

CEE6501 — Lecture 3.2

The Direct Stiffness Method (DSM) for Trusses

Learning Objectives

By the end of this lecture, you will be able to:

- Construct the **local-to-global transformation** for a truss member
- Compute element stiffness in global coordinates:

$$\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$$

- Assemble the **global stiffness matrix** \mathbf{K} using scatter-add
- Explain why an **unsupported structure** leads to a **singular stiffness matrix**
- Apply boundary conditions through **DOF partitioning** and solve for displacements
- Recover **member axial forces** from global displacements for design

Agenda

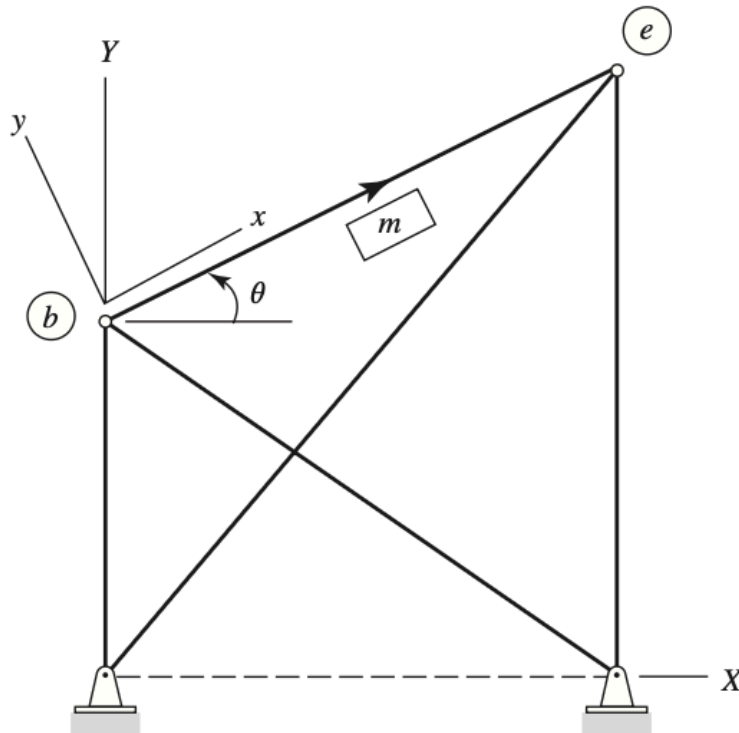
1. Local-to-global transformation using direction cosines
2. Member stiffness relations in the global coordinate system
3. Assembly of the global stiffness matrix \mathbf{K} (manual)
4. Assembly of the global stiffness matrix \mathbf{K} (DSM)
5. Constraints and supports (boundary conditions)
6. Partitioning into free and restrained DOFs
7. Solving for global displacements and reactions
8. Element force recovery
9. DSM summary

Big idea:

- A truss is a network of axial springs.
- Each element contributes stiffness to shared DOFs.
- Assembly is adding contributions into the right global rows/columns.

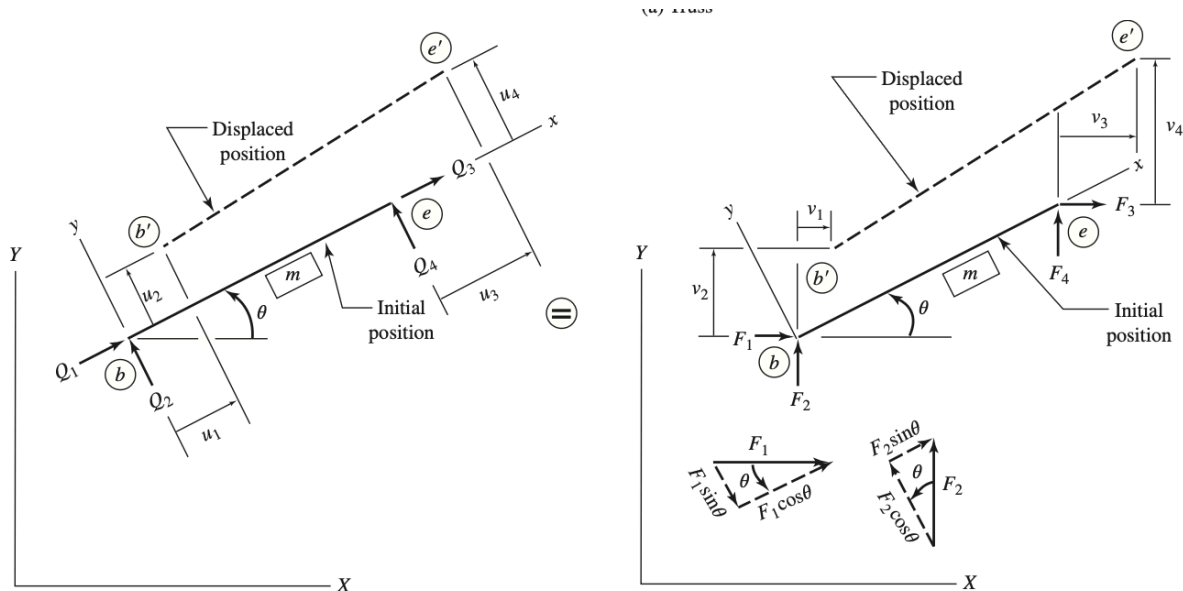
Part 1 — Local to Global Transformation

Generic Truss Element in a Structure



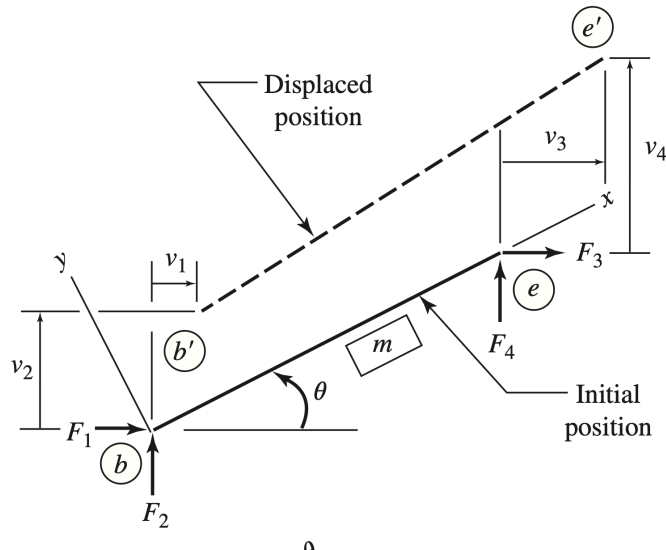
- A truss member is embedded in a **global coordinate system** (X, Y)
- The element stiffness was derived in a **local coordinate system** (x, y) aligned with the member
- The member orientation is defined by an angle θ , measured **counterclockwise** from global X to local x
- Structural assembly requires **transforming forces and displacements** between local and global coordinates

Local/Global Perspective



- **Local coordinate system (left):** forces Q , displacements u
- **Global coordinate system (right):** forces F , displacements v

Global (\mathbf{F}) \rightarrow Local (\mathbf{Q}) Force Transformation: Trigonometry

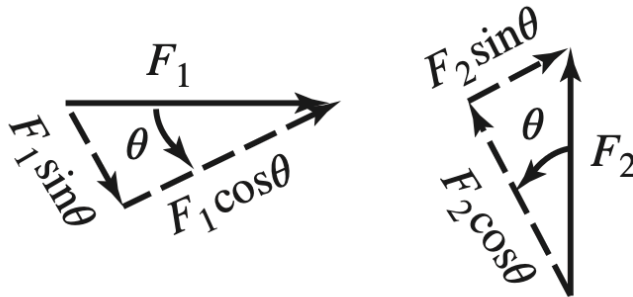


At node b (start node):

- $Q_1 = F_1 \cos \theta + F_2 \sin \theta$
- $Q_2 = -F_1 \sin \theta + F_2 \cos \theta$

At node e (end node):

- $Q_3 = F_3 \cos \theta + F_4 \sin \theta$
- $Q_4 = -F_3 \sin \theta + F_4 \cos \theta$



Global (\mathbf{F}) \rightarrow Local (\mathbf{Q}) Force Transformation: Matrix Form

- Local member forces \mathbf{Q} are obtained by **rotating** global nodal forces \mathbf{F} into the member's local coordinate system
- Each 2×2 block applies a **rotation by θ** at a node
- The transformation changes **direction only**, not force magnitude
- This operation is a **pure coordinate transformation**

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix}}_T \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\mathbf{Q} = \mathbf{T}\mathbf{F}$$

Direction Cosines: Rotation Terms

- Direction cosines define the **orientation of a truss member** in the global (X, Y) coordinate system
- The angle θ is measured **counterclockwise** from the global X axis to the local x axis
- Computed directly from the **nodal coordinates** of the element (b = start node, e = end node)

$$\cos \theta = \frac{X_e - X_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}}, \quad \sin \theta = \frac{Y_e - Y_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}}$$

- The denominator is the **member length** L
- Once computed, $\cos \theta$ and $\sin \theta$ are **reused throughout the element formulation**

Global (\mathbf{v}) \rightarrow Local (\mathbf{u}) Displacements

- Nodal displacements are transformed using the **same rotation matrix** as forces
- Displacements and forces transform identically because they are defined along the **same directions**
- This is a **pure coordinate rotation**, not a change in deformation

$$\mathbf{u} = \mathbf{T}\mathbf{v}$$

- \mathbf{v} : global displacement vector
- \mathbf{u} : local displacement vector

Local (\mathbf{Q}) \rightarrow Global (\mathbf{F}) Force: Transformation Equations

- This is the **reverse of the global \rightarrow local process**
- Local member forces are **rotated back** into the global (X, Y) directions

At node b (start node):

$$F_1 = Q_1 \cos \theta - Q_2 \sin \theta, \quad F_2 = Q_1 \sin \theta + Q_2 \cos \theta$$

At node e (end node):

$$F_3 = Q_3 \cos \theta - Q_4 \sin \theta, \quad F_4 = Q_3 \sin \theta + Q_4 \cos \theta$$

Local (\mathbf{Q}) \rightarrow Global (\mathbf{F}) Force: Matrix Form

- Local member forces are mapped to global nodal forces using the **transpose** of the global \rightarrow local transformation

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{bmatrix}}_{\mathbf{T}^T} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

$$\mathbf{F} = \mathbf{T}^T \mathbf{Q}$$

Recall (Global \rightarrow Local):

$$\mathbf{T} = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix}$$

Properties of the Transformation Matrix

$$\mathbf{T}^{-1} = \mathbf{T}^T$$

The transformation matrix is **orthogonal**, which greatly simplifies operations involving the stiffness transformations.

Summary — Local \leftrightarrow Global Transformations

Direction cosines (member orientation):

$$\cos \theta = \frac{X_e - X_b}{L}, \quad \sin \theta = \frac{Y_e - Y_b}{L}$$

Global \rightarrow Local (forces or displacements):

$$\mathbf{Q} = \mathbf{T}\mathbf{F}, \quad \mathbf{u} = \mathbf{T}\mathbf{v}$$

Local \rightarrow Global (forces or displacements):

$$\mathbf{F} = \mathbf{T}^\top \mathbf{Q}, \quad \mathbf{v} = \mathbf{T}^\top \mathbf{u}$$

$$\mathbf{T} = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix}$$

- \mathbf{T} is a **pure rotation matrix** (no translation)
- $\mathbf{T}^{-1} = \mathbf{T}^\top$ (orthogonal)

Part 2 — Member Stiffness in the Global Coordinate System

Goal

- We have derived the **local stiffness relation** (Lecture 3.1 today):

$$Q = ku$$

- We also know how to **transform forces and displacements** between local and global systems
- Objective: express the **member stiffness relation entirely in global coordinates**

Transformation Chain — From Local to Global (Step-by-Step)

Step 1 — Local force–displacement relation

$$\mathbf{Q} = \mathbf{k} \mathbf{u}$$

The element stiffness matrix \mathbf{k} relates **local nodal displacements** \mathbf{u} to the corresponding **local nodal forces** \mathbf{Q} .

Step 2 — Transform local forces to global forces

$$\mathbf{F} = \mathbf{T}^\top \mathbf{Q}$$

Global nodal forces are obtained by rotating the local force vector into the global coordinate system.

Substitute the local stiffness relation from Step 1, $\mathbf{Q} = \mathbf{k} \mathbf{u}$, into the force transformation:

$$\mathbf{F} = \mathbf{T}^\top \mathbf{Q} \implies \mathbf{F} = \mathbf{T}^\top (\mathbf{k} \mathbf{u})$$

Step 3 — Transform global displacements to local displacements

$$\mathbf{u} = \mathbf{T} \mathbf{v}$$

Substitute the displacement transformation into the previous expression:

$$\mathbf{F} = \mathbf{T}^\top \mathbf{k} \mathbf{u} \implies \mathbf{F} = \mathbf{T}^\top \mathbf{k} (\mathbf{T} \mathbf{v})$$

Step 4 — Rearrange into global stiffness form

$$\mathbf{F} = (\mathbf{T}^\top \mathbf{k} \mathbf{T}) \mathbf{v}$$

Step 5 — Final global stiffness relation

Define the global element stiffness matrix:

$$\mathbf{K} = \mathbf{T}^T \mathbf{k} \mathbf{T}$$

This is the element stiffness relation used directly in global assembly for the Direct Stiffness Method.

textbook notation:

$$\boxed{\mathbf{F} = \mathbf{K} \mathbf{v}}$$

our notation (interchangeable):

$$\boxed{\mathbf{f} = \mathbf{K} \mathbf{u}}$$

Calculating the Global Stiffness Matrix, \mathbf{K}

The global element stiffness matrix is obtained by **rotating the local axial stiffness** into the global coordinate system:

$$\mathbf{K} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{bmatrix} \cdot \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix}$$

Carrying out the matrix multiplication yields the **closed-form global stiffness matrix**:

$$\mathbf{K} = \frac{EA}{L} \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta & -\cos^2 \theta & -\cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta & -\cos \theta \sin \theta & -\sin^2 \theta \\ -\cos^2 \theta & -\cos \theta \sin \theta & \cos^2 \theta & \cos \theta \sin \theta \\ -\cos \theta \sin \theta & -\sin^2 \theta & \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$$

Key Observations — Global Stiffness Matrix, \mathbf{K}

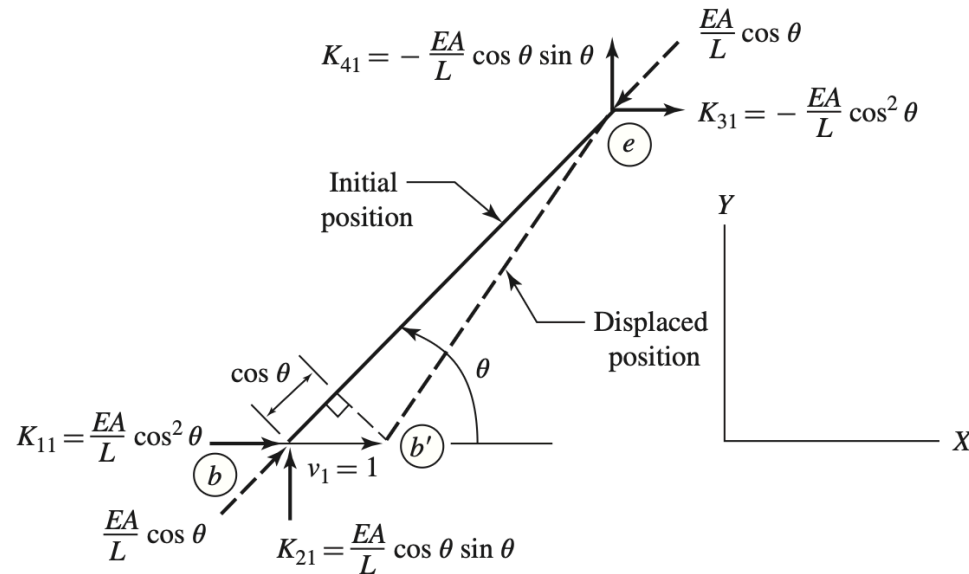
- The member global stiffness matrix \mathbf{K} is **symmetric**, just like the local stiffness matrix
- \mathbf{K} represents the same physical behavior, but expressed in the **global** (X, Y) **coordinate system**
- Each coefficient K_{ij} is the **force at global DOF i** required to produce a **unit displacement at global DOF j** , with all other displacements fixed

Direct Calculation of Global Stiffness Matrix, \mathbf{K}

- One *could* derive \mathbf{K} directly by:
 - Applying **unit global displacements** to a generic inclined truss member
 - Evaluating the **global end forces** required to produce each unit displacement in global coordinates
- The j **th column of \mathbf{K}** gives the global nodal force pattern caused by $v_j = 1$
- This approach is **theoretically equivalent** to the transformation-based derivation and provides a clear physical interpretation of \mathbf{K} .
- However, it is **significantly more labor-intensive**, and is mainly useful as a **verification tool**, rather than for routine analysis.

Example: First Column of \mathbf{K}

$$u_a = v_1 \cos \theta = 1 \cos \theta = \cos \theta$$



(a) First Column of \mathbf{K} ($v_1 = 1, v_2 = v_3 = v_4 = 0$)

- Impose $v_1 = 1$, all other global displacements zero
- Project the resulting axial deformation onto the member axis
- Resolve the axial force back into global components

You recover the **first column of \mathbf{K}** , which should exactly match $\mathbf{T}^T \mathbf{k} \mathbf{T}$.

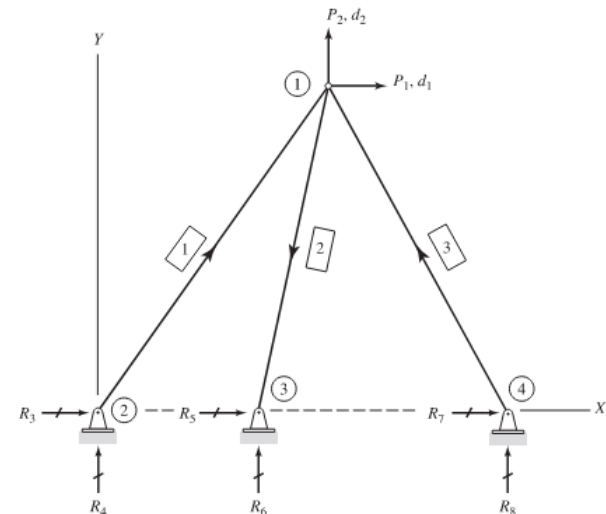
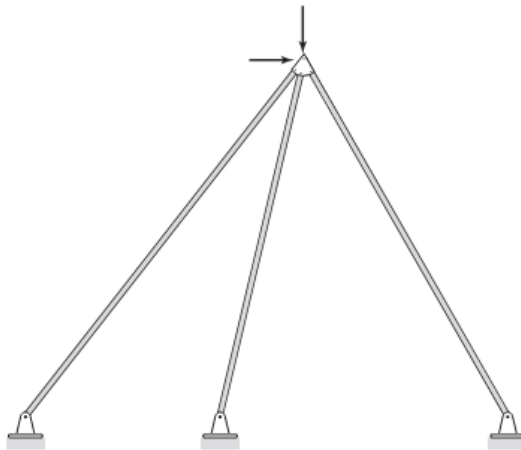
Part 3 — Assembling the Global Structure Stiffness Matrix

Manual Method

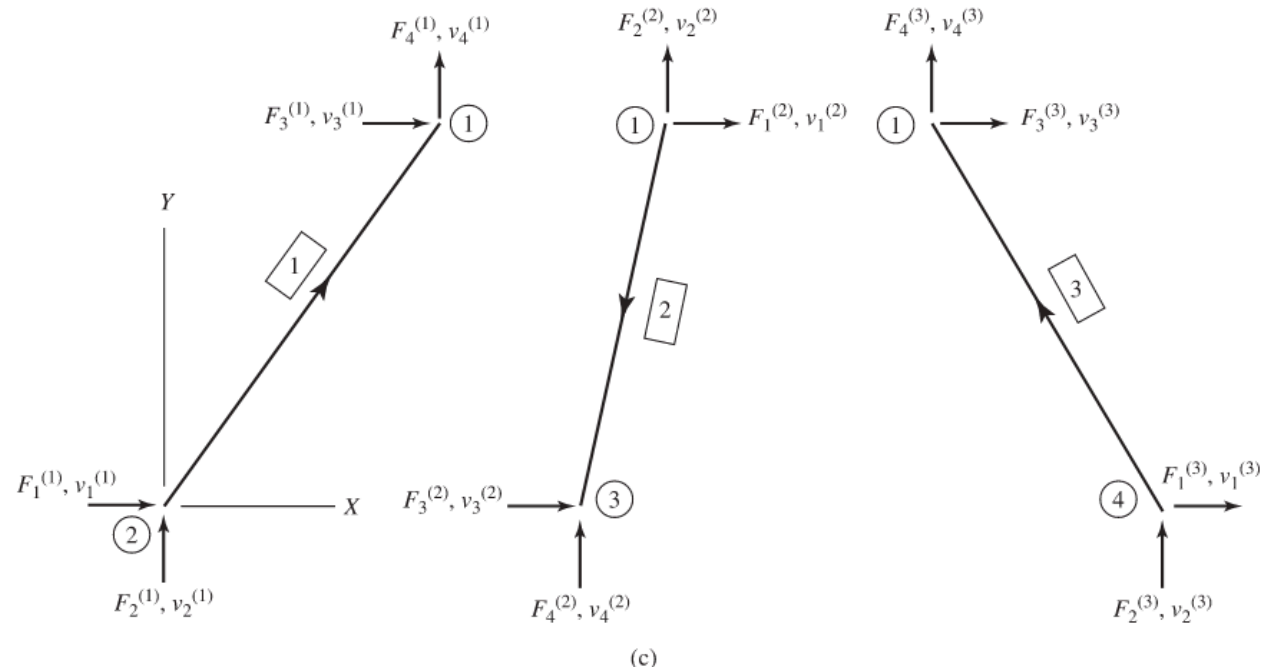
Example Structure

We now move from **element-level** to a **complete truss structure** behavior.

- Structure composed of **3 axial truss elements**
- **4 nodes**, each with (X, Y) translational DOFs, node 1 at the top
- Total system size: **8 global degrees of freedom**
 - node 1: DOF (1,2)
 - node 2: DOF (3,4)
 - node 3: DOF (5,6)
 - node 4: DOF (7,8)



Element-Level View of the Structure



- Nodes **2, 3, and 4** are **pinned** (no displacement)
- **Node 1** is free to move
- All three elements are connected at **node 1**
- A displacement at node 1 induces forces in **all connected members**

Element Forces and Notation

Local forces are defined **per element**:

- Superscript (e) \rightarrow element number
- Subscript (i) \rightarrow local DOF index

Examples:

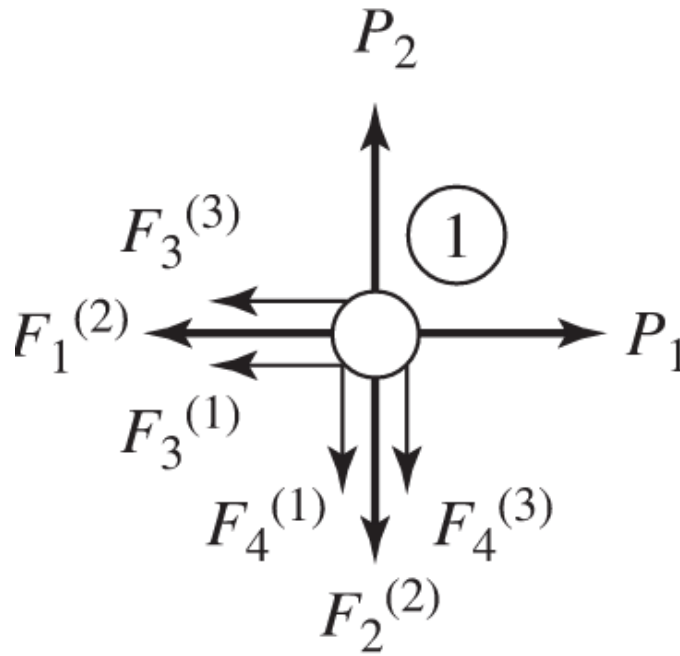
- $F_3^{(1)}$ \rightarrow force at local DOF 3 in **element 1**
- $F_1^{(2)}$ \rightarrow force at local DOF 1 in **element 2**

Equilibrium Equations at Node 1

Because all elements share **node 1**, equilibrium at this node couples the response of all members.

P = external applied loaded

F = internal forces

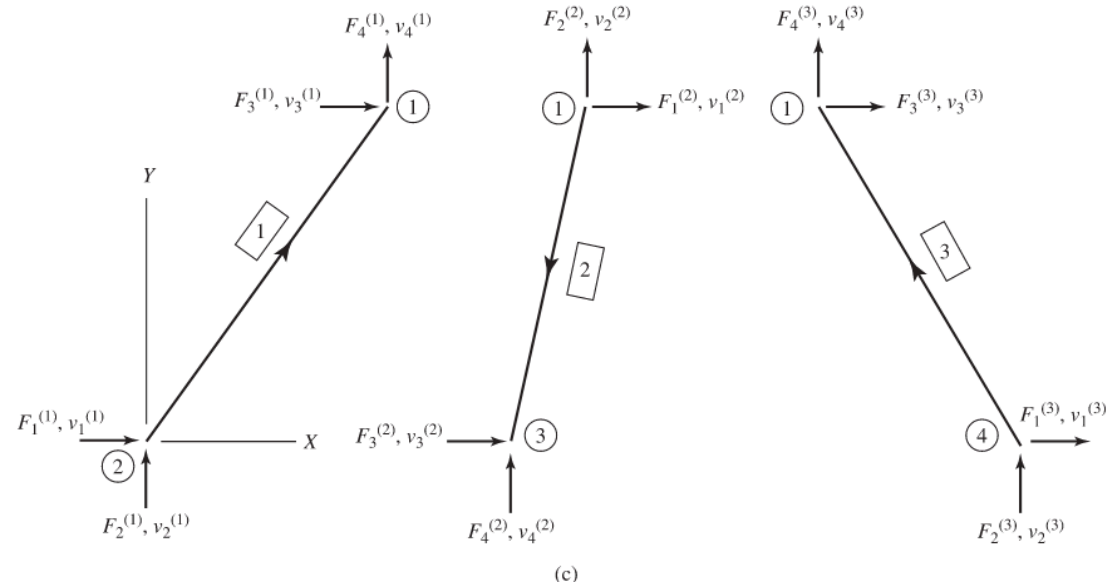


Force equilibrium at node 1:

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$

$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

Compatibility Equations



d_1 and d_2 are global displacements in X and Y at node 1

$$\text{Member (1) : } v_1^{(1)} = v_2^{(1)} = 0, \quad v_3^{(1)} = d_1, \quad v_4^{(1)} = d_2$$

$$\text{Member (2) : } v_1^{(2)} = d_1, \quad v_2^{(2)} = d_2, \quad v_3^{(2)} = v_4^{(2)} = 0$$

$$\text{Member (3) : } v_1^{(3)} = v_2^{(3)} = 0, \quad v_3^{(3)} = d_1, \quad v_4^{(3)} = d_2$$

Member 1 — Force–Displacement Relations

Compatibility (Member 1): $v_1^{(1)} = v_2^{(1)} = 0$, $v_3^{(1)} = d_1$, $v_4^{(1)} = d_2$

Element-level global stiffness expression:

$$\begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \\ F_4^{(1)} \end{Bmatrix} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & \boxed{K_{33}^{(1)}} & \boxed{K_{34}^{(1)}} \\ K_{41}^{(1)} & K_{42}^{(1)} & \boxed{K_{43}^{(1)}} & \boxed{K_{44}^{(1)}} \end{bmatrix} \begin{Bmatrix} v_1^{(1)} = 0 \\ v_2^{(1)} = 0 \\ v_3^{(1)} = d_1 \\ v_4^{(1)} = d_2 \end{Bmatrix}$$

The forces acting at **end node of member 1** (Global node 1) are:

$$F_3^{(1)} = K_{33}^{(1)} d_1 + K_{34}^{(1)} d_2$$

$$F_4^{(1)} = K_{43}^{(1)} d_1 + K_{44}^{(1)} d_2$$

Member 2 — Force–Displacement Relations

Compatibility (Member 2): $v_1^{(2)} = d_1$, $v_2^{(2)} = d_2$, $v_3^{(2)} = v_4^{(2)} = 0$

Element-level global stiffness expression:

$$\begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \\ F_4^{(2)} \end{Bmatrix} = \begin{bmatrix} \boxed{K_{11}^{(2)}} & \boxed{K_{12}^{(2)}} & K_{13}^{(2)} & K_{14}^{(2)} \\ \boxed{K_{21}^{(2)}} & \boxed{K_{22}^{(2)}} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{bmatrix} \begin{Bmatrix} v_1^{(2)} = d_1 \\ v_2^{(2)} = d_2 \\ v_3^{(2)} = 0 \\ v_4^{(2)} = 0 \end{Bmatrix}$$

The forces acting at **start node of member 2** (Global node 1) are:

$$F_1^{(2)} = K_{11}^{(2)} d_1 + K_{12}^{(2)} d_2$$

$$F_2^{(2)} = K_{21}^{(2)} d_1 + K_{22}^{(2)} d_2$$

Member 3 — Force–Displacement Relations

Compatibility (Member 3): $v_1^{(3)} = v_2^{(3)} = 0$, $v_3^{(3)} = d_1$, $v_4^{(3)} = d_2$

Element-level global stiffness expression:

$$\begin{Bmatrix} F_1^{(3)} \\ F_2^{(3)} \\ F_3^{(3)} \\ F_4^{(3)} \end{Bmatrix} = \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & \boxed{K_{33}^{(3)}} & \boxed{K_{34}^{(3)}} \\ K_{41}^{(3)} & K_{42}^{(3)} & \boxed{K_{43}^{(3)}} & \boxed{K_{44}^{(3)}} \end{bmatrix} \begin{Bmatrix} v_1^{(3)} = 0 \\ v_2^{(3)} = 0 \\ v_3^{(3)} = d_1 \\ v_4^{(3)} = d_2 \end{Bmatrix}$$

The forces acting at **end node of member 3** (Global node 1) are:

$$F_3^{(3)} = K_{33}^{(3)} d_1 + K_{34}^{(3)} d_2$$

$$F_4^{(3)} = K_{43}^{(3)} d_1 + K_{44}^{(3)} d_2$$

Global Stiffness Matrix for Free DOFs

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$

$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

Substituting element-level force expressions (from last 3 slides) into the node 1 force equilibrium gives:

$$P_1 = (K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)})d_1 + (K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)})d_2$$

$$P_2 = (K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)})d_1 + (K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)})d_2$$

These equations can be written compactly as:

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{bmatrix}}_{K_s} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\boxed{P = K_s d}$$

Physical Interpretation of the Structure Stiffness Matrix

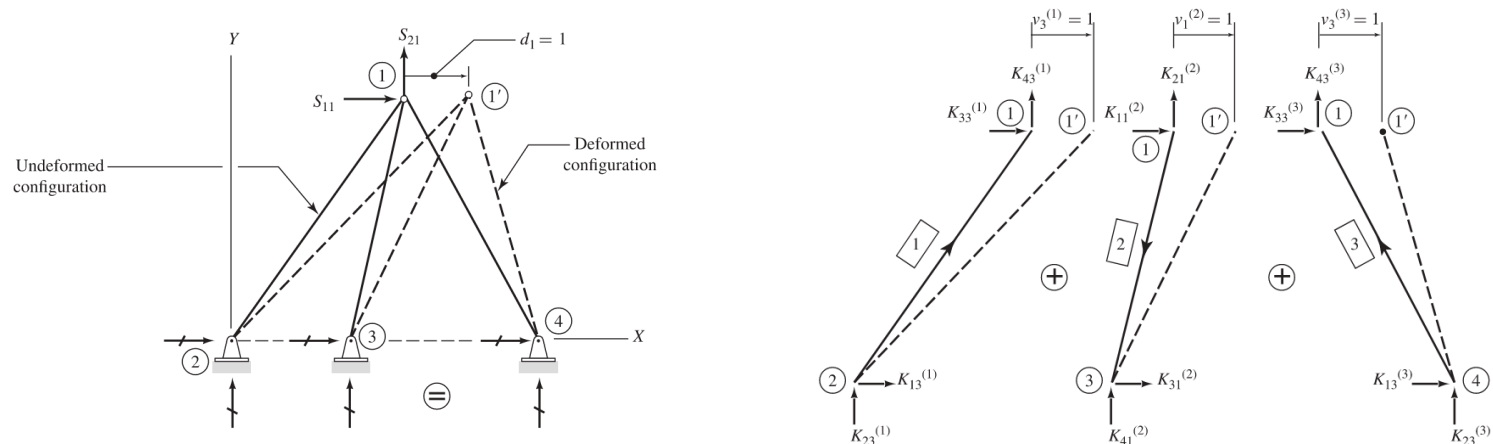
- The **structure stiffness matrix** \mathbf{K}_s has the same physical meaning as an element stiffness matrix, but at the **structure (joint) level**
- A stiffness coefficient $K_{s,ij}$ represents:
 - The **joint force** at DOF i
 - Required to cause a **unit displacement** at DOF j
 - While **all other joint displacements are zero**
- Equivalently:
 - Each **column** of \mathbf{K}_s corresponds to a unit displacement pattern
 - The column entries are the **resulting joint forces** needed to enforce that displacement

Example — First Column of \mathbf{K}_s

- To obtain the **first column** of \mathbf{K}_s :
 - Impose a **unit displacement** at the first free DOF:

$$d_1 = 1, \quad d_2 = 0$$

- All other joint displacements are held fixed
- The resulting joint forces (DOF 1, 2) define the **first column** of \mathbf{K}_s



Part 4 — Assembling the Global Structure Stiffness Matrix

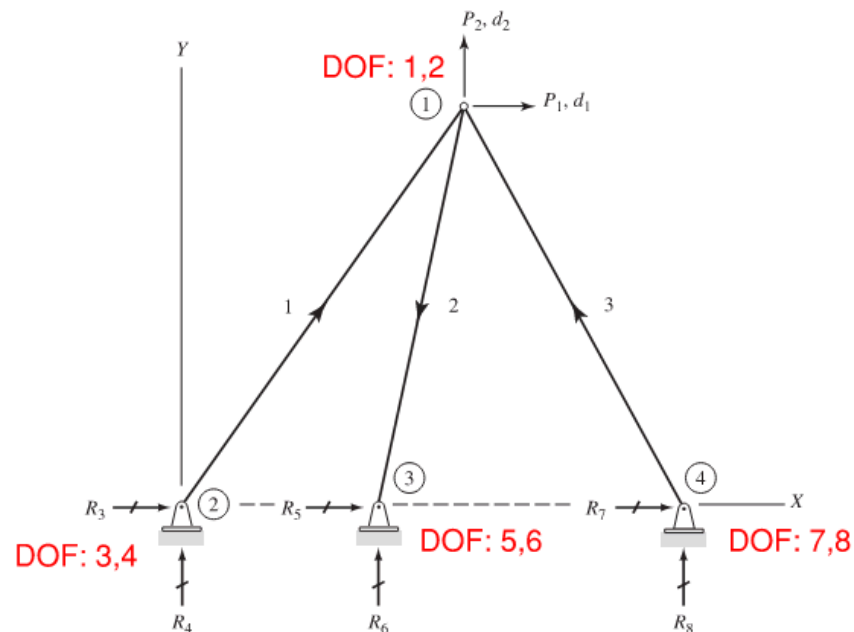
Direct Stiffness Method

Why Not the Manual (Equilibrium + Compatibility) Approach?

- Becomes **tedious** very quickly
- Requires manually tracking:
 - Free vs constrained DOFs
 - Compatibility relationships
- Hard to implement **programmatically**
- Does **not scale** to larger or changing structures

Key idea: We want a method that is systematic, scalable, and algorithmic.

Reminder — Global Degrees of Freedom



- Each node contributes two DOFs: (X, Y)
- 4 nodes → **8 global DOFs**
- Consistent numbering is essential for assembly

Map Element DOFs to Global DOFs

$$\mathbf{K}_1 = \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \left[\begin{array}{cccc} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{array} \right] \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \end{array} \quad
 \mathbf{K}_2 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{array} \right] \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array} \end{array} \quad
 \mathbf{K}_3 = \begin{array}{c} \begin{array}{cccc} 7 & 8 & 1 & 2 \end{array} \\ \left[\begin{array}{cccc} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{array} \right] \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} \end{array}$$

Each element has **4 local DOFs** (1, 2, 3, 4), which must be mapped to the correct **global DOFs** before assembly.

For this structure:

- **Element 1:** local (1, 2, 3, 4) \rightarrow global (3, 4, 1, 2)
- **Element 2:** local (1, 2, 3, 4) \rightarrow global (1, 2, 5, 6)
- **Element 3:** local (1, 2, 3, 4) \rightarrow global (7, 8, 1, 2)

This mapping determines **which rows and columns** of the global stiffness matrix each element stiffness contributes to.

Method 1: Assemble a Structure Matrix for the Free DOFs

$$\mathbf{K}_1 = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{bmatrix} \quad \mathbf{K}_2 = \begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{bmatrix} \quad \mathbf{K}_3 = \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{bmatrix}$$

In this example, only **global DOFs 1 and 2** are free; all other DOFs are constrained.

- Contributions from **each element** that map to DOFs 1 and 2 are **summed**
- This produces a reduced **structure stiffness matrix** for the free DOFs
- The result matches the stiffness matrix obtained earlier using the **manual equilibrium and compatibility** approach

For example, the (1, 1) entry of the structure stiffness matrix is (matches what we did before):

$$K_{s,11} = K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)}$$

Method 2: Assemble the Full Global Stiffness Matrix

When assembling the **global structure stiffness matrix**, do **not** worry yet about which DOFs are constrained or free.

- **Assemble all DOFs first** into a single global matrix
- **Apply boundary conditions later** (supports, prescribed displacements)
- This separation is what makes the method **general, systematic, and scalable**

For the structure shown, there are **8 global DOFs**, so 8×8 stiffness matrix:

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} \end{bmatrix}$$

Each entry K_{ij} represents the **combined stiffness contribution** from all elements that connect global DOF j to global DOF i .

Element 1 — Contribution to Global Stiffness Matrix

Local → Global DOF mapping (Element 1):

$$(1, 2, 3, 4) \rightarrow (3, 4, 1, 2)$$

$$\mathbf{K}_1 = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix}$$

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} \end{bmatrix}$$

Element 2 — Contribution to Global Stiffness Matrix

Local → Global DOF mapping (Element 2):

$$(1, 2, 3, 4) \rightarrow (1, 2, 5, 6)$$

$$\mathbf{K}_2 = \begin{matrix} & \begin{matrix} 1 & 2 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{bmatrix} \end{matrix}$$

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} \end{bmatrix}$$

Element 3 — Contribution to Global Stiffness Matrix

Local → Global DOF mapping (Element 3):

$$(1, 2, 3, 4) \rightarrow (7, 8, 1, 2)$$

$$\mathbf{K}_3 = \begin{array}{cc} & \begin{array}{cc} 7 & 8 \end{array} \\ \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} & \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix} \end{array}$$

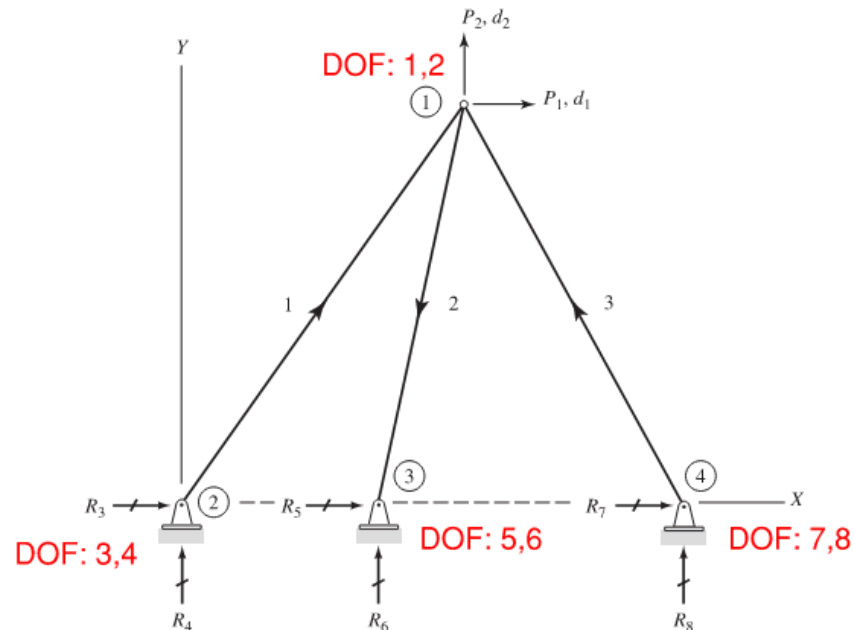
$$\mathbf{K} = \begin{bmatrix} \boxed{K_{11}} & \boxed{K_{12}} & K_{13} & K_{14} & K_{15} & K_{16} & \boxed{K_{17}} & \boxed{K_{18}} \\ \boxed{K_{21}} & \boxed{K_{22}} & K_{23} & K_{24} & K_{25} & K_{26} & \boxed{K_{27}} & \boxed{K_{28}} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\ \boxed{K_{71}} & \boxed{K_{72}} & K_{73} & K_{74} & K_{75} & K_{76} & \boxed{K_{77}} & \boxed{K_{78}} \\ \boxed{K_{81}} & \boxed{K_{82}} & K_{83} & K_{84} & K_{85} & K_{86} & \boxed{K_{87}} & \boxed{K_{88}} \end{bmatrix}$$

Final Global Structure Stiffness Matrix

- Each K_{ij} may include **multiple element contributions**
- The matrix is **symmetric**
- Diagonal terms, K_{ii} , must be non-zero and positive for stability

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & 0 & 0 & 0 & 0 \\ K_{41} & K_{42} & K_{43} & K_{44} & 0 & 0 & 0 & 0 \\ K_{51} & K_{52} & 0 & 0 & K_{55} & K_{56} & 0 & 0 \\ K_{61} & K_{62} & 0 & 0 & K_{65} & K_{66} & 0 & 0 \\ K_{71} & K_{72} & 0 & 0 & 0 & 0 & K_{77} & K_{78} \\ K_{81} & K_{82} & 0 & 0 & 0 & 0 & K_{87} & K_{88} \end{bmatrix}$$

Interpreting Zeros in the Stiffness Matrix



- Zero entry \rightarrow no stiffness coupling
- Example: $K_{37} = 0$
- DOF 3 and DOF 7 share no connecting element

Part 5 — Constraints and Supports

Inverting the Global Stiffness Matrix

You might think: *we have \mathbf{K} , so we can just solve*

$$\mathbf{K} \mathbf{u} = \mathbf{f}$$

and we're done.

However, for a **free (unconstrained) structure**, this step is **not yet valid**.

At this stage—before applying supports or boundary conditions—the global stiffness matrix typically contains **rigid-body modes** (motions that produce no strain energy). As a result:

- \mathbf{K} is **singular** (not invertible)
- the structure still has **rigid-body motion** because we haven't applied supports / boundary conditions yet

Important note (about our 8 DOF example):

In our 8×8 formulation, the ability to invert **K** is an **artifact of how the matrix is set up**. The assembly already embeds enough *support-like stiffness*—intentionally or unintentionally—to suppress rigid-body motion.

As a result, the structure represented by this **K** is **not truly free**, even though we have not yet explicitly applied boundary conditions.

In general, however, a global stiffness matrix assembled for an unconstrained structure **will be singular**.

This is why the proper and robust approach is always to apply supports explicitly (via partitioning) before solving.

```

In [1]: import numpy as np

# Construct a singular 6x6 matrix
# Last row = sum of first two rows → linear dependence
K = np.array([
    [ 2, -1,  0,  0,  0,  0],
    [-1,  2, -1,  0,  0,  0],
    [ 0, -1,  2, -1,  0,  0],
    [ 0,  0, -1,  2, -1,  0],
    [ 0,  0,  0, -1,  2, -1],
    [ 1,  1, -1,  0,  0,  0] # dependent row
], dtype=float)

try:
    np.linalg.inv(K)
except np.linalg.LinAlgError as e:
    print("Inversion failed:", e)

```

Inversion failed: Singular matrix

Why do we get a singular stiffness matrix?

Mathematically, an unconstrained structure admits non-zero displacement vectors $\mathbf{u} \neq \mathbf{0}$ such that

$$\mathbf{K}\mathbf{u} = \mathbf{0}$$

These displacement patterns are **rigid-body (zero-energy) modes**, meaning \mathbf{K} has a **non-trivial nullspace** and is therefore **singular**.

What does this mean in practice?

As a result, the structure can move without inducing member deformation:

- **Rigid translation in X**
- **Rigid translation in Y**
- **Rigid-body rotation** (geometry-dependent)

These motions are **rigid-body modes**, not structural deformations.

Because rigid-body modes produce zero internal force, **\mathbf{K}** cannot uniquely map applied loads to displacements. **Supports (boundary conditions)** remove these modes and render **\mathbf{K}** invertible.

Boundary conditions: how supports enter the model

Supports and constraints are enforced by specifying **known displacements** (a.k.a. prescribed DOFs).

Common idealized supports (2D truss):

- **Pin support:** Prescribes both translations: $u_{ix} = 0$ and $u_{iy} = 0$
- **Roller support:** Prescribes one translation and allows the other:

Prescribed displacement (settlement / actuation):

- You can prescribe a non-zero value, e.g., $u_{iy} = -2$ mm at a support.

Why this fixes singularity:

- Constraints remove rigid-body modes by preventing global translations/rotations.
- The remaining free DOFs correspond to true structural deformation.

What we do next (workflow)

We do **not** modify the element --> global matrix formulation to handle supports.

Instead, the Direct Stiffness Method proceeds like this:

1. Assemble the **full** global matrix \mathbf{K} and global load vector \mathbf{F}
2. Identify which DOFs are **free** vs **restrained**
3. Enforce boundary conditions by **partitioning** (or equivalently, by eliminating restrained DOFs)
4. Solve only for the unknown (free) displacements
5. Recover reactions at restrained DOFs

So supports are handled cleanly at the **system equation** level — via partitioning (next section).

Part 6 — Partitioning the Matrix

Free vs Restrained DOFs

Core idea — Partition DOFs (free vs restrained)

Separate the global degrees of freedom into two sets:

- **Free DOFs** f : unknown displacements (to be solved)
- **Restrained DOFs** r : prescribed displacements (from supports or constraints)

Reorder the global displacement and force vectors so that all free DOFs come first:

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_f \\ \mathbf{u}_r \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{F}_f \\ \mathbf{F}_r \end{bmatrix}$$

Apply the **same reordering** to the global stiffness matrix to obtain a partitioned system:

$$\begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fr} \\ \mathbf{K}_{rf} & \mathbf{K}_{rr} \end{bmatrix} \begin{bmatrix} \mathbf{u}_f \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} \mathbf{F}_f \\ \mathbf{F}_r \end{bmatrix}$$

Note: In our **8×8 truss example**, the system already appears partitioned because the two free global DOFs were labeled **1 and 2**, placing them in the correct positions from the start.

To illustrate the **general case**, the next slides use a separate **6×6 example** in which free and restrained DOFs are initially **intermixed**. The goal there is to show how the system is **reordered** so that all free DOFs appear first.

Step 0 — Original ordering (free DOFs = 3, 5)

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \boxed{u_3} \\ u_4 \\ \boxed{u_5} \\ u_6 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ \boxed{F_3} \\ F_4 \\ \boxed{F_5} \\ F_6 \end{bmatrix}$$

The corresponding global stiffness matrix has rows and columns associated with the same DOFs. Rows and columns for free DOFs (3 and 5) are highlighted.

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & \boxed{K_{13}} & K_{14} & \boxed{K_{15}} & K_{16} \\ K_{21} & K_{22} & \boxed{K_{23}} & K_{24} & \boxed{K_{25}} & K_{26} \\ \boxed{K_{31}} & \boxed{K_{32}} & \boxed{K_{33}} & \boxed{K_{34}} & \boxed{K_{35}} & \boxed{K_{36}} \\ K_{41} & K_{42} & \boxed{K_{43}} & K_{44} & \boxed{K_{45}} & K_{46} \\ \boxed{K_{51}} & \boxed{K_{52}} & \boxed{K_{53}} & \boxed{K_{54}} & \boxed{K_{55}} & \boxed{K_{56}} \\ K_{61} & K_{62} & \boxed{K_{63}} & K_{64} & \boxed{K_{65}} & K_{66} \end{bmatrix}$$

Step 1 — Bring DOF 3 to the top (swap DOFs 1 and 3)

Displacement and force vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \boxed{u_3} \\ u_4 \\ \boxed{u_5} \\ u_6 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{u_3} \\ u_2 \\ u_1 \\ u_4 \\ \boxed{u_5} \\ u_6 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ \boxed{F_3} \\ F_4 \\ \boxed{F_5} \\ F_6 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{F_3} \\ F_2 \\ F_1 \\ F_4 \\ \boxed{F_5} \\ F_6 \end{bmatrix}$$

Stiffness matrix (rows 1 \leftrightarrow 3, columns 1 \leftrightarrow 3)

$$\mathbf{K}^{(1)} = \begin{bmatrix} \boxed{K_{33}} & \boxed{K_{32}} & \boxed{K_{31}} & \boxed{K_{34}} & \boxed{K_{35}} & \boxed{K_{36}} \\ \boxed{K_{23}} & K_{22} & K_{21} & K_{24} & \boxed{K_{25}} & K_{26} \\ \boxed{K_{13}} & K_{12} & K_{11} & K_{14} & \boxed{K_{15}} & K_{16} \\ \boxed{K_{43}} & K_{42} & K_{41} & K_{44} & \boxed{K_{45}} & K_{46} \\ \boxed{K_{53}} & \boxed{K_{52}} & \boxed{K_{51}} & \boxed{K_{54}} & \boxed{K_{55}} & \boxed{K_{56}} \\ \boxed{K_{63}} & K_{62} & K_{61} & K_{64} & \boxed{K_{65}} & K_{66} \end{bmatrix}$$

Step 2 — Bring DOF 5 to position 2 (swap DOFs 2 and 5)

Displacement and force vectors

$$\mathbf{u} = \begin{bmatrix} \boxed{u_3} \\ u_2 \\ u_1 \\ u_4 \\ \boxed{u_5} \\ u_6 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{u_3} \\ \boxed{u_5} \\ u_1 \\ u_4 \\ u_2 \\ u_6 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \boxed{F_3} \\ F_2 \\ F_1 \\ F_4 \\ \boxed{F_5} \\ F_6 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{F_3} \\ \boxed{F_5} \\ F_1 \\ F_4 \\ F_2 \\ F_6 \end{bmatrix}$$

Stiffness matrix (rows 2 \leftrightarrow 5, columns 2 \leftrightarrow 5)

$$\mathbf{K}^{(2)} = \begin{bmatrix} \boxed{K_{33}} & \boxed{K_{35}} & \boxed{K_{31}} & \boxed{K_{34}} & \boxed{K_{32}} & \boxed{K_{36}} \\ \boxed{K_{53}} & \boxed{K_{55}} & \boxed{K_{51}} & \boxed{K_{54}} & \boxed{K_{52}} & \boxed{K_{56}} \\ \boxed{K_{13}} & \boxed{K_{15}} & K_{11} & K_{14} & K_{12} & K_{16} \\ \boxed{K_{43}} & \boxed{K_{45}} & K_{41} & K_{44} & K_{42} & K_{46} \\ \boxed{K_{23}} & \boxed{K_{25}} & K_{21} & K_{24} & K_{22} & K_{26} \\ \boxed{K_{63}} & \boxed{K_{65}} & K_{61} & K_{64} & K_{62} & K_{66} \end{bmatrix}$$

Partitioned form (read directly)

After reordering, the free DOFs come first. For our example, the free set is $f = \{3, 5\}$ and the restrained set is $r = \{1, 4, 2, 6\}$, so

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_f \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} u_3 \\ u_5 \\ u_1 \\ u_4 \\ u_2 \\ u_6 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{F}_f \\ \mathbf{F}_r \end{bmatrix} = \begin{bmatrix} F_3 \\ F_5 \\ F_1 \\ F_4 \\ F_2 \\ F_6 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fr} \\ \mathbf{K}_{rf} & \mathbf{K}_{rr} \end{bmatrix} = \begin{bmatrix} K_{33} & K_{35} & K_{31} & K_{34} & K_{32} & K_{36} \\ K_{53} & K_{55} & K_{51} & K_{54} & K_{52} & K_{56} \\ K_{13} & K_{15} & K_{11} & K_{14} & K_{12} & K_{16} \\ K_{43} & K_{45} & K_{41} & K_{44} & K_{42} & K_{46} \\ K_{23} & K_{25} & K_{21} & K_{24} & K_{22} & K_{26} \\ K_{63} & K_{65} & K_{61} & K_{64} & K_{62} & K_{66} \end{bmatrix}$$

Here the dashed line shows the split between free and restrained DOFs:

- top-left block: \mathbf{K}_{ff} (free–free)
- top-right block: \mathbf{K}_{fr} (free–restrained)
- bottom-left block: \mathbf{K}_{rf} (restrained–free)
- bottom-right block: \mathbf{K}_{rr} (restrained–restrained)

Part 7 — Solving Global Displacements and Forces

free node displacements and restraint forces

Expand the Partitioned System

Starting from the block system

$$\begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fr} \\ \mathbf{K}_{rf} & \mathbf{K}_{rr} \end{bmatrix} \begin{bmatrix} \mathbf{u}_f \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} \mathbf{F}_f \\ \mathbf{F}_r \end{bmatrix}$$

expanding the matrix–vector multiplication gives two coupled equations:

$$\mathbf{K}_{ff}\mathbf{u}_f + \mathbf{K}_{fr}\mathbf{u}_r = \mathbf{F}_f \quad (1)$$

$$\mathbf{K}_{rf}\mathbf{u}_f + \mathbf{K}_{rr}\mathbf{u}_r = \mathbf{F}_r \quad (2)$$

The first equation governs equilibrium at the **free DOFs**,
and the second governs equilibrium at the **restrained DOFs**.

Solving Equation (1) for the Free DOFs

We are interested in solving for the **unknown displacements** at the free DOFs, \mathbf{u}_f .

Starting from Equation (1): $\mathbf{K}_{ff}\mathbf{u}_f + \mathbf{K}_{fr}\mathbf{u}_r = \mathbf{F}_f$

Rearrange to isolate the unknowns:

$$\mathbf{K}_{ff}\mathbf{u}_f = \mathbf{F}_f - \mathbf{K}_{fr}\mathbf{u}_r$$

Provided that \mathbf{K}_{ff} is invertible, the solution is:

$$\mathbf{u}_f = \mathbf{K}_{ff}^{-1} \left(\mathbf{F}_f - \mathbf{K}_{fr}\mathbf{u}_r \right)$$

Typically, the restrained displacements \mathbf{u}_r are **known** from boundary conditions (often $\mathbf{u}_r = \mathbf{0}$). In that common case, the expression simplifies to:

$$\mathbf{u}_f = \mathbf{K}_{ff}^{-1} \mathbf{F}_f$$

Solve in practice to obtain the global displacements at the free DOFs.

Solving Equation (2) for the Restrained Forces

Once the free displacements \mathbf{u}_f have been computed, we can determine the **forces at the restrained DOFs** (support reactions).

Starting from Equation (2): $\mathbf{K}_{rf}\mathbf{u}_f + \mathbf{K}_{rr}\mathbf{u}_r = \mathbf{F}_r$

At the restrained DOFs, the displacements \mathbf{u}_r are **known** from the boundary conditions (often $\mathbf{u}_r = \mathbf{0}$). Substituting these known values gives a direct expression for the reaction forces:

$$\boxed{\mathbf{F}_r = \mathbf{K}_{rf}\mathbf{u}_f + \mathbf{K}_{rr}\mathbf{u}_r}$$

In the common case where the supports are fixed and $\mathbf{u}_r = \mathbf{0}$, this simplifies to:

$$\boxed{\mathbf{F}_r = \mathbf{K}_{rf}\mathbf{u}_f}$$

This step **does not require solving a linear system** (i.e., no inverse needed).

The reaction forces are obtained directly by evaluating this expression after \mathbf{u}_f is known.

Summary — What We Have Solved

At this point, the global system has been fully resolved.

We have determined the two previously unknown quantities:

- **Free nodal displacements, \mathbf{u}_f**
(obtained by solving the reduced system)
- **Support reaction forces, \mathbf{F}_r**
(computed directly once \mathbf{u}_f is known)

Together, these results provide the complete displacement and force state of the structure, satisfying equilibrium, compatibility, and the imposed boundary conditions.

Sanity checks after solving

1) Stability

- If the structure is properly constrained, \mathbf{K}_{ff} should be non-singular
- If it is still singular, you likely have:
 - insufficient supports (rigid-body mode remains), or
 - a disconnected node / element connectivity error

2) Equilibrium

- Reconstruct the full displacement vector \mathbf{u} by inserting \mathbf{u}_f (just calculated) and \mathbf{u}_r
- Compute $\mathbf{K}\mathbf{u}$ and compare to \mathbf{F}
- Differences should be near numerical tolerance (floating point error)

3) Reactions

- Check reaction directions and magnitudes against intuition
- Sum of reactions should balance applied loads (global equilibrium)

Part 8 — Recover Element Forces

Back to Local

Up to this point, we have solved for the **global nodal displacements** of the structure. While this information is essential for equilibrium, it is **not** what we design members for.

Structural design is performed at the **element level**, using:

- axial forces
 - stresses
 - internal force resultants
- expressed in each element's **local coordinate system**.

This step bridges analysis and design.

Why Go Back to the Element Level?

- Global displacements describe how the **structure moves**
- Local element forces describe how **members carry load**
- Design checks (strength, buckling, serviceability) are based on **element forces**, not global DOFs

Therefore, we must recover **element-level displacements and forces** from the global solution.

Element Force Recovery — Overview

For each element, we perform the following steps:

1. **Extract element global displacements**

Assemble the element displacement vector from the global solution.

2. **Transform to local coordinates**

Convert global displacements to the element's local axis.

3. **Compute local end forces**

Use the element stiffness matrix to compute internal forces.

4. **Evaluate axial force**

Extract the axial force used directly in design.

Step 1 — Extract Element Global Displacements

From the global displacement vector \mathbf{u} , collect the DOFs associated with element e :

$$\mathbf{u}_e = \begin{bmatrix} u_{i,x} \\ u_{i,y} \\ u_{j,x} \\ u_{j,y} \end{bmatrix}$$

This vector contains the **global displacements** of the element's end nodes.

Step 2 — Transform to Local Coordinates

Transform the element displacement vector to the local coordinate system:

$$\mathbf{u}'_e = \mathbf{T}_e \mathbf{u}_e$$

where \mathbf{T}_e is the **element transformation matrix**, for that specific element defined by the element orientation.

This step aligns the displacements with the element's axial direction.

Step 3 — Compute Local End Forces

Use the local stiffness matrix to compute the internal end forces:

$$\mathbf{f}'_e = \mathbf{k}'_e \mathbf{u}'_e$$

These are the **local element forces** (axial forces at each end). For a truss element you should get the axial force in the element

Step 4 — Compute Axial Stress

Once the axial force is known, the **axial stress** follows as:

$$\sigma = \frac{N}{A}$$

This is what you design for:

- axial stress limits
- tension vs. compression
- buckling checks for compression members

Step 5 (Optional) — Member End Forces in Global Coordinates

This step is **not required for design**, but it can be useful for interpretation and post-processing.

So far, member forces have been computed in the **local coordinate system**, which is where axial behavior is defined.

In some cases, however, it is helpful to express the **member end forces in the global (X, Y) frame**, for example:

- to visualize force flow in the structure
- to check nodal equilibrium in global directions
- to compare with externally applied loads

The local member force vector can be rotated back to global coordinates as:

$$\mathbf{f}_e = \mathbf{T}_e^T \mathbf{f}'_e$$

For a **perfectly horizontal member**, the local and global axes coincide, and this transformation leaves the forces unchanged.

Part 9 — DSM Summary

Forward Pass — Structural Analysis

1. Define geometry

- Nodes, members, connectivity

2. Number global DOFs

- Assign consistent global displacement indices

3. For each member

- Compute geometry: L , θ , c , s
- Build transformation matrix \mathbf{T}
- Compute global element stiffness:

$$\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$$

4. Assemble global stiffness matrix

- Scatter-add element contributions into \mathbf{K}

Forward Pass — Structural Analysis, cont...

5. Apply boundary conditions

- Partition DOFs into free (f) and restrained (r)

6. Solve for unknown displacements

$$\mathbf{u}_f = \mathbf{K}_{ff}^{-1}(\mathbf{F}_f - \mathbf{K}_{fr}\mathbf{u}_r)$$

7. Recover support reactions

$$\mathbf{F}_r = \mathbf{K}_{rf}\mathbf{u}_f + \mathbf{K}_{rr}\mathbf{u}_r$$

Backward Pass — Element Recovery and Design

8. Extract element global displacement vectors

- For each member, collect the relevant entries from \mathbf{u} to form \mathbf{u}_e

9. Transform displacements to local coordinates

$$\mathbf{u}' = \mathbf{T} \mathbf{u}_e$$

10. Compute local element end forces

$$\mathbf{f}' = \mathbf{k}' \mathbf{u}'$$

11. Compute axial force and stress (design quantities)

$$\sigma = \frac{N}{A}$$

12. (Optional) Express element forces in global coordinates

$$\mathbf{f} = \mathbf{T}^\top \mathbf{f}'$$

Why DSM Is Powerful

- **Static indeterminacy is not an issue**
 - Any number of members can be handled
- **Equilibrium is enforced automatically**
 - Through the global stiffness equations
- **Compatibility is built in**
 - Via shared nodal displacements
- The method scales cleanly to:
 - large structures
 - complex geometries
 - numerical implementation

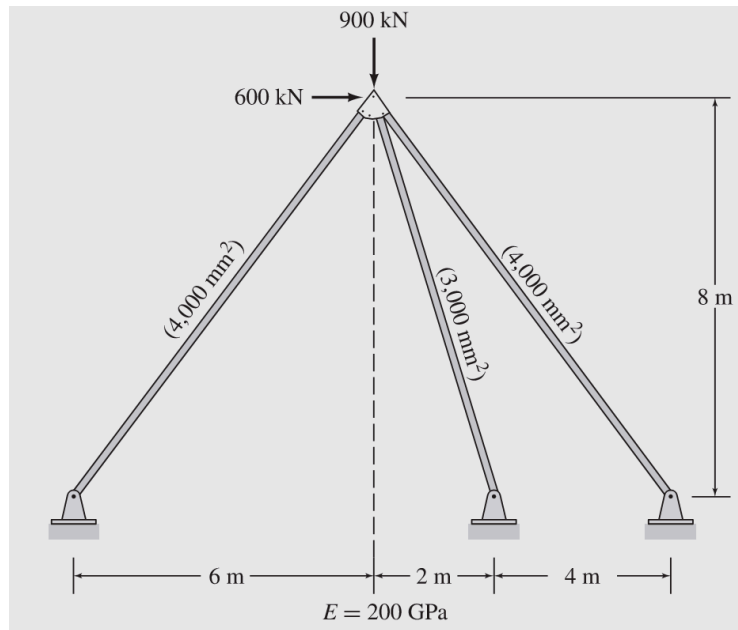
DSM replaces manual equilibrium and compatibility equations with a single, systematic matrix framework.

Class Problem (Time Permitting)

This problem will form the basis of **this week's homework**.

Begin by working through the setup **by hand**:

- define the geometry and connectivity,
- number the global degrees of freedom,
- setup the local element stiffness matrices (calculate transformation matrices).



Node numbering:

- Node 1: bottom-left support
- Node 2: middle support
- Node 3: top node
- Node 4: right support

Wrap-Up

Today you built the DSM pipeline for trusses:

- local bar stiffness \rightarrow transformation \rightarrow global element stiffness
- assembly \rightarrow constraints \rightarrow solve \rightarrow member force recovery

Next Lecture: implement DSM in Python for a worked truss example and discuss efficiency (sparsity/bandedness).