CEE6501 — Lecture 6.3

Fixed End Forces (FEFs)

Learning Objectives

By the end of this lecture, you will be able to:

- Distinguish between joint loads and member loads in beam systems
- Define fixed-end forces (FEFs) and interpret their physical meaning
- Explain why member loading requires modification of $\mathbf{Q} = \mathbf{ku}$
- Conceptually decompose indeterminate beam problems using FEFs and superposition
- Convert member loads into equivalent nodal forces for DSM analysis
- Incorporate FEFs into the global system of equations and solve for displacements and reactions
- Apply the full DSM workflow including FEFs for beam analysis

Agenda

Part 1 - What are fixed-end forces (FEFs)
Part 2 - Why member loading requires FEFs
Part 3 - Indeterminate beam solution using FEFs and superposition
Part 4 - Incorporating FEFs into the DSM formulation
Part 5 - Full DSM workflow with FEFs

Part 1 — What are Fixed End Forces?

Forces and moments that develop at the ends of a member due to external loading, assuming both ends are **fully fixed** (no translation or rotation).

Joint Loads vs Member Loads

- **Joint loads** → applied directly at nodes
- **Member loads** → applied between nodes
    - distributed loads
    - point loads along a span
    - applied moments

In deriving the beam stiffness matrix, we assumed:

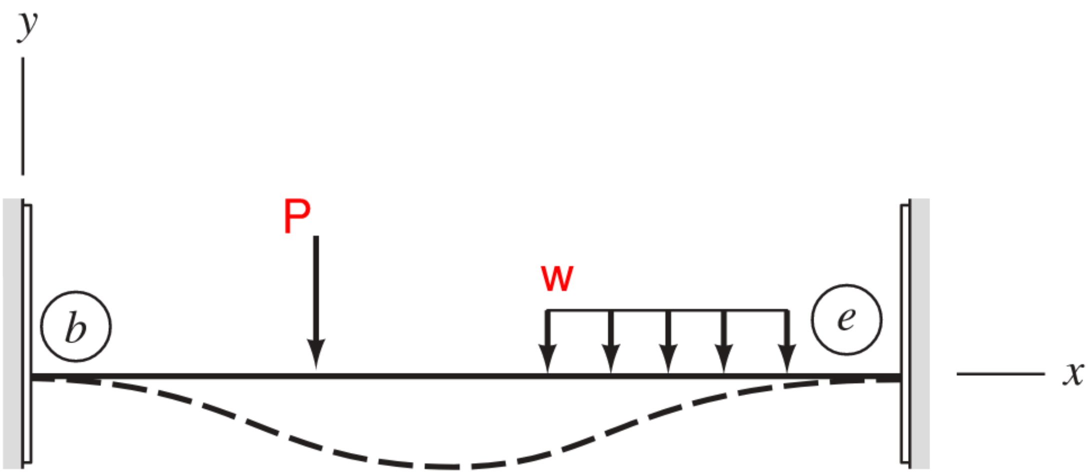
- loads act only at nodes
- no loading within the element

Now we relax this assumption.

Part 2 — Why Do We Need FEFs?

A Simple Case

Consider a **fully fixed beam** subjected to loading along its span.



All DOFs are restrained:

$$\boldsymbol{u} = \boldsymbol{0}$$

From typical local element force-displacement relationship:

$$\boldsymbol{Q} = \boldsymbol{k}\boldsymbol{u}$$

we obtain:

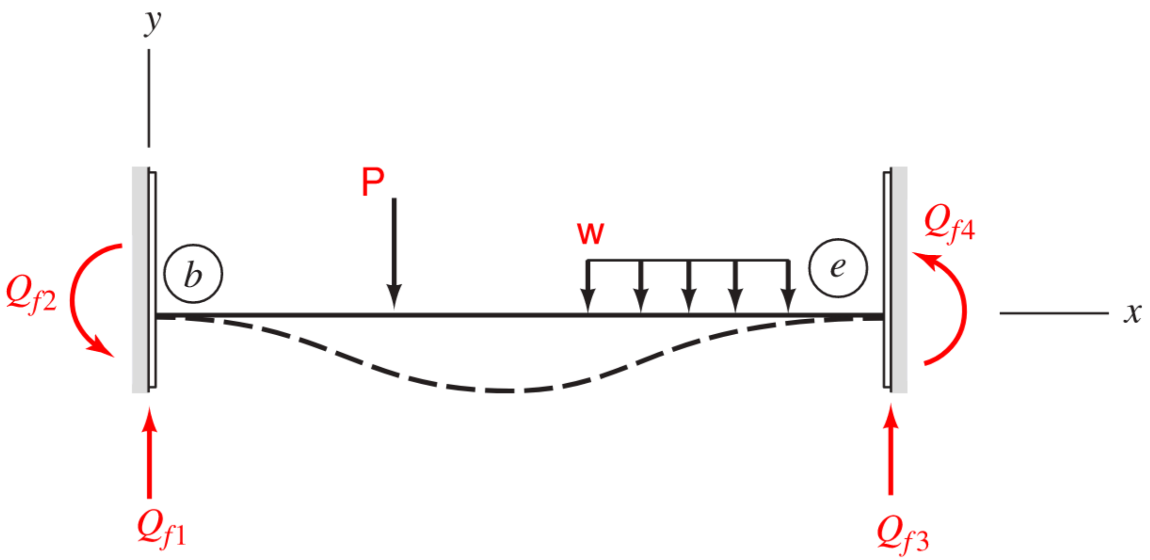
$$\boldsymbol{Q} = \boldsymbol{0}$$

❌ Clearly incorrect — reactions exist even with zero displacement.

Corrected Local Element Force-Displacement Relationship

$$\boldsymbol{Q} = \boldsymbol{k}\boldsymbol{u} + \boldsymbol{Q}_f$$

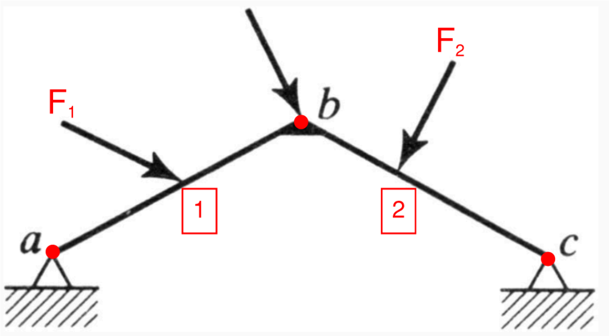
- $\boldsymbol{Q}_f$ = **fixed-end force vector**
- Captures effects of **loads between nodes**

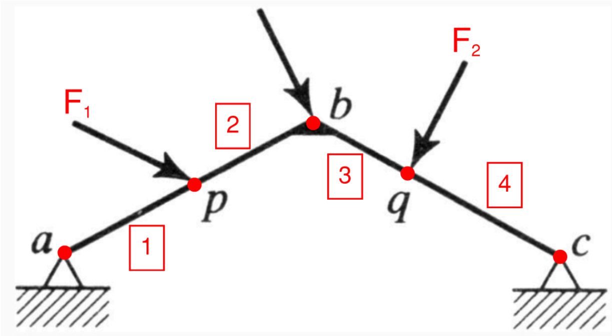


These forces are required whenever member loads are present

Avoiding FEFs with Artificial Nodes

One approach is to insert **artificial nodes** at load locations.





- Convert member loads → joint loads
- Enforce equilibrium at new nodes
- Maintain compatibility between segments

Artificial Nodes for Point Loads and Distributed Loads

This works for **point loads**, but:

- increases number of elements and DOFs in the solution
- not computationally efficient for general use

For **distributed loads**:

- Infinite number of points carry load
- Cannot represent exactly with finite nodes

Alternative: **lump distributed loads into point forces**

- preserves total force (equilibrium)
- but does not accurately represent local **internal shear and moment distributions**

Key Idea

We need a method that:

- accounts for **loads within elements**
- preserves **exact equilibrium**
- does not increase DOFs

Part 3 - Indeterminate Beam Structural Analysis

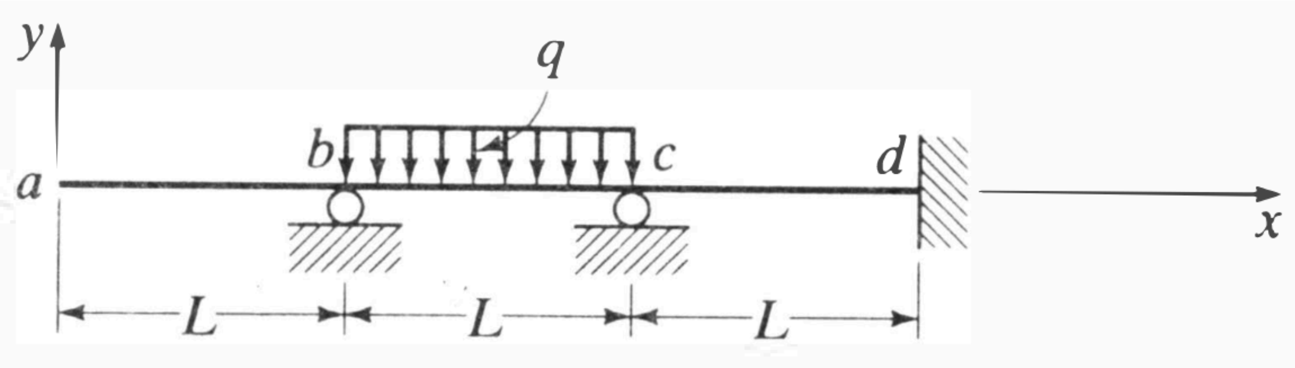
Solving Indeterminate Beams (Role of FEFs)

Recall from structural analysis:

- Indeterminate structures are solved by enforcing:
  - **equilibrium**
  - **compatibility**
- For beams with **member loads**, we conceptually:
  - split the problem into **simpler systems**
    - one with **fixed-end forces (FEFs)** (all DOFs restrained)
    - one with **equivalent nodal loads** (to restore deformation)
  - solve each system separately
  - **superimpose the results** to obtain the final response

Indeterminate Beam

Same beam as in Lecture 6.2, only loaded with distributed load



We analyze the beam by splitting it into **two systems**, then summing the results.

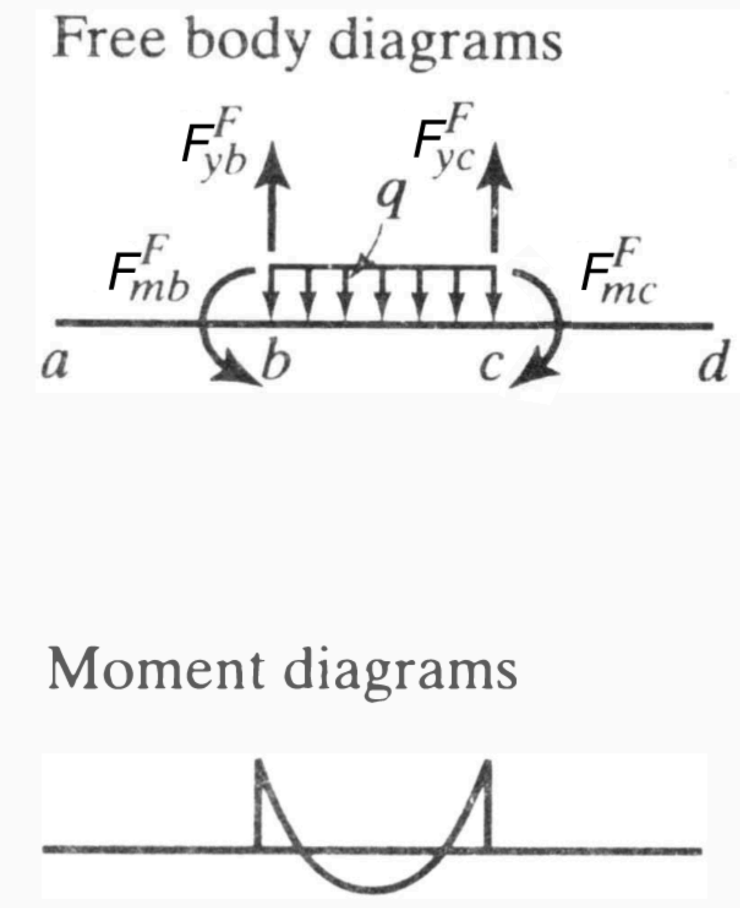
Stage 1 — Clamp the Beam (Zero DOFs)

isolate effect of **member loading only**

- Add **fictitious constraints** at element ends
- Beam becomes **fully fixed**

Stage 2 — Compute Fixed-End Forces

- Solve the **fully fixed beam** under loading
- Use known closed-form solutions



Example (uniform load):

$$F^F_{yb} = \frac{qL}{2}, \quad F^F_{mb} = \frac{qL^2}{12}$$

$$F^F_{yc} = \frac{qL}{2}, \quad F^F_{mc} = -\frac{qL^2}{12}$$

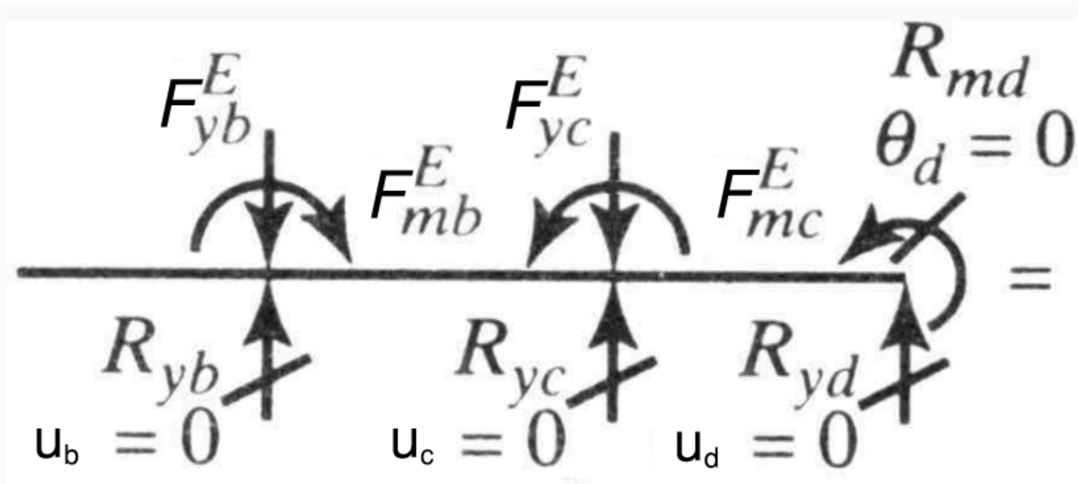
These are the **fixed-end forces**, $\boldsymbol{F}^F$.

Stage 3 — Remove Constraints (Equivalent Loads)

- Remove fictitious constraints
- Apply **equal and opposite forces** at nodes:

$$\boldsymbol{F}^E = -\boldsymbol{F}^F$$

- These are called **equivalent nodal loads**
- converts member loading → nodal loading

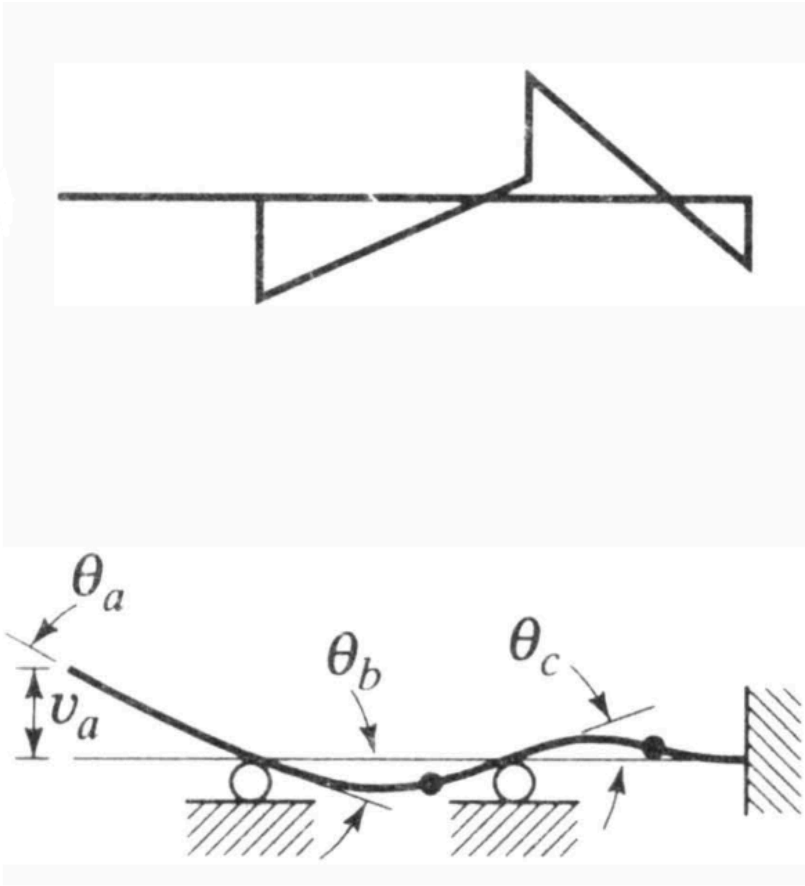


Stage 4 — Solve for Deformations Under Equivalent Nodal Loads

Now solve:

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}^E$$

Using standard DSM solution



Stage 5 — Superimpose and Recover the Real System

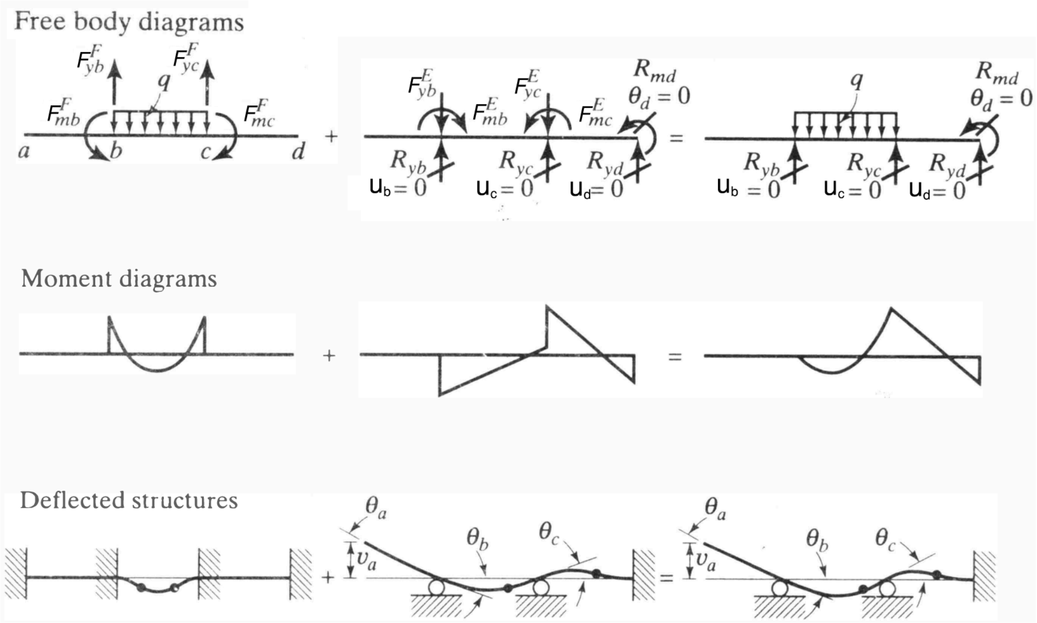
Superimpose:

- **Stage 1:** fixed-end solution
- **Stage 4:** deformation solution

The **fictitious forces cancel** in the sum

Final system satisfies:

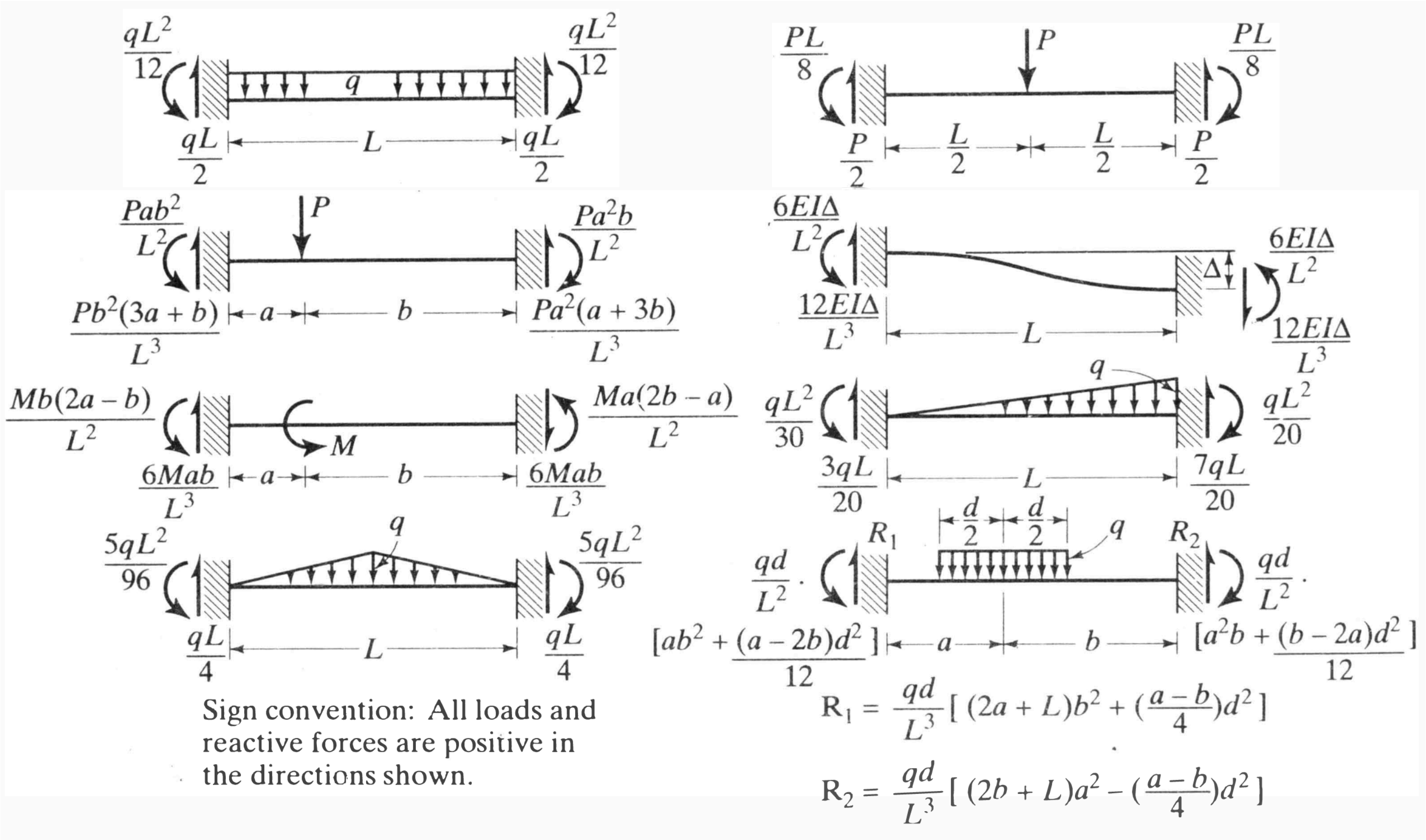
- **equilibrium**
- **compatibility**



List of Typical FEFs

The internal forces and displacements of the fixed-end part of the problem must be obtained by some means

Don't need to calculate these each time as there are closed form solutions for typical cases.


qL^2/12
q
L
qL/2
PL/8
P
L/2
P/2
Pab^2/L^2
Pa^2b/L^2
a
b
Pb^2(3a + b)/L^3
Pa^2(a + 3b)/L^3
6EIΔ/L^2
Δ
12EIΔ/L^3
Mb(2a – b)/L^2
M
Ma(2b – a)/L^2
6Mab/L^3
qL^2/30
qL^2/20
3qL/20
7qL/20
5qL^2/96
qL/4
d/2
R1
R2
qd/L^2 · [ab^2 + (a – 2b)d^2/12]
qd/L^2 · [a^2b + (b – 2a)d^2/12]
R1 = qd/L^3 [(2a + L)b^2 + ((a – b)/4)d^2]
R2 = qd/L^3 [(2b + L)a^2 – ((a – b)/4)d^2]
Sign convention: All loads and reactive forces are positive in the directions shown.

Part 4 — DSM Setup Incorporating FEFs

Local Element System

As we already mentioned:

$$\boldsymbol{Q} = \boldsymbol{k}\boldsymbol{u} + \boldsymbol{Q}_f$$

- $\boldsymbol{k}\boldsymbol{u}$ → local response due to deformation
- $\boldsymbol{Q}_f$ → local fixed end forces due to member loading

Total response = deformation + load effects

Global Structure

The FEFs formulation allows standard DSM procedure, but with an additional FEF vector

- FEFs act as **equivalent nodal forces**
- Added to global force vector during assembly

switching to this notation in Global analysis (consistent with Truss Notation from Lecture 4), to not confuse with partitioned notation

$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{u} + \boldsymbol{F}^F$$

- $\boldsymbol{K}\boldsymbol{u}$ → global response due to deformation, assembled stiffness matrix
- $\boldsymbol{F}^F$ → global fixed end forces due to member loading

Physically, this states that, in the absence of any nodal displacement, i.e., $\boldsymbol{u} = 0$, $\boldsymbol{F}$ would be equal to the vector of fixed end forces, $\boldsymbol{F}^{\boldsymbol{F}}$.

Conceptually, this formation is useful because it helps to keep clear the distinction between any real nodal loads and reaction components of $\boldsymbol{F}$ and the fictitious components that comprise $\boldsymbol{F}^{\boldsymbol{F}}$.

Partitioned Matrix Form

Bring F^F to the other side of the equation, and partition along with the rest of the vectors and matrices.

$$\left[\begin{array}{c:c} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fr} \\ \hdashline \boldsymbol{K}_{rf} & \boldsymbol{K}_{rr} \end{array}\right] \left\{\begin{array}{c} \boldsymbol{u}_f \\ \hdashline \boldsymbol{u}_r \end{array}\right\} = \left\{\begin{array}{c} \boldsymbol{F}_f \\ \hdashline \boldsymbol{F}_r \end{array}\right\} - \left\{\begin{array}{c} \boldsymbol{F}_f^F \\ \hdashline \boldsymbol{F}_r^F \end{array}\right\}$$

Solving for $\boldsymbol{u}_f$ with FEFs

We are interested in solving for the **unknown displacements** at the free DOFs, $\boldsymbol{u}_f$.

Starting from:

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f + \boldsymbol{K}_{fr}\boldsymbol{u}_r = \boldsymbol{F}_f - \boldsymbol{F}_f^F$$

Rearrange to isolate the unknowns:

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f = \boldsymbol{F}_f - \boldsymbol{F}_f^F - \boldsymbol{K}_{fr}\boldsymbol{u}_r$$

Provided that $\boldsymbol{K}_{ff}$ is invertible, the solution is:

$$\boxed{\boldsymbol{u}_f = \boldsymbol{K}_{ff}^{-1}\left(\boldsymbol{F}_f - \boldsymbol{F}_f^F - \boldsymbol{K}_{fr}\boldsymbol{u}_r\right)}$$

Solving for $\boldsymbol{F}_r$ with FEFs

Once the free displacements $\boldsymbol{u}_f$ have been computed, we can determine the **forces at the restrained DOFs** (support reactions).

Starting from:

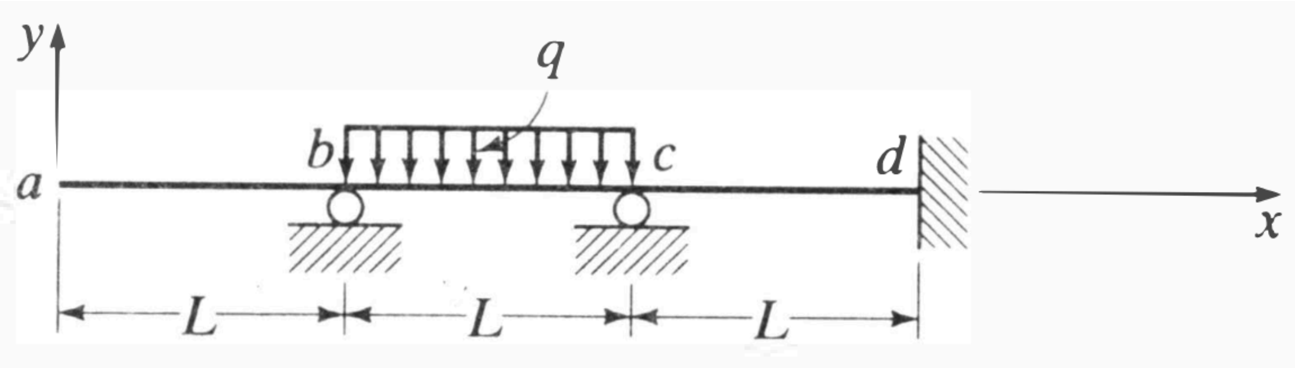
$$\boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r = \boldsymbol{F}_r - \boldsymbol{F}_r^F$$

At the restrained DOFs, the displacements $\boldsymbol{u}_r$ are **known** from the boundary conditions (often $\boldsymbol{u}_r = \boldsymbol{0}$). Substituting these known values gives a direct expression for the reaction forces:

$$\boxed{\boldsymbol{F}_r = \boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r + \boldsymbol{F}_r^F}$$

Example Structure

Same beam as in Lecture 6.2, only loaded with distributed load



FEFs for Middle Span

DOF 3: $F_{yb}^F = \frac{qL}{2}$

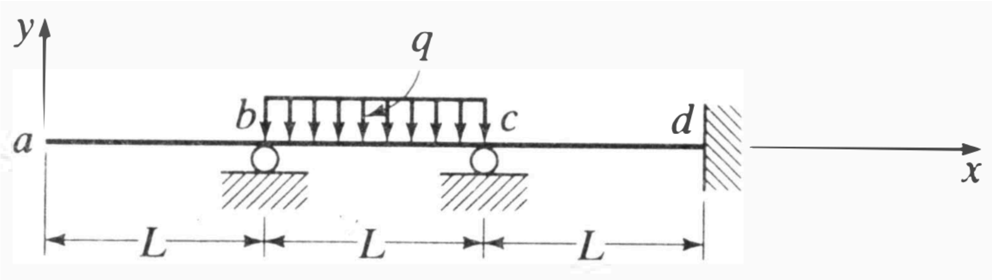
DOF 4: $F_{mb}^F = \frac{qL^2}{12}$

DOF 5: $F_{yc}^F = \frac{qL}{2}$

DOF 6: $F_{mc}^F = -\frac{qL^2}{12}$

DSM Setup for Example Structure (Partitioned)

$$
\left[\begin{array}{c:c} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fr} \\ \hdashline \boldsymbol{K}_{rf} & \boldsymbol{K}_{rr} \end{array}\right]
\left\{\begin{array}{c} u_a \\ \theta_a \\ \theta_b \\ \theta_c \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{array}\right\}
=
\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hdashline R_{yb} \\ R_{yc} \\ R_{yd} \\ R_{md} \end{array}\right\}
-
\left\{\begin{array}{c} 0 \\ 0 \\ \frac{qL^2}{12} \\ -\frac{qL^2}{12} \\ \hdashline \frac{qL}{2} \\ \frac{qL}{2} \\ 0 \\ 0 \end{array}\right\}
$$



Part 5 — DSM Full Procedure

Forward Pass — Structural Analysis

1. **Defining Structure**
    - Node numbering and coordinates
    - Global DOF numbering
    - Element connectivity
    - Restraints and Applied Forces
2. **Assemble global stiffness matrix**
    - Scatter-add element contributions into $\boldsymbol{K}$
    - This assumes beam element is aligned with global axis (no transformation)
3. **Generate applied force vector**
    - Scatter-add force and moment contributions into $\boldsymbol{F}$

Forward Pass — Structural Analysis, cont...

4. **Generate FEF vectors**
    - Fix all beam segments
    - identify what case, calculate closed form FEFs
    - Scatter-add FEFs into $\boldsymbol{F}^{\boldsymbol{F}}$
5. **Apply boundary conditions**
    - Partition DOFs into free (f) and restrained (r)
6. **Solve for unknown displacements**

$$\boldsymbol{u}_f = \boldsymbol{K}_{ff}^{-1}\left(\boldsymbol{F}_f - \boldsymbol{F}_f^F - \boldsymbol{K}_{fr}\boldsymbol{u}_r\right)$$

7. **Recover support reactions**

$$\boldsymbol{F}_r = \boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r + \boldsymbol{F}_r^F$$

Backward Pass — Element Recovery and Design

7. **Extract element global displacement vectors**
    - For each member, collect the relevant entries from $\boldsymbol{u}$ to form $\boldsymbol{u}'$
    - This assumes beam element is aligned with global axis (no transformation)
8. **Extract element FEF vectors**
    - For each member, collect the relevant entries from $\boldsymbol{F}^{\boldsymbol{F}}$ to form $\boldsymbol{Q}_f$
    - This assumes beam element is aligned with global axis (no transformation)
9. **Compute local element end forces and moments**
    - Add element-level FEFs to deformation-based forces and moments

$$\boldsymbol{f}' = \boldsymbol{k}'\,\boldsymbol{u}' + \boldsymbol{Q}_f$$

10. **Compute shear stress and bending stress (design quantities)**

$$\tau_{\text{shear}} = \frac{VQ}{It}$$
$$\sigma_{\text{bending}} = \pm\frac{My}{I}$$

11. **Plot Deformed Shape**
    - Use the nodal displacement vector $\boldsymbol{u}'$
    - Interpolate along each element using appropriate shape functions

$$u(x) = \sum_{i=1}^{4} N_i(x)\, u_i'$$

Wrap-Up

- Member loads cannot be handled using $\mathbf{Q} = \mathbf{k}\mathbf{u}$ alone
- Fixed-end forces capture the effect of loads within elements
- FEFs allow us to convert member loads into equivalent nodal forces
- The problem is solved by:
    - computing FEFs (fixed system)
    - solving for deformations (DSM system)
    - superimposing results
- The standard DSM workflow extends naturally by including the FEF vector