

CEE6501 — Lecture 6.2

2D Beam Element Stiffness Matrix

Learning Objectives

By the end of this lecture, you will be able to:

- Define the **local DOFs**, ordering, and sign conventions for a 2D Euler–Bernoulli beam element
- Write the **local displacement and force vectors** and interpret the relation $\mathbf{Q} = \mathbf{k}\mathbf{u}$
- Use the **unit displacement method** to compute stiffness coefficients (and interpret each column physically)
- Assemble and recognize the **standard closed-form 4×4 beam stiffness matrix**
- Use **beam shape functions** to evaluate and plot a **continuous deformed shape** from nodal DOFs

Agenda

1. **Part 1** — Beam Element Stiffness Relations
2. **Part 2** — Derive Column 1 of the Beam Element Stiffness Matrix
3. **Part 3** — Complete 4×4 Beam Stiffness Matrix
4. **Part 4** — Beam Deformation Patterns

Part 1 — Beam Element Stiffness Relations

Beam Element Response

The **member stiffness relations** express the end forces of a beam element (including shear forces and bending moments) as functions of the **end displacements** (including transverse displacements and rotations).

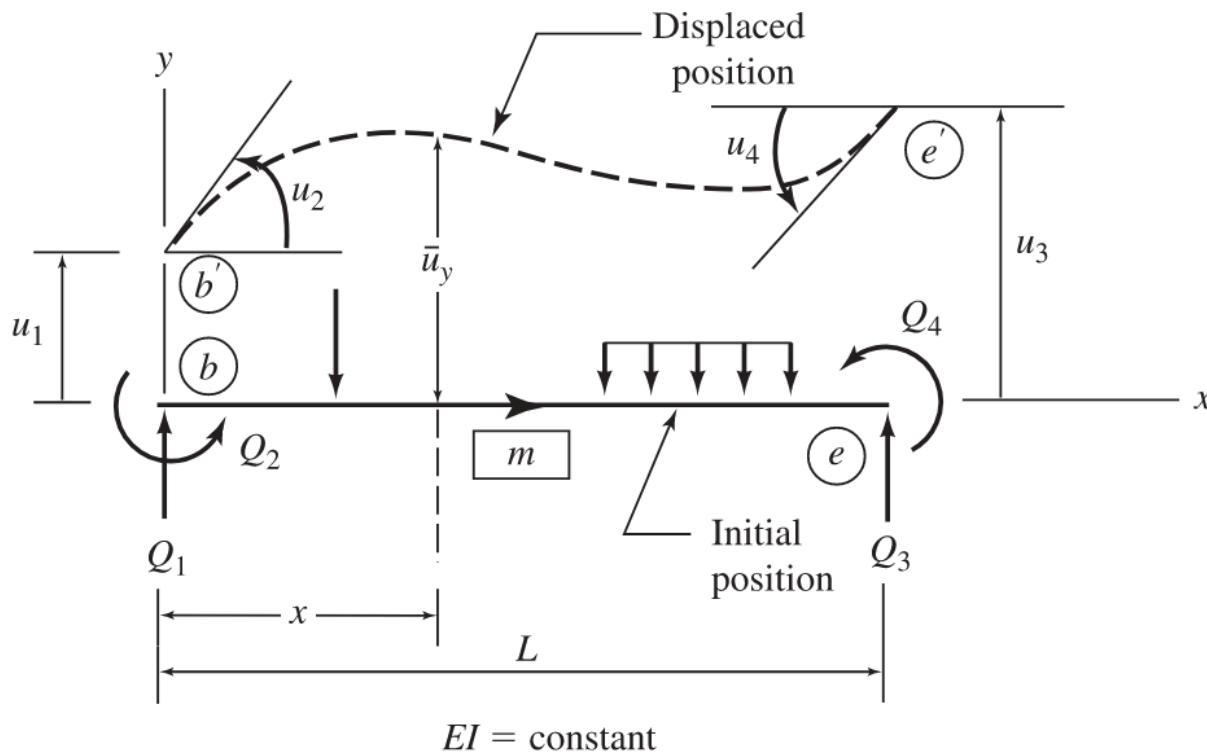
When a beam element is subjected to external loading:

- The member **deforms** (bending)
- **Internal shear forces** develop at the ends
- **Bending moments** are induced at the ends

These internal forces are fully determined by the **displacements and rotations at the element ends**.

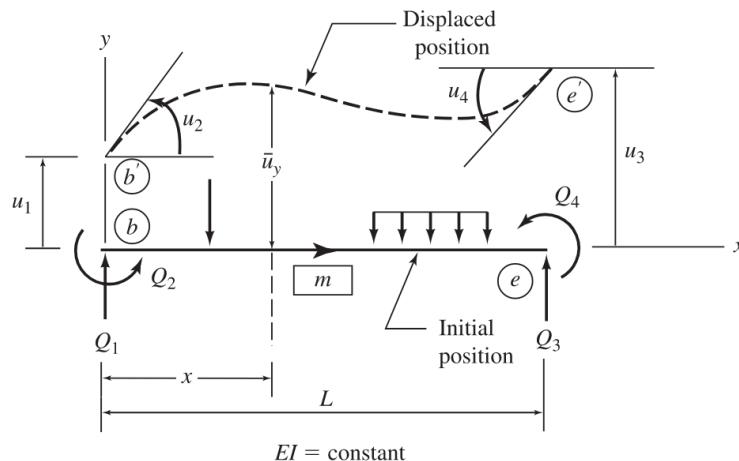
Generic Displacement for a 2D Beam Element

We define all quantities in the **local coordinate system**, with origin at the left end b , and ending at node e .



2D Beam Element DOF Numbering

Degrees of freedom are ordered **left → right**, with **translation first**, then **rotation**.



- **DOF 1:** u_1 — node b , local y
- **DOF 2:** u_2 — node b , θ
- **DOF 3:** u_3 — node e , local y
- **DOF 4:** u_4 — node e , θ

Sign conventions:

- $u > 0 \rightarrow$ upward (local y direction)
- $\theta > 0 \rightarrow$ counterclockwise
- Forces follow the same order: $[V_b, M_b, V_e, M_e]^T$

Local displacement and force vectors

Local displacement vector (including rotations):

$$\boldsymbol{u} = \begin{Bmatrix} u_b \\ \theta_b \\ u_e \\ \theta_e \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Local nodal force vector (including moments):

$$\boldsymbol{Q} = \begin{Bmatrix} V_b \\ M_b \\ V_e \\ M_e \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

We seek:

$$\boldsymbol{Q} = \boldsymbol{k} \boldsymbol{u}$$

Four equations (one per DOF)

Recall: this is similar in spirit to the truss element definition.

$$Q_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4$$

$$Q_2 = k_{21}u_1 + k_{22}u_2 + k_{23}u_3 + k_{24}u_4$$

$$Q_3 = k_{31}u_1 + k_{32}u_2 + k_{33}u_3 + k_{34}u_4$$

$$Q_4 = k_{41}u_1 + k_{42}u_2 + k_{43}u_3 + k_{44}u_4$$

Each equation expresses **force equilibrium at a single local degree of freedom**.

For a linear elastic element, the force at any DOF is a **linear combination** of all DOF displacements:

- displacing one DOF can induce forces at *all* DOFs
- the proportionality constants are the stiffness coefficients k_{ij}

Same equations in matrix form

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Part 2 — Derive Column 1 of the Beam Element Stiffness Matrix

Unit Displacement Method

Definition:

k_{ij} = force at DOF i due to a unit displacement at DOF j ,
with all other DOFs fixed.

Each column of \mathbf{k} is built by:

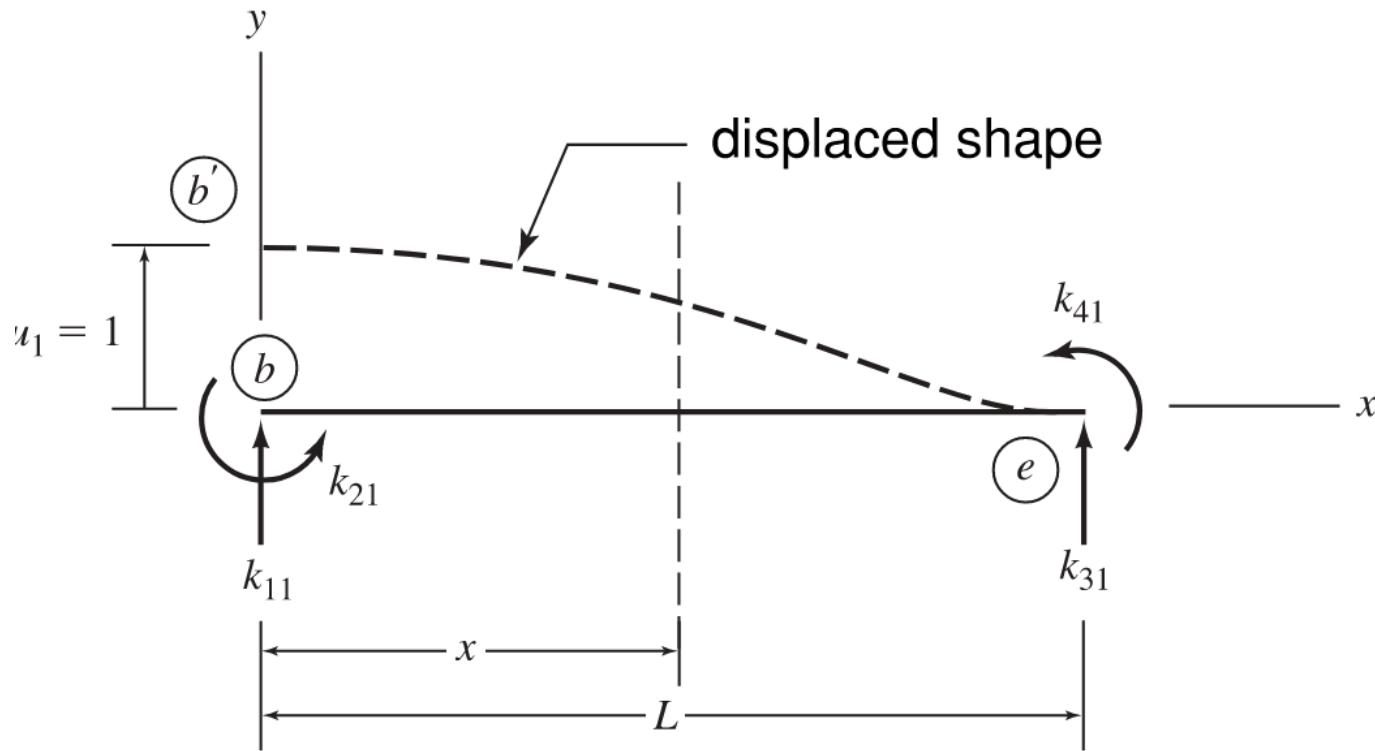
- impose a **unit displacement/rotation** at one DOF
- hold all other DOFs fixed
- record the resulting nodal force pattern

Column 1: impose $u_1 = 1$ ($u_2 = u_3 = u_4 = 0$)

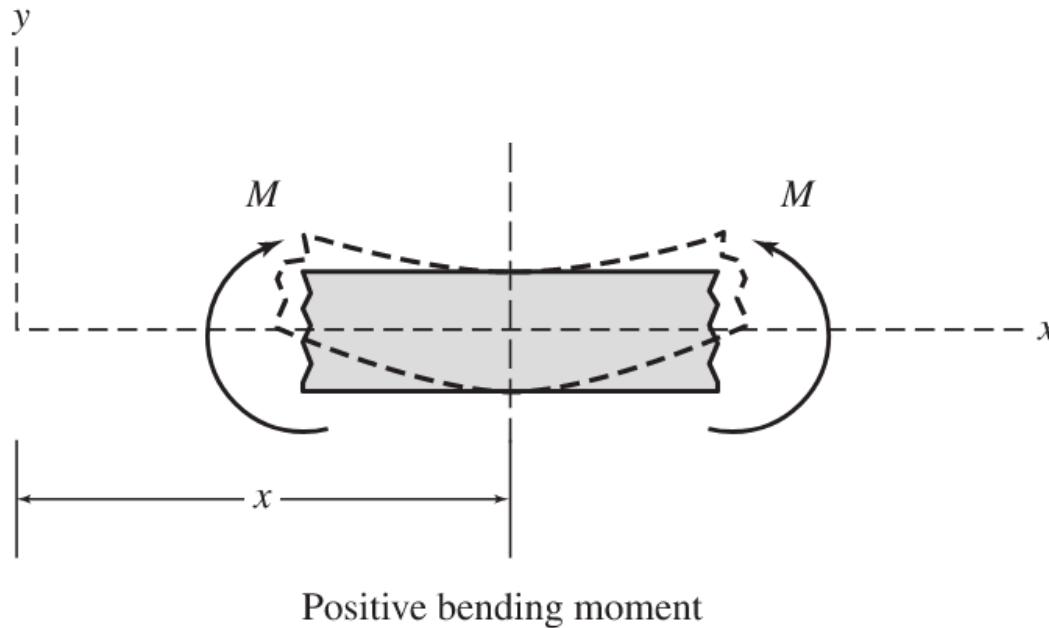
$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

- k_{11} : force at DOF 1 due to unit displacement at DOF 1
- k_{21} : force at DOF 2 due to unit displacement at DOF 1
- k_{31} : force at DOF 3 due to unit displacement at DOF 1
- k_{41} : force at DOF 4 due to unit displacement at DOF 1

Column 1: impose $u_1 = 1$ ($u_2 = u_3 = u_4 = 0$)



Sign Convention for Derivation

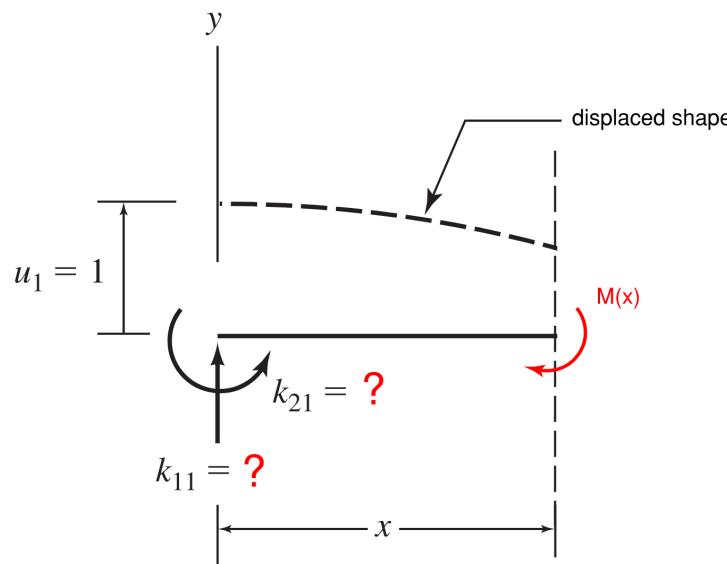


Governing Equation (Beam Bending)

$$\frac{d^2u}{dx^2} = \frac{M(x)}{EI} \quad (1)$$

- $u(x)$ = transverse displacement
- $M(x)$ = bending moment
- EI = flexural rigidity

Step 1 — Express Internal Moment as a Function of x



Cut the beam at a distance x from node b .

Using equilibrium:

$$M(x) = -k_{21} + k_{11}x \quad (2)$$

- k_{21} : end moment at node b
- k_{11} : end shear at node b

Step 2 — Substitute into Governing Equation

Substitute Eq. (2) into the **moment term** of Eq. (1):

$$\frac{d^2u}{dx^2} = \frac{M(x)}{EI} \xrightarrow{\text{Eq. (2)}} \frac{d^2u}{dx^2} = \frac{1}{EI}(-k_{21} + k_{11}x) \quad (3)$$

Step 3 — Integrate to Obtain Rotation $\theta(x)$

Start from Eq. (3):

$$\frac{d^2u}{dx^2} = \frac{1}{EI}(-k_{21} + k_{11}x) \quad (3)$$

Integrate both sides with respect to x :

$$\int \frac{d^2u}{dx^2} dx = \int \frac{1}{EI}(-k_{21} + k_{11}x) dx \quad (4a)$$

Left-hand side:

$$\int \frac{d^2u}{dx^2} dx = \frac{du}{dx} = \theta(x) \quad (4b)$$

Right-hand side (term-by-term):

$$\int \frac{1}{EI}(-k_{21} + k_{11}x) dx = \frac{1}{EI} \left(\int -k_{21} dx + \int k_{11}x dx \right) \quad (4c)$$

Compute:

$$\int -k_{21} dx = -k_{21}x \quad \int k_{11}x dx = \frac{k_{11}}{2}x^2 \quad (4d)$$

Include integration constant C_1 :

$$\theta(x) = \frac{du}{dx} = \frac{1}{EI} \left(-k_{21}x + \frac{k_{11}}{2}x^2 \right) + C_1 \quad (4)$$

Step 4 — Integrate to Obtain Displacement $u(x)$

Start from Eq. (4):

$$\frac{du}{dx} = \frac{1}{EI} \left(-k_{21}x + \frac{k_{11}}{2}x^2 \right) + C_1 \quad (4)$$

Integrate both sides with respect to x :

$$\int \frac{du}{dx} dx = \int \left[\frac{1}{EI} \left(-k_{21}x + \frac{k_{11}}{2}x^2 \right) + C_1 \right] dx \quad (5a)$$

Left-hand side:

$$\int \frac{du}{dx} dx = u(x) \quad (5b)$$

Right-hand side (compact form):

$$u(x) = \frac{1}{EI} \left(\int -k_{21}x \, dx + \int \frac{k_{11}}{2}x^2 \, dx \right) + \int C_1 \, dx \quad (5c)$$

Compute:

$$\int -k_{21}x \, dx = -\frac{k_{21}}{2}x^2 \quad \int \frac{k_{11}}{2}x^2 \, dx = \frac{k_{11}}{6}x^3 \quad \int C_1 \, dx = C_1x \quad (5d)$$

Include integration constant C_2 :

$$u(x) = \frac{1}{EI} \left(-\frac{k_{21}}{2}x^2 + \frac{k_{11}}{6}x^3 \right) + C_1x + C_2 \quad (5)$$

Step 5 — Apply Boundary Conditions

We enforce **unit displacement at node b** (0 elsewhere):

At $x = 0$:

$$\begin{aligned}\theta(0) &= 0 \\ u(0) &= 1\end{aligned}$$

At $x = L$:

$$\begin{aligned}\theta(L) &= 0 \\ u(L) &= 0\end{aligned}$$

Step 6 — Solve for Constants

Substitute $x = 0$ into Eqs. (4)–(5):

- From $\theta(0) = 0$: all terms with x vanish $\Rightarrow C_1 = 0$
- From $u(0) = 1$: all terms with x vanish $\Rightarrow C_2 = 1$

Updated Eqs. (4)–(5):

$$\theta(x) = \frac{1}{EI} \left(-k_{21}x + \frac{k_{11}}{2}x^2 \right) \quad (6)$$

$$u(x) = \frac{1}{EI} \left(-\frac{k_{21}}{2}x^2 + \frac{k_{11}}{6}x^3 \right) + 1 \quad (7)$$

Step 7 — Apply Boundary Condition at $x = L$

Substitute $\theta(L) = 0$ into Eq. (6):

$$0 = \frac{1}{EI} \left(-k_{21}L + \frac{k_{11}}{2}L^2 \right)$$

Cancel EI and solve:

$$k_{21} = \frac{k_{11}L}{2} \tag{9}$$

Step 8 — Apply Final Condition

Substitute $u(L) = 0$ into Eq. (7):

$$0 = \frac{1}{EI} \left(-\frac{k_{21}}{2} L^2 + \frac{k_{11}}{6} L^3 \right) + 1$$

Substitute $k_{21} = \frac{k_{11}L}{2}$ and solve:

$$k_{11} = \frac{12EI}{L^3}$$

Step 9 — Solve for k_{21}

Back-substitute k_{11} into Eq. (9):

$$k_{21} = \frac{6EI}{L^2}$$

Step 10 — Solve Remaining Stiffness Terms k_{31} and k_{41}

Use equilibrium of the beam element free-body diagram (with k_{11} and k_{21} known).

Vertical force equilibrium

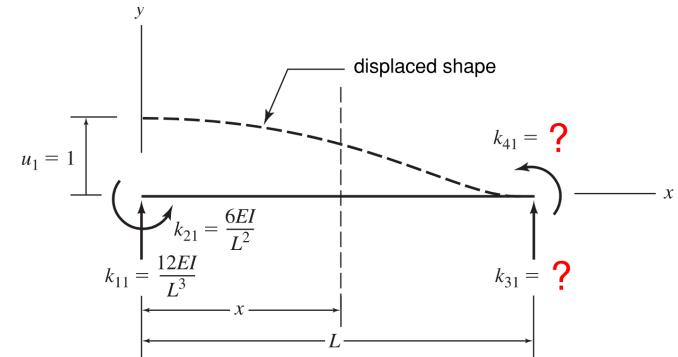
$$\sum F_y = 0 : \quad \frac{12EI}{L^3} + k_{31} = 0$$

$$k_{31} = -\frac{12EI}{L^3}$$

Moment equilibrium about node b

$$\begin{aligned} \sum M_b = 0 : \quad & \frac{6EI}{L^2} - \left(\frac{12EI}{L^3} \right) L + k_{41} \\ & = 0 \end{aligned}$$

$$k_{41} = \frac{6EI}{L^2}$$



Free-body diagram used for $\sum F_y = 0$ and
 $\sum M_b = 0$.

Step 11 (Optional) — Displacement Shape Function

Substitute k_{11} and k_{21} into Eq. (7):

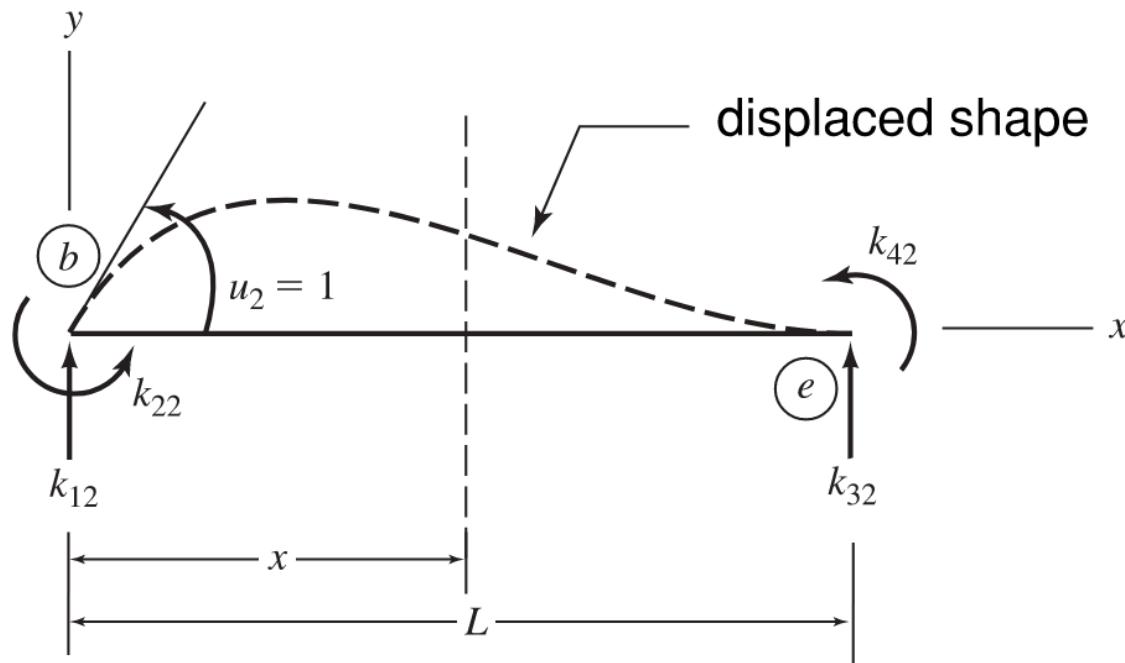
$$u(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

Column 1 of the 4×4 Stiffness Matrix

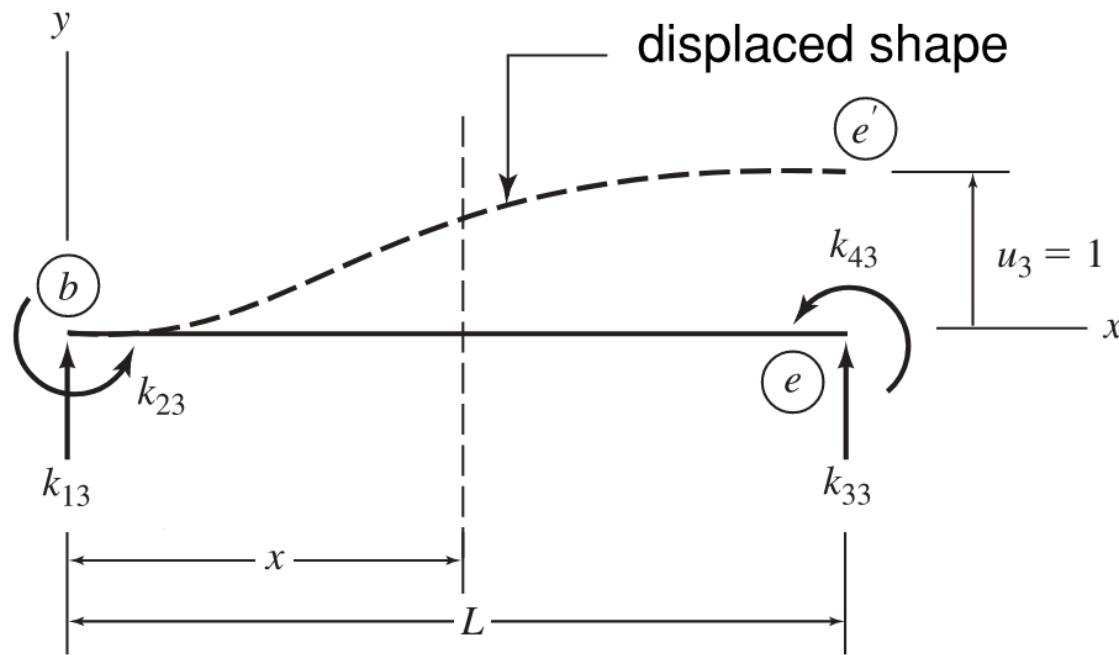
$$\mathbf{k} = \begin{bmatrix} \frac{12EI}{L^3} & k_{12} & k_{13} & k_{14} \\ \frac{6EI}{L^2} & k_{22} & k_{23} & k_{24} \\ -\frac{12EI}{L^3} & k_{32} & k_{33} & k_{34} \\ \frac{6EI}{L^2} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

Part 3 — Complete 4×4 Beam Stiffness Matrix

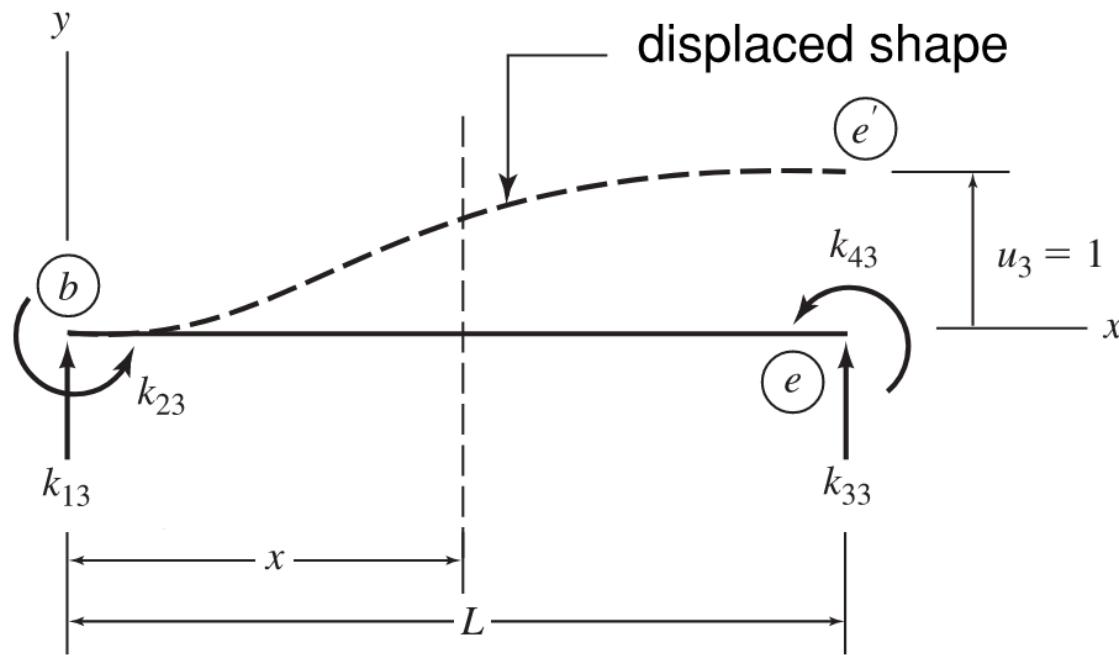
Column 2: impose $u_2 = 1$ ($u_1 = u_3 = u_4 = 0$)



Column 3: impose $u_3 = 1$ ($u_1 = u_2 = u_4 = 0$)



Column 3: impose $u_3 = 1$ ($u_1 = u_2 = u_4 = 0$)



Complete 4×4 Beam Element Stiffness Matrix

$$\mathbf{k} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Part 4 - Beam Deformation Patterns

Displacement Field

A 2D beam element has **4 DOFs**, so the displacement field is written as a superposition:

$$u(x) = N_1(x) u_1 + N_2(x) u_2 + N_3(x) u_3 + N_4(x) u_4$$

Here, $u(x)$ represents the **transverse displacement of the beam in the local y -direction**.

Each $N_i(x)$ is a **shape function** (a deformation pattern) associated with one DOF:

- $N_1(x)$: unit transverse displacement at node b
- $N_2(x)$: unit rotation at node b
- $N_3(x)$: unit transverse displacement at node e
- $N_4(x)$: unit rotation at node e

The 4 Shape Functions

We derived N_1 earlier:

$$N_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

The shape functions for the other DOFs:

$$N_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2}$$

$$N_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$N_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

Connection to the Finite Element Method (FEM)

The shape functions we derived are exactly those used in the **Finite Element Method (FEM)** for beam elements.

- In FEM, the displacement field is approximated as:

$$u(x) = \sum_{i=1}^4 N_i(x) u_i$$

- The unknowns are the same **nodal DOFs** $\{u_1, u_2, u_3, u_4\}$
- Shape functions provide a **continuous interpolation** within the element

In **CEE6501**, we focus on the **Direct Stiffness Method (DSM)** and derive the stiffness matrix using structural mechanics.

But for the FEM-based derivation of the beam stiffness matrix, see:

- **Kassimali**, Section 5.3

Plotting a Deformed Shape

While we do not explicitly use shape functions in the **DSM formulation**, they are very useful for **visualizing structural response**.

Given element DOFs

$$\mathbf{u} = \{u_1, u_2, u_3, u_4\}^T,$$

we can evaluate the displacement field

$$u(x) = \sum_{i=1}^4 N_i(x) u_i$$

at many points along $x \in [0, L]$ to obtain a smooth deformation curve.

Why This Is Useful

- Visualize the **continuous beam deflection** (not just nodal values)
- Check **sign conventions** and DOF interpretation
- Construct smooth **deformed shapes for multi-element structures**

In [34]:

```
import numpy as np
import matplotlib.pyplot as plt

def shape_functions(x, L):
    """
    Return the 4 Hermite beam shape functions evaluated at x.
    x can be a scalar or a numpy array.
    """
    xi = x / L
    N1 = 1 - 3*xi**2 + 2*xi**3
    N2 = L * (xi - 2*xi**2 + xi**3)
    N3 = 3*xi**2 - 2*xi**3
    N4 = L * (-xi**2 + xi**3)
    return N1, N2, N3, N4

def beam_deflection(x, L, u1, u2, u3, u4):
    """
    Compute beam transverse deflection u(x) from nodal DOFs.
    DOFs: u1 (disp at b), u2 (rot at b), u3 (disp at e), u4 (rot at e)
    """
    N1, N2, N3, N4 = shape_functions(x, L)
    return N1*u1 + N2*u2 + N3*u3 + N4*u4
```

```
In [35]: def plot_beam(L, u1, u2, u3, u4, n=200):
    """
        Plot beam deformed shape given nodal DOFs.
    """
    x = np.linspace(0, L, n)
    u = beam_deflection(x, L, u1, u2, u3, u4)

    plt.figure(figsize=(6, 3)) # smaller plot

    # undeformed shape
    plt.plot(x, np.zeros_like(x), linestyle="--", label="undeformed")
    plt.scatter([0, L], [0, 0]) # nodes (no legend)

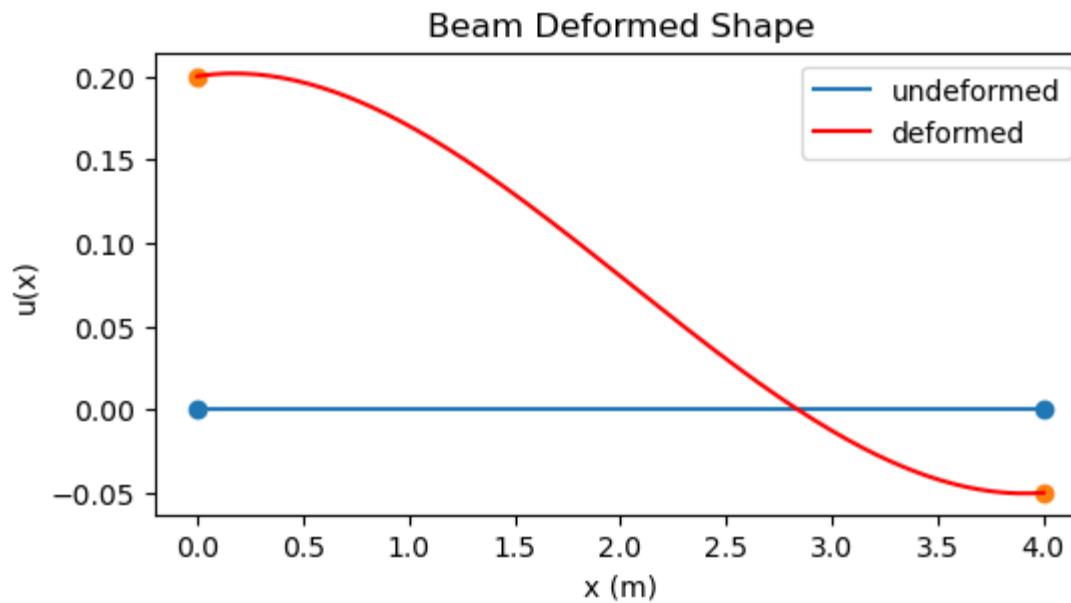
    # deformed shape
    plt.plot(x, u, color="red", label="deformed")
    plt.scatter([0, L], [u1, u3]) # nodes (no legend)

    plt.xlabel("x (m)")
    plt.ylabel("u(x)")
    plt.title("Beam Deformed Shape")
    plt.legend()
    plt.show()
```

```
In [36]: L = 4.0 # element length
```

```
# Example nodal DOFs (pick anything to test)
u1 = 0.2 # displacement at node b
u2 = 0.02 # rotation at node b (rad)
u3 = -0.05 # displacement at node e
u4 = 0.01 # rotation at node e (rad)

plot_beam(L, u1, u2, u3, u4)
```



Wrap-Up

Today you:

- Defined local DOFs and the member stiffness relation for a 2D beam element
- Derived **Column 1** of the 4×4 beam stiffness matrix using the unit displacement method
- Stated the final closed-form 4×4 beam element stiffness matrix

Next: use this beam element inside **plane frames**, and introduce fixed-end forces for member loading.