CEE6501 — Lecture 2.1

Matrix Representation and Operations

Learning Objectives

By the end of this lecture, you will:

- Understand matrices as **linear mappings** and as data structures
- Use consistent **notation** for scalars, vectors, and matrices
- Interpret matrix–vector and matrix–matrix products
- Reason about **dimensions, structure, and compatibility**
- Connect special matrix structure (symmetric/triangular/diagonal) to efficient solution strategies

Part 1 — Scalars, Vectors, and Matrices

> *What mathematical objects are we working with?*

Scalars

A **scalar** is a single numerical value:

$$\boldsymbol{a} \in \mathbb{R}$$

Scalars have magnitude but no direction or internal structure.

Vectors

A **vector** is an ordered collection of scalars:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

Vectors are treated as **column objects** by default.

Matrices

A **matrix** is a rectangular array of scalars:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

A matrix can be interpreted as a **linear mapping** from one vector space to another.

Part 2 — Notation Conventions

> *How do we write linear algebra unambiguously in this class?*

Why Notation Matters

Consistent notation:

- Makes dimensions immediately visible
- Prevents algebraic errors
- Allows equations to be read without ambiguity

Scalars

Scalars are written in **lowercase italic**:

$$a,\ b,\ c \in \mathbb{R}$$

Vectors

Vectors are written in **bold lowercase**:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

You may also see:

- $\{x\}$ (brace notation in some textbooks)

Matrices

Matrices are written in **bold uppercase**:

$$[A] = \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

You may also see:

- $[A]$ (square-bracket notation in some textbooks)

Part 3 — Matrix Indexing

> *How do we refer to individual entries precisely?*

Order (Size) of a Matrix

A matrix with m rows and n columns has size:

$$\mathbf{A} \in \mathbb{R}^{m \times n}$$

We say $\mathbf{A}$ is of order:

$$m \times n$$

Matrix Elements

Each entry is called an **element**. The element in row i, column j is:

$$(\mathbf{A})_{ij} = a_{ij}$$

- First subscript i → row
- Second subscript j → column

Meaning of Indices

For $\mathbf{A} \in \mathbb{R}^{m \times n}$:

- $i = 1, 2, \ldots, m$ (rows)
- $j = 1, 2, \ldots, n$ (columns)

So a_{ij} is the element in the i-th row and j-th column.

Example: A $\mathbf{4} \times \mathbf{3}$ Matrix

$$\mathbf{D} = \begin{bmatrix} 8 & 26 & 0 \\ 33 & 5 & 37 \\ 12 & 23 & 2 \\ 7 & 29 & 14 \end{bmatrix}$$

- Order: 4×3
- Rows: $i = 1, \ldots, 4$
- Columns: $j = 1, \ldots, 3$

Referring to Individual Elements

Elements of $\mathbf{D}$ are d_{ij}.

Examples:

- $d_{13} = 0$
- $d_{31} = 12$
- $d_{42} = 29$

Part 4 — Types of Matrices

> *Matrix structure is not cosmetic — it reflects physics, modeling choices, and solver strategy.*

Why Matrix Types Matter

In matrix structural analysis, matrix *structure* tells us:

- which DOFs are coupled
- which solvers we can use
- how expensive a computation will be

We will see the same matrix appear in multiple forms:

- stiffness matrices
- mass matrices
- constraint and penalty matrices

Column Matrix (Vector)

Definition

A matrix with a single column ($n = 1$), commonly called a **vector**:

$$\mathbf{x} \in \mathbb{R}^{m\times 1}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Column Matrix — Structural Interpretation

Column matrices (vectors) are the **primary carriers of information** in matrix structural analysis. Inputs and Outputs.

They represent:

- **Displacements $\mathbf{u}$** — the unknown DOFs we solve for
- **Loads $\mathbf{f}$** — the forces driving the system
- **Reactions** — forces at constrained DOFs

All structural analysis reduces to:

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

How to read this:

- each entry corresponds to **one degree of freedom**
- vectors define *what is unknown* and *what is applied*

Row Matrix

Definition

A matrix with a single row ($m = 1$):

$$\mathbf{c} \in \mathbb{R}^{1\times n}$$

$$\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

Row Matrix — Structural Interpretation

Row matrices act as **operators on DOF vectors**.

They take a **column vector input** and return a **scalar quantity**.

Let the displacement vector be:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Let the selector vector be: $\mathbf{s_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$ $\mathbf{s_2} = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}$

DOF selection, s_1

$$\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = u_2$$

DOF combination, s_2

$$\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = u_1 - u_2$$

Interpretation:

- column vectors **store DOF values**
- row matrices **query or combine DOFs**
- output is a **single scalar condition**

Square Matrix

Definition

A matrix with the same number of rows and columns ($m = n$):

$$\mathbf{A} \in \mathbb{R}^{n \times n}$$

The **main diagonal** contains $a_{11}, a_{22}, \ldots, a_{nn}$.

Square Matrix — Structural Meaning

Square matrices are the **heart of structural analysis**.

Direct stiffness method:

- **# equations = # unknown DOFs**
- **→ square global system**

$$\mathbf{Ku} = \mathbf{f}, \qquad \mathbf{K} \in \mathbb{R}^{n\times n}$$

How to read $\mathbf{K}$:

- rows → equilibrium at DOFs
- columns → DOF influence

Why it matters:

- only square systems can be **solved**
- enable factorization and eigenanalysis

> *Not square → model is incomplete, over-constrained, or ill-posed.*

Symmetric Matrix

Definition

A square matrix where the entries are mirrored about the main diagonal:

$$a_{ij} = a_{ji} \quad \Leftrightarrow \quad \mathbf{A}^T = \mathbf{A}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Symmetric Matrix — Structural Meaning

(special case of square matrices)

Symmetry reflects **reciprocity and energy consistency**:

- **Reciprocity**: If moving DOF A causes a force at DOF B, then moving DOF B causes the same force at DOF A. The influence between two DOFs goes both ways

$$k_{ij} = k_{ji}$$

- **Energy consistency**: The structure behaves like a spring that stores energy. The work done does not depend on the order in which displacements are applied — only on the final configuration.

Why symmetry matters:

- store only half the matrix
- faster solvers (Cholesky, LDL^T)

In this course, stiffness matrices are symmetric for all situations:

- linear elastic analysis
- material nonlinearity (elastic, energy-based)
- geometric nonlinearity (conservative)

Triangular Matrices

Definition

A matrix where all entries on one side of the main diagonal are zero.

Lower triangular:

$$a_{ij} = 0 \ (j > i)$$

Upper triangular:

$$a_{ij} = 0 \ (j < i)$$

$$\mathbf{L} = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Triangular Matrices — Structural Meaning

Triangular matrices appear in matrix structural analysis when a large system is **broken into simpler steps**.

Instead of solving

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

all at once, we factor the stiffness matrix:

$$\mathbf{K} = \mathbf{L}\mathbf{U} \quad \text{or} \quad \mathbf{K} = \mathbf{L}\mathbf{D}\mathbf{L}^T$$

This turns one difficult problem into **two easy ones**.

How to think about it:

- **Forward substitution** → uses the lower triangular matrix $\mathbf{L}$
- **Back substitution** → uses the upper triangular matrix $\mathbf{U}$

We will return to this in detail when we study **matrix solvers** in the next section.

Diagonal Matrix

Definition

A matrix where all off-diagonal entries are zero:

$$a_{ij} = 0 \quad \text{for } i \neq j$$

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Diagonal Matrix — Structural Interpretation

Diagonal matrices represent **uncoupled degrees of freedom**.

In matrix structural analysis, diagonal matrices commonly appear as:

- **Lumped mass matrices** in dynamics (each DOF has its own inertia)
- **Penalty stiffness matrices** for enforcing boundary conditions
- **Diagonal preconditioners** in iterative solvers (e.g., Jacobi, CG)

Interpretation:

- each diagonal term acts on **one DOF only**
- no force or displacement coupling between DOFs

Identity (Unit) Matrix

Definition

A matrix where all diagonal entries are 1 and all off-diagonal entries are 0:

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Ix} = \mathbf{x}$$

Identity Matrix — Structural Interpretation

The identity matrix represents a **neutral operation** on a DOF vector.

In matrix structural analysis, it appears in:

- **Penalty methods**: $\mathbf{K} + \alpha \mathbf{I}$ for constraints
- **Regularization** of ill-conditioned stiffness matrices
- **Incremental–iterative solvers** (Newton updates)
- **Eigenvalue problems** and modal normalization

Interpretation:

- multiplying by $\mathbf{I}$ leaves DOFs unchanged
- adding $\alpha \mathbf{I}$ stiffens DOFs *without introducing coupling*

Null (Zero) Matrix

Definition

A matrix where all entries equal to zero:

$$o_{ij} = 0$$

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Null Matrix — Structural Interpretation

Zero matrices encode **absence of coupling** between DOF sets.

In matrix structural analysis, they arise in:

- **Partitioned stiffness matrices**:

$$\begin{bmatrix} \mathbf{K}_{ff} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{cc} \end{bmatrix}$$

- Free vs constrained DOF separation
- Multi-component or multi-physics models before coupling

Interpretation:

- no force transfer between DOF groups
- modeling assumption of independent subsystems

Part 5 — Matrix Compatibility for Operations

When do operations make sense?

Addition

Addition requires identical dimensions:

$$\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$$

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$$

Example: Matrix Addition (Numerical)

We can add matrices **only if** they have the same shape. The result is elementwise:

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$$

In [9]:
```
import numpy as np

A = np.array([[1, 2],
              [3, 4]])
B = np.array([[10, 20],
              [30, 40]])

print('A =\n', A)
print('B =\n', B)
print('A + B =\n', A + B)
print('shape(A), shape(B) =', A.shape, B.shape)
```

```
A =
 [[1 2]
 [3 4]]
B =
 [[10 20]
 [30 40]]
A + B =
 [[11 22]
 [33 44]]
shape(A), shape(B) = (2, 2) (2, 2)
```

Multiplication Compatibility

Matrix multiplication requires inner dimensions to match:

$$\mathbf{A} \in \mathbb{R}^{m\times n}, \quad \mathbf{B} \in \mathbb{R}^{n\times p}$$

$$\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m\times p}$$

Example: Multiplication Compatibility (Shapes)

If $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$, then $\mathbf{AB}$ is defined and is in $\mathbb{R}^{m\times p}$.

```
In [10]: import numpy as np

         A = np.random.randint(0, 10, (2, 3))  # 2x3
         B = np.random.randint(0, 10, (3, 4))  # 3x4

         C = A @ B  # 2x4

         print('shape(A), shape(B), shape(C) =', A.shape, B.shape, C.shape)
         print('\nA=\n', A)
         print('\nB=\n', B)
         print('\nA @ B=\n', C)
```

```
shape(A), shape(B), shape(C) = (2, 3) (3, 4) (2, 4)

A=
 [[2 1 7]
 [3 2 4]]

B=
 [[9 6 9 5]
 [6 5 1 5]
 [6 2 1 5]]

A @ B=
 [[66 31 26 50]
 [63 36 33 45]]
```

Part 6 — Matrix–Vector Multiplication

What does a matrix do to a vector?

Linear Mapping

A matrix defines a linear transformation:

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Forward problem: given $\mathbf{A}$ and $\mathbf{x}$, compute $\mathbf{y}$.

Component Form = Row Dot Product

Each output component is:

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

This is the **dot product** between row i of $\mathbf{A}$ and the vector $\mathbf{x}$:

$$y_i = \big(\mathrm{row}_i(\mathbf{A})\big) \cdot \mathbf{x}$$

Annotated Figure (TO DO): Computing One Component y_i

For a fixed row index i, the formula

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

means: **take row i of $\mathbf{A}$**, multiply elementwise by $\mathbf{x}$, then sum.

```
In [11]: import numpy as np
         # Example sizes
         A = np.array([[2, 1, -1],
                       [0, 3,  2],
                       [4, -2, 1]], dtype=float)
         x = np.array([1, 2, -1], dtype=float)

         # Choose which component y_i to illustrate
         i = 1  # second row (0-index)
         y = A @ x

         row_i = A[i, :]
         print('A =\n', A)
         print('x =', x)
         print('Row i =', row_i)
         print('Elementwise product row_i * x =', row_i * x)
         print('y = A @ x =', y)
         print(f'y_{i+1} (1-indexed) = sum_j a_{{{i+1}j}} x_j =', y[i])
```

```
A =
 [[ 2.  1. -1.]
 [ 0.  3.  2.]
 [ 4. -2.  1.]]
x = [ 1.  2. -1.]
Row i = [0. 3. 2.]
Elementwise product row_i * x = [ 0.  6. -2.]
y = A @ x = [ 5.  4. -1.]
y_2 (1-indexed) = sum_j a_{2j} x_j = 4.0
```

Same Product, Column Interpretation

Let $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ (columns). Then:

$$\mathbf{A}\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

So $\mathbf{A}\mathbf{x}$ is a **linear combination of the columns of $\mathbf{A}$**.

Column Interpretation — Numerical Example

We will compute $\mathbf{y} = \mathbf{A}\mathbf{x}$ two equivalent ways:

1. as a matrix–vector product, and
2. as a **linear combination of the columns** of $\mathbf{A}$.

Step 1 — Choose $\mathbf{A}$ and $\mathbf{x}$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Our goal is to compute:

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Step 2 — Write $\mathbf{A}$ by its columns

Let $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2]$, where:

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The weights come from $\mathbf{x}$:

$$x_1 = 4, \quad x_2 = -2$$

Step 3 — Form the linear combination

$$\mathbf{Ax} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = 4\begin{bmatrix} 2 \\ -1 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Step 4 — Compute the weighted columns and sum

$$4\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}, \qquad (-2)\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

Add them:

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 8 \\ -4 \end{bmatrix} + \begin{bmatrix} -2 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \end{bmatrix}$$

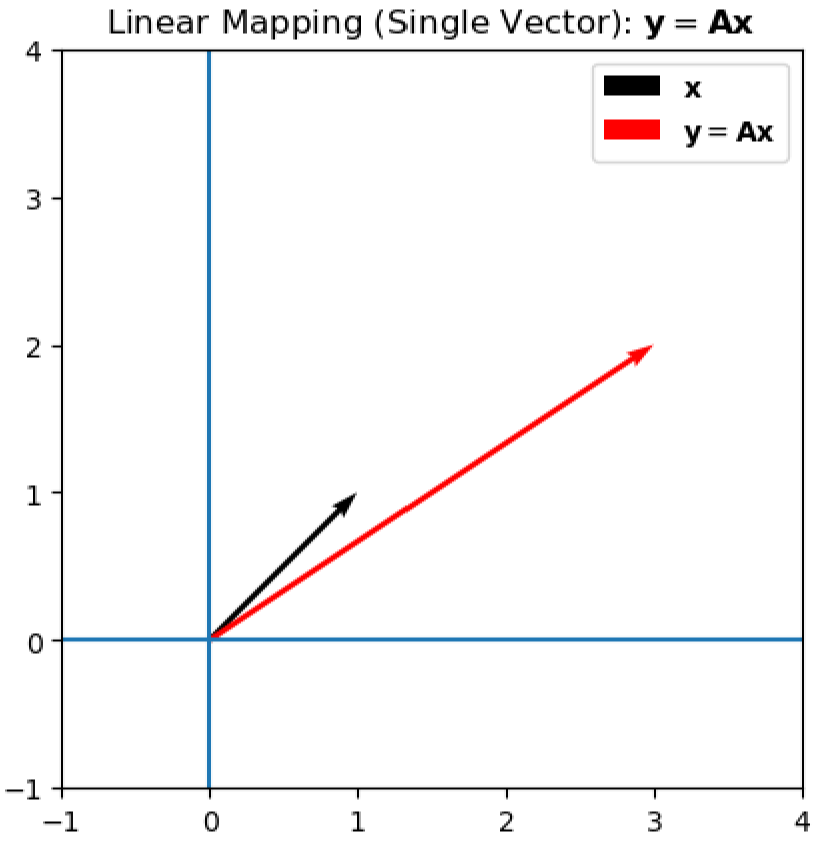
Conclusion: $\mathbf{A}\mathbf{x}$ is a weighted sum of the columns of $\mathbf{A}$.

Visual: Linear Mapping of a Single Vector in $\mathbb{R}^2$

We choose a matrix $\mathbf{A}$ and a vector $\mathbf{x}$, then plot:

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

In [12]:
```
import numpy as np
import matplotlib.pyplot as plt

A = np.array([[ 2,  1],
              [-1,  3]], dtype=float)

# single input vector
x = np.array([1, 1], dtype=float)
y = A @ x

plt.figure()
plt.axhline(0)
plt.axvline(0)

# Use quiver for labeled vectors
plt.quiver(0, 0, x[0], x[1], angles='xy', scale_units='xy', scale=1, label=r'$
plt.quiver(0, 0, y[0], y[1], angles='xy', color='red', scale_units='xy', scale

plt.gca().set_aspect('equal', adjustable='box')
plt.xlim(-1, 4)
plt.ylim(-1, 4)
plt.legend()
plt.title(r'Linear Mapping (Single Vector): $\mathbf{y}=\mathbf{A}\mathbf{x}$'
plt.show()

print('A =\n', A)
print('x =', x)
print('y = A @ x =', y)
```



```
A =
 [[ 2.  1.]
 [-1.  3.]]
x = [1. 1.]
y = A @ x = [3. 2.]
```

Same Equation, Different Questions

$$\mathbf{y} = \mathbf{A}\mathbf{x} \quad \Longleftrightarrow \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

- **Forward problem:** given $\mathbf{A}$ and $\mathbf{x}$, compute $\mathbf{y}$
- **Inverse problem:** given $\mathbf{A}$ and $\mathbf{b}$, solve for $\mathbf{x}$

Example: From Linear Equations to $\mathbf{Ax} = \mathbf{b}$

Start with a system in unknowns x_1, x_2:

$$\begin{aligned} 2x_1 + 1x_2 &= 5 \\ -1x_1 + 3x_2 &= 4 \end{aligned}$$

Group coefficients, unknowns, and constants:

$$\underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 5 \\ 4 \end{bmatrix}}_{\mathbf{b}}$$

```
In [13]: import numpy as np

         A = np.array([[ 2, 1],
                       [-1, 3]], dtype=float)
         b = np.array([5, 4], dtype=float)

         # Solve A x = b (preferred over explicit inversion)
         x = np.linalg.solve(A, b)

         print('A=\n', A)
         print('b=\n', b)
         print('x=\n', x)
         print('\nCheck: A @ x =', A @ x)
         print('Residual ||A x - b|| =', np.linalg.norm(A @ x - b))
```

```
A=
 [[ 2.  1.]
 [-1.  3.]]
b=
 [5. 4.]
x=
 [1.57142857 1.85714286]

Check: A @ x = [5. 4.]
Residual ||A x - b|| = 4.440892098500626e-16
```

Part 7 — Matrix–Matrix Multiplication

Multiplication composes linear transformations

Component Definition (Row–Column Dot Product)

$$(\mathbf{AB})_{ij} = \sum_k a_{ik} b_{kj}$$

Interpretation:

- Fix i (a row of $\mathbf{A}$)
- Fix j (a column of $\mathbf{B}$)
- Take a dot product over k

Annotated Figure (TO DO): Row i of $\mathbf{A}$ with Column j of $\mathbf{B}$

To compute a single entry $(\mathbf{AB})_{ij}$, you combine:

- **row** i from $\mathbf{A}$
- **column** j from $\mathbf{B}$

USE THE KASSIMALI FIGURE FOR THIS

Example: Non-Commutativity

In general:

$$\mathbf{AB} \neq \mathbf{BA}$$

Even when both products are defined, they can produce different results.

In [14]:

```
import numpy as np

A = np.array([[1, 2],
              [0, 1]], dtype=float)
B = np.array([[2, 0],
              [3, 1]], dtype=float)

AB = A @ B
BA = B @ A

print('A=\n', A)
print('B=\n', B)
print('\nA @ B=\n', AB)
print('\nB @ A=\n', BA)
print('\nAB equals BA?', np.allclose(AB, BA))
```

```
A=
 [[1. 2.]
 [0. 1.]]
B=
 [[2. 0.]
 [3. 1.]]

A @ B=
 [[8. 2.]
 [3. 1.]]

B @ A=
 [[2. 4.]
 [3. 7.]]

AB equals BA? False
```

Part 8 — Special Matrix Operations

Useful transformations and special matrices

Transpose

Transpose swaps rows and columns:

$$(\mathbf{A}^T)_{ij} = a_{ji}$$

If $\mathbf{A} \in \mathbb{R}^{m\times n}$, then $\mathbf{A}^T \in \mathbb{R}^{n\times m}$.

Inverse

Need a lot here

From Representation to Solution

> *Next: solving systems efficiently (LU, Cholesky, LDL^T)*