CEE6501 — Lecture 6.2

Beams (Stiffness Matrix Definition)

Learning Objectives

By the end of this lecture, you will be able to:

- Text
- Text

Agenda

1. Text
2. Text

Part 1 — Beam Element Stiffness Relations

Beam Element Response

The **member stiffness relations** express the forces at the ends of a beam element (including shear forces and bending moments) as functions of the **end displacements** (including transverse displacements and rotations).

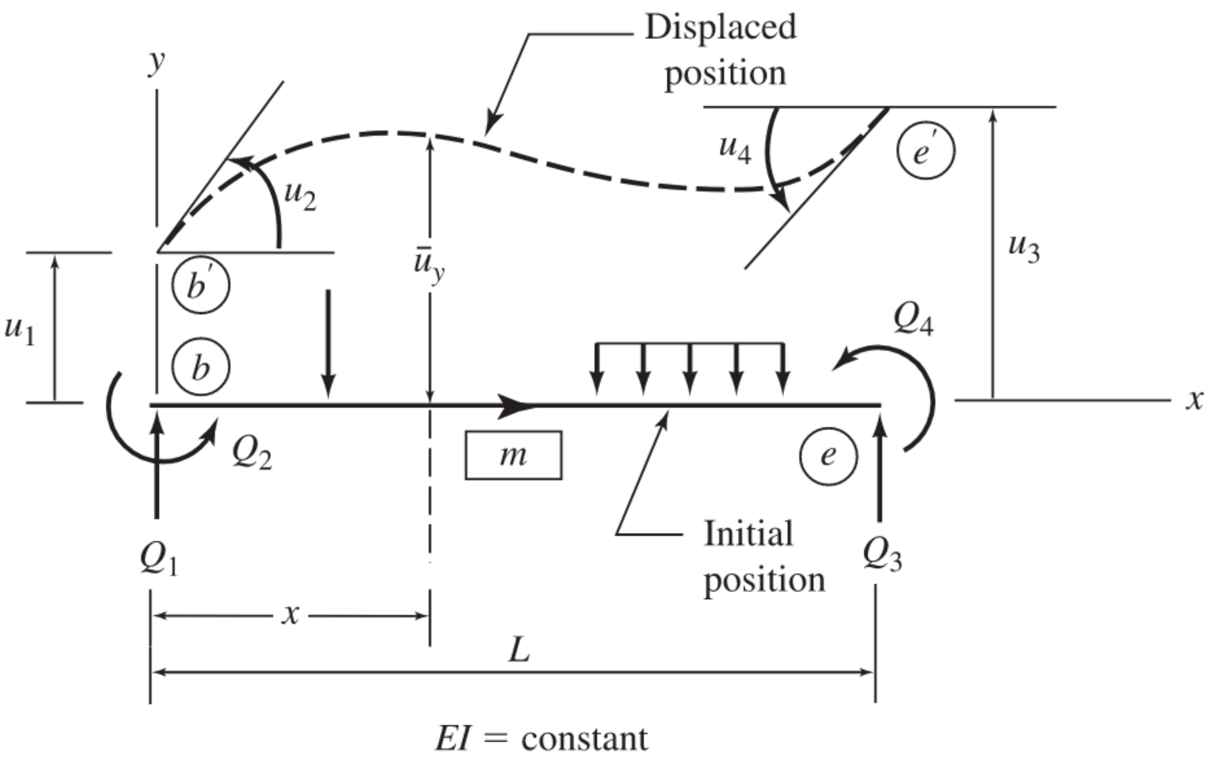
When a beam element is subjected to external loading:

- The member **deforms** (bending)
- **Internal shear forces** develop at the ends
- **Bending moments** are induced at the ends

These internal forces are fully determined by the **displacements and rotations at the element ends**.

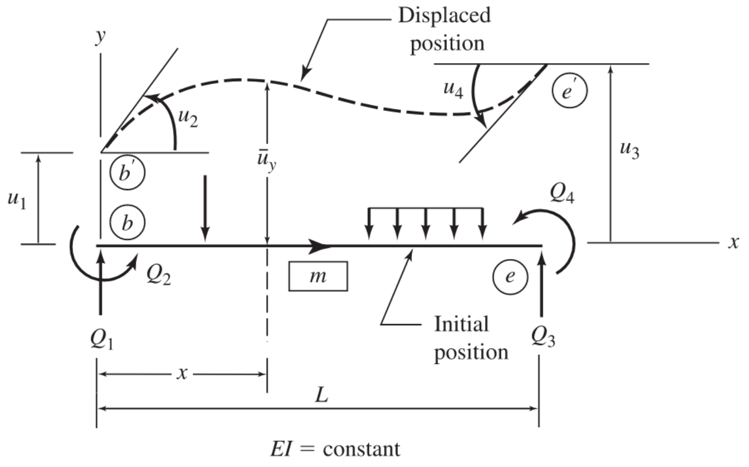
Generic Displacement for a 2D Beam Element

We define all quantities in the **local coordinate system**, with origin at the left end b, and ending at node e.



2D Beam Element DOF Numbering

Degrees of freedom are ordered **left → right**, with **translation first**, then **rotation**.



- **DOF 1:** u_1 — node 1, local y
- **DOF 2:** u_2 — node 1, θ
- **DOF 3:** u_3 — node 2, local y
- **DOF 4:** u_4 — node 2, θ

Sign conventions:

- $u > 0$ → upward (local y direction)
- $\theta > 0$ → counterclockwise
- Forces follow the same order: $[V_b,\ M_b,\ V_e,\ M_e]^T$

Local displacement and force vectors

Local displacement vector (including rotations):

$$\boldsymbol{u} = \begin{Bmatrix} u_b \\ \theta_b \\ u_e \\ \theta_e \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Local nodal force vector (including moments):

$$\boldsymbol{Q} = \begin{Bmatrix} V_b \\ M_b \\ V_e \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

We seek:

$$\boldsymbol{Q} = \boldsymbol{k}\,\boldsymbol{u}$$

Four equations (one per DOF)

recall, this form is similar to the definition for a truss element.

$$Q_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4$$

$$Q_2 = k_{21}u_1 + k_{22}u_2 + k_{23}u_3 + k_{24}u_4$$

$$Q_3 = k_{31}u_1 + k_{32}u_2 + k_{33}u_3 + k_{34}u_4$$

$$Q_4 = k_{41}u_1 + k_{42}u_1 + k_{43}u_3 + k_{44}u_4$$

Each equation expresses **force equilibrium at a single local degree of freedom**.

For a linear elastic element, the force at any DOF is a **linear combination of all DOF displacements**:

- displacing one DOF can induce forces at *all* DOFs
- the proportionality constants are the stiffness coefficients k_{ij}

Same equations in matrix form

This form is similar to a truss element.

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Part 2 — The Beam Element Stiffness Matrix

Unit Displacement Method Derivation

Recall:

$$k_{ij} = \text{force at DOF } i \text{ due to a unit displacement at DOF } j,$$
$$\text{with all other DOFs held fixed.}$$

Each column of k is built by:

- Impose a **unit displacement** at one DOF
- Hold all the other DOF fixed
- Record the resulting nodal force pattern

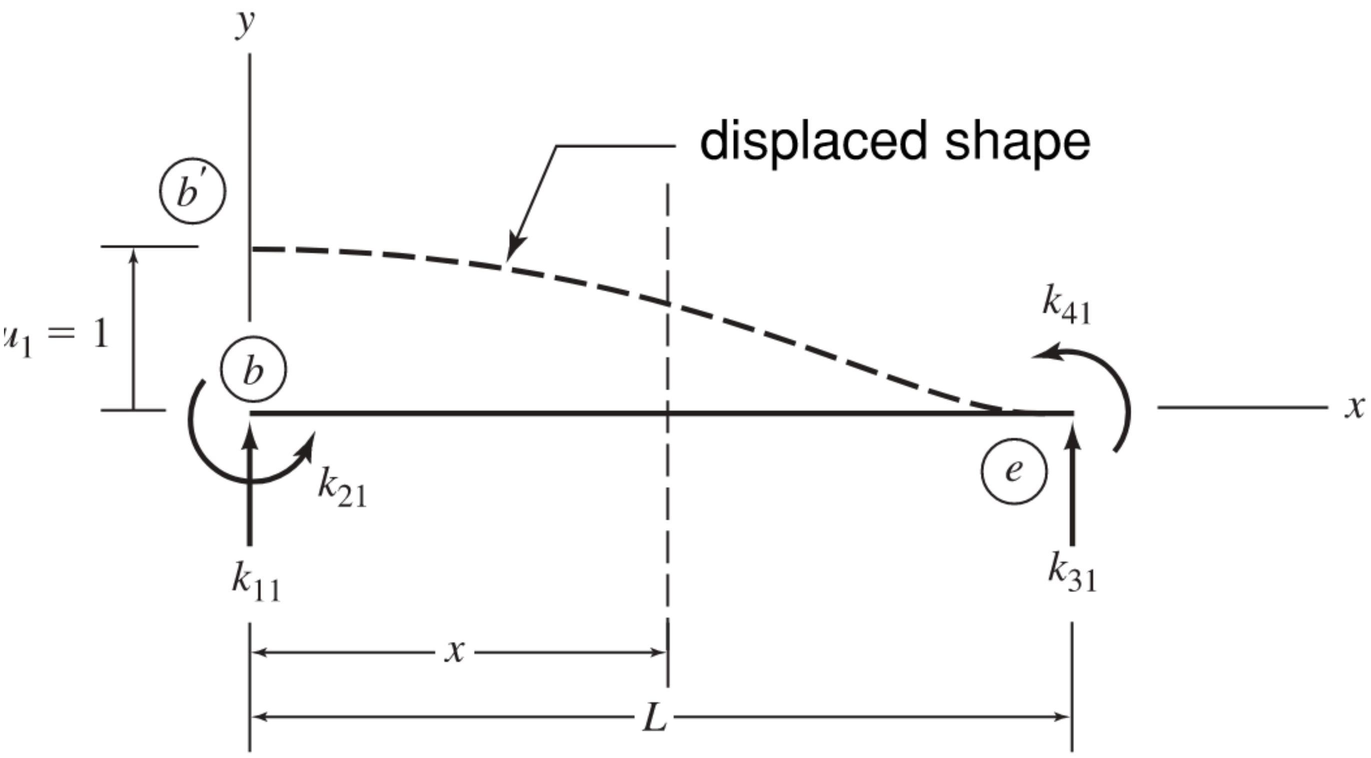
We will do this for DOFs 1–4.

Column 1: impose $u_1 = 1$ $(u_2 = u_3 = u_4 = 0)$

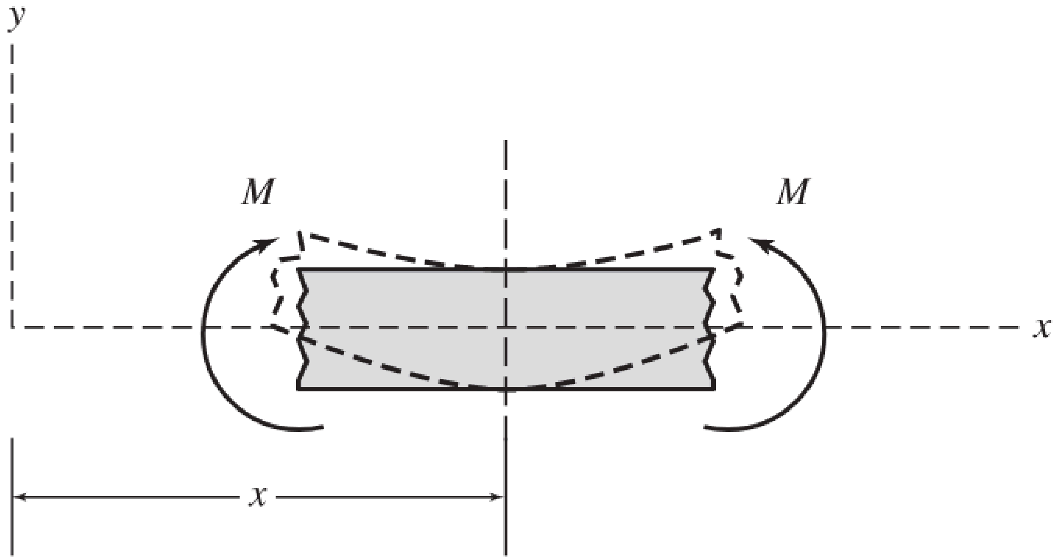
$$\begin{bmatrix} \boxed{k_{11}} & k_{12} & k_{13} & k_{14} \\ \boxed{k_{21}} & k_{22} & k_{23} & k_{24} \\ \boxed{k_{31}} & k_{32} & k_{33} & k_{34} \\ \boxed{k_{41}} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

- k_{11} $(i = 1,\ j = 1)$
  force at **DOF 1** due to unit displacement at **DOF 1**
- k_{21} $(i = 2,\ j = 1)$
  force at **DOF 2** due to unit displacement at **DOF 1**
- k_{31} $(i = 3,\ j = 1)$
  force at **DOF 3** due to unit displacement at **DOF 1**
- k_{41} $(i = 4,\ j = 1)$
  force at **DOF 4** due to unit displacement at **DOF 1**

Column 1: impose $u_1 = 1$ $(u_2 = u_3 = u_4 = 0)$



Sign Convention for Derivation



Positive bending moment

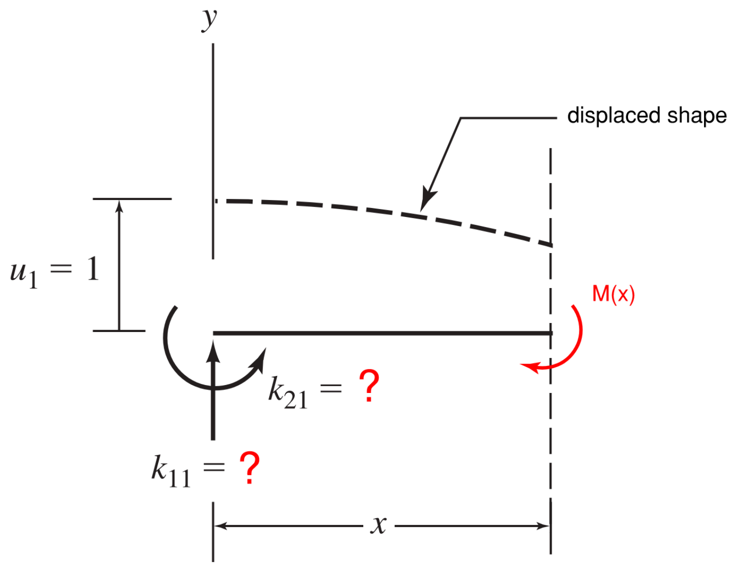
Governing Equation (Beam Bending)

From mechanics of materials, the beam bending equation is:

$$\frac{d^2u}{dx^2} = \frac{M(x)}{EI} \tag{1}$$

- $u(x)$ = transverse displacement
- $M(x)$ = bending moment
- EI = flexural rigidity

Step 1 — Express Internal Moment as a Function of x



Cut the beam at a distance x from node b.

Using equilibrium:

$$M(x) = -k_{21} + k_{11}x \tag{2}$$

- k_{21} → moment at node b
- k_{11} → shear at node b

Step 2 — Substitute into Governing Equation

Substitute the expression for the **bending moment** from Eq. (2) into the moment term of Eq. (1):

$$\frac{d^2u}{dx^2} = \frac{M(x)}{EI} \quad \xrightarrow{\text{Eq. (2)}} \quad \frac{d^2u}{dx^2} = \frac{1}{EI}(-k_{21} + k_{11}x) \tag{3}$$

Step 3 — Integrate to Obtain Rotation $\theta(x)$

Start from Eq. (3):

$$\frac{d^2u}{dx^2} = \frac{1}{EI}(-k_{21} + k_{11}x) \tag{3}$$

Integrate **both sides** with respect to x:

$$\int \frac{d^2u}{dx^2}\,dx = \int \frac{1}{EI}(-k_{21} + k_{11}x)\ dx \tag{4a}$$

Left-hand side:

$$\int \frac{d^2u}{dx^2}\,dx = \frac{du}{dx} = \theta(x) \tag{4b}$$

Right-hand side (integrate term-by-term, and pull out constants $1/(EI)$):

$$\int \frac{1}{EI}(-k_{21} + k_{11}x)\ dx = \frac{1}{EI}\left(\int -k_{21}\, dx + \int k_{11}x\, dx\right) \tag{4c}$$

Compute each integral:

$$\int -k_{21}\, dx = -k_{21}x \qquad \int k_{11}x\, dx = k_{11}\frac{x^2}{2} \tag{4d}$$

Include the integration constant:

$$\theta(x) = \frac{du}{dx} = \frac{1}{EI}\left(-k_{21}x + \frac{k_{11}}{2}x^2\right) + C_1 \tag{4}$$

Step 4 — Integrate to Obtain Displacement $u(x)$

Start from Eq. (4):

$$\frac{du}{dx} = \frac{1}{EI}\left(-k_{21}x + \frac{k_{11}}{2}x^2\right) + C_1 \tag{4}$$

Integrate **both sides** with respect to x:

$$\int \frac{du}{dx}\,dx = \int \left[\frac{1}{EI}\left(-k_{21}x + \frac{k_{11}}{2}x^2\right) + C_1\right] dx \tag{5a}$$

Left-hand side:

$$\int \frac{du}{dx}\,dx = u(x) \tag{5b}$$

Right-hand side (integrate term-by-term):

$$u(x) = \frac{1}{EI}\left(\int -k_{21}x\,dx + \int \frac{k_{11}}{2}x^2\,dx\right) + \int C_1\,dx \tag{5c}$$

Compute each integral:

$$\int -k_{21}x\,dx = -\frac{k_{21}}{2}x^2 \qquad \int \frac{k_{11}}{2}x^2\,dx = \frac{k_{11}}{6}x^3 \qquad \int C_1\,dx = C_1 x \tag{5d}$$

Include the second integration constant:

$$u(x) = \frac{1}{EI}\left(-\frac{k_{21}}{2}x^2 + \frac{k_{11}}{6}x^3\right) + C_1 x + C_2 \tag{5}$$

Step 5 — Apply Boundary Conditions

We enforce **unit displacement at node** b (0 elsewhere):

At $x = 0$:

$$\theta(0) = 0$$
$$u(0) = 1$$

At $x = L$:

$$\theta(L) = 0$$
$$u(L) = 0$$

Step 6 — Solve for Constants

Substitute $x = 0$ into Eqs. (4)–(5):

- From $\theta(0) = 0$: all terms with x vanish $\Rightarrow C_1 = 0$
- From $u(0) = 1$: all terms with x vanish $\Rightarrow C_2 = 1$

Updated Eqs. (4)–(5):

$$\theta(x) = \frac{1}{EI}\left(-k_{21}x + \frac{k_{11}}{2}x^2\right) \tag{6}$$

$$u(x) = \frac{1}{EI}\left(-\frac{k_{21}}{2}x^2 + \frac{k_{11}}{6}x^3\right) + 1 \tag{7}$$

Step 7 — Apply Boundary Conditions at $x = L$

Substitute $\theta(L) = 0$ into Eq. (6):

$$0 = \frac{1}{EI}\left(-k_{21}L + \frac{k_{11}}{2}L^2\right)$$

Multiply both sides by EI (nonzero), so it cancels:

$$0 = -k_{21}L + \frac{k_{11}}{2}L^2$$

Rearrange to solve for k_{21}:

$$k_{21}L = \frac{k_{11}}{2}L^2$$

Divide by L:

$$k_{21} = \frac{k_{11}}{2}L \tag{9}$$

Step 8 — Apply Final Condition

Substitute $u(L) = 0$ into Eq. (7):

$$0 = \frac{1}{EI}\left(-\frac{k_{21}}{2}L^2 + \frac{k_{11}}{6}L^3\right) + 1$$

Rearrange:

$$-EI = -\frac{k_{21}}{2}L^2 + \frac{k_{11}}{6}L^3$$

Now substitute Eq. (9), $k_{21} = \dfrac{k_{11}L}{2}$:

$$-EI = -\frac{1}{2}\left(\frac{k_{11}L}{2}\right)L^2 + \frac{k_{11}}{6}L^3$$

Solve for k_{11}:

$$\boxed{k_{11} = \frac{12EI}{L^3}}$$

Step 9 — Final Coefficient

Back-substitute k_{11} into Eq. (9):

$$\boxed{k_{21} = \frac{6EI}{L^2}}$$

Step 10 — Solve Remaining Stiffness Terms k_{31} and k_{41}

Use equilibrium of the **beam element free-body diagram** (with the previously found terms $k_{11} = \dfrac{12EI}{L^3}$ and $k_{21} = \dfrac{6EI}{L^2}$).

(1) Vertical force equilibrium

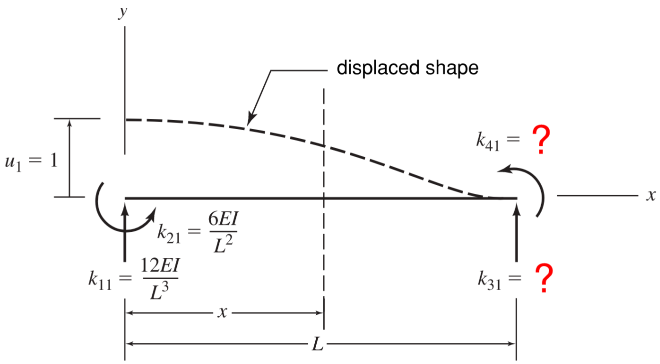
$$\sum F_y = 0: \qquad \frac{12EI}{L^3} + k_{31} = 0$$

$$\boxed{k_{31} = -\frac{12EI}{L^3}}$$

(2) Moment equilibrium about end b

$$\sum M_b = 0: \qquad \frac{6EI}{L^2} - \left(\frac{12EI}{L^3}\right) L + k_{41} = 0$$

$$\boxed{k_{41} = \frac{6EI}{L^2}}$$



Free-body diagram used to enforce $\sum F_y = 0$ and $\sum M_b = 0$.

Step 11 (Optional) — Displacement Shape Function

Substitute the solved coefficients k_{11} and k_{21} into Eq. (7):

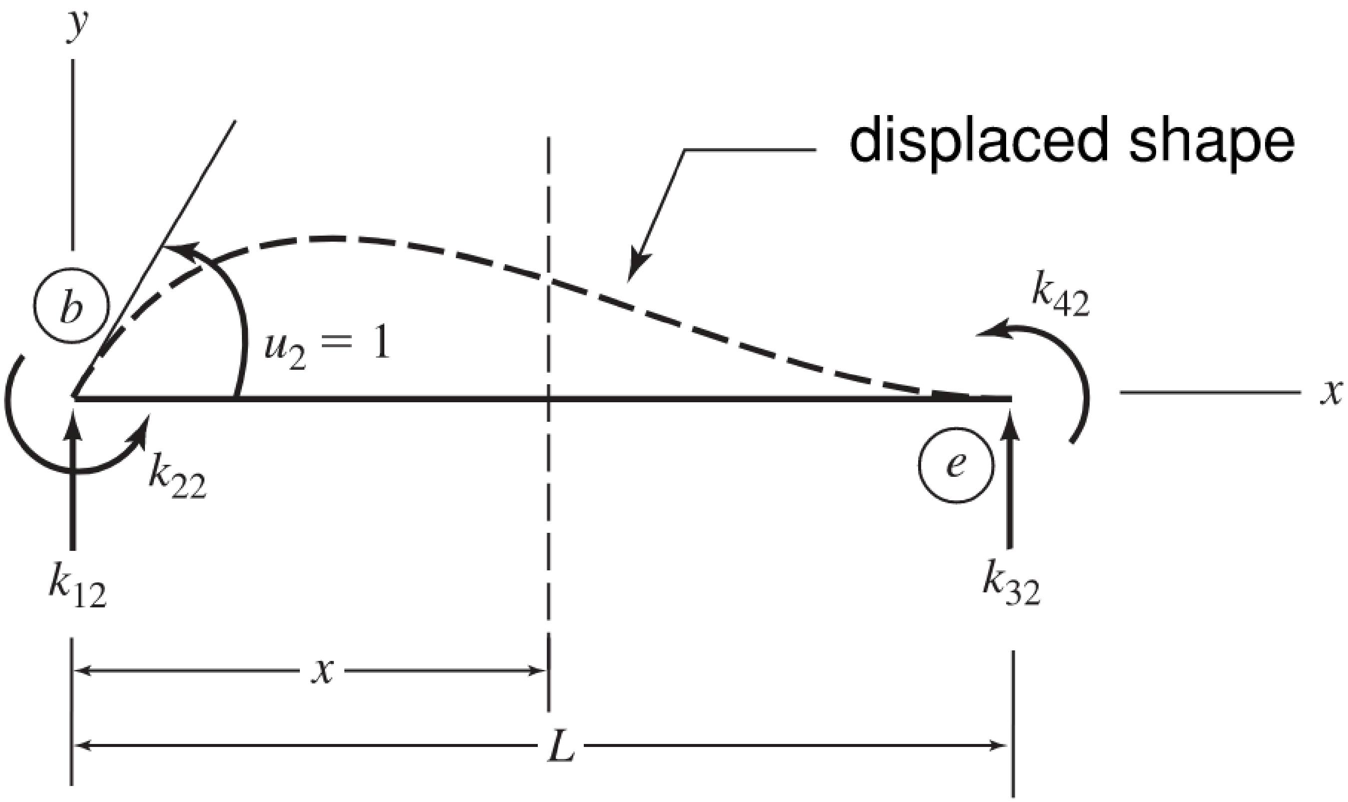
$$u(x) = \frac{1}{EI}\left(-\frac{k_{21}}{2}x^2 + \frac{k_{11}}{6}x^3\right) + 1 \quad \xrightarrow{k_{11},\, k_{21}} \quad u(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

1st Column of 4x4 Stiffness Matrix

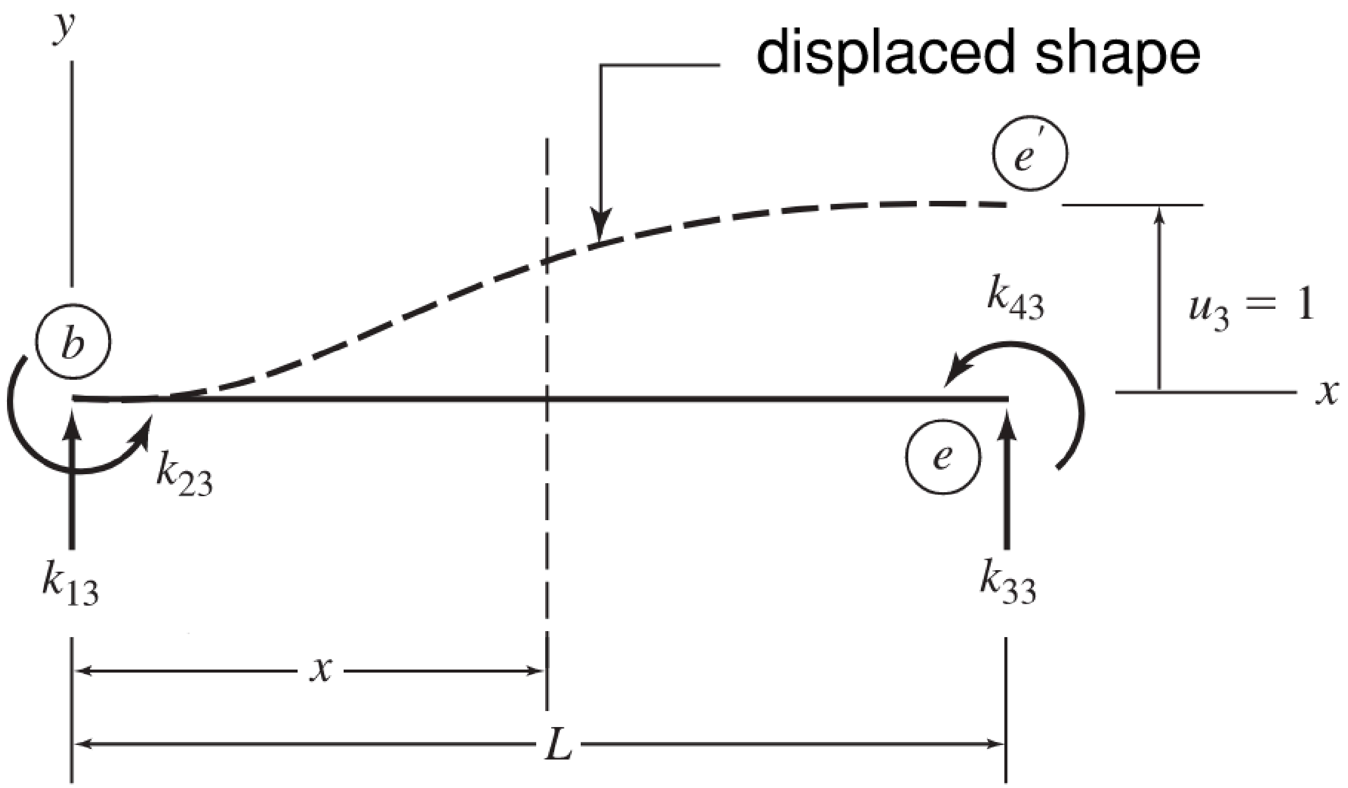
$$k = \begin{bmatrix} \boxed{\dfrac{12EI}{L^3}} & k_{12} & k_{13} & k_{14} \\ \boxed{\dfrac{6EI}{L^2}} & k_{22} & k_{23} & k_{24} \\ \boxed{-\dfrac{12EI}{L^3}} & k_{32} & k_{33} & k_{34} \\ \boxed{\dfrac{6EI}{L^2}} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

Part 3 — Complete 4×4 Beam Stiffness Matrix

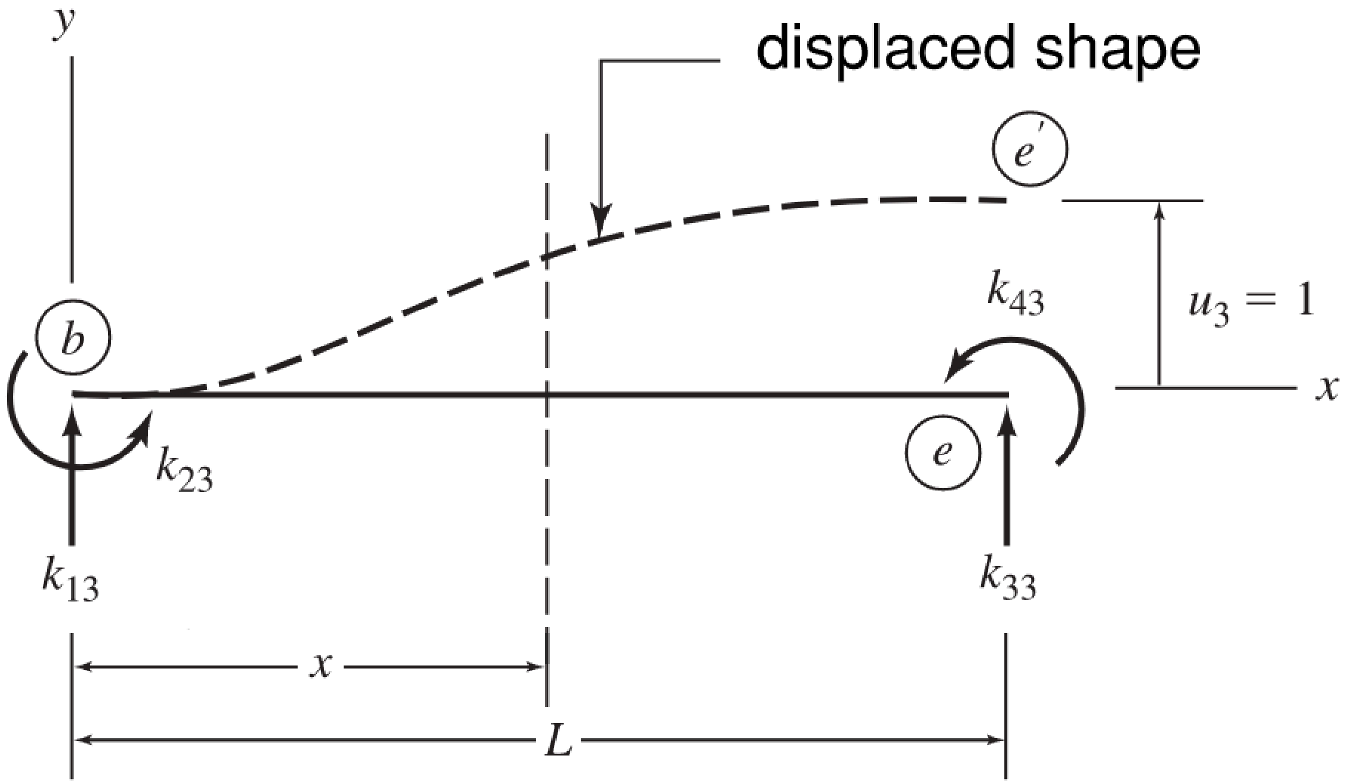
Column 2: impose $u_2 = 1$ ($u_1 = u_3 = u_4 = 0$)



Column 3: impose $u_3 = 1$ ($u_1 = u_2 = u_4 = 0$)



Column 4: impose $u_4 = 1$ ($u_1 = u_2 = u_3 = 0$)



Wrap-Up

Today you started

Next Lecture: continue