

CEE6501 — Lecture 4.1

The Direct Stiffness Method (DSM) for Trusses
(PART 2)

Learning Objectives

By the end of this lecture, you will be able to:

- Construct the **local-to-global transformation** for a truss member
- Compute element stiffness in global coordinates:

$$\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$$

- Assemble the **global stiffness matrix \mathbf{K}** using scatter-add
- Explain why an **unsupported structure** leads to a **singular stiffness matrix**
- Apply boundary conditions through **DOF partitioning** and solve for displacements
- Recover **member axial forces** from global displacements for design

Agenda

1. Review
2. Assembly of the global stiffness matrix \mathbf{K} (DSM)
3. Constraints and supports (boundary conditions)
4. Partitioning into free and restrained DOFs
5. Solving for global displacements and reactions
6. Element force recovery
7. DSM summary

Big idea:

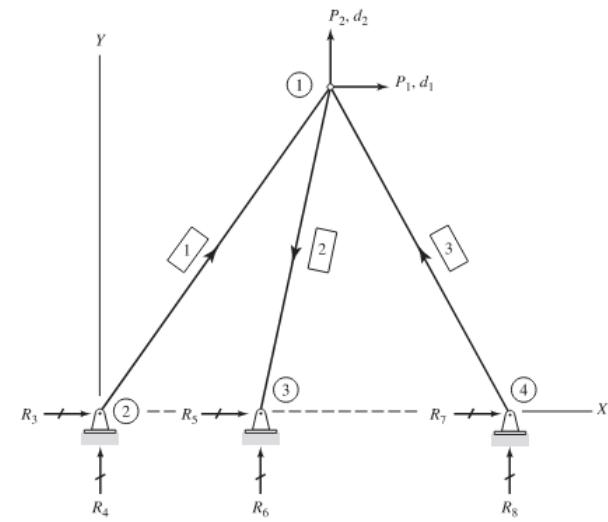
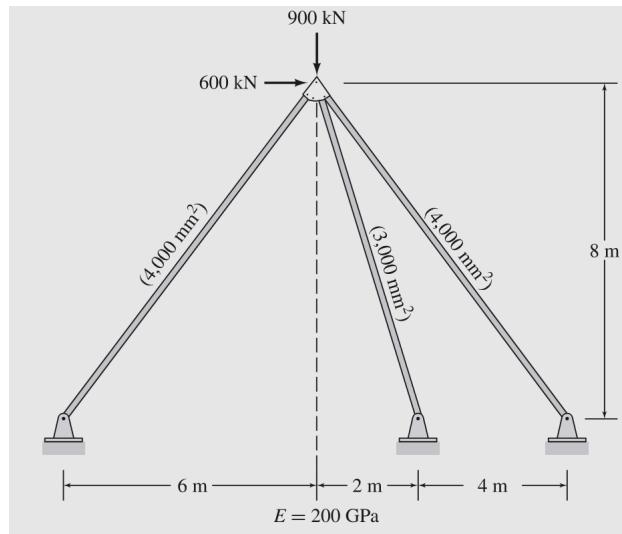
- A truss is a network of axial springs.
- Each element contributes stiffness to shared DOFs.
- Assembly is adding contributions into the right global rows/columns.

Part 1 — Assembling the Global Structure Stiffness Matrix

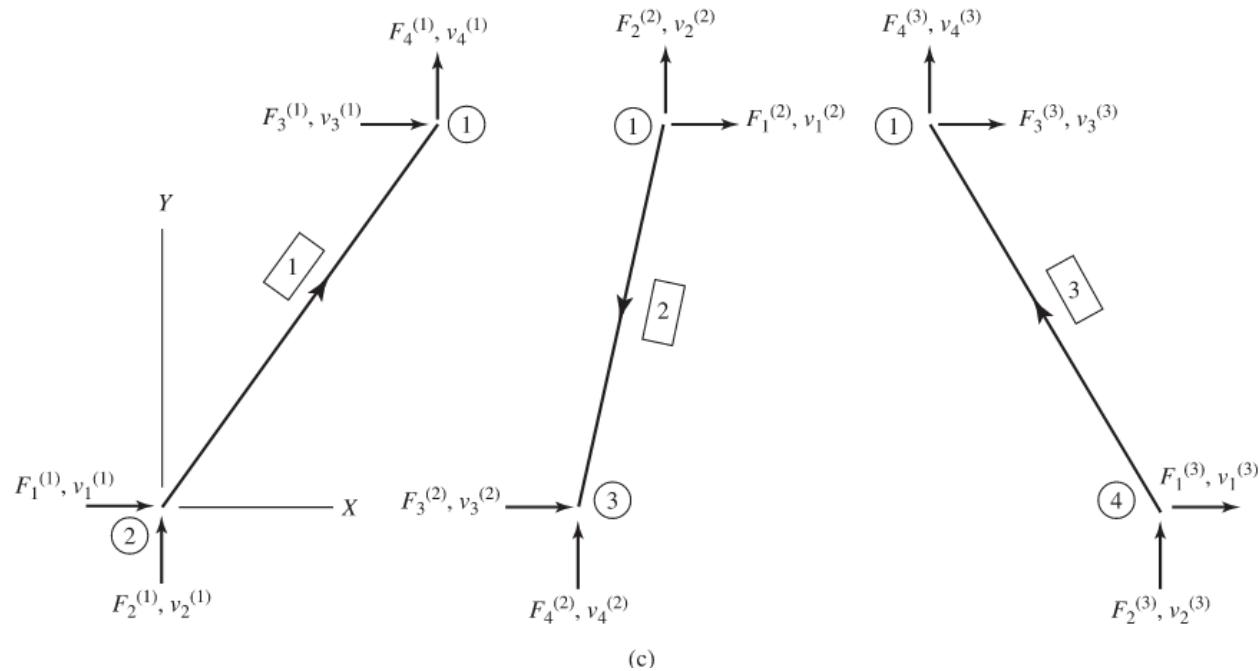
Review what we did last lecture

Example Structure

- Structure composed of **3 axial truss elements**
- 4 nodes**, each with (X, Y) translational DOFs, node 1 at the top
- Total system size: **8 global degrees of freedom**
 - node 1: DOF (1,2)
 - node 2: DOF (3,4)
 - node 3: DOF (5,6)
 - node 4: DOF (7,8)



Element-Level View of the Structure



$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$

$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

Summary of Force–Displacement Relations at Free Node

We derived these equations using displacement compatibility coupled with member-level equilibrium (see last lecture for more details on this process)

The forces acting at **end node of member 1** (Global node 1) are:

$$F_3^{(1)} = K_{33}^{(1)}d_1 + K_{34}^{(1)}d_2$$

$$F_4^{(1)} = K_{43}^{(1)}d_1 + K_{44}^{(1)}d_2$$

The forces acting at **start node of member 2** (Global node 1) are:

$$F_1^{(2)} = K_{11}^{(2)}d_1 + K_{12}^{(2)}d_2$$

$$F_2^{(2)} = K_{21}^{(2)}d_1 + K_{22}^{(2)}d_2$$

The forces acting at **end node of member 3** (Global node 1) are:

$$F_3^{(3)} = K_{33}^{(3)}d_1 + K_{34}^{(3)}d_2$$

$$F_4^{(3)} = K_{43}^{(3)}d_1 + K_{44}^{(3)}d_2$$

Global Stiffness Matrix for Free DOFs, \mathbf{K}_{ff}

$$P_1 = (K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)})d_1 + (K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)})d_2$$

$$P_2 = (K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)})d_1 + (K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)})d_2$$

These equations can be written compactly as:

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{bmatrix}}_{\mathbf{K}_{ff}} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$\mathbf{P} = \mathbf{K}_{ff} \mathbf{d}$

Now we can solve for unknown displacements in the 2 free DOFs

Why notation K_{ff} ? Later...to match partitioning node

Physical Interpretation of the Structure Stiffness Matrix

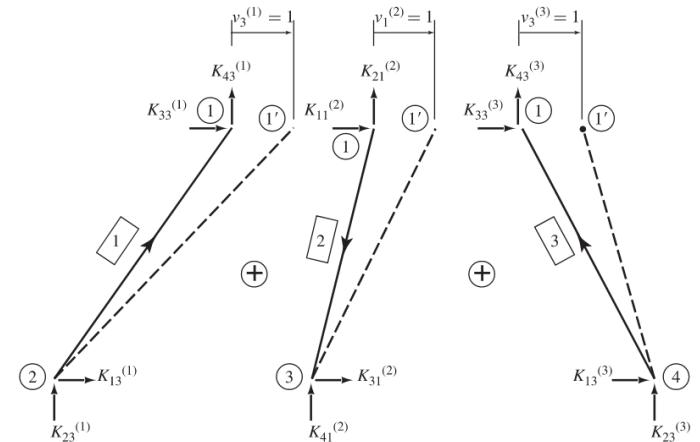
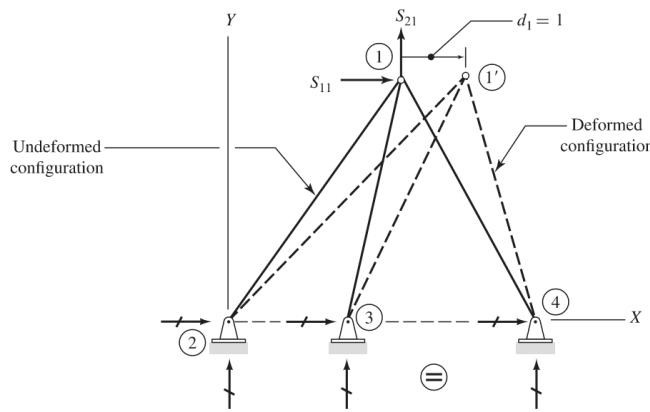
- The **structure stiffness matrix** \mathbf{K}_{ff} has the same physical meaning as an element stiffness matrix, but at the **structure (joint) level**
- A stiffness coefficient $K_{ff,ij}$ represents:
 - The **joint force** at DOF i
 - Required to cause a **unit displacement** at DOF j
 - While **all other joint displacements are zero**
- Equivalently:
 - Each **column** of \mathbf{K}_{ff} corresponds to a unit displacement pattern
 - The column entries are the **resulting joint forces** needed to enforce that displacement

Unit Displacement Method - Derive First Column of \mathbf{K}_{ff} - Unit Displacement Method

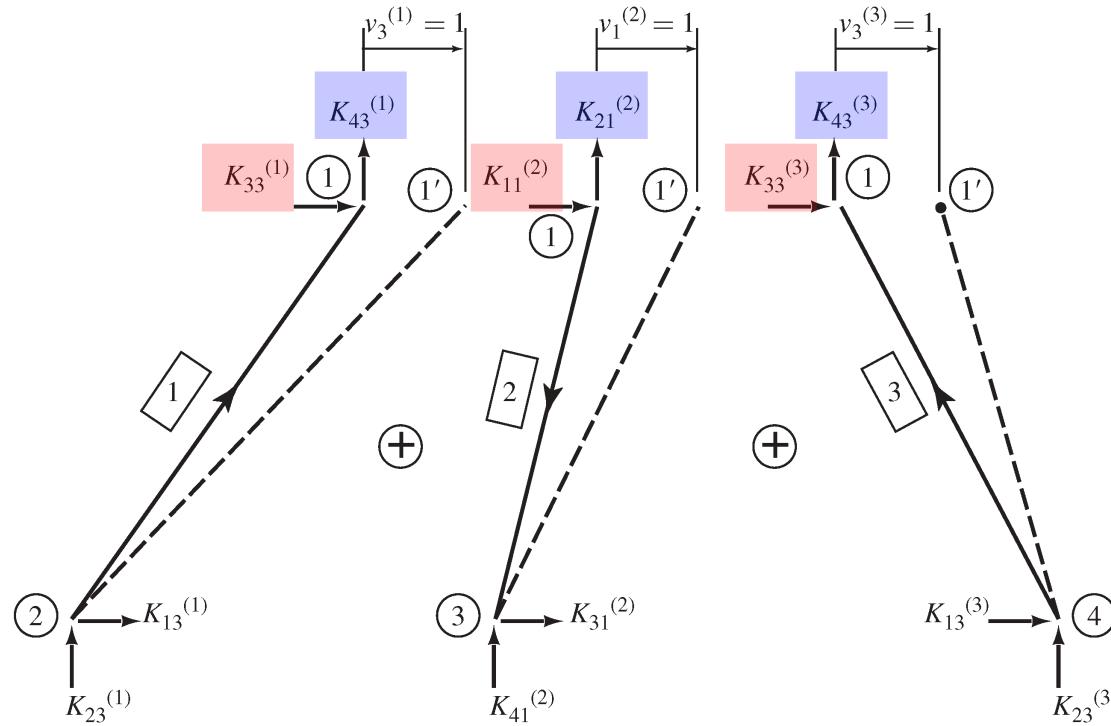
- To obtain the **first column** of \mathbf{K}_{ff} :
 - Impose a **unit displacement** at the first free DOF:

$$d_1 = 1, \quad d_2 = 0$$

- All other joint displacements are held fixed
- The resulting joint forces (DOF 1, 2) define the **first column** of \mathbf{K}_{ff}



red = in direction of displacement



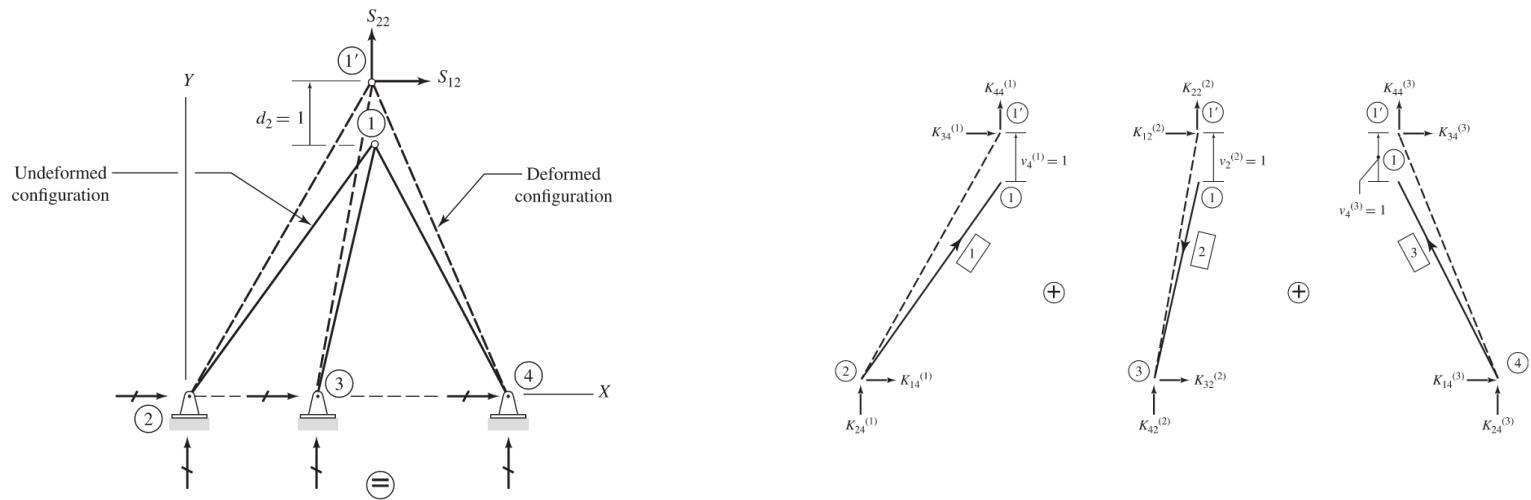
$$\begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{bmatrix}$$

Unit Displacement Method - Derive Second Column of \mathbf{K}_{ff}

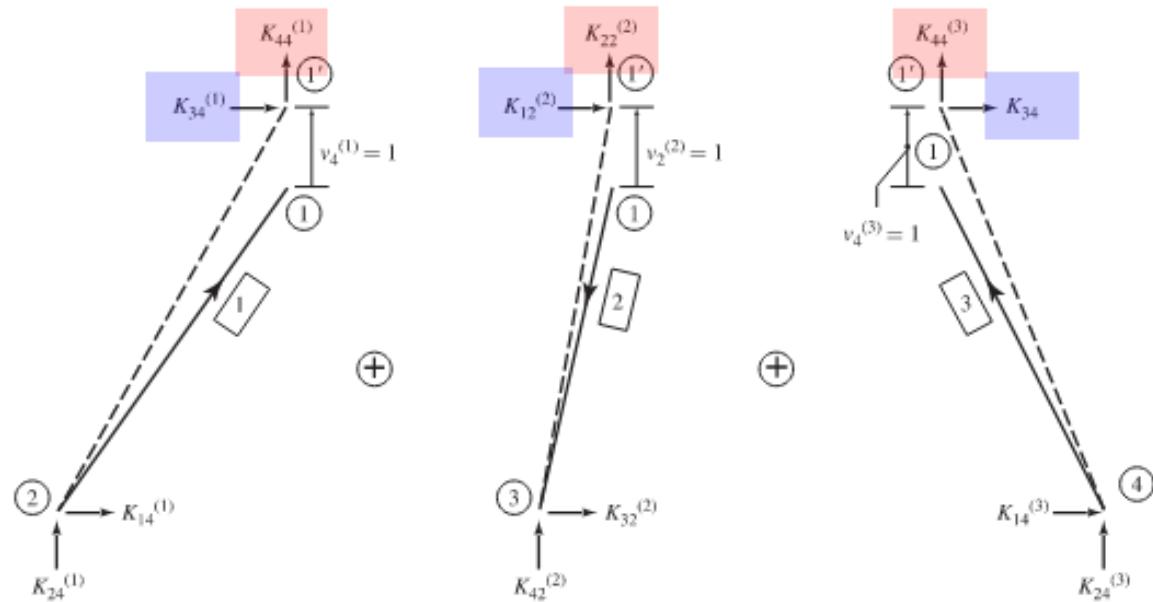
- To obtain the **second column** of \mathbf{K}_{ff} :
 - Impose a **unit displacement** at the first free DOF:

$$d_1 = 0, \quad d_2 = 1$$

- All other joint displacements are held fixed
- The resulting joint forces (DOF 1, 2) define the **second column** of \mathbf{K}_{ff}

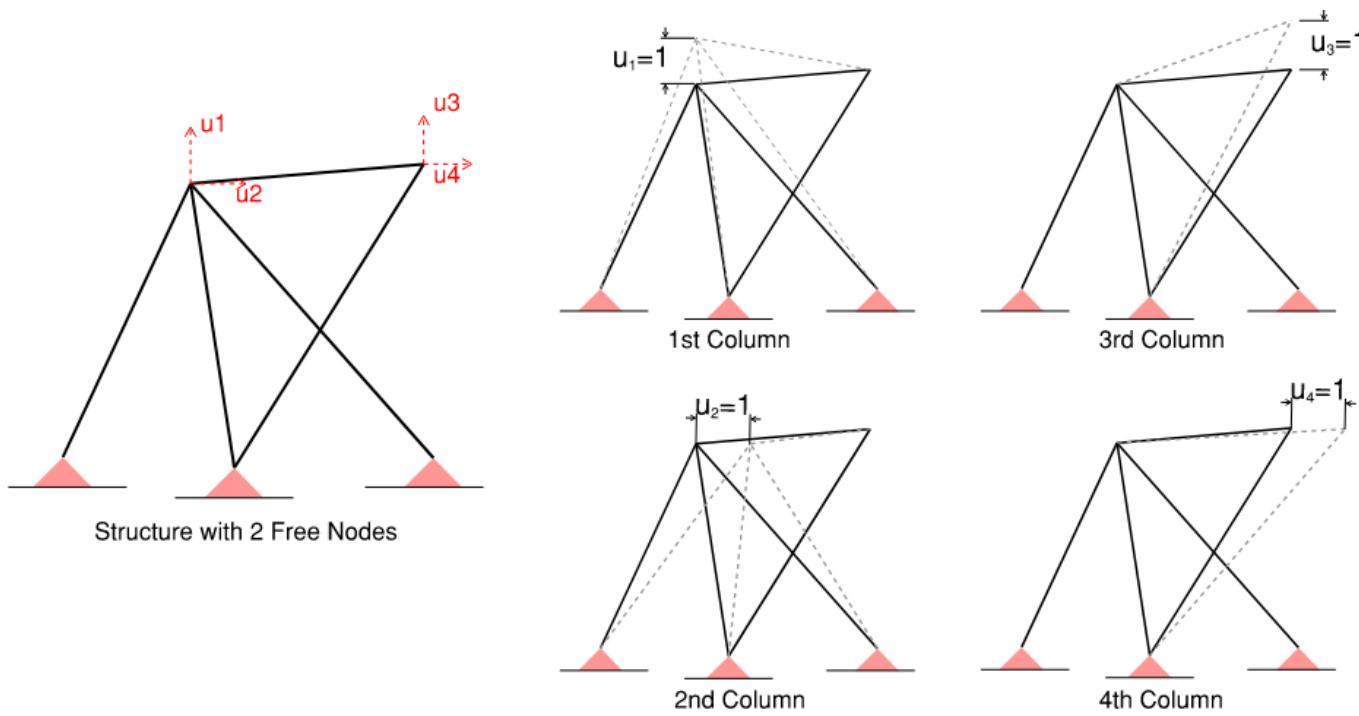


red = in direction of displacement



$$\begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & \boxed{K_{34}^{(1)} + K_{12}^{(2)} + K_{34}} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & \boxed{K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)}} \end{bmatrix}$$

Unit Displacement Method for More Complex Structures



Force equilibrium and compatibility relationships are even harder to set up!

Part 4 — Assembling the Global Structure Stiffness Matrix

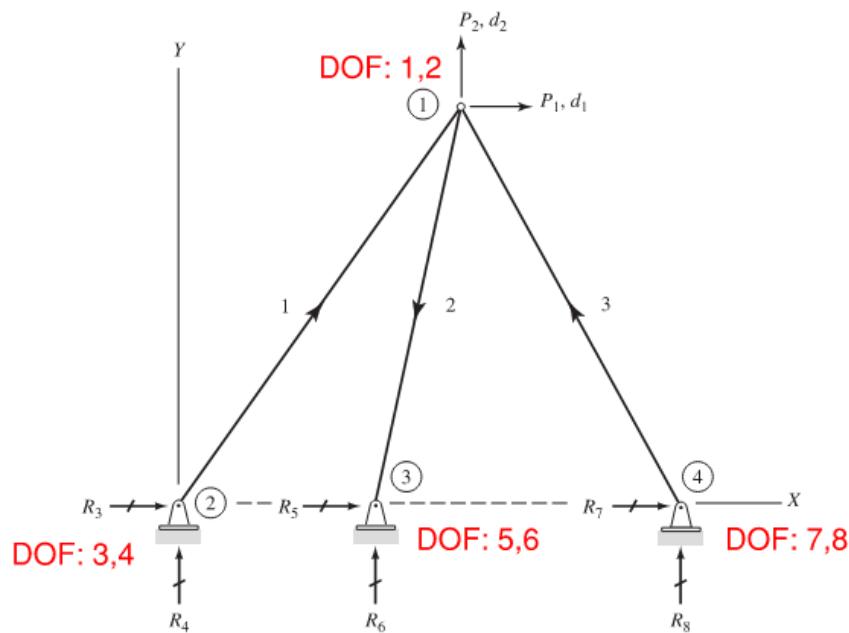
Direct Stiffness Method, Scatter Add

Why Not the Manual (Equilibrium + Compatibility) Approach?

- Becomes **tedious** very quickly
- Requires manually tracking:
 - Free vs constrained DOFs
 - Compatibility relationships
- Hard to implement **programmatically**
- Does **not scale** to larger or changing structures

Key idea: We want a method that is systematic, scalable, and algorithmic.

Reminder — Global Degrees of Freedom



- Each node contributes two DOFs: (*X*, *Y*)
- 4 nodes → **8 global DOFs**
- Consistent numbering is essential for assembly

Map Element DOFs to Global DOFs

$$\mathbf{K}_1 = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix}$$

$$\mathbf{K}_2 = \begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}$$

$$\mathbf{K}_3 = \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix} \begin{matrix} 7 \\ 8 \\ 1 \\ 2 \end{matrix}$$

Each element has **4 local DOFs** (1, 2, 3, 4), which must be mapped to the correct **global DOFs** before assembly.

For this structure:

- **Element 1:** local (1, 2, 3, 4) → global (3, 4, 1, 2)
- **Element 2:** local (1, 2, 3, 4) → global (1, 2, 5, 6)
- **Element 3:** local (1, 2, 3, 4) → global (7, 8, 1, 2)

This mapping determines **which rows and columns** of the global stiffness matrix each element stiffness contributes to.

Method 1: Assemble a Structure Matrix for the Free DOFs

$$\mathbf{K}_1 = \begin{bmatrix} 3 & 4 & 1 & 2 \\ K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{bmatrix}$$

$$\mathbf{K}_2 = \begin{bmatrix} 1 & 2 & 5 & 6 \\ K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{bmatrix}$$

$$\mathbf{K}_3 = \begin{bmatrix} 7 & 8 & 1 & 2 \\ K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 1 & 2 \\ K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & [K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)}] \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & [K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)}] \end{bmatrix}$$

In this example, only **global DOFs 1 and 2** are free; all other DOFs are constrained.

- Contributions from **each element** that map to DOFs 1 and 2 are **summed**
- This produces a reduced **structure stiffness matrix** for the free DOFs
- The result matches the stiffness matrix obtained earlier using the **manual equilibrium and compatibility** approach

For example, the (1, 1) entry of the structure stiffness matrix is (matches manual):

$$K_{ff,11} = K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)}$$

Method 2: Assemble the Full Global Stiffness Matrix

When assembling the **global structure stiffness matrix**, do **not** worry yet about which DOFs are constrained or free.

- **Assemble all DOFs first** into a single global matrix
- **Apply boundary conditions later** (supports, prescribed displacements)
- This separation is what makes the method **general, systematic, and scalable**

For the structure shown, there are **8 global DOFs**, so 8×8 stiffness matrix:

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} \end{bmatrix}$$

Each entry K_{ij} represents the **combined stiffness contribution** from all elements that connect global DOF j to global DOF i .

Element 1 — Contribution to Global Stiffness Matrix

Local → Global DOF mapping (Element 1):

$$(1, 2, 3, 4) \rightarrow (3, 4, 1, 2)$$

$$\mathbf{K}_1 = \begin{bmatrix} 3 & 4 & 1 & 2 \\ K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} \end{bmatrix}$$

Element 2 — Contribution to Global Stiffness Matrix

Local → Global DOF mapping (Element 2):

$$(1, 2, 3, 4) \rightarrow (1, 2, 5, 6)$$

$$\mathbf{K}_2 = \begin{bmatrix} 1 & 2 & 5 & 6 \\ K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 5 \\ 6 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} \end{bmatrix}$$

Element 3 — Contribution to Global Stiffness Matrix

Local → Global DOF mapping (Element 3):

$$(1, 2, 3, 4) \rightarrow (7, 8, 1, 2)$$

$$\mathbf{K}_3 = \begin{bmatrix} 7 & 8 & 1 & 2 \\ K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix}$$

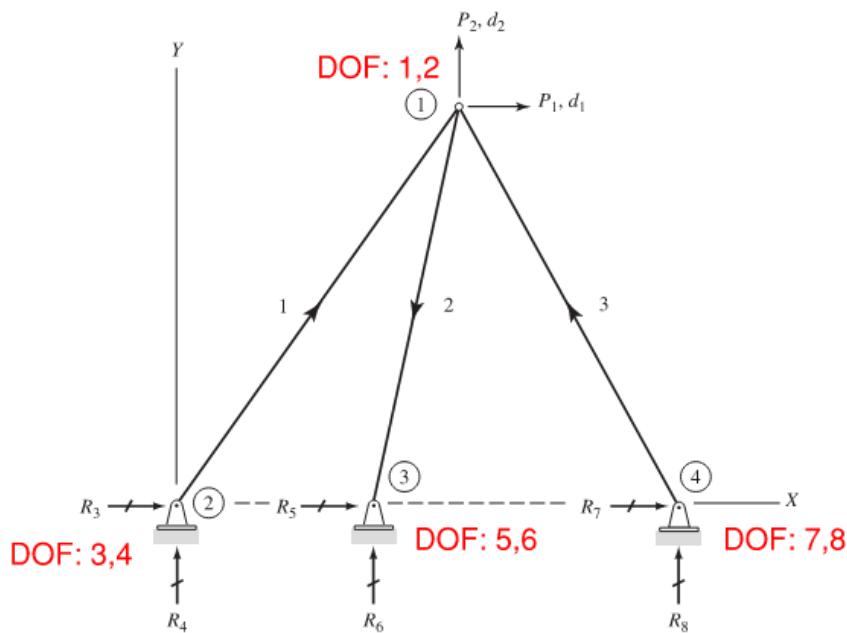
$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} \end{bmatrix}$$

Final Global Structure Stiffness Matrix

- Each K_{ij} may include **multiple element contributions**
- The matrix is **symmetric**
- Diagonal terms, K_{ii} , must be non-zero and positive for stability

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & 0 & 0 & 0 & 0 \\ K_{41} & K_{42} & K_{43} & K_{44} & 0 & 0 & 0 & 0 \\ K_{51} & K_{52} & 0 & 0 & K_{55} & K_{56} & 0 & 0 \\ K_{61} & K_{62} & 0 & 0 & K_{65} & K_{66} & 0 & 0 \\ K_{71} & K_{72} & 0 & 0 & 0 & 0 & K_{77} & K_{78} \\ K_{81} & K_{82} & 0 & 0 & 0 & 0 & K_{87} & K_{88} \end{bmatrix}$$

Interpreting Zeros in the Stiffness Matrix



- Zero entry \rightarrow no stiffness coupling
- Example: $K_{37} = 0$
- DOF 3 and DOF 7 share no connecting element

Part 5 — Constraints and Supports

Inverting the Global Stiffness Matrix

You might think: *we have \mathbf{K} , so we can just solve*

$$\mathbf{K} \mathbf{u} = \mathbf{f}$$

and we're done.

However, for a **free (unconstrained) structure**, this step is **not yet valid**.

At this stage—before applying supports or boundary conditions—the global stiffness matrix typically contains **rigid-body modes** (motions that produce no strain energy). As a result:

- \mathbf{K} is **singular** (not invertible)
- the structure still has **rigid-body motion** because we haven't applied supports / boundary conditions yet

Important note (about our 8 DOF example):

In our 8×8 formulation, the ability to invert \mathbf{K} is an **artifact of how the matrix is set up**. The assembly already embeds enough *support-like stiffness*—intentionally or unintentionally—to suppress rigid-body motion.

As a result, the structure represented by this \mathbf{K} is **not truly free**, even though we have not yet explicitly applied boundary conditions.

In general, however, a global stiffness matrix assembled for an unconstrained structure **will be singular**.

This is why the proper and robust approach is always to apply supports explicitly (via partitioning) before solving.

In [4]:

```
import numpy as np

# Construct a singular 6x6 matrix
# Last row = sum of first two rows → linear dependence
K = np.array([
    [ 2, -1,  0,  0,  0,  0],
    [-1,  2, -1,  0,  0,  0],
    [ 0, -1,  2, -1,  0,  0],
    [ 0,  0, -1,  2, -1,  0],
    [ 0,  0,  0, -1,  2, -1],
    [ 1,  1, -1,  0,  0,  0] # dependent row
], dtype=float)

try:
    np.linalg.inv(K)
except np.linalg.LinAlgError as e:
    print("Inversion failed:", e)
```

Inversion failed: Singular matrix

Why do we get a singular stiffness matrix?

Mathematically, an unconstrained structure admits non-zero displacement vectors $\mathbf{u} \neq \mathbf{0}$ such that

$$\mathbf{K}\mathbf{u} = \mathbf{0}$$

These displacement patterns are **rigid-body (zero-energy) modes**, meaning \mathbf{K} has a **non-trivial nullspace** and is therefore **singular**.

What does this mean in practice?

As a result, the structure can move without inducing member deformation:

- **Rigid translation in X**
- **Rigid translation in Y**
- **Rigid-body rotation** (geometry-dependent)

These motions are **rigid-body modes**, not structural deformations.

Because rigid-body modes produce zero internal force, \mathbf{K} cannot uniquely map applied loads to displacements. **Supports (boundary conditions)** remove these modes and render \mathbf{K} invertible.

Boundary conditions: how supports enter the model

Supports and constraints are enforced by specifying **known displacements** (a.k.a. prescribed DOFs).

Common idealized supports (2D truss):

- **Pin support:** Prescribes both translations: $u_{ix} = 0$ and $u_{iy} = 0$
- **Roller support:** Prescribes one translation and allows the other:

Prescribed displacement (settlement / actuation):

- You can prescribe a non-zero value, e.g., $u_{iy} = -2$ mm at a support.

Why this fixes singularity:

- Constraints remove rigid-body modes by preventing global translations/rotations.
- The remaining free DOFs correspond to true structural deformation.

What we do next (workflow)

We do **not** modify the element --> global matrix formulation to handle supports.

Instead, the Direct Stiffness Method proceeds like this:

1. Assemble the **full** global matrix \mathbf{K} and global load vector \mathbf{F}
2. Identify which DOFs are **free** vs **restrained**
3. Enforce boundary conditions by **partitioning** (or equivalently, by eliminating restrained DOFs)
4. Solve only for the unknown (free) displacements
5. Recover reactions at restrained DOFs

So supports are handled cleanly at the **system equation** level — via partitioning (next section).

Part 6 — Partitioning the Matrix

Free vs Restrained DOFs

Core idea — Partition DOFs (free vs restrained)

Separate the global degrees of freedom into two sets:

- **Free DOFs** f : unknown displacements (to be solved)
- **Restrained DOFs** r : prescribed displacements (from supports or constraints)

Reorder the global displacement and force vectors so that all free DOFs come first:

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_r \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_f \\ \boldsymbol{F}_r \end{bmatrix}$$

Apply the **same reordering** to the global stiffness matrix to obtain a partitioned system:

$$\begin{bmatrix} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fr} \\ \boldsymbol{K}_{rf} & \boldsymbol{K}_{rr} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_r \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_f \\ \boldsymbol{F}_r \end{bmatrix}$$

Physical meaning K_{ff} etc..

Note: In our **8×8 truss example**, the system already appears partitioned because the two free global DOFs were labeled **1 and 2**, placing them in the correct positions from the start.

To illustrate the **general case**, the next slides use a separate **6×6 example** in which free and restrained DOFs are initially **intermixed**. The goal there is to show how the system is **reordered** so that all free DOFs appear first.

Step 0 — Original ordering (free DOFs = 3, 5)

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \boxed{u_3} \\ u_4 \\ \boxed{u_5} \\ u_6 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ \boxed{F_3} \\ F_4 \\ \boxed{F_5} \\ F_6 \end{bmatrix}$$

The corresponding global stiffness matrix has rows and columns associated with the same DOFs. Rows and columns for free DOFs (3 and 5) are highlighted.

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & \boxed{K_{13}} & K_{14} & \boxed{K_{15}} & K_{16} \\ K_{21} & K_{22} & \boxed{K_{23}} & K_{24} & \boxed{K_{25}} & K_{26} \\ \boxed{K_{31}} & \boxed{K_{32}} & \boxed{K_{33}} & \boxed{K_{34}} & \boxed{K_{35}} & \boxed{K_{36}} \\ K_{41} & K_{42} & \boxed{K_{43}} & K_{44} & \boxed{K_{45}} & K_{46} \\ \boxed{K_{51}} & \boxed{K_{52}} & \boxed{K_{53}} & \boxed{K_{54}} & \boxed{K_{55}} & \boxed{K_{56}} \\ K_{61} & K_{62} & \boxed{K_{63}} & K_{64} & \boxed{K_{65}} & K_{66} \end{bmatrix}$$

Step 1 — Bring DOF 3 to the top (swap DOFs 1 and 3)

Displacement and force vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \boxed{u_3} \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{u_3} \\ u_2 \\ u_1 \\ u_4 \\ \boxed{u_5} \\ u_6 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ \boxed{F_3} \\ F_4 \\ \boxed{F_5} \\ F_6 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{F_3} \\ F_2 \\ F_1 \\ F_4 \\ \boxed{F_5} \\ F_6 \end{bmatrix}$$

Stiffness matrix (rows 1 \leftrightarrow 3, columns 1 \leftrightarrow 3)

$$\mathbf{K}^{(1)} = \begin{bmatrix} K_{33} & K_{32} & K_{31} & K_{34} & K_{35} & K_{36} \\ \boxed{K_{23}} & K_{22} & K_{21} & K_{24} & \boxed{K_{25}} & K_{26} \\ \boxed{K_{13}} & K_{12} & K_{11} & K_{14} & \boxed{K_{15}} & K_{16} \\ \boxed{K_{43}} & K_{42} & K_{41} & K_{44} & \boxed{K_{45}} & K_{46} \\ \boxed{K_{53}} & \boxed{K_{52}} & \boxed{K_{51}} & \boxed{K_{54}} & \boxed{K_{55}} & \boxed{K_{56}} \\ \boxed{K_{63}} & K_{62} & K_{61} & K_{64} & \boxed{K_{65}} & K_{66} \end{bmatrix}$$

Step 2 — Bring DOF 5 to position 2 (swap DOFs 2 and 5)

Displacement and force vectors

$$\mathbf{u} = \begin{bmatrix} u_3 \\ u_2 \\ u_1 \\ u_4 \\ \boxed{u_5} \\ u_6 \end{bmatrix} \longrightarrow \begin{bmatrix} u_3 \\ \boxed{u_5} \\ u_1 \\ u_4 \\ u_2 \\ u_6 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_3 \\ F_2 \\ F_1 \\ F_4 \\ \boxed{F_5} \\ F_6 \end{bmatrix} \longrightarrow \begin{bmatrix} F_3 \\ \boxed{F_5} \\ F_1 \\ F_4 \\ F_2 \\ F_6 \end{bmatrix}$$

Stiffness matrix (rows 2 \leftrightarrow 5, columns 2 \leftrightarrow 5)

$$\mathbf{K}^{(2)} = \begin{bmatrix} K_{33} & K_{35} & K_{31} & K_{34} & K_{32} & K_{36} \\ K_{53} & K_{55} & K_{51} & K_{54} & K_{52} & K_{56} \\ \boxed{K_{13}} & \boxed{K_{15}} & K_{11} & K_{14} & K_{12} & K_{16} \\ \boxed{K_{43}} & \boxed{K_{45}} & K_{41} & K_{44} & K_{42} & K_{46} \\ \boxed{K_{23}} & \boxed{K_{25}} & K_{21} & K_{24} & K_{22} & K_{26} \\ \boxed{K_{63}} & \boxed{K_{65}} & K_{61} & K_{64} & K_{62} & K_{66} \end{bmatrix}$$

Partitioned form (read directly)

After reordering, the free DOFs come first. For our example, the free set is $f = \{3, 5\}$ and the restrained set is $r = \{1, 4, 2, 6\}$, so

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_f \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} u_3 \\ u_5 \\ u_1 \\ u_4 \\ u_2 \\ u_6 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{F}_f \\ \mathbf{F}_r \end{bmatrix} = \begin{bmatrix} F_3 \\ F_5 \\ F_1 \\ F_4 \\ F_2 \\ F_6 \end{bmatrix}$$

$$\mathbf{K} = \left[\begin{array}{c|c} \mathbf{K}_{ff} & \mathbf{K}_{fr} \\ \hline \mathbf{K}_{rf} & \mathbf{K}_{rr} \end{array} \right] = \left[\begin{array}{cc|ccccc} K_{33} & K_{35} & K_{31} & K_{34} & K_{32} & K_{36} \\ K_{53} & K_{55} & K_{51} & K_{54} & K_{52} & K_{56} \\ \hline K_{13} & K_{15} & K_{11} & K_{14} & K_{12} & K_{16} \\ K_{43} & K_{45} & K_{41} & K_{44} & K_{42} & K_{46} \\ K_{23} & K_{25} & K_{21} & K_{24} & K_{22} & K_{26} \\ K_{63} & K_{65} & K_{61} & K_{64} & K_{62} & K_{66} \end{array} \right]$$

Here the dashed line shows the split between free and restrained DOFs:

- top-left block: \mathbf{K}_{ff} (free–free)
- top-right block: \mathbf{K}_{fr} (free–restrained)
- bottom-left block: \mathbf{K}_{rf} (restrained–free)
- bottom-right block: \mathbf{K}_{rr} (restrained–restrained)

Part 7 — Solving Global Displacements and Forces

free node displacements and restraint forces

Expand the Partitioned System

Starting from the block system

$$\begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fr} \\ \mathbf{K}_{rf} & \mathbf{K}_{rr} \end{bmatrix} \begin{bmatrix} \mathbf{u}_f \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} \mathbf{F}_f \\ \mathbf{F}_r \end{bmatrix}$$

expanding the matrix–vector multiplication gives two coupled equations:

$$\mathbf{K}_{ff}\mathbf{u}_f + \mathbf{K}_{fr}\mathbf{u}_r = \mathbf{F}_f \tag{1}$$

$$\mathbf{K}_{rf}\mathbf{u}_f + \mathbf{K}_{rr}\mathbf{u}_r = \mathbf{F}_r \tag{2}$$

The first equation governs equilibrium at the **free DOFs**,
and the second governs equilibrium at the **restrained DOFs**.

Solving Equation (1) for the Free DOFs

We are interested in solving for the **unknown displacements** at the free DOFs, \mathbf{u}_f .

Starting from Equation (1): $\mathbf{K}_{ff}\mathbf{u}_f + \mathbf{K}_{fr}\mathbf{u}_r = \mathbf{F}_f$

Rearrange to isolate the unknowns:

$$\mathbf{K}_{ff}\mathbf{u}_f = \mathbf{F}_f - \mathbf{K}_{fr}\mathbf{u}_r$$

Provided that \mathbf{K}_{ff} is invertible, the solution is:

$$\boxed{\mathbf{u}_f = \mathbf{K}_{ff}^{-1}(\mathbf{F}_f - \mathbf{K}_{fr}\mathbf{u}_r)}$$

Typically, the restrained displacements \mathbf{u}_r are **known** from boundary conditions (often $\mathbf{u}_r = \mathbf{0}$). In that common case, the expression simplifies to:

$$\boxed{\mathbf{u}_f = \mathbf{K}_{ff}^{-1}\mathbf{F}_f}$$

Solve in practice to obtain the global displacements at the free DOFs.

Solving Equation (2) for the Restrained Forces

Once the free displacements \mathbf{u}_f have been computed, we can determine the **forces at the restrained DOFs** (support reactions).

Starting from Equation (2): $\mathbf{K}_{rf}\mathbf{u}_f + \mathbf{K}_{rr}\mathbf{u}_r = \mathbf{F}_r$

At the restrained DOFs, the displacements \mathbf{u}_r are **known** from the boundary conditions (often $\mathbf{u}_r = \mathbf{0}$). Substituting these known values gives a direct expression for the reaction forces:

$$\boxed{\mathbf{F}_r = \mathbf{K}_{rf}\mathbf{u}_f + \mathbf{K}_{rr}\mathbf{u}_r}$$

In the common case where the supports are fixed and $\mathbf{u}_r = \mathbf{0}$, this simplifies to:

$$\boxed{\mathbf{F}_r = \mathbf{K}_{rf}\mathbf{u}_f}$$

This step **does not require solving a linear system** (i.e., no inverse needed).

The reaction forces are obtained directly by evaluating this expression after \mathbf{u}_f is known.

Summary — What We Have Solved

At this point, the global system has been fully resolved.

We have determined the two previously unknown quantities:

- **Free nodal displacements, \mathbf{u}_f**
(obtained by solving the reduced system)
- **Support reaction forces, \mathbf{F}_r**
(computed directly once \mathbf{u}_f is known)

Together, these results provide the complete displacement and force state of the structure, satisfying equilibrium, compatibility, and the imposed boundary conditions.

Sanity checks after solving

1) Stability

- If the structure is properly constrained, \mathbf{K}_{ff} should be non-singular
- If it is still singular, you likely have:
 - insufficient supports (rigid-body mode remains), or
 - a disconnected node / element connectivity error

2) Equilibrium

- Reconstruct the full displacement vector \mathbf{u} by inserting \mathbf{u}_f (just calculated) and \mathbf{u}_r
- Compute $\mathbf{K}\mathbf{u}$ and compare to \mathbf{F}
- Differences should be near numerical tolerance (floating point error)

3) Reactions

- Check reaction directions and magnitudes against intuition
- Sum of reactions should balance applied loads (global equilibrium)

Part 8 — Recover Element Forces

Back to Local

Up to this point, we have solved for the **global nodal displacements** of the structure. While this information is essential for equilibrium, it is **not** what we design members for.

Structural design is performed at the **element level**, using:

- axial forces
- stresses
- internal force resultants

expressed in each element's **local coordinate system**.

This step bridges analysis and design.

Why Go Back to the Element Level?

- Global displacements describe how the **structure moves**
- Local element forces describe how **members carry load**
- Design checks (strength, buckling, serviceability) are based on **element forces**, not global DOFs

Therefore, we must recover **element-level displacements and forces** from the global solution.

Element Force Recovery — Overview

For each element, we perform the following steps:

- 1. Extract element global displacements**

Assemble the element displacement vector from the global solution.

- 2. Transform to local coordinates**

Convert global displacements to the element's local axis.

- 3. Compute local end forces**

Use the element stiffness matrix to compute internal forces.

- 4. Evaluate axial force**

Extract the axial force used directly in design.

Step 1 — Extract Element Global Displacements

From the global displacement vector \mathbf{u} , collect the DOFs associated with element e :

$$\mathbf{u}_e = \begin{bmatrix} u_{i,x} \\ u_{i,y} \\ u_{j,x} \\ u_{j,y} \end{bmatrix}$$

This vector contains the **global displacements** of the element's end nodes.

Step 2 — Transform to Local Coordinates

Transform the element displacement vector to the local coordinate system:

$$\mathbf{u}'_e = \mathbf{T}_e \mathbf{u}_e$$

where \mathbf{T}_e is the **element transformation matrix**, for that specific element defined by the element orientation.

This step aligns the displacements with the element's axial direction.

Step 3 — Compute Local End Forces

Use the local stiffness matrix to compute the internal end forces:

$$\mathbf{f}'_e = \mathbf{k}'_e \mathbf{u}'_e$$

These are the **local element forces** (axial forces at each end). For a truss element you should get the axial force in the element

Step 4 — Compute Axial Stress

Once the axial force is known, the **axial stress** follows as:

$$\sigma = \frac{N}{A}$$

This is what you design for:

- axial stress limits
- tension vs. compression
- buckling checks for compression members

Step 5 (Optional) — Member End Forces in Global Coordinates

This step is **not required for design**, but it can be useful for interpretation and post-processing.

So far, member forces have been computed in the **local coordinate system**, which is where axial behavior is defined.

In some cases, however, it is helpful to express the **member end forces in the global (X, Y) frame**, for example:

- to visualize force flow in the structure
- to check nodal equilibrium in global directions
- to compare with externally applied loads

The local member force vector can be rotated back to global coordinates as:

$$\mathbf{f}_e = \mathbf{T}_e^T \mathbf{f}'_e$$

For a **perfectly horizontal member**, the local and global axes coincide, and this transformation leaves the forces unchanged.

Part 9 — DSM Summary

Forward Pass — Structural Analysis

1. Defining Structure

- Node numbering and coordinates
- Global DOF numbering
- Element connectivity
- Restraints and Applied Forces

2. Element-level

- Compute geometry: L, c, s
- Build transformation matrix, \mathbf{T}
- Compute global element stiffness:

$$\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$$

3. Assemble global stiffness matrix

- Scatter-add element contributions into \mathbf{K}

Forward Pass — Structural Analysis, cont...

4. Apply boundary conditions

- Partition DOFs into free (f) and restrained (r)

5. Solve for unknown displacements

$$\mathbf{u}_f = \mathbf{K}_{ff}^{-1} (\mathbf{F}_f - \mathbf{K}_{fr} \mathbf{u}_r)$$

6. Recover support reactions

$$\mathbf{F}_r = \mathbf{K}_{rf} \mathbf{u}_f + \mathbf{K}_{rr} \mathbf{u}_r$$

Backward Pass — Element Recovery and Design

7. Extract element global displacement vectors

- For each member, collect the relevant entries from \mathbf{u} to form \mathbf{u}_e

8. Transform displacements to local coordinates

$$\mathbf{u}' = \mathbf{T} \mathbf{u}_e$$

9. Compute local element end forces

$$\mathbf{f}' = \mathbf{k}' \mathbf{u}'$$

10. Compute axial force and stress (design quantities)

$$\sigma = \frac{N}{A}$$

11. (Optional) Express element forces in global coordinates

$$\mathbf{f} = \mathbf{T}^\top \mathbf{f}'$$

Why DSM Is Powerful

- **Static indeterminacy is not an issue**
 - Any number of members can be handled
- **Equilibrium is enforced automatically**
 - Through the global stiffness equations
- **Compatibility is built in**
 - Via shared nodal displacements
- The method scales cleanly to:
 - large structures
 - complex geometries
 - numerical implementation

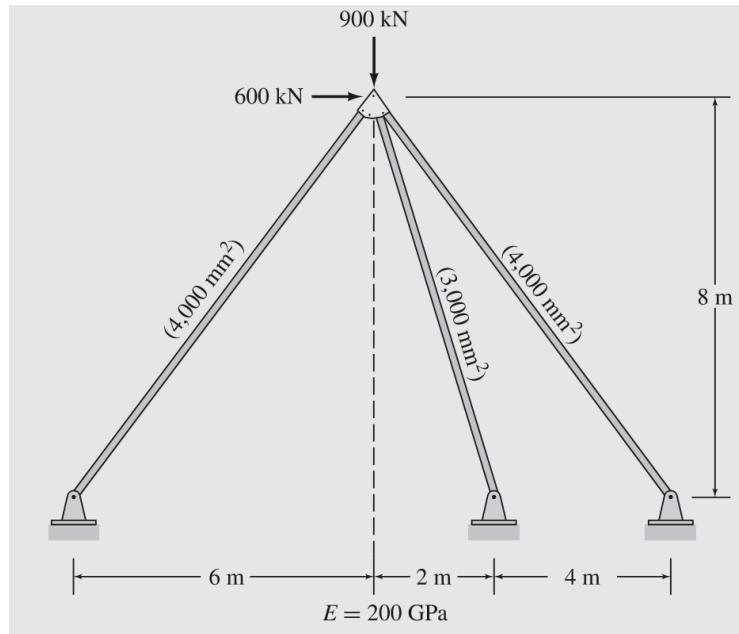
DSM replaces manual equilibrium and compatibility equations with a single, systematic matrix framework.

Class Problem (Time Permitting)

This problem will form the basis of **this week's homework**.

Begin by working through the setup **by hand**:

- define the geometry and connectivity,
- number the global degrees of freedom,
- setup the local element stiffness matrices (calculate transformation matrices).



Node numbering:

- Node 1: bottom-left support
- Node 2: middle support
- Node 3: top node
- Node 4: right support

Wrap-Up

Today you built the DSM pipeline for trusses:

- local bar stiffness → transformation → global element stiffness
- assembly → constraints → solve → member force recovery

Next Lecture: implement DSM in Python for a worked truss example and discuss efficiency (sparsity/bandedness).