

# CEE6501 — Lecture 7.1

## Introduction to 2D Frame Analysis

# Learning Objectives

By the end of this lecture, you will be able to:

- Describe how a 2D frame extends the beam element by including axial deformation
- Define and number DOFs, loads, and reactions for a 2D frame model
- Write the global equilibrium equation, including fixed-end force contributions
- Compute local member forces using  $\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$
- Resolve inclined member loads into parallel/perpendicular components for FEF calculations

# Agenda

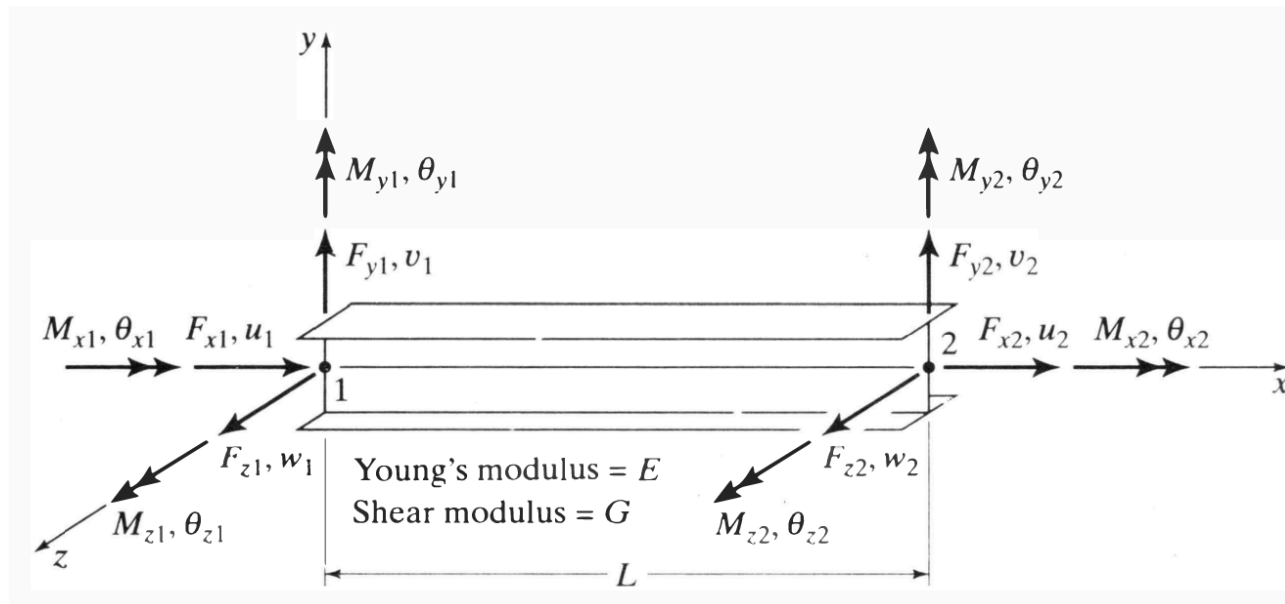
- Part 1 — Roadmap: truss → beam → 2D frame → 3D frame
- Part 2 — 2D frame analytical model
- Part 3 — Numbering DOFs, loads, reactions
- Part 4 — Setting up the global system of equations
- Part 5 — Local stiffness relations
- Part 6 — 2D frame element stiffness matrix
- Part 7 — Member forces (local) + fixed-end forces
- Part 8 — Worked example (Kassimali §6.2)
- Part 9 — In-class exercise

# Part 1 — DSM Element Roadmap

*From trusses to 2D beams to 2D frames to 3D frames*

## The 3D Frame Element (Where We Are Headed)

- A general 3D frame member has **6 DOFs per node** (12 per element)
  - translations:  $u_x, u_y, u_z$
  - rotations:  $\theta_x, \theta_y, \theta_z$
- This single element can represent **axial**, **torsion**, and **bending** behavior



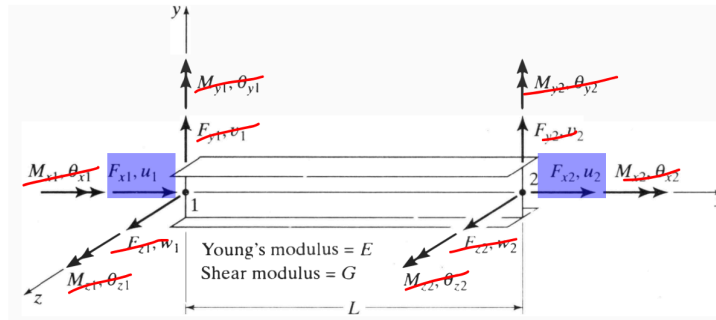
## Key Idea: Superposition

You can think of the 3D frame element as a **superposition** of four behaviors:

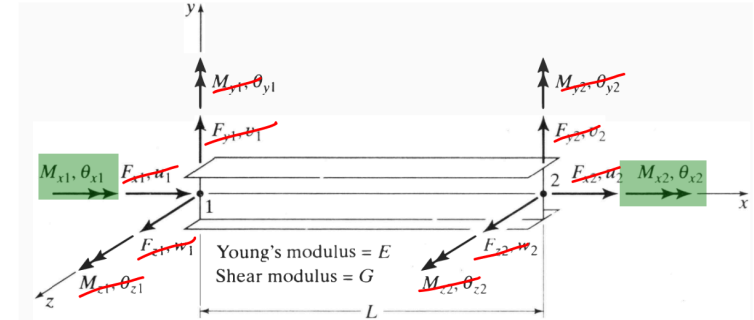
1. **Axial** deformation (truss-like)
2. **Torsion** about the member axis
3. **Bending** about one principal axis (Z-axis)
4. **Bending** about the other principal axis (Y-axis)

# The Four Behaviors

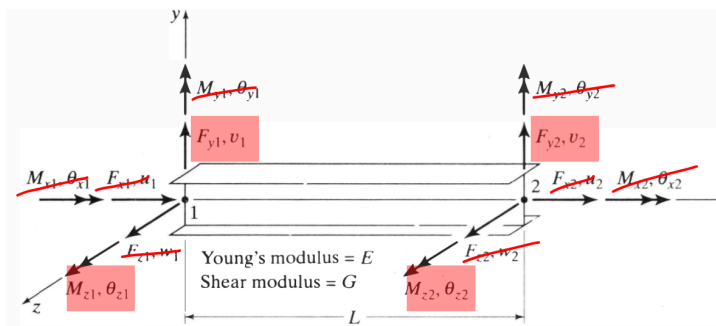
## (1) Axial



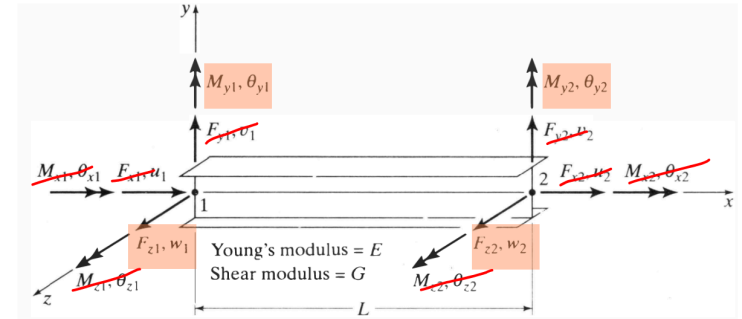
## (2) Torsion



## (3) Bending about Z-axis



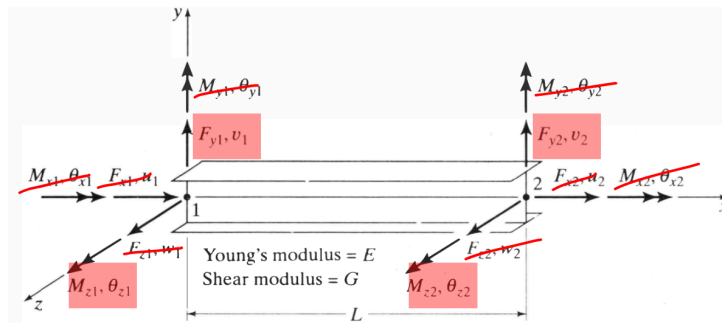
## (4) Bending about Y-axis



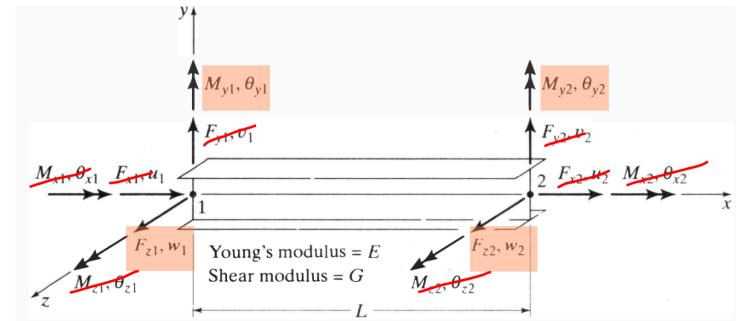
## What We Derived Last Lecture

- The **2D beam element** governing planar bending behavior (cases (3) and (4))
- Once derived for one principal axis, the formulation extends directly to the other (identical mathematics, different axis)

### (3) Bending about Z-axis

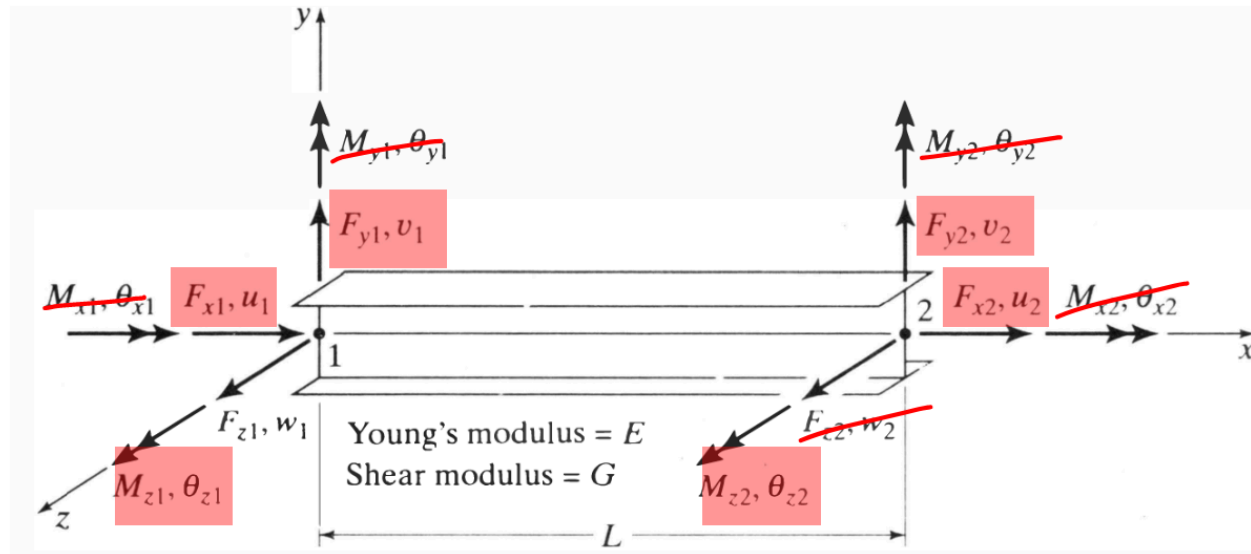


### (4) Bending about Y-axis



# What We Will Derive Today

- The DSM formulation for the **2D frame element**
- A combination of **axial + bending** behavior, specifically **bending about the  $Z$ -axis** (case (1) + (3))
- Once derived for one principal axis, the formulation extends directly to the other (identical mathematics, different axis)



## Part 2 — 2D Frame Analytical Model

*Representing a frame structure as **members + joints** for stiffness-based analysis.*

## Key Conceptual Shift (Beam $\rightarrow$ Frame)

The major differences from the pure beam formulation:

- We now include **axial deformation** in addition to bending.
- Because axial effects are included, a **frame element may be inclined** at any angle.
- A beam element (in its simplest form) was required to align with the **global X-axis**.
- A frame element therefore requires a **local coordinate system** and transformation to global coordinates.

The 2D frame element is a true generalization of the beam element.

## Frame Definition (2D Idealization)

A 2D frame member is modeled as:

- A long, straight prismatic member
- Loaded in a single plane (e.g., the  $XY$ -plane)

We adopt the **Euler–Bernoulli beam assumptions**:

- Cross-sections remain **plane**
- Cross-sections remain **perpendicular to the centerline**
- **Shear deformation is neglected**

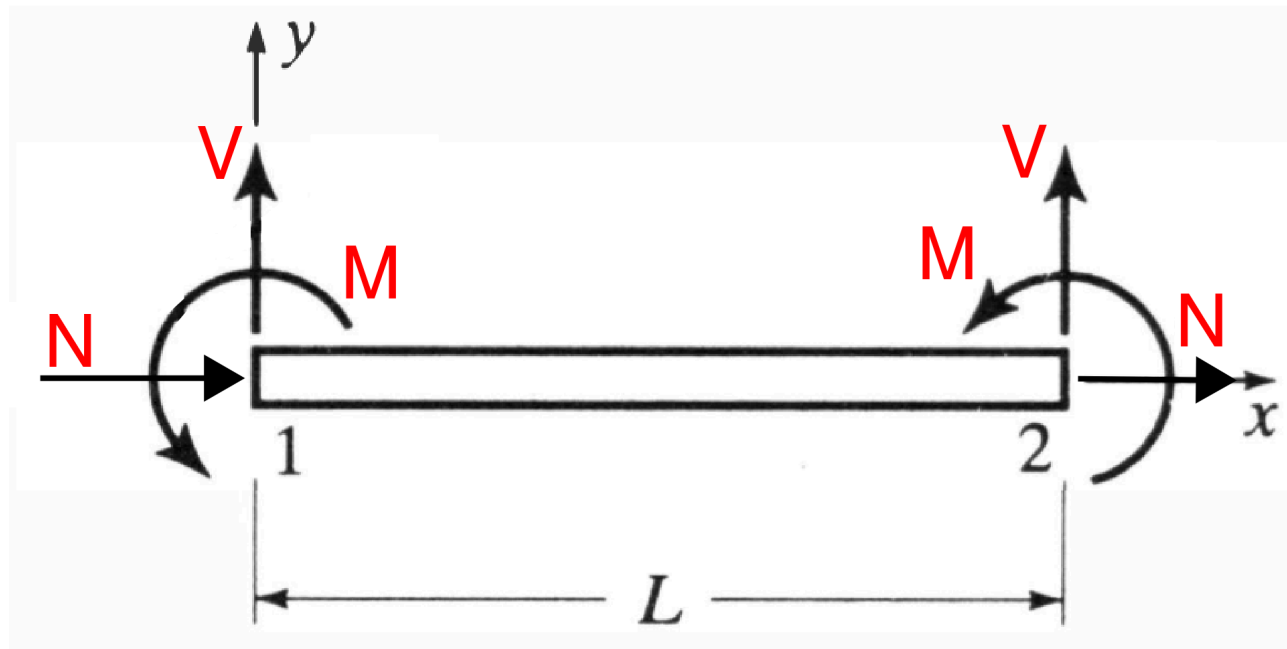
Additionally (for the frame formulation):

- **Axial deformation is included**

## Consequences of This Idealization

Each member can develop:

- Axial force  $N$ : **Parallel** to the centroidal axis (axial)
- Shear force  $V$ : **Perpendicular** to the centroidal axis (shear)
- Bending moment  $M$ : In the  $XY$ -plane (about the  $Z$ -axis, into the page)



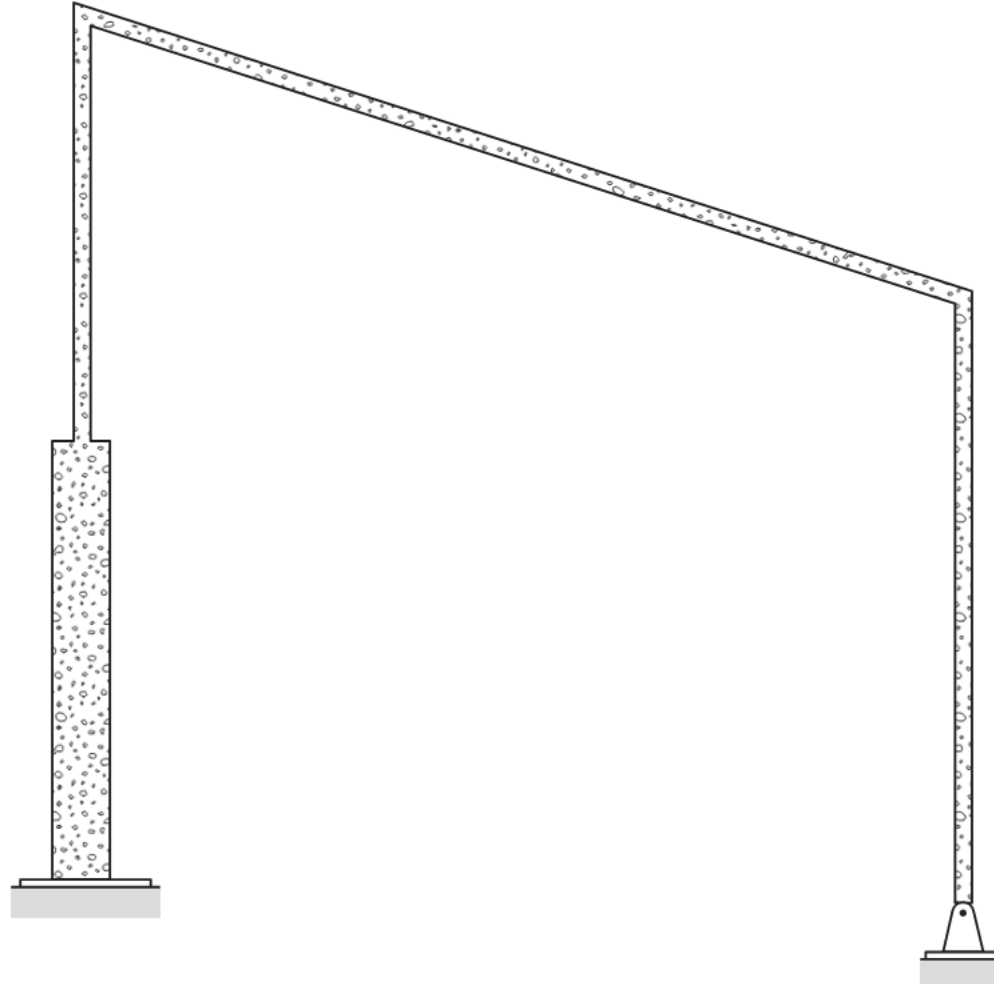
## Analytical Model for the DSM

To apply the DSM, we idealize the structure as:

- Straight, prismatic members (constant  $A$ ,  $E$ ,  $I$ )
- Connected at discrete **joints (nodes)**
- Rigid joint connections (for now)
- Unknown reactions acting **only at joints**
- External loads applied anywhere along members (converted to FEFs)

This converts a continuous structural system into a **finite system of degrees of freedom** suitable for matrix analysis.

# Example Frame Structure

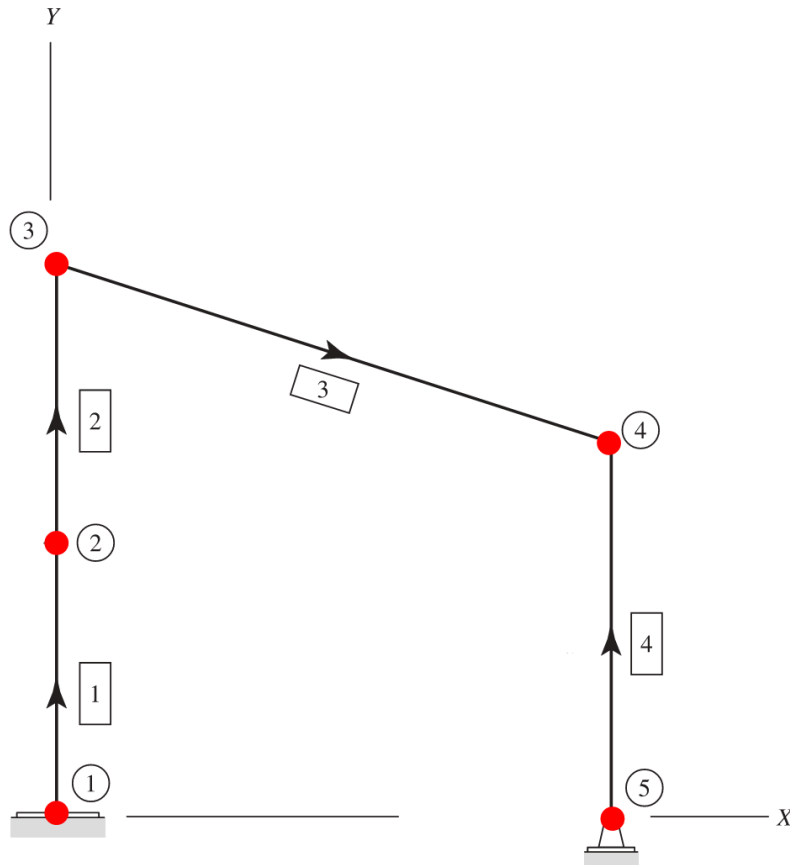


## Discretize into Members and Joints

Even if the structure is physically continuous, we often insert joints so that:

- **Support reactions** occur at joints (not mid-member)
- Each member has **constant properties** (e.g., constant  $EI$ )
- **Member loads** can be easily turned into fixed-end forces (more in Lecture 6.3)

# Example Frame Discretization



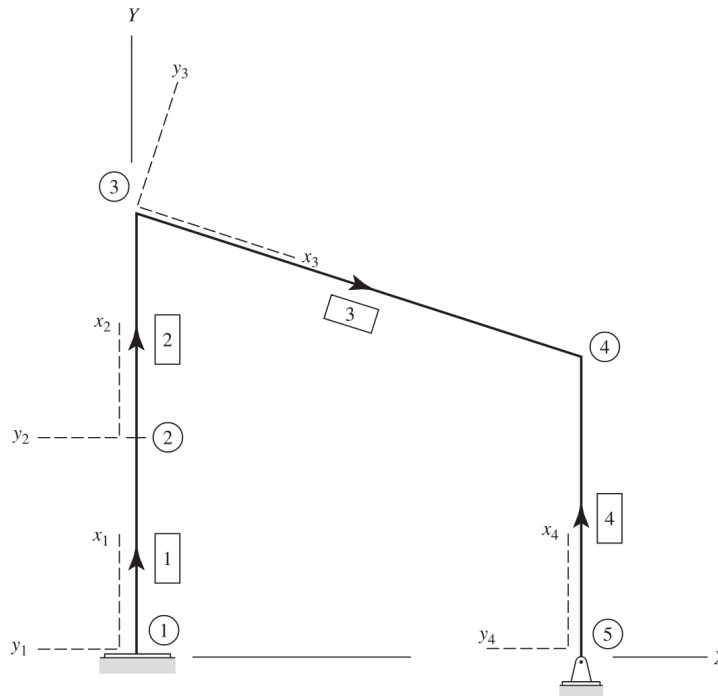
## Joints

- **Joint 1:** Fixed support
- **Joint 2:** Free,  $\Delta$  in  $EI$
- **Joint 3:** Free
- **Joint 4:** Free
- **Joint 5:** Pin

## Elements

- **Element 1:** Joint 1  $\rightarrow$  Joint 2
- **Element 2:** Joint 2  $\rightarrow$  Joint 3
- **Element 3:** Joint 3  $\rightarrow$  Joint 4
- **Element 4:** Joint 5  $\rightarrow$  Joint 4

# Global and Local Coordinate Systems



## Global Coordinate System

- $X$ : horizontal (positive to the right)
- $Y$ : vertical (positive upward)
- All loads and reactions lie in the  $X$ – $Y$  plane

## Local Coordinate System (per member)

- Origin at the start node
- $x$ : along the member (start  $\rightarrow$  end)
- $y$ : perpendicular to the member (in the frame plane)
- Right-handed system; positive rotation about  $z$  is counterclockwise

## Part 3 — Numbering DOFs, Loads, Reactions

## DOFs per Joint (2D Frame Model)

Each joint can have up to **three** types of deformations:

1. horizontal translation:  $u_x$  (along global  $X$ )
2. vertical translation:  $u_y$  (along global  $Y$ )
3. rotation:  $\theta$  (about global  $Z$ )

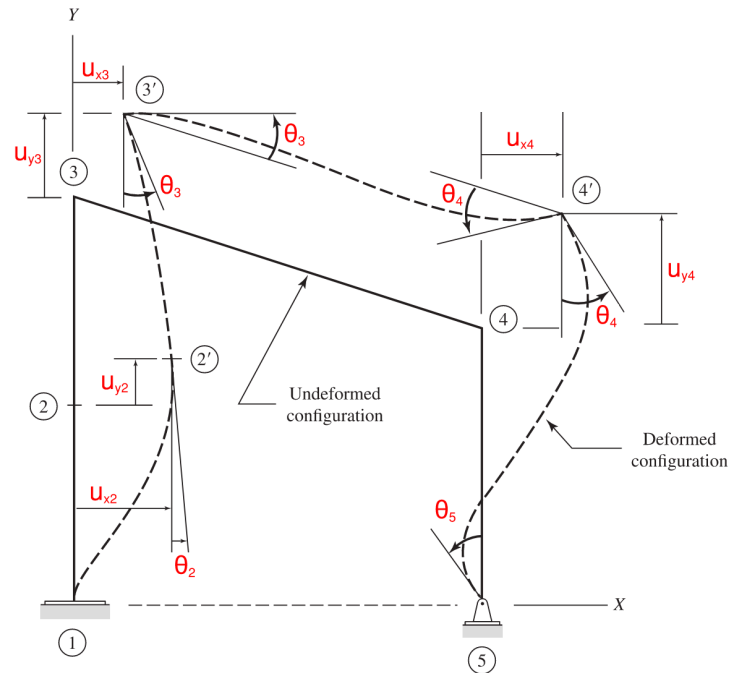
At a joint  $i$ , we collect DOFs as:

$$\mathbf{u}_i = \begin{Bmatrix} u_{x,i} \\ u_{y,i} \\ \theta_i \end{Bmatrix} = \begin{Bmatrix} u_{1,i} \\ u_{2,i} \\ u_{3,i} \end{Bmatrix}$$

Sign conventions:

- $u_x$  is positive **right**
- $u_y$  is positive **up**
- $\theta$  is positive **counterclockwise**

## Example: Deformed Shape, Showing $u$ and $\theta$



From the support types:

- node 1: fixed support
- node 2: free
- node 3: free
- node 4: free
- node 5: pin

## Counting Degrees of Freedom

For a frame, each free joint has two translational and one rotational DOFs:

$$N_{\text{CJT}} = 3 \quad (u_x, u_y, \theta)$$

So:

$$N_{\text{DOF}} = 3j - r$$

This is the number of **independent joint displacements** that must be solved for.

## DOF Numbering Convention (Recommended)

We adopt a **systematic, equation-based numbering scheme**:

- Number joints sequentially
- At each joint:
  1. number horizontal displacement  $u_x$  first
  2. number vertical displacement  $u_y$  next
  3. then number rotation  $\theta$
- Continue this pattern for all joints (regardless of support type)

With this convention (1-based indexing):

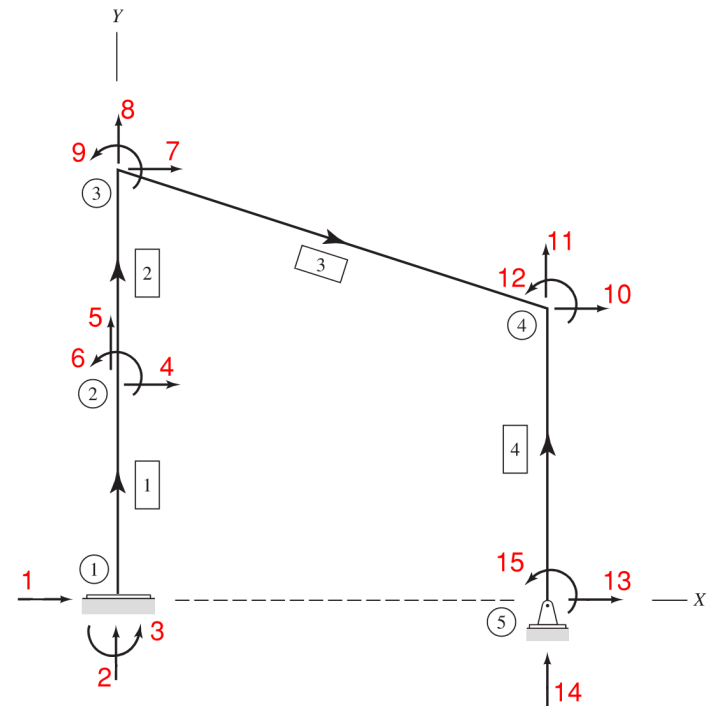
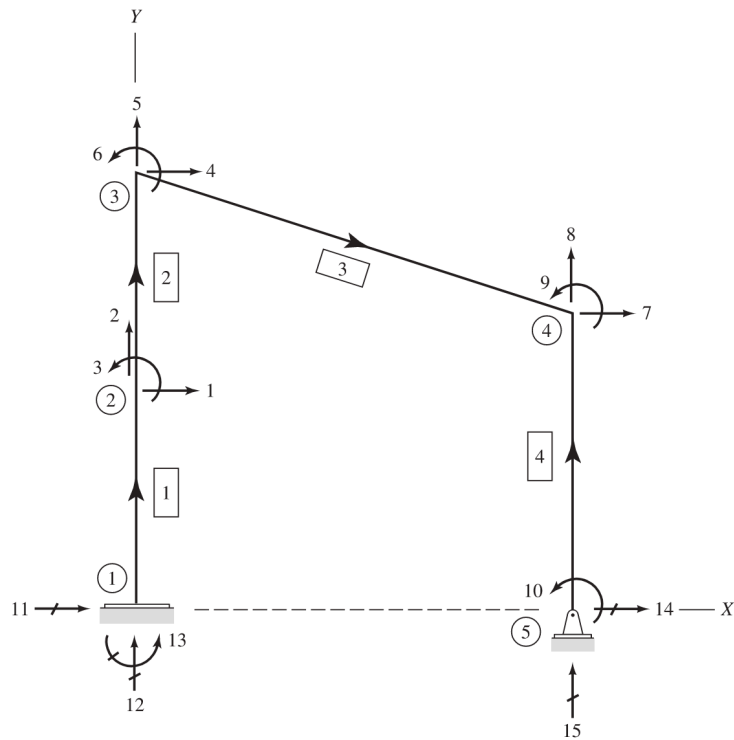
$$\text{DOF}(j, u_x) = 3j - 2$$

$$\text{DOF}(j, u_y) = 3j - 1$$

$$\text{DOF}(j, \theta) = 3j$$

This produces a fully predictable mapping between joint index and DOF number.

# Textbook vs. Our Numbering



## Joint Loads vs. Member Loads

- **Joint loads:** forces/moments applied at joints
- **Member loads:** loads applied between joints (distributed loads, point loads on a span, a couple, etc.)

Here, the term **load** is used in a broad sense to mean either a **force** or a **moment**, applied in the direction of a DOF.

## Loads and Reactions Correspond to Global DOF Numbering

Each joint has up to three DOFs ( $u_x, u_y, \theta$ ).

To every DOF, there corresponds a **generalized load**:

- If the DOF is **free** → an external load may be applied
- If the DOF is **restrained** → an unknown reaction will develop

For a 2D frame element:

- $u_x \leftrightarrow$  Axial force  $F_x$  (horizontal force)
- $u_y \leftrightarrow$  Shear force  $F_y$  (vertical force)
- $\theta_z \leftrightarrow$  Bending moment  $M_z$

## Part 4 — Setting Up the Global System of Equations

## Global Equilibrium (Including Fixed-End Forces)

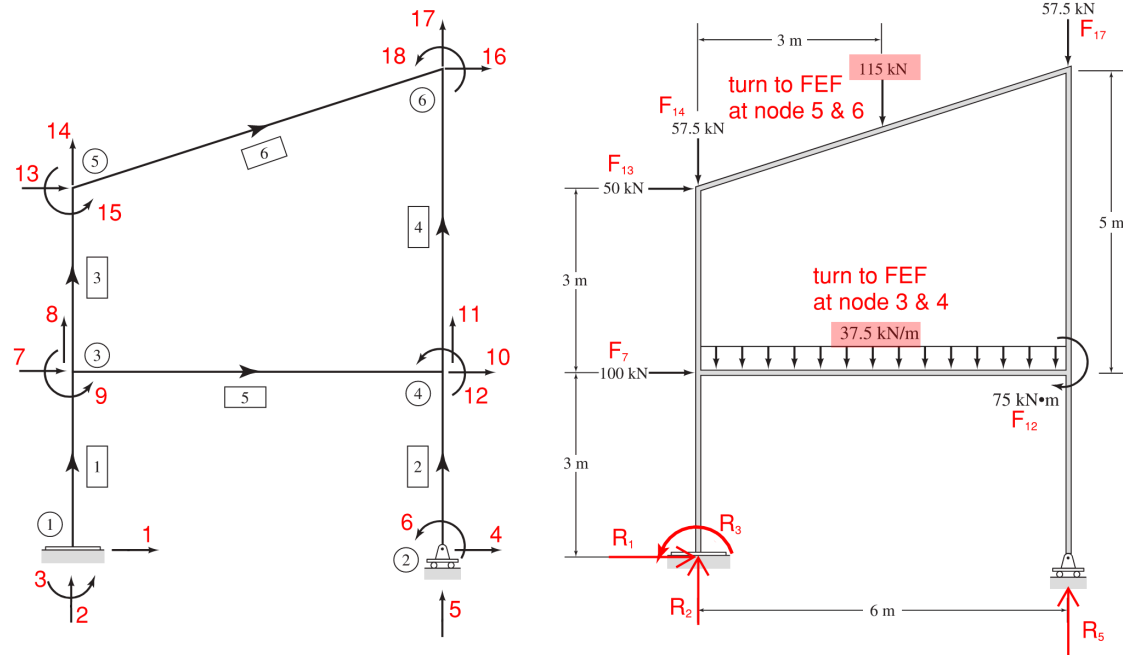
Once the frame is discretized and DOFs are numbered:

$$\mathbf{K}\mathbf{u} = \mathbf{f} - \mathbf{f}^F$$

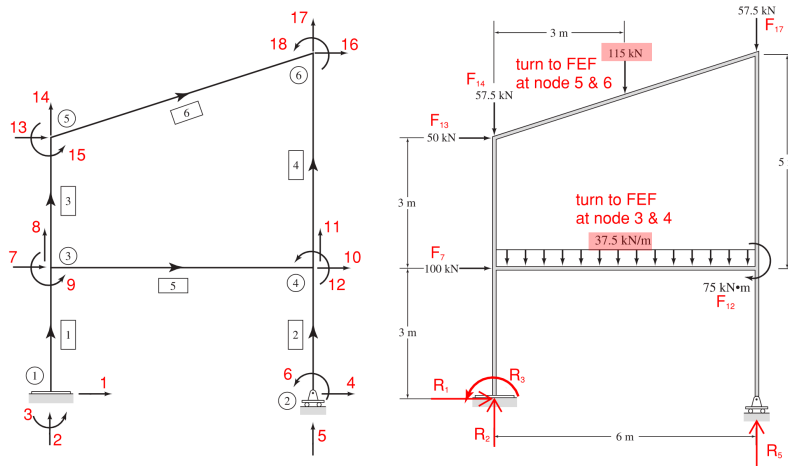
- $\mathbf{u}$  collects joint translations and rotations at the selected coordinates
- $\mathbf{f}$  collects the corresponding joint forces and moments
- $\mathbf{f}^F$  collects the corresponding fixed-end joint forces and moments
- $\mathbf{K}$  comes from **assembling frame element stiffness matrices**

# Example Structure

A frame with 6 joints (3 DOFs per joint)



We will derive fixed-end forces (FEFs) for frames later in the lecture, when discussing local member forces.



$$\mathbf{K}_{18 \times 18} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ 0 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \\ u_{17} \\ u_{18} \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ 0 \\ R_5 \\ 0 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ -75 \\ 50 \\ -57.5 \\ 0 \\ 0 \\ -57.5 \\ 0 \end{Bmatrix} - \{F^F\}$$

# Part 5 — 2D Frame Element Local Stiffness Relations

*Local Element Mechanics*

## Frame Element Response

The **member stiffness relations** express the end forces of a frame element as functions of the **end displacements**.

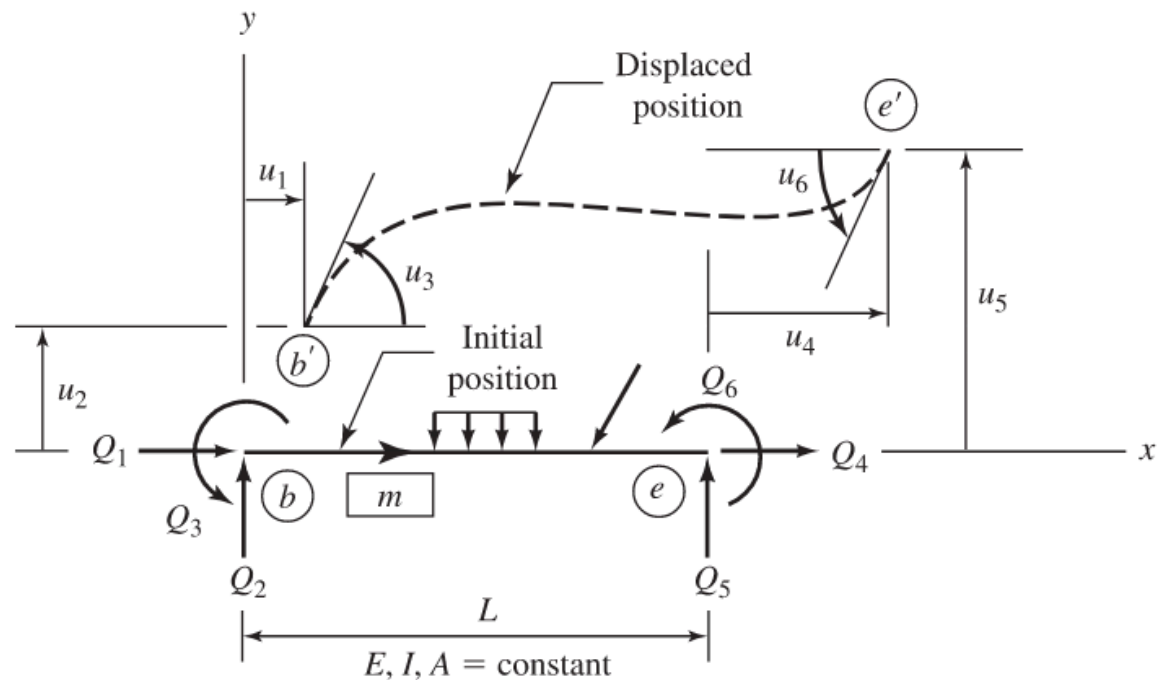
When a frame element is subjected to external loading:

- **Axial forces** develop at the ends
- **Shear forces** develop at the ends
- **Bending moments** are induced at the ends

These loads are fully determined by the **displacements and rotations at the element ends**.

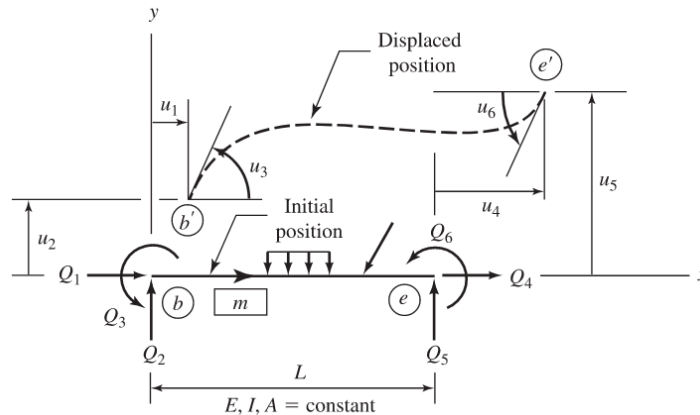
## Generic Displacement for a 2D Frame Element

We define all quantities in the **local coordinate system**, with origin at the left end  $b$ , and ending at node  $e$ .



## 2D Frame Element DOF Numbering (Local)

Degrees of freedom are ordered from **left** → **right** node.



- **DOF 1:**  $u_1$  — node  $b$ , local  $x$
- **DOF 2:**  $u_2$  — node  $b$ , local  $y$
- **DOF 3:**  $u_3$  — node  $b$ ,  $\theta$
- **DOF 4:**  $u_4$  — node  $e$ , local  $x$
- **DOF 5:**  $u_5$  — node  $e$ , local  $y$
- **DOF 6:**  $u_6$  — node  $e$ ,  $\theta$

### Sign conventions:

- $+u_x \rightarrow$  right (local  $x$  direction)
- $+u_y \rightarrow$  up (local  $y$  direction)
- $+\theta \rightarrow$  counterclockwise
- Forces follow the same order and convention

## Relating Local Displacement and Force

$$\mathbf{Q} = \mathbf{k} \mathbf{u} + \mathbf{Q}_f$$

- $\mathbf{Q}$  denotes the  $6 \times 1$  member end-force vector
- $\mathbf{u}$  denotes the  $6 \times 1$  member end-displacement vector
- $\mathbf{k}$  represents the  $6 \times 6$  member local stiffness matrix
- $\mathbf{Q}_f$  is the  $6 \times 1$  member fixed-end force vector in the local coordinate system

## 6 Linear Equations (One per DOF)

For a linear elastic element, the force at any DOF is a **linear combination** of the force induced by all DOF displacements:

- Displacing one DOF can induce forces at *all* DOFs
- The proportionality constants are the stiffness coefficients  $k_{ij}$

Each equation expresses **force equilibrium at a single local degree of freedom**.

For example, DOF 1:

$$Q_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4 + k_{15}u_5 + k_{16}u_6 + Q_{f,1}$$

Rather than writing out all 6 equations:

$$Q_i = \sum_{j=1}^6 (k_{ij} u_j) + Q_{fi}, \quad i = 1, 2, \dots, 6$$

## Same Equations in Matrix Form

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} + \begin{Bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \\ Q_{f5} \\ Q_{f6} \end{Bmatrix}$$

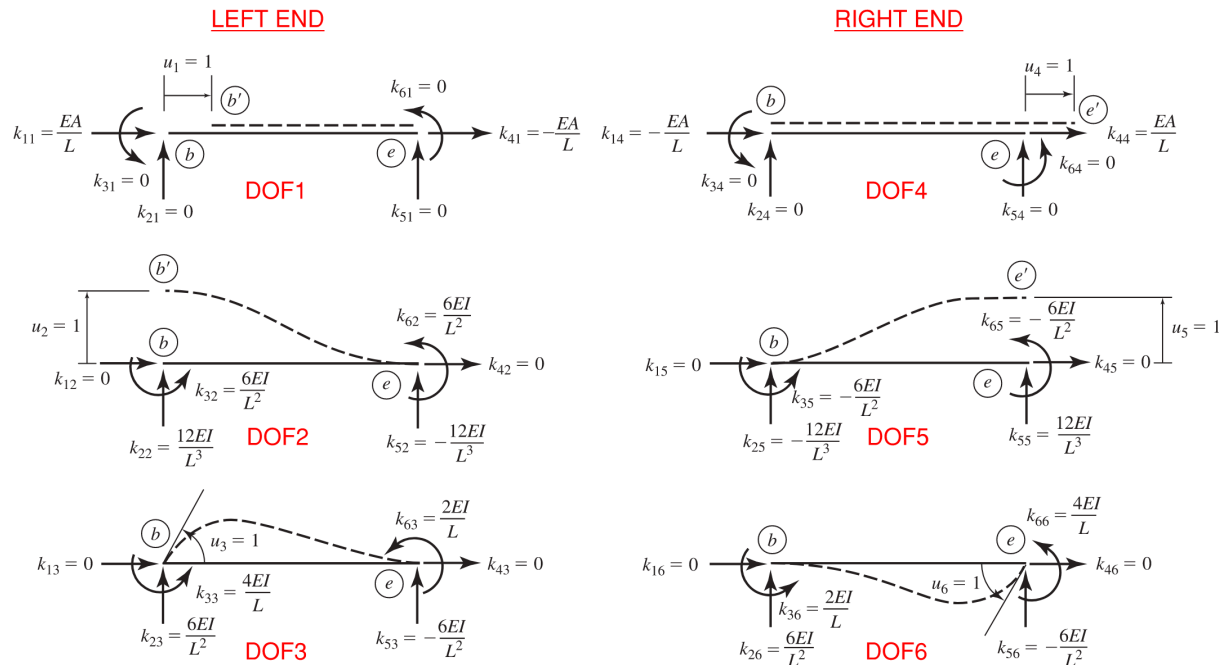
## Part 6 — 2D Frame Element Stiffness Matrix

*Local Coordinates*

# Unit Displacement Method

You can derive the 2D frame element from scratch using the unit displacement method.

$k_{ij}$  = force at DOF  $i$  due to a unit displacement at DOF  $j$ ,  
with all other DOFs fixed.



## Combination of Truss and Beam Stiffnesses

We have already derived the **axial (truss)** and **bending (beam)** stiffness matrices.

For a 2D frame element (local DOF ordering,  $(u_1, v_1, \theta_1, u_2, v_2, \theta_2)$ ):

- Axial effects act in **DOFs 1 and 4**
- Bending effects act in **DOFs 2, 3, 5, 6**
- In local coordinates, these effects are **decoupled**

Therefore, the frame stiffness matrix is obtained by **superposing** the two contributions:

## Axial + Bending Stiffness

$$\mathbf{k}_{\text{frame}} = \underbrace{\frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{axial}} + \underbrace{\frac{EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}}_{\text{bending}}$$

## 2D Frame Stiffness Matrix

$$\mathbf{k}_{\text{frame}} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

## 2D Frame Stiffness Matrix (Factored)

$$\mathbf{k}_{\text{frame}} = \frac{EI}{L^3} \begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}$$

# Part 7 — Member Forces

*Local Coordinates*

# Fixed-End Forces in Frames

For frame members, there are two common loading situations:

## 1. Load perpendicular to the member axis

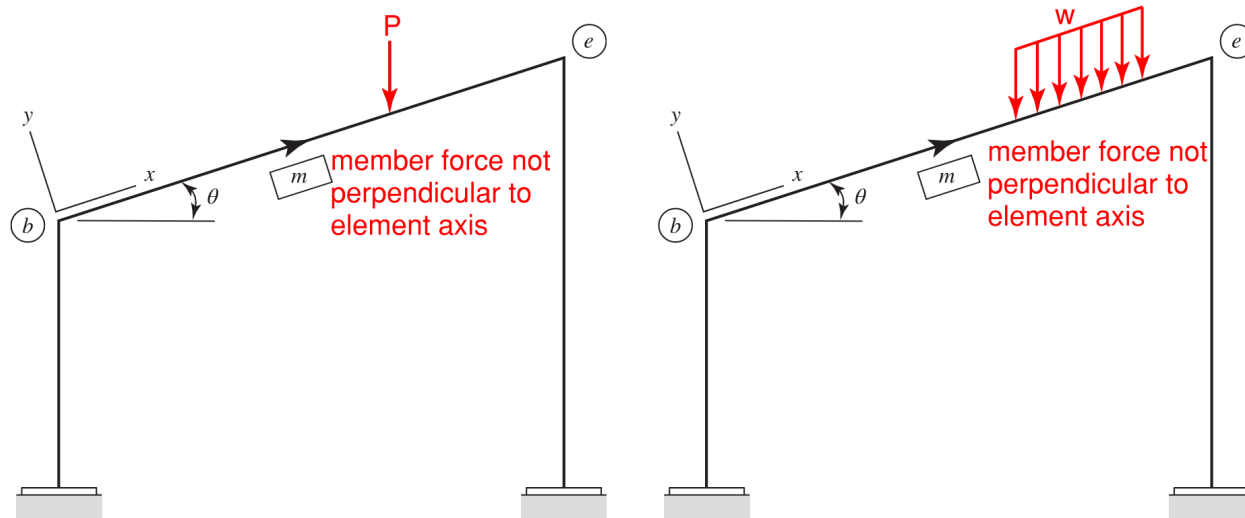
- Produces **shear** ( $V$ ) and **moment** ( $M$ )
- No axial force
- Covered in the previous lecture

## 2. Load not perpendicular (inclined member or load)

- Must resolve into components
- Produces:
  - Axial force ( $N$ )
  - Shear ( $V$ )
  - Moment ( $M$ )

# Inclined Member Forces

- In **beams**:
  - Loads are applied **perpendicular** to the member axis
  - No need for resolution
- In **frames**:
  - Members and loads may be **inclined**
  - Forces must be **resolved into local components**



# Resolving Applied Forces

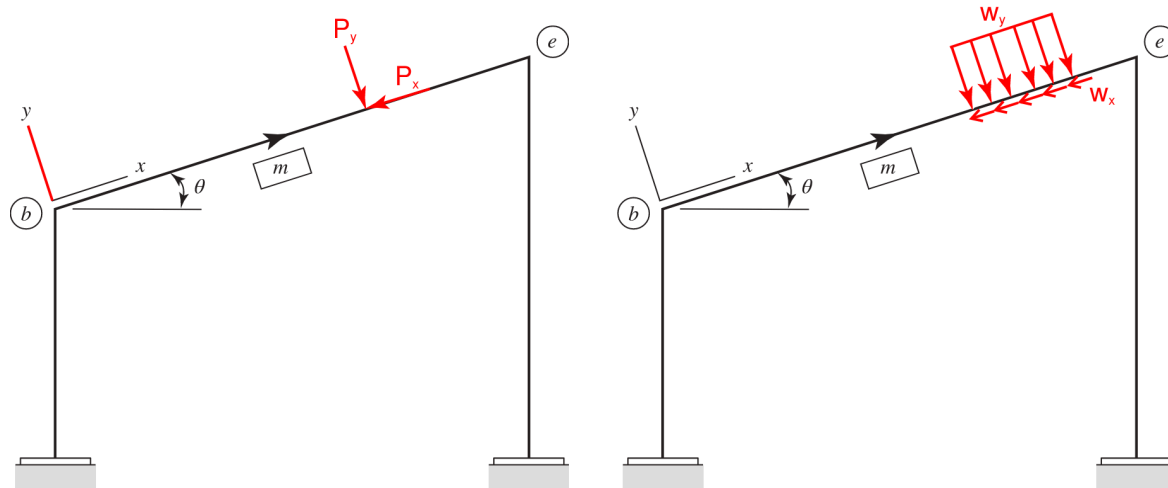
The load is resolved into the element's local coordinate system using the element's global orientation angle  $\theta$ .

Point load:

$$P_x = P \sin \theta, \quad P_y = P \cos \theta$$

Distributed load:

$$w_x = w \sin \theta, \quad w_y = w \cos \theta$$



## Two Components $\rightarrow$ Two Effects

- **Perpendicular component ( $P_y, w_y$ ):**
  - Use standard beam FEF formulas
  - Produces **shear and bending moment**
- **Parallel component ( $P_x, w_x$ ):**
  - Produces **axial force**
  - Acts along the member axis

## Sign Convention

We define the following sign conventions for applied loads:

- Perpendicular loads: "Down" on the beam (opposite the local +y axis) is positive.
- Parallel loads: Forces opposite the local +x axis are positive.

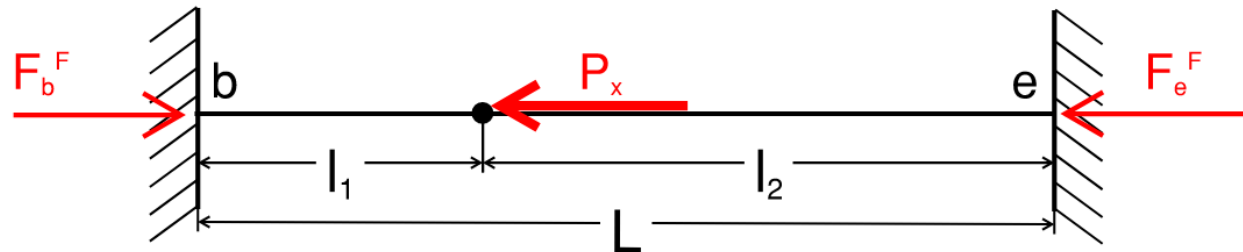
If the element orientation angle  $\theta$  is defined correctly, the resolved load components  $(f_x, f_y)$  will automatically have the correct signs. No manual sign adjustments are required after resolving the force.

## Axial Fixed-End Forces

- Correspond to **axial DOFs**:
  - $Q_1$  and  $Q_4$
- Question:
  - What are the **fixed-end axial forces** at each node due to loads **parallel to the member axis**?

The following expressions come from equilibrium of an **axially loaded member**. See Kassimali — Section 6.2 for the full derivation.

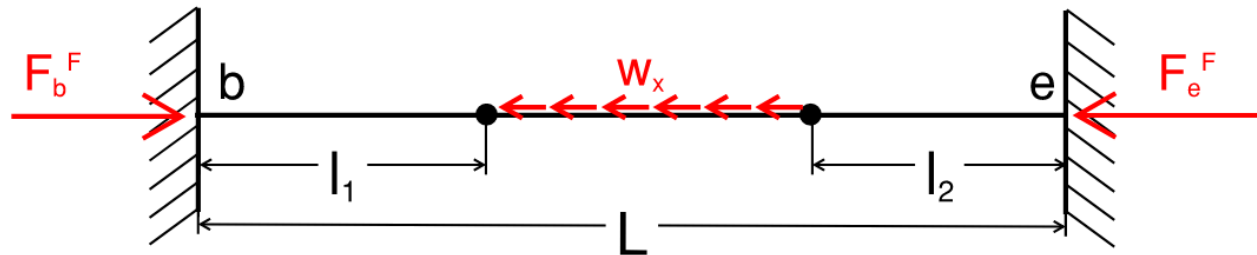
## Axial FEF — Point Load



$$F_b^F = P_x \frac{l_2}{L}, \quad F_e^F = P_x \frac{l_1}{L}$$

$$l_2 = L - l_1$$

## Axial FEF — Distributed Load



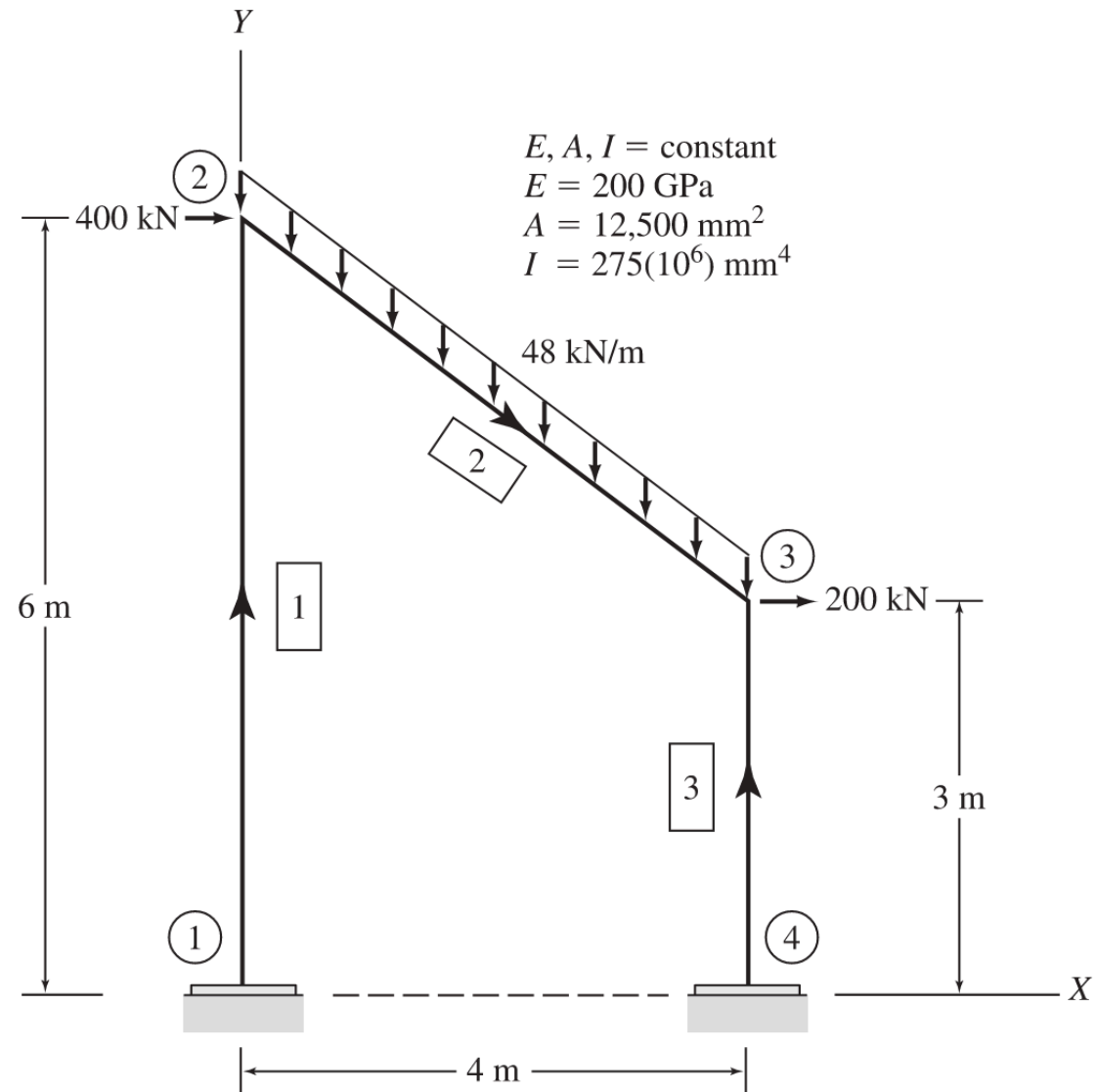
$$F_b^F = \frac{w_x}{2L} (L - l_1 - l_2)(L - l_1 + l_2)$$

$$F_e^F = \frac{w_x}{2L} (L - l_1 - l_2)(L + l_1 - l_2)$$

Key takeaway:

- **Inclined member loads → axial + shear + moment**
- Must account for all three in frame analysis as fixed-end forces

## Part 8 — Example (6.2 in Kassimali)



**1. Compute the local stiffness matrix.**

For member 2, use:

$$E = 200 \text{ GPa}, \quad A = 12,500 \times 10^{-6} \text{ m}^2, \quad I = 275 \times 10^{-6} \text{ m}^4,$$

$$L = \sqrt{4^2 + 3^2} = 5 \text{ m}.$$

```

In [1]: import numpy as np
np.set_printoptions(precision=1, suppress=True)

def frame_element_kl(E, A, I, L):
    """
    Return 6x6 *local* stiffness matrix for a 2D frame element
    """

    factor = (E * I) / (L**3)

    kl = factor * np.array(
        [
            [ (A*L**2)/I, 0.0, 0.0, -(A*L**2)/I, 0.0, 0.0 ],
            [ 0.0, 12.0, 6.0*L, 0.0, -12.0, 6.0 ],
            [ 0.0, 6.0*L, 4.0*L**2, 0.0, -6.0*L, 2.0 ],
            [ -(A*L**2)/I, 0.0, 0.0, (A*L**2)/I, 0.0, 0.0 ],
            [ 0.0, -12.0, -6.0*L, 0.0, 12.0, -6.0 ],
            [ 0.0, 6.0*L, 2.0*L**2, 0.0, -6.0*L, 4.0 ],
        ],
        dtype=float,
    )

    return kl

```

```

In [2]: # --- properties in kN and m ---
E = 200e6          # kN/m^2 (converted from 200e9 N/m^2)
A = 12500e-6       # m^2 (converted from 12500 mm^2)
I = 275e-6         # m^4 (from 275×10^6 mm^4)
L = 5.0           # m

k_local = frame_element_kl(E, A, I, L)

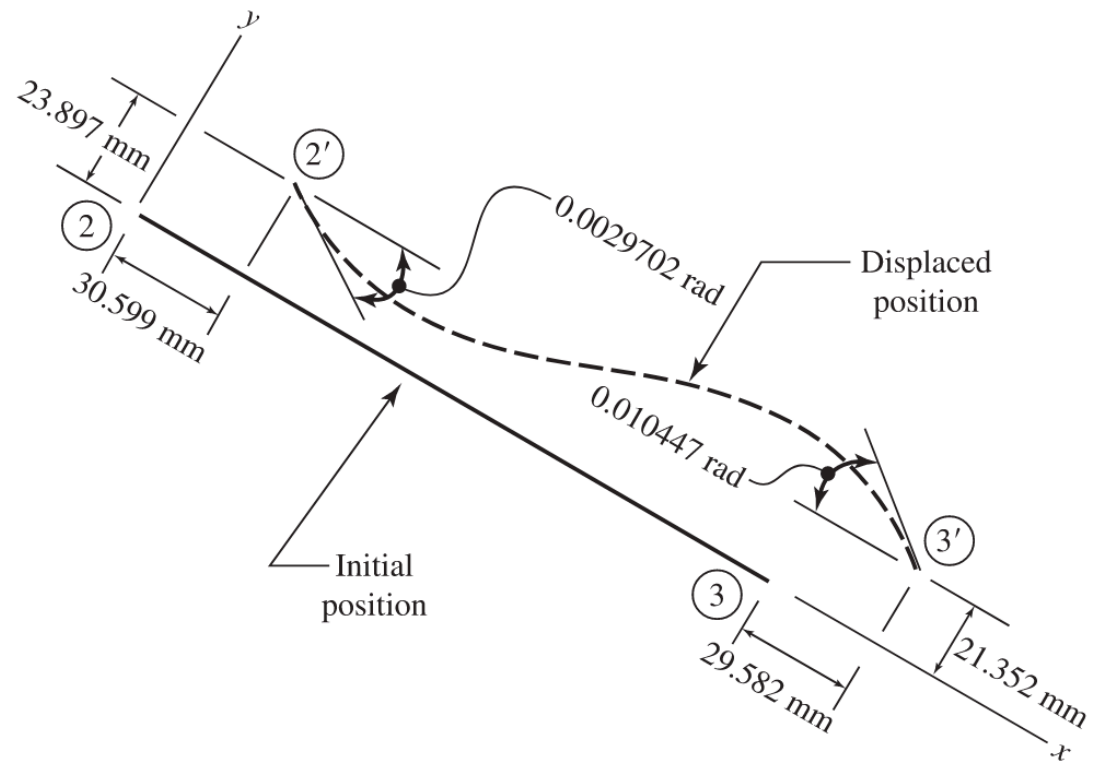
np.set_printoptions(precision=3, suppress=True)
print(k_local)

[[ 500000.         0.         0. -500000.         0.         0.]
 [         0.      5280.     13200.         0.     -5280.     13200.]
 [         0.     13200.     44000.         0.    -13200.     22000.]
 [-500000.         0.         0.  500000.         0.         0.]
 [         0.     -5280.    -13200.         0.      5280.    -13200.]
 [         0.     13200.     22000.         0.    -13200.     44000.]]

```

2. Given the following local displacement pattern, calculate the local member forces.

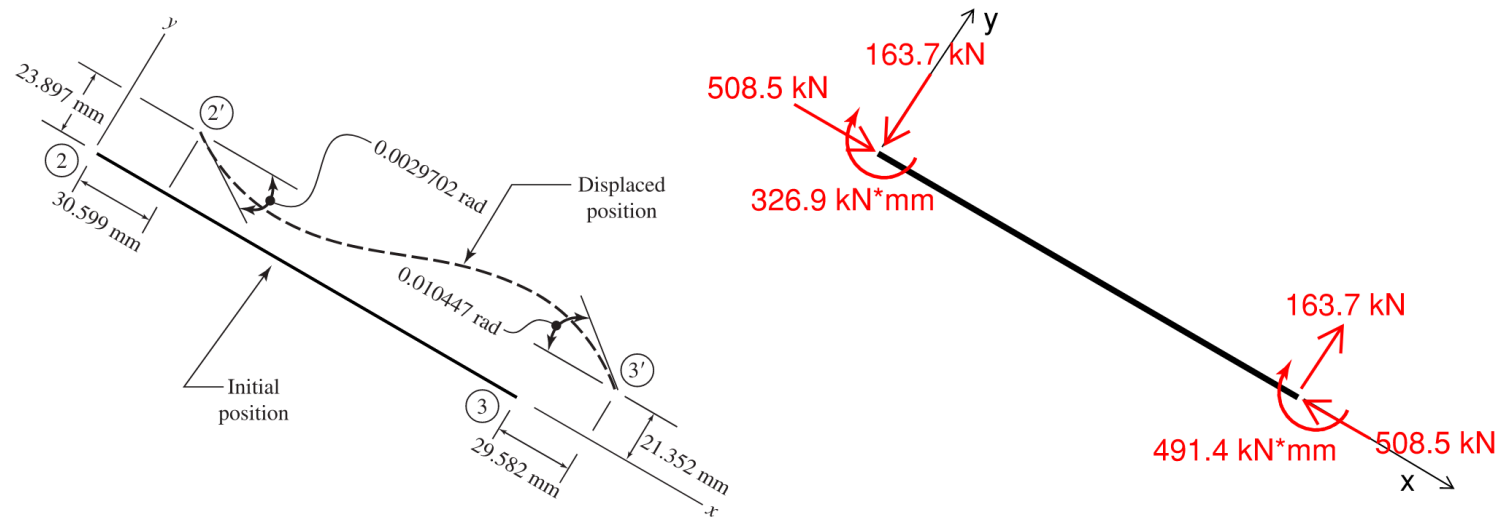
$$\mathbf{Q} = \mathbf{k}_{local} \mathbf{u}$$



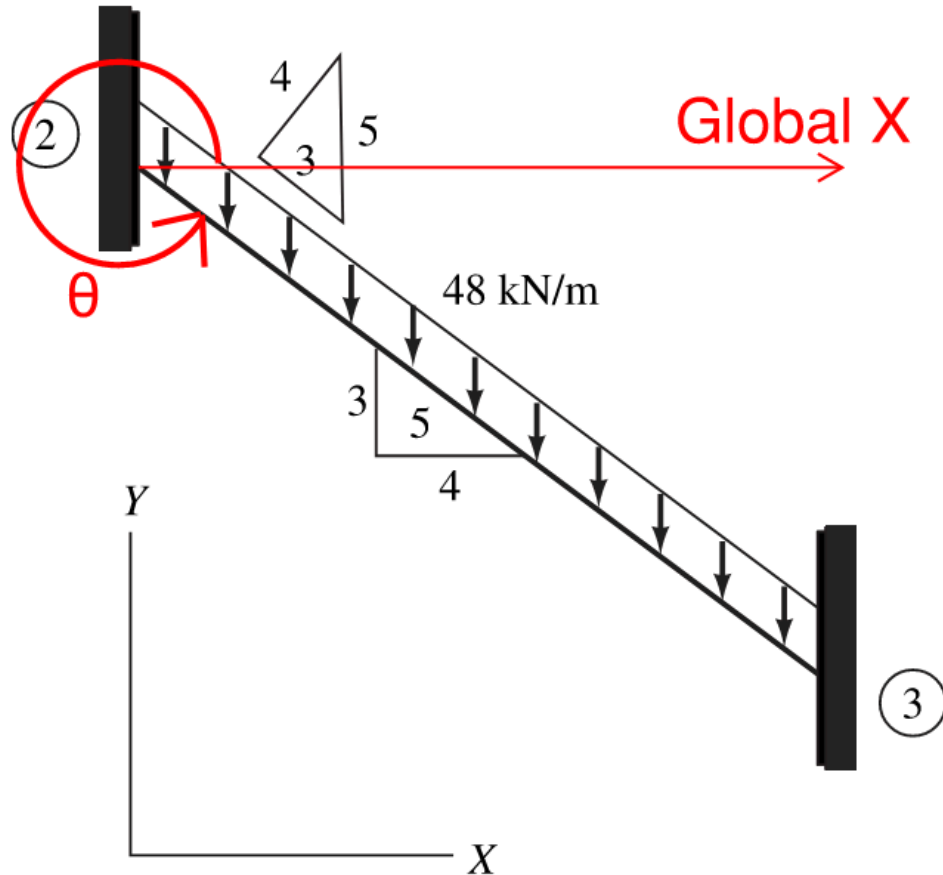
```
In [3]: # displacement vector (units: m, rad)
u = np.array([
    0.030599,
    0.023897,
    -0.0029702, # clockwise rotation is negative
    0.029582,
    0.021352,
    -0.010447 # clockwise rotation is negative
], dtype=float)

Q = k_local @ u
print(Q)

[ 508.5   -163.669 -326.929 -508.5    163.669 -491.418]
```



3. Given the applied member load, turn this into equivalent fixed-end forces (FEFs).



```

In [4]: import numpy as np

def resolve_load_to_local(f_global_down, theta):
    """
    Resolve a *vertical downward* load (kN or kN/m)
    into local (x, y) components using angle theta (degrees).

    Convention:
        f_x = f * sin(theta)
        f_y = f * cos(theta)

    Returns:
        (f_x, f_y) in the same units as the input
    """
    theta = np.radians(theta)  # Convert degrees to radians

    f_x = f_global_down * np.sin(theta)
    f_y = f_global_down * np.cos(theta)
    return f_x, f_y

```

```

In [5]: theta = np.arctan(-3/4) # radians
        theta = np.degrees(theta)
        print(f"theta: {theta:.2f}")

        # Same as
        theta = 270 + np.degrees(np.arctan(4/3))
        print(f"theta: {theta:.2f}")

        w = 48.0

        w_x, w_y = resolve_load_to_local(w, theta)
        print(f"w_x (kN): {w_x:.2f}")
        print(f"w_y (kN): {w_y:.2f}")

        theta: -36.87
        theta: 323.13
        w_x (kN): -28.80
        w_y (kN): 38.40

```

Perpendicular Component ( $w_y$ )  $\rightarrow$  Fixed-End Forces

For a **uniform load perpendicular to the member axis**, use standard beam fixed-end force relationships.

$$\{Q^F\}_y = \begin{Bmatrix} 0 \\ \frac{w_y L}{2} \\ \frac{w_y L^2}{12} \\ 0 \\ \frac{w_y L}{2} \\ -\frac{w_y L^2}{12} \end{Bmatrix}$$

- Produces **shear** and **bending moment**
- Acts along DOFs 2, 3, 5, 6

```
In [6]: def fef_uniform_perpendicular(wy, L):
        """
        Fixed-end forces for a fixed-fixed member under a uniform transverse
        acting in the local y-direction (signed).

        DOF order: [F1, F2, F3, F4, F5, F6] = [u_i, v_i, th_i, u_j, v_j, th_j]
        """
        F1 = 0.0
        F2 = wy * L / 2
        F3 = wy * L**2 / 12
        F4 = 0.0
        F5 = wy * L / 2
        F6 = -wy * L**2 / 12
        return np.array([F1, F2, F3, F4, F5, F6], dtype=float)
```

Parallel Component ( $w_x$ )  $\rightarrow$  Fixed-End Forces

For a **uniform load parallel to the member axis**, the equivalent nodal forces are purely axial:

$$\{Q^F\}_x = \begin{Bmatrix} \frac{w_x L}{2} \\ 0 \\ 0 \\ \frac{w_x L}{2} \\ 0 \\ 0 \end{Bmatrix}$$

- Produces **axial force only**
- Acts along DOFs 1 and 4

```

In [7]: def fef_uniform_parallel(wx, L, l1, l2):
        """
        Fixed-end forces for a distributed axial load wx (kN/m)

        Parameters
        -----
        wx : Axial distributed load (kN/m)
        L : Member length (m)
        l1 : Distance from node i to start of load (m)
        l2 : Distance from node j to end of load (m)
        """

        Fb = (wx / (2 * L)) * (L - l1 - l2) * (L - l1 + l2)
        Fe = (wx / (2 * L)) * (L - l1 - l2) * (L + l1 - l2)

        F1 = Fb
        F2 = 0.0
        F3 = 0.0
        F4 = Fe
        F5 = 0.0
        F6 = 0.0

        return np.array([F1, F2, F3, F4, F5, F6], dtype=float)

```

```

In [8]: # --- perpendicular FEF (beam part) ---
Qf_y = fef_uniform_perpendicular(wy=w_y, L=L)
# --- parallel FEF (axial part) ---
Qf_x = fef_uniform_parallel(wx=w_x, L=L, l1=0.0, l2=0.0)
# --- total FEF vector ---
Qf = Qf_x + Qf_y

print("FEF from perpendicular component:")
print(Qf_y, "\n")

print("FEF from parallel component:")
print(Qf_x, "\n")

print("Total local FEF vector [F1..F6]:")
print(Qf)

```

```

FEF from perpendicular component:
[  0.  96.  80.   0.  96. -80.]

```

```

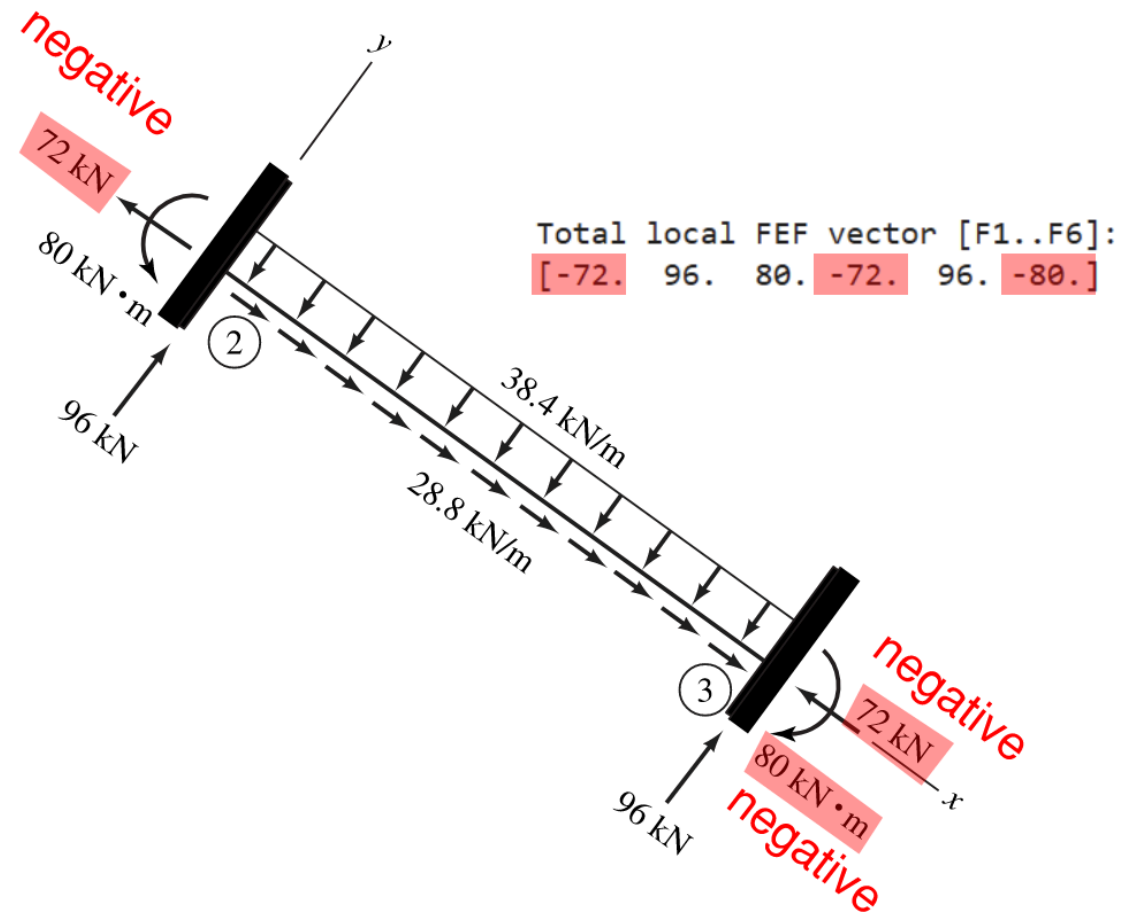
FEF from parallel component:
[-72.   0.   0. -72.   0.   0.]

```

```

Total local FEF vector [F1..F6]:
[-72.  96.  80. -72.  96. -80.]

```

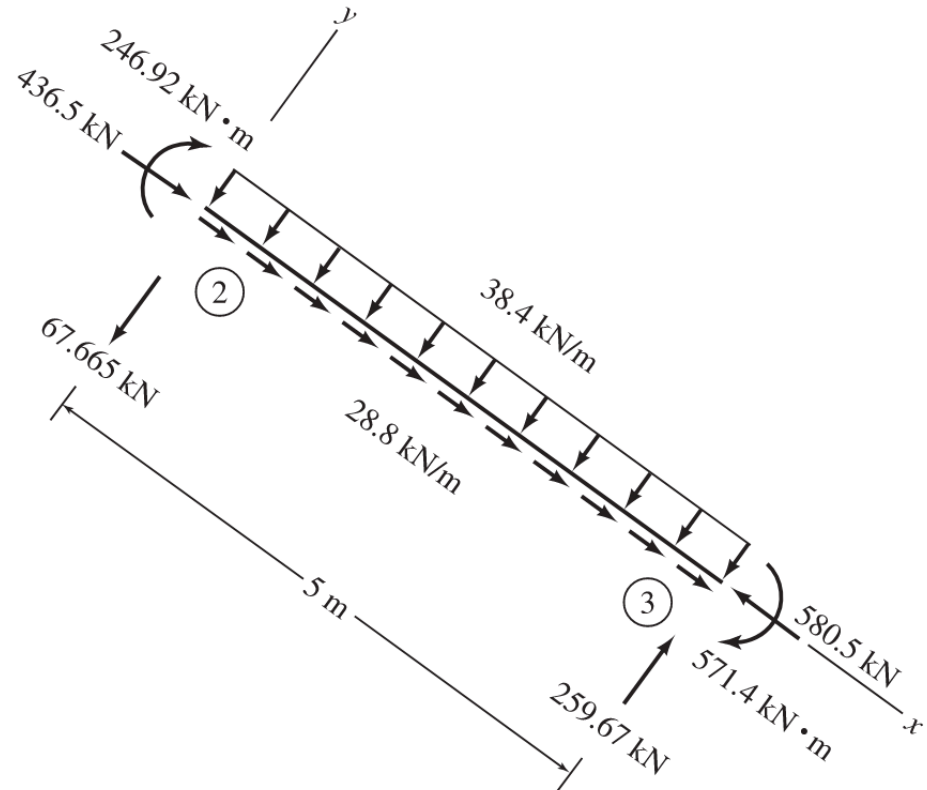


4. Calculate the complete local element end forces

$$\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}^F$$

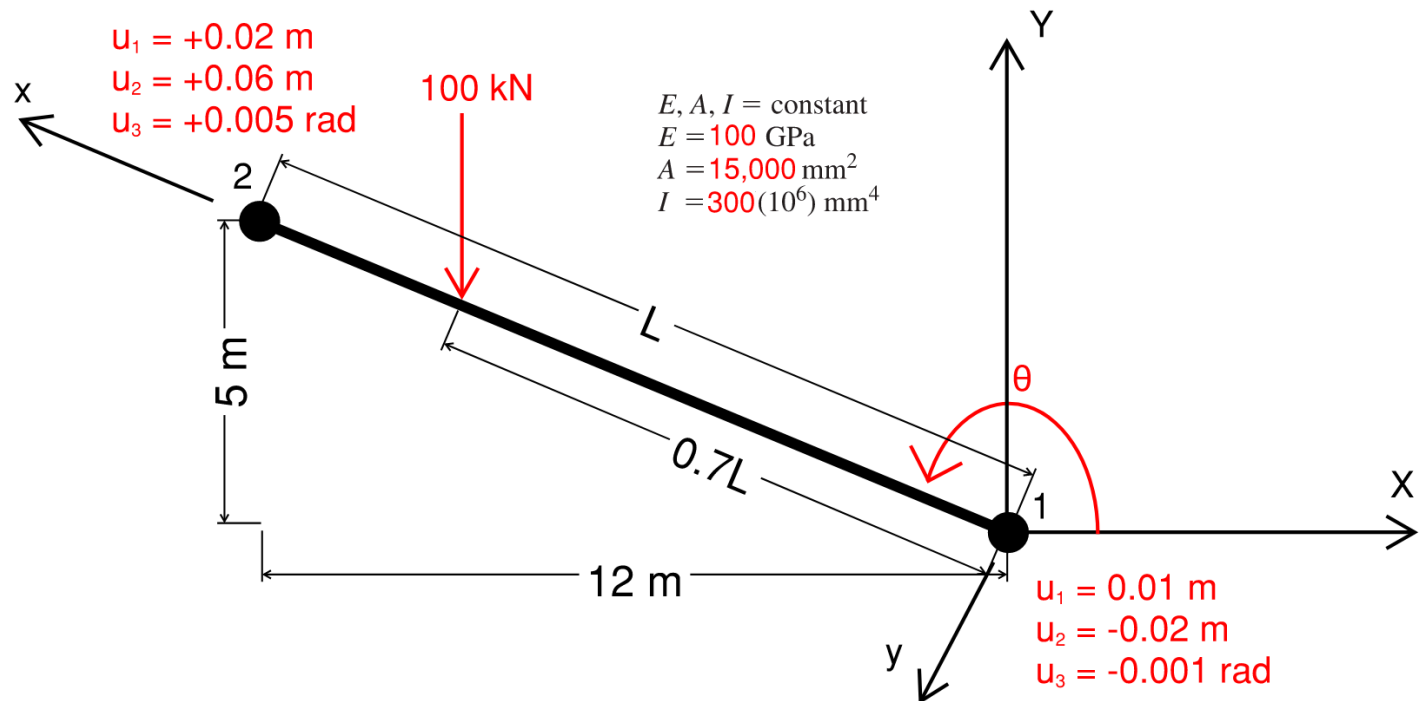
```
In [9]: Q_total = Q + Qf  
print(Q_total)
```

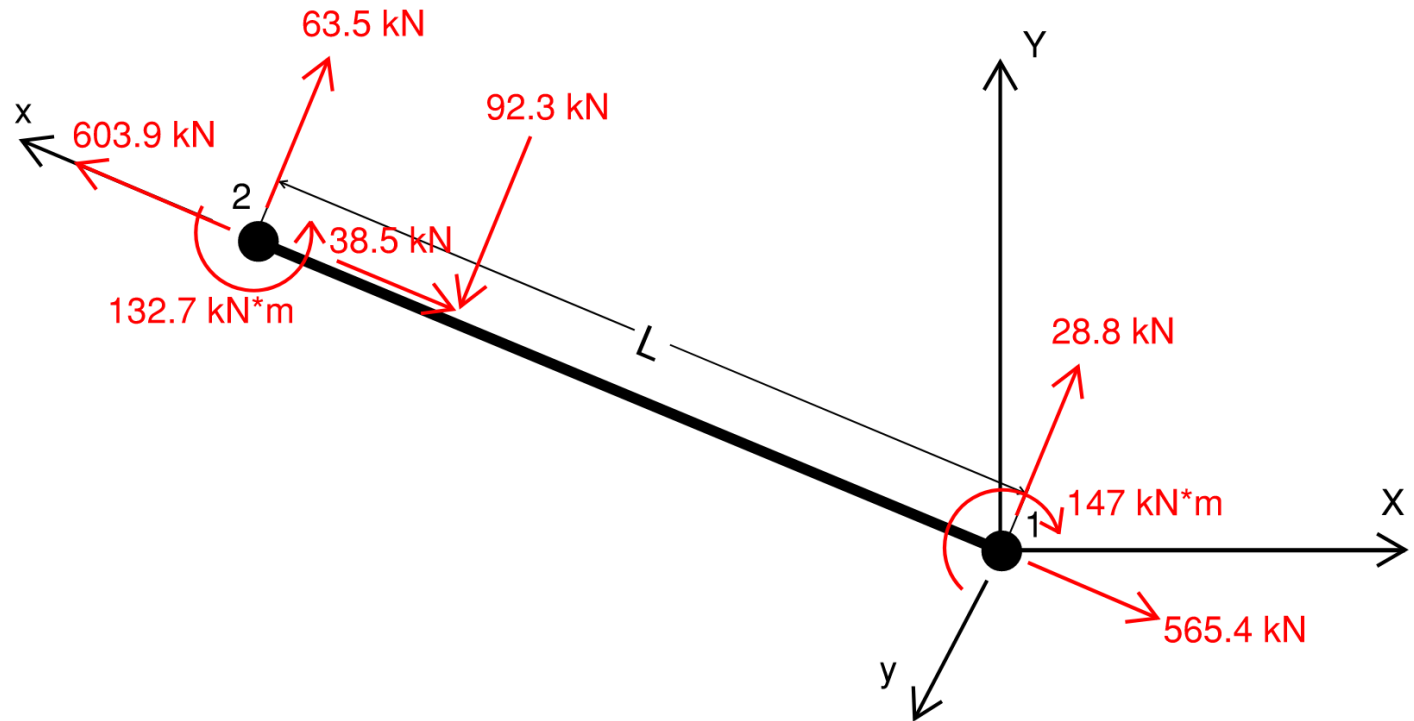
```
[ 436.5    -67.669 -246.929 -580.5    259.669 -571.418]
```



## Part 9 — In-Class Exercise

update:  $u_1 = 0.015$





# Wrap-Up

In this lecture, we:

- Introduced the 2D frame analytical model and DOF conventions
- Reviewed local frame stiffness relations and the 2D frame stiffness matrix
- Worked through a complete example, including load resolution and fixed-end forces

Next lecture:

- Transforming local frame stiffness to global
- DSM for frame structures