

# CEE6501 — Lecture 6.2

## 2D Beam Element Stiffness Matrix

# Learning Objectives

By the end of this lecture, you will be able to:

- Define the **local DOFs**, ordering, and sign conventions for a 2D Euler–Bernoulli beam element
- Write the **local displacement and force vectors** and interpret the relation  $\mathbf{Q} = \mathbf{k}\mathbf{u}$
- Use the **unit displacement method** to compute stiffness coefficients (and interpret each column physically)
- Assemble and recognize the **standard closed-form 4×4 beam stiffness matrix**
- Use **beam shape functions** to evaluate and plot a **continuous deformed shape** from nodal DOFs

# Agenda

1. **Part 1** — Beam Element Stiffness Relations
2. **Part 2** — Derive Column 1 of the Beam Element Stiffness Matrix
3. **Part 3** — Complete  $4 \times 4$  Beam Stiffness Matrix
4. **Part 4** — Beam Deformation Patterns

# Part 1 — Beam Element Stiffness Relations

## Beam Element Response

The **member stiffness relations** express the end forces of a beam element (including shear forces and bending moments) as functions of the **end displacements** (including transverse displacements and rotations).

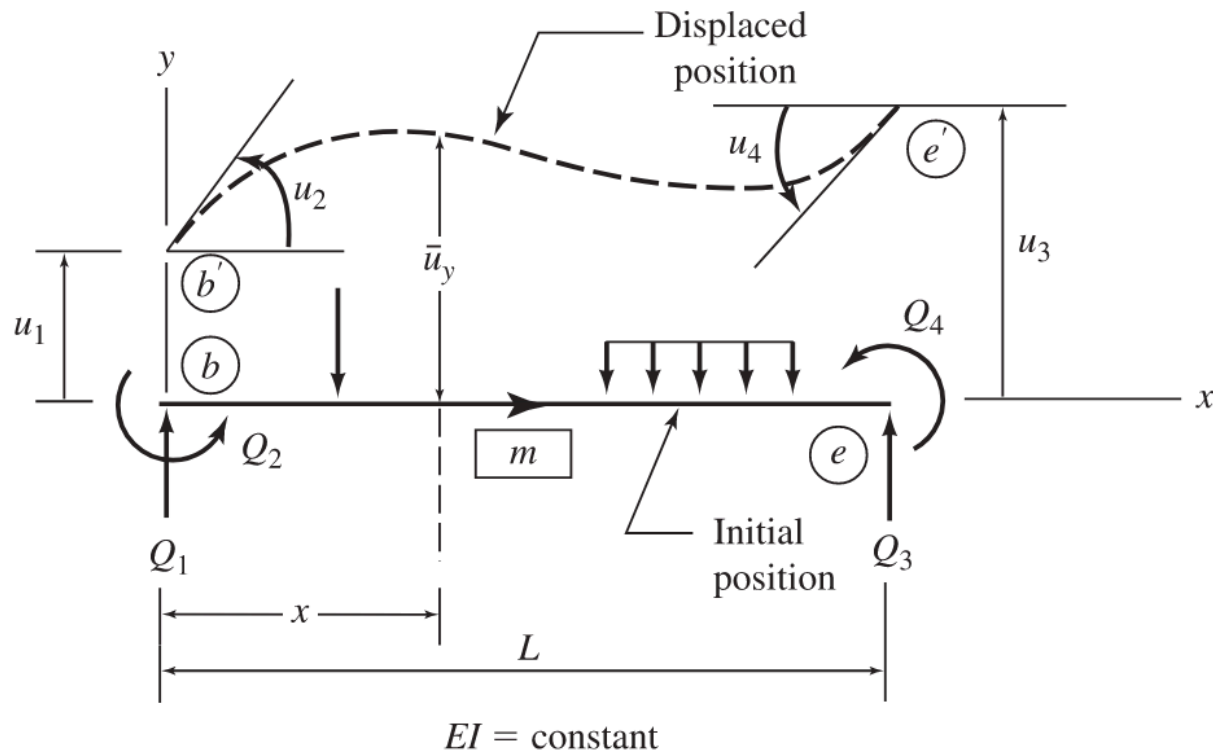
When a beam element is subjected to external loading:

- The member **deforms** (bending)
- **Internal shear forces** develop at the ends
- **Bending moments** are induced at the ends

These internal forces are fully determined by the **displacements and rotations at the element ends**.

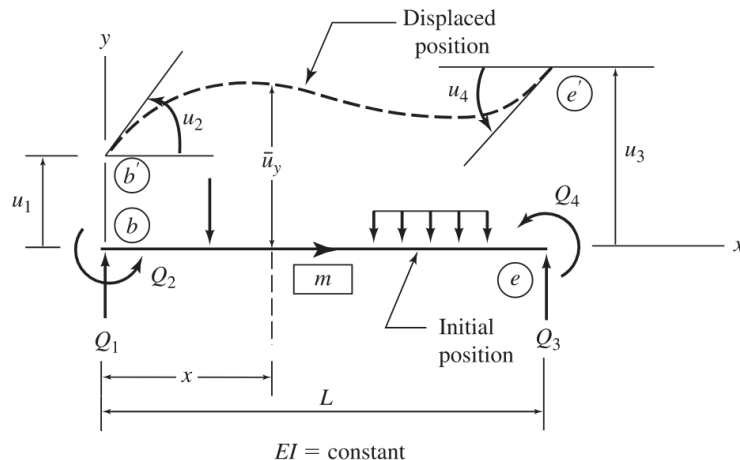
# Generic Displacement for a 2D Beam Element

We define all quantities in the **local coordinate system**, with origin at the left end  $b$ , and ending at node  $e$ .



## 2D Beam Element DOF Numbering

Degrees of freedom are ordered **left** → **right**, with **translation first**, then **rotation**.



- **DOF 1:**  $u_1$  — node  $b$ , local  $y$
- **DOF 2:**  $u_2$  — node  $b$ ,  $\theta$
- **DOF 3:**  $u_3$  — node  $e$ , local  $y$
- **DOF 4:**  $u_4$  — node  $e$ ,  $\theta$

### Sign conventions:

- $u > 0 \rightarrow$  upward (local  $y$  direction)
- $\theta > 0 \rightarrow$  counterclockwise
- Forces follow the same order:  $[V_b, M_b, V_e, M_e]^T$

# Local displacement and force vectors

Local displacement vector (including rotations):

$$\mathbf{u} = \begin{Bmatrix} u_b \\ \theta_b \\ u_e \\ \theta_e \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Local nodal force vector (including moments):

$$\mathbf{Q} = \begin{Bmatrix} V_b \\ M_b \\ V_e \\ M_e \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

We seek:

$$\mathbf{Q} = \mathbf{k} \mathbf{u}$$



## Four equations (one per DOF)

Recall: this is similar in spirit to the truss element definition.

$$Q_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4$$

$$Q_2 = k_{21}u_1 + k_{22}u_2 + k_{23}u_3 + k_{24}u_4$$

$$Q_3 = k_{31}u_1 + k_{32}u_2 + k_{33}u_3 + k_{34}u_4$$

$$Q_4 = k_{41}u_1 + k_{42}u_2 + k_{43}u_3 + k_{44}u_4$$

Each equation expresses **force equilibrium at a single local degree of freedom**.

For a linear elastic element, the force at any DOF is a **linear combination** of all DOF displacements:

- displacing one DOF can induce forces at *all* DOFs
- the proportionality constants are the stiffness coefficients  $k_{ij}$

Same equations in matrix form

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

## Part 2 — Derive Column 1 of the Beam Element Stiffness Matrix

## Unit Displacement Method

Definition:

$k_{ij}$  = force at DOF  $i$  due to a unit displacement at DOF  $j$ ,  
with all other DOFs fixed.

Each column of  $\mathbf{k}$  is built by:

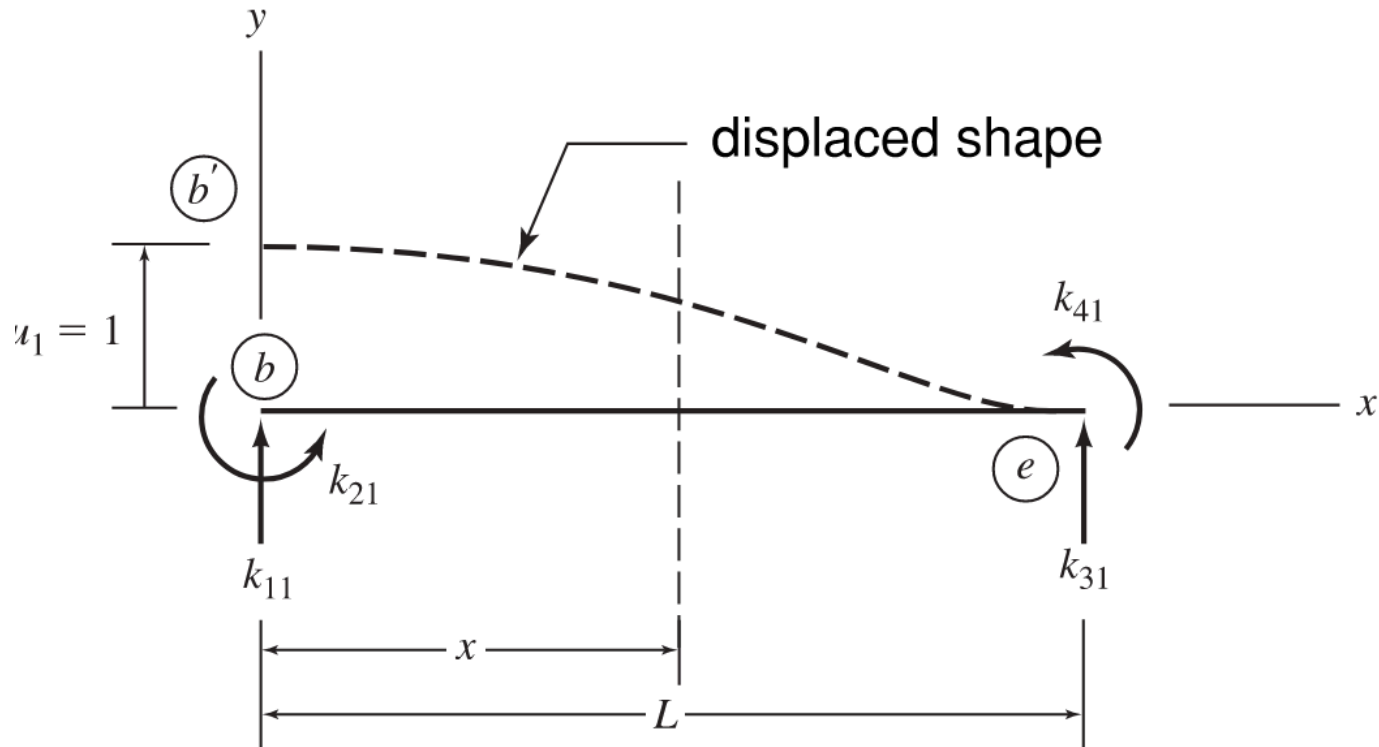
- impose a **unit displacement/rotation** at one DOF
- hold all other DOFs fixed
- record the resulting nodal force pattern

Column 1: impose  $u_1 = 1$  ( $u_2 = u_3 = u_4 = 0$ )

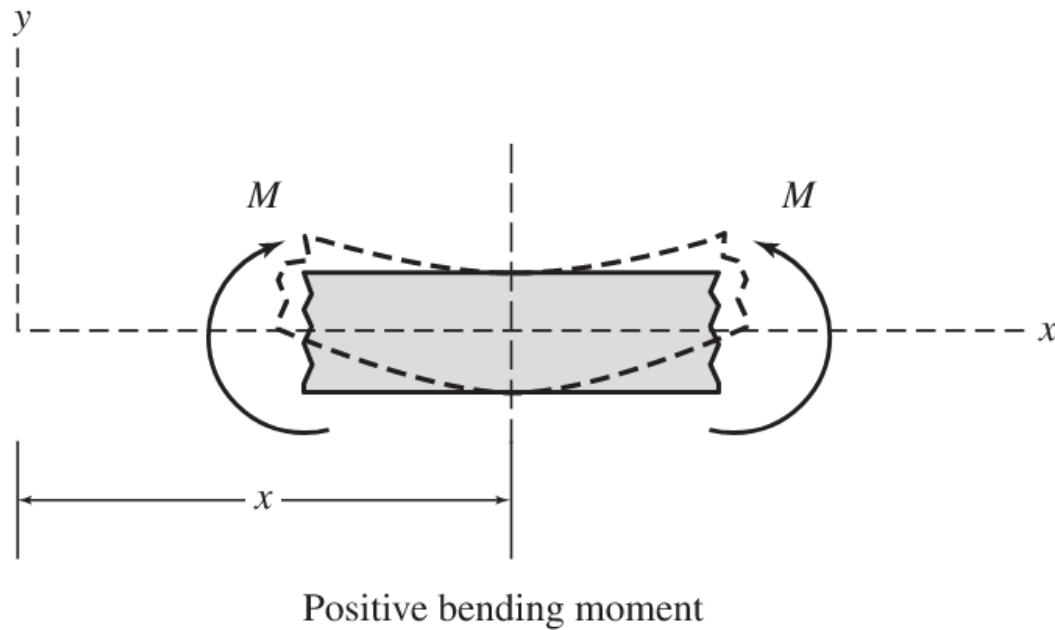
$$\begin{bmatrix} \boxed{k_{11}} & k_{12} & k_{13} & k_{14} \\ \boxed{k_{21}} & k_{22} & k_{23} & k_{24} \\ \boxed{k_{31}} & k_{32} & k_{33} & k_{34} \\ \boxed{k_{41}} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

- $k_{11}$ : force at DOF 1 due to unit displacement at DOF 1
- $k_{21}$ : force at DOF 2 due to unit displacement at DOF 1
- $k_{31}$ : force at DOF 3 due to unit displacement at DOF 1
- $k_{41}$ : force at DOF 4 due to unit displacement at DOF 1

Column 1: impose  $u_1 = 1$  ( $u_2 = u_3 = u_4 = 0$ )



## Sign Convention for Derivation



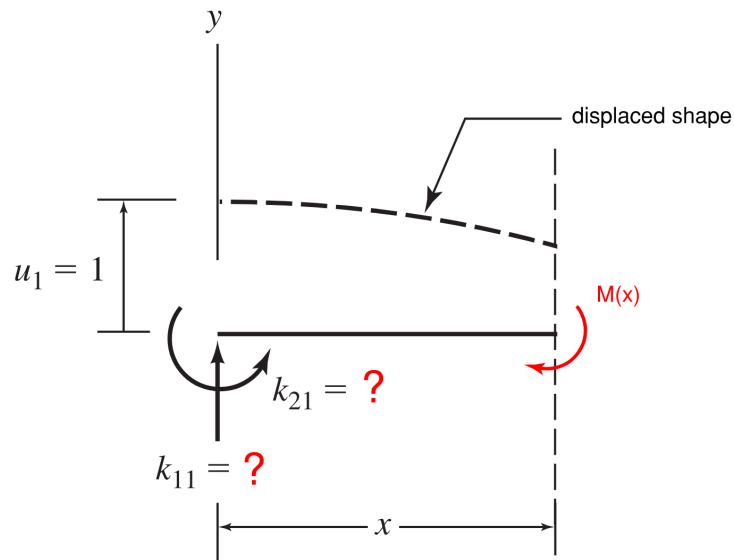
## Governing Equation (Beam Bending)

$$\frac{d^2u}{dx^2} = \frac{M(x)}{EI} \quad (1)$$

- $u(x)$  = transverse displacement
- $M(x)$  = bending moment
- $EI$  = flexural rigidity



## Step 1 — Express Internal Moment as a Function of $x$



Cut the beam at a distance  $x$  from node  $b$ .

Using equilibrium:

$$M(x) = -k_{21} + k_{11}x \quad (2)$$

- $k_{21}$ : end moment at node  $b$
- $k_{11}$ : end shear at node  $b$

## Step 2 — Substitute into Governing Equation

Substitute Eq. (2) into the **moment term** of Eq. (1):

$$\frac{d^2u}{dx^2} = \frac{M(x)}{EI} \xrightarrow{\text{Eq. (2)}} \frac{d^2u}{dx^2} = \frac{1}{EI}(-k_{21} + k_{11}x) \quad (3)$$

## Step 3 — Integrate to Obtain Rotation $\theta(x)$

Start from Eq. (3):

$$\frac{d^2u}{dx^2} = \frac{1}{EI}(-k_{21} + k_{11}x) \quad (3)$$

Integrate both sides with respect to  $x$ :

$$\int \frac{d^2u}{dx^2} dx = \int \frac{1}{EI}(-k_{21} + k_{11}x) dx \quad (4a)$$

Left-hand side:

$$\int \frac{d^2u}{dx^2} dx = \frac{du}{dx} = \theta(x) \quad (4b)$$

Right-hand side (term-by-term):

$$\int \frac{1}{EI} (-k_{21} + k_{11}x) dx = \frac{1}{EI} \left( \int -k_{21} dx + \int k_{11}x dx \right) \quad (4c)$$

Compute:

$$\int -k_{21} dx = -k_{21}x \quad \int k_{11}x dx = \frac{k_{11}}{2}x^2 \quad (4d)$$

Include integration constant  $C_1$ :

$$\theta(x) = \frac{du}{dx} = \frac{1}{EI} \left( -k_{21}x + \frac{k_{11}}{2}x^2 \right) + C_1 \quad (4)$$

## Step 4 — Integrate to Obtain Displacement $u(x)$

Start from Eq. (4):

$$\frac{du}{dx} = \frac{1}{EI} \left( -k_{21}x + \frac{k_{11}}{2}x^2 \right) + C_1 \quad (4)$$

Integrate both sides with respect to  $x$ :

$$\int \frac{du}{dx} dx = \int \left[ \frac{1}{EI} \left( -k_{21}x + \frac{k_{11}}{2}x^2 \right) + C_1 \right] dx \quad (5a)$$

Left-hand side:

$$\int \frac{du}{dx} dx = u(x) \quad (5b)$$

Right-hand side (compact form):

$$u(x) = \frac{1}{EI} \left( \int -k_{21}x \, dx + \int \frac{k_{11}}{2}x^2 \, dx \right) + \int C_1 \, dx \quad (5c)$$

Compute:

$$\int -k_{21}x \, dx = -\frac{k_{21}}{2}x^2 \quad \int \frac{k_{11}}{2}x^2 \, dx = \frac{k_{11}}{6}x^3 \quad \int C_1 \, dx = C_1x \quad (5d)$$

Include integration constant  $C_2$ :

$$u(x) = \frac{1}{EI} \left( -\frac{k_{21}}{2}x^2 + \frac{k_{11}}{6}x^3 \right) + C_1x + C_2 \quad (5)$$

## Step 5 — Apply Boundary Conditions

We enforce **unit displacement at node  $b$**  (0 elsewhere):

At  $x = 0$ :

$$\theta(0) = 0$$

$$u(0) = 1$$

At  $x = L$ :

$$\theta(L) = 0$$

$$u(L) = 0$$

## Step 6 — Solve for Constants

Substitute  $x = 0$  into Eqs. (4)–(5):

- From  $\theta(0) = 0$ : all terms with  $x$  vanish  $\Rightarrow C_1 = 0$
- From  $u(0) = 1$ : all terms with  $x$  vanish  $\Rightarrow C_2 = 1$

Updated Eqs. (4)–(5):

$$\theta(x) = \frac{1}{EI} \left( -k_{21}x + \frac{k_{11}}{2}x^2 \right) \quad (6)$$

$$u(x) = \frac{1}{EI} \left( -\frac{k_{21}}{2}x^2 + \frac{k_{11}}{6}x^3 \right) + 1 \quad (7)$$



## Step 7 — Apply Boundary Condition at $x = L$

Substitute  $\theta(L) = 0$  into Eq. (6):

$$0 = \frac{1}{EI} \left( -k_{21}L + \frac{k_{11}}{2}L^2 \right)$$

Cancel  $EI$  and solve:

$$k_{21} = \frac{k_{11}L}{2} \quad (9)$$

## Step 8 — Apply Final Condition

Substitute  $u(L) = 0$  into Eq. (7):

$$0 = \frac{1}{EI} \left( -\frac{k_{21}}{2} L^2 + \frac{k_{11}}{6} L^3 \right) + 1$$

Substitute  $k_{21} = \frac{k_{11}L}{2}$  and solve:

$$\boxed{k_{11} = \frac{12EI}{L^3}}$$

## Step 9 — Solve for $k_{21}$

Back-substitute  $k_{11}$  into Eq. (9):

$$k_{21} = \frac{6EI}{L^2}$$

## Step 10 — Solve Remaining Stiffness Terms $k_{31}$ and $k_{41}$

Use equilibrium of the beam element free-body diagram (with  $k_{11}$  and  $k_{21}$  known).

### Vertical force equilibrium

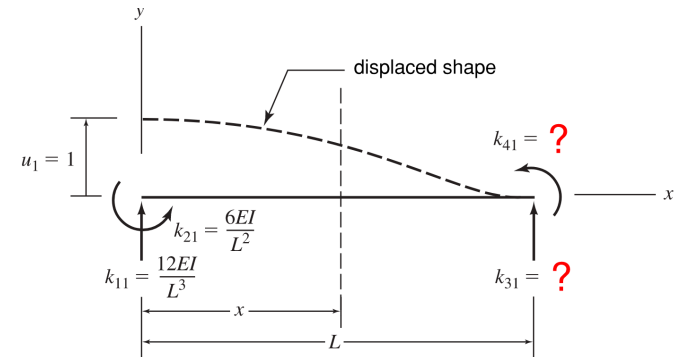
$$\sum F_y = 0 : \quad \frac{12EI}{L^3} + k_{31} = 0$$

$$\boxed{k_{31} = -\frac{12EI}{L^3}}$$

### Moment equilibrium about node $b$

$$\sum M_b = 0 : \quad \frac{6EI}{L^2} - \left( \frac{12EI}{L^3} \right) L + k_{41} = 0$$

$$\boxed{k_{41} = \frac{6EI}{L^2}}$$



Free-body diagram used for  $\sum F_y = 0$  and  $\sum M_b = 0$ .

## Step 11 (Optional) — Displacement Shape Function

Substitute  $k_{11}$  and  $k_{21}$  into Eq. (7):

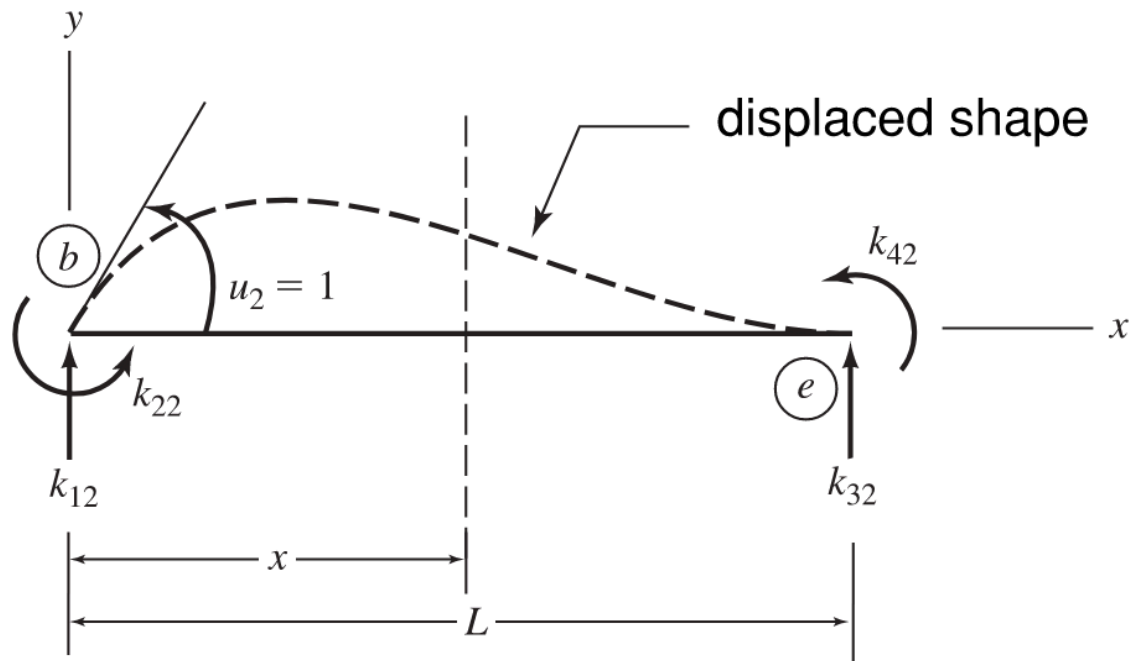
$$u(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

## Column 1 of the 4×4 Stiffness Matrix

$$\mathbf{k} = \begin{bmatrix} \boxed{\frac{12EI}{L^3}} & k_{12} & k_{13} & k_{14} \\ \boxed{\frac{6EI}{L^2}} & k_{22} & k_{23} & k_{24} \\ \boxed{-\frac{12EI}{L^3}} & k_{32} & k_{33} & k_{34} \\ \boxed{\frac{6EI}{L^2}} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

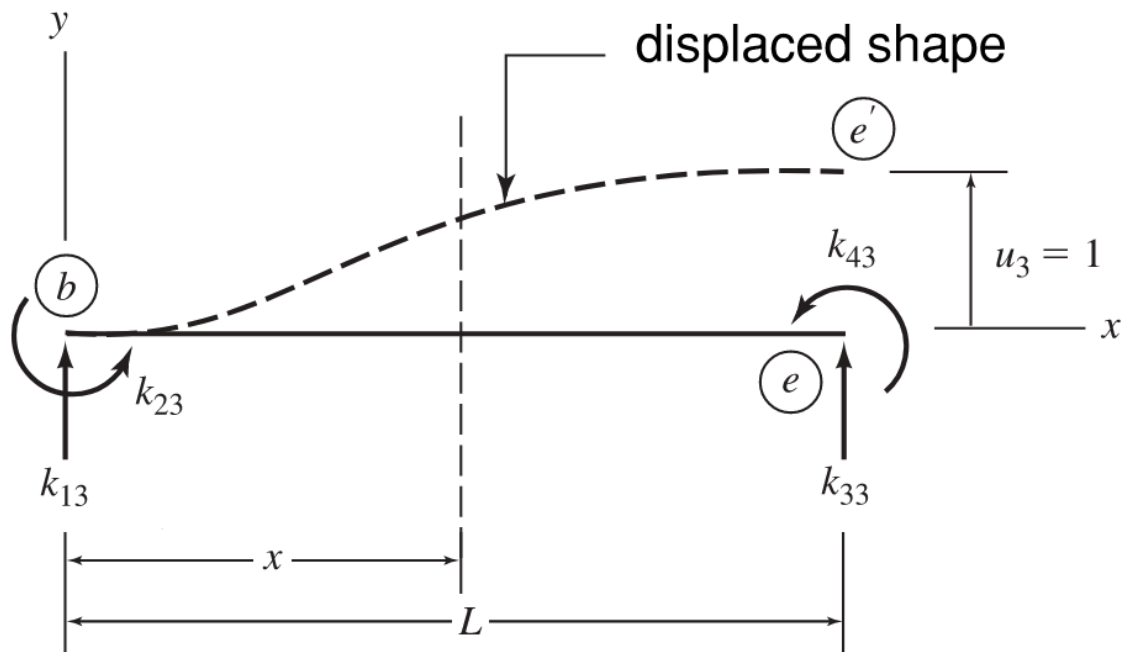
## Part 3 — Complete $4 \times 4$ Beam Stiffness Matrix

Column 2: impose  $u_2 = 1$  ( $u_1 = u_3 = u_4 = 0$ )

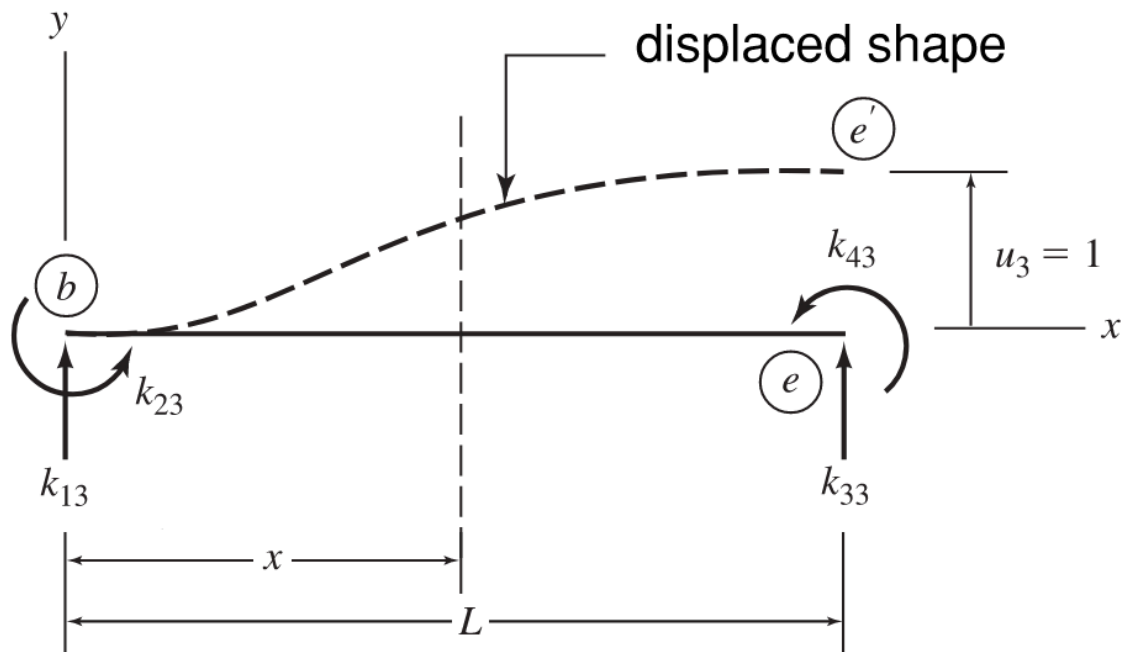




Column 3: impose  $u_3 = 1$  ( $u_1 = u_2 = u_4 = 0$ )



Column 3: impose  $u_3 = 1$  ( $u_1 = u_2 = u_4 = 0$ )



## Complete 4×4 Beam Element Stiffness Matrix

$$\mathbf{k} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

# Part 4 - Beam Deformation Patterns

## Displacement Field

A 2D beam element has **4 DOFs**, so the displacement field is written as a superposition:

$$u(x) = N_1(x) u_1 + N_2(x) u_2 + N_3(x) u_3 + N_4(x) u_4$$

Here,  $u(x)$  represents the **transverse displacement of the beam in the local  $y$ -direction**.

Each  $N_i(x)$  is a **shape function** (a deformation pattern) associated with one DOF:

- $N_1(x)$ : unit transverse displacement at node  $b$
- $N_2(x)$ : unit rotation at node  $b$
- $N_3(x)$ : unit transverse displacement at node  $e$
- $N_4(x)$ : unit rotation at node  $e$

## The 4 Shape Functions

We derived  $N_1$  earlier:

$$N_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

The shape functions for the other DOFs:

$$N_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2}$$

$$N_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$N_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

## Connection to the Finite Element Method (FEM)

The shape functions we derived are exactly those used in the **Finite Element Method (FEM)** for beam elements.

- In FEM, the displacement field is approximated as:

$$u(x) = \sum_{i=1}^4 N_i(x) u_i$$

- The unknowns are the same **nodal DOFs**  $\{u_1, u_2, u_3, u_4\}$
- Shape functions provide a **continuous interpolation** within the element

In **CEE6501**, we focus on the **Direct Stiffness Method (DSM)** and derive the stiffness matrix using structural mechanics.

But for the FEM-based derivation of the beam stiffness matrix, see:

- **Kassimali**, Section 5.3



## Plotting a Deformed Shape

While we do not explicitly use shape functions in the **DSM formulation**, they are very useful for **visualizing structural response**.

Given element DOFs

$$\mathbf{u} = \{u_1, u_2, u_3, u_4\}^T,$$

we can evaluate the displacement field

$$u(x) = \sum_{i=1}^4 N_i(x) u_i$$

at many points along  $x \in [0, L]$  to obtain a smooth deformation curve.

## Why This Is Useful

- Visualize the **continuous beam deflection** (not just nodal values)
- Check **sign conventions** and DOF interpretation
- Construct smooth **deformed shapes for multi-element structures**

```

In [34]: import numpy as np
import matplotlib.pyplot as plt

def shape_functions(x, L):
    """
    Return the 4 Hermite beam shape functions evaluated at x.
    x can be a scalar or a numpy array.
    """
    xi = x / L
    N1 = 1 - 3*xi**2 + 2*xi**3
    N2 = L * (xi - 2*xi**2 + xi**3)
    N3 = 3*xi**2 - 2*xi**3
    N4 = L * (-xi**2 + xi**3)
    return N1, N2, N3, N4

def beam_deflection(x, L, u1, u2, u3, u4):
    """
    Compute beam transverse deflection u(x) from nodal DOFs.
    DOFs: u1 (disp at b), u2 (rot at b), u3 (disp at e), u4 (rot at e)
    """
    N1, N2, N3, N4 = shape_functions(x, L)
    return N1*u1 + N2*u2 + N3*u3 + N4*u4

```

```

In [35]: def plot_beam(L, u1, u2, u3, u4, n=200):
        """
        Plot beam deformed shape given nodal DOFs.
        """
        x = np.linspace(0, L, n)
        u = beam_deflection(x, L, u1, u2, u3, u4)

        plt.figure(figsize=(6, 3)) # smaller plot

        # undeformed shape
        plt.plot(x, np.zeros_like(x), linestyle="--", label="undeformed")
        plt.scatter([0, L], [0, 0]) # nodes (no Legend)

        # deformed shape
        plt.plot(x, u, color="red", label="deformed")
        plt.scatter([0, L], [u1, u3]) # nodes (no Legend)

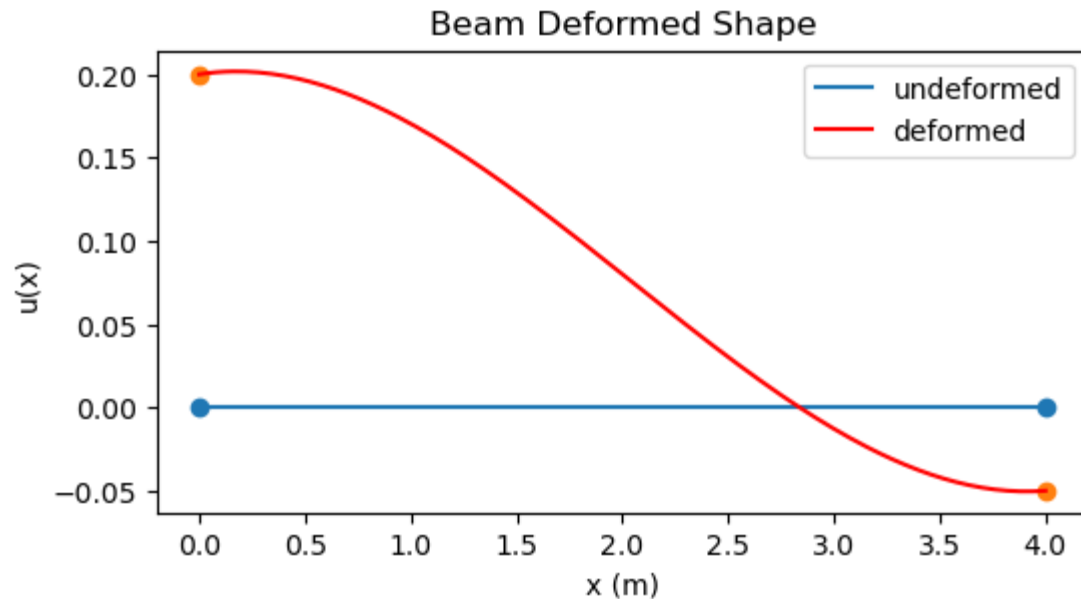
        plt.xlabel("x (m)")
        plt.ylabel("u(x)")
        plt.title("Beam Deformed Shape")
        plt.legend()
        plt.show()

```

```
In [36]: L = 4.0 # element length

# Example nodal DOFs (pick anything to test)
u1 = 0.2 # displacement at node b
u2 = 0.02 # rotation at node b (rad)
u3 = -0.05 # displacement at node e
u4 = 0.01 # rotation at node e (rad)

plot_beam(L, u1, u2, u3, u4)
```



# Wrap-Up

Today you:

- Defined local DOFs and the member stiffness relation for a 2D beam element
- Derived **Column 1** of the  $4 \times 4$  beam stiffness matrix using the unit displacement method
- Stated the final closed-form  $4 \times 4$  beam element stiffness matrix

**Next:** use this beam element inside **plane frames**, and introduce fixed-end forces for member loading.