CEE6501 — Lecture 6.2

2D Beam Element Stiffness Matrix

Learning Objectives

By the end of this lecture, you will be able to:

- Define the local DOFs and sign conventions for a 2D Euler–Bernoulli beam element
- Interpret the member stiffness relation $\mathbf{Q} = \mathbf{k}\mathbf{u}$ in a physical sense
- Compute stiffness coefficients using the unit displacement method and understand their meaning
- Recognize and use the standard 4×4 beam element stiffness matrix
- Evaluate and visualize beam deformation using shape functions and nodal DOFs
- Understand how beam elements are assembled into a global DSM system

Agenda

Part 1 — Beam element stiffness relations and local DOFs
Part 2 — Derive Column 1 (unit displacement method)
Part 3 — Complete the 4×4 beam stiffness matrix
Part 4 — Beam deformation patterns and shape functions
Part 5 — DSM setup and element-to-global mapping

Part 1 — Beam Element Stiffness Relations

Beam Element Response

The **member stiffness relations** express the end forces of a beam element (including shear forces and bending moments) as functions of the **end displacements** (including transverse displacements and rotations).

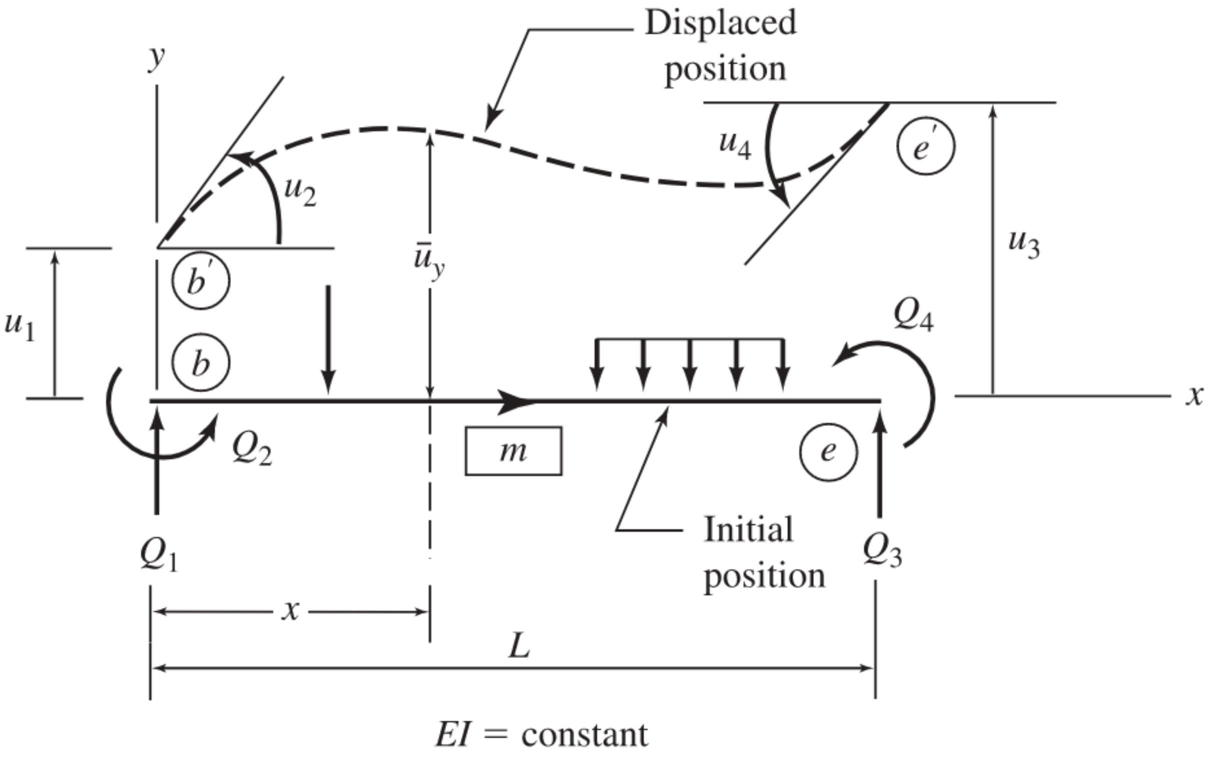
When a beam element is subjected to external loading:

- The member **deforms** (bending)
- **Internal shear forces** develop at the ends
- **Bending moments** are induced at the ends

These internal forces are fully determined by the **displacements and rotations at the element ends**.

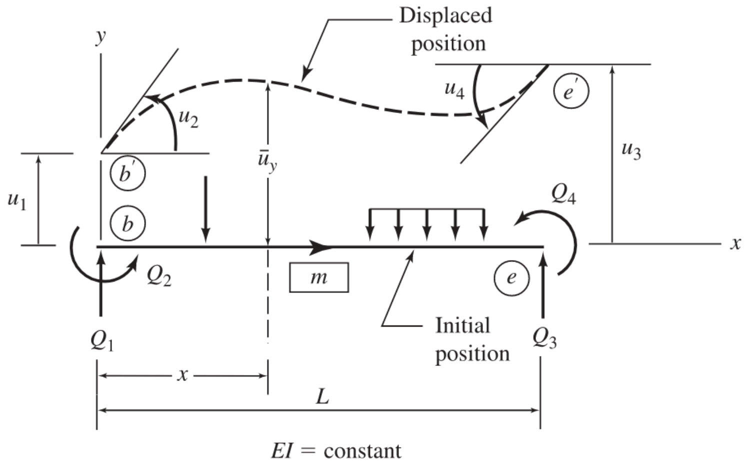
Generic Displacement for a 2D Beam Element

We define all quantities in the **local coordinate system**, with origin at the left end b, and ending at node e.



2D Beam Element DOF Numbering

Degrees of freedom are ordered **left → right**, with **translation first**, then **rotation**.



- **DOF 1:** u_1 — node b, local y
- **DOF 2:** u_2 — node b, θ
- **DOF 3:** u_3 — node e, local y
- **DOF 4:** u_4 — node e, θ

Sign conventions:

- $+u$ → upward (local y direction)
- $+\theta$ → counterclockwise
- Forces follow the same order: $[V_b,\ M_b,\ V_e,\ M_e]^T$

Local displacement and force vectors

Local displacement vector (including rotations):

$$\boldsymbol{u} = \begin{Bmatrix} u_b \\ \theta_b \\ u_e \\ \theta_e \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Local nodal force vector (including moments):

$$\boldsymbol{Q} = \begin{Bmatrix} V_b \\ M_b \\ V_e \\ M_e \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

We seek:

$$\boldsymbol{Q} = \boldsymbol{k}\,\boldsymbol{u}$$

Four equations (one per DOF)

Recall: this is similar in spirit to the truss element definition.

$$Q_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4$$

$$Q_2 = k_{21}u_1 + k_{22}u_2 + k_{23}u_3 + k_{24}u_4$$

$$Q_3 = k_{31}u_1 + k_{32}u_2 + k_{33}u_3 + k_{34}u_4$$

$$Q_4 = k_{41}u_1 + k_{42}u_2 + k_{43}u_3 + k_{44}u_4$$

Each equation expresses **force equilibrium at a single local degree of freedom**.

For a linear elastic element, the force at any DOF is a **linear combination** of all DOF displacements:

- displacing one DOF can induce forces at *all* DOFs
- the proportionality constants are the stiffness coefficients k_{ij}

Same equations in matrix form

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Part 2 — Derive Column 1 of the Beam Element Stiffness Matrix

Unit Displacement Method

Definition:

$$k_{ij} = \text{force at DOF } i \text{ due to a unit displacement at DOF } j,\\ \text{with all other DOFs fixed.}$$

Each column of $\boldsymbol{k}$ is built by:

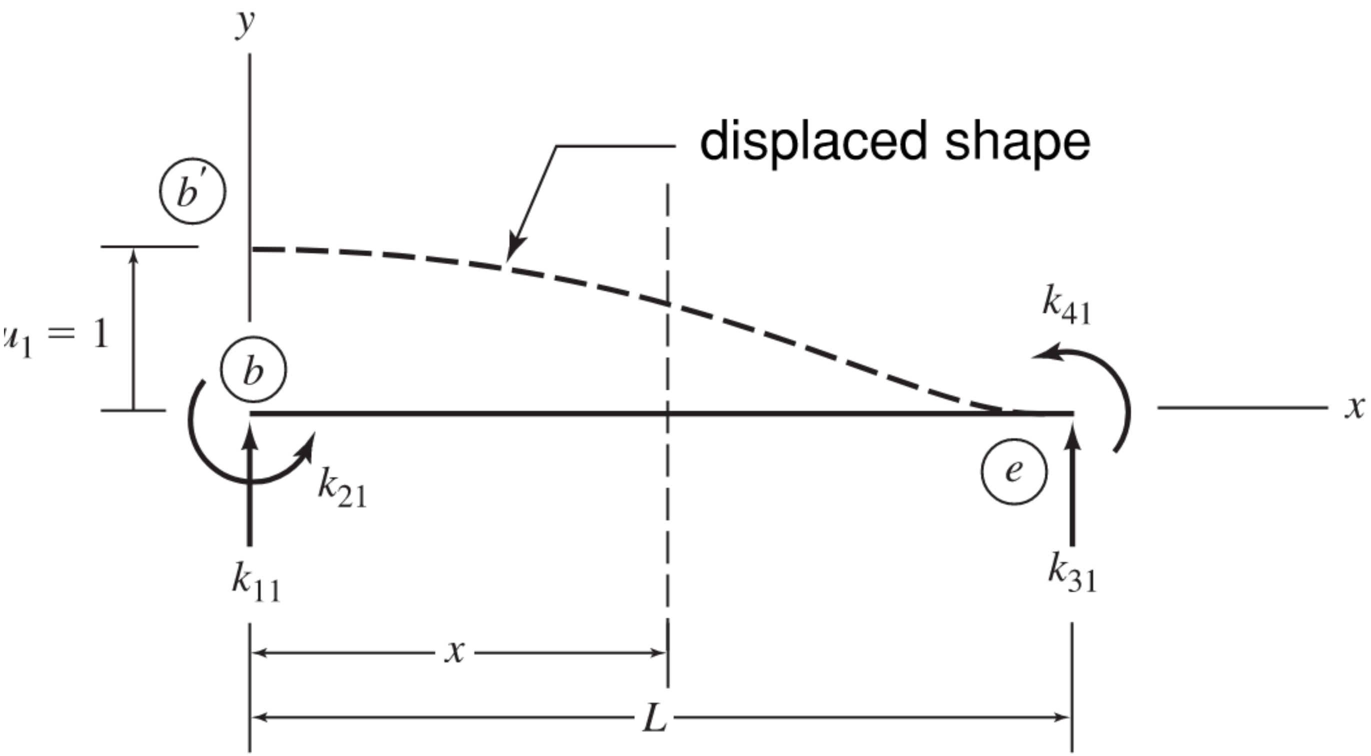
- impose a **unit displacement/rotation** at one DOF
- hold all other DOFs fixed
- record the resulting nodal force pattern

Column 1: impose $u_1 = 1$ ($u_2 = u_3 = u_4 = 0$)

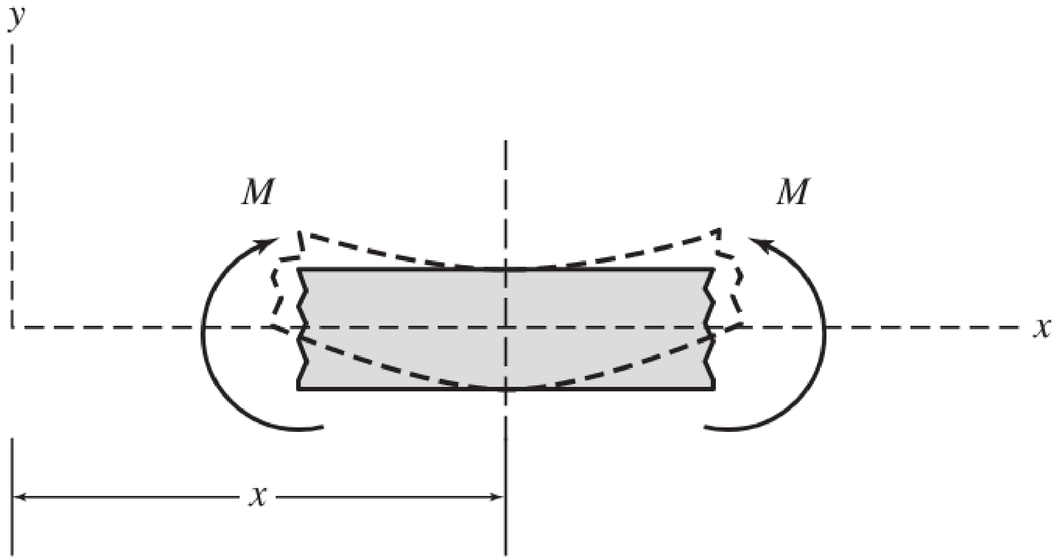
$$\begin{bmatrix} \boxed{k_{11}} & k_{12} & k_{13} & k_{14} \\ \boxed{k_{21}} & k_{22} & k_{23} & k_{24} \\ \boxed{k_{31}} & k_{32} & k_{33} & k_{34} \\ \boxed{k_{41}} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

- k_{11}: force at DOF 1 due to unit displacement at DOF 1
- k_{21}: force at DOF 2 due to unit displacement at DOF 1
- k_{31}: force at DOF 3 due to unit displacement at DOF 1
- k_{41}: force at DOF 4 due to unit displacement at DOF 1

Column 1: impose $u_1 = 1$ $(u_2 = u_3 = u_4 = 0)$



Sign Convention for Derivation



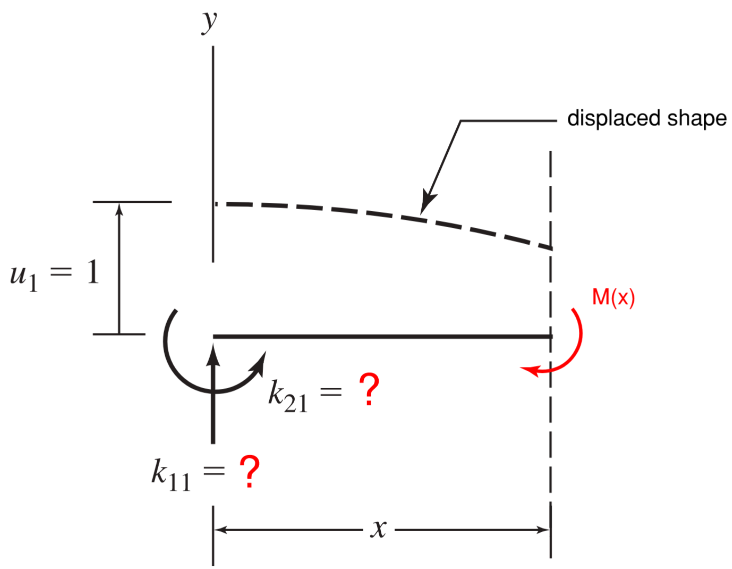
Positive bending moment

Governing Equation (Beam Bending)

$$\frac{d^2u}{dx^2} = \frac{M(x)}{EI} \tag{1}$$

- $u(x)$ = transverse displacement
- $M(x)$ = bending moment
- EI = flexural rigidity

Step 1 — Express Internal Moment as a Function of x



Cut the beam at a distance x from node b.

Using equilibrium:

$$M(x) = -k_{21} + k_{11}x \tag{2}$$

- k_{21}: end moment at node b
- k_{11}: end shear at node b

Step 2 — Substitute into Governing Equation

Substitute Eq. (2) into the **moment term** of Eq. (1):

$$\frac{d^2u}{dx^2} = \frac{M(x)}{EI} \quad \xrightarrow{\text{Eq. (2)}} \quad \frac{d^2u}{dx^2} = \frac{1}{EI}(-k_{21} + k_{11}x) \tag{3}$$

Step 3 — Integrate to Obtain Rotation $\theta(x)$

Start from Eq. (3):

$$\frac{d^2u}{dx^2} = \frac{1}{EI}(-k_{21} + k_{11}x) \tag{3}$$

Integrate both sides with respect to x:

$$\int \frac{d^2u}{dx^2}\,dx = \int \frac{1}{EI}(-k_{21} + k_{11}x)\ dx \tag{4a}$$

Left-hand side:

$$\int \frac{d^2u}{dx^2}\,dx = \frac{du}{dx} = \theta(x) \tag{4b}$$

Right-hand side (term-by-term):

$$\int \frac{1}{EI}(-k_{21} + k_{11}x)\ dx = \frac{1}{EI}\left(\int -k_{21}\,dx + \int k_{11}x\,dx\right) \tag{4c}$$

Compute:

$$\int -k_{21}\,dx = -k_{21}x \qquad \int k_{11}x\,dx = \frac{k_{11}}{2}x^2 \tag{4d}$$

Include integration constant C_1:

$$\theta(x) = \frac{du}{dx} = \frac{1}{EI}\left(-k_{21}x + \frac{k_{11}}{2}x^2\right) + C_1 \tag{4}$$

Step 4 — Integrate to Obtain Displacement $u(x)$

Start from Eq. (4):

$$\frac{du}{dx} = \frac{1}{EI}\left(-k_{21}x + \frac{k_{11}}{2}x^2\right) + C_1 \tag{4}$$

Integrate both sides with respect to x:

$$\int \frac{du}{dx}\,dx = \int \left[\frac{1}{EI}\left(-k_{21}x + \frac{k_{11}}{2}x^2\right) + C_1\right] dx \tag{5a}$$

Left-hand side:

$$\int \frac{du}{dx}\,dx = u(x) \tag{5b}$$

Right-hand side (compact form):

$$u(x) = \frac{1}{EI}\left(\int -k_{21}x\,dx + \int \frac{k_{11}}{2}x^2\,dx\right) + \int C_1\,dx \tag{5c}$$

Compute:

$$\int -k_{21}x\,dx = -\frac{k_{21}}{2}x^2 \qquad \int \frac{k_{11}}{2}x^2\,dx = \frac{k_{11}}{6}x^3 \qquad \int C_1\,dx = C_1x \tag{5d}$$

Include integration constant C_2:

$$u(x) = \frac{1}{EI}\left(-\frac{k_{21}}{2}x^2 + \frac{k_{11}}{6}x^3\right) + C_1x + C_2 \tag{5}$$

Step 5 — Apply Boundary Conditions

We enforce **unit displacement at node** b (0 elsewhere):

At $x = 0$:

$$\begin{aligned} \theta(0) &= 0 \\ u(0) &= 1 \end{aligned}$$

At $x = L$:

$$\begin{aligned} \theta(L) &= 0 \\ u(L) &= 0 \end{aligned}$$

Step 6 — Solve for Constants

Substitute $x = 0$ into Eqs. (4)–(5):

- From $\theta(0) = 0$: all terms with x vanish $\Rightarrow C_1 = 0$
- From $u(0) = 1$: all terms with x vanish $\Rightarrow C_2 = 1$

Updated Eqs. (4)–(5):

$$\theta(x) = \frac{1}{EI}\left(-k_{21}x + \frac{k_{11}}{2}x^2\right) \tag{6}$$

$$u(x) = \frac{1}{EI}\left(-\frac{k_{21}}{2}x^2 + \frac{k_{11}}{6}x^3\right) + 1 \tag{7}$$

Step 7 — Apply Boundary Condition at $x = L$

Substitute $\theta(L) = 0$ into Eq. (6):

$$0 = \frac{1}{EI}\left(-k_{21}L + \frac{k_{11}}{2}L^2\right)$$

Cancel EI and solve:

$$k_{21} = \frac{k_{11}L}{2} \tag{8}$$

Step 8 — Apply Final Condition

Substitute $u(L) = 0$ into Eq. (7):

$$0 = \frac{1}{EI}\left(-\frac{k_{21}}{2}L^2 + \frac{k_{11}}{6}L^3\right) + 1 \tag{9}$$

Substitute Eq. (8) into Eq. (9) and solve:

$$\boxed{k_{11} = \frac{12EI}{L^3}}$$

Step 9 — Solve for $\boldsymbol{k_{21}}$

Back-substitute k_{11} into Eq. (8):

$$\boxed{k_{21} = \frac{6EI}{L^2}}$$

Step 10 — Solve Remaining Stiffness Terms k_{31} and k_{41}

Use equilibrium of the beam element free-body diagram (with k_{11} and k_{21} known).

Vertical force equilibrium

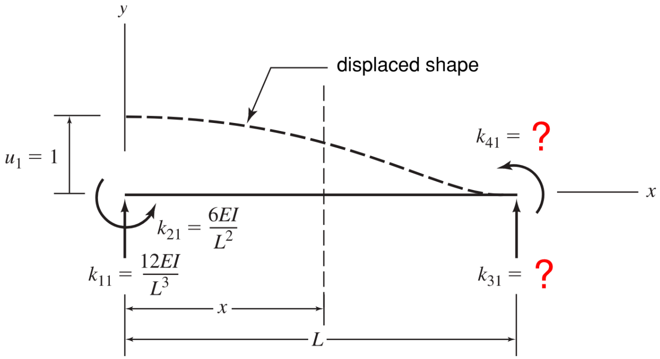
$$\sum F_y = 0 : \qquad \frac{12EI}{L^3} + k_{31} = 0$$

$$\boxed{k_{31} = -\frac{12EI}{L^3}}$$

Moment equilibrium about node b

$$\sum M_b = 0 : \qquad \frac{6EI}{L^2} - \left(\frac{12EI}{L^3}\right) L + k_{41} = 0$$

$$\boxed{k_{41} = \frac{6EI}{L^2}}$$



Free-body diagram used for $\sum F_y = 0$ and $\sum M_b = 0$.

Step 11 (Optional) — Displacement Shape Function

Substitute k_{11} and k_{21} into Eq. (7):

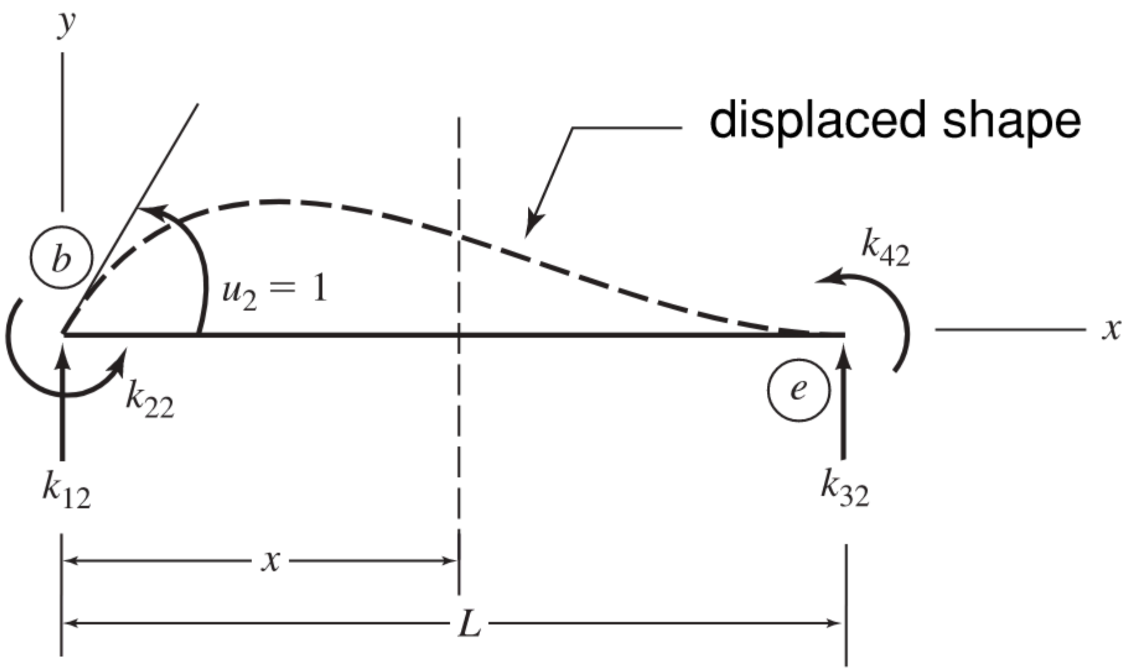
$$u(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

Column 1 of the 4×4 Stiffness Matrix

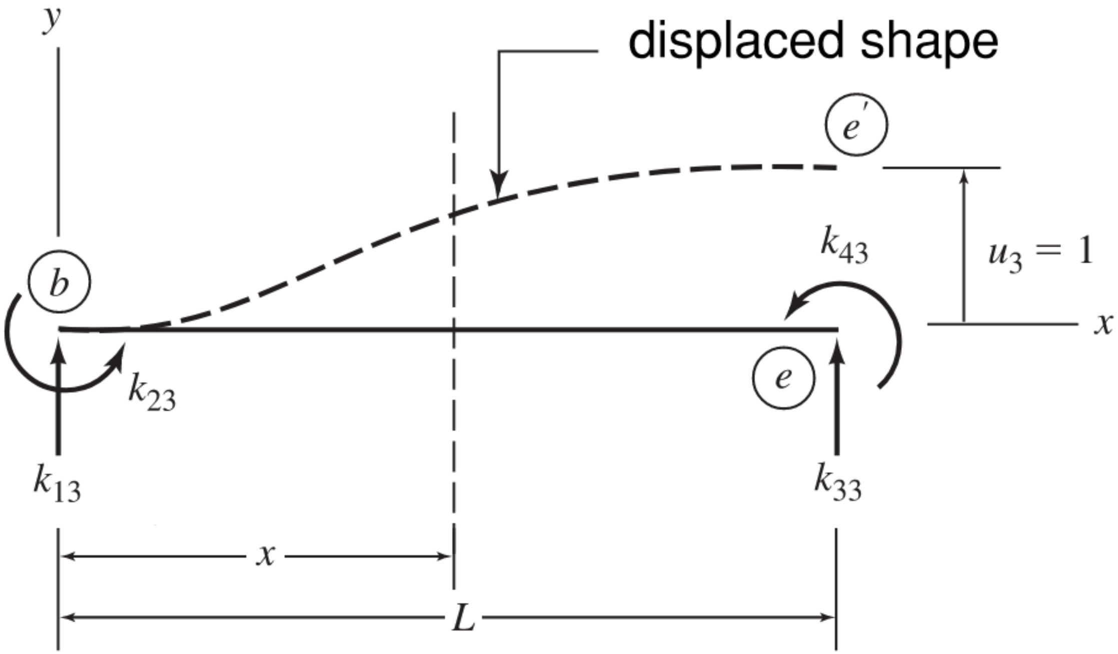
$$\boldsymbol{k} = \begin{bmatrix} \boxed{\dfrac{12EI}{L^3}} & k_{12} & k_{13} & k_{14} \\ \boxed{\dfrac{6EI}{L^2}} & k_{22} & k_{23} & k_{24} \\ \boxed{-\dfrac{12EI}{L^3}} & k_{32} & k_{33} & k_{34} \\ \boxed{\dfrac{6EI}{L^2}} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

Part 3 — Complete 4×4 Beam Stiffness Matrix

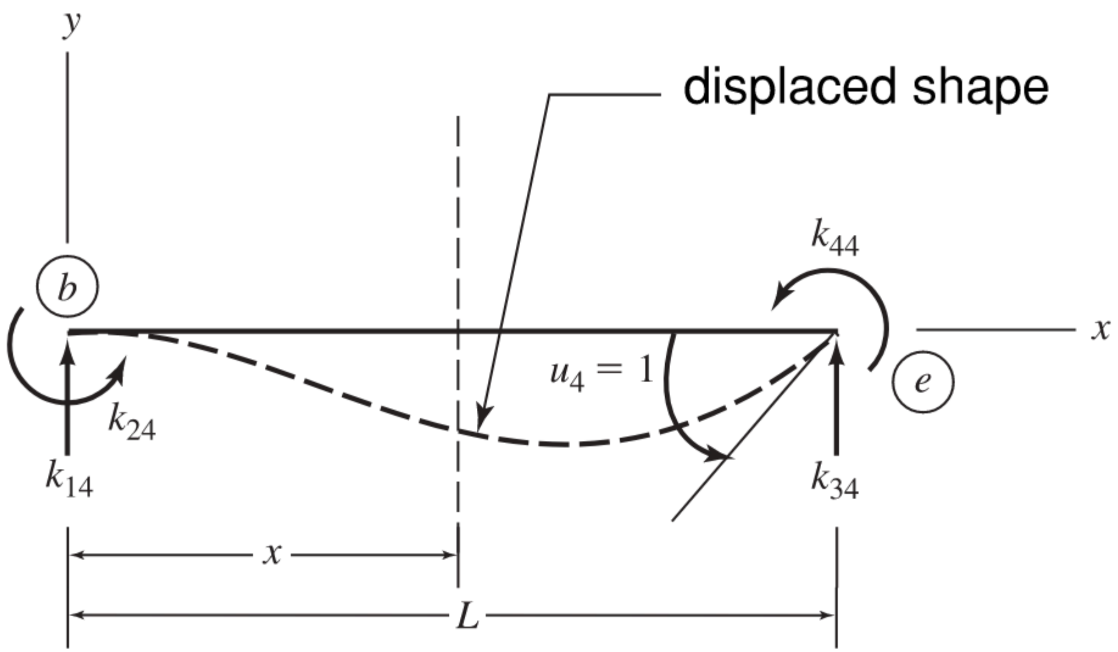
Column 2: impose $u_2 = 1$ ($u_1 = u_3 = u_4 = 0$)



Column 3: impose $u_3 = 1$ ($u_1 = u_2 = u_4 = 0$)



Column 4: impose $u_4 = 1$ ($u_1 = u_2 = u_3 = 0$)



Complete 4×4 Beam Element Stiffness Matrix

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Part 4 - Beam Deformation Patterns

Displacement Field

A 2D beam element has **4 DOFs**, so the displacement field is written as a superposition:

$$u(x) = N_1(x)\, u_1 + N_2(x)\, u_2 + N_3(x)\, u_3 + N_4(x)\, u_4$$

Here, $u(x)$ represents the **transverse displacement of the beam in the local y-direction**.

Each $N_i(x)$ is a **shape function** (a deformation pattern) associated with one DOF:

- $N_1(x)$: unit transverse displacement at node b
- $N_2(x)$: unit rotation at node b
- $N_3(x)$: unit transverse displacement at node e
- $N_4(x)$: unit rotation at node e

The 4 Shape Functions

We derived N_1 earlier:

$$N_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

The shape functions for the other DOFs:

$$N_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2}$$

$$N_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$N_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

Connection to the Finite Element Method (FEM)

The shape functions we derived are exactly those used in the **Finite Element Method (FEM)** for beam elements.

- In FEM, the displacement field is approximated as:

$$u(x) = \sum_{i=1}^{4} N_i(x)\, u_i$$

- The unknowns are the same **nodal DOFs** $\{u_1, u_2, u_3, u_4\}$
- Shape functions provide a **continuous interpolation** within the element

In **CEE6501**, we focus on the **Direct Stiffness Method (DSM)** and derive the stiffness matrix using structural mechanics.

But for the FEM-based derivation of the beam stiffness matrix, see:

- **Kassimali**, Section 5.3

Plotting a Deformed Shape

While we do not explicitly use shape functions in the **DSM formulation**, they are very useful for **visualizing structural response**.

Given element DOFs

$$\boldsymbol{u} = \{u_1,\ u_2,\ u_3,\ u_4\}^T,$$

we can evaluate the displacement field

$$u(x) = \sum_{i=1}^{4} N_i(x)\, u_i$$

at many points along $x \in [0, L]$ to obtain a smooth deformation curve.

Why This Is Useful

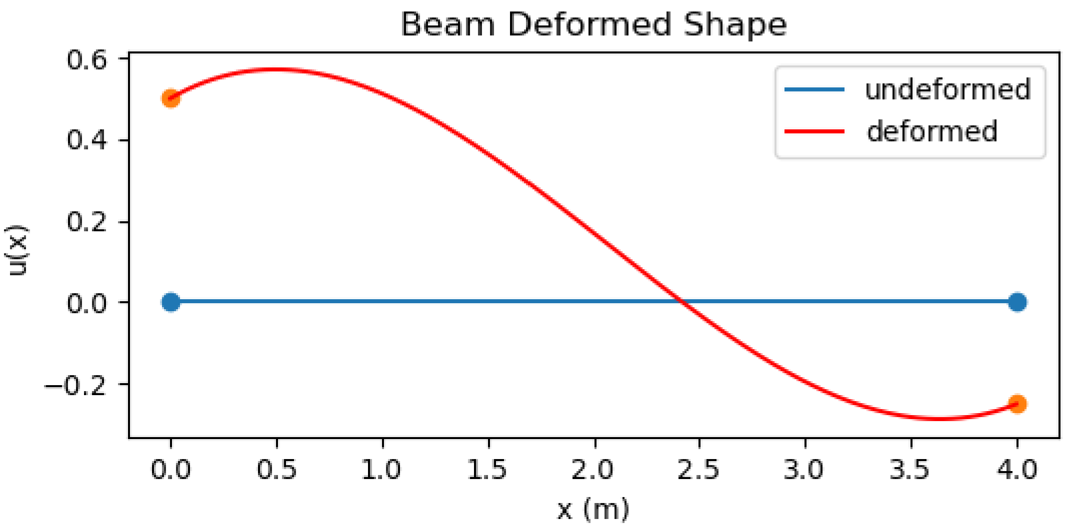
- Visualize the **continuous beam deflection** (not just nodal values)
- Check **sign conventions** and DOF interpretation
- Construct smooth **deformed shapes for multi-element structures**

In [25]:

```
import numpy as np
import matplotlib.pyplot as plt

def shape_functions(x, L):
    xi = x / L
    N1 = 1 - 3*xi**2 + 2*xi**3
    N2 = L * (xi - 2*xi**2 + xi**3)
    N3 = 3*xi**2 - 2*xi**3
    N4 = L * (-xi**2 + xi**3)
    return N1, N2, N3, N4

def beam_deflection(x, L, u1, u2, u3, u4):
    """
    Compute beam transverse deflection u(x) from nodal DOFs.
    DOFs: u1 (disp at b), u2 (rot at b), u3 (disp at e), u4 (rot at e)
    """
    N1, N2, N3, N4 = shape_functions(x, L)
    return N1*u1 + N2*u2 + N3*u3 + N4*u4
```

In [26]:
```
def plot_beam(L, u1, u2, u3, u4, n=200):
    x = np.linspace(0, L, n)
    u = beam_deflection(x, L, u1, u2, u3, u4)

    plt.figure(figsize=(6, 2.5))  # smaller plot

    # undeformed shape
    plt.plot(x, np.zeros_like(x), linestyle="-", label="undeformed")
    plt.scatter([0, L], [0, 0])  # nodes (no legend)

    # deformed shape
    plt.plot(x, u, color="red", label="deformed")
    plt.scatter([0, L], [u1, u3])  # nodes (no legend)

    plt.xlabel("x (m)")
    plt.ylabel("u(x)")
    plt.title("Beam Deformed Shape")
    plt.legend()
    plt.show()
```

In [27]:
```
L = 4.0  # element length

# Example nodal DOFs (pick anything to test)
u1 = 0.5  # displacement at node b
u2 = +0.3   # rotation at node b (rad)
u3 = -0.25  # displacement at node e
u4 = 0.21   # rotation at node e (rad)

plot_beam(L, u1, u2, u3, u4)
```



Part 5 - DSM Setup

Global System (Same as Trusses)

The overall system still follows the standard **Direct Stiffness Method**:

$$\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f}$$

Separating **free** and **restrained** DOFs:

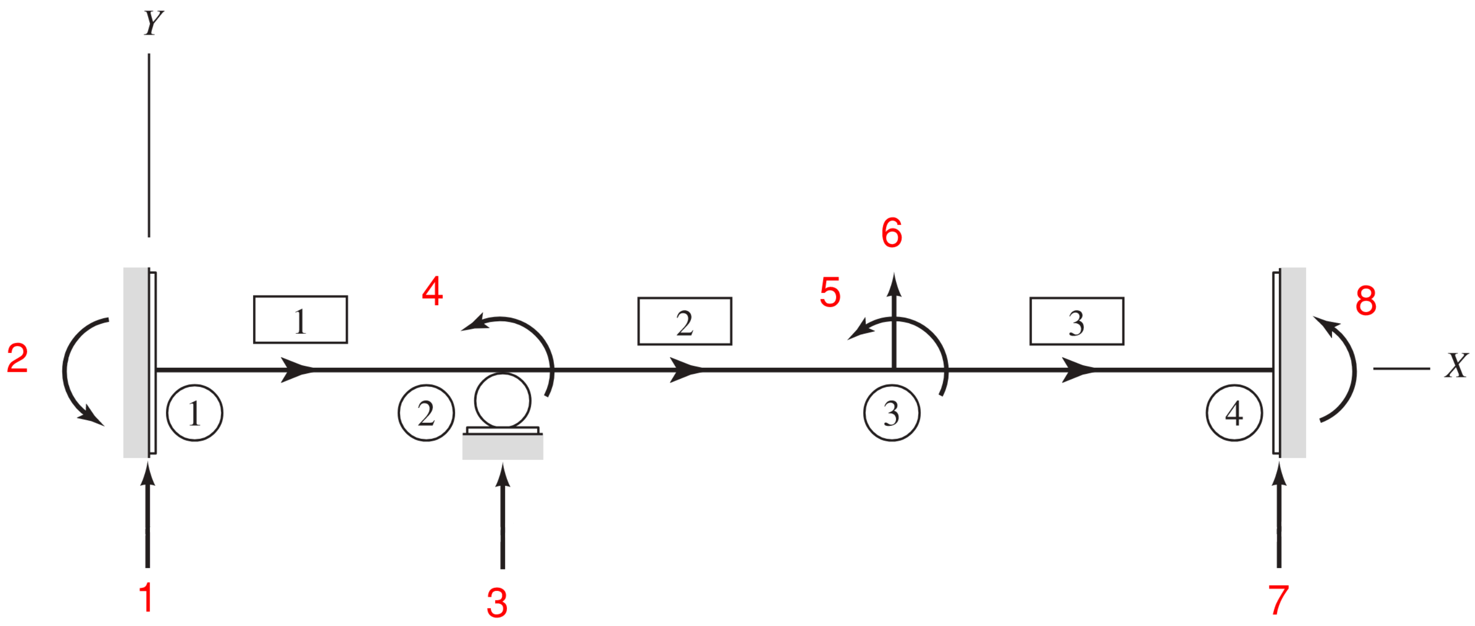
$$\left[\begin{array}{c:c} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fr} \\ \hdashline \boldsymbol{K}_{rf} & \boldsymbol{K}_{rr} \end{array}\right] \left\{\begin{array}{c} \boldsymbol{u}_f \\ \hdashline \boldsymbol{u}_r \end{array}\right\} = \left\{\begin{array}{c} \boldsymbol{F}_f \\ \hdashline \boldsymbol{F}_r \end{array}\right\}$$

Here the dashed line shows the split between free and restrained DOFs:

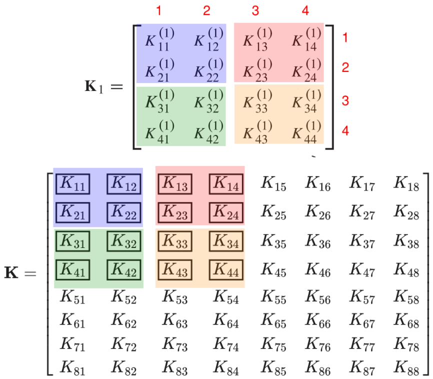
- top-left block: $\boldsymbol{K}_{ff}$ (free–free)
- top-right block: $\boldsymbol{K}_{fr}$ (free–restrained)
- bottom-left block: $\boldsymbol{K}_{rf}$ (restrained–free)
- bottom-right block: $\boldsymbol{K}_{rr}$ (restrained–restrained)

Example Structure 1

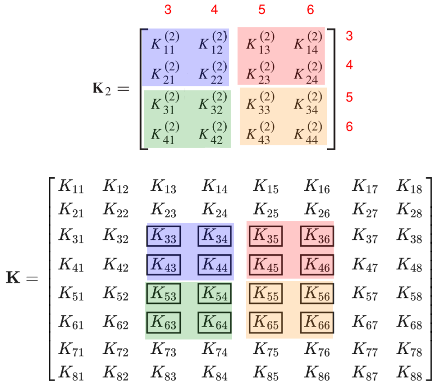
Same beam as in 6.1



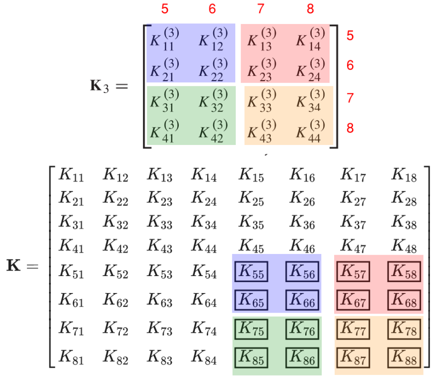
DSM Assembly — Element-to-Global Mapping (Unpartitioned)



Element 1 mapping



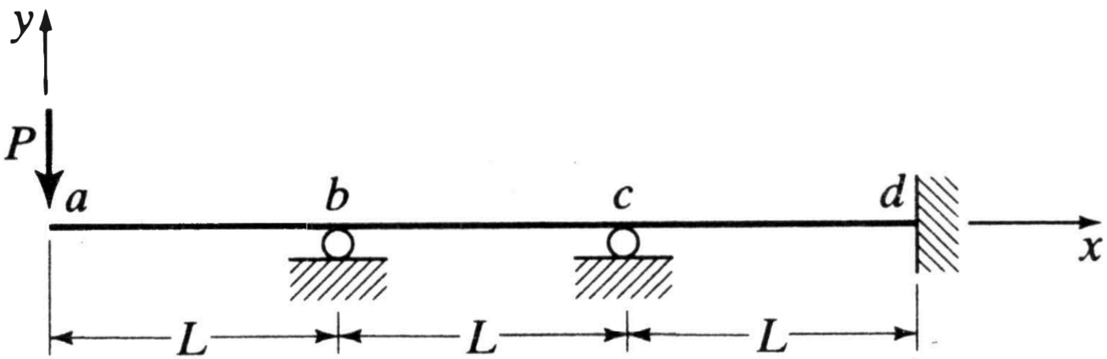
Element 2 mapping

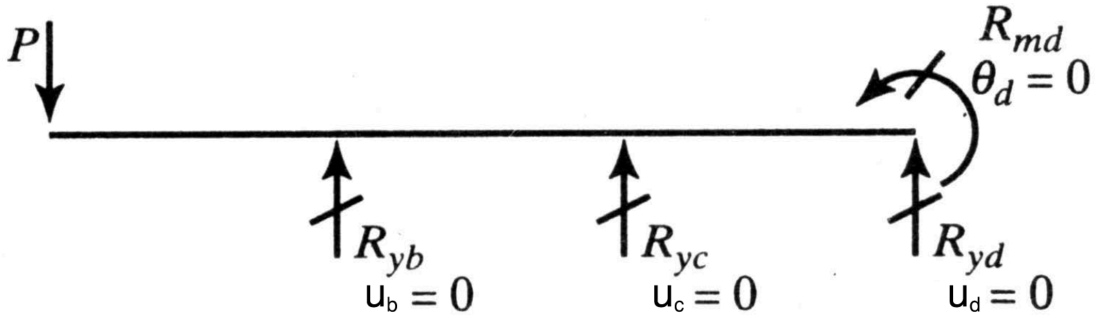


Element 3 mapping

DOFs 4,5,6 are free, partition the matrix in that way

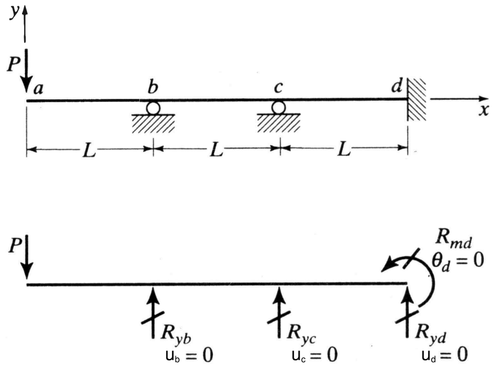
Example Structure 2





DSM Assembly — Element-to-Global Mapping (Partitioned System)

$$
\left[\begin{array}{c:c} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fr} \\ \hdashline \boldsymbol{K}_{rf} & \boldsymbol{K}_{rr} \end{array}\right]
\left\{\begin{array}{c} u_a \\ \theta_a \\ \theta_b \\ \theta_c \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{array}\right\}
=
\left\{\begin{array}{c} -P \\ 0 \\ 0 \\ 0 \\ \hdashline R_{yb} \\ R_{yc} \\ R_{yd} \\ R_{md} \end{array}\right\}
$$



Free DOFs (Global Indexing):

- u_a → DOF 1
- θ_a → DOF 2
- θ_b → DOF 4
- θ_c → DOF 6

Reactions (Global Indexing):

- R_{yb} → DOF 3
- R_{yc} → DOF 5
- R_{yd} → DOF 7
- R_{md} → DOF 8

Solving for $\boldsymbol{u}_f$ (Review)

We are interested in solving for the **unknown displacements** at the free DOFs, $\boldsymbol{u}_f$.

Starting from:

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f + \boldsymbol{K}_{fr}\boldsymbol{u}_r = \boldsymbol{F}_f$$

Rearrange to isolate the unknowns:

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f = \boldsymbol{F}_f - \boldsymbol{K}_{fr}\boldsymbol{u}_r$$

Provided that $\boldsymbol{K}_{ff}$ is invertible, the solution is:

$$\boxed{\boldsymbol{u}_f = \boldsymbol{K}_{ff}^{-1}\left(\boldsymbol{F}_f - \boldsymbol{K}_{fr}\boldsymbol{u}_r\right)}$$

Solving for $\boldsymbol{F}_r$ (Review)

Once the free displacements $\boldsymbol{u}_f$ have been computed, we can determine the **forces at the restrained DOFs** (support reactions).

Starting from:

$$\boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r = \boldsymbol{F}_r$$

At the restrained DOFs, the displacements $\boldsymbol{u}_r$ are **known** from the boundary conditions (often $\boldsymbol{u}_r = \boldsymbol{0}$). Substituting these known values gives a direct expression for the reaction forces:

$$\boxed{\boldsymbol{F}_r = \boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r}$$

Wrap-Up

In this lecture, we:

- Defined the local beam DOFs and the local displacement and force vectors
- Built stiffness coefficients using the unit displacement method (column-by-column)
- Arrived at the standard 4×4 Euler–Bernoulli beam element stiffness matrix
- Introduced shape functions to reconstruct a smooth deflected shape $u(x)$ from nodal DOFs
- Reviewed DSM system setup and partitioning into free and restrained DOFs

Next lecture:

- Introduce member loads and convert them to equivalent nodal loads
- Solve for nodal displacements, then compute reactions and internal member forces
- Complete DSM workflow