CEE6501 — Lecture 6.3

Fixed End Forces (FEFs)

Learning Objectives

By the end of this lecture, you will be able to:

- Distinguish between **joint loads** and **member loads**
- Define and interpret **fixed-end forces (FEFs)**
- Explain why $\boldsymbol{Q} = \boldsymbol{k}\boldsymbol{u}$ is insufficient for member loading
- Incorporate FEFs into the beam element formulation

Agenda

1. Part 1 — What are Fixed End Forces
2. Part 2 — Why Do We Need FEFs
3. Part 3 — Incorporating FEFs into DSM

Part 1 — What are Fixed End Forces

Definition

Fixed-end forces (FEFs) are:

The forces and moments that develop at the ends of a member due to external loading, assuming both ends are **fully fixed** (no translation or rotation).

Joint Loads vs Member Loads

- **Joint loads** → applied directly at nodes
- **Member loads** → applied between nodes
    - distributed loads
    - point loads along a span
    - applied moments

In deriving the beam stiffness matrix, we assumed:

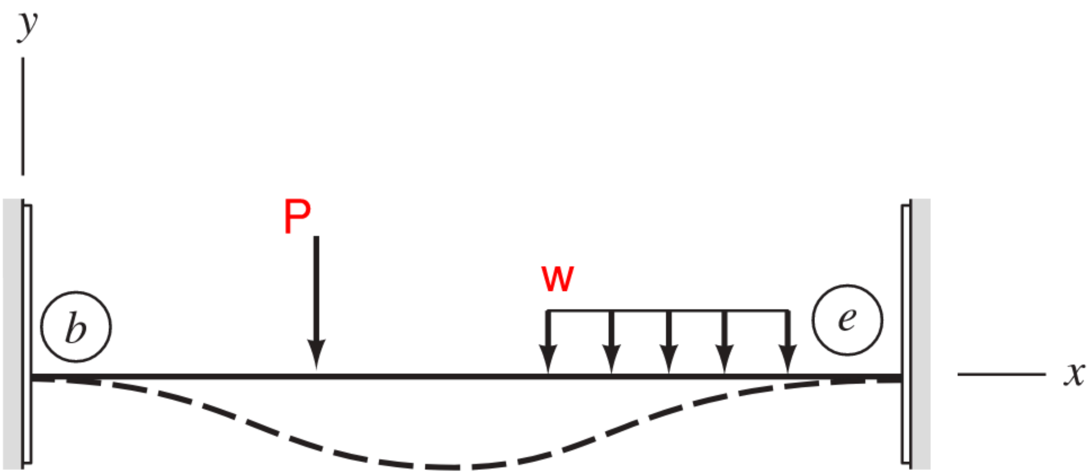
- loads act only at nodes
- no loading within the element

➡️ Now we relax this assumption.

Part 2 — Why Do We Need FEFs?

A Simple Case

Consider a **fully fixed beam** subjected to loading along its span.



All DOFs are restrained:

$$\boldsymbol{u} = \boldsymbol{0}$$

From typical local element force-displacement relationship:

$$\boldsymbol{Q} = \boldsymbol{k}\boldsymbol{u}$$

we obtain:

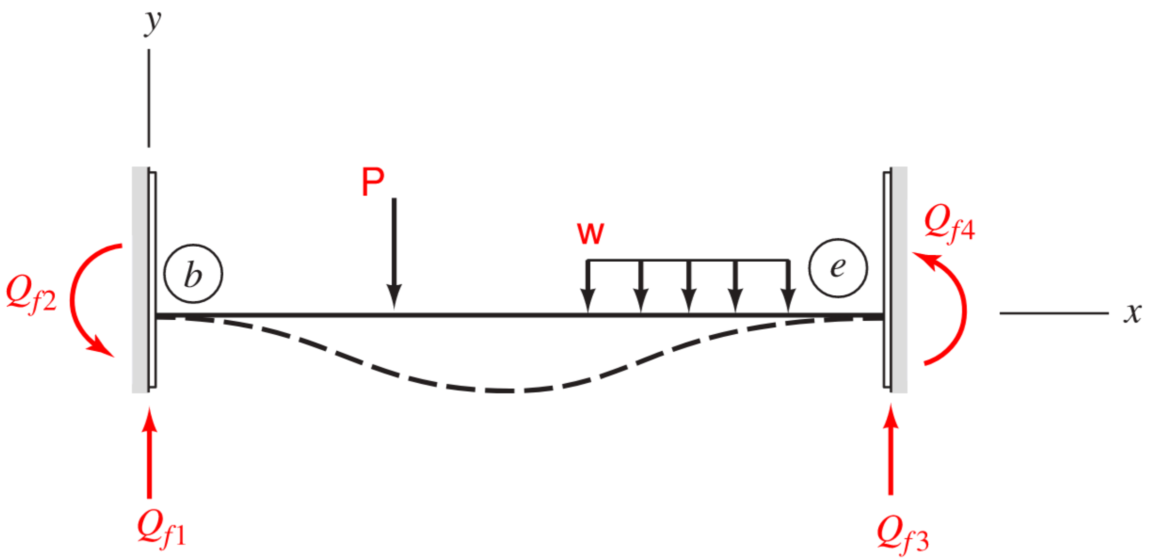
$$\boldsymbol{Q} = \boldsymbol{0}$$

❌ Clearly incorrect — reactions exist even with zero displacement.

Corrected Local Element Force-Displacement Relationship

$$\boldsymbol{Q} = \boldsymbol{k}\boldsymbol{u} + \boldsymbol{Q}_f$$

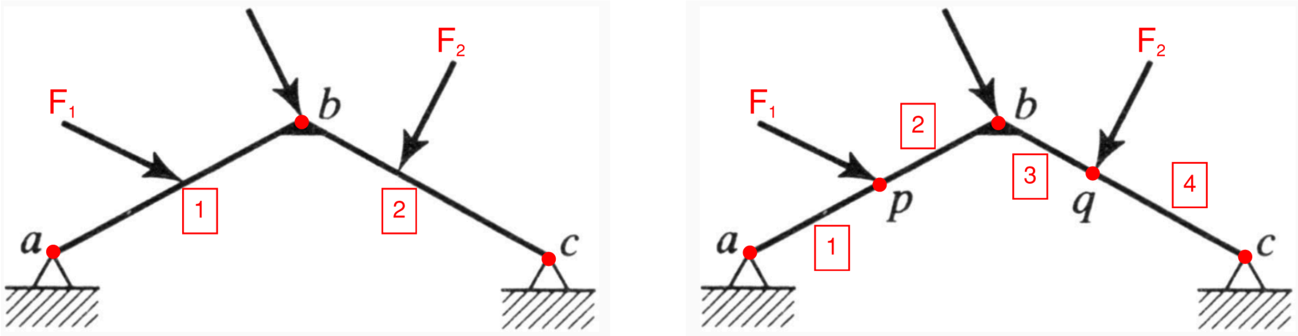
- $\boldsymbol{Q}_f$ = **fixed-end force vector**
- Captures effects of **loads between nodes**



These forces are required whenever member loads are present

Avoiding FEFs with Artificial Nodes

One approach is to insert **artificial nodes** at load locations.



- Convert member loads → joint loads
- Enforce equilibrium at new nodes
- Maintain compatibility between segments

Artificial Nodes for Point Loads and Distributed Loads

This works for **point loads**, but:

- increases number of elements and DOFs in the solution
- not computationally efficient for general use

For **distributed loads**:

- Infinite number of points carry load
- Cannot represent exactly with finite nodes

Alternative: **lump distributed loads into point forces**

- preserves total force (equilibrium)
- but does not accurately represent local **internal shear and moment distributions**

Key Idea

We need a method that:

- accounts for **loads within elements**
- preserves **exact equilibrium**
- does not increase DOFs

Part 3 - Indeterminate Beam Structural Analysis

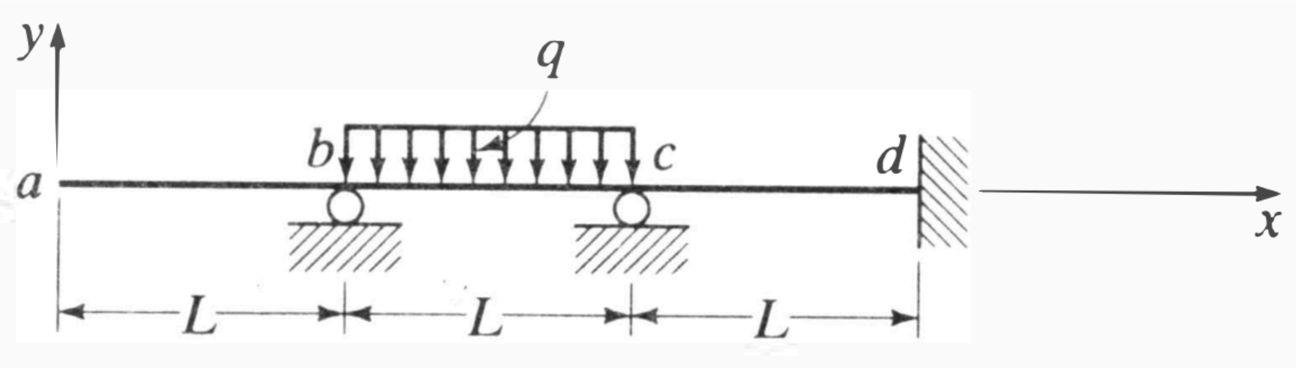
Solving Indeterminate Beams (Role of FEFs)

Recall from structural analysis:

- Indeterminate structures are solved by enforcing:
  - **equilibrium**
  - **compatibility**
- For beams with **member loads**, we conceptually:
  - split the problem into **simpler systems**
    - one with **fixed-end forces (FEFs)** (all DOFs restrained)
    - one with **equivalent nodal loads** (to restore deformation)
  - solve each system separately
  - **superimpose the results** to obtain the final response

Indeterminate Beam

Same beam as in Lecture 6.2, only loaded with distributed load

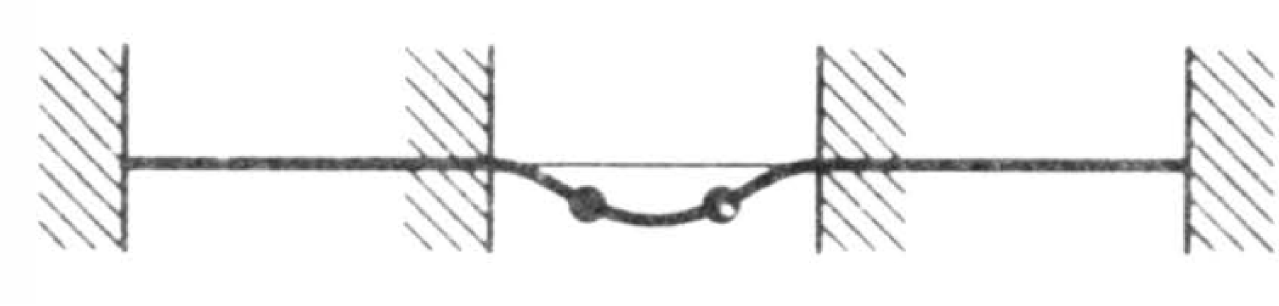


We analyze the beam by splitting it into **two systems**, then summing the results.

Stage 1 — Clamp the Beam (Zero DOFs)

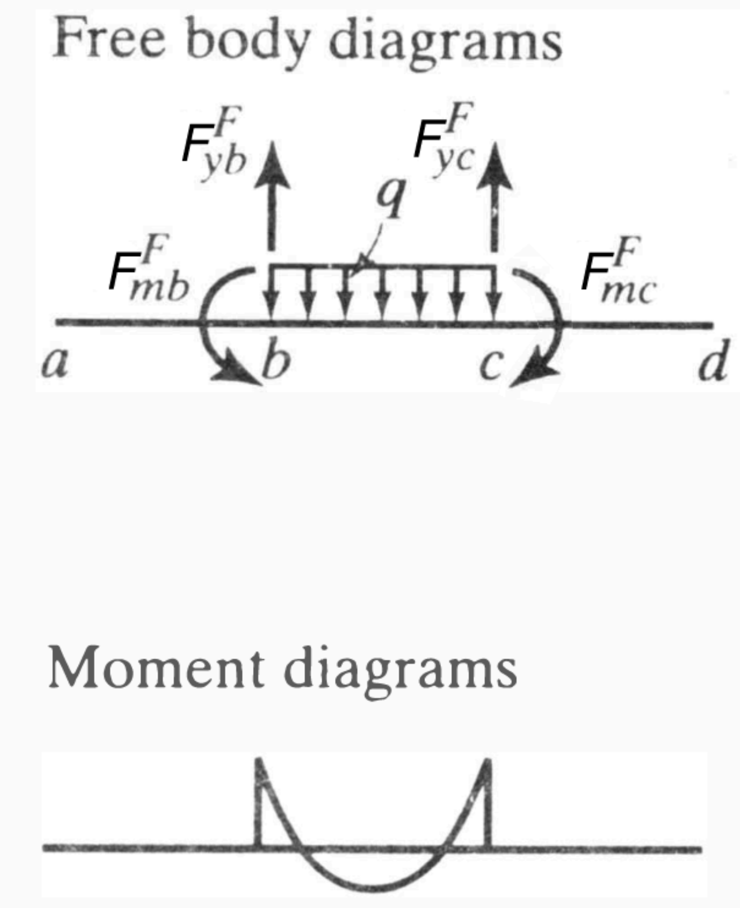
isolate effect of **member loading only**

- Add **fictitious constraints** at element ends
- Beam becomes **fully fixed**



Stage 2 — Compute Fixed-End Forces

- Solve the **fully fixed beam** under loading
- Use known closed-form solutions



Example (uniform load):

$$F_{yb}^F = \frac{qL}{2}, \quad F_{mb}^F = \frac{qL^2}{12}$$

$$F_{yc}^F = \frac{qL}{2}, \quad F_{mc}^F = -\frac{qL^2}{12}$$

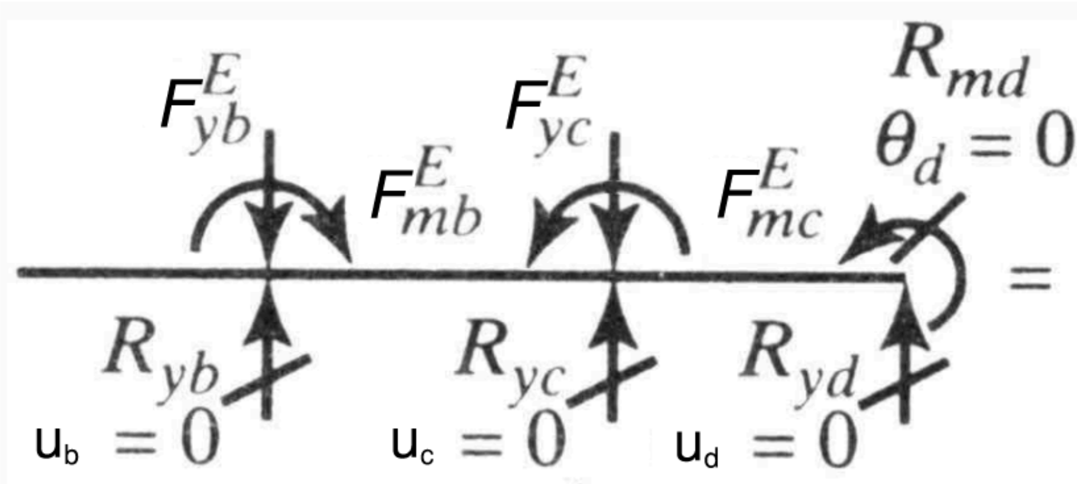
These are the **fixed-end forces**, $\boldsymbol{F}^F$.

Stage 3 — Remove Constraints (Equivalent Loads)

- Remove fictitious constraints
- Apply **equal and opposite forces** at nodes:

$$\boldsymbol{F}^E = -\boldsymbol{F}^F$$

- These are called **equivalent nodal loads**
- converts member loading → nodal loading

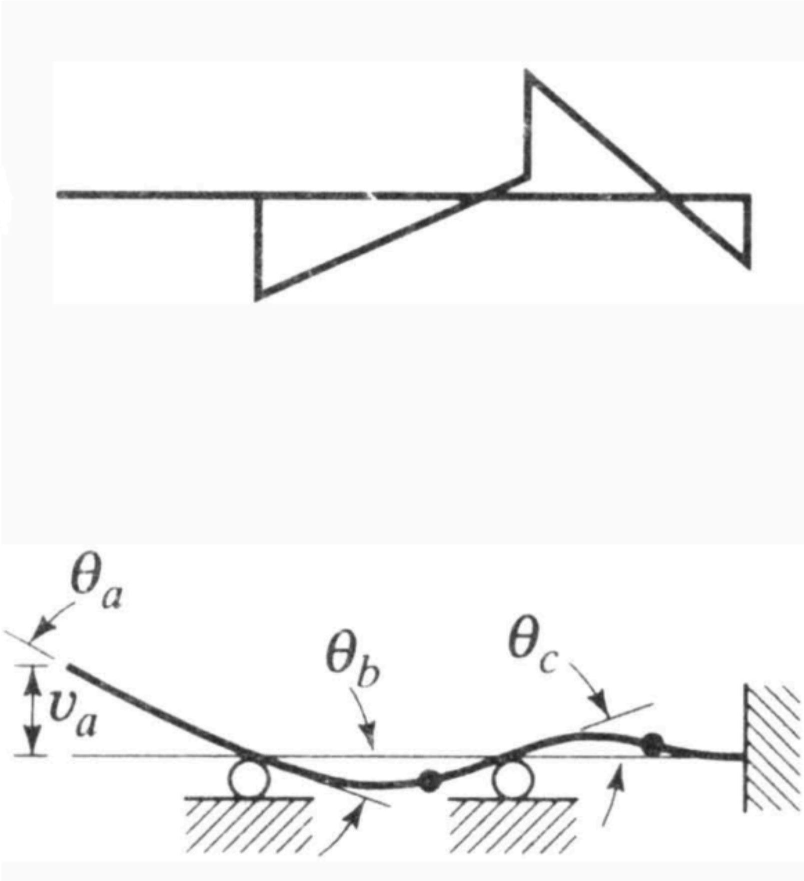


Stage 4 — Solve for Deformations Under Equivalent Nodal Loads

Now solve:

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}^E$$

Standard DSM solution



Stage 5 — Superimpose and Recover the Real System

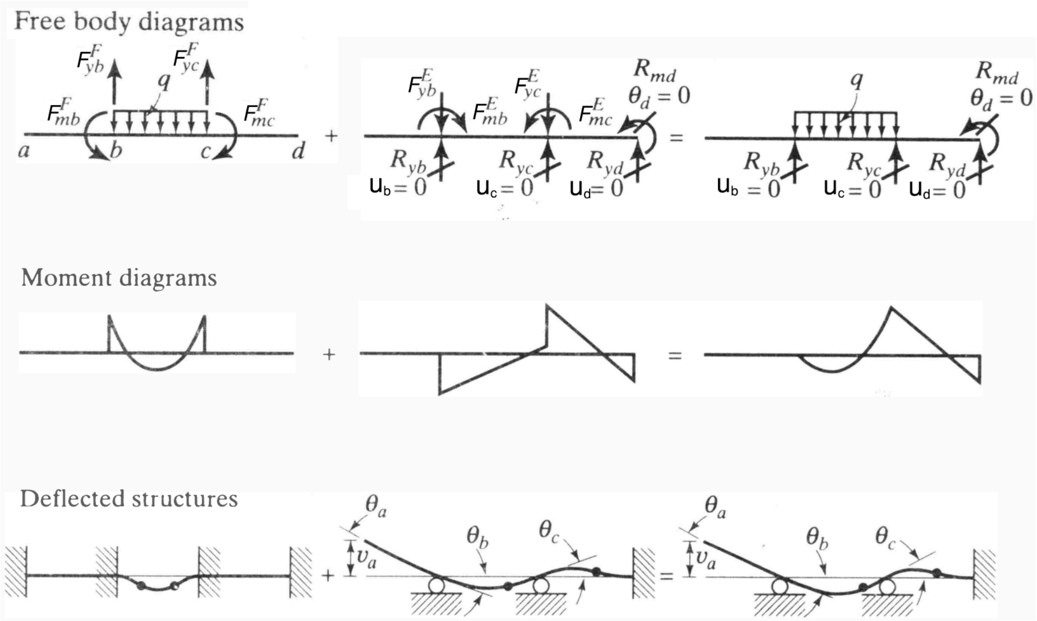
Superimpose:

- **Stage 1:** fixed-end solution
- **Stage 4:** deformation solution

The **fictitious forces cancel** in the sum

Final system satisfies:

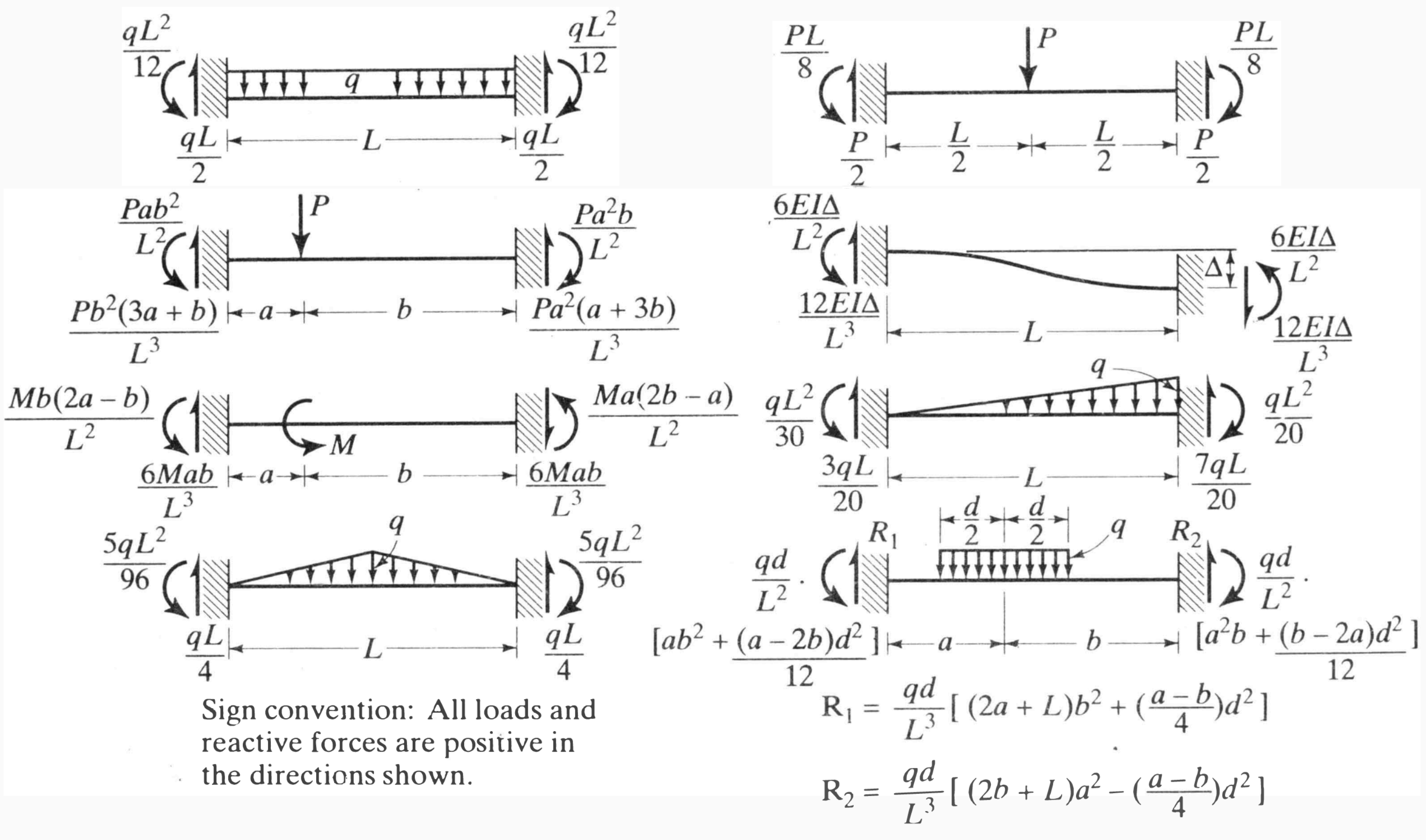
- **equilibrium**
- **compatibility**



List of Typical FEFs

the internal forces and displacements of the fixed-end part of the problem must be obtained by some means

You don't need to calculate each time. There are typical cases


$\frac{qL^2}{12}$
q
$\frac{qL^2}{12}$
$\frac{qL}{2}$
L
$\frac{qL}{2}$
$\frac{PL}{8}$
P
$\frac{PL}{8}$
$\frac{P}{2}$
$\frac{L}{2}$
$\frac{L}{2}$
$\frac{P}{2}$
$\frac{Pab^2}{L^2}$
P
$\frac{Pa^2b}{L^2}$
$\frac{Pb^2(3a+b)}{L^3}$
a
b
$\frac{Pa^2(a+3b)}{L^3}$
$\frac{6EI\Delta}{L^2}$
Δ
$\frac{6EI\Delta}{L^2}$
$\frac{12EI\Delta}{L^3}$
L
$\frac{12EI\Delta}{L^3}$
$\frac{Mb(2a-b)}{L^2}$
M
$\frac{Ma(2b-a)}{L^2}$
$\frac{6Mab}{L^3}$
a
b
$\frac{6Mab}{L^3}$
q
$\frac{qL^2}{30}$
$\frac{qL^2}{20}$
$\frac{3qL}{20}$
L
$\frac{7qL}{20}$
$\frac{5qL^2}{96}$
q
$\frac{5qL^2}{96}$
$\frac{qL}{4}$
L
$\frac{qL}{4}$
$\frac{d}{2}$
$\frac{d}{2}$
R_1
q
R_2
$\frac{qd}{L^2}\cdot\left[ab^2 + \frac{(a-2b)d^2}{12}\right]$
a
b
$\frac{qd}{L^2}\cdot\left[a^2b + \frac{(b-2a)d^2}{12}\right]$
Sign convention: All loads and reactive forces are positive in the directions shown.
$R_1 = \frac{qd}{L^3}\left[(2a+L)b^2 + \left(\frac{a-b}{4}\right)d^2\right]$
$R_2 = \frac{qd}{L^3}\left[(2b+L)a^2 - \left(\frac{a-b}{4}\right)d^2\right]$

Part 4 — DSM Setup Incorporating FEFs

Local Element System

As we already mentioned:

$$\boldsymbol{Q} = \boldsymbol{k}\boldsymbol{u} + \boldsymbol{Q}_f$$

- $\boldsymbol{k}\boldsymbol{u}$ → local response due to deformation
- $\boldsymbol{Q}_f$ → local fixed end forces due to member loading

Total response = deformation + load effects

Global Structure

The FEFs formulation allows standard DSM procedure, but with an additional FEF vector

- FEFs act as **equivalent nodal forces**
- Added to global force vector during assembly

switching to this notation in Global analysis (consistent with Truss Notation from Lecture 4), to not confuse with partitioned notation

$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{u} + \boldsymbol{F}^F$$

- $\boldsymbol{K}\boldsymbol{u}$ → global response due to deformation, assembled stiffness matrix
- $\boldsymbol{F}^F$ → global fixed end forces due to member loading

Partitioned Matrix Form

Bring F^F to the other side of the equation

$$\left[\begin{array}{c:c} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fr} \\ \hdashline \boldsymbol{K}_{rf} & \boldsymbol{K}_{rr} \end{array}\right] \left\{\begin{array}{c} \boldsymbol{u}_f \\ \hdashline \boldsymbol{u}_r \end{array}\right\} = \left\{\begin{array}{c} \boldsymbol{F}_f \\ \hdashline \boldsymbol{F}_r \end{array}\right\} - \left\{\begin{array}{c} \boldsymbol{F}_f^F \\ \hdashline \boldsymbol{F}_r^F \end{array}\right\}$$

Solving for $\boldsymbol{u}_f$ with FEFs

We are interested in solving for the **unknown displacements** at the free DOFs, $\boldsymbol{u}_f$.

Starting from:

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f + \boldsymbol{K}_{fr}\boldsymbol{u}_r = \boldsymbol{F}_f - \boldsymbol{F}_f^F$$

Rearrange to isolate the unknowns:

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f = \boldsymbol{F}_f - \boldsymbol{F}_f^F - \boldsymbol{K}_{fr}\boldsymbol{u}_r$$

Provided that $\boldsymbol{K}_{ff}$ is invertible, the solution is:

$$\boxed{\boldsymbol{u}_f = \boldsymbol{K}_{ff}^{-1}\left(\boldsymbol{F}_f - \boldsymbol{F}_f^F - \boldsymbol{K}_{fr}\boldsymbol{u}_r\right)}$$

Solving for $\boldsymbol{F_r}$ with FEFs

Once the free displacements $\boldsymbol{u}_f$ have been computed, we can determine the **forces at the restrained DOFs** (support reactions).

Starting from:

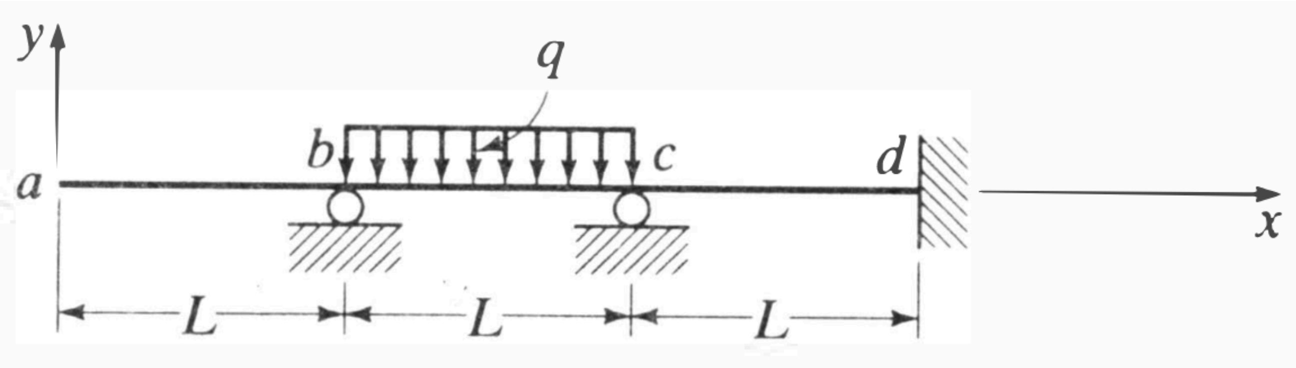
$$\boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r = \boldsymbol{F}_r - \boldsymbol{F}_r^F$$

At the restrained DOFs, the displacements $\boldsymbol{u}_r$ are **known** from the boundary conditions (often $\boldsymbol{u}_r = \boldsymbol{0}$). Substituting these known values gives a direct expression for the reaction forces:

$$\boxed{\boldsymbol{F}_r = \boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r + \boldsymbol{F}_r^F}$$

Example Structure

Same beam as in Lecture 6.2, only loaded with distributed load



FEFs for Middle Span

$$F_{yb}^{F} = \frac{qL}{2}, \quad F_{mb}^{F} = \frac{qL^2}{12}$$

$$F_{yc}^{F} = \frac{qL}{2}, \quad F_{mc}^{F} = -\frac{qL^2}{12}$$

DSM Setup for Example Structure

$$
\left[\begin{array}{c:c}
\boldsymbol{K}_{ff} & \boldsymbol{K}_{fr} \\
\hdashline
\boldsymbol{K}_{rf} & \boldsymbol{K}_{rr}
\end{array}\right]
\left\{\begin{array}{c}
u_a \\ \theta_a \\ \theta_b \\ \theta_c \\ \hdashline 0 \\ 0 \\ 0 \\ 0
\end{array}\right\}
=
\left\{\begin{array}{c}
0 \\ 0 \\ 0 \\ 0 \\ \hdashline R_{yb} \\ R_{yc} \\ R_{yd} \\ R_{md}
\end{array}\right\}
-
\left\{\begin{array}{c}
0 \\ 0 \\ \frac{qL^2}{12} \\ -\frac{qL^2}{12} \\ \hdashline \frac{qL}{2} \\ \frac{qL}{2} \\ 0 \\ 0
\end{array}\right\}
$$

Part 5 — DSM Full Procedure

Forward Pass — Structural Analysis

1. **Defining Structure**
    - Node numbering and coordinates
    - Global DOF numbering
    - Element connectivity
    - Restraints and Applied Forces
2. **Assemble global stiffness matrix and force vector**
    - Scatter-add element contributions into $\boldsymbol{K}$
    - This assumes beam element is aligned with global axis (no transformation)
3. **Generate applied force and FEF vectors**
    - Scatter-add force and moment contributions into $\boldsymbol{F}$
    - Fix all beam segments
    - identify what case, calculate closed form FEFs
    - Scatter-add FEFs into $\boldsymbol{F}^{\boldsymbol{F}}$

Forward Pass — Structural Analysis, cont...

5. **Apply boundary conditions**
   - Partition DOFs into free (f) and restrained (r)
6. **Solve for unknown displacements**

$$\boldsymbol{u}_f = \boldsymbol{K}_{ff}^{-1}\left(\boldsymbol{F}_f - \boldsymbol{F}_f^F - \boldsymbol{K}_{fr}\boldsymbol{u}_r\right)$$

7. **Recover support reactions**

$$\boldsymbol{F}_r = \boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r + \boldsymbol{F}_r^F$$

Backward Pass — Element Recovery and Design

7. **Extract element global displacement vectors**
    - For each member, collect the relevant entries from $\boldsymbol{u}$ to form $\boldsymbol{u}'$
    - This assumes beam element is aligned with global axis (no transformation)

8. **Extract element FEF vectors**
    - For each member, collect the relevant entries from $\boldsymbol{F}^{\boldsymbol{F}}$ to form $\boldsymbol{Q}_f$
    - This assumes beam element is aligned with global axis (no transformation)

9. **Compute local element end forces and moments**

$$\boldsymbol{f}' = \boldsymbol{k}'\,\boldsymbol{u}' + \boldsymbol{Q}_f$$

10. **Compute axial stress and bending stress (design quantities)**

$$\sigma_{\text{axial}} = \frac{N}{A}$$

$$\sigma_{\text{bending}} = \pm\frac{My}{I}$$

Wrap-Up

Next Lecture