

# CEE6501 — Lecture 3.1

## Local Behavior of an Axial Element

# Learning Objectives

By the end of this lecture, you will be able to:

- Define local degrees of freedom and sign conventions for a truss element
- Explain axial deformation and force–displacement behavior of an axial member
- Derive the local  $2 \times 2$  stiffness matrix for an axial element
- Extend the formulation to the local  $4 \times 4$  truss stiffness matrix
- Interpret key stiffness matrix properties, including symmetry and rigid-body motion

# Agenda

1. Introductory definitions & degrees of freedom
2. Planar trusses as a structural system
3. Truss stability & static determinacy
4. Global and local coordinate systems
5. Axial element kinematics (local behavior)
6. Deriving the local  $2 \times 2$  axial stiffness matrix
7. Deriving the local  $4 \times 4$  truss stiffness matrix
8. In-class exercise — local truss stiffness (Python)

# Part 1 — Introductory Definitions & Concepts

## Degrees of Freedom (DOFs)

A **degree of freedom (DOF)** is an **independent displacement component** used to describe a structure's motion.

The set of DOFs is the **minimum set of joint displacement components** needed to uniquely describe the deformed configuration under arbitrary loading.

## What DOFs Represent

In matrix analysis, DOFs are:

- the **unknown displacement components** we solve for
- the **locations/directions** where nodal loads are applied
- the coordinates used to describe the structure's deformation

Once DOFs are defined, the response is written compactly in matrix form.

## DOFs Depend on the Structural Model

The structural model determines **what motion is allowed**:

- **Trusses:** joint translations only
- **Frames:** joint translations **and rotations**
- **Higher-order models:** additional deformation modes

So “DOFs” are not universal — they depend on the assumptions built into the model.

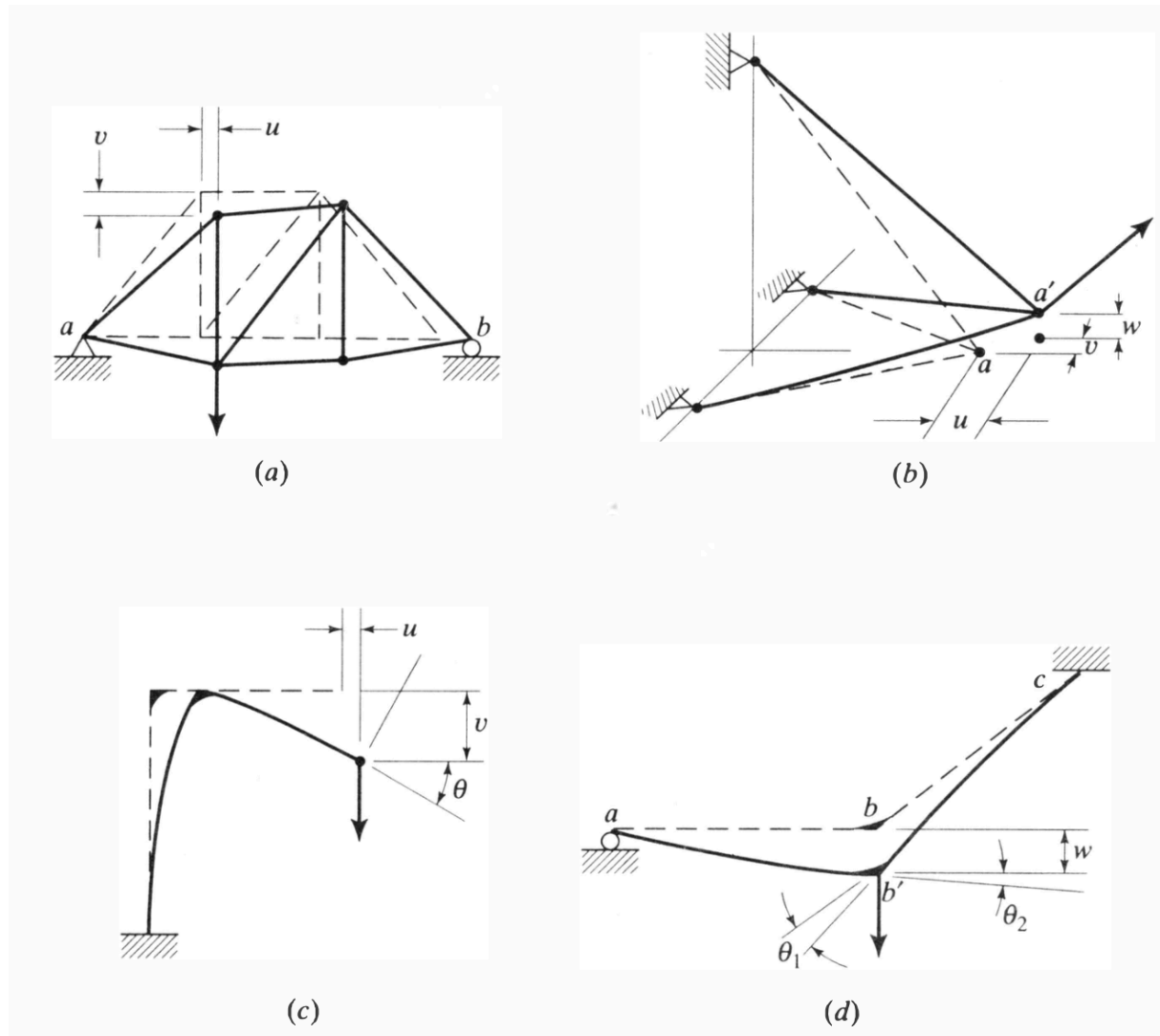


Figure 2.1 Joint displacements. (a) Pin-jointed plane truss. (b) Pin-jointed space truss. (c) Plane frame (in-plane loading). (d) Plane frame (out-of-plane loading).

## Global vs Local Viewpoint

In structural analysis we constantly switch between two perspectives:

- **Global:** how the *entire structure* moves and equilibrates
- **Local:** how an *individual element* deforms internally

This distinction is fundamental to the matrix stiffness method.

## Global–Local Workflow

The stiffness method follows a consistent pattern:

### **local element behavior**

- assemble to a **global system**
- solve for **global displacements**
- recover **local deformations**
- compute **member forces / stresses**

We solve the structure **globally**, but evaluate and design **locally**.

## Notation: Global vs Local

We use notation to distinguish viewpoints:

- **Global (structure):**  $\mathbf{u}$ ,  $\mathbf{f}$ ,  $\mathbf{K}$
- **Local (element):**  $\mathbf{u}'$ ,  $\mathbf{f}'$ ,  $\mathbf{k}'$

The prime ( $'$ ) indicates quantities defined in an **element's local coordinate system**.

## Part 2 — Planar Trusses as a Structural System

## What Is a Plane Truss?

A **plane truss** is a two-dimensional framework of straight members that:

- lie entirely in a single plane
- connect through **frictionless pin joints**
- carry **axial force only** (no bending or shear)
- are loaded only at the joints

**Key consequence:** each member is in **tension** or **compression** only.

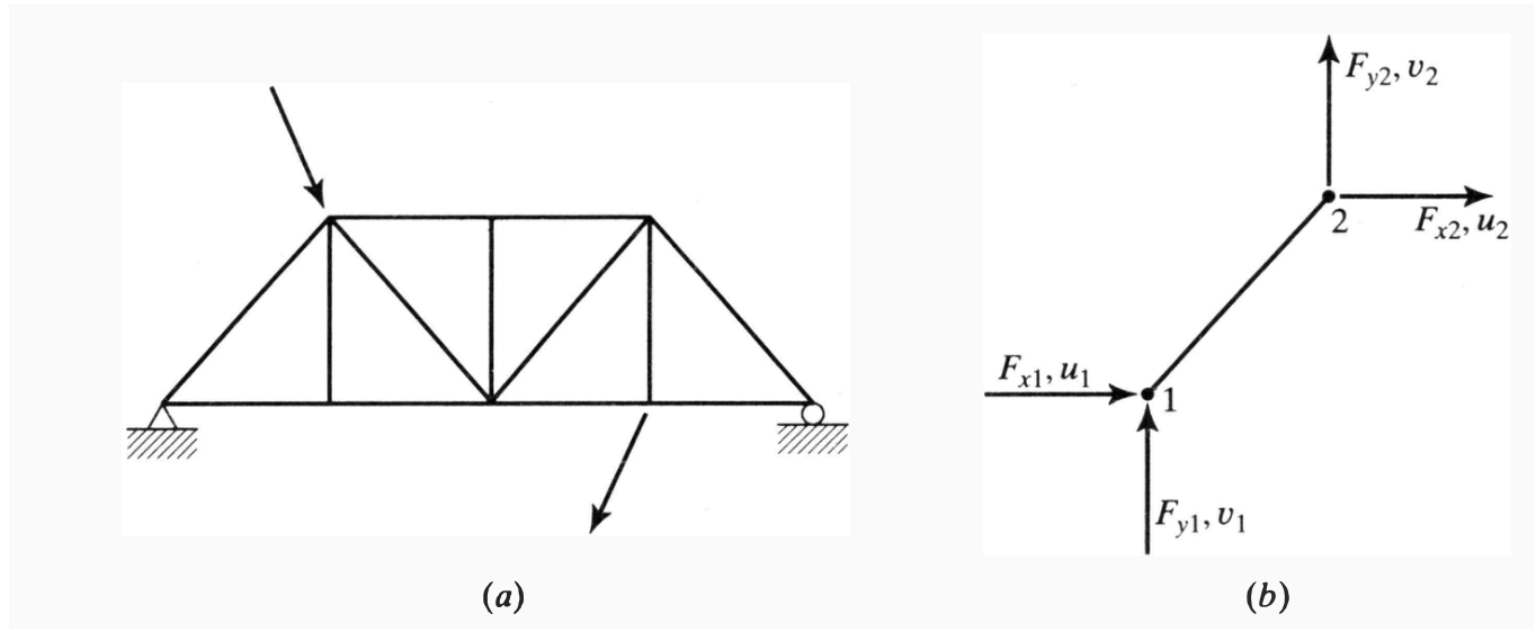


Figure. (a) Idealized planar truss (pin-jointed members). (b) Typical truss member carrying axial force.

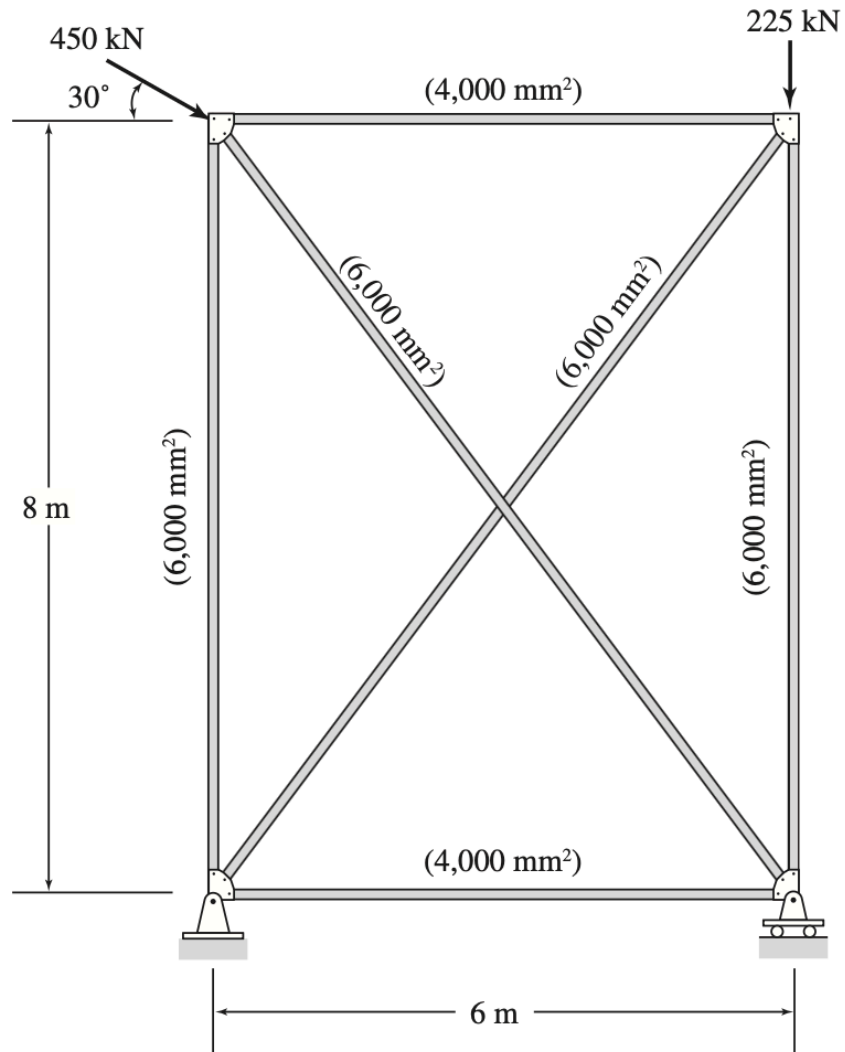


Figure. Idealized planar truss showing member properties, loads, and supports.

## Why Trusses Are Ideal for Matrix Analysis

Because members carry **axial force only**:

- Each member behaves like a **1D axial element**
- Force–displacement relations are **linear and simple**
- The local behavior can be derived cleanly and assembled into a system

This makes trusses an excellent first application of the **matrix stiffness method**.

## Components of a Truss Analysis Model

A truss analysis (analytical) model has four ingredients:

- **Nodes (joints):** where displacements are defined
- **Elements (members):** connect nodes and carry axial force
- **Supports:** restrain specific displacement components
- **Applied forces:** external loads acting at nodes

### What we identify on the line diagram of a truss:

- **Nodes** (circled numbers)
- **Members** (element connectivity between nodes)
- **Applied loads** (known forces at DOF locations)
- **Supports** (restrained displacement components)
- **Free DOFs** (unknown joint translations)

Next we make the DOFs explicit and store them in a displacement vector.

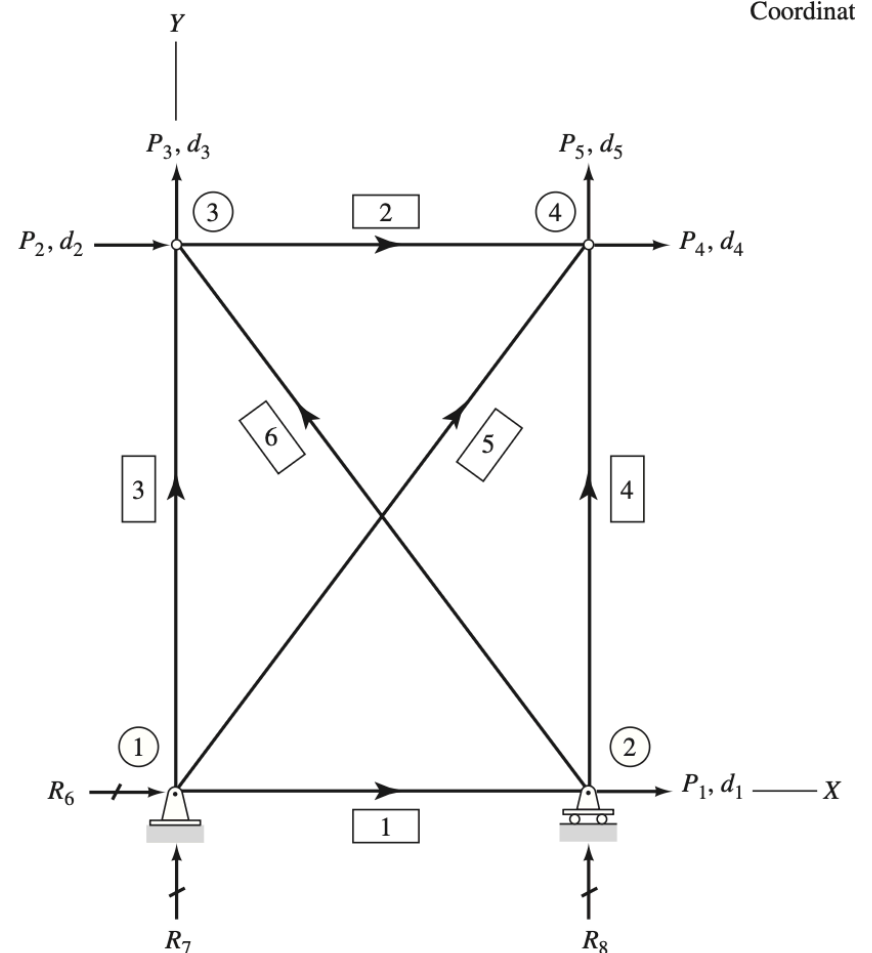


Figure. Components of a planar truss analysis model.

## DOFs for a 2D Truss Node

For a planar truss joint, the possible displacement components are:

- $u_x$  — translation in global  $+x$
- $u_y$  — translation in global  $+y$

Each **free joint** therefore has **two DOFs**.

A truss with  $j$  joints has up to  $2j$  joint displacement components (*before supports are applied*).

## Free vs Restrained Joints

Not all joints are free to move.

- **Free joints**
  - displacements are **unknown**
  - contribute DOFs
- **Restrained joints (supports)**
  - displacements are **prescribed** (often zero)
  - remove DOFs

Supports eliminate specific displacement components in  $x$  and/or  $y$ .

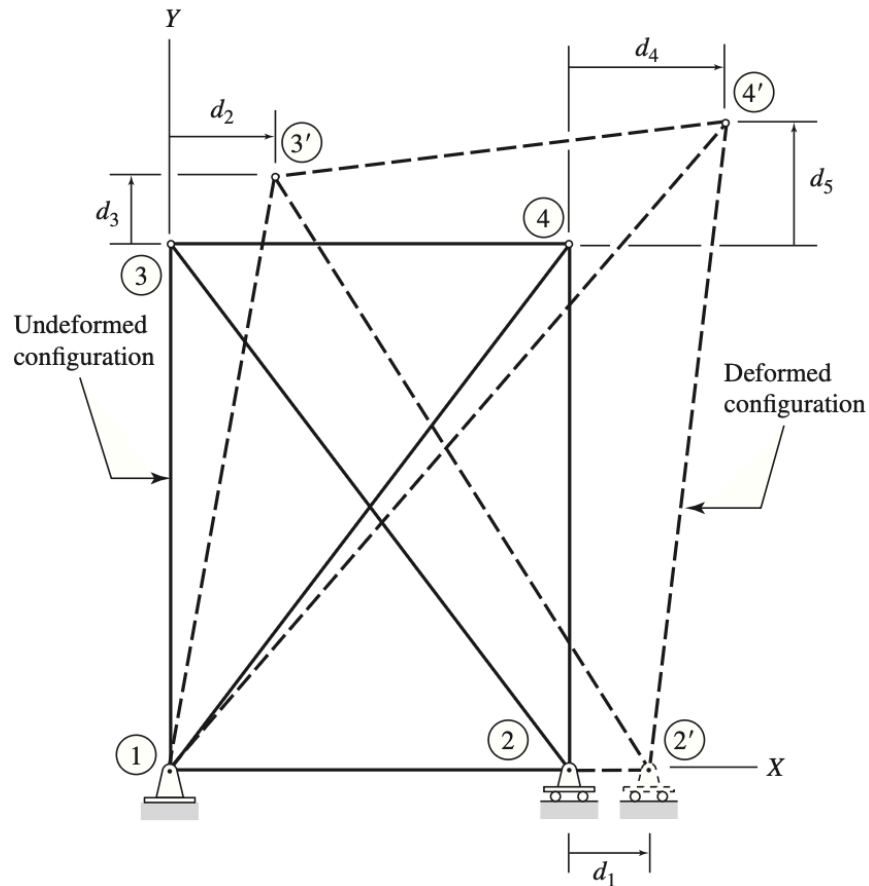
## Restrained Degrees of Freedom

Each support restrains one or more displacement components:

- **Pin support**
  - restrains  $u_x$  and  $u_y$
  - contributes **2** restraints
- **Roller support**
  - restrains **one** displacement component
  - contributes **1** restraint

We count **restrained displacement components**, not supports.

Let  $r$  be the total number of restrained joint displacement components.



The DOFs are the **independent joint translations** that describe the deformed shape.

- $d_1$ – $d_5$  are defined in the global  $x$ – $y$  system
- Positive directions follow the global axes
- Collect them into a displacement vector  $\{d\}$

Figure. Truss joint displacements defined in the global coordinate system.

# Structural Vectors: Displacements and Loads

Recall the global equilibrium relation:

$$\mathbf{K} \mathbf{u} = \mathbf{f}$$

In matrix form, we collect the structure's unknowns and knowns into vectors.

## Displacement vector

- $\mathbf{u}$  — global displacement vector
  - $\mathbf{d} \rightarrow$  **free degrees of freedom** (unknown joint displacements)
  - $\mathbf{0} \rightarrow$  **restrained degrees of freedom** (prescribed displacements)

## Force vector

- $\mathbf{f}$  — global force vector
  - $\mathbf{P} \rightarrow$  **applied nodal loads** at free DOFs
  - $\mathbf{R} \rightarrow$  **reaction forces** at restrained coordinates

**Applied load vector (free DOFs)**

$$\mathbf{P} = \{P_1, P_2, P_3, P_4, P_5\}$$

$$= \{0, 389.7, -225, 0, -225\}$$

- $P_1 = 0$
- $P_2 = 389.7$  kN (positive  $x$ )
- $P_3 = -225$  kN (negative  $y$ )
- $P_4 = 0$
- $P_5 = -225$  kN (negative  $y$ )

Each load component corresponds directly to a numbered **degree of freedom**.

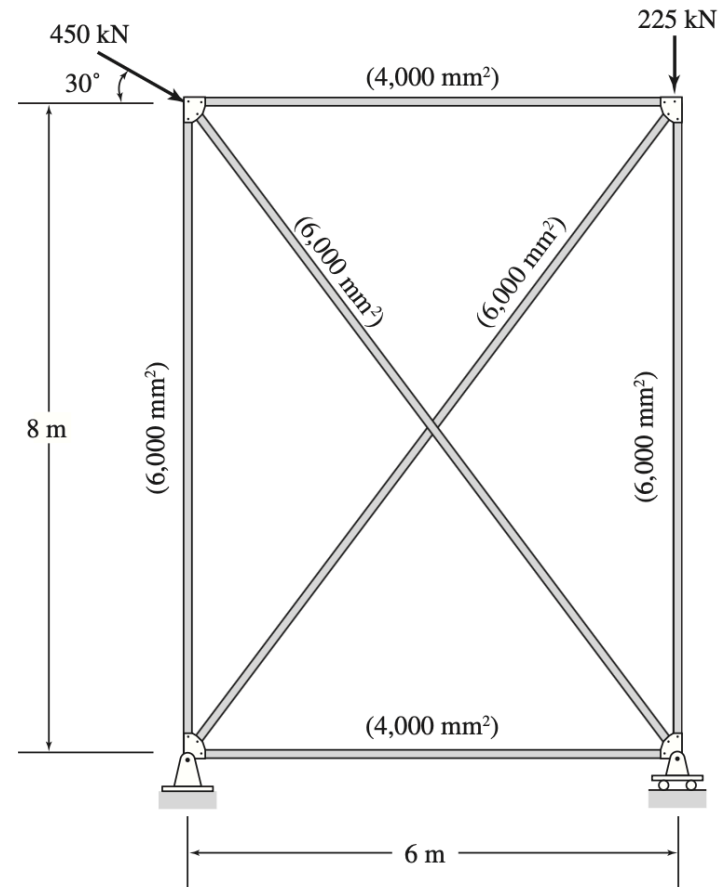
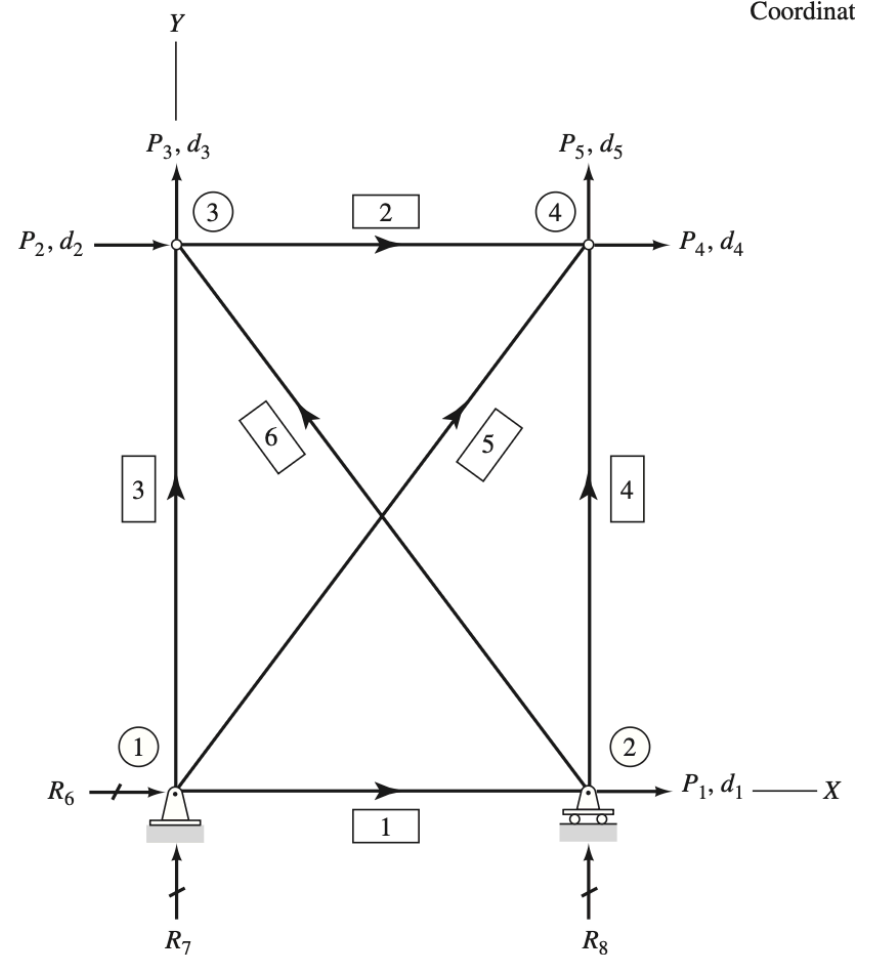


Figure. Applied nodal loads.

The global equilibrium is written as:

$$\mathbf{K} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 389.7 \\ -225 \\ 0 \\ -225 \\ R_6 \\ R_7 \\ R_8 \end{Bmatrix} \text{ kN}$$



Components of a planar truss analysis model.

## In-Class Exercise - Structural Vectors

1. **Number the joints** (nodes) on the figure.
2. **Number the members** (elements).
3. **Assign global DOF numbers** at each node.
4. **Identify DOF types:**
  - *Circle* free (unknown) DOFs
  - *Box* DOFs with applied loads
  - *Underline* restrained DOFs (reaction forces)
5. **Write the global displacement vector  $\{u\}$ .**
6. **Write the global force vector  $\{f\}$ .**

*(Based on your DOF numbering; use variables where unknown and insert known values where prescribed.)*

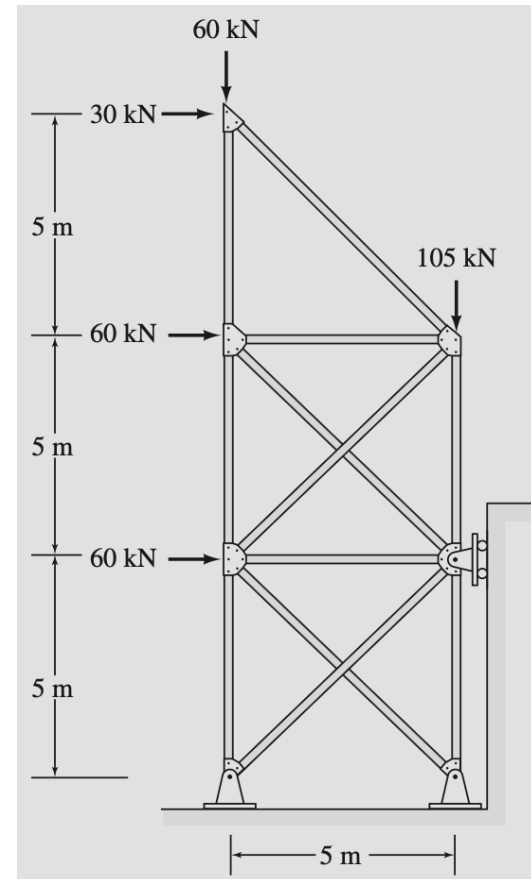


Figure. Truss structure.

# Answers — Structural Vectors

$$\mathbf{u} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_5 \\ u_6 \\ 0 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \end{Bmatrix} \quad \mathbf{f} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ 60 \\ 0 \\ r_7 \\ 0 \\ 60 \\ 0 \\ 0 \\ -105 \\ 30 \\ 60 \end{Bmatrix}$$

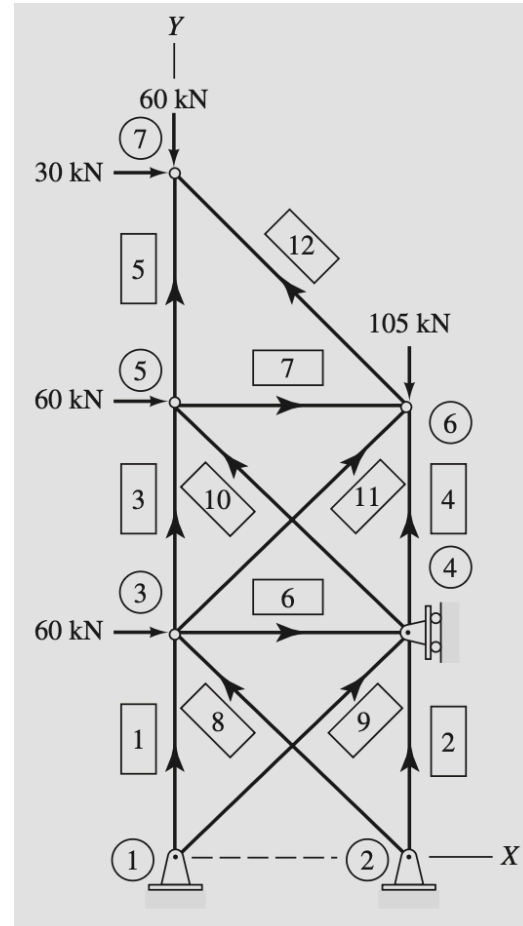


Figure. Truss structure numbered.

## Part 3 - Truss Stability and Determinacy

## Counting Degrees of Freedom

Let:

- $j$  = number of joints
- $r$  = number of restrained joint displacement components
- $N_{\text{CJT}}$  = DOFs per free joint

Then the number of structural degrees of freedom is:

$$N_{\text{DOF}} = N_{\text{CJT}} j - r$$

## DOFs for a Planar Truss

For a planar truss, each free joint has two translational DOFs:

$$N_{\text{CJT}} = 2 \quad (u_x, u_y)$$

So:

$$\boxed{N_{\text{DOF}} = 2j - r}$$

This is the number of **independent joint displacements** that must be solved for.

## Stability vs Static Determinacy

- **Stability:** does the structure prevent rigid-body motion?
- **Static determinacy:** can forces be found from equilibrium alone?

These are related, but **not the same**.

We will separate:

- **stability** (rigid-body motion)
- **external determinacy** (reactions from global equilibrium)
- **internal determinacy** (member forces from joint equilibrium)

## Stability Requirement (Rigid-Body Motion)

A free planar structure has **three rigid-body motions**:

- translation in  $x$
- translation in  $y$
- rotation in the plane

To prevent rigid-body motion, the supports must restrain at least:

$$r \geq 3$$

This is a **stability requirement**, not a determinacy condition.

- **Stability** asks whether the structure can resist rigid-body motion and admit an equilibrium configuration.
- **Determinacy** is a separate question, asked **only after stability is ensured**, and concerns whether forces can be found from equilibrium alone.

## Global Equilibrium (What Statics Provides)

For a planar structure, global equilibrium provides **three equations**:

- $\sum F_x = 0$
- $\sum F_y = 0$
- $\sum M = 0$

These equations apply to the **entire structure** treated as a rigid body.

They govern **reaction forces only** (external equilibrium).

## External (Global) Static Indeterminacy

Let  $r$  be the number of reaction components.

Global equilibrium provides **3 equations** in 2D.

So if:

$$r > 3$$

then reactions cannot be determined from statics alone.

The structure is **externally statically indeterminate**.

## Internal Static Determinacy for a Planar Truss (Counting)

At each joint, equilibrium gives two equations:

$$\sum F_x = 0, \quad \sum F_y = 0$$

Across  $j$  joints, that is  **$2j$  joint equilibrium equations**.

Unknown forces are:

- $m$  member axial forces
- $r$  reaction components

A necessary counting condition for solving forces by equilibrium is thus:

$$m + r = 2j$$

# Interpreting the Internal Counting Condition

The comparison

$$m + r \quad \text{vs.} \quad 2j$$

compares **unknown forces** to **joint equilibrium equations**:

- $m + r < 2j$   
→ **unstable / mechanism** (not enough constraints; a deformation mode exists)
- $m + r = 2j$   
→ **statically determinate by counting** (*if geometry is stable*)
- $m + r > 2j$   
→ **statically indeterminate by counting** (redundant member/support forces)

This is a **counting test** — it does not guarantee stability.

## What the Counting Condition Does *Not* Guarantee

Counting compares equations and unknowns, but it does **not**:

- detect rigid-body motion
- detect geometric mechanisms
- guarantee a unique solution

So passing a counting test is **necessary**, not sufficient.

## Possible Outcomes

A structure can therefore be:

- **Stable but statically indeterminate**
  - stable (no rigid-body motion)
  - but has redundancies:  $m + r > 2j$  or  $r > 3$
  - forces depend on deformation compatibility
- **Unstable but satisfy  $m + r = 2j$** 
  - the count matches, but the geometry forms a mechanism
  - equilibrium equations exist, but the structure can move without resistance

## Why the Direct Stiffness Method is Powerful

Equilibrium alone does **not** enforce compatibility.

The Direct Stiffness Method solves by enforcing:

- equilibrium at joints
- compatibility of joint displacements
- member force–deformation relations

We can still solve indeterminate cases when:

$$r > 3 \quad \text{or} \quad m + r > 2j$$

If a structure is **unstable**, it appears as a **singular stiffness matrix  $\mathbf{K}$** .

# In-Class Exercise — Stability & Determinacy

## Questions (work in pairs):

1. How many **joints** ( $j$ ) does this structure have?
2. How many **members** ( $m$ ) are present?
3. How many **reaction components** ( $r$ ) are provided by the supports?
4. Does the structure satisfy the **stability requirement**?
5. Based on counting, is the structure **statically determinate or indeterminate**?
6. If indeterminate, is the indeterminacy **external, internal, or both**?

*No force calculations — focus on counting and concepts.*

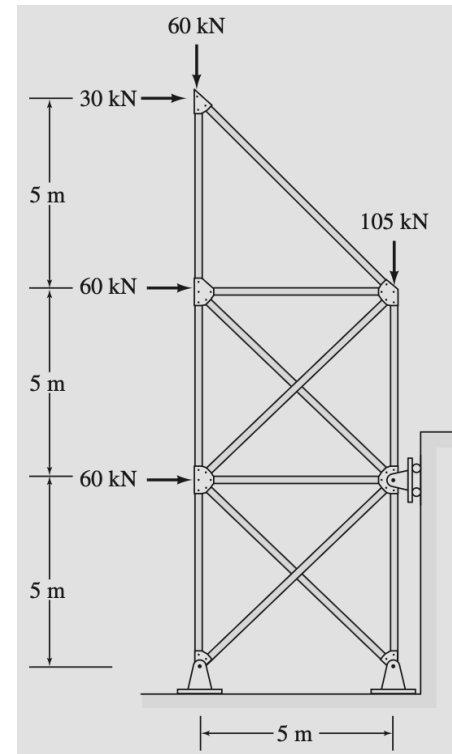


Figure. Truss used for stability and determinacy assessment.

## ✓ Answers — Stability & Determinacy

- **Number of joints:**  $j = 7$ :
- **Number of members:**  $m = 12$ :
- **Number of reaction components**  $r = 2 + 2 + 1 = 5$ :
- **Stability requirement:**  
For a planar structure, external stability requires

$$r \geq 3$$

Since  $5 \geq 3$ , the structure **is stable**.

- **Static determinacy (counting test):**

For a planar truss, compare

$$m + r \quad \text{and} \quad 2j$$

$$m + r = 12 + 5 = 17, \quad 2j = 2(7) = 14$$

Since (  $17 > 14$  ), the structure is **statically indeterminate**.

- **Type of indeterminacy:**

- Total indeterminacy:

$$D_t = m + r - 2j = 3$$

- External indeterminacy:

$$D_e = r - 3 = 2$$

- Internal indeterminacy:

$$D_i = D_t - D_e = 1$$

👉 The structure is **indeterminate both externally and internally**.

### **External indeterminacy**

There are **more reaction components than required for global equilibrium**, so the support reactions cannot be determined using statics alone.

### **Internal indeterminacy**

The structure contains **more members than are required for equilibrium**, meaning at least one **redundant load path** exists and some member forces cannot be found using statics alone.

### **Another way to think about indeterminacy:**

Indeterminacy often means there is **more than one load path** available for forces to travel through the structure.

How the load is shared between these paths depends on **stiffness and deformation compatibility**, not just equilibrium.

✅ This is **not a problem** for the **Direct Stiffness Method**: stiffness-based formulations resolve redundancy automatically by enforcing **compatibility and equilibrium simultaneously**.

# Part 4 — Global and Local Coordinate Systems

## Why Local Coordinates?

Local coordinates simplify element behavior:

- axial deformation occurs **along the member axis**
- stress–strain relations are simplest in that direction
- local coordinates separate **element behavior** from **global geometry**

## Coordinate and Sign Conventions

- Global axes:  $+x$  to the right,  $+y$  upward
- Positive axial force: **tension**
- Local  $+x'$  axis: defined from **start node** to **end node**

Consistent conventions are essential for correct assembly.

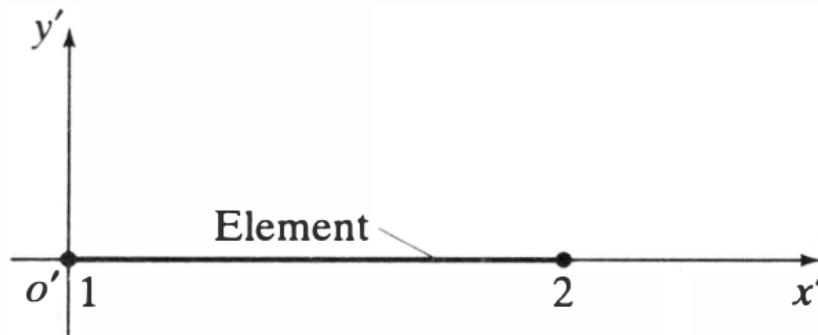
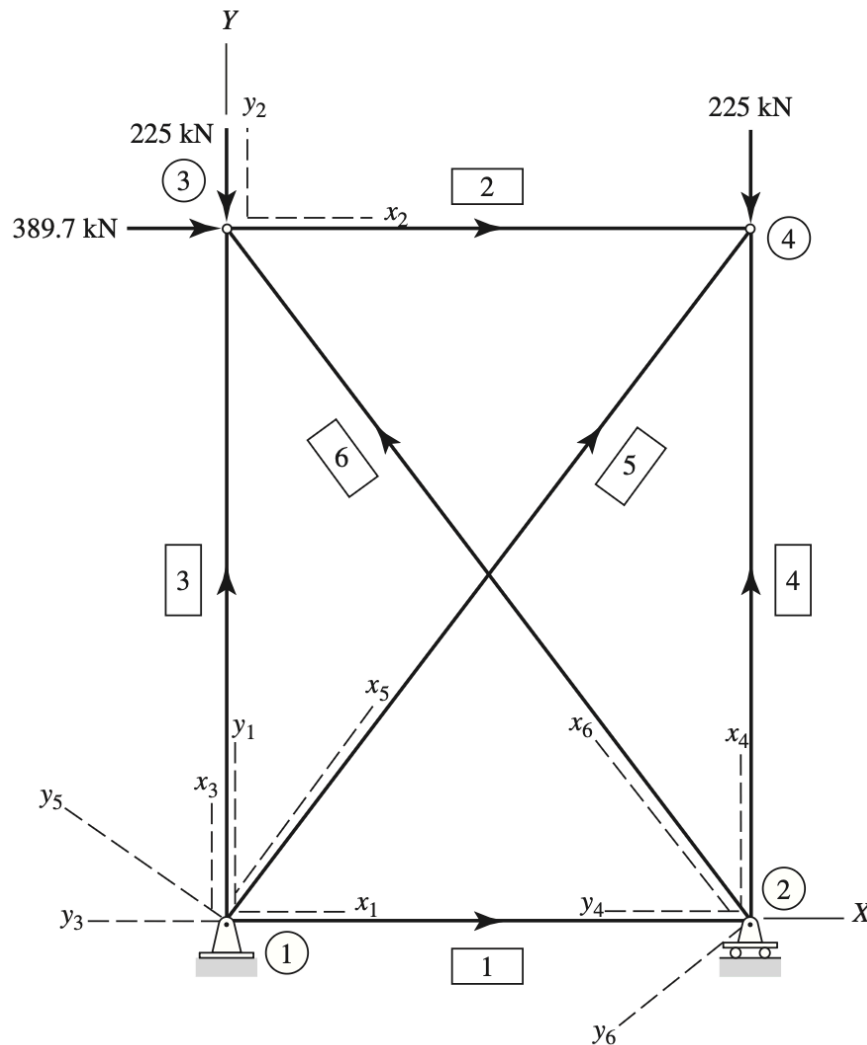


Figure. Definition of local element axis.



Each truss member is assigned its own **local coordinate system**.

- The local axis  $x'$  is aligned with the member
- The positive direction is defined from start node to end node
- Axial deformation and force are expressed in this system

Figure. Local element axes superimposed on a truss structure.

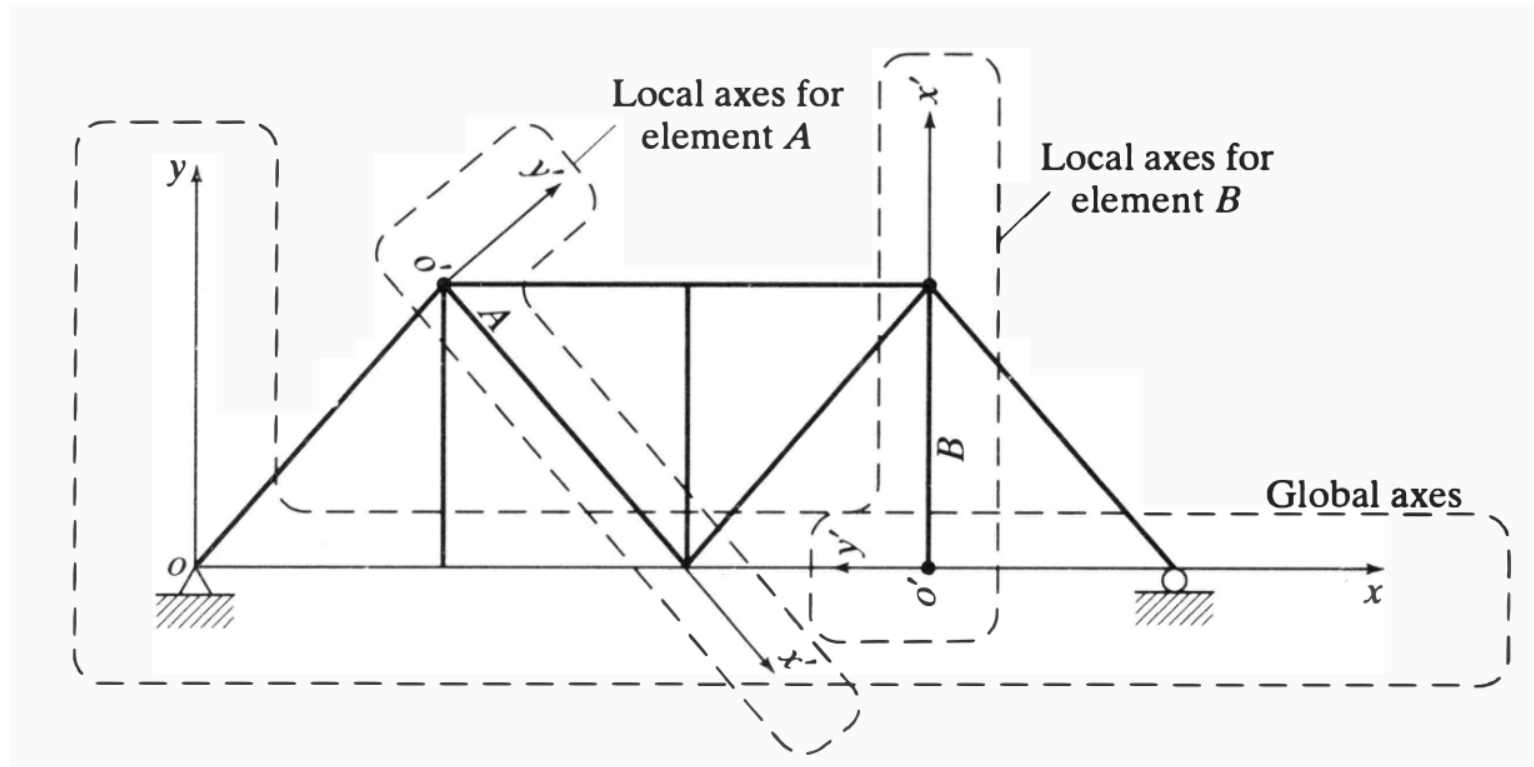


Figure. Relationship between global and local coordinate systems.

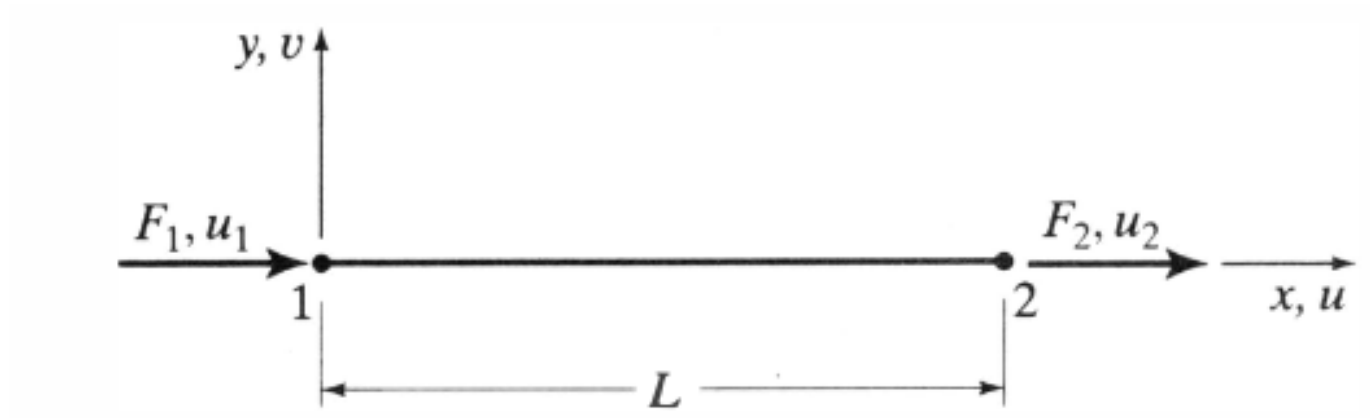
## Part 5 — Axial Element Kinematics

## Axial truss element (local view)

- Straight, prismatic truss member
- Length  $L$ , area  $A$ , Young's modulus  $E$
- Two nodes define the element: node 1 and node 2
- Two **local axial DOFs**:  $u_1$  and  $u_2$  (one at each node)
- Local  $x$  axis is aligned with the member (positive to the right)

**Note:** In the following section it is assumed we are only talking about local level behavior, hence the ' notation will not be used to distinguish between local and global matrices

## Local degrees of freedom



- Axial displacement at node 1:  $u_1$
- Axial displacement at node 2:  $u_2$

**In this section:** axial DOFs only ( $u_1, u_2$ ).

*Note: transverse DOFs ( $v_1, v_2$ ) appear later when we build the full  $4 \times 4$  local truss matrix.*

## Axial kinematics

The axial deformation is the relative displacement of the ends:

$$\delta = u_2 - u_1$$

- If  $u_2 > u_1$  then  $\delta > 0 \rightarrow$  **elongation (tension)**
- If  $u_2 < u_1$  then  $\delta < 0 \rightarrow$  **shortening (compression)**

## Free-body diagram (axial)

- Axial end force at node 1:  $F_1$
- Axial end force at node 2:  $F_2$

Equilibrium along the member axis:

$$\sum F_x = 0$$

➡ The end forces are equal and opposite.

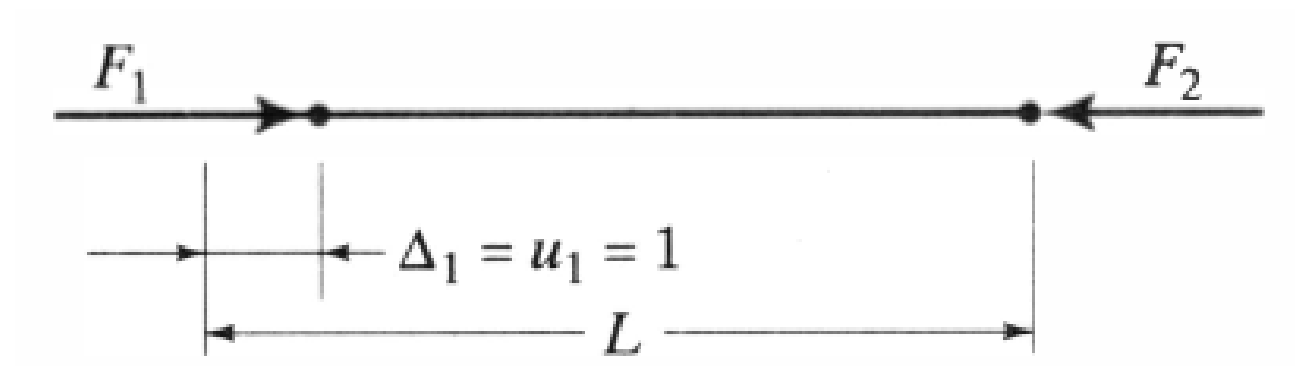
Kinematics examples (what do  $\delta$  and  $F$  look like?)

**Compression example:**  $u_1 = 1, u_2 = 0$

$$\delta = 0 - 1 = -1 < 0 \Rightarrow \text{compression}$$

Take positive as pointing to the right:

$$+F_1 - F_2 = 0 \Rightarrow F_1 = F_2$$

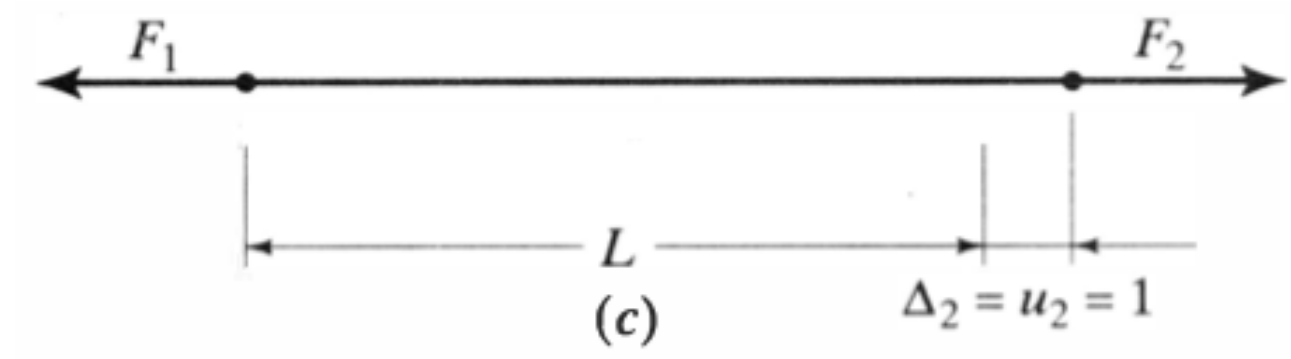


**Tension example:**  $u_1 = 0, u_2 = 1$

$$\delta = 1 - 0 = 1 > 0 \Rightarrow \text{tension}$$

Take positive as pointing to the right:

$$-F_1 + F_2 = 0 \Rightarrow F_1 = F_2$$



## Part 6 — Deriving Local 2x2 Stiffness Matrix

# Axial constitutive law (linear elasticity)

Hooke's law (uniaxial, linear elastic):

$$E = \frac{\sigma}{\varepsilon}$$

Strain (Engineering strain, small deformation,  $L = 0$ ):

$$\varepsilon = \frac{\delta}{L} = \frac{u_2 - u_1}{L}$$

Stress definition:

$$\sigma = \frac{F}{A}$$

Axial force:

$$F = \frac{EA}{L}(u_2 - u_1)$$

*Notes: In geometric nonlinearity, the " $L$ " and kinematics change; in material nonlinearity, the  $\sigma$ - $\varepsilon$  law changes.*

## End forces in terms of $u_1, u_2$

Using the same sign convention (positive to the right):

- If  $u_1 > u_2$  (compression), then  $F_1 > 0$  ( $F_2 < 0$ )

$$F_1 = \frac{EA}{L}(u_1 - u_2)$$

- If  $u_2 > u_1$  (tension), then  $F_2 > 0$  ( $F_1 < 0$ )

$$F_2 = \frac{EA}{L}(u_2 - u_1)$$

## Local axial stiffness matrix (2×2)

Multiply out the end-force equations:

$$F_1 = \frac{EA}{L} \cdot u_1 - \frac{EA}{L} \cdot u_2$$

$$F_2 = -\frac{EA}{L} \cdot u_1 + \frac{EA}{L} \cdot u_2$$

Collect in matrix form:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Scenario 1 (unit displacement):  $u_1 = 1, u_2 = 0$

$$\delta = u_2 - u_1 = 0 - 1 = -1 \quad (\text{compression})$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

So:

$$F_1 = +EA/L$$

$$F_2 = -EA/L$$

*This matches the "compression example" we looked at in Part 5*

Scenario 2:  $u_1 = 5$ ,  $u_2 = 3$

$$\delta = u_2 - u_1 = 3 - 5 = -2 \quad (\text{compression})$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 5 \\ 3 \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} 2 \\ -2 \end{Bmatrix}$$

So:

$$F_1 = +2EA/L$$

$$F_2 = -2EA/L$$

*Double the amount of strain, double the amount of force.*

Scenario 3:  $u_1 = 1, u_2 = -1$

$$\delta = u_2 - u_1 = -1 - 1 = -2 \quad (\text{compression})$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} 2 \\ -2 \end{Bmatrix}$$

So:

$$F_1 = +2EA/L$$

$$F_2 = -2EA/L$$

*Same relative deformation ( $\delta = -2$ )  $\Rightarrow$  same end-force pattern.*

Scenario 4:  $u_1 = -2$ ,  $u_2 = -1$

$$\delta = u_2 - u_1 = -1 - (-2) = +1 \quad (\text{tension})$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -2 \\ -1 \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

So:

$$F_1 = -EA/L$$

$$F_2 = +EA/L$$

*Same example we looked at in Part 5, tension where  $u = 1$ .*

Scenario 5 (rigid-body translation):  $u_1 = 3, u_2 = 3$

$$\delta = u_2 - u_1 = 3 - 3 = 0 \quad (\text{no deformation})$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 3 \\ -3 \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

So:

$$F_1 = 0$$

$$F_2 = 0$$

*Rigid translation produces **no internal axial force**.*

```

In [3]: # Axial 2x2 stiffness: one editable scenario
import numpy as np

# Choose stiffness scale (set EA_over_L = 1 for clean numbers)
EA_over_L = 1.0
k = EA_over_L * np.array([[1.0, -1.0],
                           [-1.0, 1.0]])

# --- Edit these ---
u1 = 1.0
u2 = 0.0

u = np.array([u1, u2])
f = k @ u
delta = u2 - u1

state = "tension" if delta > 0 else ("compression" if delta < 0 else "ri

# print("k =\n", k)
print(f"u = [u1, u2] = {u}")
print(f"delta = u2 - u1 = {delta}  ->  {state}")
print(f"f = [F1, F2] = {f}")

u = [u1, u2] = [1. 0.]
delta = u2 - u1 = -1.0  ->  compression
f = [F1, F2] = [ 1. -1.]

```

## IMPORTANT: Definition of a stiffness coefficient $k_{ij}$

$k_{ij}$  = force at DOF  $i$  due to a unit displacement at DOF  $j$ ,  
with all other DOFs held fixed.

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

*Here we only have 2 DOFs. Later (beams / frames), this interpretation becomes more important.*

## Building the $2 \times 2$ stiffness matrix (unit displacement method)

Each column of  $k$  is built by:

- Impose a **unit displacement** at one DOF
- Hold all the other DOF fixed
- Record the resulting nodal force pattern

This suppresses rigid-body motion while defining the columns.

Column 1: impose  $u_1 = 1$  ( $u_2 = 0$ )

$$\begin{bmatrix} \boxed{k_{11}} & k_{12} \\ \boxed{k_{21}} & k_{22} \end{bmatrix}$$

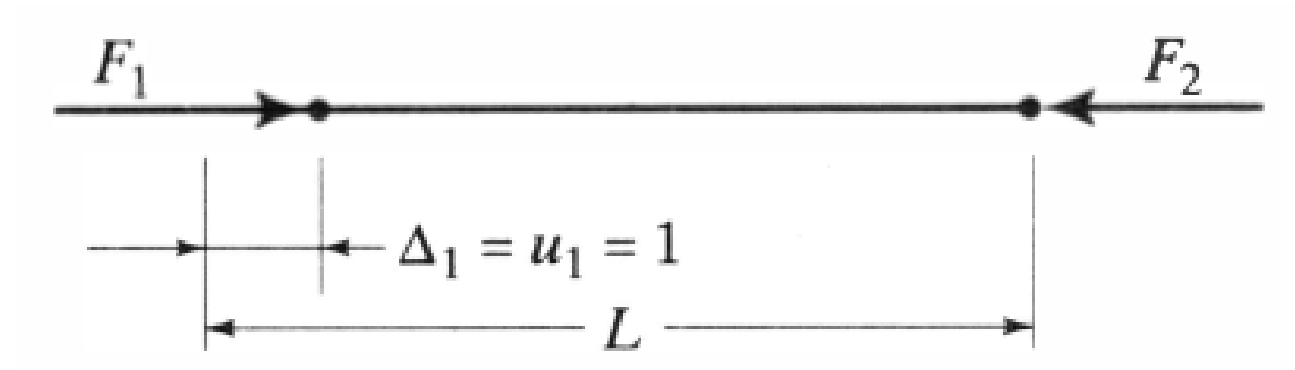
$k_{11}$  ( $i = 1, j = 1$ ):

- $i = 1$ : force at DOF 1
- $j = 1$ : due to a unit displacement at DOF 1 ( $u_1 = 1$ )
- all other displacements are 0 (i.e.,  $u_2 = 0$ )

$k_{21}$  ( $i = 2, j = 1$ ):

- $i = 2$ : force at DOF 2
- $j = 1$ : due to a unit displacement at DOF 1 ( $u_1 = 1$ )
- all other displacements are 0 (i.e.,  $u_2 = 0$ )

Column 1: impose  $u_1 = 1$  ( $u_2 = 0$ )



The resulting nodal force vector is:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{u_1=1, u_2=0} = \frac{EA}{L} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

This force vector **defines the first column** of the local stiffness matrix:

$$\begin{Bmatrix} k_{11} \\ k_{21} \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Column 2: impose  $u_2 = 1$  ( $u_1 = 0$ )

$$\begin{bmatrix} k_{11} & \boxed{k_{12}} \\ k_{21} & \boxed{k_{22}} \end{bmatrix}$$

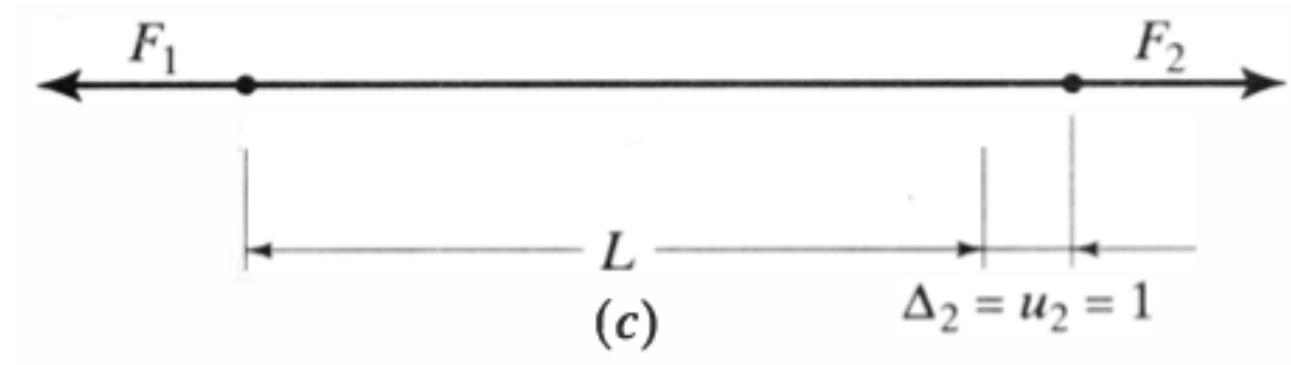
$k_{12}$  ( $i = 1, j = 2$ ):

- $i = 1$ : force at DOF 1
- $j = 2$ : due to a unit displacement at DOF 2 ( $u_2 = 1$ )
- all other displacements are 0 (i.e.,  $u_1 = 0$ )

$k_{22}$  ( $i = 2, j = 2$ ):

- $i = 2$ : force at DOF 2
- $j = 2$ : due to a unit displacement at DOF 2 ( $u_2 = 1$ )
- all other displacements are 0 (i.e.,  $u_1 = 0$ )

Column 2: impose  $u_2 = 1$  ( $u_1 = 0$ )



The resulting nodal force vector is:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{u_1=0, u_2=1} = \frac{EA}{L} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

This force vector **defines the first column** of the local stiffness matrix:

$$\begin{Bmatrix} k_{12} \\ k_{22} \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

## Equilibrium and column sums

$$\mathbf{k} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Check each column is self-equilibrated:

$$\sum F_x = 0$$

➡ The entries in each column sum to zero.

## Symmetry (reciprocity)

$$k_{12} = k_{21} = -\frac{EA}{L}$$

- **Case 1 ( $k_{12}$ ):** response to  $u_2 = 1$  (with  $u_1 = 0$ )  
→ the element is **stretched**, producing a negative force (left) at DOF 1
- **Case 2 ( $k_{21}$ ):** response to  $u_1 = 1$  (with  $u_2 = 0$ )  
→ the element is **compressed**, producing a negative force (left) at DOF 2

In both cases, the magnitude of the resisting force is the same.

Mathematically, this follows from **energy reciprocity**:

the work done by DOF 1 on DOF 2 equals the work done by DOF 2 on DOF 1.

➡ **Symmetry reflects that the element's response depends only on relative deformation, not on which end is displaced.**

## Why the element stiffness matrix is singular?

$$\det(\mathbf{k}) = 0$$

Matrix meaning: there is some **nonzero/non-trivial displacement vector** that produces **zero force**.

Physical meaning: the element alone is not anchored — it can undergo **rigid-body translation** (we showed this in scenario 5)

This is not a problem at the local level: stability is enforced only when we assemble the **full structure** and apply supports in the form of DOF restraints.

## Rigid-body motion (worked out)

Let  $u_1 = u_2 = c$ :

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} c \\ c \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} c - c \\ -c + c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

➡ No internal force unless there is **relative displacement**, translation to the right.

## Key takeaways

- The 2×2 axial stiffness matrix enforces **equilibrium + compatibility** at the element level
- Symmetry reflects **reciprocity** ( $k_{12} = k_{21}$ )
- Singularity reflects **rigid-body freedom** (no force if  $u_1 = u_2$ )
- Supports/constraints are applied at the **global structure level**, not the element level

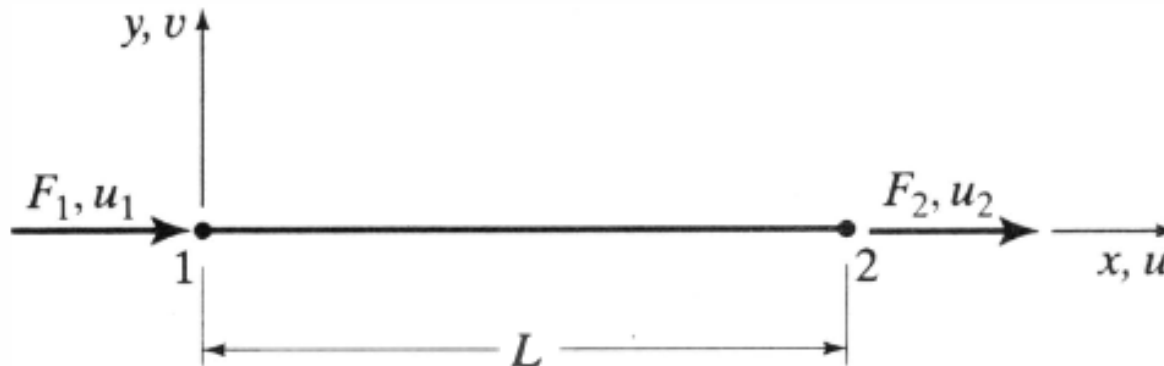
➡ This 2×2 matrix is the fundamental building block of truss stiffness assembly.

## Part 7 — Deriving Local $4 \times 4$ Stiffness Matrix

## Why do we need a 4×4 matrix?

In a 2D truss, each node has **two translational degrees of freedom**:

- $u$  : displacement in the local  $x$  direction (along the member axis)
- $v$  : displacement in the local  $y$  direction (perpendicular to the member)

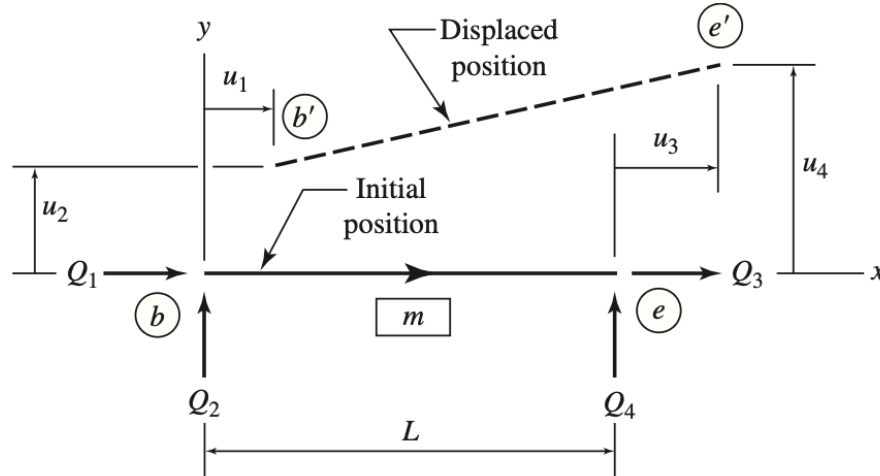


In **Part 6**, we derived a  **$2 \times 2$  stiffness matrix** by considering only the axial ( $x$ -direction) behavior of the element.

To describe the **complete local behavior** of a truss element, we must account for **both translations at both nodes**.

This leads to a  **$4 \times 4$  local stiffness relation**, with **2 DOFs per node** and **4 DOFs total**.

## Generic displacement for a 2D truss element



### Local DOF numbering (for 2x2):

- **DOF 1:**  $u_1$  — node 1, local  $x$
- **DOF 2:**  $u_2$  — node 2, local  $x$

### Local DOF numbering (for 4x4):

- **DOF 1:**  $u_1$  — node 1, local  $x$
- **DOF 2:**  $u_2$  — node 1, local  $y$
- **DOF 3:**  $u_3$  — node 2, local  $x$
- **DOF 4:**  $u_4$  — node 2, local  $y$

# Local displacement and force vectors

Local displacement vector:

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Local nodal force vector (using  $Q$  to stay consistent with figures):

$$\mathbf{F} = \mathbf{Q} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

We seek:

$$\mathbf{Q} = \mathbf{k} \mathbf{u}$$

## Four equations (one per DOF)

$$Q_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4$$

$$Q_2 = k_{21}u_1 + k_{22}u_2 + k_{23}u_3 + k_{24}u_4$$

$$Q_3 = k_{31}u_1 + k_{32}u_2 + k_{33}u_3 + k_{34}u_4$$

$$Q_4 = k_{41}u_1 + k_{42}u_2 + k_{43}u_3 + k_{44}u_4$$

Each equation expresses **force equilibrium at a single local degree of freedom**.

For a linear elastic element, the force at any DOF is a **linear combination of all DOF displacements**:

- displacing one DOF can induce forces at *all* DOFs
- the proportionality constants are the stiffness coefficients  $k_{ij}$

Same equations in matrix form

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

## Building the 4×4 stiffness matrix (unit displacement method)

Recall:

$k_{ij}$  = force at DOF  $i$  due to a unit displacement at DOF  $j$ ,  
with all other DOFs held fixed.

Each column of  $k$  is built by:

- Impose a **unit displacement** at one DOF
- Hold all the other DOF fixed
- Record the resulting nodal force pattern

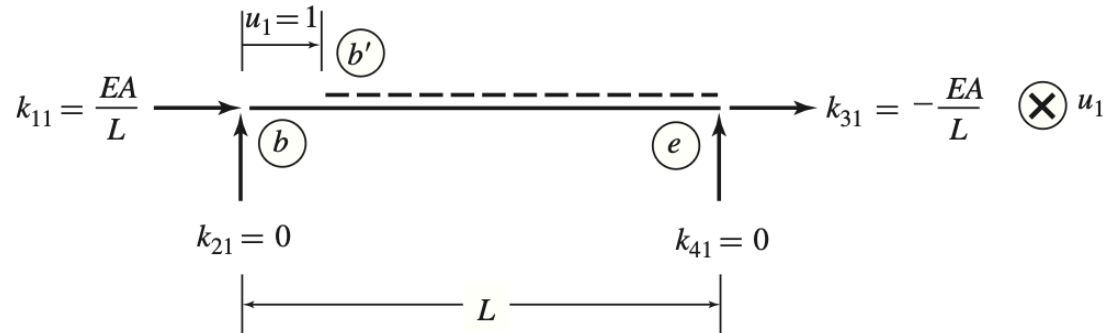
We will do this for DOFs 1–4.

Column 1: impose  $u_1 = 1$  ( $u_2 = u_3 = u_4 = 0$ )

$$\begin{bmatrix} \boxed{k_{11}} & k_{12} & k_{13} & k_{14} \\ \boxed{k_{21}} & k_{22} & k_{23} & k_{24} \\ \boxed{k_{31}} & k_{32} & k_{33} & k_{34} \\ \boxed{k_{41}} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

- $k_{11}$  ( $i = 1, j = 1$ )  
force at **DOF 1** due to unit displacement at **DOF 1**
- $k_{21}$  ( $i = 2, j = 1$ )  
force at **DOF 2** due to unit displacement at **DOF 1**
- $k_{31}$  ( $i = 3, j = 1$ )  
force at **DOF 3** due to unit displacement at **DOF 1**
- $k_{41}$  ( $i = 4, j = 1$ )  
force at **DOF 4** due to unit displacement at **DOF 1**

Column 1: impose  $u_1 = 1$  ( $u_2 = u_3 = u_4 = 0$ )



**Axial equilibrium**

$$\sum F_x = 0 \Rightarrow k_{11} + k_{31} = 0 \Rightarrow k_{31} = -k_{11}$$

**Transverse and moment equilibrium**

$$\sum F_y = 0 \Rightarrow k_{21} + k_{41} = 0, \quad \sum M_e = 0 \Rightarrow k_{21}L = 0$$

Since  $L \neq 0$ :

$$k_{21} = 0 \Rightarrow k_{41} = 0$$

## Column 1: stiffness terms

The resulting nodal force vector  $Q$  is:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}_{u_1=1, u_{2,3,4}=0} = \frac{EA}{L} \begin{Bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} k_{11} \\ k_{21} \\ k_{31} \\ k_{41} \end{Bmatrix}$$

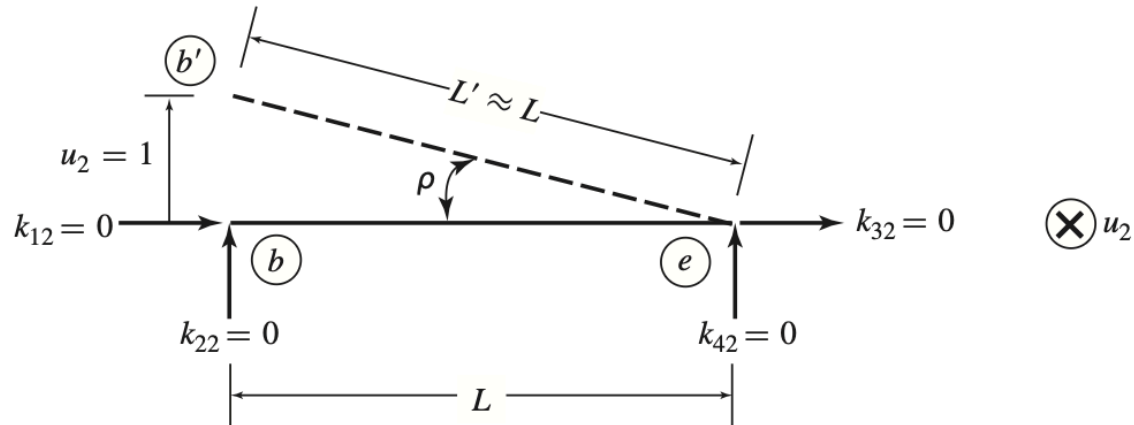
This force vector **defines the first column** of the local stiffness matrix:

Column 2: impose  $u_2 = 1$  ( $u_1 = u_3 = u_4 = 0$ )

$$\begin{bmatrix} k_{11} & \boxed{k_{12}} & k_{13} & k_{14} \\ k_{21} & \boxed{k_{22}} & k_{23} & k_{24} \\ k_{31} & \boxed{k_{32}} & k_{33} & k_{34} \\ k_{41} & \boxed{k_{42}} & k_{43} & k_{44} \end{bmatrix}$$

- $k_{12}$  ( $i = 1, j = 2$ )  
force at **DOF 1** due to unit displacement at **DOF 2**
- $k_{22}$  ( $i = 2, j = 2$ )  
force at **DOF 2** due to unit displacement at **DOF 2**
- $k_{32}$  ( $i = 3, j = 2$ )  
force at **DOF 3** due to unit displacement at **DOF 2**
- $k_{42}$  ( $i = 4, j = 2$ )  
force at **DOF 4** due to unit displacement at **DOF 2**

Column 2: impose  $u_2 = 1$  ( $u_1 = u_3 = u_4 = 0$ )



Because the displacement is **small compared to the member length** ( $u_2 \ll L$ ), the change in length is negligible:

$$L' \approx L \Rightarrow \delta \approx 0$$

Thus a transverse unit displacement produces **no axial strain**.

**Axial equilibrium**

$$\sum F_x = 0 \Rightarrow k_{12} + k_{32} = 0$$

$$\delta = 0 \Rightarrow k_{12} = 0 \Rightarrow k_{32} = 0$$

**Vertical and moment equilibrium**

$$\sum F_y = 0 \Rightarrow k_{22} + k_{42} = 0$$

$$\Rightarrow k_{22} = -k_{42}$$

A nonzero pair  $(k_{22}, k_{42})$  would form a **force couple**, violating:

$$\sum M_e = 0$$

Therefore:

$$k_{22} = k_{42} = 0$$

## Column 2: stiffness terms

The resulting nodal force vector  $\mathbf{Q}$  is:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}_{u_2=1, u_{1,3,4}=0} = \frac{EA}{L} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} k_{12} \\ k_{22} \\ k_{32} \\ k_{42} \end{Bmatrix}$$

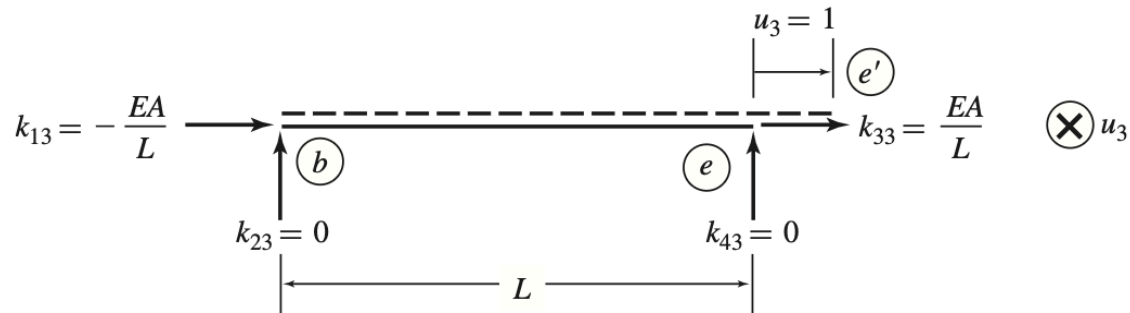
This force vector **defines the second column** of the local stiffness matrix.

Column 3: impose  $u_3 = 1$  ( $u_1 = u_2 = u_4 = 0$ )

$$\begin{bmatrix} k_{11} & k_{12} & \boxed{k_{13}} & k_{14} \\ k_{21} & k_{22} & \boxed{k_{23}} & k_{24} \\ k_{31} & k_{32} & \boxed{k_{33}} & k_{34} \\ k_{41} & k_{42} & \boxed{k_{43}} & k_{44} \end{bmatrix}$$

- $k_{13}$  ( $i = 1, j = 3$ )  
force at **DOF 1** due to unit displacement at **DOF 3**
- $k_{23}$  ( $i = 2, j = 3$ )  
force at **DOF 2** due to unit displacement at **DOF 3**
- $k_{33}$  ( $i = 3, j = 3$ )  
force at **DOF 3** due to unit displacement at **DOF 3**
- $k_{43}$  ( $i = 4, j = 3$ )  
force at **DOF 4** due to unit displacement at **DOF 3**

Column 3: impose  $u_3 = 1$  ( $u_1 = u_2 = u_4 = 0$ )



The resulting nodal force vector  $\mathbf{Q}$  is:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}_{u_3=1, u_{1,2,4}=0} = \frac{EA}{L} \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} k_{13} \\ k_{23} \\ k_{33} \\ k_{43} \end{Bmatrix}$$

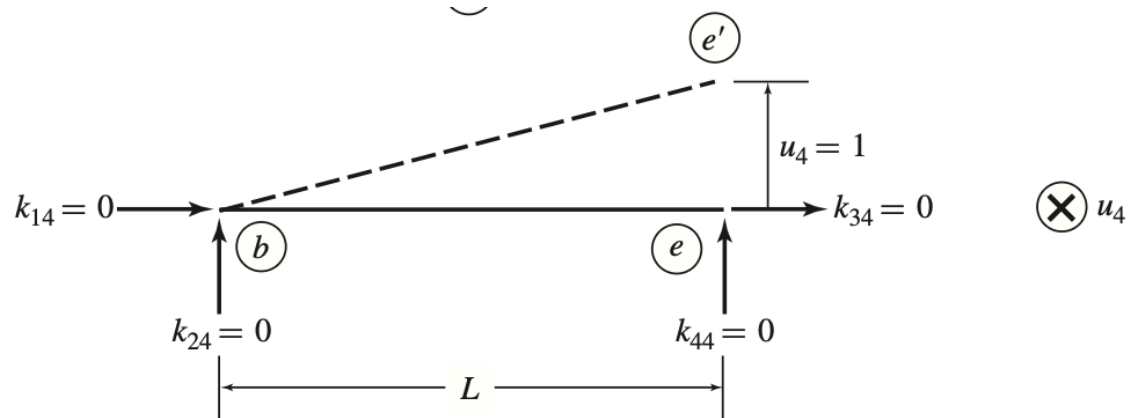
This force vector **defines the third column** of the local stiffness matrix.

Column 4: impose  $u_4 = 1$  ( $u_1 = u_2 = u_3 = 0$ )

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & \boxed{k_{14}} \\ k_{21} & k_{22} & k_{23} & \boxed{k_{24}} \\ k_{31} & k_{32} & k_{33} & \boxed{k_{34}} \\ k_{41} & k_{42} & k_{43} & \boxed{k_{44}} \end{bmatrix}$$

- $k_{14}$  ( $i = 1, j = 4$ )  
force at **DOF 1** due to unit displacement at **DOF 4** ( $u_4 = 1$ )
- $k_{24}$  ( $i = 2, j = 4$ )  
force at **DOF 2** due to unit displacement at **DOF 4**
- $k_{34}$  ( $i = 3, j = 4$ )  
force at **DOF 3** due to unit displacement at **DOF 4**
- $k_{44}$  ( $i = 4, j = 4$ )  
force at **DOF 4** due to unit displacement at **DOF 4**

Column 4: impose  $u_4 = 1$  ( $u_1 = u_2 = u_3 = 0$ )



The resulting nodal force vector  $Q$  is:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}_{u_4=1, u_{1,2,3}=0} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} k_{14} \\ k_{24} \\ k_{34} \\ k_{44} \end{Bmatrix}$$

This force vector **defines the fourth column** of the local stiffness matrix.

# Summary — Local 4×4 Truss Stiffness Matrix

For a 2D truss element in **local coordinates**, the full element stiffness matrix is:

$$\mathbf{k} = \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

## Key observations

- Only the **axial DOFs** ( $u_1, u_3$ ) contribute stiffness
- Transverse DOFs ( $u_2, u_4$ ) produce **no internal force** for a truss element
- The matrix is:
  - **symmetric** (reciprocity)
  - **singular** (rigid-body motion possible at the element level)

➡ This local matrix is the **building block** for assembling the global stiffness matrix, after transformation to global coordinates.

## Part 8 - In-Class Exercise

## Local Truss Stiffness Matrix Exercise

Work in pairs. Goal: **compute** the local nodal force vector and verify the key properties of the **local 4x4** stiffness matrix.

Given

Use:

- $E = 200000$ , MPa
- $A = 4,000$  mm<sup>2</sup>
- $L = 2.0$  m

Local DOF order and stiffness matrix (as per derivation in class):

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix}, \quad \mathbf{k} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Tasks

1. **Build** the local stiffness matrix  $\mathbf{k}$  in Python (print the matrix).
2. For each displacement case below, compute:  $\mathbf{f} = \mathbf{k}\mathbf{u}$
3. **For each displacement case**, print the following:
  - Displacement vector (mm)
  - Nodal force vector (N)
  - Axial deformation (mm)
  - Axial state (compression, tension, rigid body)
4. **Plot** the original and displaced element shape for each case.

## Displacement cases (local)

Use **mm** for all displacements. Each case uses **all four DOFs**.

- **Case A - Axial extension + transverse offset:**  
 $u_1 = 0, \quad u_2 = 200, \quad u_3 = 100, \quad u_4 = 200$
- **Case B — Mixed axial translation + transverse skew:**  
 $u_1 = 300, \quad u_2 = 0, \quad u_3 = 300, \quad u_4 = 100$
- **Case C — Axial shortening + opposite transverse motion:**  
 $u_1 = 200, \quad u_2 = 100, \quad u_3 = 0, \quad u_4 = -100$

# Looking Ahead

➡ Next (Lecture 3.2):

- Build the transformation matrix from local to global
- Rotate element stiffness into global coordinates
- Assemble the global stiffness matrix for a truss
- Apply supports, solve for displacements, and recover member forces