

CEE6501 — Lecture 7.1

Introduction to 2D Frame Analysis

Learning Objectives

By the end of this lecture, you will be able to:

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Agenda

Part 1 — Roadmap: truss → beam → 3D frame

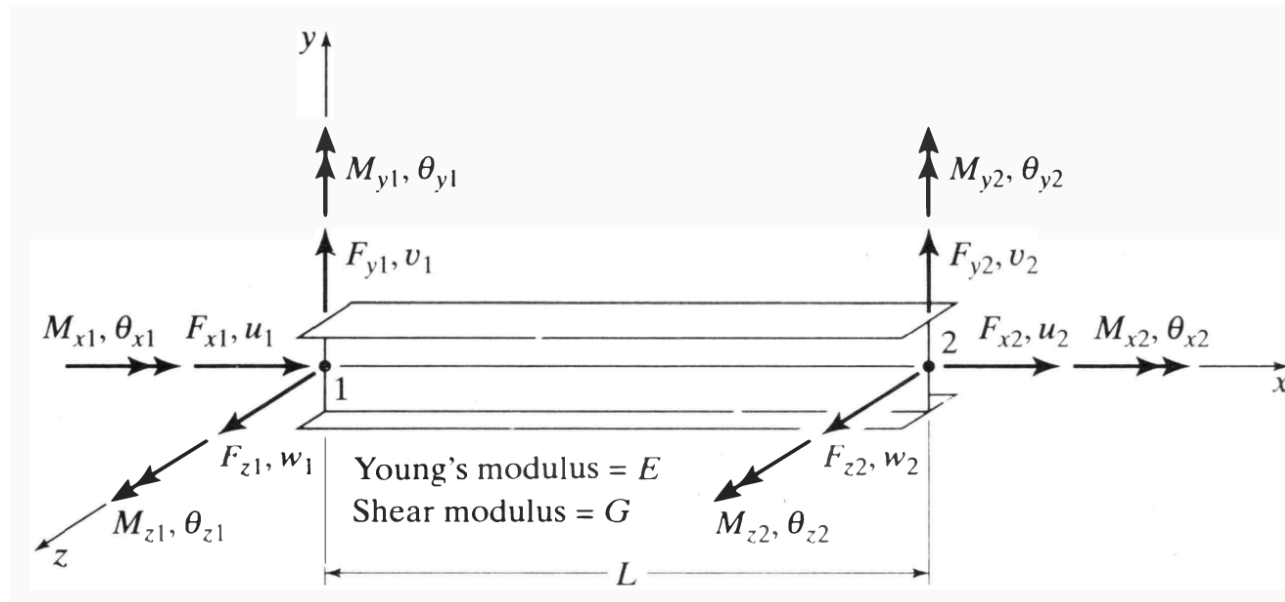
Part 2 —

Part 1 — DSM Element Roadmap

From Trusses to 2D Beams to 2D Frames to 3D Frames

The 3D Frame Element (Where We Are Headed)

- A general 3D frame member has **6 DOFs per node** (12 per element)
 - translations: u_x, u_y, u_z
 - rotations: $\theta_x, \theta_y, \theta_z$
- This single element can represent **axial**, **torsion**, and **bending** behavior



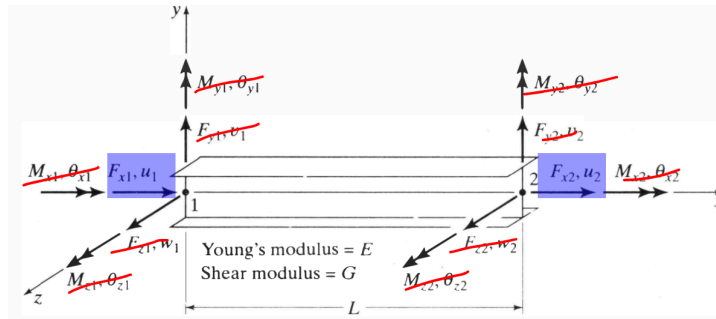
Key idea: Superposition

You can think of the 3D frame element as a **superposition** of four behaviors:

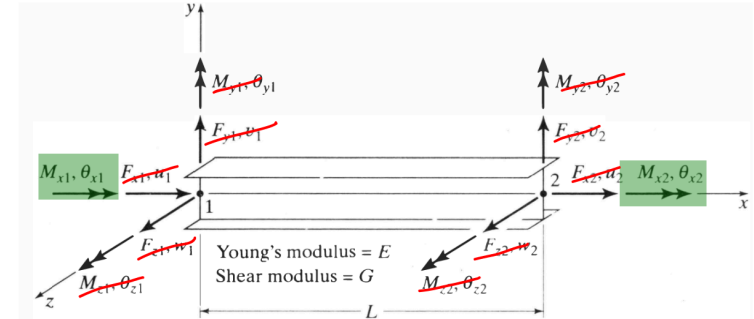
1. **Axial** deformation (truss-like)
2. **Torsion** about the member axis
3. **Bending** about one principal axis (Z - axis)
4. **Bending** about the other principal axis (Y - Axis)

The Four Behaviors

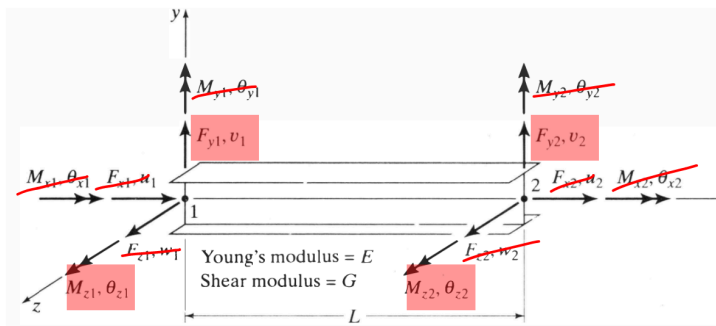
(1) Axial



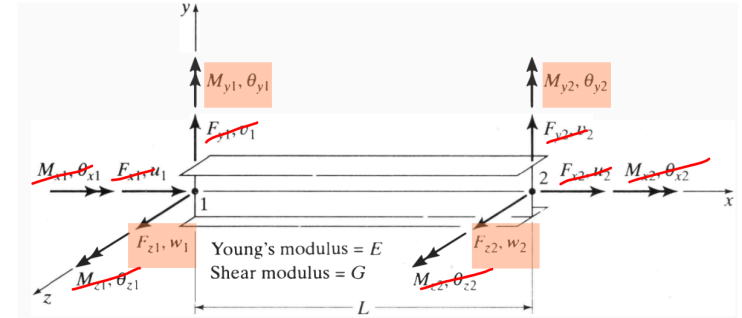
(2) Torsion



(3) Bending about Z-axis



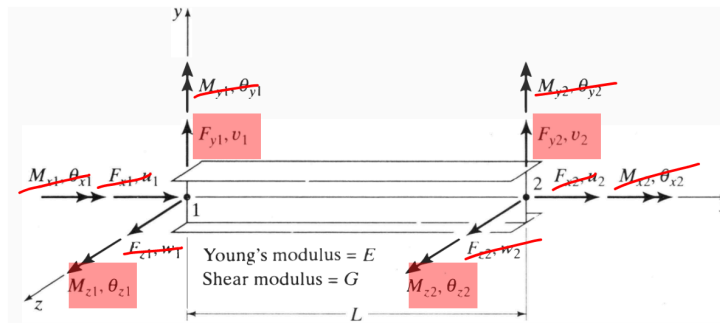
(4) Bending about Y-axis



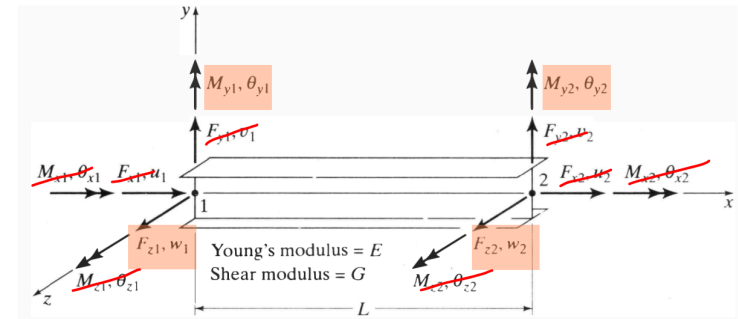
What We Derived Last Lecture

- The **2D beam element** governing planar bending behavior, case (3) and (4)
- Once derived for one principal axis, the formulation extends directly to the other (identical mathematics, different axis)

(3) Bending about Z-axis

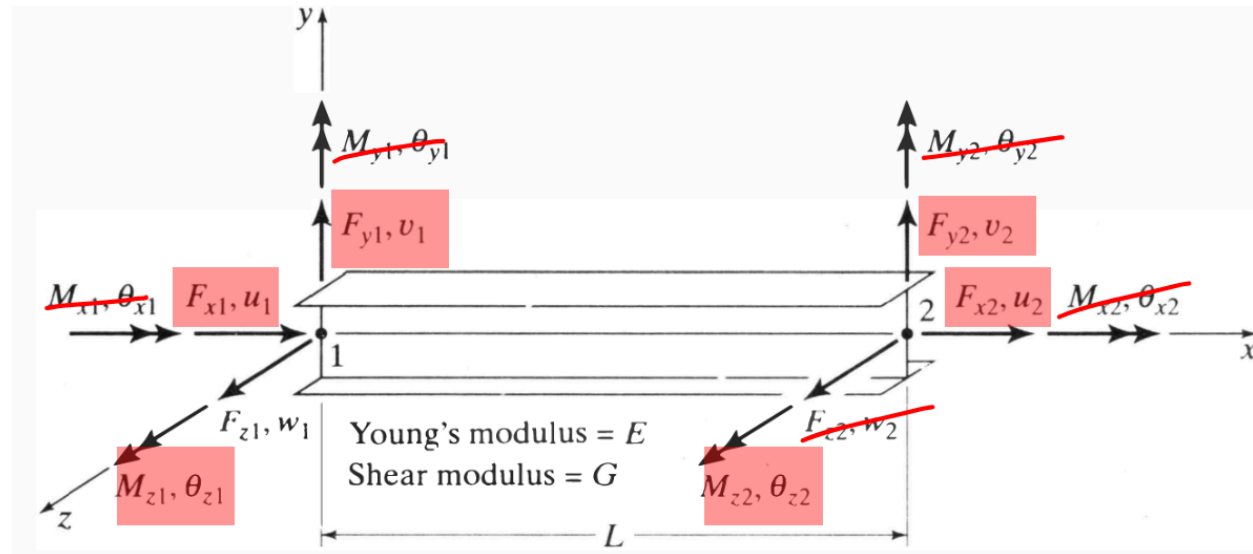


(4) Bending about Y-axis



What We Will Derive Today

- The DSM formulation for the **2D frame element**
- A combination of bending + axial behaviour **bending about the Z -axis**, case (1) + (3)
- Once derived for one principal axis, the formulation extends directly to the other (identical mathematics, different axis)



Part 2 — 2D Frame Analytical Model

*Representing a frame structure as **members** + **joints** for stiffness-based analysis.*

Key Conceptual Shift (Beam \rightarrow Frame)

The major difference from the pure beam formulation:

- We now include **axial deformation** in addition to bending.
- Because axial effects are included, a **frame element may be inclined** at any angle.
- A beam element (in its simplest form) was required to align with the **global X-axis**.
- A frame element therefore requires a **local coordinate system** and transformation to global coordinates.

The 2D frame element is a true generalization of the beam element.

Frame Definition (2D Idealization)

A 2D frame member is modeled as:

- A long, straight prismatic member
- Loaded in a single plane (e.g., the XY -plane)

We adopt the **Euler–Bernoulli beam assumptions**:

- Cross-sections remain **plane**
- Cross-sections remain **perpendicular to the centerline**
- **Shear deformation is neglected**

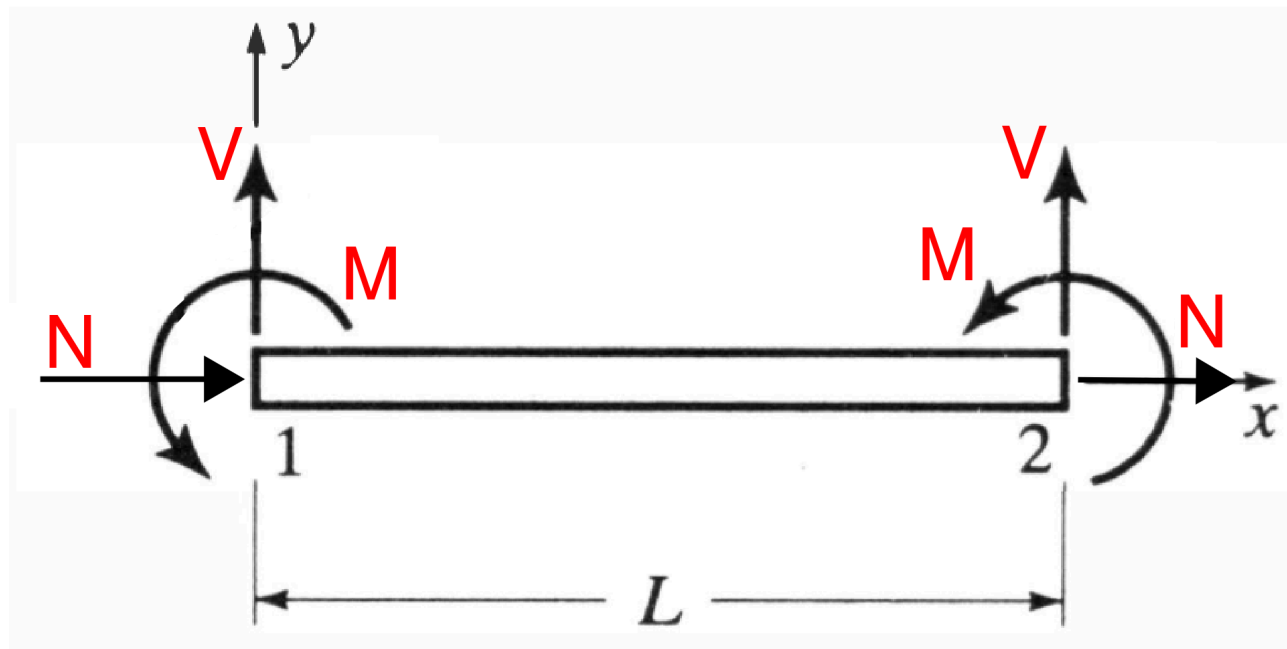
Additionally (for the frame formulation):

- **Axial deformation is included**

Consequences of This Idealization

Each member can develop:

- Axial force N : **Parallel** to the centroidal axis (axial)
- Shear force V : **Perpendicular** to the centroidal axis (shear)
- Bending moment M : In the XY -plane (about the Z -axis, into the page)



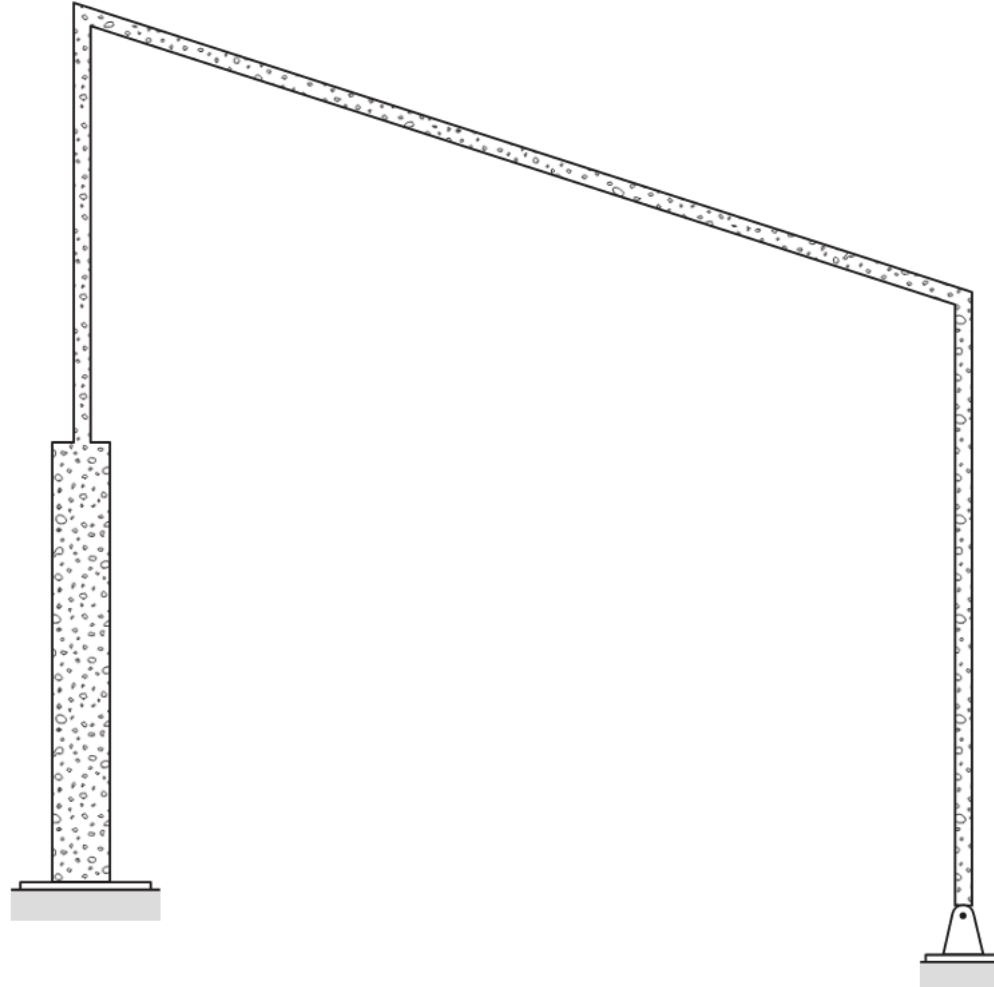
Analytical Model for the DSM

To apply the DSM, we idealize the structure as:

- Straight, prismatic members (constant A , E , I)
- Connected at discrete **joints (nodes)**
- Rigid joint connections (for now)
- Unknown reactions acting **only at joints**
- External loads applied anywhere along members (converted to FEFs)

This converts a continuous structural system into a **finite system of degrees of freedom** suitable for matrix analysis.

Example Frame Structure

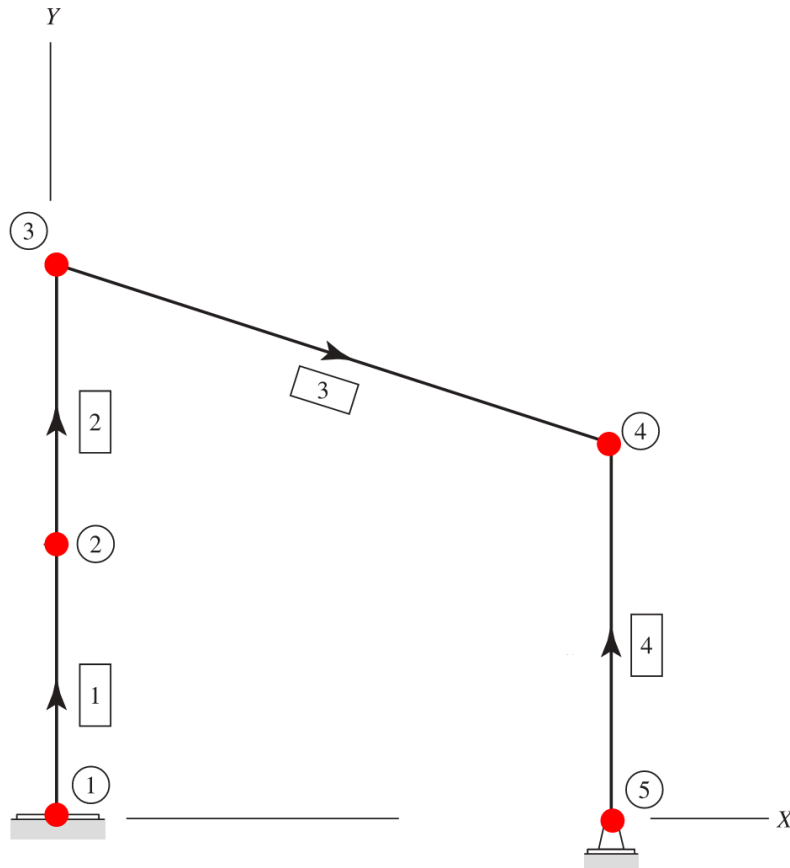


Discretize into Members and Joints

Even if the structure is physically continuous, we often insert joints so that:

- **support reactions** occur at joints (not mid-member)
- each member has **constant properties** (e.g., constant EI)
- **member loads** can be easily turned into Fixed-End Forces (more in Lecture 6.3)

Example Frame Discretization



Joints

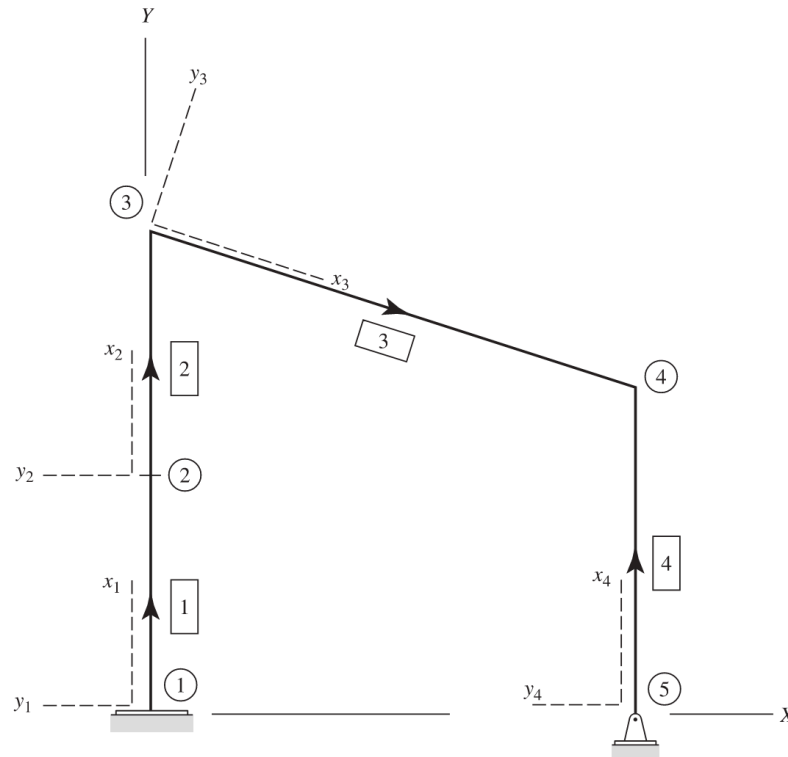
- **Joint 1:** Fixed support
- **Joint 2:** Free, Δ in EI
- **Joint 3:** Free
- **Joint 4:** Free
- **Joint 5:** Pin

Elements

- **Element 1:** Joint 1 \rightarrow Joint 2
- **Element 2:** Joint 2 \rightarrow Joint 3
- **Element 3:** Joint 3 \rightarrow Joint 4
- **Element 4:** Joint 5 \rightarrow Joint 4

Part 3 — Coordinate Systems

Global and Local Coordinate Systems



Global

- X axis along the beam (positive to the right)
- Y axis vertical (positive upward)
- loads and reactions lie in the X - Y plane

Local

- origin at the **left end** of the member
- x axis along the member centroidal axis
- y axis vertical (positive upward)

Part 4 — Numbering DOFs, Loads, Reactions

DOFs per Joint (2D Frame Model)

So each joint can have up to **three** types of deformations:

1. horizontal translation: u (along global X)
2. vertical translation: u (along global Y)
3. rotation: θ (about global Z)

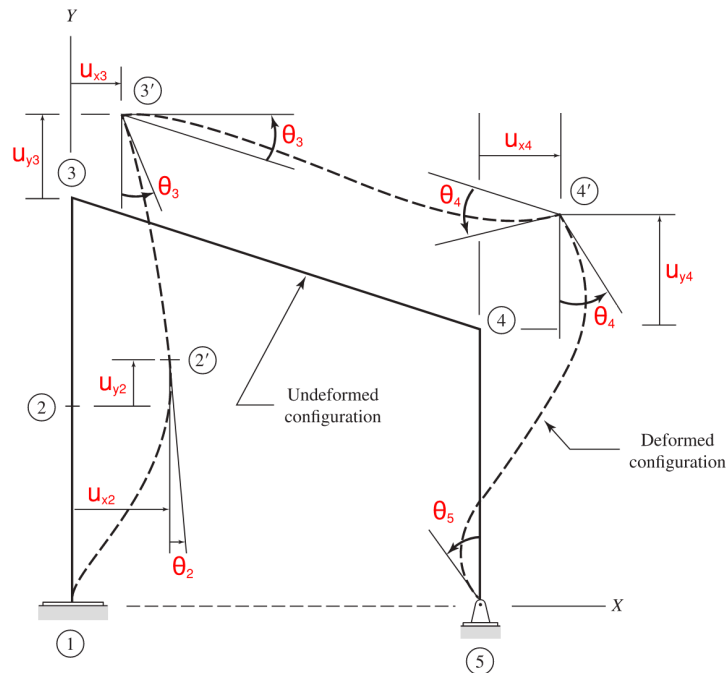
At a joint, i , we collect beam DOFs as:

$$\mathbf{u}_i = \begin{Bmatrix} u_{x,i} \\ u_{y,i} \\ \theta_i \end{Bmatrix} = \begin{Bmatrix} u_{1,i} \\ u_{2,i} \\ u_{3,i} \end{Bmatrix}$$

Sign conventions:

- u_x is positive **right**
- u_y is positive **up**
- θ is positive **counterclockwise**

Example: Deformed Shape, Showing u and θ



From the support types:

- node 1: fixed support
- node 2: free
- node 3: free
- node 4: free
- node 5: pin

Counting Degrees of Freedom

For a frame, each free joint has two translational and one rotational DOFs:

$$N_{\text{CJT}} = 3 \quad (u_x, u_y, \theta)$$

So:

$$\boxed{N_{\text{DOF}} = 3j - r}$$

This is the number of **independent joint displacements** that must be solved for.

DOF Numbering Convention (Recommended)

We adopt a **systematic, equation-based numbering scheme**:

- Number joints sequentially
- At each joint:
 1. number horizontal displacement u_x first
 2. number vertical displacement u_y next
 3. then number rotation θ
- Continue this pattern for all joints (regardless of support type)

With this convention (1-based indexing):

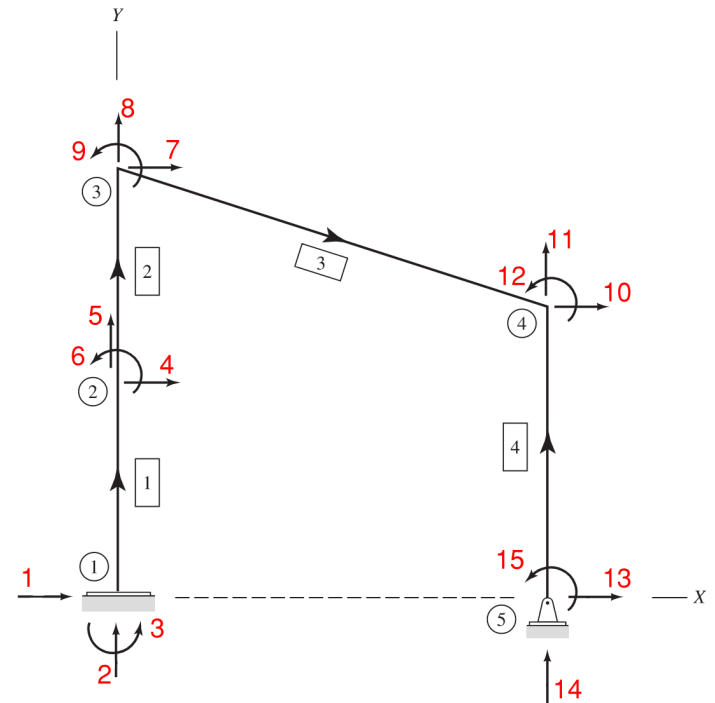
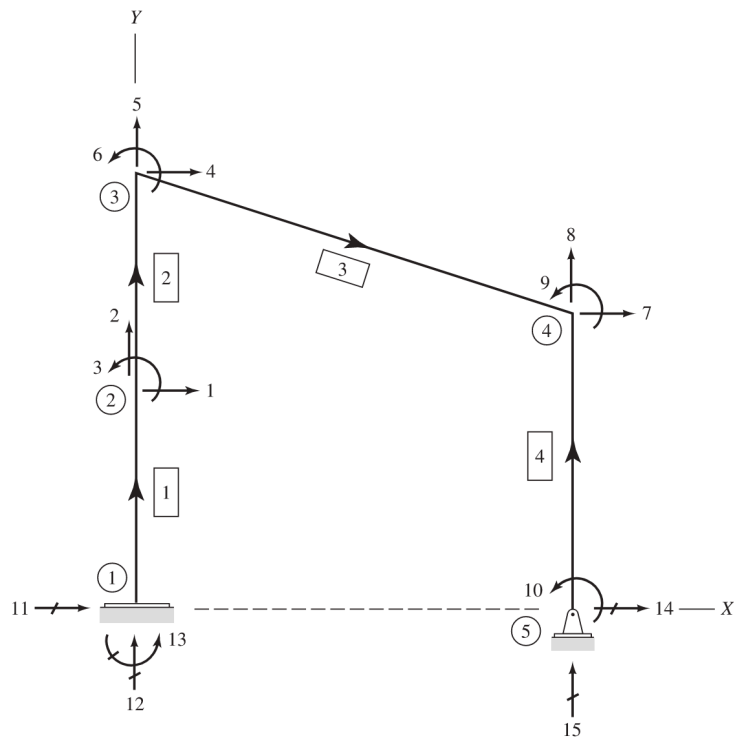
$$\text{DOF}(j, u_x) = 3j - 2$$

$$\text{DOF}(j, u_y) = 3j - 1$$

$$\text{DOF}(j, \theta) = 3j$$

This produces a fully predictable mapping between joint index and DOF number.

Textbook vs. Our Numbering



Joint Loads vs Member Loads

- **Joint loads:** forces/moments applied at joints
- **Member loads:** loads applied between joints (distributed loads, point loads on a span, a couple, etc.)

Here, the term **load** is used in a broad sense to mean either a **force** or a **moment**, applied in the direction of a DOF.

Loads and Reactions Correspond to Global Beam DOF Numbering

Each beam joint has up to three DOFs (u_x, u_y, θ).

To every DOF, there corresponds a **generalized load**.

- If the DOF is **free** → an external load may be applied
- If the DOF is **restrained** → an unknown reaction will develop

For a 2D frame element:

- $u_x \longleftrightarrow$ Axial force F_x (horizontal force)
- $u_y \longleftrightarrow$ Shear force F_y (vertical force)
- $\theta_z \longleftrightarrow$ Bending moment M_z

Part 5 — Putting It All Together

Global System Equation

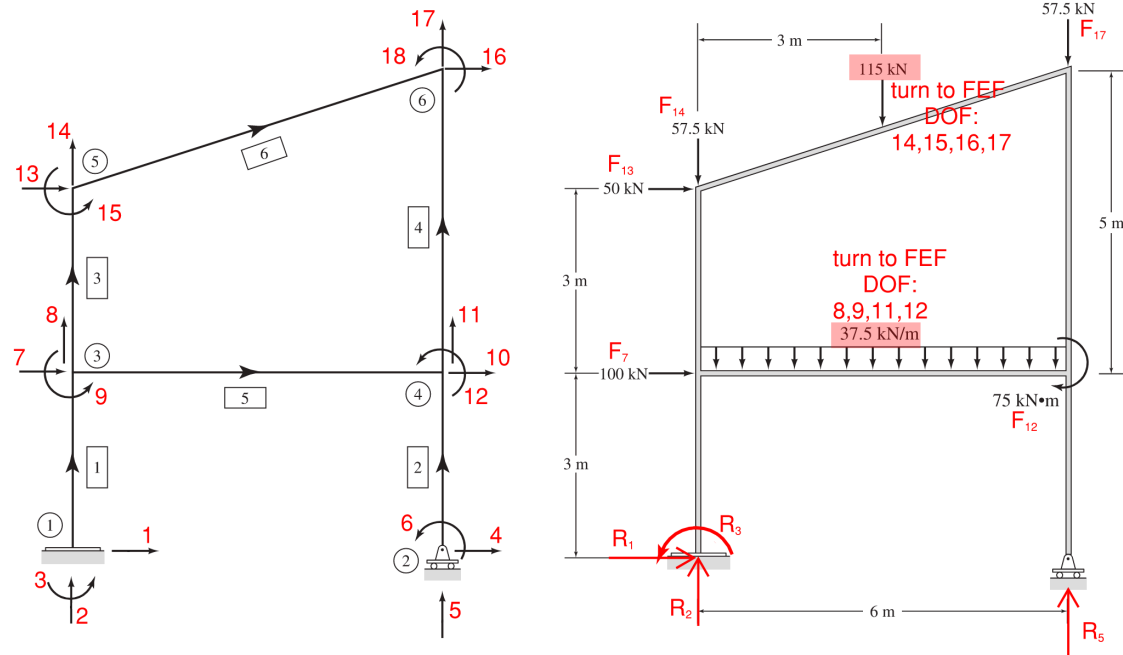
Once the beam is discretized and DOFs are numbered:

$$\mathbf{K}\mathbf{u} = \mathbf{f} - \mathbf{f}^F$$

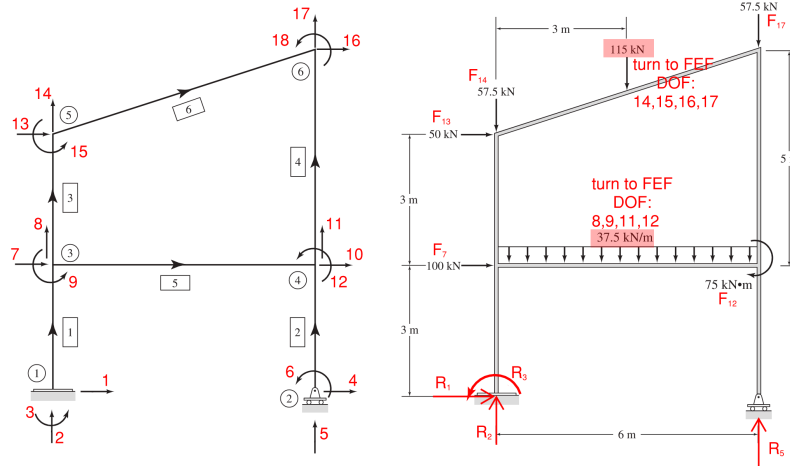
- \mathbf{u} collects joint translations and rotations at the selected coordinates
- \mathbf{f} collects the corresponding joint forces and moments
- \mathbf{f}^F collects the corresponding fixed end joint forces and moments
- \mathbf{K} comes from **assembling beam element stiffness matrices**

Example Structure

A frame with 6 joints (3 DOFs per joint)



For FEF forces, look at standard cases: (1) distributed load, (2) point load in center of span



$$\mathbf{K}_{18 \times 18} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \\ u_{17} \\ u_{18} \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ 0 \\ R_5 \\ 0 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ -75 \\ 50 \\ -57.5 \\ 0 \\ 0 \\ -57.5 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F_8^F \\ F_9^F \\ 0 \\ F_{11}^F \\ F_{12}^F \\ 0 \\ F_{14}^F \\ F_{15}^F \\ 0 \\ F_{17}^F \\ F_{18}^F \end{Bmatrix}$$

Part 6 — 2D Frame Element Stiffness Relations

Local Behavior

Frame Element Response

The **member stiffness relations** express the end forces of a beam element as functions of the **end displacements**

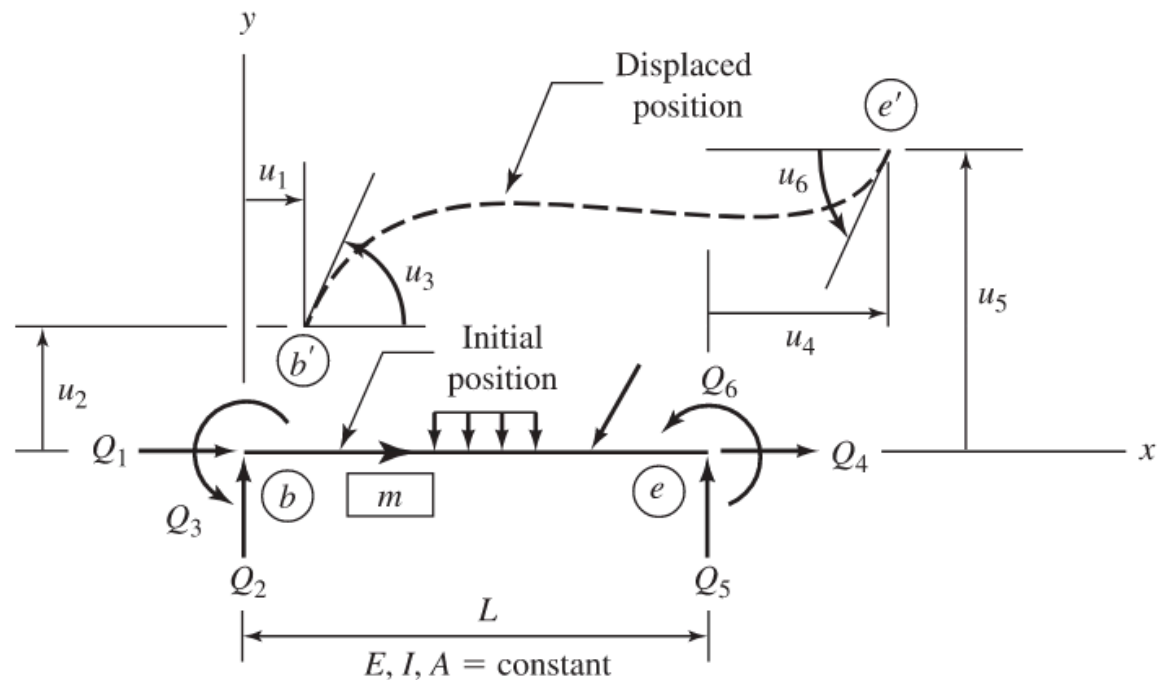
When a beam element is subjected to external loading:

- **Axial forces** develop at the ends
- **Shear forces** develop at the ends
- **Bending moments** are induced at the ends

These loads are fully determined by the **displacements and rotations at the element ends**.

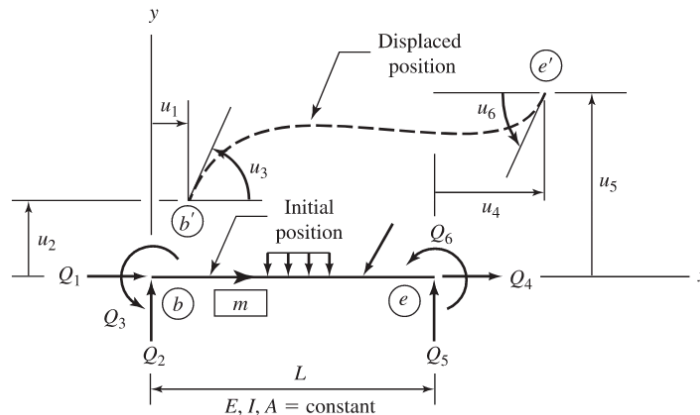
Generic Displacement for a 2D Frame Element

We define all quantities in the **local coordinate system**, with origin at the left end b , and ending at node e .



2D Beam Element DOF Numbering

Degrees of freedom are ordered from **left** → **right** node



- **DOF 1:** u_1 — node b , local x
- **DOF 2:** u_2 — node b , local y
- **DOF 3:** u_3 — node b , θ
- **DOF 4:** u_4 — node e , local x
- **DOF 5:** u_5 — node e , local y
- **DOF 6:** u_6 — node e , θ

Sign conventions:

- $+u_x \rightarrow$ right (local x direction)
- $+u_y \rightarrow$ up (local y direction)
- $+\theta \rightarrow$ counterclockwise
- Forces follow the same order and convention

Relating Local Displacement and Force

$$\mathbf{Q} = \mathbf{k} \mathbf{u} + \mathbf{Q}_f$$

- \mathbf{Q} denotes the 6×1 member end-force vector
- \mathbf{u} denote the 6×1 member end-displacement vectors
- \mathbf{k} represents the 6×6 member local stiffness matrix
- \mathbf{Q}_f is the 6×1 member fixed-end force vector in the local coordinate system.

6 Linear Equations (one per DOF)

For a linear elastic element, the force at any DOF is a **linear combination** of the force induced by all DOF displacements:

- displacing one DOF can induce forces at *all* DOFs
- the proportionality constants are the stiffness coefficients k_{ij}

Each equation expresses **force equilibrium at a single local degree of freedom**.

For example DOF 1:

$$Q_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4 + Q_{f,1}$$

Rather than writing out all 6 equations:

$$Q_i = \sum_{j=1}^6 (k_{ij} u_j) + Q_{fi}, \quad i = 1, 2, \dots, 6$$

Same Equations in Matrix Form

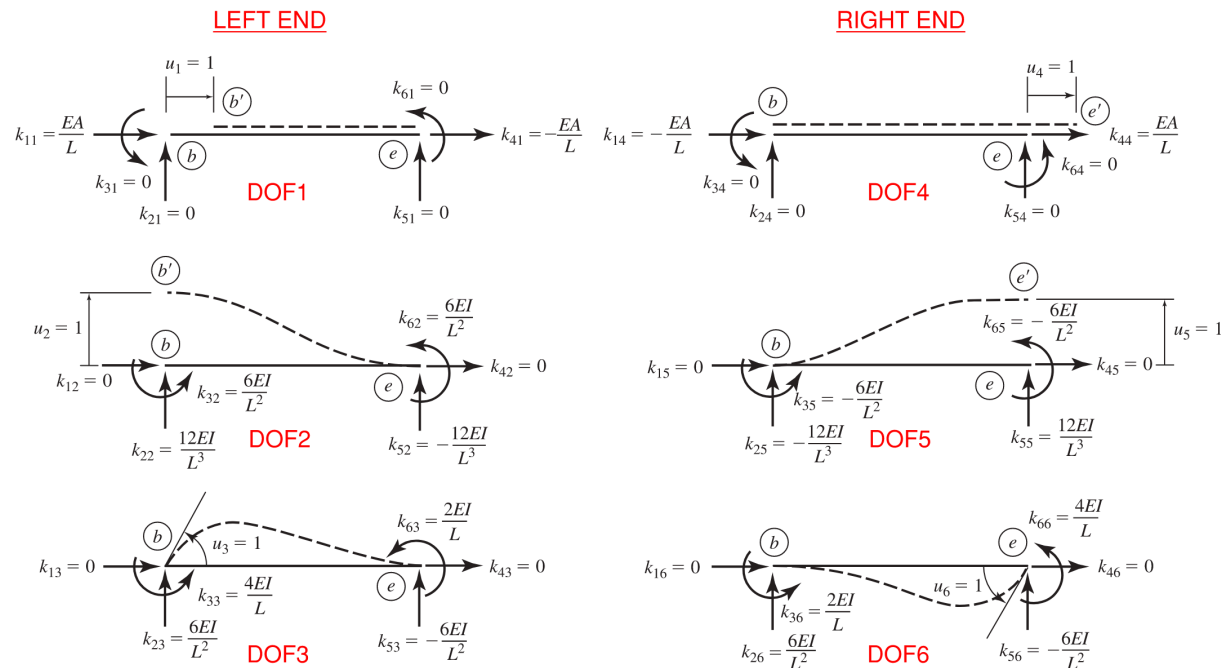
$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} + \begin{Bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \\ Q_{f5} \\ Q_{f6} \end{Bmatrix}$$

Part 6 - 2D Frame Element Stiffness Matrix

Unit Displacement Method

You can derive 2D frame element from scratch using unit displacement method.

k_{ij} = force at DOF i due to a unit displacement at DOF j ,
with all other DOFs fixed.



Combination of Truss and Beam Stiffnesses

We have already derived the **axial (truss)** and **bending (beam)** stiffness matrices.

For a 2D frame element (local DOF ordering $(u_1, v_1, \theta_1, u_2, v_2, \theta_2)$):

- Axial effects act in **DOFs 1 and 4**
- Bending effects act in **DOFs 2, 3, 5, 6**
- In local coordinates, these effects are **decoupled**

Therefore, the frame stiffness matrix is obtained by **superposing** the two contributions:

Axial + Bending Stiffness

$$\mathbf{k}_{\text{frame}} = \underbrace{\frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{axial}} + \underbrace{\frac{EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}}_{\text{bending}}$$

2D Frame Stiffness Matrix

$$\mathbf{k}_{\text{frame}} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

2D Frame Stiffness Matrix (Factored)

$$\mathbf{k}_{\text{frame}} = \frac{EI}{L^3} \begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}$$

Test

Wrap-Up

In this lecture, we:

Next lecture: