

CEE6501 — Lecture 6.3

Fixed End Forces (FEFs)

Learning Objectives

By the end of this lecture, you will be able to:

- Distinguish between joint loads and member loads in beam systems
- Define fixed-end forces (FEFs) and interpret their physical meaning
- Explain why member loading requires modification of $\mathbf{Q} = \mathbf{k}\mathbf{u}$
- Conceptually decompose indeterminate beam problems using FEFs and superposition
- Convert member loads into equivalent nodal forces for DSM analysis
- Incorporate FEFs into the global system of equations and solve for displacements and reactions
- Apply the full DSM workflow including FEFs for beam analysis

Agenda

- Part 1 - What are fixed-end forces (FEFs)
- Part 2 - Why member loading requires FEFs
- Part 3 - Indeterminate beam solution using FEFs and superposition
- Part 4 - Incorporating FEFs into the DSM formulation
- Part 5 - Full DSM workflow with FEFs

Part 1 — What are Fixed End Forces?

Forces and moments that develop at the ends of a member due to external loading, assuming both ends are **fully fixed** (no translation or rotation).

Joint Loads vs Member Loads

- **Joint loads** → applied directly at nodes
- **Member loads** → applied between nodes
 - distributed loads
 - point loads along a span
 - applied moments

In deriving the beam stiffness matrix, we assumed:

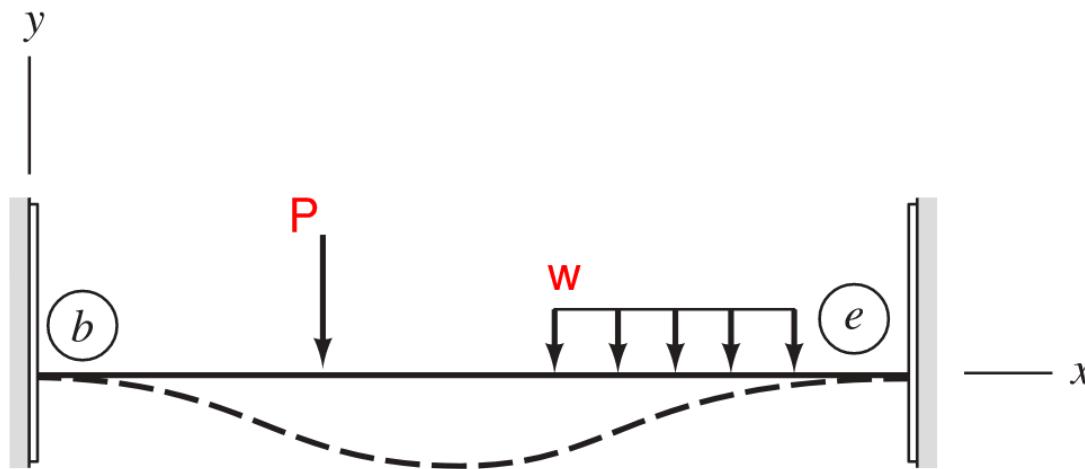
- loads act only at nodes
- no loading within the element

Now we relax this assumption.

Part 2 — Why Do We Need FEFs?

A Simple Case

Consider a **fully fixed beam** subjected to loading along its span.



All DOFs are restrained:

$$\mathbf{u} = \mathbf{0}$$

From typical local element force-displacement relationship:

$$\mathbf{Q} = \mathbf{k}\mathbf{u}$$

we obtain:

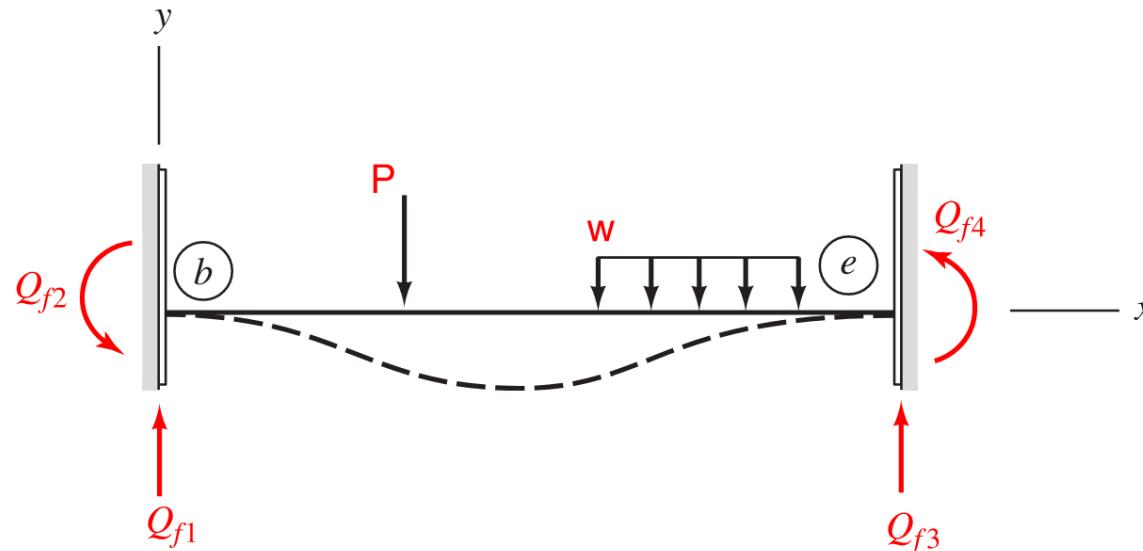
$$\mathbf{Q} = \mathbf{0}$$

✗ Clearly incorrect — reactions exist even with zero displacement.

Corrected Local Element Force-Displacement Relationship

$$\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$$

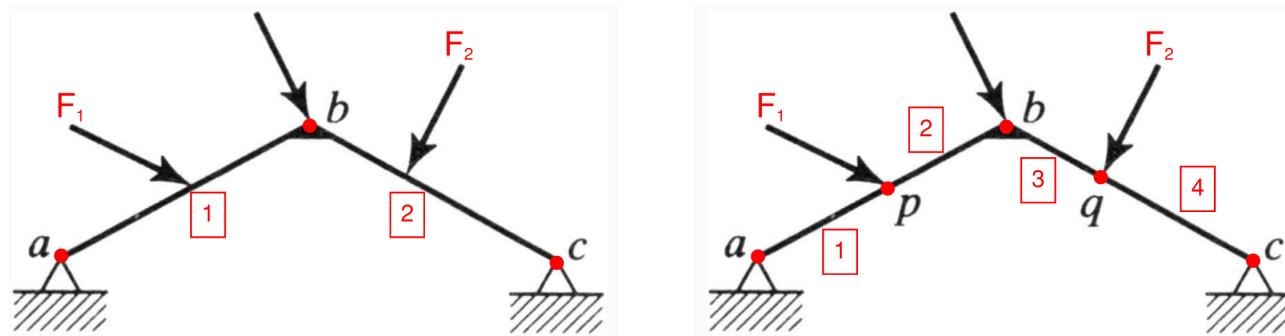
- \mathbf{Q}_f = **fixed-end force vector**
- Captures effects of **loads between nodes**



These forces are required whenever member loads are present

Avoiding FEFs with Artificial Nodes

One approach is to insert **artificial nodes** at load locations.



- Convert member loads → joint loads
- Enforce equilibrium at new nodes
- Maintain compatibility between segments

Artificial Nodes for Point Loads and Distributed Loads

This works for **point loads**, but:

- increases number of elements and DOFs in the solution
- not computationally efficient for general use

For **distributed loads**:

- Infinite number of points carry load
- Cannot represent exactly with finite nodes

Alternative: **lump distributed loads into point forces**

- preserves total force (equilibrium)
- but does not accurately represent local **internal shear and moment distributions**

Key Idea

We need a method that:

- accounts for **loads within elements**
- preserves **exact equilibrium**
- does not increase DOFs

Part 3 - Indeterminate Beam Structural Analysis

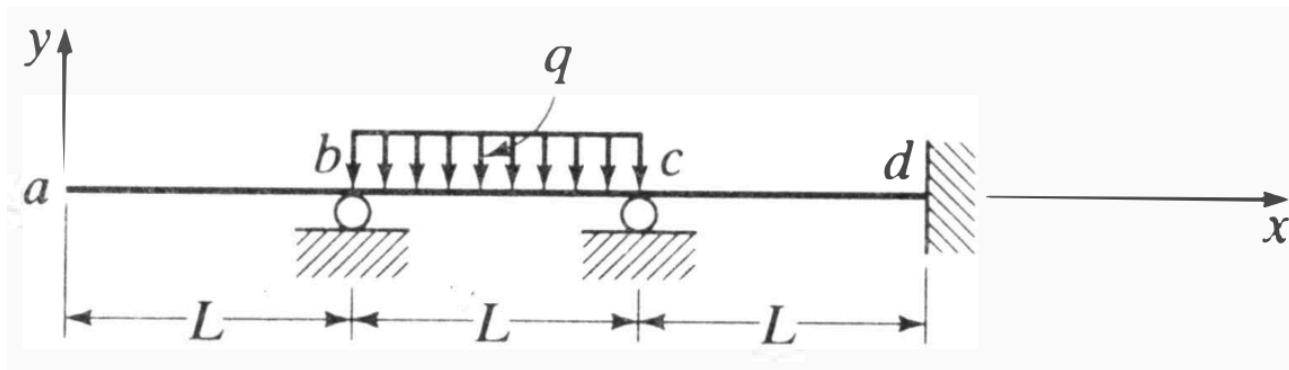
Solving Indeterminate Beams (Role of FEFs)

Recall from structural analysis:

- Indeterminate structures are solved by enforcing:
 - **equilibrium**
 - **compatibility**
- For beams with **member loads**, we conceptually:
 - split the problem into **simpler systems**
 - one with **fixed-end forces (FEFs)** (all DOFs restrained)
 - one with **equivalent nodal loads** (to restore deformation)
 - solve each system separately
 - **superimpose the results** to obtain the final response

Indeterminate Beam

Same beam as in Lecture 6.2, only loaded with distributed load



We analyze the beam by splitting it into **two systems**, then summing the results.

Stage 1 — Clamp the Beam (Zero DOFs)

isolate effect of **member loading only**

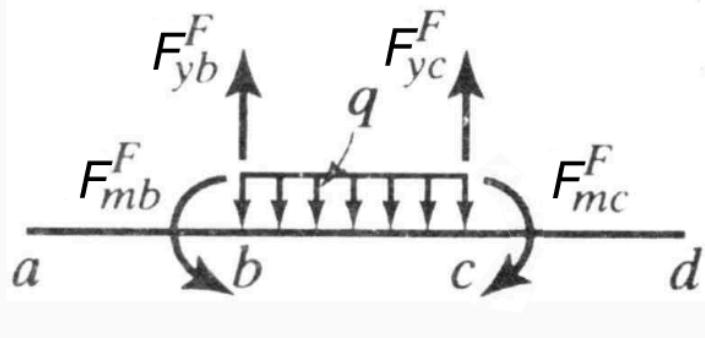
- Add **fictitious constraints** at element ends
- Beam becomes **fully fixed**



Stage 2 — Compute Fixed-End Forces

- Solve the **fully fixed beam** under loading
- Use known closed-form solutions

Free body diagrams



Example (uniform load):

$$F_{yb}^F = \frac{qL}{2}, \quad F_{mb}^F = \frac{qL^2}{12}$$

$$F_{yc}^F = \frac{qL}{2}, \quad F_{mc}^F = -\frac{qL^2}{12}$$

Moment diagrams



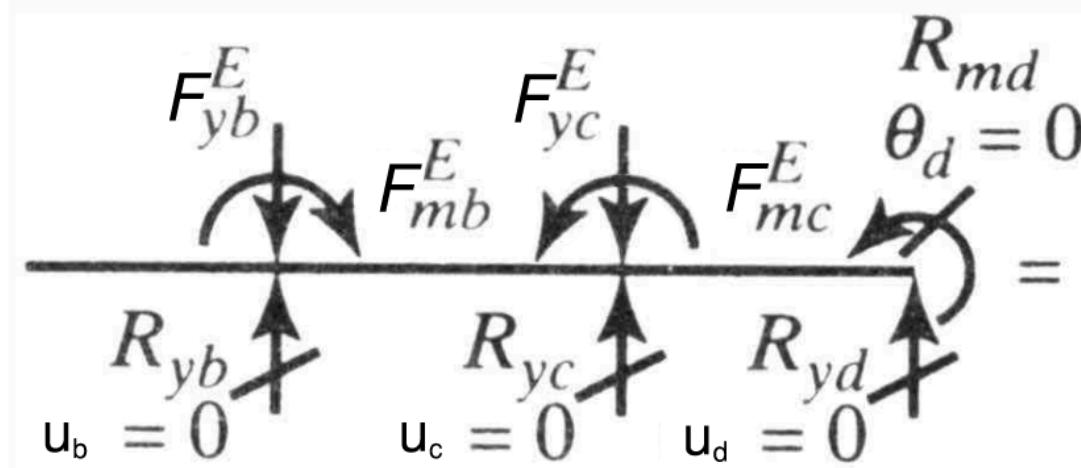
These are the **fixed-end forces**, \mathbf{F}^F .

Stage 3 — Remove Constraints (Equivalent Loads)

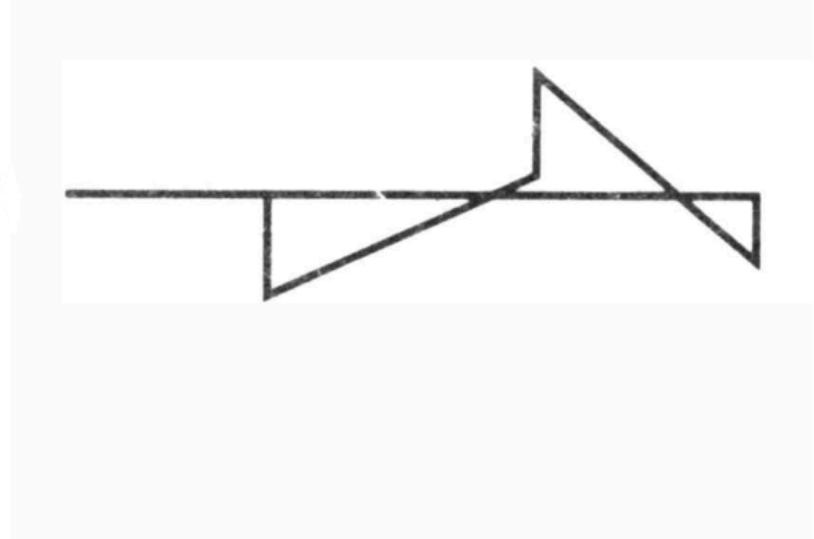
- Remove fictitious constraints
- Apply **equal and opposite forces** at nodes:

$$\mathbf{F}^E = -\mathbf{F}^F$$

- These are called **equivalent nodal loads**
- converts member loading \rightarrow nodal loading



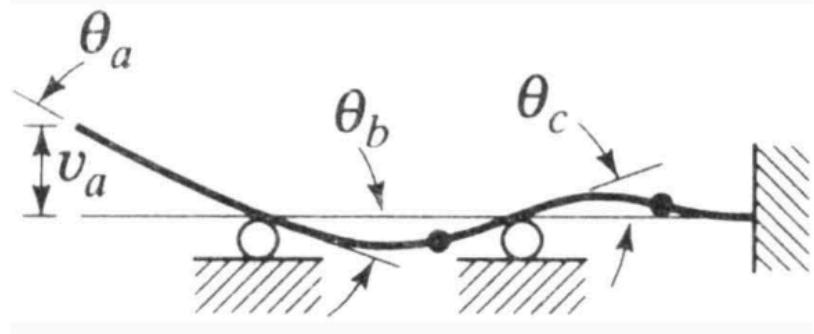
Stage 4 — Solve for Deformations Under Equivalent Nodal Loads



Now solve:

$$\mathbf{K}\mathbf{u} = \mathbf{F}^E$$

Using standard DSM solution



Stage 5 — Superimpose and Recover the Real System

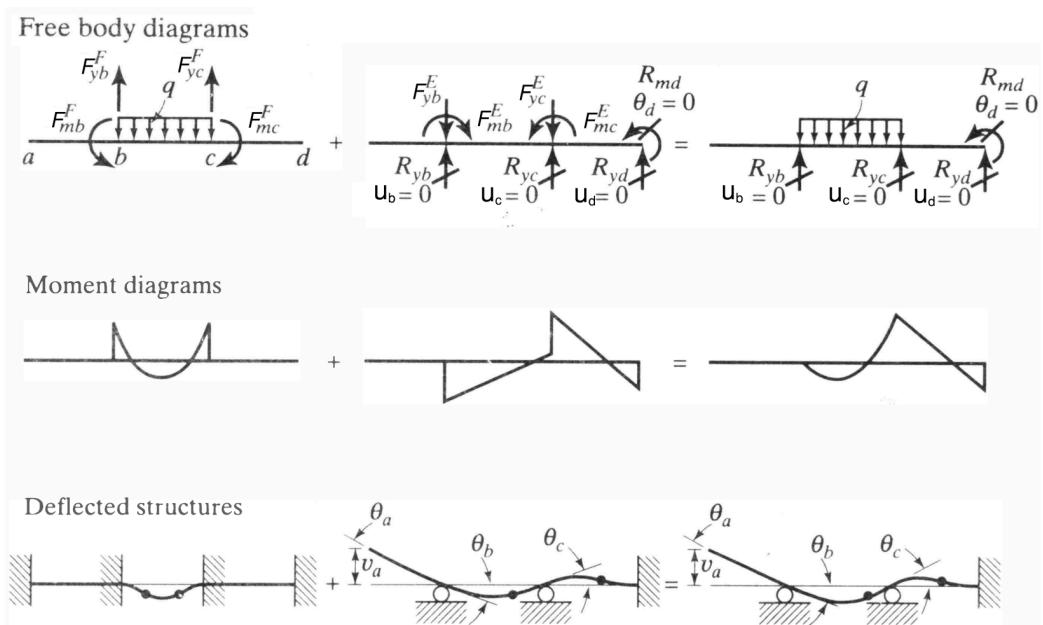
Superimpose:

- **Stage 1:** fixed-end solution
- **Stage 4:** deformation solution

The **fictitious forces cancel** in the sum

Final system satisfies:

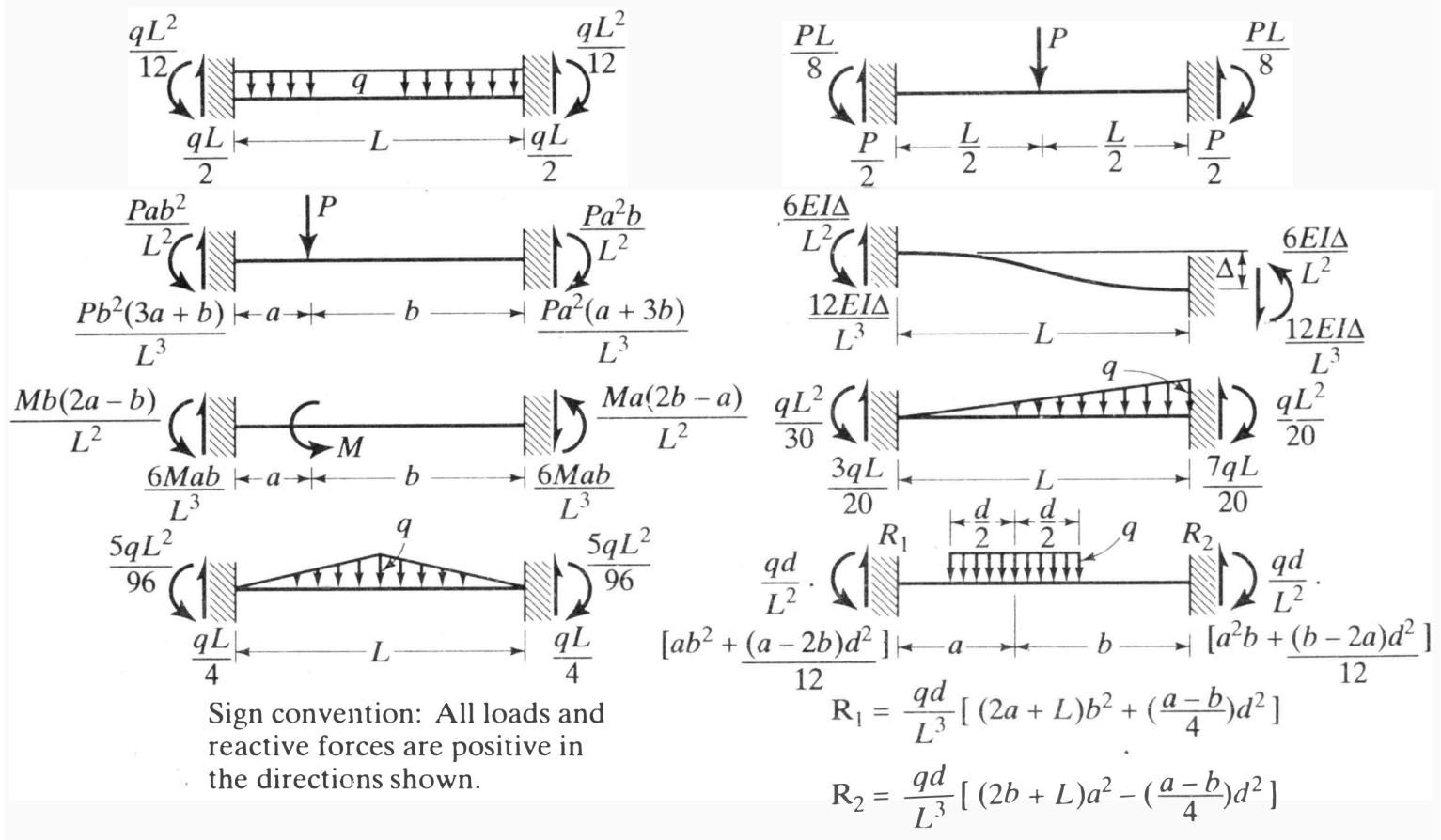
- **equilibrium**
- **compatibility**



List of Typical FEFs

The internal forces and displacements of the fixed-end part of the problem must be obtained by some means

Don't need to calculate these each time as there are closed form solutions for typical cases.



Part 4 — DSM Setup Incorporating FEFs

Local Element System

As we already mentioned:

$$\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$$

- $\mathbf{k}\mathbf{u}$ → local response due to deformation
- \mathbf{Q}_f → local fixed end forces due to member loading

Total response = deformation + load effects

Global Structure

The FEFs formulation allows standard DSM procedure, but with an additional FEF vector

- FEFs act as **equivalent nodal forces**
- Added to global force vector during assembly

switching to this notation in Global analysis (consistent with Truss Notation from Lecture 4), to not confuse with partitioned notation

$$\mathbf{F} = \mathbf{K}\mathbf{u} + \mathbf{F}^F$$

- $\mathbf{K}\mathbf{u}$ → global response due to deformation, assembled stiffness matrix
- \mathbf{F}^F → global fixed end forces due to member loading

Partitioned Matrix Form

Bring F^F to the other side of the equation

$$\left[\begin{array}{c|c} \mathbf{K}_{ff} & \mathbf{K}_{fr} \\ \hline \mathbf{K}_{rf} & \mathbf{K}_{rr} \end{array} \right] \left\{ \begin{array}{c} \mathbf{u}_f \\ \hline \mathbf{u}_r \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{F}_f \\ \hline \mathbf{F}_r \end{array} \right\} - \left\{ \begin{array}{c} \mathbf{F}_f^F \\ \hline \mathbf{F}_r^F \end{array} \right\}$$

Solving for \mathbf{u}_f with FEFs

We are interested in solving for the **unknown displacements** at the free DOFs, \mathbf{u}_f .

Starting from:

$$\mathbf{K}_{ff}\mathbf{u}_f + \mathbf{K}_{fr}\mathbf{u}_r = \mathbf{F}_f - \mathbf{F}_f^F$$

Rearrange to isolate the unknowns:

$$\mathbf{K}_{ff}\mathbf{u}_f = \mathbf{F}_f - \mathbf{F}_f^F - \mathbf{K}_{fr}\mathbf{u}_r$$

Provided that \mathbf{K}_{ff} is invertible, the solution is:

$$\boxed{\mathbf{u}_f = \mathbf{K}_{ff}^{-1}(\mathbf{F}_f - \mathbf{F}_f^F - \mathbf{K}_{fr}\mathbf{u}_r)}$$

Solving for \mathbf{F}_r with FEFs

Once the free displacements \mathbf{u}_f have been computed, we can determine the **forces at the restrained DOFs** (support reactions).

Starting from:

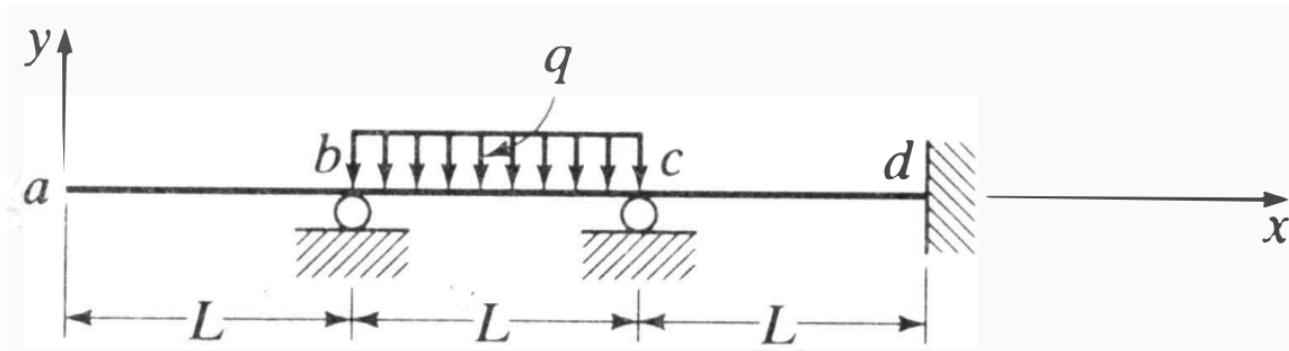
$$\mathbf{K}_{rf}\mathbf{u}_f + \mathbf{K}_{rr}\mathbf{u}_r = \mathbf{F}_r - \mathbf{F}_r^F$$

At the restrained DOFs, the displacements \mathbf{u}_r are **known** from the boundary conditions (often $\mathbf{u}_r = \mathbf{0}$). Substituting these known values gives a direct expression for the reaction forces:

$$\boxed{\mathbf{F}_r = \mathbf{K}_{rf}\mathbf{u}_f + \mathbf{K}_{rr}\mathbf{u}_r + \mathbf{F}_r^F}$$

Example Structure

Same beam as in Lecture 6.2, only loaded with distributed load



FEFs for Middle Span

$$\text{DOF 3: } F_{yb}^F = \frac{qL}{2}$$

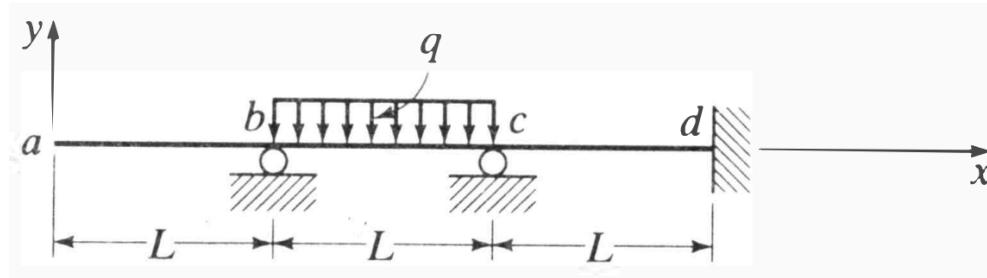
$$\text{DOF 4: } F_{mb}^F = \frac{qL^2}{12}$$

$$\text{DOF 5: } F_{yc}^F = \frac{qL}{2}$$

$$\text{DOF 6: } F_{mc}^F = -\frac{qL^2}{12}$$

DSM Setup for Example Structure (Partitioned)

$$\left[\begin{array}{c|c} \mathbf{K}_{ff} & \mathbf{K}_{fr} \\ \hline \mathbf{K}_{rf} & \mathbf{K}_{rr} \end{array} \right] \left\{ \begin{array}{l} u_a \\ \theta_a \\ \theta_b \\ \theta_c \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\} = \left\{ \begin{array}{l} 0 \\ 0 \\ 0 \\ 0 \\ R_{yb} \\ R_{yc} \\ R_{yd} \\ R_{md} \end{array} \right\} - \left\{ \begin{array}{l} 0 \\ 0 \\ \frac{qL^2}{12} \\ -\frac{qL^2}{12} \\ \frac{qL}{2} \\ \frac{qL}{2} \\ 0 \\ 0 \end{array} \right\}$$



Part 5 — DSM Full Procedure

Forward Pass — Structural Analysis

1. Defining Structure

- Node numbering and coordinates
- Global DOF numbering
- Element connectivity
- Restraints and Applied Forces

2. Assemble global stiffness matrix and force vector

- Scatter-add element contributions into \mathbf{K}
- This assumes beam element is aligned with global axis (no transformation)

3. Generate applied force and FEF vectors

- Scatter-add force and moment contributions into \mathbf{F}
- Fix all beam segments
- identify what case, calculate closed form FEFs
- Scatter-add FEFs into \mathbf{F}^F

Forward Pass — Structural Analysis, cont...

5. Apply boundary conditions

- Partition DOFs into free (f) and restrained (r)

6. Solve for unknown displacements

$$\mathbf{u}_f = \mathbf{K}_{ff}^{-1} (\mathbf{F}_f - \mathbf{F}_f^F - \mathbf{K}_{fr} \mathbf{u}_r)$$

7. Recover support reactions

$$\mathbf{F}_r = \mathbf{K}_{rf} \mathbf{u}_f + \mathbf{K}_{rr} \mathbf{u}_r + \mathbf{F}_r^F$$

Backward Pass — Element Recovery and Design

7. Extract element global displacement vectors

- For each member, collect the relevant entries from \mathbf{u} to form \mathbf{u}'
- This assumes beam element is aligned with global axis (no transformation)

8. Extract element FEF vectors

- For each member, collect the relevant entries from \mathbf{F}^F to form \mathbf{Q}_f
- This assumes beam element is aligned with global axis (no transformation)

9. Compute local element end forces and moments

$$\mathbf{f}' = \mathbf{k}' \mathbf{u}' + \mathbf{Q}_f$$

10. Compute axial stress and bending stress (design quantities)

$$\sigma_{\text{axial}} = \frac{N}{A}$$

$$\sigma_{\text{bending}} = \pm \frac{My}{I}$$

Wrap-Up

- Member loads cannot be handled using $\mathbf{Q} = \mathbf{k}\mathbf{u}$ alone
- Fixed-end forces capture the effect of loads within elements
- FEFs allow us to convert member loads into equivalent nodal forces
- The problem is solved by:
 - computing FEFs (fixed system)
 - solving for deformations (DSM system)
 - superimposing results
- The standard DSM workflow extends naturally by including the FEF vector