CEE6501 — Lecture 6.1

Introduction to 2D Beam Analysis

Learning Objectives

By the end of this lecture, you will be able to:

- Define the 2D Euler–Bernoulli beam idealization used in DSM
- Identify beam joint DOFs (u, θ) and apply the sign convention
- Distinguish joint loads vs member loads, and where each enters the stiffness method
- Create a beam line diagram with joint numbering, DOF numbering, and restrained coordinates
- Write and interpret the global system $\mathbf{Ku} = \mathbf{f}$

Agenda

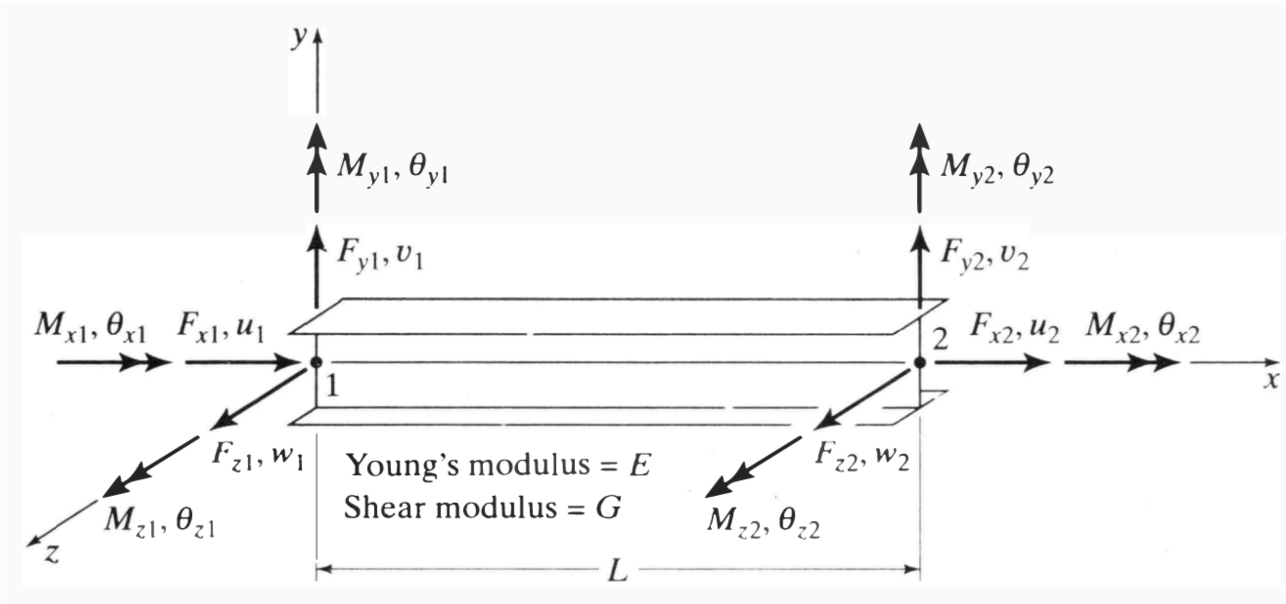
Part 1 — Roadmap: truss → beam → 3D frame
Part 2 — Beam analytical model (members + joints)
Part 3 — Global vs local coordinate systems
Part 4 — Degrees of freedom (DOFs) and sign conventions
Part 5 — DOF counting and numbering
Part 6 — Joint loads, member loads, and reactions
Part 7 — Global system: $\mathbf{Ku} = \mathbf{f}$
Part 8 — Relating moment to rotation (beam–truss analogy)

Part 1 — Roadmap: From Trusses to Beams to 3D Frames

> *Expanding our element library: axial-only → bending → full 3D frame.*

The 3D Frame Element (Where We Are Headed)

- A general 3D frame member has **6 DOFs per node** (12 per element)
  - translations: u_x, u_y, u_z
  - rotations: $\theta_x, \theta_y, \theta_z$
- This single element can represent **axial**, **torsion**, and **bending** behavior



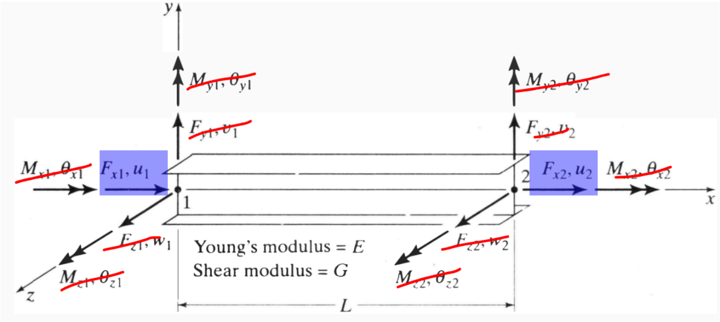
Key idea: Superposition

You can think of the 3D frame element as a **superposition** of four behaviors:

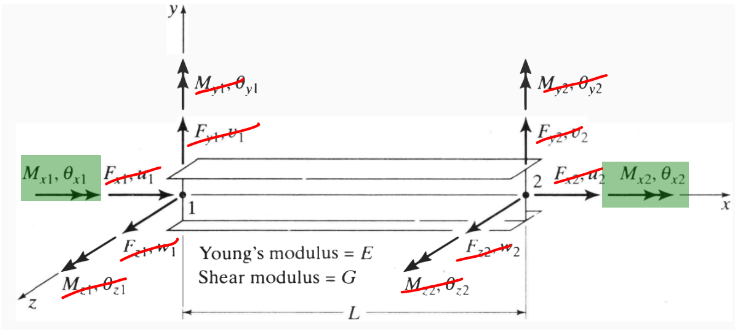
1. **Axial** deformation (truss-like)
2. **Torsion** about the member axis
3. **Bending** about one principal axis (Z - axis)
4. **Bending** about the other principal axis (Y - Axis)

The Four Behaviors

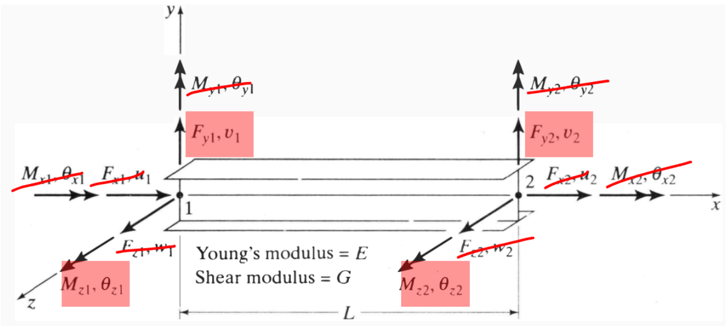
(1) Axial



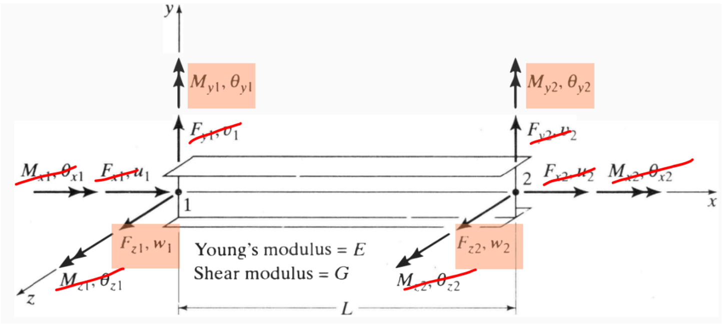
(2) Torsion



(3) Bending about Z-axis



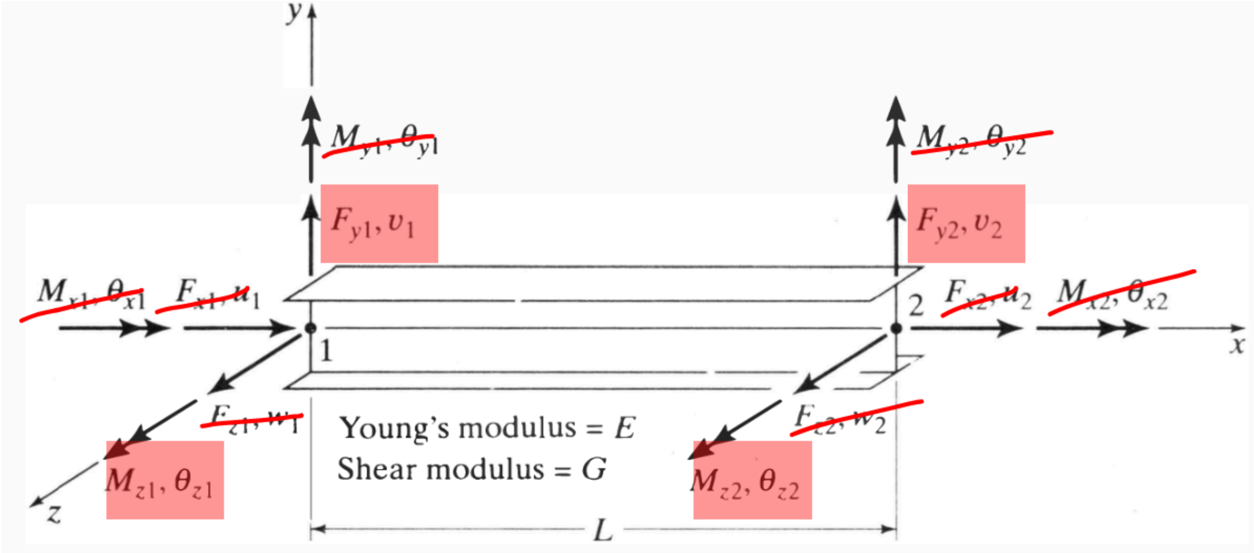
(4) Bending about Y-axis



What We Derive First

- The **2D beam element** governing planar bending behavior, case (3) and (4)
- Once derived for one principal axis, the formulation extends directly to the other (identical mathematics, different axis)

Today we start by developing the DSM formulation for a beam **bending about the Z-axis**



Part 2 — 2D Beam Analytical Model

> *How we represent a continuous beam as members + joints for stiffness analysis.*

Beam Definition (2D Bending Idealization)

A **beam** is modeled as:

- a long, straight member
- loaded in a single plane (e.g., XY-plane)
- deformation dominated by **bending**

We adopt the **Euler–Bernoulli beam assumption**:

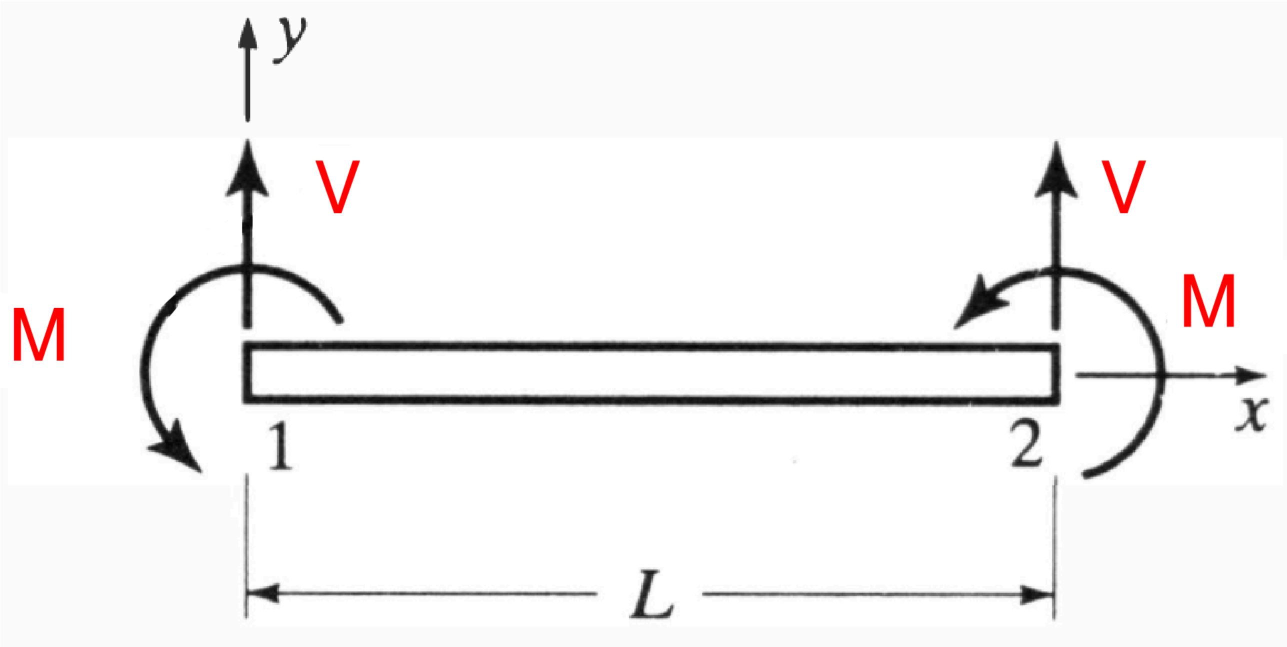
- cross-sections remain **plane and perpendicular** to the centerline
- **shear deformation is neglected**

Additionally (for this formulation):

- **axial deformation is neglected** ($u_x \approx 0$)

Consequence (for this model):

- primary internal actions are **shear** and **bending moment**
- only forces **perpendicular** to the centroidal axis of the member (shear)
- moments in XY-plane, or about Z-axis (into the page)

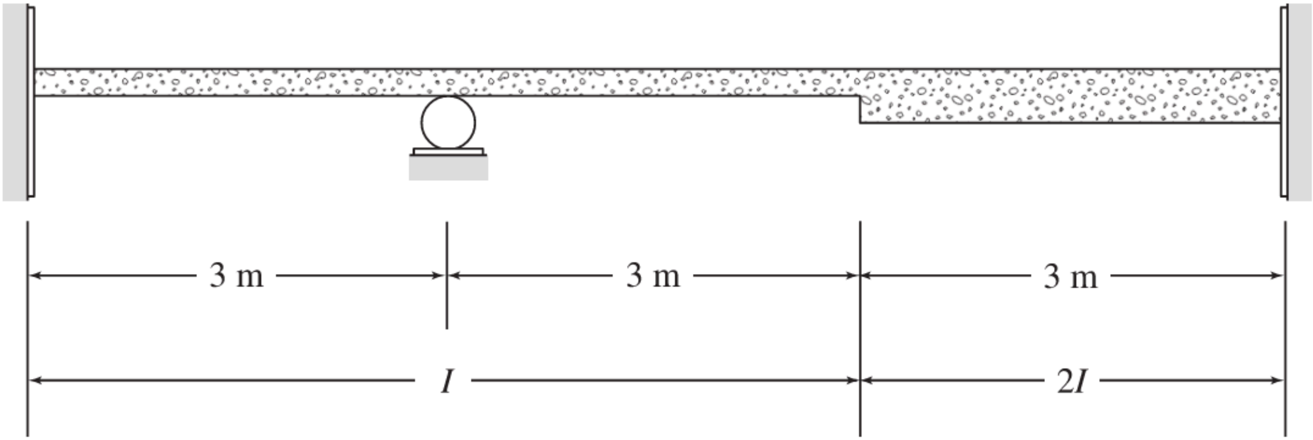


Analytical Model:

For the matrix stiffness method, a continuous beam is modeled as:

- straight prismatic (constant cross-section) members
- connected at joints (nodes)
- with unknown reactions assumed to act **only at joints**
- applied loads **anywhere** along beam

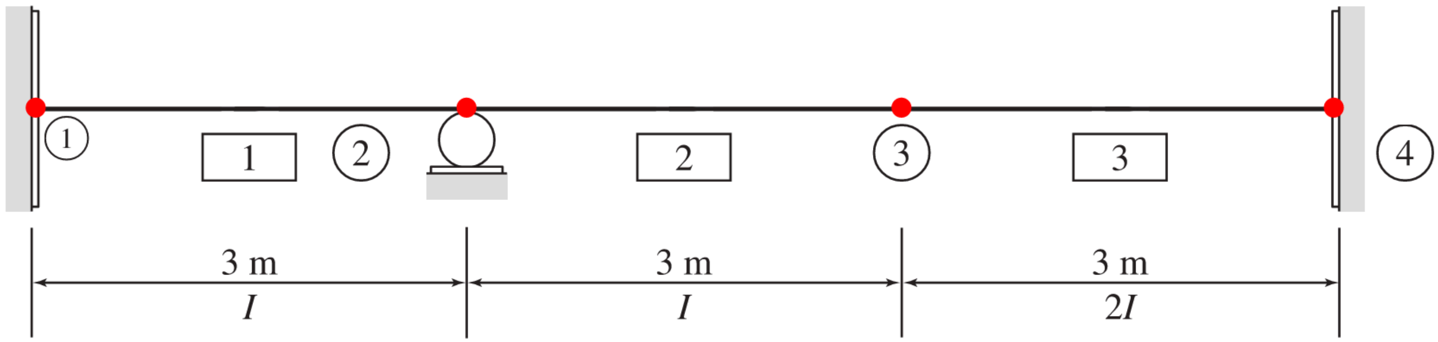
Example Beam Structure



Discretize into Members and Joints

Even if the structure is physically continuous, we often insert joints so that:

- **support reactions** occur at joints (not mid-member)
- each member has **constant properties** (e.g., constant EI)
- **member loads** can be easily turned into Fixed-End Forces (more in Lecture 6.3)



Joints

- **Joint 1:** Fixed support
- **Joint 2:** Pin support
- **Joint 3:** Change in cross-section (flexural rigidity transitions from I to $2I$)
- **Joint 4:** Fixed support

Elements

- **Element 1:** Joint 1 → Joint 2
- **Element 2:** Joint 2 → Joint 3
- **Element 3:** Joint 3 → Joint 4

Part 3 — Coordinate Systems

> *Global vs local*

Global Coordinate System (Beams)

Use a right-handed XYZ system:

- X axis along the beam (positive to the right)
- Y axis vertical (positive upward)
- loads and reactions lie in the X–Y plane

Practical note: It’s convenient to place the origin at the leftmost joint.

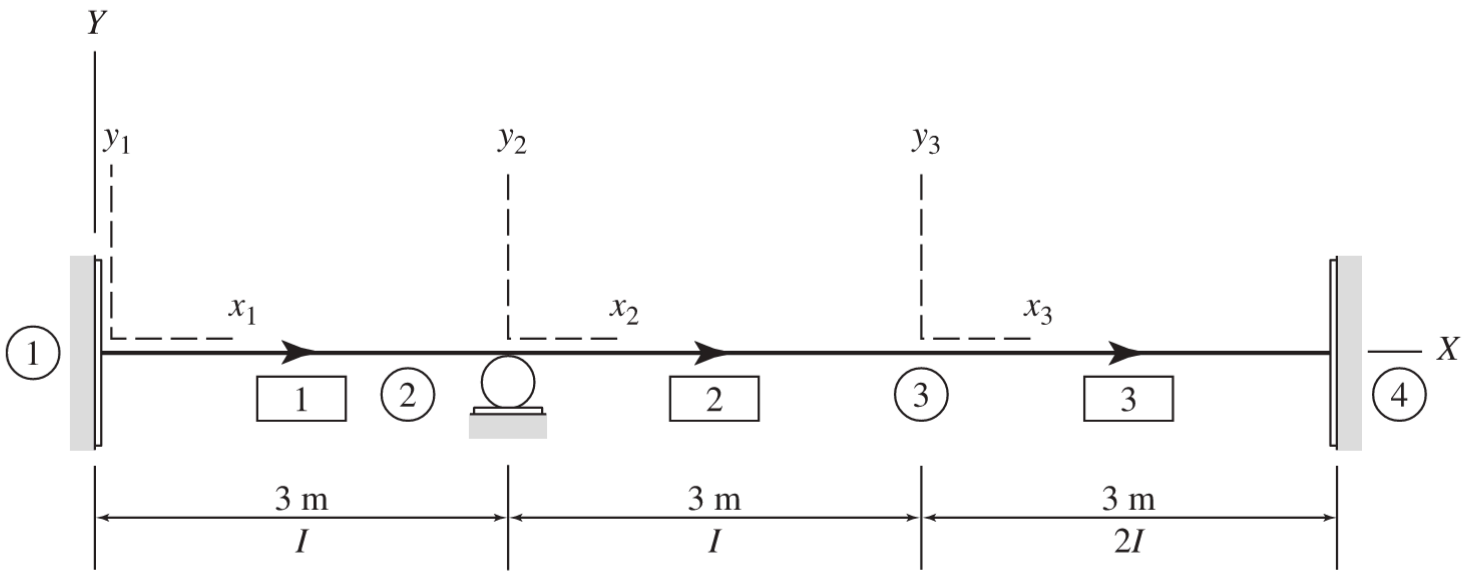
Why this simplifies programming:

- all joints lie on the X axis in the undeformed configuration
- So each joint location can be specified by **one coordinate**: X_i
- Geometry is essentially 1D (along the beam), even though bending is 2D

Local Coordinate System (Per Member)

Define a right-handed local system xyz for each member:

- origin at the **left end** of the member
- x axis along the member centroidal axis
- y axis vertical (positive upward)


Y
y_1
y_2
y_3
x_1
x_2
x_3
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3 m
3 m
I
I
2I

Bending-Only Beam Elements (This Week)

- We use a **4-DOF beam formulation**: $\{u_y,\ \theta\}$ at each node
- Assumes **pure bending** (no axial deformation)

Key requirement:

- Each element must be **aligned with the global axis**
- ⇒ local and global coordinate systems **coincide**

Implication for DSM:

- No coordinate transformation is required

$$\mathbf{k} = \mathbf{k}^{'}$$

- Forces and displacements are already expressed in the global system
- Element stiffness matrices can be **assembled directly into the global system**

Part 4 — Degrees of Freedom (DOFs)

> *Beams introduce rotations as unknowns.*

DOFs per Joint (2D Beam Model)

Axial deformation is neglected, translations in the global X direction are taken as **zero**

So each joint can have up to **two** types of deformations:

1. vertical translation: u (along global Y)
2. rotation: θ (about global Z)

DOF Vector and Sign Conventions

At a joint i, we collect beam DOFs as:

$$\mathbf{u}_i = \begin{Bmatrix} u_i \\ \theta_i \end{Bmatrix} = \begin{Bmatrix} u_{i,1} \\ u_{i,2} \end{Bmatrix}$$

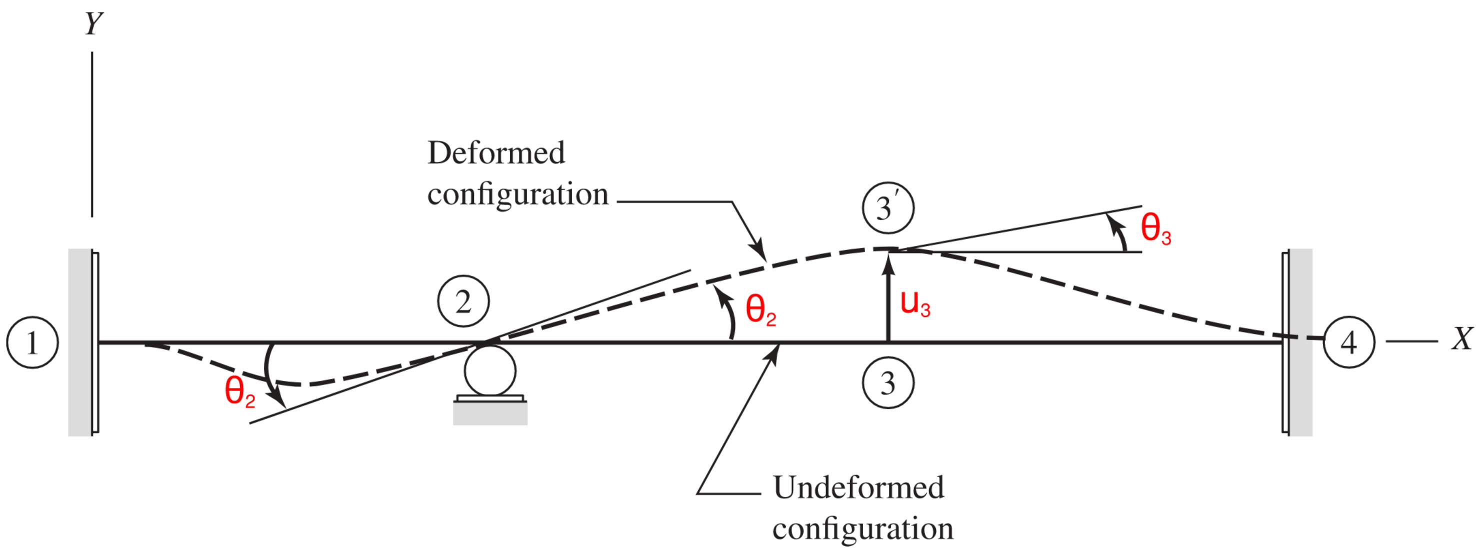
Sign conventions:

- u is positive **upward**
- θ is positive **counterclockwise**

Example: Deformed Shape, Showing u and θ

From the support types:

- node 1: (fixed support): $u = 0$ and $\theta = 0$ (no free DOFs)
- node 2: roller support (vertical restraint): $u = 0$ but θ is free
- node 3: free joint: both u and θ are free
- node 4: (fixed support): $u = 0$ and $\theta = 0$ (no free DOFs)



Part 5 — DOF Counting and Numbering

> *Before you build $\mathbf{K}$, DOF bookkeeping.*

Counting Degrees of Freedom

Let:

- j = number of joints
- r = number of restrained joint displacement components
- N_{CJT} = DOFs per free joint

Then the number of structural degrees of freedom is:

$$N_{\mathrm{DOF}} = N_{\mathrm{CJT}}\, j - r$$

For a planar beam, each free joint has one translational and one rotational DOFs:

$$N_{\text{CJT}} = 2 \quad (u_y, \theta)$$

So:

$$\boxed{N_{\text{DOF}} = 2j - r}$$

This is the number of **independent joint displacements** that must be solved for.

DOF Numbering Convention (Recommended)

We adopt a **systematic, equation-based numbering scheme**:

- Number joints sequentially (e.g., left → right)
- At each joint:
    1. number vertical displacement u first
    2. then number rotation θ
- Continue this pattern for all joints (regardless of support type)

With this convention (1-based indexing):

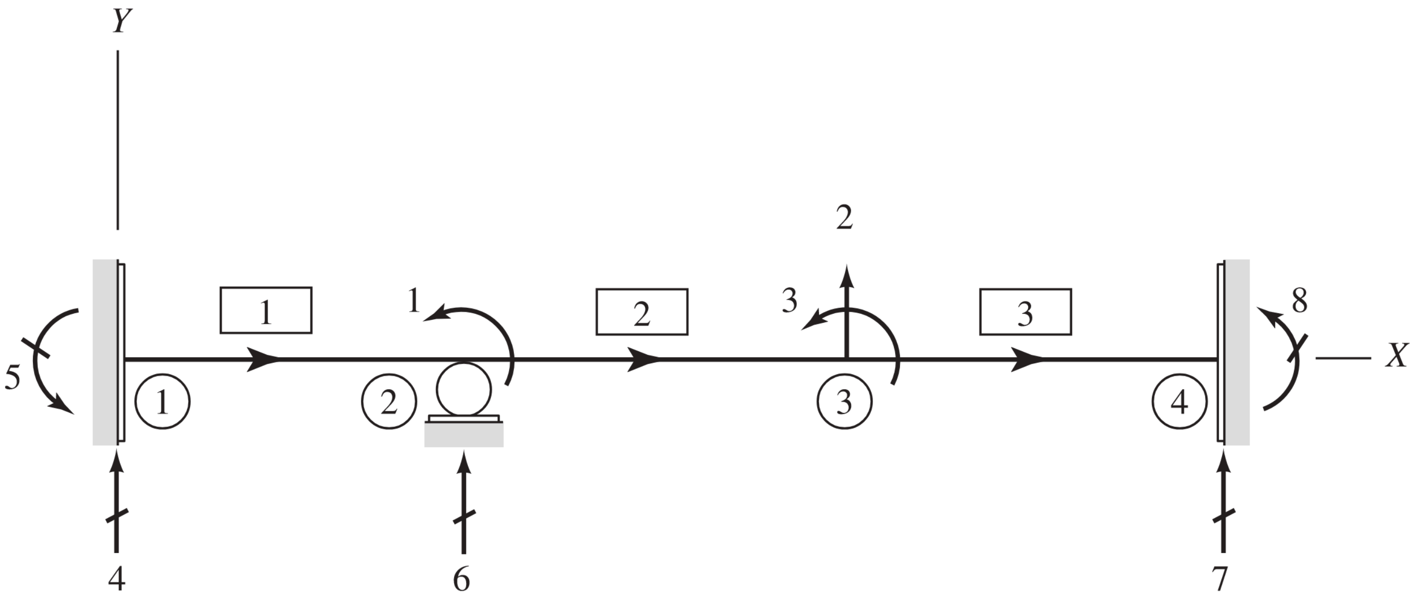
$$\mathrm{DOF}(j, u) = 2j - 1$$

$$\mathrm{DOF}(j, \theta) = 2j$$

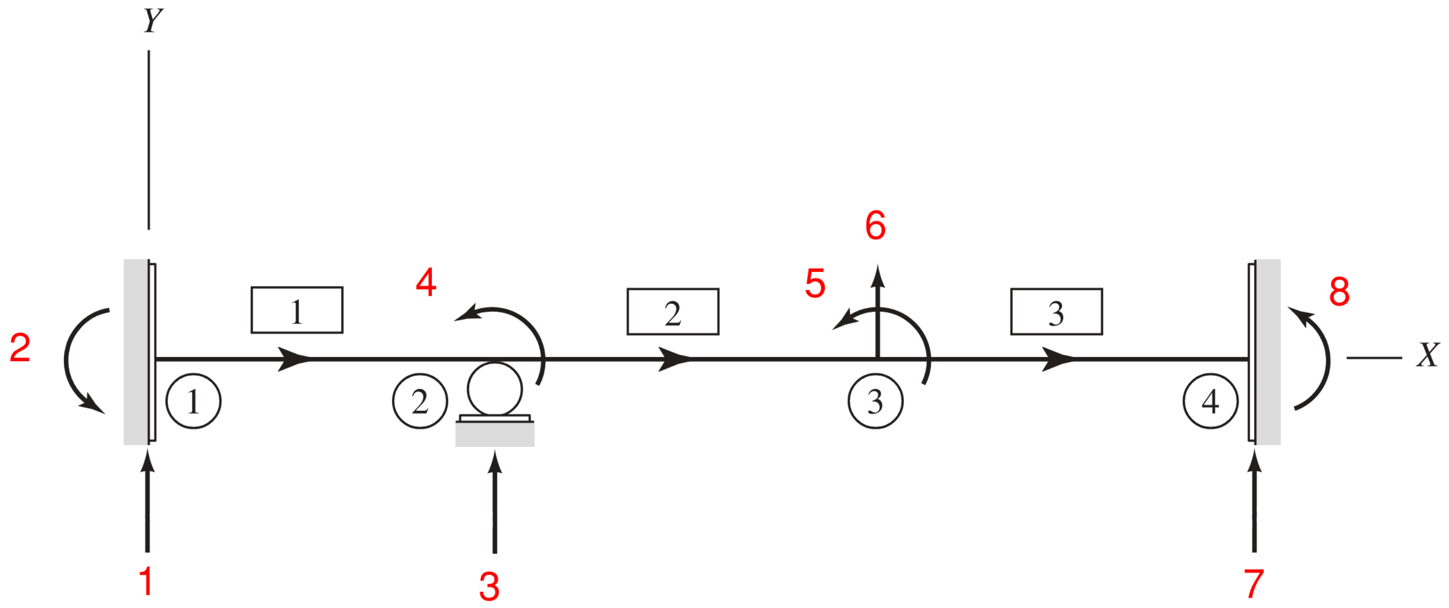
This produces a fully predictable mapping between joint index and DOF number.

> ***Note:*** *Some textbooks (e.g., Kassimali) recommend numbering all free DOFs first and restrained DOFs afterward. While mathematically equivalent, the above convention is often easier to implement and debug, because DOF numbers follow a direct formula rather than a bookkeeping pass.*

Textbook Numbering



Our Numbering



Part 6 — Loads and Reactions

> *Beams can have loads at joints and along members.*

Joint Loads vs Member Loads

- **Joint loads**: forces/moments applied at joints
- **Member loads**: loads applied between joints (distributed loads, point loads on a span, a couple, etc.)

In beam analysis, both occur.

For now, we will **start with joint loads only** to mirror our truss workflow.

Loads that Correspond to Beam DOFs

Each beam joint has up to two DOFs (u, θ), and each DOF has a corresponding **generalized load**:

- For vertical translation u → a **vertical force** in the Y direction
- For rotation θ → a **nodal moment** about the Z axis

Here, the term **load** is used in a broad sense to mean either a **force** or a **moment**, applied in the direction of a DOF.

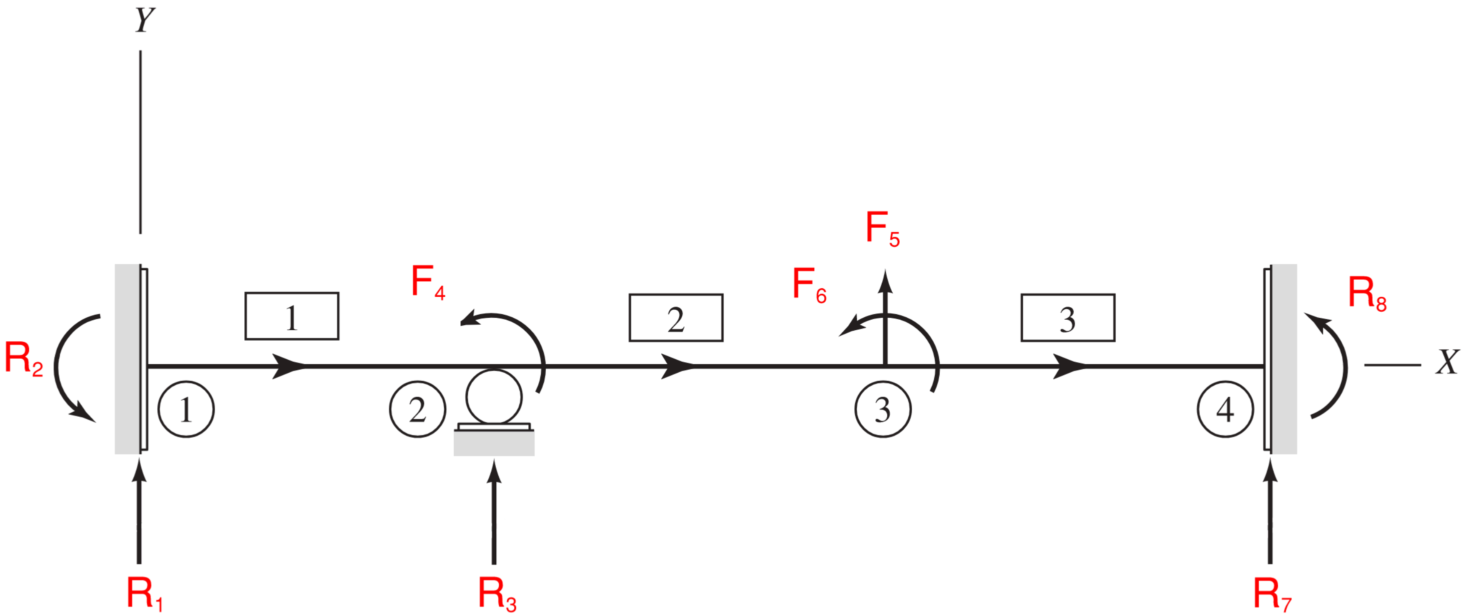
Rule of thumb:
A joint load may be applied in the direction of each degree of freedom.

Support Reactions

A reaction can develop at each **restrained coordinate**:

- restrained u → vertical reaction force
- restrained θ → reaction moment

So if a beam has N_R restrained coordinates, it can develop up to N_R **reactions**.


Y
X
R2
R1
R3
R7
R8
F4
F5
F6
1
2
3
4

Part 7 — Putting It All Together

Global System Equation

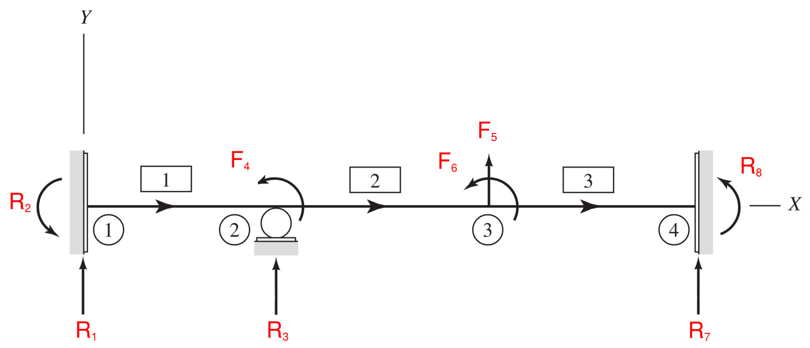
Once the beam is discretized and DOFs are numbered:

$$\boxed{\mathbf{K}\mathbf{u} = \mathbf{f}}$$

- $\mathbf{u}$ collects joint translations and rotations at the selected coordinates
- $\mathbf{f}$ collects the corresponding joint forces and moments
- $\mathbf{K}$ comes from **assembling beam element stiffness matrices**

Example Structure

For the beam shown (4 joints, 2 DOFs per joint), the assembled global system is



$$[K]\begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ u_5 \\ u_6 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ F_4 \\ F_5 \\ F_6 \\ R_7 \\ R_8 \end{Bmatrix}$$

where:

- R_1, R_2, R_3, R_7, R_8 = support reactions (forces or moments at restrained DOFs)
- F_4, F_5, F_6 = applied joint loads (forces or moments at free DOFs)
- $[K]$ = 8×8 global stiffness matrix assembled from Elements 1–3.

Part 8 — Relating Moment to Rotation

Before deriving the beam stiffness matrix, it is helpful to recall the analogy between **axial deformation in a truss** and **bending deformation in a beam**.

Recall: Truss (Axial Deformation)

For a prismatic bar in axial tension/compression:

$$F = \frac{EA}{L}\,u$$

- F = axial force
- u = axial displacement
- E = Young's modulus
- A = cross-sectional area
- L = member length

The axial stiffness is therefore:

$$k_{\text{axial}} = \frac{EA}{L}$$

Beam: Bending and Rotation

For bending of a prismatic beam, the fundamental curvature relation from flexural theory:

$$\kappa = \frac{M}{EI}$$

where:

- M = bending moment
- E = Young's modulus
- I = second moment of area

Curvature from Displacement

Curvature is the second derivative of transverse displacement:

$$\kappa = \frac{d^2u}{dx^2}$$

Therefore,

$$\frac{d^2u}{dx^2} = \frac{M}{EI}$$

Connecting to Rotation

Rotation is the **slope** of the deflection curve:

$$\theta = \frac{du}{dx}$$

So curvature is the derivative of rotation,

$$\kappa = \frac{d\theta}{dx} = \frac{d^2u}{dx^2}$$

Substituting the bending relation:

$$\frac{d\theta}{dx} = \frac{M}{EI}$$

Integrating Along the Member

Integrate over a beam segment of length L:

$$\Delta\theta = \int_0^L \frac{M(x)}{EI}\,dx$$

For the simple case of **constant bending moment** ($M(x) = M$):

$$\Delta\theta = \frac{ML}{EI}$$

Rearranging gives a stiffness-like relationship:

$$M = \frac{EI}{L}\,\Delta\theta$$

Analogy: Truss (Axial) vs Beam (Bending)

The stiffness relationships for trusses and beams follow the same structural pattern.

Axial Bar (Truss)	Beam (Bending)
$u = \dfrac{FL}{EA}$	$\Delta\theta = \dfrac{ML}{EI}$
$F = \dfrac{EA}{L}\,u$	$M = \dfrac{EI}{L}\,\Delta\theta$

Both have the general form:

$$\text{Force-like quantity} = (\text{stiffness}) \times \text{deformation}$$

Key Insight

- In a **truss**, axial force is proportional to axial displacement.
- In a **beam**, bending moment is proportional to rotation.

This analogy allows us to anticipate the structure of the **beam element stiffness matrix**:

- Axial DOFs → relate forces to translations via $\frac{EA}{L}$
- Rotational DOFs → relate moments to rotations via $\frac{EI}{L}$

The beam stiffness matrix will therefore contain terms that connect **moments and rotations** in exactly the same systematic way that the truss matrix connects **forces and displacements**.

Wrap-Up

In this lecture, we:

- discretized a beam into members + joints
- defined global/local coordinate systems (and when transformations are unnecessary)
- introduced beam DOFs (u, θ) and sign conventions
- counted and numbered DOFs, identifying restrained coordinates
- connected loads/reactions to DOFs and formed $\mathbf{Ku} = \mathbf{f}$
- related moment ↔ rotation using the truss–beam analogy

Next lecture:

- derive the beam element stiffness matrix
- assemble beam systems using DSM workflow