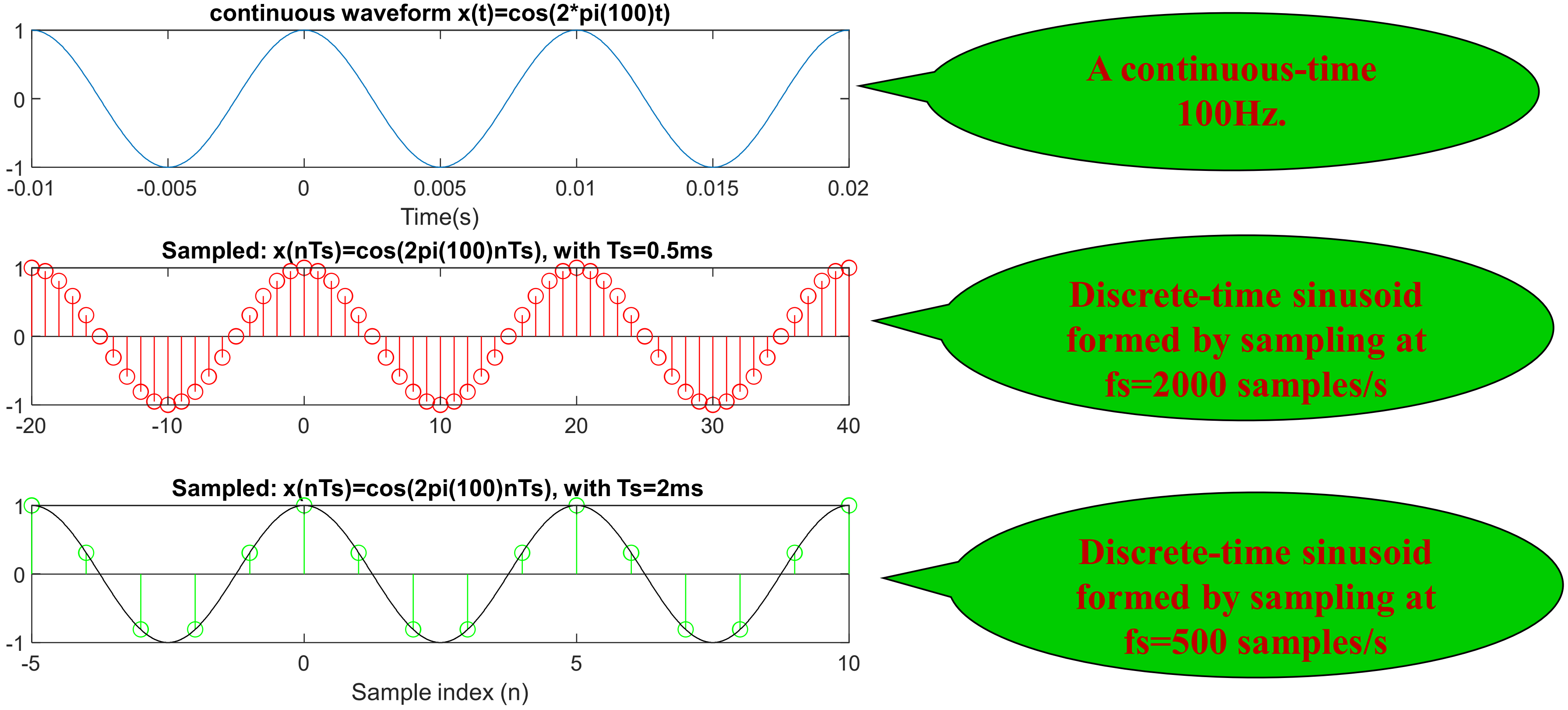
- The processing and analysis of signals using digital processors is performed in the discrete-time domain.
- This, naturally, requires the signals that evolve continuously over time to be **sampled**.
- The sampling process consists of acquiring samples of the signal (typically at a periodic rate).



T_a - Sampling period. $f_a = 1/T_a$ - Sampling frequency.

Sampling

- A key question is how many samples per second are needed to adequately represent a continuous time signal?



Sampling

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- Let's first define the normalized frequency

$$\hat{w} \triangleq w_0 T_s = \frac{w_0}{f_s}$$

Normalized radian frequency

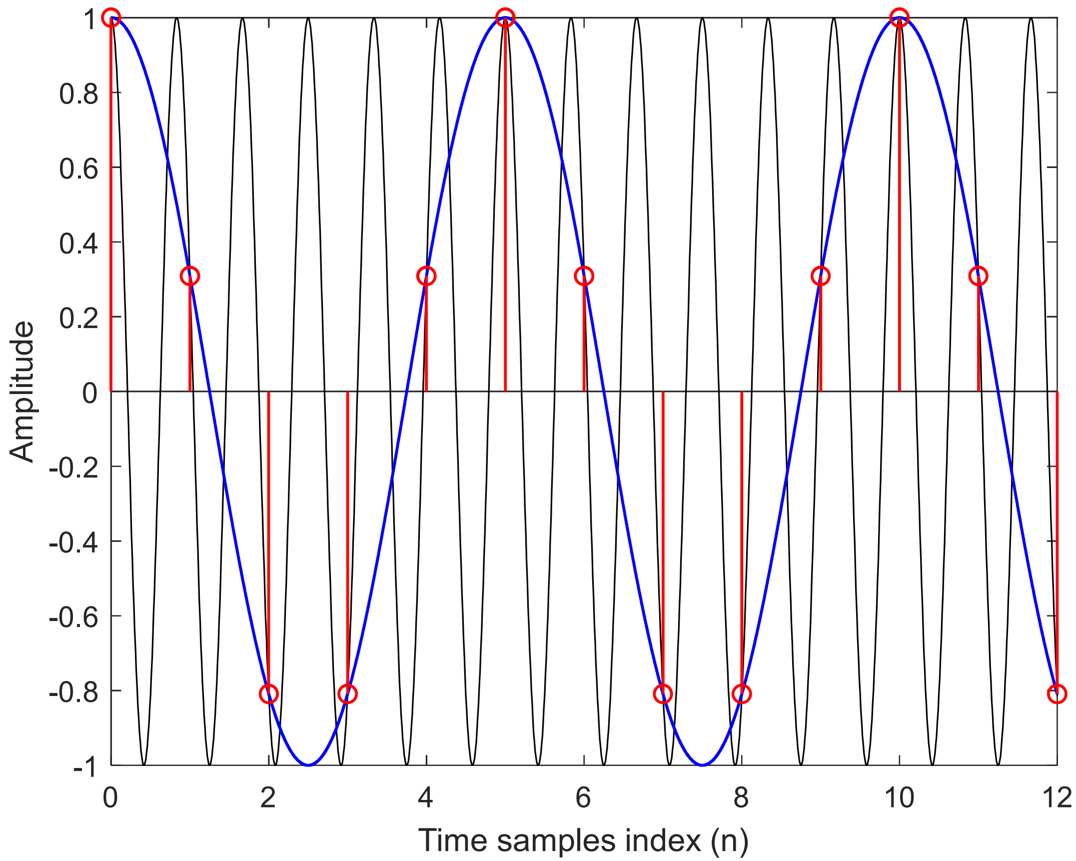
$$\hat{f} \triangleq \frac{f_0}{f_s}$$

Normalized frequency

- Let's also define the concept of aliases. A simple definition of the word alias would be "*two names for the same person or thing*"
- In the scope of sampling the concept of alias can be introduced by showing that two different discrete-time sinusoid can define the same signal values.
- The sampled signal discussed in the previous slide $x_1(n) = \cos(0.4\pi n)$ is identical to $x_2(n) = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n)$ because $2\pi n$ is an integer number of periods of the cosine function.
  - Since $x_1(n) = x_2(n), \ \forall n \in \mathbb{N}$, the frequency 2.4π is an alias of 0.4π

Sampling

- The following figure illustrate the aliasing concept.



Blue: $x_1(n) = \cos(0.4\pi n)$ **Black:** $x_2(n) = \cos(2.4\pi n)$

Illustration of the aliasing: two signals drawn through the same samples

The samples belong to two different cosine signals with different frequencies, but the cosine have the same values at n=0,1,2,3….

Aliasing occurs when a high frequency is indistinguishable from a low frequency after sampling

Sampling

- In general, sampled signal

$$x(nT_s) = \cos(2\pi f_0 T_s n) = \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

is indistinguishable from each of the signals

$$\cos\left(2\pi \frac{f_0}{f_s} n + 2\pi nk\right) = \left(2\pi \frac{f_0 + kf_s}{f_s} n\right),\ k \in \mathbb{Z}, n \in \mathbb{N}$$

- Therefore, in general, what happens at frequency f_0 of the original unsampled signal will have to be reflected in the frequencies $f_0 + kf_s$ of the sampled signal, since we cannot distinguish them.

Sampling

- The spectrum contains all the aliases at the following frequencies

$$\hat{w} = \frac{w_0}{f_s} + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots \qquad \text{or} \qquad \hat{f} = \frac{f_0}{f_s} + k, \quad k = 0, \pm 1, \pm 2, \ldots$$

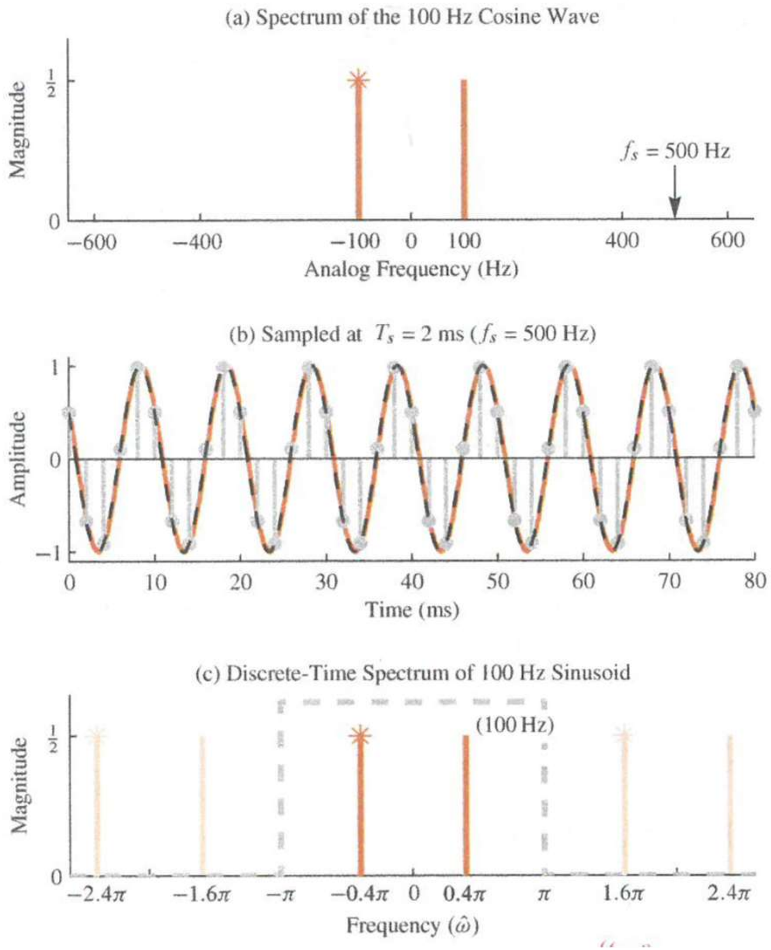
$$\hat{w} = -\frac{w_0}{f_s} + 2\pi k \quad k = 0, \pm 1, \pm 2, \ldots \qquad \hat{f} = -\frac{f_0}{f_s} + k \quad k = 0, \pm 1, \pm 2, \ldots$$

- If $\left|\frac{f_0}{f_s}\right| < \frac{1}{2}$ it is possible to unambiguously determine the frequency f_0 from the signal samples.
- In general, it is always possible to find a k so that $\left|\frac{f_0 + kf_s}{f_s}\right| < \frac{1}{2}$.
- When reproducing the tone, it is this frequency between $-f_s / 2$ and $+f_s / 2$ that will be heard.

Sampling

- Let's see an example of sampling a continuous-time 100Hz sinusoid

$$x(t) = \cos\left(2\pi(100)t + \pi/3\right)$$



Oversampling at f_s=500samples/s

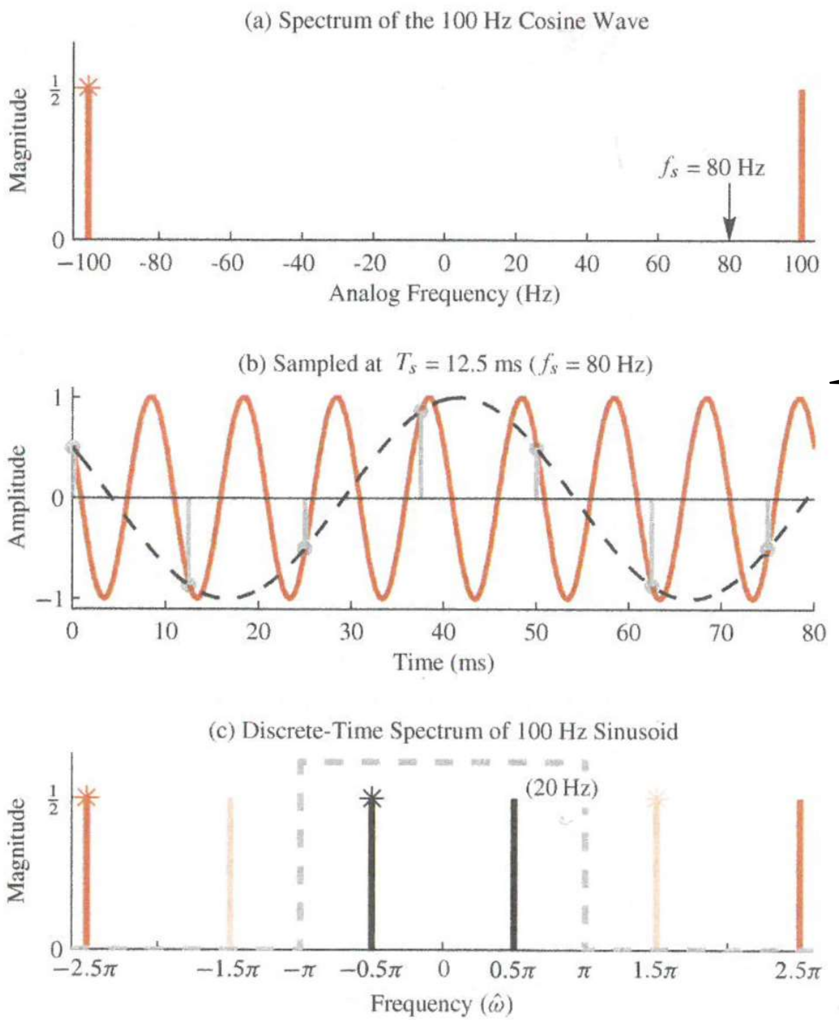
Samples x(n) as grey dots and the reconstructed as dashed black line. Orange is the original

Spectrum with the original $\pm f_o$ along with two aliases

$$\hat{w} = 0.4\pi + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\hat{w} = -0.4\pi + 2\pi k \quad k = 0, \pm 1, \pm 2, \ldots$$

Sampling



Undersampling at f_s=80 samples/s

Samples x(n) as grey dots and the reconstructed as dashed black line. Orange is the original

Spectrum with the original $\pm f_o$ along with two aliases

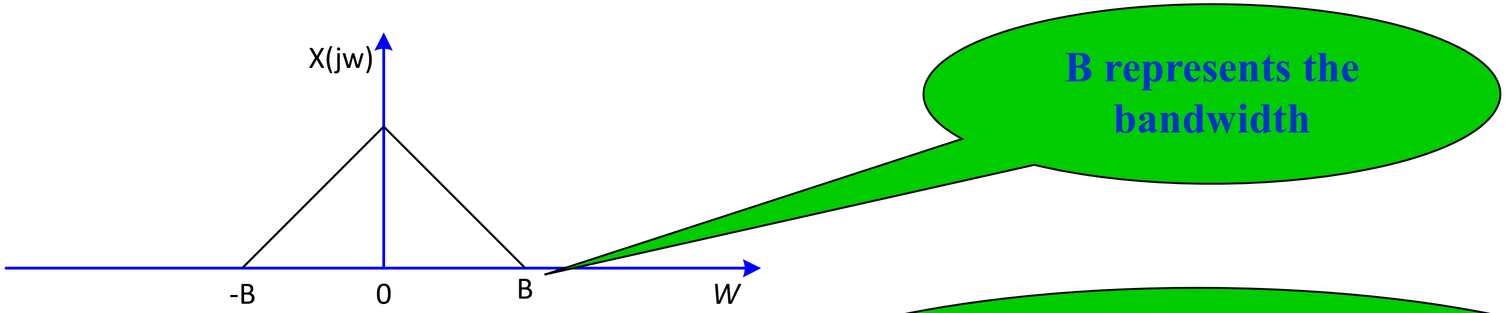
$$\hat{w} = 2.5\pi + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\hat{w} = -2.5\pi + 2\pi k \quad k = 0, \pm 1, \pm 2, \ldots$$

What can you conclude?

Sampling

- Let's now consider a general signal $x(t)$ with limited bandwidth



$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$$

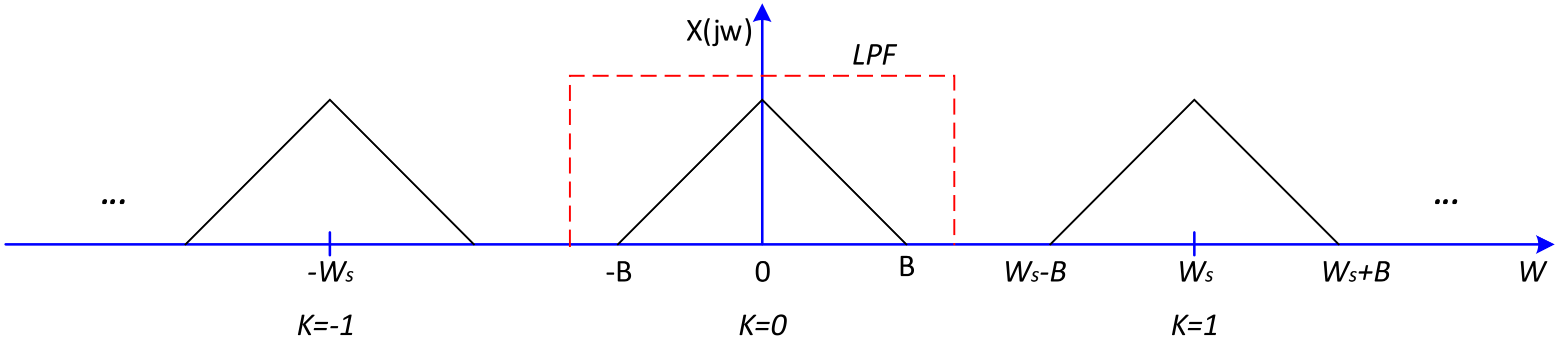
It is the Fourier transform . Note that the Fourier discrete transform can be considered an approximation of the Fourier transform

- Is can be shown that the Fourier transform of the sampled signal, less than a possible scale factor, is given by

$$X(jw) = \sum_{k=-\infty}^{+\infty} X\left(j(w - kw_s)\right) = \sum_{k=-\infty}^{+\infty} X\left(j2\pi(f - kf_s)\right)$$

Sampling

- From the previous expression we can see that the spectrum of the samples signal is the sum of all translations of $X(jw)$ by multiples of the sampling frequency



- It can be seen that the translations of $X(jw)$ do not overlap if

$$w_s - B > B \quad ,\text{i.e, } w_s > 2B$$

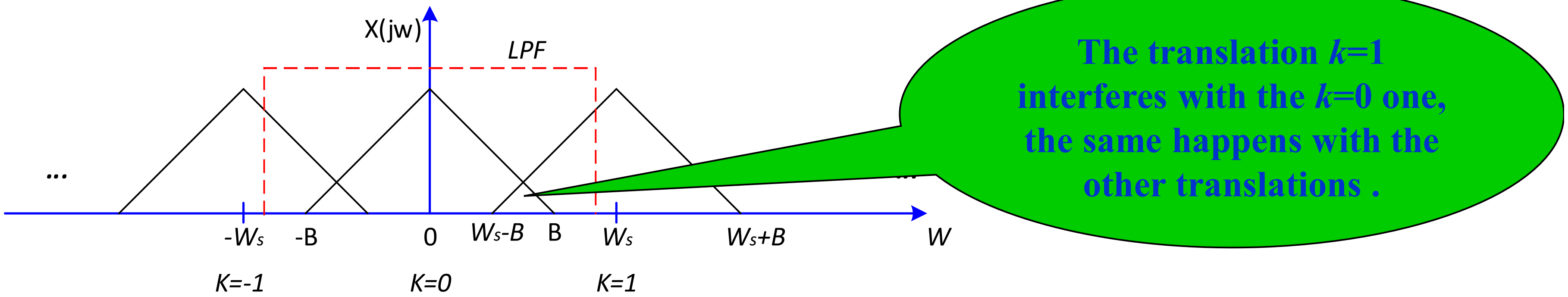
Known as Nyquist frequency

- In this case, as there is no overlap, a low-pass filtering (LPF) would eliminate all translations with $k \neq 0$, and thus perfectly recovering the original signal.

Sampling

- Shannon Sampling Theorem
  - A continuous-time signal $x(\mathrm{t})$ with frequencies no higher that B can be reconstructed exactly from its samples $x(\mathrm{nT_s})$, if the samples are taken at a rate $f_s=1/T_s$ that is greater that 2B

- What happens, then, when sampling does not meet the Nyquist Criterion?

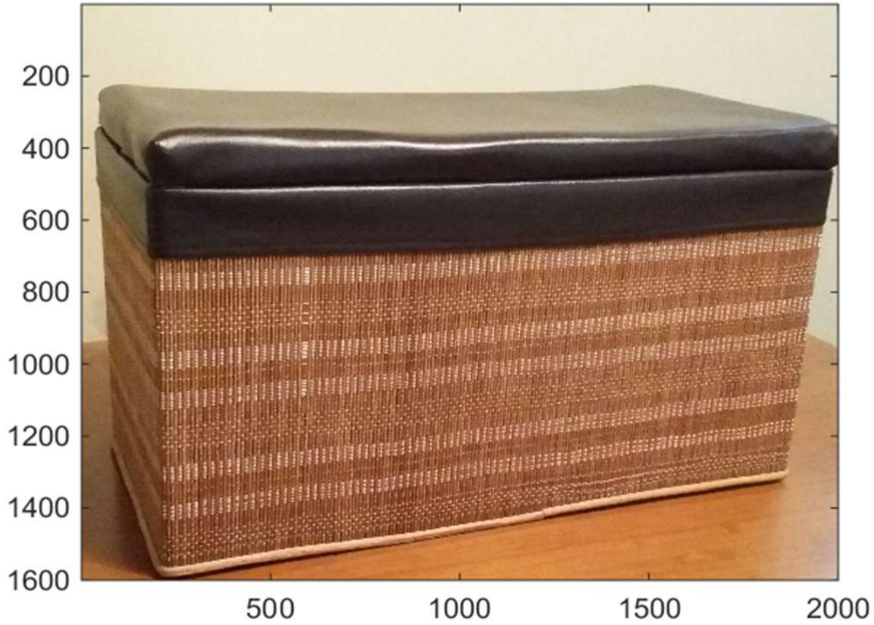


- If $w_s \leq 2B$ the translations will overlap and the original signal may not be recovered perfectly.

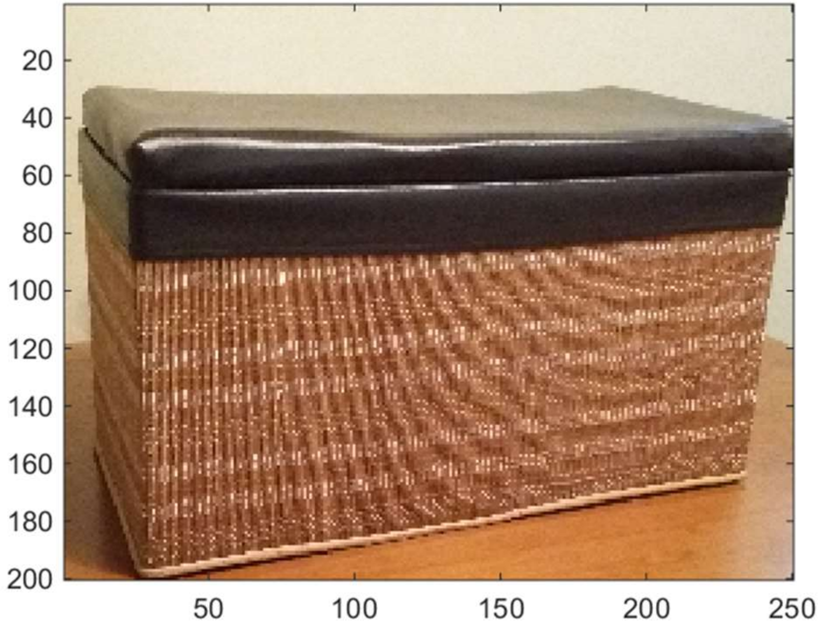
Sampling

- Example of an image submitted to under sampling:

Original Image

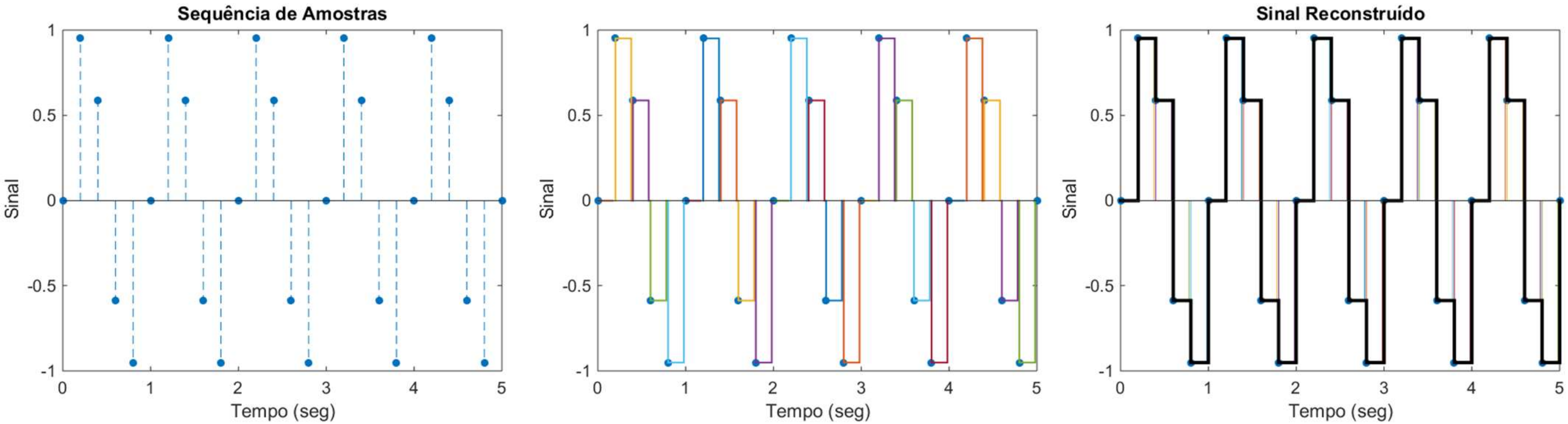


Under-sampled image (8x8 times)



Reconstruction of Sampled Signals

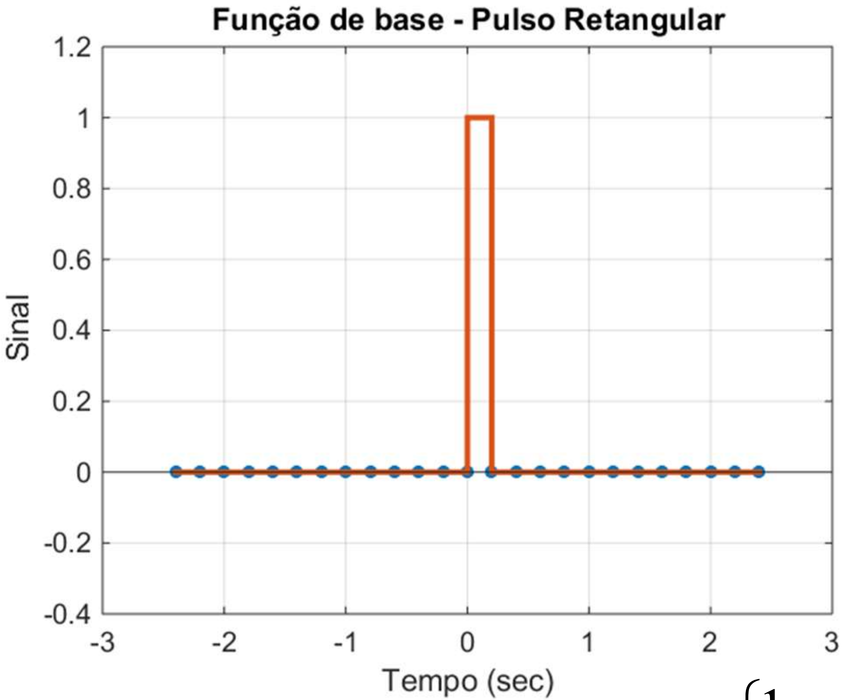
- Let $x(n), n = 1, \ldots, N$, be the sequence of samples of a signal, with sampling period T_s. How can the original continuous-time signal be reconstructed from those samples?
- It is easy to verify that this can not be performed using a pulse (pedestal) for each sample:

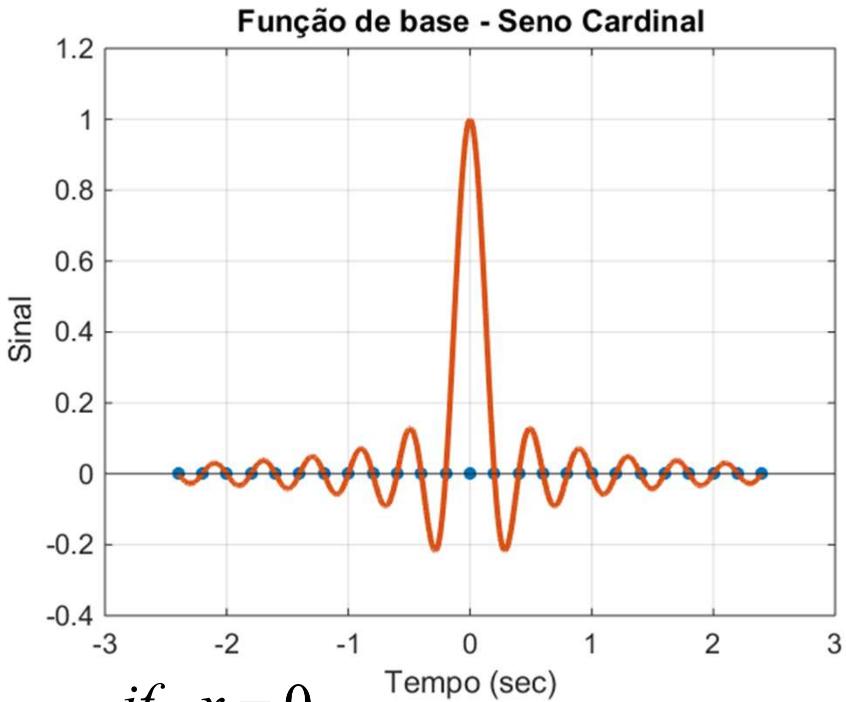


- The resulting signal clearly has a maximum frequency greater than half the sampling frequency.

2. Reconstruction of Sampled Signals

- The ideal reconstruction should consider the **cardinal sine** function (sinc) as the basis function, instead of the rectangular pulses.

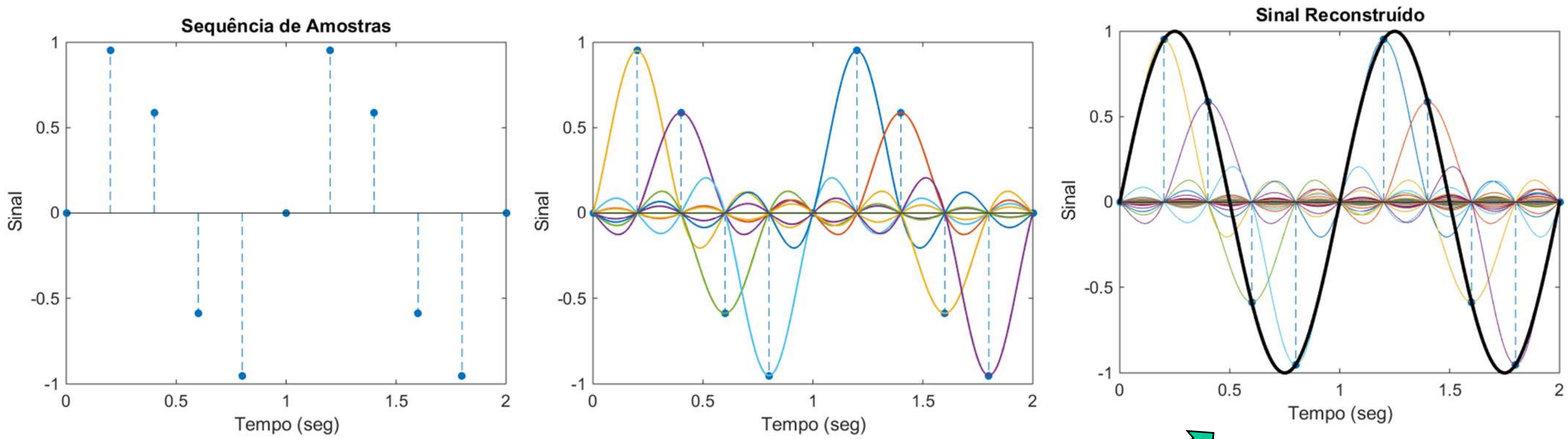




$$\sin c(x) = \begin{cases} 1, & if \quad x = 0 \\ \dfrac{\sin(\pi x)}{\pi x} & if \quad x \neq 0 \end{cases}$$

Reconstruction of Sampled Signals

- The ideal reconstruction should consider the **cardinal sine** function as the basis function, instead of the rectangular pulses.



- The reconstructed signal is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}\left(\frac{t}{T_s} - n\right)$$