Dynamic Portfolio Selection in Arbitrage*

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Abstract

This paper derives the optimal dynamic strategy for arbitrageurs with a finite horizon and non-myopic preferences facing a mean-reverting arbitrage opportunity (e.g. an equity pairs trade). We find that intertemporal hedging demands play an important role in determining how aggressively arbitrageurs trade against the mispricing and account for a large fraction of the total allocation to the arbitrage opportunity. While arbitrageurs typically bet against the mispricing, we analytically show that there is a critical level of mispricing beyond which further divergence precipitates a *reduction* in the allocation. When applied to Siamese twin shares our optimal strategy delivers a significant improvement in the realized Sharpe ratio and welfare relative to a simple threshold rule.

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The law of one price asserts that - in a frictionless market - securities with identical payoffs must trade at identical prices. If this were not the case, arbitrageurs could generate a riskless profit by supplying (demanding) the expensive (cheap) asset until the mispricing was eliminated. Of course, real world markets are not frictionless, and the prices of securities with identical payoffs may significantly diverge for extended periods of time.[1] Arbitrageurs can earn potentially attractive profits by taking offsetting positions in these relatively mispriced securities, but a worsening of the mispricing can result in sizable capital losses. Understanding these risks and how they impact the behavior of rational arbitrageurs is crucial to developing a more nuanced view of how market efficiency is enforced, and is the subject of this paper.

Although researchers have identified numerous examples of seeming violations of market efficiency in relative asset pricing, e.g. closed-end funds (Malkiel, 1977), stock index futures (Brennan and Schwartz, 1990), Siamese twins (Froot and Dabora, 1999), tech stock carve-outs (Lamont and Thaler, 2003), treasuries (Cornell and Shapiro, 1989), and stubs (Mitchell, Pulvino and Stafford, 2002), relatively little is known about the trading strategies that arbitrageurs would use to optimally exploit these mispricings.[2] Unlike textbook arbitrages, which generate riskless profits and require no capital commitments, exploiting real-world mispricings requires the assumption of some combination of *horizon* and *divergence risk*. These two risks represent the uncertainty about whether the mispricing will converge before the positions must be closed (or profits reported) and the possibility of a worsening in the mispricing prior to its elimination. De Long et al. (1990) and Shleifer and Vishny (1997) were the first to emphasize that these risks may play a crucial role in limiting the size of positions that arbitrageurs are willing to take, contributing to the persistence of mispricings in equilibrium.

While a complete treatment of optimal arbitrage and price formation requires a general equilibrium framework, we take on the more modest goal of examining the arbitrageur's strategy in partial equilibrium. Consequently, we begin by pre-specifying a process for the evolution of the mispricing that is sufficiently rich to capture the presence of both horizon and divergence risk in common arbitrage trades. This simplification allows us to derive a variety of novel, analytical results concerning the optimal behavior of rational arbitrageurs with a wide range of preference specifications. The optimal strategies we derive are of interest for two main reasons. First, given the central role of arbitrageurs in asset pricing, the optimal strategies allow arbitrageurs to limit losses and preserve the amount of capital available to enforce market efficiency. Second, the conclusions of existing empirical studies examining the risk-return characteristics of strategies exploiting suspected deviations from market efficiency may be affected when the optimal strategy is utilized in place of the commonly employed rules of thumb.[3]

[1]The persistence of arbitrage opportunities is typically attributable to the presence of agency problems, risk or transactions costs. See for example, De Long et al (1990), Shleifer and Vishny (1997), and Basak and Croitoru (2000).

[2]In the context of absolute asset pricing, Shiller (1981, 1984) was the first to provide evidence of deviations of price from fundamental value. Similarly, Black (1986) highlights the role of "noise" in price formation.

[3]Balvers et al. (2000) consider simple contrarian investment to exploit mean-reversion in national stock indices, i.e. strategies that give 100% weight to the national stock index that is furthest below the mean. Gatev, Goetzmann and Rouwenhorst (2006), consider a threshold trading rule where an investor opens a long-short position when the

To capture the presence of horizon and divergence risk, we model the dynamics of the mispricing using a mean-reverting stochastic process. Under this process, although the mispricing is guaranteed to be eliminated at some future date, the timing of convergence, as well as the maximum magnitude of the mispricing prior to convergence, are uncertain. With this assumption, we are able to derive the arbitrageur's optimal dynamic portfolio policy for a set of general, non-myopic preference specifications, including CRRA utility defined over wealth at a finite horizon and Epstein-Zin utility defined over intermediate cash flows (e.g. fees). This allows us to analytically examine the role of intertemporal hedging demands in arbitrage activities and represents a novel contribution relative to the commonly assumed log utility specification (e.g. Rashes (2000), Xiong (2001)), under which hedging demands are absent. Furthermore, unlike previous work examining the behavior of non-myopic arbitrageurs (e.g. Liu and Longstaff (2004)), our results do not require any numerical simulations. The price of our model's analytical tractability is that we are forced to specialize to a partial equilibrium setting, typical in the portfolio choice literature.[4]

The availability of an analytical expression for the optimal allocation to the risky arbitrage opportunity enables us to examine the arbitrageur's response to shocks in the underlying mispricing. We find that, in the presence of horizon and divergence risk, there is a critical level of the mispricing beyond which further divergence in the mispricing precipitates a reduction in the allocation. Although a divergence in the mispricing results in an improvement in the instantaneous investment opportunity set and should induce added participation by rational arbitrageurs, this effect can be more than offset by the combination of the loss in wealth and the nearing of the evaluation date, which reduce the arbitrageur's effective risk-bearing capacity. The complex tradeoff between these two effects leads to the creation of a time-varying boundary, outside of which continued divergence of the mispricing induces rational arbitrageurs to cut their losses and decrease their allocations to the mispricing. Although the numerical solution to Xiong's (2001) stationary general equilibrium model also exhibited this feature, this paper is the first to analytically describe the conditions under which this destabilizing trading takes place. This finding highlights that investor short-termism may play a key role in equilibrium price formation in crisis situations.

Our simple benchmark framework can be extended to include multiple risky spreads, and accommodate the presence of flexibly specified fund flows. Consistent with Shleifer and Vishny (1997), we show that, when flows chase performance, the arbitrageur attenuates the intensity of his trading against the mispricing to offset the increase in the volatility of wealth caused by the component of fund flow correlated with past performance. Finally, a comparison of the optimal trading strategy to a simple threshold rule proposed in Gatev, Goetzmann and Rouwenhorst (2006) indicates that the optimal strategy delivers a considerable improvement in welfare and the realized Sharpe ratio when the mispricing is mean-reverting.

In order to obtain a parsimonious representation of the arbitrageur's portfolio selection problem,

return spread is two standard deviations away from the mean, and closes it when it converges. Mitchell, Pulvino and Stafford (2002) utilize a similar strategy in their test of efficiency in the market for stubs.

[4] General equilibrium models examining price formation in the presence of arbitrage trading include Campbell and Kyle (1993), Xiong (2001), Dumas et al. (2005), Kondor (2006).

we assume that there are only two traded assets: a riskless bond and a mean-reverting arbitrage opportunity, representing the price differential (i.e. spread) between two close substitutes.[5] Candidate spreads can be constructed either on the basis of economic arguments (e.g. the law of one price) or statistical methods (e.g. cointegration), and are best interpreted as a long-short relative value trades (e.g. Siamese twin shares) in which the offsetting positions are assumed to cancel the underlying exposure to fluctuations in the fundamental values of the two securities. This allows us to abstract from the determinants of fair value, and simply focus on the risk-return characteristics of the strategies seeking to profit from temporary departures from efficiency in relative asset pricing.[6]

The central assumption of our model is that the arbitrage opportunity is described by an Ornstein-Uhlenbeck process (henceforth "OU"). The OU process captures the two key features of a real-world mispricing: the convergence times are uncertain and the mispricing can diverge arbitrarily far from its mean prior to convergence.[7] This assumption represents a crucial difference between our paper and Liu and Longstaff's (2004) partial equilibrium examination of convergence trading strategies, where the mispricing is modelled using a Brownian bridge. The *ex ante* knowledge of the parameters of a Brownian bridge requires that the agent have perfect information about the magnitude of the mispricing at some future date, or - equivalently - the date on which the mispricing will be eliminated. Due to this feature, their model is restricted to settings with perfect inside information or finitely lived assets. Our OU assumption, on the other hand, implies that the arbitrageur faces uncertainty about the magnitude of the mispricing at all future dates, which is characteristic of most relative value trades. As a consequence of the non-vanishing degree of uncertainty, risk-averse agents take finite positions in the mispricing even in the absence of margin constraints, unlike in the case of the Brownian bridge.

The optimal trading strategies we derive are self-financing and can be interpreted as the optimal trading rules for a fund which is not subject to withdrawals but also cannot raise additional assets (i.e. a closed-end fund). The dynamics of the optimal allocation to the arbitrage opportunity are driven by two factors: the necessity of maintaining wealth (equity) above zero and the proximity of the arbitrageur's terminal evaluation date, which affects his appetite for risk. For CRRA arbitrageurs with utility defined over wealth at time T, we find that – holding the magnitude of the mispricing fixed – the size of the position in the mispricing decreases (increases) as the terminal date approaches when the arbitrageur's risk aversion coefficient is greater (smaller) than implied by log utility. In the knife-edge case of log utility, the arbitrageur's allocation is myopic and does not depend on the proximity of the terminal date. A decomposition of the optimal risky asset holding

[5] To the extent that the process driving the mispricing is orthogonal to the price processes of other assets in which the trader might be invested, the optimal spread trading rule will not be significantly affected by their inclusion in the model, and so they are omitted for parsimony. Consequently, our model should be interpreted as if any systematic risk of the arbitrage opportunity has been hedged, as would likely be the case in a realistic scenario.

[6] The model can also be applied to *value investing*, as shown in Appendix D, where we consider the case of mean-reversion in the log price of an asset around its fair value - the canonical model considered in Balvers, et al. (2000).

[7] Boguslavsky and Boguslavskaya (2004) also utilize an OU process to describe an arbitrage opportunity.

into a myopic, timing component and an intertemporal hedging component reveals that hedging demands play an important role in the total allocation at all horizons, contrary to Rashes's (2000) conjecture.

When applied to trading in the Siamese twin shares of Royal Dutch-Shell Transport and Unilever (Unilever PLC and Unilever NL), the optimal trading rule delivers Sharpe ratios of 0.50 and 0.61, respectively, for arbitrageurs with $\gamma = 10$. This attractive risk-return tradeoff is accompanied by highly non-normal returns, which feature some rare but significant losses. These findings are consistent with our theoretical characterization of the distribution of instantaneous returns delivered by the optimal strategy, and confirm that arbitrage – in the sense of exploiting temporary mispricings by the market – can be highly risky, despite being lucrative in theory.

Lastly, we compare the performance of the optimal trading rule relative to the threshold rule adopted by Gatev, Goetzmann and Rouwenhorst (2006) in their examination of pairs trading. The threshold rule calls for opening a long-short position in a pair of stocks when the spread in their normalized price series (i.e. total return indices) deviates more than two standard deviations away from its mean, and unwinding it when the mispricing is eliminated. Although this choice of threshold rule is clearly arbitrary, variants of it are commonly employed in empirical studies of the profitability of strategies that exploit deviations in relative asset pricing. Using the Siamese twins, we find that the GGR rule results in a lower Sharpe ratio for Unilever than the optimal rule and that the threshold rule leads to bankruptcy for the Royal Dutch trade. Using simulated data, we find that the optimal rule offers a significant improvement in terms of realized Sharpe ratios and accumulated wealth when the mispricing is highly mean-reverting. Conversely, when the rate of mean-reversion is low and the mispricing is long-lived, the performance of the threshold rule is comparable to and sometimes better than the performance of the optimal strategy. Moreover, we find that the optimal trading rule is highly sensitive to estimation errors in the parameters of the process describing the arbitrage opportunity.

The remainder of the paper is organized as follows: in Section I, we introduce the assumptions of our model and derive the optimal policy and value functions for arbitrageurs with CRRA and Epstein-Zin preferences. Section II characterizes the dynamics of wealth and the optimal allocation to the arbitrage opportunity, deriving a bound for the region in which the arbitrageur's trading is stabilizing. In Section III, we extend the model to allow for multiple mean-reverting spreads and performance-based fund flows. In Section IV, we compare the performance of the optimal strategy relative to the Gatev, Goetzmann and Rouwenhorst (2006) threshold rule, examine the robustness of the optimal policy rule to estimation risk and illustrate our model through an empirical application to the trading of the Siamese twin shares of Royal Dutch-Shell and Unilever. Section V concludes.

I The Model

Efficiency in the pricing of related securities is enforced by arbitrageurs who buy (sell) the relatively cheap (expensive) asset, with the goal of profiting from the elimination of the mispricing.

Such trades – known as relative value trades – are aimed at exploiting deviations from the law of one price, and are a central mechanism in enforcing market equilibrium. Despite their importance, very little is known about the characteristics of the optimal strategies employed by rational arbitrageurs in exploiting these deviations from market efficiency. To address this question we develop a model of relative value trading in a finite-horizon, partial equilibrium portfolio choice setting. Our key model assumptions concern the structure of investor preferences and the process characterizing the evolution of the value of the arbitrage opportunity, and are discussed in the next two sections.

A Investor Preferences

We consider two alternative preferences structures for the arbitrageur in our continuous-time model. In the first, we assume that the agent has constant relative risk aversion and maximizes the discounted utility of terminal wealth. The arbitrageur's value function at time t - denoted by V_t - takes the form:

$$V_t = \sup E_t \left[e^{-\beta(T-t)} \frac{W_T^{1-\gamma}}{1-\gamma} \right] \tag{1}$$

The second preference structure we consider is the recursive utility of Epstein and Zin (1989, 1991), which allows the elasticity of intertemporal substitution and the coefficient of relative risk aversion to vary independently. Under this preference specification, the value function of the arbitrageur is given by:

$$V_t = \sup E_t \left[\int_t^T f(C_s, J_s) ds \right] \tag{2}$$

where $f(C_s, J_s)$ is the normalized aggregator for the continuous-time Epstein-Zin utility function:

$$f(C_t, J_t) = \frac{\beta(1-\gamma)}{1-\frac{1}{\psi}} \cdot J_t \cdot \left[\left(\frac{C_t}{((1-\gamma)J_t)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\psi}} - 1 \right] \tag{3}$$

The aggregator depends on the instantaneous consumption (e.g. cash flow), C_t, and the continuation value of the problem; β is the rate of time preference, ψ is the elasticity of intertemporal substitution and γ is the coefficient of relative risk aversion. Epstein-Zin utility has traditionally been used in stationary infinite-horizon settings (e.g. Campbell and Viceira (1999, 2001)), and requires a minor adaptation in order to be consistent in a finite-horizon setting. In particular, we require that the value function at the horizon date T equal the consumption value of total wealth.

By employing these general, non-myopic preference specifications we are able to significantly generalize existing results on arbitrage trading. For example, Rashes (2000) and Xiong (2001) only consider arbitrageurs with log utility, which precludes their models from being able to make predictions regarding intertemporal hedging demands. As we show later, intertemporal hedging demands in arbitrage activities turn out to be quantitatively significant and to play an important role in determining when arbitrageurs trade against (or with) the mispricing. Furthermore, in contrast to the existing literature – both partial and general equilibrium – our model delivers analytically tractable solutions for the optimal, dynamic portfolio allocation, even with non-myopic preferences.

Our choice of preference structures is driven by economic intuition regarding the incentives of real-life arbitrageurs. In particular, we find it plausible that the representative arbitrageur is a proprietary trading desk or delegated money manager with a fixed investment horizon. It seems likely that such investors would only be interested in the distribution of wealth at a finite horizon, e.g. at the end of the fiscal year, rather than the value of a long-dated consumption stream. However, the decision to model arbitrageurs as finite-horizon CRRA investors neglects the role of management fees, which are often collected by arbitrageurs. To capture this feature, we also consider the Epstein-Zin model specialized to the case of a unit elasticity of intertemporal substitution.[8] As is well known, in this case the agent's consumption to wealth ratio is constant, which we exploit as a model of a flat management fee, collected (and consumed) as a continuous stream rather than as a lump-sum payment.

B The Arbitrage Opportunity

The arbitrage opportunity in our model is interpretable as a long/short relative value trade, and its magnitude is measured by the price differential, i.e. spread, between the prices of two related securities (e.g. Siamese twin shares). It can be exploited by buying the undervalued security and shorting the overvalued security, effectively taking a bet on the convergence of the pricing differential. Unlike a "value" trade, where investors attempt to profit from the reversion of an asset's price to some measure of fundamental value, the arbitrage trade is a non-directional strategy in which the long and short positions offset the underlying exposure to fluctuations in the fundamental values of the two securities. In other words, the underlying premise of the our arbitrage strategy is a violation of the law of one price, rather than a directional view regarding an asset's price. Although we devote the remainder of the paper to an analysis of optimal trading strategies for relative value investors (i.e. arbitrageurs), Appendix D provides a complete solution to the "value" investor's problem.[9]

Investable arbitrage opportunities can be identified using various methods, which can generally be subdivided into economically- and statistically-motivated. The former emphasize economic links between the underlying securities (e.g. claims on common cash flows), and are motivated by the law of one price, which states that assets with identical payoffs in all states of the world should command identical prices.[10] The statistical methods, on the other hand, identify mispricings

[8] In the special case of a unit elasticity of intertemporal substitution ($\psi = 1$) the aggregator takes the somewhat simpler form:

$$f(C_t, J_t) = \beta(1-\gamma) \cdot J_t \cdot \left[\log C_t - \frac{1}{1-\gamma} \log\left((1-\gamma)J_t\right) \right]$$

[9] Our decision to relegate the model of value investing to the appendix is motivated by a desire to avoid having to specify an auxiliary model determining an asset's fair value.

[10] For a textbook treatment, see Ingersoll (1987).

as deviations from common stochastic trends (fundamental value) using cointegration methods (Engle and Granger (1987)).[11] Because the testing of hypotheses involving cointegrating vectors is known to be subject to numerous pitfalls (Elliot, 1998), we use the law of one price to propose candidate arbitrage trades and subsequently verify that the relevant price spreads are stationary. This approach allows us retain the greatest amount of economic intuition, while confirming the consistency of the data with our proposed model.

Unlike textbook arbitrages, which yield riskless profits, exploiting mispricings in the real-world exposes the arbitrageur to two important dimensions of risk. These risks are present in essentially all relative value trades and include the uncertainty about the timing at which the mispricing will be eliminated (*horizon risk*) and the potential for the mispricing to diverge arbitrarily far from its mean prior to convergence (*divergence risk*). To capture these two forms of risk we model the mispricing using a mean-reverting (Ornstein-Uhlenbeck) process. Under this assumption – and unlike in the Brownian bridge model of Liu and Longstaff (2004) – the value of the mispricing is uncertain at *all* future dates.[12] Because the Brownian bridge model requires the arbitrageur to have perfect information about the magnitude of the mispricing at some future date, or - equivalently - the date on which the mispricing will be eliminated, it applies only to cases with private information or finitely lived assets. The absence of this restriction under the OU process makes our model more broadly applicable, and complements the convergence trading results provided by Liu and Longstaff (2004).

Moreover, the OU assumption provides a convenient framework for quantifying *horizon* and *divergence risk.* Horizon risk can be measured by the uncertainty over the length of time that will elapse prior to the spread's return to its long-run mean, and divergence risk – by the variance of the distribution of the running maximum (minimum) of the spread between its current value and its first hitting time to the long-run mean.

C Optimal Portfolio Choice

We allow the arbitrageur to invest in a riskless asset and the mean-reverting spread. We denote the prices of the two assets by B_t and S_t, respectively. Their dynamics are given by

$$dB_t = rB_t dt \tag{4}$$

$$dS_t = \kappa(\overline{S} - S_t)dt + \sigma dZ \tag{5}$$

Because the value of the spread represents the price of a long-short portfolio, buying (shorting) one unit of the spread is equivalent to buying (shorting) one unit of the overvalued security and shorting (buying) one unit of the undervalued security. In the special case with $\overline{S} = 0$, when $S_t > 0$

[11] In fact, the intuition provided by the authors for the nature of the cointegration residual is identical to the intuition underpinning relative value strategies - "deviations from equilibrium are stationary, with finite variance, even though the series themselves are non-stationary and have infinite variance." Alexander (1999) and Alexander and Dimitriu (2002) provide examples of using cointegration analysis to identify mean-reverting spreads.

[12] A Brownian bridge is a continuous-time stochastic process, whose value is pinned down at two points in time - typically the start date, $t = 0$, and a future date, $t = T$.

the arbitrageur shorts the spread asset - effectively going long the undervalued security and short the overvalued security - and invests the proceeds in the riskless asset. The situation is reversed when $S_t < 0$. Due to the mean-reversion in the price differential these positions can be expected to generate positive revenues as the spread converges to its long-run mean, $\overline{S}$. In both cases, we assume there are no margin constraints, giving the agent full use of the short proceeds.[13] Furthermore, we abstract from the inclusion of transaction costs (bid-ask spreads, price impact, etc.) and focus on the optimal portfolio choice decision of the agent in a *frictionless*, continuous-time setting.

Although we allow the spread, S_t, to have an arbitrary long-run mean of $\overline{S}$, economic theory suggests that - in applications where the underlying securities are perfect substitutes - the long-run mean spread will be equal to zero ($\overline{S} = 0$). In some circumstances, however, the anticipated mean of the spread in prices may be different from zero, for example, if the dividend payments on the two securities are not perfectly synchronized.

In order to emphasize the generality of our solutions we consider two cases of investor preferences. In the first specification, the agent has constant relative risk aversion preferences over wealth at a terminal horizon. In the second scenario, the investor's preferences take the Epstein-Zin form (with $\psi = 1$) and utility is defined over intermediate consumption (e.g. management fees). If we denote the number of units of the spread and riskless assets held by the agent by N_t and M_t, respectively, the budget constraints for the two problems can be written as:

$$dW_t = N_t dS_t + M_t dB_t - C_t \mathbf{1}[C_t > 0]dt \tag{6}$$

where $\mathbf{1}[C_t > 0]$ is an indicator variable for whether intermediate consumption is taking place. Suppressing the time subscripts on the asset prices and number of shares of each asset in the portfolio, the evolution of wealth satisfies:

$$\begin{aligned} dW &= NdS + \frac{W - NS}{B} dB - C\mathbf{1}[C > 0]dt \\ &= \big(r(W - NS) + \kappa(\overline{S} - S)N - C\mathbf{1}[C > 0]\big)\, dt + \sigma N dZ \end{aligned} \tag{7}$$

Given these budget constraints, we can derive optimal dynamic portfolio strategies under the two preference specifications. Utilizing the standard methods of stochastic control theory, we find the Hamilton-Jacobi-Bellman (HJB) equation for each problem and derive a partial differential equation for the value function.[14] This derivation also yields the optimal policy functions in terms of the derivatives of the value function. In particular, we find that the optimal number of units of the

[13] In the Liu and Longstaff (2004) framework, margin constraints are necessary to prevent infinite positions because the convergence date is known and fixed. In our model, horizon risk causes the investor to endogenously limit the size of investment positions.

[14] A complete analytical solution of each of the stochastic dynamic programming problems can be found in the appendix. Appendix A solves the problem for an agent with utility of terminal wealth, and Appendix B considers the analogous problem for an agent whose utility is defined over intermediate consumption.

spread the agent holds is given by:

$$N = -\left(\frac{V_W}{V_{WW}}\right)\left(\frac{\kappa(\overline{S}-S)-rS}{\sigma^2}\right) - \frac{V_{SW}}{V_{WW}} \tag{8}$$

Within the first term, the first factor is the absolute risk aversion factor of the value function, and the second factor is the instantaneous Sharpe ratio of the spread asset divided by its standard deviation. The second term represents the intertemporal hedging demand. In addition, the optimal consumption rule for an agent whose utility is defined over intermediate consumption must satisfy the first-order condition:

$$f_C(C_t, V_t) \;\; = \;\; V_W \tag{9}$$

This condition states that the agent's marginal utility from immediate consumption equals the marginal utility from delayed consumption.

For the CRRA investor with utility over only terminal wealth, the value function takes the form:

$$V(S,W,\tau) = \begin{cases} e^{-\beta\tau}\log W + e^{-\beta\tau}\left(A(\tau)\cdot S^2 + B(\tau)\cdot S + C(\tau)\right) & \gamma = 1 \\ e^{-\beta\tau}\frac{W^{1-\gamma}}{1-\gamma}\exp\left(A(\tau)\cdot S^2 + B(\tau)\cdot S + C(\tau)\right) & \gamma \neq 1 \end{cases} \tag{10}$$

where we have defined τ to be equal to the time remaining from the current period to the horizon date ($\tau = T - t$). The value function for the Epstein-Zin investor with intermediate consumption is identical up to one normalization factor (see Appendix B). Additionally, in order to match the boundary conditions on the value function, we impose that the coefficient functions, $\{A(\tau), B(\tau), C(\tau)\}$, satisfy the following boundary condition: $A(0) = B(0) = C(0) = 0$. In Appendices A and B, we present a complete derivation of the optimal policy functions and verify that the proposed form of the value function is correct. We collect our results in the following two theorems.

Theorem 1 (Optimal Portfolio Strategy for Terminal Wealth Problem) *The optimal portfolio allocation of an agent with constant relative risk aversion with utility defined over terminal wealth,* (1)*, is given by:*

$$N(W,S,\tau) = \begin{cases} \left(\frac{\kappa(\overline{S}-S)-rS}{\sigma^2}\right)W & \gamma = 1 \\ \left(\frac{\kappa(\overline{S}-S)-rS}{\gamma\sigma^2} + \frac{2A(\tau)S+B(\tau)}{\gamma}\right)W & \gamma \neq 1 \end{cases} \tag{11}$$

The coefficient functions $A(\tau)$, $B(\tau)$ and $C(\tau)$ depend on the time remaining to the horizon and the parameters of the underlying model. They are given in closed form in Appendix A.

Analogous derivations for the case of an agent with Epstein-Zin preferences over intermediate consumption and unit elasticity of intertemporal substitution lead to the following result.

Theorem 2 (Optimal Portfolio Strategy for Intermediate Consumption Problem)

The optimal portfolio allocation and the corresponding value function for the maximization problem of an agent with Epstein-Zin utility with unit elasticity of intertemporal substitution ($\psi = 1$) defined over intermediate consumption have the same form as the optimal policy and value function of an agent with constant relative risk aversion whose utility is defined over terminal wealth. The optimal policy function for consumption yields a constant consumption-to-wealth ratio equal to the subjective rate of time preference (β). The functional form of the coefficient functions $A(\tau)$, $B(\tau)$ and $C(\tau)$ is only slightly affected and can be found in Appendix B.

It is worthwhile to briefly pause here and contrast our results with those derived by Kim and Omberg (1996), in their model with a mean-reverting, instantaneous Sharpe ratio. Although the mean-reversion assumption represents a close parallel between the two models, it is applied to two different quantities – the instantaneous *expected equity return* in Kim and Omberg (1996) and the *spread in prices* between two related securities in our model. As a result of this difference, their model is best thought of as a model of market timing, whereas ours is about the optimal exploitation of relative mispricings. Furthermore, because their framework focuses on the returns of a single risky asset, they have no conceptual equivalent to divergence risk and horizon risk, which are the central features of our model. While our assumptions lead to a similar mathematical formalism, i.e. a system of differential equations involving a Riccati equation, neither model's foundational assumption is implied by the other. Specifically, our mean-reverting price spread does not imply an OU process for expected returns, nor do OU processes for expected returns on two related assets, imply that their price differential will be OU.

II Optimal Arbitrage Strategies and Their Properties

Unlike existing papers on arbitrage trading, our model yields closed-form solutions for the optimal portfolio strategies and value functions for a wide range of non-myopic preference specifications. This allows us to present a novel, analytical characterization of intertemporal hedging demands in arbitrage activities that was mechanically absent from papers with myopic arbitrageurs (Rashes (2000), Xiong (2001)), and impossible to obtain under the Brownian bridge specification for the arbitrage opportunity (Liu and Longstaff (2004)). We begin this section by examining the relative magnitudes of the myopic and intertemporal hedging components of the allocation to the arbitrage opportunity. We then turn to a characterization of the distribution of returns on wealth and, finally, we conclude by examining the conditions under which the arbitrageur acts in a "destabilizing" manner relative to the mispricing, decreasing his position as the mispricing worsens.[15] While this type of destabilizing behavior was observed by Xiong (2001), this paper is the first to provide a closed-form representation for the region in which it takes place.

[15] Because our model is cast in partial equilibrium the arbitrageur's trading has no effect on the evolution of the mispricing. Despite this fact we characterize the arbitrageur's actions as "stabilizing" and "destabilizing," depending on the the effect they would likely have on the mispricing in general equilibrium.

A Intertemporal Hedging Demands

The optimal policy function, (11), can be decomposed into two components. The first component, $(\frac{\kappa(\overline{S}-S)-rS}{\gamma\sigma^2}\cdot W)$, represents myopic demand and responds solely to the instantaneous magnitude of the mispricing, while the second component $(\frac{2A(\tau)S+B(\tau)}{\gamma}\cdot W)$ represents the intertemporal hedging demand – the investor's additional demand for the asset due to covariation of the wealth process with the attractiveness of the available investment opportunities. The sign and magnitude of the intertemporal hedging demand are horizon-dependent and are determined by the coefficient functions, $A(\tau)$ and $B(\tau)$, which are of the hyperbolic trigonometric class. Their precise form depends on the arbitrageur's risk aversion and the parameters of the process describing the arbitrage opportunity.[16] Interestingly, it is possible to show that the $B(\tau)$ term represents an adjustment for a non-zero long-run mean in the spread asset, and is equal to zero for all values of τ, so long as the mean value of the mispricing, $\overline{S}$, is equal to zero.[17] To simplify the ensuing analysis we specialize to the case of $\overline{S}=0$.

For plausible parameter values of κ and r the functional form of $A(\cdot)$ in its argument – the time remaining to the horizon, τ – is either hyperbolic cotangent ($\gamma<1$) or hyperbolic tangent ($\gamma>1$). In other words, for investors who are more (less) risk averse than log utility, $A(\tau)$ is non-positive (non-negative) and decreasing (increasing) in τ. Intuitively, because the arbitrage is mean-reverting, a divergence in the mispricing coincides with an improvement in the instantaneous investment opportunity set. This causes investors who are risk-averters ($\gamma>1$) to desire a greater allocation to the arbitrage opportunity than implied by the myopic component, and vice versa. Moreover, for a given magnitude of the spread, S_t, the degree of aggressiveness with which arbitrageurs seek to trade against the mispricing depends on the time remaining to the evaluation date, τ.[18] In general, as τ increases the magnitude of the hedging demand increases, since the arbitrageur has more time to exploit the variation in investment opportunities. Consequently, risk-averse arbitrageurs ($\gamma>1$) who face infrequent performance evaluation are predicted to be more aggressive in trading against the mispricing than equally risk-averse arbitrageurs who face more frequent performance evaluation. This prediction coincides with the intuition that agency conflicts which endogenously shorten arbitrageurs' horizons are likely to limit the robustness of arbitrage and result in less efficient markets. Also, as expected, the intertemporal hedging demand disappears altogether in the knife-edge case of log utility.

To fix intuitions about the sign and monotonicity of the hedging demands, Figure 1 plots the intertemporal hedging component of the optimal allocation as a function of the time remaining to the evaluation date, τ, for two values of relative risk aversion. Specifically, the left (right) panel plots the hedging demand of an arbitrageur who is less (more) risk averse than implied by log utility. Because we are interested in the horizon dynamics of the hedging component, we normalize

[16] See derivations in Appendix A and B for details.

[17] Campbell and Viceira (1999) obtain an analogous result in their model of long-horizon portfolio choice.

[18] The intertemporal hedging demand term vanishes when $\tau=0$, since the coefficient functions satisfy the boundary condition, $A(\tau)=B(\tau)=0$.

wealth to one and hold the spread fixed at one standard deviation above its long-run mean.[19] Under these circumstances, the myopic component of the optimal allocation is negative. This indicates that the hedging demand dampens the myopic allocation of risk-tolerant arbitrageurs ($\gamma < 1$), and amplifies the myopic allocation of risk-averse arbitrageurs ($\gamma > 1$). Since the myopic allocation is horizon invariant, the plots indicate that the magnitude of this effect increases with the arbitrageurs' horizon. Each plot also examines the magnitude of the intertemporal hedging demand as a function of the rate of mean reversion of the arbitrage opportunity, measured using the half-life of its shocks ($\lambda_{1/2}$). The decline in the shock half-life corresponds to the presence of more mean-reversion and induces greater covariation between the arbitrageur's marginal utility and the investment opportunity set. Consequently, the magnitude of the intertemporal hedging demand increases as the shock half-life declines.

Finally, having examined the comparative statics of the intertemporal hedging demand, we would like to assess its relative importance in the arbitrageur's total allocation to the arbitrage opportunity. To simplify this analysis, we specialize once again to the case where the long-run mean of the spread is zero, yielding $B(\tau) = 0$ for all τ. This allows us to write the fraction of the total demand, N, due to the myopic term as,

$$\frac{N_{myo}}{N_{total}} = \frac{-\frac{\kappa+r}{\gamma\sigma^2}}{-\frac{\kappa+r}{\gamma\sigma^2} + \frac{2A(\tau)}{\gamma}} = \left(1 - \frac{2\sigma^2}{\kappa + r} \cdot A(\tau)\right)^{-1} \tag{12}$$

So long as the sign of the intertemporal hedging term is the same as the sign of the myopic term, the above quantity can be interpreted as a fraction of total demand. This condition is met when $\gamma > 1$, which is the maintained assumption of the ensuing approximations. In other areas of the parameter space the hedging term may be positive and/or divergent in magnitude. When $\gamma > 1$, $A(\tau)$ is non-positive and converges to zero as $\tau \to 0$, which indicates that the fraction of total demand due to the myopic term is decreasing in τ and attains unity when $\tau = 0$, as expected. The lower bound on the fraction of total demand due to the myopic term is attained as $\tau \to \infty$, i.e. as the time to the terminal date becomes infinite.[20]

To derive a lower bound on the fraction of total demand due to the myopic term we note that for all reasonably mean-reverting processes, $\kappa \gg r$. To see this, suppose that the annualized value of the riskless rate, r, is set to its sample average over the period 1/1962-1/2006 of 6.28%. In order for κ to be equal to the daily value of the riskless rate implied by the above calibration, the half-life of shocks to the OU process would have to roughly be equal to 11 years.[21] With this half-life, however, it would be essentially impossible to distinguish a mean-reverting process from a random walk given a plausible value for the standard deviation of the shocks. For values of γ such

[19] The spread, S, is fixed at a value of $\frac{\sigma}{\sqrt{2\kappa}}$, i.e. one standard deviation above its long run mean of zero, under the stationary distribution. The remaining parameters of the OU process were chosen to closely match those found in the empirical calibrations of Section IV.

[20] The existence of a finite bound is guaranteed by the fact that $A(\tau)$ is an affine transformation of the hyperbolic tangent.

[21] Setting $\kappa = \frac{0.0626}{365}$, the shock half-life is given by: $\lambda = \frac{\ln 2}{\kappa} \approx 4026$ days

that $\frac{\kappa}{\gamma} \gg r$, a simple manipulation shows that the lower bound of $A(\tau)$ is approximately equal to $\frac{\kappa}{2\sigma^2}(1-\sqrt{\gamma})$. Substituting this expression into (12) we find that:

$$\lim_{\tau \to \infty} \frac{N_{myo}}{N_{total}} \approx \left(1 - \frac{\kappa}{\kappa + r}(1-\sqrt{\gamma})\right)^{-1} \approx \frac{1}{\sqrt{\gamma}} \tag{13}$$

We can therefore conclude that, as a function of τ, the fraction of total demand for the arbitrage opportunity due to the intertemporal hedging term varies in the interval $\left[0, 1 - \frac{1}{\sqrt{\gamma}}\right]$, with the myopic demand accounting for the balance. This limiting result suggests that, contrary to Rashes's (2000) conjecture, the hedging component plays an important role in the total risky asset holding. This assertion is further validated by the empirical exercises in Section IV where we decompose the total risky asset demand into its two constituents.

B The Wealth Process

Having solved for the optimal consumption and portfolio policy functions in closed form, we next explore the equilibrium distribution of returns on wealth generated by trading a mean-reverting spread. Along the optimal path derived in the previous section, wealth evolves according to the following stochastic differential equation:

$$\begin{aligned} \frac{dW}{W} &= \left(1 - \frac{NS}{W}\right) r dt - \beta \mathbf{1}[C > 0] dt + \frac{N}{W} dS \\ &= \left((1 - \psi(S,\tau)S)r - \beta \mathbf{1}[C > 0]\right) dt + \psi(S,\tau) dS \end{aligned} \tag{14}$$

where:

$$\psi(S,\tau) = \left(\frac{\kappa(\overline{S} - S) - rS}{\gamma\sigma^2} + \frac{2A(\tau)S + B(\tau)}{\gamma}\right) \tag{15}$$

The function $\psi(S,\tau)$ measures the instantaneous volatility of returns on wealth and can be naturally interpreted as the partial effect of an infinitesimal change in wealth on the optimal holding of the spread asset, $\frac{\partial N}{\partial W}$. Because $\psi(\cdot)$ depends on both the magnitude of the spread, S, and the time remaining to the terminal date, τ, it is easy to see that the volatility of the instantaneous return on wealth will be time-varying. A simple application of Ito's Lemma indicates that the volatility of the instantaneous return on wealth, σ_W, satisfies,

$$d\sigma_W = \frac{\partial \psi(S,\tau)}{\partial S} dS - \left(\frac{\partial \psi(S,\tau)}{\partial \tau} - \frac{\sigma^2}{2} \cdot \frac{\partial^2 \psi(S,\tau)}{\partial S^2}\right) dt \tag{16}$$

By examining the evolution of the instantaneous correlation between this process and the wealth process, we are able to deduce some of the key qualitative features of the unconditional return distribution.

Since there is only one Brownian diffusion in our model, the processes governing the evolution of wealth and its instantaneous volatility will be perfectly correlated. The sign of the instantaneous correlation, $\rho(S,\tau)$, however, will vary over time and be determined by the sign of the product of

$\psi(S,\tau)$ and $\phi(\tau) \equiv \frac{\partial \psi(S,\tau)}{\partial S}$, where:

$$\phi(\tau) \equiv \frac{\partial \psi(S,\tau)}{\partial S} = \frac{1}{W}\frac{\partial N}{\partial S} = \frac{\partial^2 N}{\partial S \partial W} = \left(\frac{2A(\tau)}{\gamma} - \frac{\kappa + r}{\gamma\sigma^2}\right) \tag{17}$$

To see that $\phi(\tau) < 0$, it is sufficient to note that when $\overline{S} = 0$, the optimal allocation to the mean-reverting mispricing, (11), can be truncated to:

$$N(S,\tau) = \begin{cases} -\left(\frac{\kappa+r}{\sigma^2}\right) SW & \gamma = 1 \\ -\left(\frac{\kappa+r}{\gamma\sigma^2} - \frac{2A(\tau)}{\gamma}\right) SW \equiv \phi(\tau) \cdot SW & \gamma \neq 1 \end{cases} \tag{18}$$

Holding the agent's wealth, W, and the magnitude of the spread, S, constant one would intuitively expect a positive (negative) spread to result in a negative (positive) allocation to the spread asset - as indeed is the case when $\phi(\tau) < 0$. This result also follows immediately when $\gamma > 1$, since this condition is sufficient to ensure that $A(\tau) \leq 0$. The sign of $\psi(S,\tau)$, on the other hand, is time-varying, which is also easiest seen by specializing to the case of $\overline{S} = 0$, which allows us to set $B(\tau) = 0$:

$$\psi(S,\tau)|_{\overline{S}=0} = \left(\frac{2A(\tau)}{\gamma} - \frac{\kappa + r}{\gamma\sigma^2}\right) \cdot S = \phi(\tau) \cdot S \tag{19}$$

In this case, the instantaneous correlation between $\frac{dW}{W}$ and $d\sigma_W$, denoted by $\rho(S,\tau)$, will alternate in sign:

$$\rho(S,\tau) = \frac{\psi(S,\tau) \cdot \left(\sigma \cdot \frac{\partial \psi(S,\tau)}{\partial S}\right)}{\sqrt{\psi(S,\tau)^2 \cdot \left(\sigma \cdot \frac{\partial \psi(S,\tau)}{\partial S}\right)^2}} = \frac{S}{|S|} \tag{20}$$

and will be positive (negative) when S is above (below) its long-run mean of zero. Because the sign of $\rho(S,\tau)$ is equally likely to be positive or negative when $\overline{S} = 0$, the unconditional distribution of returns on wealth will be an equally-weighted mixture of two distributions - one with negative skewness and one with positive skewness. In turn, it is reasonable to expect that the unconditional distribution will have heavy tails, but no skewness, when $\overline{S} = 0$.

The situation is somewhat more complicated in the case where $\overline{S} \neq 0$. Here it is convenient to rewrite the product of $\phi(\tau)$ and $\psi(S,\tau)$, representing the covariance of the instantaneous return and variance processes, as follows:

$$\phi(\tau) \cdot \psi(S,\tau) = \phi(\tau)^2 \cdot S + \phi(\tau) \cdot \left(\frac{\kappa\overline{S} + \sigma^2 B(\tau)}{\gamma\sigma^2}\right) \tag{21}$$

The first term is the same as before and is equally likely to be positive or negative; it only generates excess kurtosis in the unconditional distribution of returns. The sign of the second term will depend on the signs of $B(\tau)$ and $\overline{S}$, and will generally spend an unequal amount of time being positive and negative, inducing skewness in the realized returns. Consequently, whenever $\overline{S}$ is different from zero, it is reasonable to expect that the time-series of returns delivered by a strategy seeking to exploit a mean-reverting mispricing will be highly non-normal.

A related question concerns the characteristics of the distribution of long-horizon returns, i.e. the distribution of realized returns that an arbitrageur with utility defined over a finite horizon can expect to see from repeated investment in mean-reverting arbitrages. To explore this issue we conduct a Monte Carlo simulation in which we generate 25,000 paths of a realistically calibrated OU process spanning 250 days and examine the distribution of the time T returns generated as a result of applying the optimal investment rule derived in the previous section. We find that long-horizon returns are highly variable, with a standard deviation of 27.17%, and exhibit significant kurtosis (Figure 2). Moreover, the strategy generates a loss in 2.69% of the simulations and underperforms investing in the riskless asset along 3.84% of the simulated paths. In our simulation the OU shock half-life is set to 80 days, confirming the intuition that, when the time remaining to the consumption date is comparable to the shock half-life, attempts to exploit a mean-reverting mispricing may underperform other, less-risky strategies.

C Do Arbitrageurs Always Arbitrage?

Next we analyze the comparative statics of the optimal allocation to the spread asset. In particular, we are interested in determining the direction in which an arbitrageur trades in response to a shock to the value of the spread asset. If an arbitrageur increases his position in the spread asset in response to an adverse shock, his trading is likely to have a stabilizing effect on the mispricing, contributing to its elimination in equilibrium. Conversely, if the arbitrageur decreases his position in response to the adverse shock, his trading will tend to exacerbate the mispricing. Indeed, the solution to Xiong's (2001) equilibrium model of arbitrage trading indicates that sometimes arbitrageurs do not arbitrage. He finds that, if the mispricing is sufficiently wide, a divergence in the mispricing can result in the decline of the total allocation, as the wealth effect dominates the improvement in the investment opportunity set. However, because his solution was based on numerical methods, he was unable to characterize the conditions under which arbitrageurs cease to trade against the mispricing. The tractability of our partial equilibrium model, on the other hand, allows us to derive precise, analytical conditions for the time-varying envelope within which arbitrageurs trade against the mispricing. An analysis of this result yields novel insights into the situations under the law of one price is most likely to be violated.

To derive the boundary of the stabilization region, we begin with the optimal policy function. Equation (11) indicates that the allocation to the arbitrage opportunity, $N(S, W, \tau)$ depends on the product of the spread, S, and the agent's wealth, W. Hence, although an adverse shock increases the magnitude of the deviation of the spread from its long-run mean and makes the investment opportunity more attractive, it has a countervailing effect on the arbitrageur's wealth. To determine which of these effects dominates we use Ito's Lemma to derive the process for $dN(S, W, \tau)$ and examine the loading of the process on the spread innovations, dS,

$$\begin{aligned} dN(S,W,\tau) &= \frac{\partial N}{\partial S}dS + \frac{1}{2}\frac{\partial^2 N}{\partial S^2}(dS)^2 + \frac{\partial N}{\partial W}dW + \frac{1}{2}\frac{\partial^2 N}{\partial W^2}(dW)^2 + \frac{\partial^2 N}{\partial S \partial W}(dS \cdot dW) - \frac{\partial N}{\partial \tau}dt \\ &= \left(\left(\phi(\tau)\sigma^2 + r\left(1 - \frac{NS}{W}\right) - \frac{C}{W}\mathbf{1}\left[C > 0\right]\right)N - \left(\frac{2A'(\tau)S + B'(\tau)}{\gamma}\right)\right)dt + \\ &\quad + \left(\phi(\tau) + \left(\frac{N}{W}\right)^2\right)W\,dS \end{aligned} \tag{22}$$

The process for $dN(S,W,\tau)$ consists of two components: a deterministic component reflecting the horizon effect, and a stochastic component reflecting responses to shocks in the value of the spread asset. Although this separation is not exact, since the dS process itself has a deterministic component, this form is most convenient for developing intuition for the agent's responses to changes in the mispricing.[22]

Since the arbitrageur's wealth is strictly positive, the direction of the response to a change in the magnitude of the spread, dS, is governed by the sign of $\phi(\tau)+\left(\frac{N}{W}\right)^2$. In the previous section, we showed that the first term in the sum, $\phi(\tau)$, is strictly negative, implying that the sign of the factor controlling the direction of the agent's response to shocks to the spread asset will be time-varying and dependent on the magnitude of S, through $\frac{N}{W}$. To see this most clearly, let us specialize to the case of $\overline{S} = 0$ and consider the circumstances under which the arbitrageur responds in a stabilizing manner to changes in the magnitude of the spread. When $S < 0$ (and $N > 0$) the arbitrageur's response is considered to be stabilizing if a negative shock to the spread ($dS < 0$) leads him to increase his long position ($dN > 0$). Symmetrically, when $S > 0$ (and $N < 0$) the arbitrageur's response is stabilizing when a divergence in the spread ($dS > 0$) leads him to increase his short position ($dN < 0$). In other words, in order for the agent's trading to be instantaneously stabilizing we must have, $dN \cdot dS < 0$, or equivalently,

$$\phi(\tau) + \left(\frac{N}{W}\right)^2 = \phi(\tau) + \phi(\tau)^2 S^2 < 0 \tag{23}$$

Therefore, the arbitrageur's trading only helps to eliminate the mispricing when its magnitude is not too large, and satisfies,

$$|S| < \sqrt{-\frac{1}{\phi(\tau)}} = \sqrt{\frac{\gamma\sigma^2}{(\kappa + r) - 2\sigma^2 A(\tau)}} \tag{24}$$

When the magnitude of S exceeds the critical value given by $\sqrt{-\frac{1}{\phi(\tau)}}$ the arbitrageur's trading becomes destabilizing, in that he attempts to decrease his position precisely as the mispricing worsens. The *stabilization region* is bounded by a time-varying envelope determined by $\phi(\tau)$. Because $A(\tau)$ is non-positive (non-negative) and monotonically increasing (decreasing) towards

[22] In principle one could substitute out the dS process and consider the response of the allocation to shocks in dZ.

zero, the stabilization region becomes progressively looser (tighter) as $\tau \to 0$ for agents with $\gamma > 1$ ($\gamma < 1$). The critical value is plotted as a function of τ in Figure 3 for two values of risk aversion and two mean-reversion rates. To emphasize the width of the stabilization region, the critical value is expressed as a multiple of the spread's standard deviation under the stationary distribution (i.e. a Z-score). The modest values of the Z-scores in the figure immediately indicate that destabilizing behavior is not an extremely rare event. For example, for an arbitrageur with a risk aversion coefficient of two, the mean magnitude of the stabilization bound is roughly 1.75, which implies that at any given point in time, the probability that the arbitrageur will act in a destabilizing manner is roughly 8%. Destabilizing behavior is also more likely to be observed for arbitrageurs who are closer to risk-neutral ($\gamma \to 0$), particularly when the time remaining to the evaluation date is short. As one would expect, the stabilization region is independent of the agent's instantaneous wealth due to the homogeneity of the optimization problem in W. The generalized version of these results, for an arbitrary value of $\overline{S}$, is presented in the following theorem.

Theorem 3 (Stabilization Region) *In the general case when $\overline{S} \neq 0$ the range of values of S for which the arbitrageur's response to an adverse shock is stabilizing - i.e. the agent trades against the spread, increasing his position as the spread widens - is determined by a time-varying envelope determined by both $A(\tau)$ and $B(\tau)$. The boundary of the stabilization region is determined by the following inequality:*

$$\left| \phi(\tau)S + \frac{\kappa\overline{S} + \sigma^2 B(\tau)}{\gamma\sigma^2} \right| < \sqrt{-\phi(\tau)} \tag{25}$$

As long as the spread is within the stabilization region, the improvement in investment opportunities from a divergence of the spread away from its long-run mean outweighs the negative wealth effect and the arbitrageur increases his position, N, in the mean-reverting asset. When the spread is outside of the stabilization region, the wealth effect dominates, leading the agent to curb his position despite an improvement in investment opportunities. Our finding of a compact stabilization region has novel implications for the comparative analysis of the optimality of closed- vs. open-end structures for carrying out arbitrage activities (Stein (2005)). Although closed-end funds are isolated from the types of client fund withdrawals that impose a serious limit to arbitrage on open-end investment vehicles, their optimal strategy may involve functionally similar behavior, including liquidating poorly performing arbitrage trades. Furthermore, we find that this result is robust to the addition of flexibly specified fund flows.

III Extensions

Our model of trading in mean-reverting spreads and its partial equilibrium implications for understanding the potentially destabilizing effects rational arbitrageurs can have on asset markets allows for two immediate extensions. The first – along the lines of the portfolio choice literature – concerns the inclusion of multiple spreads; and the second – along the lines of the "limits to arbitrage" literature – concerns the inclusion of exogenous fund flows in the spirit of Shleifer and Vishny (1997).

A Multiple Spreads

The model presented in Section I assumed the existence of only one mean-reverting spread asset. Clearly, this assumption is restrictive in that a robust spread construction methodology used to construct the first asset is likely to be used elsewhere, if only to assure greater diversification. It turns out that we can extend the analytical form of our solution to the case with N ($N > 1$) spreads in a few stylized scenarios. The first of these scenarios is the case of N uncorrelated spreads. In this case, the lack of cross-sectional correlations eliminates all the cross-partial effects in the optimization problem, yielding a Bellman equation which is separable in the state variables (i.e. spreads). In turn, the value function satisfying the optimality condition is also separable, and takes on the form:

$$V(S_1 \dots S_N, W, \tau) = e^{-\beta\tau} \cdot \frac{W_T^{1-\gamma}}{1-\gamma} \cdot \prod_{i=1}^{N} \exp\left(A_i(\tau)S_i^2 + B_i(\tau)S_i + C_i(\tau)\right) \tag{26}$$

The form of the value function also implies that the policy functions determining the optimal holding of each spread will be independent of all other spreads. This is quite intuitive – the statistical independence of the spreads effectively allows the arbitrageur to consider them separately after accounting for their individual volatilities.[23] Analogously, the introduction of other non-spread assets such as stocks and bonds does not affect optimal investment in the spread as long as their returns processes are uncorrelated with the spread process.

If the spreads are correlated, however, the problem becomes intractable except in the special case where the mean reversion parameters κ_i are identical for all of the spreads i. The tractability of this solution is owed to the fact that any linear combination of OU processes with identical mean-reversion coefficients is itself a mean-reverting process. This feature effectively allows us to diagonalize the covariance structure of the problem without sacrificing the OU nature of the underlying assets, as would be the case if the κ_i varied across the spreads. The resulting, diagonalized system is very similar to the case of the N uncorrelated spreads, with the important modification that the drift on any given spread now depends on the deviation of all N spreads from their long-run means.

B Fund Flows and Liquidity Crunches

Another interesting model extension concerns the inclusion of fund flows. As Shleifer and Vishny (1997) point out in their seminal paper, delegated managers are not only exposed to the financial fluctuations of asset prices but also to their clients' desires to contribute or withdraw

[23] The derivation of this solution is algebraically tedious and is omitted from the paper, but is available from the authors upon request.

funds. Paradoxically, clients are most likely to withdraw funds after performance has been poor (i.e. spreads have been widening) and investment opportunities are the best. There is substantial empirical evidence that investors chase performance when selecting mutual funds (Ippolito (1992), Chevalier and Ellison (1997), Sirri and Tufano (1998)), and it seems reasonable to assume they do the same for hedge funds or other investment vehicles typically associated with arbitrage activities (Agarwal, Daniel and Naik (2004)).

Although fund flows have been the subject of extensive empirical analysis, relatively little is known about their statistical nature, aside from their positive correlation with recent performance. We therefore choose to model the fund flow process, F_t, as being comprised of two components - a component which is directly proportional to lagged performance and a component which is uncorrelated with lagged performance.[24] Aside from this assumption, and the associated modification to the wealth process, we maintain the model setup presented in Section I without any change.

In the presence of fund flows the evolution of wealth under management will depend not only on performance, denoted by Π_t, but also on fund flows, F_t. We therefore have:

$$\begin{aligned} d\Pi &= \tilde{N}dS + (W - \tilde{N}S)rdt && (27) \\ dF &= fd\Pi + \sigma_f W dZ_f && (28) \\ dW &= d\Pi + dF = (1+f)d\Pi + \sigma_f W dZ_f && (29) \end{aligned}$$

where $\tilde{N}$ is the optimal policy rule chosen by a fund manager facing fund flows of the type described above, and $E[dZ_f dZ] = 0$. In this setup, the fund flow magnifies the effect of performance on wealth under management, with each dollar in performance generating a fund flow of f dollars, and adds an idiosyncratic wealth shock of $\sigma_f W dZ_f$. The size of the idiosyncratic shocks to wealth due to fund flows is proportional to wealth to assure that wealth under management stays strictly positive. The proposed fund flow model also conveniently captures the possibility of a liquidity crunch, whereby the magnitude of the correlation between the fund flow, F, and realized performance, Π, increases after periods of extreme negative performance. A simple calculation reveals that the instantaneous correlation between the fund flow and performance is given by:

$$\rho_{\Pi,F} = \frac{f\sigma^2\tilde{N}^2}{\sqrt{(\sigma^2\tilde{N}^2)(f^2\sigma^2\tilde{N}^2 + \sigma_f^2 W^2)}} = \frac{f}{\sqrt{f^2 + \frac{\sigma_f^2}{\sigma^2}\cdot\left(\frac{\tilde{N}}{W}\right)^{-2}}} \qquad (30)$$

Aside from having the desirable properties of being strictly positive and scale-independent in wealth, the correlation is increasing in the magnitude of $\frac{\tilde{N}}{W}$, which becomes large outside of the stabilization region. Consequently, the correlation between fund flows and performance increases as the spread diverges from its long-run mean. A fund that has experienced losses due to the divergence of the mispricing is also likely to experience outflows from client withdrawals. Our model therefore

[24] We have also explored an alternative formulation of the fund flows in which the free parameters of the model are the volatility of fund flows and its correlation of with performance. A full, analytical characterization of the solution to that problem is available from the authors upon request.

parsimoniously captures the possibility of a liquidity crunch in which, after sustaining extreme losses, old investors exit the market when the investment opportunity set is most attractive. The model, however, also has the symmetric and somewhat less appealing implication that when the mispricing is at its worst, small corrections tend to induce positive fund flows. This ancillary implication is to be expected given the symmetric structure of the model.

Using a solution strategy analogous to the one used in Section I, we can characterize the optimal portfolio rule for an investor with isoelastic preferences over wealth at a future date, T. We present this result as the next theorem.

Theorem 4 (Optimal Portfolio Strategy for Terminal Wealth Problem with Fund Flows) *The optimal portfolio allocation of an agent with constant relative risk aversion with utility defined over terminal wealth,* (1)*, in the presence of fund flows is given by:*

$$\tilde{N}(S,\tau) = \left(\frac{1}{1+f}\right) \cdot N(S,\tau) \tag{31}$$

where $N(S,\tau)$ is the optimal policy function in the problem without fund flows. The value function depends on the time remaining to the horizon, and the parameters of the underlying model, including the proportionality coefficient f and the volatility of the idiosyncratic fund flow component σ_f. It is given in closed form in Appendix C.

The intuition behind this elegant solution is simple. The performance-chasing component of fund flows increases the volatility of wealth by a factor of $(1+f)$, causing a manager who anticipates this flow to commensurately decrease the amount of risk taken on by the underlying strategy. Unsurprisingly, the idiosyncratic component of fund flows does not affect the functional form of the optimal policy rule.

IV Robustness and Empirical Applications

Having solved for the trading rule which optimally exploits mean-reversion in prices, we are in position to examine its robustness and efficacy. First, we introduce our methodology for constructing mean-reverting price spreads and describe some potential alternatives. Next, using simulations, we compare the performance of the optimal strategy to the threshold rule proposed by Gatev, Goetzmann and Rouwenhorst (2006), and verify the robustness of our strategy to estimation risk in the rate of mean-reversion, κ. Finally, we examine the performance of the optimal portfolio policy in a relative value trade involving the Siamese twin shares of Unilever PLC and Unilever NV. Although a more detailed empirical investigation is beyond the scope of this paper, our strategy can be applied in the trading of the closed-end fund discount (Gasbarro, Johnson and Zumwalt (2003)), yield curve butterflies, and the swap and credit spreads, all of which exhibit mean-reverting tendencies.

A Spread Construction

The key input for constructing the spread asset are the security-specific total return indices, constructed by compounding the daily total returns of each asset. Each of these series is normalized to have an initial level equal to 1. For example, the total return index for asset i is given by:

$$P_{i,t} = \left(\frac{1}{P_{i,1}}\right) \cdot \left(P_{i,1} \cdot \prod_{j=1}^{t-1}(1 + R_{i,j+1})\right) \tag{32}$$

Total return indices account for intermediate dividend flows and splits, and yield the price series of a portfolio initially comprised of $\frac{1}{P_{i,1}}$ shares of asset i. The price spread is then constructed by taking a linear combination of the total return indices. Although we deduce the vector of linear weights on the basis of the law of one price, one could also estimate it via a cointegrating regression (Engle and Granger (1987)). For example, since the equities of Royal Dutch and Shell represent claims on identical cash flows, the law of one price predicts that the difference in their total return indices should exactly equal zero. In the presence of market frictions and potential inefficiencies, however, it seems sensible to expect that their difference, if not always zero, will mean-revert around this level. A similar approach drives the construction of stub trades in Mitchell, Pulvino and Stafford (2002). Consequently, the law of one price allows us to not only identify potentially interesting pairs, but also pre-specifies the relevant weighting vector on the total return indices.[25]

After constructing the proposed mean-reverting spread process, we test it for stationarity by estimating whether its rate of mean-reversion is statistically different from zero. Since the conditional distribution of the OU process is known in closed form, it is straightforward to obtain closed-form solutions for the maximum-likelihood (ML) estimators for the long-run mean ($\hat{\bar{S}}$), rate of mean reversion ($\hat{\kappa}$) and instantaneous volatility ($\hat{\sigma}$). We present these formulas and their derivation in Appendix E. In order to address the presence of non-standard sampling distributions and avoid complications arising from finite samples, we test the null hypothesis of no mean reversion by running a Monte Carlo bootstrap experiment. In particular, we simulate a large set of paths for a white noise process with instantaneous volatility $\hat{\sigma}$ and length equal to our sample size. For each of these paths we compute the values of our estimators and construct confidence sets of desired size. Whenever our estimate of $\hat{\kappa}$ is outside of this confidence set, we can reject the null of a white noise process and assume that the data conforms to our proposed model of an arbitrage opportunity.

B Comparison with Threshold Rules in Simulation

Market efficiency and the law of one price can be tested by examining the trading profits to long-short strategies exploiting deviations in the pricing of close substitutes. In the presence of violations of the law of one price, one should be able to identify strategies that lead to statistically significant (risk-adjusted) profits. The magnitude of these profits, however, is necessarily a function

[25] Gatev, Goetzmann and Rouwenhorst (2006) adopt a purely statistical approach and identify "pairs" by minimizing the sum of squared deviations between pairs of total return indices during a twelve-month formation period.

of the adopted portfolio construction rule. A number of authors have relied on simple threshold rules, which call for establishing a long-short position in a pair of securities when the difference in their prices reaches some pre-specified level. For example, Gatev, Goetzmann and Rouwenhorst (2006) and Mitchell, Pulvino and Stafford (2002) rely on threshold rules to evaluate the profits of pairs trading and stub trading strategies.

Having derived the optimal dynamic trading strategy for exploiting a mean-reverting spread, we can now compare its performance relative to the benchmark provided by the threshold rule. In particular, we choose to focus on the threshold rule examined in Gatev, Goetzmann and Rouwenhorst (2006), henceforth GGR. The threshold rule proposed by GGR calls for opening a long-short position when the spread between the total return indices of two securities diverges by "two historical standard deviations, as estimated during the pairs formation period." The choice of the threshold is made arbitrarily and the authors make no claims regarding its optimality. Our analysis should therefore be interpreted as merely an illustrative comparison of a sensible threshold rule relative to its optimal dynamic counterpart.

Because the GGR threshold rule does not provide any guidance regarding the number of units, N_{GGR}, of the spread asset that should be purchased at the time the position is established, we will simply set N_{GGR} at the optimal spread position prescribed for a log utility investor. In turn, our comparison examines the realized returns and welfare consequences of adopting the two rules for an investor with log utility defined over terminal wealth. Comparing the threshold rule to the optimal rule of the log utility investor has the added attraction that it allows us to eliminate the horizon-dependent intertemporal hedging demands present in the optimal strategy when $\gamma \neq 1$.

We compare the performance of the two rules using simulated data. We generate 50,000 paths for the spread, assuming that it follows an Ornstein-Uhlenbeck process with mean-reversion rate κ, volatility of instantaneous innovations σ and a long-run mean, $\overline{S}$, equal to zero.[26] The optimal dynamic strategy is given by (11), and the GGR strategy is assumed to open a position in the spread, when it deviates by more than two standard deviations of the stationary distribution, $\sqrt{\frac{\sigma^2}{2\kappa}}$, from its long-run mean. Because of this requirement, along some paths the GGR strategy will not be invested in the spread at all. Consequently, we keep track of the fraction of paths along which the threshold rule is invested for at least one period, I_{GGR}, and for these paths, we compute the mean (median) number of days with a non-zero position in the spread, $\overline{\tau}_{GGR}$. Both strategies are marked-to-market each period, yielding a time-series of returns along each path, which can be used to compute a realized Sharpe ratio. We report the across-path mean of the realized Sharpe ratios for the GGR strategy, SR_{GGR}, and the optimal strategy, SR_{OPT}. All simulations and computations are carried out under the assumption that the riskless rate, r, is fixed at 5% *per annum*.

Unlike the optimal dynamic strategy, which guarantees that wealth remains positive, the threshold rule is exposed to a risk of bankruptcy, which does not vanish in the limit of continuous time, because the strategy does not adjust the spread allocation in response to changes in leverage. It is therefore also useful to examine the fraction of paths along which adoption of the threshold rule

[26] The first observations for each path is drawn randomly from the corresponding stationary distribution.

results in bankruptcy. Moreover, using paths not resulting in bankruptcy, we can compute terminal wealths, evaluate the expected utilities and back out the welfare loss, ω, from the adoption of the threshold rule relative to the optimal strategy. It is important to note, however, that this welfare loss statistic is computed using paths not resulting in bankruptcy, which imply infinitely negative utility, and consequently, understates the true magnitude of the welfare loss. We collect our results in Table I.[27]

We find that in the presence of very weak mean-reversion – when the half-life of the innovations exceeds the length of the trading period – relying on the GGR threshold rule rarely results in bankruptcy (0.44% of all trials) and leads to a very small welfare loss relative to the optimal strategy (-1.99%). The GGR strategy even delivers a superior Sharpe ratio. These results, however, are driven by the fact that, with a low rate of mean-reversion, the GGR strategy is rarely invested in the spread (only 9.51% of all trials), and when it is, the allocation – as implied by (11) – is extremely small.[28] As the mean-reversion rate increases, a progressively greater fraction of paths, I_{GGR}, involve opening a position in the spread under the GGR strategy, and the mean-number of days with an open position, $\overline{\tau}_{GGR}$, declines. Moreover, both strategies become more aggressive in their allocation to the spread, exposing the drawbacks implicit in the lack of dynamic position adjustment under the GGR threshold rule. The default rate for the strategy based on the GGR threshold rule increases dramatically, rising from 2.3% ($\kappa = 0.005$) to 11.9% ($\kappa = 0.05$), and the realized Sharpe ratio drops significantly below the Sharpe ratio for the optimal strategy. In welfare space, the comparison is even more stark as the welfare loss increases from -9.5% ($\kappa = 0.005$) to -56.1% ($\kappa = 0.05$). We therefore conclude that the GGR threshold rule is a sensible alternative to the optimal strategy, delivering comparable (and sometimes better) Sharpe ratios and low default rates, when the price spread is only weakly mean-reverting. On the other hand, with high rates of mean reversion, the lack of risk control in the threshold rule exposes it to the risk of bankruptcy and prevents it from fully capturing the profits from mean-reversion.

C Robustness to Estimation Risk

Throughout the paper we have assumed that agent has perfect knowledge of the parameters of the Ornstein-Uhlenbeck process describing the arbitrage opportunity. In reality, however, these parameters must be estimated, introducing parameter uncertainty into the problem.[29] Although deriving the optimal strategy in the presence of estimation risk is beyond the scope of this paper, we present some simulation results, which suggest that estimation risk may significantly attenuate the optimal position in the mean-reverting spread. Specifically, we examine the welfare loss and deterioration in the Sharpe ratio when the agent implements the optimal strategy using estimated

[27] We do not examine the comparative statics in the volatility of innovations to the spread process, because with $\gamma = 1$ and $\overline{S} = 0$, both strategies are invariant to the re-scaling of σ.

[28] On the other hand, the slow mean-reversion also implies that conditional on having a position open for at least one day, the position will remain open for a relatively long period of time - an average of 99 of the 126 days.

[29] A separate dimension of uncertainty is introduced by the validity of the assumption that the arbitrage opportunity is adequately described by an Ornstein-Uhlenbeck process. An agent concerned about model uncertainty would also have to consider other alternative specifications.

parameter values instead of their true values. We foreshadow the empirical application of the next section by considering estimated parameter values and errors, which are comparable to the estimates and standard errors obtained from empirical data.

We consider two values for each of the parameters of the OU process, and examine the performance of the optimal dynamic trading strategy of a log utility ($\gamma = 1$) investor using estimated values instead of true values. Our results are based on simulations comprised of 50,000 randomly generated paths, each with a length of 252 days, corresponding to one business year, and are collected in Table II. Consistent with intuition, we find that an upward (downward) bias in the estimated value of the mean-reversion (volatility) coefficient leads the agent to trade the mispricing too aggressively relative to the strategy prescribed under the true parameter values. Consequently, a significant fraction of trials result in bankruptcy, and the realized Sharpe ratio of the strategy deteriorates significantly. However, because the welfare loss statistic, ω, is computed conditional on not having defaulted, its value ends up being positive, indicating a gain from using incorrect parameter values. This simply reflects the extreme positive bias of this statistic in a setting where the agent is overly aggressive. Conditional on surviving, the agent's performance will tend to be impressive. Of course, the probability of survival is significantly diminished relative to using the correct parameter values. A 50% upward bias in the estimate of κ relative to the population value ($\kappa \in \{0.005, 0.01\}$) causes roughly 20-30% of all trials to end in bankruptcy. In the case of volatility, a 50% downward bias in the estimate of σ increases the fraction of runs ending in bankruptcy to 58%. Errors in the other direction – a downward bias in the rate of mean-reversion or an upward bias in the volatility of instantaneous innovations to the mean-reverting spread – are more sparing. They lead to less aggressive trading, and consequently, lower values of terminal wealth, as reflected by the modest negative values of ω, but they also shelter the agent from going bankrupt. In fact, under the more conservative parameter values, none of the trials result in bankruptcy. Moreover, the realized Sharpe ratios experience only modest declines relative to those attainable under the true parameter values.

As expected, errors in the estimate of the mean spread, $\overline{S}$, are extremely damaging to the performance of the strategy. The last two entries in Table II examine the effect of using a parameter estimate for the long-run mean which is one or two standard deviations (of the stationary distribution) away from the true value. We find that the incidence of default increases from zero under the true value of the long-run mean to 44% and 72%, respectively, under the two erroneous values, and is accompanied by a sharp decline in the realized Sharpe ratio.

These observations suggest that obtaining precise estimates of the parameters of the Ornstein-Uhlenbeck process describing the arbitrage opportunity will be crucial in implementing the optimal strategy. Errors in estimation that bias the strategy in favor of adopting more aggressive positions – an excessively high rate of mean-reversion, κ, or an excessively low value of the instantaneous risk, σ – are likely to result in bankruptcy. Consequently, in the empirical application presented in the next section, we seek to minimize estimation error by using a long, ten-year window to estimate the parameters of the OU process used in the trading strategy.

D Trading Siamese Twins

Siamese Twins, as discussed in Froot and Dabora (1999) and Lamont and Thaler (2003), are "firms that for historical reasons have two types of shares with fixed claims on the cash flows and assets of the firm." Given this contractual arrangement, the market capitalizations of these two share classes should be fixed at a ratio equal to the ratio in which cash flows are apportioned. It has long been known, however, that one observes significant deviations from parity implied by the law of one price. In this section we construct long-short portfolios of the two share classes, fit an OU process to the spread in the total return indices and examine the gains from applying the optimal trading strategy derived in Section I of the paper. In particular, we consider the Royal Dutch / Shell Transport and the Unilever PLC / Unilever NV pairs.[30] Daily price, dividend and total return data are obtained from CRSP and span the period from July 1962 through September 2004 for Royal Dutch and Shell and through December 2006 for Unilever. In our simulations we use a subset of this data, starting in January 1980, to avoid problems due to differential dividend policies previously reported by Froot and Dabora (1999).

To conduct the trading simulation we use a rolling 2500-day (10 year) window to estimate the parameters of the OU process and then use the resulting point estimates to trade the spread over the next 250-days.[31] Within each 2750-day epoch (estimation window + trading window), the spread process is constructed as the difference between the total return indices of the two share classes.[32] Consequently, the spread asset corresponds to a different long-short portfolio in each epoch. We override the estimate of the long-run mean, $\hat{\bar{S}}$, and set it to zero, reflecting the prediction of the law of one price that the spread between the two total return indices should be equal to zero. The annual riskless rate of return is held constant at 5%.

Panel A of Table 3 presents the results of the rolling estimation procedure. We find that in 23 (24) out of the 25 estimation windows we can reject the null of no mean reversion with a p-value smaller than 0.05 (0.10), confirming that the Royal Dutch - Shell price spread is indeed mean-reverting. The estimated rates of mean-reversion, however, vary greatly across the estimation windows and imply half-lives ranging from 45 to 203 days. The average half-life implied by the series of estimates is 85 days, suggesting that the departures of the price spread from its mean can be long-lived. The daily volatility of the spread varies in a much tighter range and is centered around 0.0457 per day.

Using this series of rolling estimates we implement the optimal trading strategy in a sequence of 250-day windows immediately following the 2500-day estimation windows for an arbitrageur with a relative risk aversion coefficient equal to ten ($\gamma = 10$). Our first valid, out-of-sample trading day is the first business day in January 1980; the last trading day corresponds to the end of September 2004.

[30] The two share classes of Royal Dutch and Shell were merged in June 2005.

[31] The parameters of the OU process are estimated via maximum likelihood; details of the estimation procedure are given in Appendix E.

[32] The total return indices are scaled by the mispricing at the beginning of the estimation window, so that if B is trading at a 30% discount relative to A, the first index value for B is 0.70 and the first index for B is 1.00

The results of the trading simulation are illustrated graphically in Figure 4. The top panel graphs the evolution of the spread, S (in blue) and the stabilization bound (in red); the middle panel graphs the total allocation to the spread scaled by wealth $\frac{N}{W}$ (in blue) and the intertemporal hedging demand scaled by wealth (in red); the bottom panel plots the base ten log of wealth accrued throughout the simulation. Although the arbitrageur's wealth grows dramatically over the life of the simulation (bottom panel), there are periods in which he experiences severe losses.[33]

The summary of strategy performance in Panel B of Table 3 indicates a mean annualized arithmetic (geometric) return of 28% (13%). However, the most extreme daily returns themselves stand at -30% and 34%, respectively. Consistent with theory, the distribution of daily returns obtained in the simulation is heavy tailed and has a kurtosis of 14.0. Despite a high annualized volatility (55%), the dynamic strategy achieves a respectable Sharpe ratio of 0.50. For comparison, we also examine the performance of the threshold rule considered in Gatev et al. (2006) that invests a *fixed* amount whenever the spread diverges at least two standard deviations from the mean, and holds the position until convergence. To ensure a fair comparison we set the fixed amount to equal the position of an optimal investor with $\gamma = 10$. Nonetheless, due to a lack of continuous rebalancing under the threshold rule, we find that it leads to arbitrageur to ruin within the first 250-day trading window.

Panel A of Table 3 also gathers the same set of statistics for Unilever. In all 28 estimation windows, the estimated parameter of mean reversion is significant at the 5% level. The mean estimate of 0.0191 for κ suggests a half-life of 48.22 days. Application of the optimal strategy results in an average arithmetic return of 43.45% and an average geometric return of 19.38%. By comparison, the threshold rule delivers an average arithmetic return of 35.27%, but an average geometric return of only 8.02%. Both strategies are highly volatile with the optimal strategy yielding volatility of 71.21% and the threshold rule yielding volatility of 75.12%. Despite their high realized volatility both strategies offer respectable risk-return tradeoffs. In particular, the optimal strategy delivers a Sharpe ratio of 0.61, which roughly represents a 25% improvement over the Sharpe ratio delivered by the threshold rule (0.47).

Figures 5 illustrates of the relative performance of the optimal and threshold rules in the case of the Unilever Siamese twins. The top panel plots the evolution of the spread, and panels two and four plot a scaled version of the arbitrageur's allocation to the spread, N/W, which takes out the effect of growth in wealth. The periods when N/W is flat for the threshold rule correspond to periods after the spread has converged, causing the investor to unwind his position and before the spread has again diverged outside the threshold. The evolution of the arbitrageur's wealth under the optimal and threshold trading rules, plotted in the third and fifth panels, respectively, highlights the disparate performance delivered by the two strategies. The terminal wealth under the optimal strategy exceeds the terminal wealth of the arbitrageur employing the threshold rule by more than a factor of ten, allowing us to conclude that while the realized performance of the

[33] In fact, a less conservative investor, e.g. with $\gamma = 5$, does go bankrupt trading the Royal Dutch-Shell pair. However, this bankruptcy is an artifact of the discretization of data at daily intervals. If the investor were in fact trading continuously, then bankruptcy would never occur.

optimal strategy departs significantly from the textbook notion of a riskless arbitrage, it delivers a reasonably attractive Sharpe ratio, and improves significantly upon the performance of the threshold rule.

V Conclusion

Although arbitrage opportunities, interpreted as deviations from equilibrium pricing, are typically transitory, they present significant risks to those seeking to exploit them. The two key dimensions of risk in trading in mean-reverting arbitrage opportunities are horizon risk and divergence risk. The first risk refers to the fact that the arbitrageur generally cannot be certain about the time at which the mispricing will be eliminated, and the second to the fact that there is a non-trivial risk of the mispricing worsening prior to convergence. To parsimoniously capture these two dimensions of risk, we model arbitrage opportunities using an Ornstein-Uhlenbeck process. We then solve for the optimal, partial-equilibrium trading strategy of a rational arbitrageur who has access to the mispricing and a riskless asset. The arbitrageur's strategy is marked-to-market continuously and is required to be self-financing. Since there are no opportunities for raising additional capital, the model imposes a strong "limits to arbitrage" condition and can be interpreted as applying to the activities of a closed-end fund.

We obtain closed-form characterizations of the policy and value functions for arbitrageurs with constant relative risk aversion preferences over terminal wealth and agents with Epstein-Zin preferences who maximize the utility of intermediate and terminal consumption for the case of unit elasticity of intertemporal substitution. Our results complement the vast empirical literature on seeming violations of market efficiency, and extend the much sparser literature on optimal investment strategies aimed at exploiting these arbitrage opportunities. We find that the optimal policy function and the arbitrageur's value function for the problem can take on a rich array of functional forms depending on the location of the model in the parameter space. In particular, we identify the conditions determining the form of the solution and link them explicitly to the relationship between the agent's risk aversion, the rate of mean-reversion and the prevailing riskless rate. As in Kim and Omberg (1996) we also document the existence of "nirvana solutions." However, we find that the conditions under which these apply are ruled out by empirical calibrations of the model. In general, the value function for the problem is exponential quadratic in the magnitude of the mispricing with coefficient functions depending on the time remaining to the terminal consumption date and other fundamental model parameters. Although we do not investigate the welfare implications of various perturbations to these parameters, such a study is made possible by the availability of a closed-form characterization of the value function.

The analytical form of the policy function allows us to explore in detail the relative importance of intertemporal hedging terms in the total allocation to the mean-reverting arbitrage, and the potential for destabilizing trade by rational arbitrageurs. For empirically plausible values of the risk aversion coefficient in excess of unity, we find that intertemporal hedging demands play an important role in determining the total allocation to the arbitrage opportunity, and account for roughly a $1 - \frac{1}{\sqrt{\gamma}}$ fraction of the total allocation when the time remaining to the consumption date is large. Furthermore, we investigate the changes in the number of units of the arbitrage opportunity optimally held by the arbitrageur in response to shocks to the mispricing. We find that although typically the arbitrageur increases his position in the arbitrage opportunity in response to a divergence in the mispricing, this is not always the case. We analytically characterize a stabilization bound beyond which adverse shocks to the mispricing lead the rational arbitrageur to unwind his position. Consequently, while the trading of arbitrageurs will typically be stabilizing, our partial equilibrium model predicts that in extreme scenarios even arbitrageurs may rationally choose to exit the market, contributing to a worsening of the deviation from equilibrium pricing. This result is generated without recourse to any frictions and simply reflects the arbitrageur's optimal decision to balance his risk exposure.

When the mispricing is highly mean-reverting, our optimal dynamic trading strategy delivers significant improvements in welfare and realized Sharpe ratios relative to a commonly employed threshold rule. This suggests that some caution should be taken in interpreting the findings of market efficiency studies, which base their conclusions on the results of trading strategies employing simple rules of thumb. Conclusions favoring market efficiency may be overturned when the optimal trading strategy is applied. The magnitude of the improvements offered by the optimal strategy, however, declines in the presence of slow mean-reversion and estimation error. Another dimension of interest, which has not been explored in this paper, is the robustness of our strategy to the misspecification of the model describing the stationary mispricing. An extension deriving the optimal dynamic strategy in the presence of model risk remains an interesting target for future research.

Lastly, it is important to re-iterate that our results are derived in a model devoid of institutional frictions. While this serves to highlight the breadth of dynamics that can be generated without adding layers of empirically relevant complications, it has the important drawback that our model provides an incomplete description of reality. For example, we assume full use of short proceeds and exclude borrowing and transaction costs. The trading simulation results presented in the paper should therefore be interpreted as an upper-bound on the potential gains that can be attained from trading in Siamese twins. Nonetheless, they provide a parsimonious illustration of the performance of the optimal strategy derived with the paper and highlight its applicability.

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Table 1: **Comparison of the optimal strategy with the Gatev et al. (2006) threshold rule.** This table compares the performance of the optimal strategy with the threshold rule proposed by Gatev, Goetzmann and Rouwenhorst (2006) using a set of 50,000 randomly generated 126-day paths. The GGR threshold rule calls for opening a position when the spread deviates from its long-run mean by two standard deviations of the stationary distribution; the position is closed when the spread reverts to its long-run mean, or at the investment horizon. To ensure comparability between the two rules we assume that, conditional on opening a position, the GGR investor chooses the same number of units of the spread, N, as the optimal investor with $\gamma = 1$. We set the long-run mean of the spread to zero, $\overline{S} = 0$, the (daily) volatility of innovations to $\sigma = 0.05$, and the riskless rate at 5% *per annum.* λ is the half-life of a shock (in days) implied by the estimate of κ. I_{GGR} reports the fraction of simulated paths along which the threshold rule prescribed a non-zero position for at least one day. $\overline{\tau}_{GGR}$ gives the mean number of days during which the threshold rule prescribed a non-zero position (medians in brackets). d_{GGR} (d_{OPT}) is the fraction of paths along which the GGR (optimal) rule leads to default. SR_{GGR} and SR_{OPT} report the across-path mean of the Sharpe ratios realized by the two strategies. ω is the welfare loss from adopting the GGR threshold rule rather than the optimal rule for an investor with log utility, conditional on no default.

κ	σ	λ	I_{GGR}	$\overline{\tau}_{GGR}$	d_{GGR}	d_{OPT}	SR_{GGR}	SR_{OPT}	ω
0.001	0.05	693.15	9.51%	98 [122]	0.44%	0.00%	1.05	0.78	-1.99%
0.005	0.05	138.63	19.34%	81 [88]	2.30%	0.00%	1.28	1.23	-9.49%
0.010	0.05	69.31	28.34%	68 [66]	3.80%	0.00%	1.37	1.51	-18.41%
0.050	0.05	13.86	65.86%	33 [29]	11.94%	>0.01%	1.06	2.68	-56.09%

Table 2: **Robustness to estimation errors.** This table examines the deterioration in performance in the optimal strategy as a result of errors in estimating the parameters of the Ornstein-Uhlenbeck process for a log ($\gamma = 1$) investor with utility defined over terminal wealth. The true parameters of the Ornstein-Uhlenbeck process are denoted by $\{\kappa, \sigma, \overline{S}\}$, and their estimated counterparts are given by $\{\hat{\kappa}, \hat{\sigma}, \hat{\overline{S}}\}$, respectively. We fix the riskless rate at 5% *per annum*, and compare the performance of the optimal strategy based on the true and estimated parameters using 50,000 randomly generated 252-day paths. We report the fraction of paths along which the optimal, dynamic strategy based on the true (estimated) parameters leads to default in d_{OPT} ($\hat{d}_{OPT}$) and the across-path mean Sharpe ratios realized by the two strategies SR_{OPT} ($\hat{SR}_{OPT}$). ω is the welfare loss/gain from using the optimal rule based on the estimated parameter values, relative to the optimal rule based on the true parameter values, conditional on no default.

κ	σ	$\overline{S}$	$\hat{\kappa}$	$\hat{\sigma}$	$\hat{\overline{S}}$	d_{OPT}	$\hat{d}_{OPT}$	SR_{OPT}	$\hat{SR}_{OPT}$	ω
0.005	0.05	0	0.0025	0.05	0	0.00%	0.00%	1.09	0.94	-10.17%
0.005	0.05	0	0.010	0.05	0	0.00%	23.73%	1.09	0.77	58.21%
0.01	0.05	0	0.005	0.05	0	0.00%	0.00%	1.38	1.18	-18.90%
0.01	0.05	0	0.020	0.05	0	0.00%	31.71%	1.38	0.67	101.65%
0.01	0.005	0	0.05	0.0025	0	0.00%	58.43%	1.38	-1.98	302.48%
0.01	0.005	0	0.05	0.010	0	0.00%	0.00%	1.38	1.08	-32.00%
0.01	0.05	0	0.05	0.025	0	1.61%	58.65%	1.38	-1.67	304.64%
0.01	0.05	0	0.05	0.10	0	1.57%	0.01%	1.37	1.09	-32.33%
0.01	0.05	0	0.05	0.05	0.3536	0.00%	44.03%	1.38	0.45	42.59%
0.01	0.05	0	0.05	0.05	0.7071	0.00%	71.72%	1.37	-0.61	89.83%

Table 3: **The Royal Dutch - Shell and Unilever spread processes and out-of-sample strategy performance for $\gamma = 10$ investors.** Panel A presents summary statistics for parameter estimates of an Ornstein-Uhlenbeck process fitted to a rolling 2500-day window of historical spread data. The first spread is constructed as the difference between the total return indices of Royal Dutch and Shell, while the second spread is constructed as the difference between the total return indices of the two classes of Unilever shares. There are 25 estimation windows in this sample for Royal Dutch-Shell, and 28 estimation windows for Unilever. $\overline{\kappa}$ and $\overline{\sigma}$ are the mean estimates of the rate of mean reversion and volatility of instantaneous shocks, respectively. $\overline{\lambda}$ reports the mean half-life implied by the series of κ estimates. The values in the square brackets report the range of estimates obtained using the rolling-window estimation procedure. The last three columns list the number of estimation periods in which the null of no mean-reversion can be rejected with a p-value smaller than 0.01, 0.05 and 0.10. The p-value for the test of no mean reversion is generated from a Monte Carlo bootstrap simulation with 5000 paths.

Panel B presents the out-of-sample performance of an investor with relative risk aversion $\gamma = 10$ maximizing the power utility of terminal wealth, and a 'GGR style' investor who invests using a threshold rule. The threshold rule is normalized so that the position the investor takes when the spread first crosses the threshold is the same position that the optimizing investor would take. The estimates of the Ornstein-Uhlenbeck process characterizing the arbitrage opportunity are constructed using a 2500-day backward looking window and are used for 250 days, after which they are re-estimated. The annualized riskless rate in the economy is fixed at 5%. μ_a and μ_g are the annualized arithmetic and geometric mean returns realized under the optimal strategy utilizing the rolling estimates. σ is the annualized standard deviation of returns, and the 'skew' and 'kurt' columns report the higher moment statistics for the time-series of realized returns. The 'min' and 'max' columns report the minimum and maximum daily returns. A 'B' entry indicates the simulation resulted in bankruptcy.

A. Parameter Estimates

First window	Last window	$\overline{\kappa}$	$\overline{\sigma}$	$\overline{\lambda}$	$p < 0.01$	$p < 0.05$	$p < 0.10$
Royal Dutch - Shell							
2/9/1970	11/3/1993	0.0094	0.0457	84.55	13	23	24
12/31/1979	10/6/2003	[0.0034, 0.0154]	[0.0034, 0.0154]	[44.88, 203.32]			
Unilever PLC - Unilever NV							
2/9/1970	10/22/1996	0.0191	0.0822	48.22	23	28	28
12/31/1979	9/27/2006	[0.0058, 0.0397]	[0.0322, 0.1125]	[17.46, 119.79]			

B. Strategy Performance

Trading date range	Trading rule	μ_a	μ_g	σ	Skew	Kurt	Min	Max
Royal Dutch - Shell								
1/2/1980-9/30/2004	Optimal	27.61%	13.12%	55.45%	0.55	14.00	-29.67%	33.95%
1/2/1980-9/30/2004	Threshold	B	B	B	B	B	B	B
Unilever PLC - Unilever NV								
1/2/1980-12/29/2006	Optimal	43.45%	19.38%	71.21%	0.96	36.23	-70.91%	66.35%
1/2/1980-12/29/2006	Threshold	35.27%	8.02%	75.12%	2.23	49.11	-64.29%	90.18%

Figure 1: **Intertemporal hedging demands.** The intertemporal hedging component of the optimal allocation is plotted as a function of the time remaining to the evaluation date, τ. The allocations are plotted under the assumption that the spread, S, is held constant at one standard deviation (of its stationary distribution) above its long-run mean of zero, and wealth, W, is normalized to one. The left panel plots the intertemporal hedging demand for an investor who is less risk averse than implied log utility ($\gamma = 0.5$), and the right panel – for an investor who is more risk averse than implied by log utility ($\gamma = 2$). The solid (dotted) line corresponds to a scenario where the half-life, $\lambda_{1/2}$, of the shock to the arbitrage opportunity is 40 days (80 days). The daily standard deviation of the OU shocks is 0.05 and the riskless rate is fixed at 5% *per annum*.

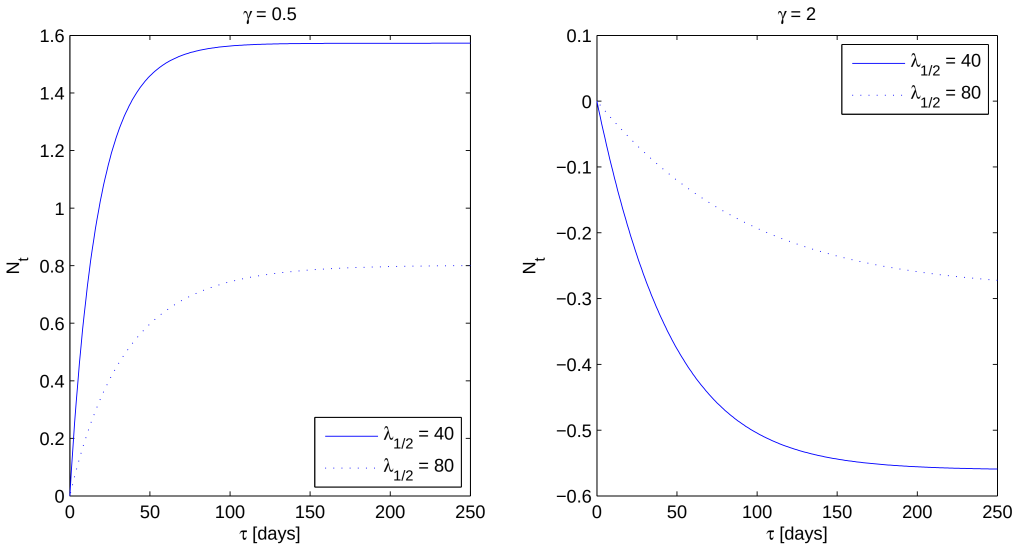


Figure 2: **Terminal wealth distribution.** This figure illustrates the terminal wealth distribution for a $\gamma = 5$ investor obtained from 25,000 Monte Carlo path simulations of a 250 period OU process. The process parameters are calibrated such that each period corresponds to one business day. The rate of mean-reversion of the OU process ($\kappa = 0.0087$) implies a shock half-life of 80 days. The daily standard deviation of the OU shocks is 0.05 and the riskless rate is fixed at 5% *per annum*. The initial wealth is normalized to 1 and the OU process is initialized at a random value drawn from its stationary distribution. The mean (median) terminal wealth is 1.45 (1.43) and the standard deviation of the annual returns is 27.17%; the return distribution has a skewness (kurtosis) coefficient of 2.62 (25.47). The vertical line delimits investment outcomes inferior to a 250 day investment in the riskless asset.

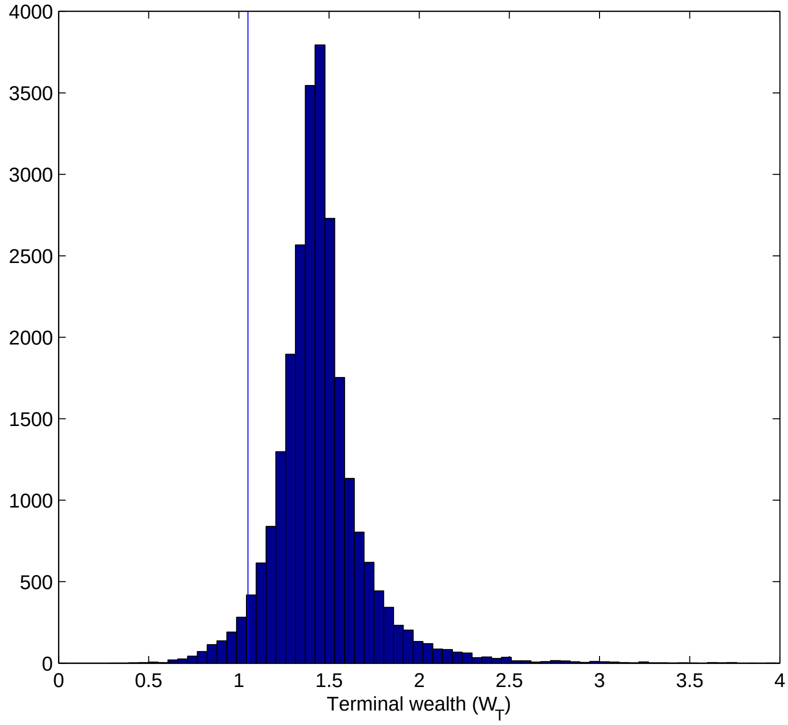


Figure 3: **Stabilization region.** The bound of the stabilization region is plotted as a function of the time remaining to the evaluation date, τ. The magnitude of the bound is represented as a multiple of the standard deviation of the spread's stationary distribution. Wealth, W, is normalized to one. The left panel plots the intertemporal hedging demand for an investor who is less risk averse than implied log utility ($\gamma = 0.5$), and the right panel – for an investor who is more risk averse than implied by log utility ($\gamma = 2$). The solid (dotted) line corresponds to a scenario where the half-life, $\lambda_{1/2}$, of the shock to the arbitrage opportunity is 40 days (80 days). The daily standard deviation of the OU shocks is 0.05 and the riskless rate is fixed at 5% *per annum*.

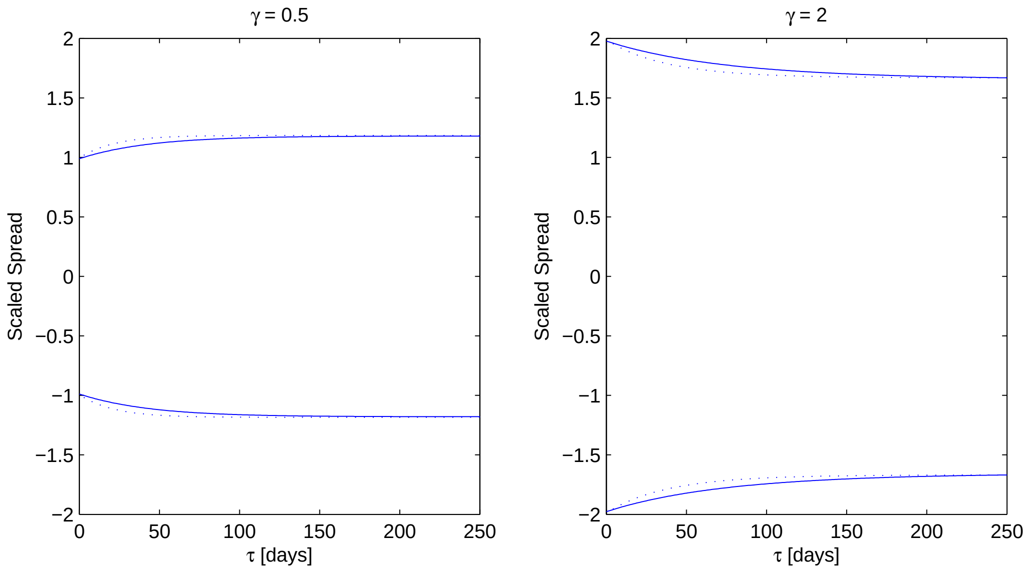


Figure 4: **Out-of-sample performance of the optimal trading rule for the Royal Dutch - Shell pair (1980-2004).** In the out-of-sample trading simulation the agent uses a 2500-day backward-looking window to estimate the parameters of the Ornstein-Uhlenbeck process describing the arbitrage opportunity, and uses those estimates to trade for the next 250 days. The top panel presents the evolution of the Royal Dutch - Shell spread asset (in blue), S_t, within each trading period, and the corresponding stabilization bound (in red). The second panel presents the optimal allocation to the spread asset (in blue) scaled by wealth, N_t/W_t, and the component of the allocation due to intertemporal hedging demands (in red). The third panel presents the evolution of the base 10 log of wealth over the lifetime of the simulation using the optimal rule, with initial wealth normalized to 1. The vertical dotted lines delimit trading periods using parameter estimates from different estimation windows.

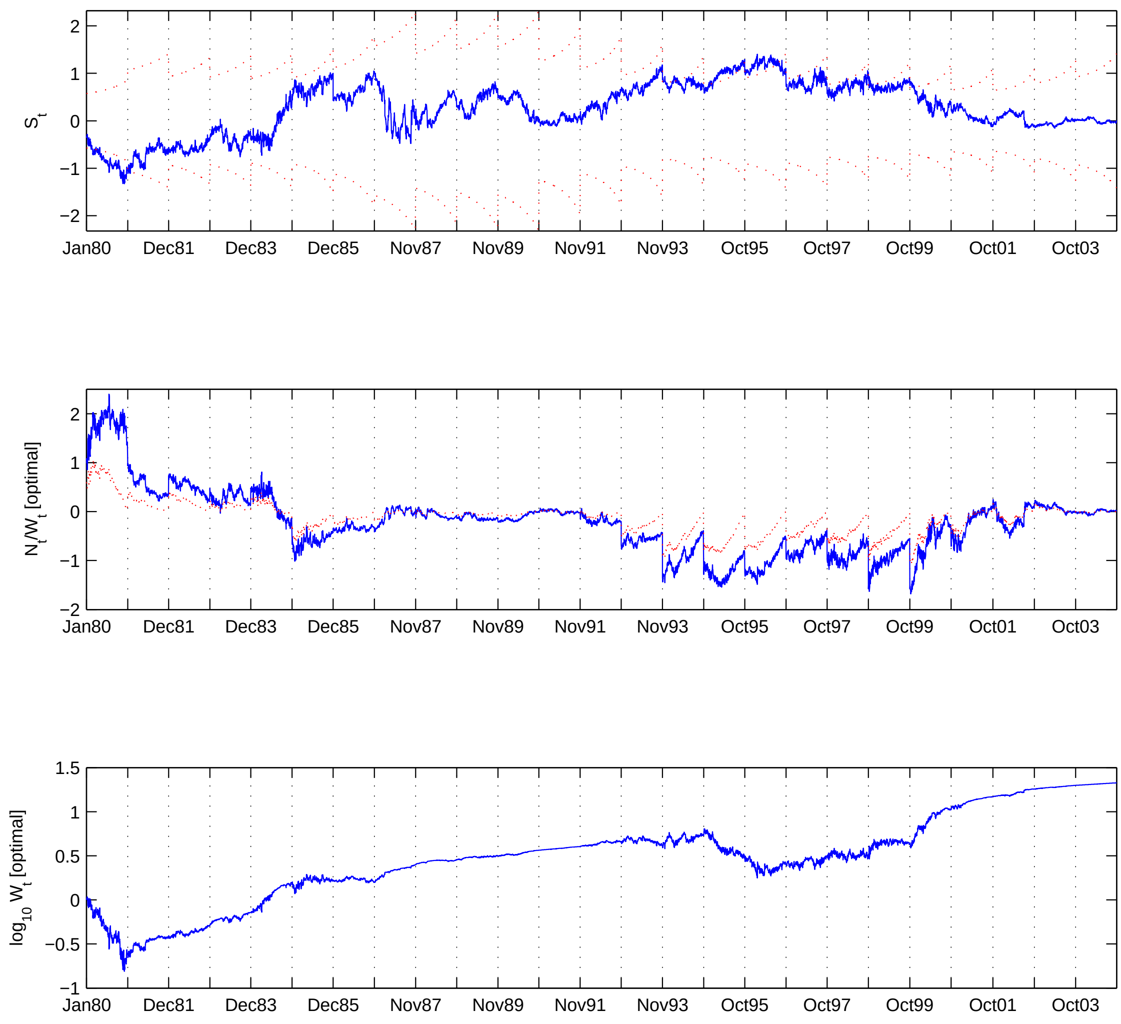
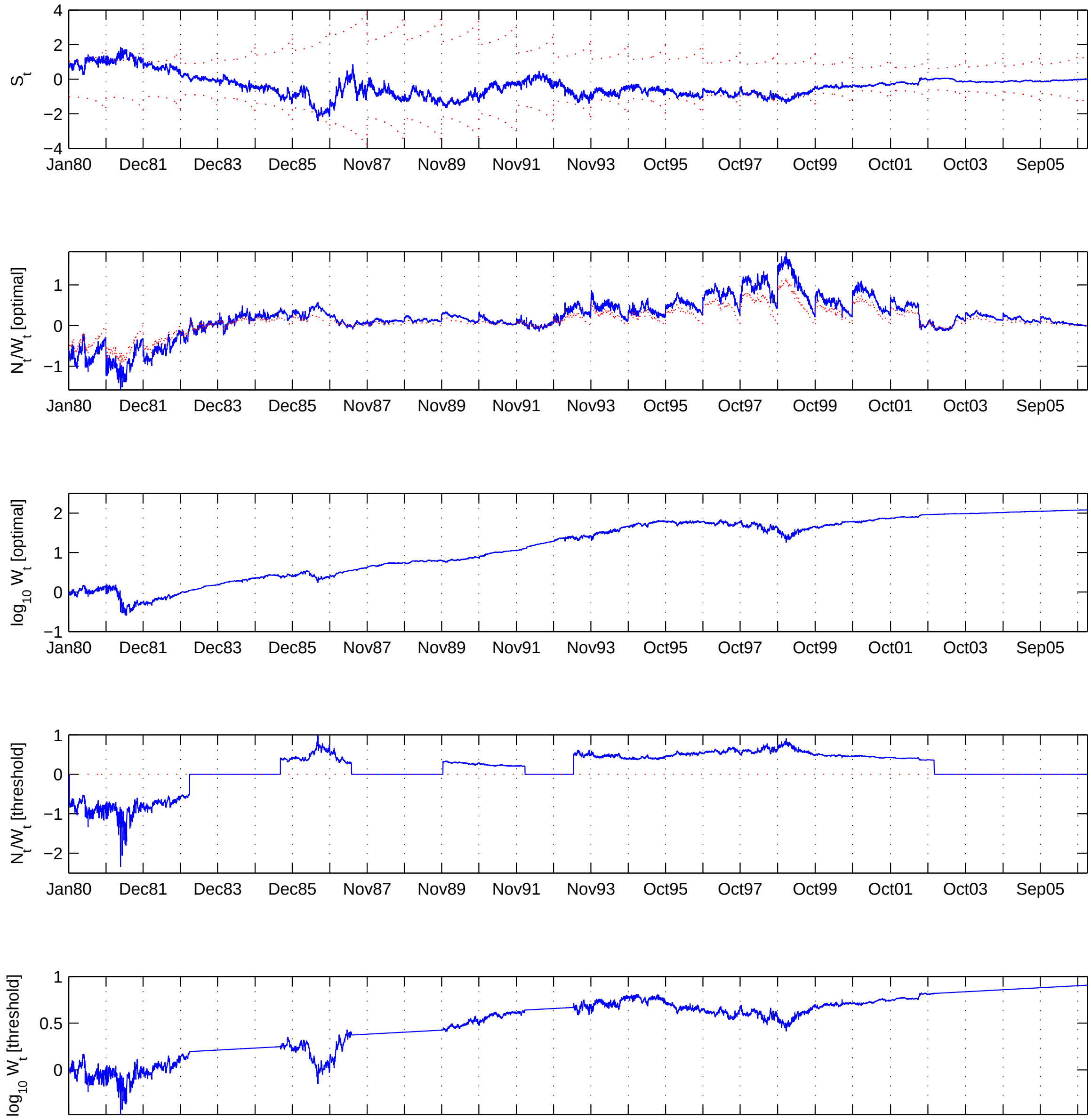


Figure 5: **Out-of-sample performance of trading rules for the Unilever pair (1980-2006).** In the out-of-sample trading simulation the agent uses a 2500-day backward-looking window to estimate the parameters of the Ornstein-Uhlenbeck process describing the arbitrage opportunity, and uses those estimates to trade for the next 250 days. The top panel presents the evolution of the Unilever spread asset (in blue), S_t, within each trading period, and the corresponding stabilization bound (in red). The second panel presents the optimal allocation to the spread asset (in blue) scaled by wealth, N_t/W_t, and the component of the allocation due to intertemporal hedging demands (in red). The third panel presents the evolution of the base 10 log of wealth over the lifetime of the simulation, with initial wealth normalized to 1. The fourth and fifth panels present the allocation and the wealth evolution under the threshold rule. The vertical dotted lines delimit trading periods using parameter estimates from different estimation windows.



Appendix To
Dynamic Portfolio Selection in Arbitrage

Jakub W. Jurek and Halla Yang*

A O-U Spread with Utility of Terminal Wealth

The price processes for the mean-reverting spread and the riskless bond are assumed to satisfy the following stochastic differential equations:

$$\begin{aligned} dS &= \kappa(\overline{S}-S)dt + \sigma dZ && (1)\\ dB &= rBdt && (2)\end{aligned}$$

If we denote by $N(\tau)$ the number of units of the spread purchased with τ periods remaining to the terminal date, T, the instantaneous budget constraint for the agent's optimization problem is:

$$\begin{aligned} dW &= NdS + \frac{W-NS}{B}dB \\ &= \left(r(W-NS) + \kappa(\overline{S}-S)N\right)dt + \sigma N dZ && (3)\end{aligned}$$

and the value function for the problem is defined as:

$$V(W(\tau), S(\tau), \tau) = \begin{cases} \max_{N(S(\tau),\tau)} E_t\left[e^{-\beta\tau}\log W_T \,\middle|\, N(\cdot)\right] & \gamma \neq 1 \\ \max_{N(S(\tau),\tau)} E_t\left[e^{-\beta\tau}\frac{W_T^{1-\gamma}}{1-\gamma} \,\middle|\, N(\cdot)\right] & \gamma \neq 1 \end{cases} \tag{4}$$

where we have defined $\tau = T - t$. By Ito's Lemma, we get:

$$\begin{aligned} dV &= V_S\, dS + V_W\, dW - \frac{\partial V}{\partial \tau}\, dt + \frac{1}{2}V_{SS}\, dS^2 + \frac{1}{2}V_{WW}\, dW^2 + V_{SW}\, dWdS \\ &= V_S\left(\kappa(\overline{S}-S)dt + \sigma dZ\right) + V_W\left(\left(r(W-NS)+\kappa(\overline{S}-S)N\right)dt + \sigma N dZ\right) + \\ &\quad - \frac{\partial V}{\partial \tau}dt + \frac{1}{2}V_{SS}\cdot\sigma^2 dt + \frac{1}{2}V_{WW}\cdot N^2\sigma^2 dt + V_{SW}\cdot N\sigma^2 dt \end{aligned}$$

The HJB equation for the problem, $E_t[dV(\cdot)] = \beta V$, therefore becomes:

$$\beta V = \max_{N(S(\tau),\tau)} V_S\kappa(\overline{S}-S) + V_W\left(r(W-NS)+\kappa(\overline{S}-S)N\right) - \frac{\partial V}{\partial \tau} + \frac{1}{2}V_{SS}\sigma^2 + \frac{1}{2}V_{WW}N^2\sigma^2 + V_{SW}N\sigma^2 \tag{5}$$

The first order condition for the optimal number of shares of the arbitrage yields the optimal portfolio allocation:

$$N = -\left(\frac{V_W}{\sigma^2 V_{WW}}\right)\left(\kappa(\overline{S}-S) - rS\right) - \frac{V_{SW}}{V_{WW}} \tag{6}$$

We postulate and verify that the value function takes on the following forms depending on the agent coefficient of relative risk aversion:

$$V(S,W,\tau) = \begin{cases} e^{-\beta\tau}\log W + e^{-\beta\tau}\left(A(\tau)\cdot S^2 + B(\tau)\cdot S + C(\tau)\right) & \gamma = 1 \\ e^{-\beta\tau}\frac{W^{1-\gamma}}{1-\gamma}\exp\left(A(\tau)\cdot S^2 + B(\tau)\cdot S + C(\tau)\right) & \gamma \neq 1 \end{cases} \tag{7}$$

where $A(\tau)$, $B(\tau)$, and $C(\tau)$ are functions that satisfy the boundary constraints:

$$A(0) = B(0) = C(0) = 0 \tag{8}$$

which implies that the optimal portfolio allocation is given by:

$$N = \begin{cases} \left(\frac{\kappa(\overline{S}-S)-rS}{\sigma^2}\right)W & \gamma = 1 \\ \left(\frac{\kappa(\overline{S}-S)-rS}{\gamma\sigma^2} + \frac{2A(\tau)S+B(\tau)}{\gamma}\right)W & \gamma \neq 1 \end{cases} \tag{9}$$

We now turn to the problem of deriving expression for the coefficient functions $A(\tau)$, $B(\tau)$ and $C(\tau)$.

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A.1 Case 1: Log-utility Investor ($\gamma = 1$)

Substituting the proposed portfolio rule into the HJB equation, along with the relevant derivatives of the value function yields the following condition:

$$\begin{aligned}\beta V &= S^2\left(\frac{(\kappa+r)^2}{2\sigma^2}-2\kappa A(\tau)-A'(\tau)\right)+S\left(-\kappa B(\tau)-\frac{\kappa(\kappa+r)\overline{S}}{\sigma^2}+2\kappa\overline{S}A(\tau)-B'(\tau)\right)+\\ &\quad +r+\sigma^2 A(\tau)+\frac{1}{2}\kappa\overline{S}\left(2B(\tau)+\frac{\kappa\overline{S}}{\sigma^2}\right)-C'(\tau)+\beta V \qquad (10)\end{aligned}$$

Since the no arbitrage condition must hold for all values of S, each of the coefficients in the above polynomial in S must be equal to zero at all points in time. This yields a system of three ODEs for the three coefficients functions, $\{A(\tau), B(\tau), C(\tau)\}$:

$$A'(\tau) = -2\kappa A(\tau)+\frac{(\kappa+r)^2}{2\sigma^2} \qquad (11)$$

$$B'(\tau) = -\kappa B(\tau)-\frac{\kappa(\kappa+r)\overline{S}}{\sigma^2}+2\kappa\overline{S}A(\tau) \qquad (12)$$

$$C'(\tau) = r+\sigma^2 A(\tau)+\frac{1}{2}\kappa\overline{S}\left(2B(\tau)+\frac{\kappa\overline{S}}{\sigma^2}\right) \qquad (13)$$

Solving the ODEs consecutively, and applying the boundary condition yields:

$$A(\tau) = \frac{(\kappa+r)^2\left(1-e^{-2\kappa\tau}\right)}{4\kappa\sigma^2} \qquad (14)$$

$$B(\tau) = -\frac{(\kappa+r)(\kappa-r+e^{-\kappa\tau}(\kappa+r))(1-e^{-\kappa\tau})}{2\kappa\sigma^2}\overline{S} \qquad (15)$$

$$\begin{aligned}C(\tau) &= -\left(\frac{\kappa+r}{2\kappa}\right)^2\sinh(\kappa\tau)e^{-k\tau}+\frac{(\kappa+r)(\kappa-3r+e^{-\kappa\tau}(\kappa+r))(1-e^{-\kappa\tau})}{4\kappa\sigma^2}\overline{S}^2+\\ &\quad +\left(\frac{\kappa^2+6\kappa r+r^2}{4\kappa}+\frac{1}{2}\left(\frac{r\overline{S}}{\sigma}\right)^2\right)\tau \qquad (16)\end{aligned}$$

A.2 Case 2: General CRRA Investor ($\gamma \neq 1$)

Substituting the proposed value function along with the corresponding portfolio rule into the optimality condition and gathering powers of S we obtain:

$$\begin{aligned}\beta &= S^2\cdot\left[\frac{2\sigma^2}{\gamma}\cdot A(\tau)^2-2\left(\frac{\kappa}{\gamma}+r\left(\frac{1-\gamma}{\gamma}\right)\right)\cdot A(\tau)+\frac{1}{2}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\kappa+r}{\sigma}\right)^2-A'(\tau)\right]\\ &\quad +S\cdot\left[\left(\frac{2}{\gamma}\cdot A(\tau)-\left(\frac{\kappa+r}{\sigma^2}\right)\left(\frac{1-\gamma}{\gamma}\right)\right)\kappa\overline{S}+\left(\frac{2\sigma^2}{\gamma}\cdot A(\tau)-\left(\frac{\kappa}{\gamma}+r\cdot\left(\frac{1-\gamma}{\gamma}\right)\right)\right)\cdot B(\tau)-B'(\tau)\right]\\ &\quad +(1-\gamma)r+\sigma^2\cdot A(\tau)+\frac{\sigma^2}{2\gamma}\cdot B(\tau)^2+\left(\frac{1}{\gamma}\cdot B(\tau)+\left(\frac{1}{2\sigma^2}\right)\left(\frac{1-\gamma}{\gamma}\right)\kappa\overline{S}\right)\kappa\overline{S}+\beta-C'(\tau). \qquad (17)\end{aligned}$$

Since this optimality condition must be satisfied for all values of S, the coefficients on the different powers of S must jointly be equal to zero. This leads to a system of three ordinary differential equations for the coefficient functions $\{A(\tau), B(\tau), C(\tau)\}$, which can be solved sequentially to obtain the final expression for the value function. The system of ODEs includes a Ricatti equation and is given by:

$$A'(\tau) = \frac{2\sigma^2}{\gamma}\cdot A(\tau)^2-2\left(\frac{\kappa}{\gamma}+r\cdot\left(\frac{1-\gamma}{\gamma}\right)\right)\cdot A(\tau)+\frac{1}{2}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\kappa+r}{\sigma}\right)^2 \qquad (18)$$

$$B'(\tau) = -\left(\frac{\kappa}{\gamma}+r\left(\frac{1-\gamma}{\gamma}\right)-\frac{2\sigma^2}{\gamma}\cdot A(\tau)\right)\cdot B(\tau)+\left(\frac{2}{\gamma}\cdot A(\tau)-\left(\frac{\kappa+r}{\sigma^2}\right)\left(\frac{1-\gamma}{\gamma}\right)\right)\kappa\overline{S} \qquad (19)$$

$$C'(\tau) = \sigma^2\cdot A(\tau)+\frac{\sigma^2}{2\gamma}\cdot B(\tau)^2+\frac{\kappa\overline{S}}{\gamma}\cdot B(\tau)+(1-\gamma)r+\left(\frac{1}{2\sigma^2}\right)\left(\frac{1-\gamma}{\gamma}\right)(\kappa\overline{S})^2 \qquad (20)$$

A.2.1 Solving for $A(\tau)$

The first differential equation in the system belongs to the class of Ricatti equations, which have the following general form:

$$\frac{dA}{d\tau} = c_1 A(\tau)^2+2c_2 A(\tau)+c_3 \qquad (21)$$

or, alternatively, written as an integral:

$$\int \frac{dA}{c_1 A(\tau)^2 + 2c_2 A(\tau) + c_3} = \int d\tau = \tau - K_1 \tag{22}$$

where the constant of integration K_1 is determined by imposing the condition that $A(0) = 0$. The functional form of the solution to a Riccati equation depends on the discriminant of the quadratic form appearing in the denominator of the left-hand side of (22) - a result which has previously been observed in the finance literature by Kim and Omberg (1996). This discriminant is given by:

$$\Delta = (2c_2)^2 - 4c_1 c_3 \tag{23}$$

It turns out, in fact, that the solution space is in fact partitioned by two conditions - the sign of the discriminant and whether the coefficient of relative risk aversion is greater or smaller than unity. Although, mathematically the second condition is an artifact of the domain restrictions of the inverse hyperbolic trigonometric functions appearing in the constant of integration, it has a clear economic counterpart in the risk-seeking (risk-averting) in the preferences of agents with $\gamma < 1$ and $\gamma = 1$. In the case at hand we have:

$$c_1 = \frac{2\sigma^2}{\gamma} \tag{24}$$

$$c_2 = -\left(\frac{\kappa}{\gamma} + r \cdot \left(\frac{1-\gamma}{\gamma}\right)\right) \tag{25}$$

$$c_3 = \frac{1}{2}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\kappa + r}{\sigma}\right)^2 \tag{26}$$

and the discriminant is given by:

$$\Delta = 4\left(\frac{\kappa^2 - r^2(1-\gamma)}{\gamma}\right) \tag{27}$$

The mathematical significance of whether γ is greater than unity is immediate, since this condition is a sufficient to ensure that the discriminant to be positive. However, this condition is not necessary. It turns out that the sign of the discriminant is uniquely determined by the relationship between the coefficient of relative risk aversion and the ratio of two model parameters, κ - the rate of mean reversion - and r - the riskless rate. We have the following partitioning of the state space:

$$\Delta < 0 \quad \leftrightarrow \quad \gamma < 1 - \left(\frac{\kappa}{r}\right)^2 \tag{28}$$

$$\Delta = 0 \quad \leftrightarrow \quad \gamma = 1 - \left(\frac{\kappa}{r}\right)^2 \tag{29}$$

$$\Delta > 0 \quad \leftrightarrow \quad \gamma > 1 - \left(\frac{\kappa}{r}\right)^2 \tag{30}$$

which we further augment by separating the cases when $\gamma < 1$ and $\gamma > 1$. If we denote the value of γ for which the discriminant is precisely equal to zero by γ_0 we arrive at the following general solution to the Riccati equation:

$$A(\tau) = \begin{cases} -\frac{c_2}{c_1} + \frac{\sqrt{-\Delta}}{2c_1}\tan\left(\frac{\sqrt{-\Delta}}{2}\tau + \tan^{-1}\left(\frac{2c_2}{\sqrt{-\Delta}}\right)\right) & \gamma \in (0, \gamma_0) \\ -\frac{c_2}{c_1}\left(1 + \frac{1}{c_2\tau - 1}\right) & \gamma = \gamma_0 \\ -\frac{c_2}{c_1} + \frac{\sqrt{\Delta}}{2c_1}\coth\left(-\frac{\sqrt{\Delta}}{2}\tau + \coth^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right) & \gamma \in (\gamma_0, 1) \\ -\frac{c_2}{c_1} + \frac{\sqrt{\Delta}}{2c_1}\tanh\left(-\frac{\sqrt{\Delta}}{2}\tau + \tanh^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right) & \gamma \in (1, \infty) \end{cases} \tag{31}$$

Note that if $\gamma_0 < 0$, i.e. when the rate of mean reversion exceeds the riskless rate, only the tanh and coth solutions apply. With this solution in hand we can proceed to solving the second ODE in the system for $B(\tau)$.

A.2.2 Solving for $B(\tau)$

Using the already introduced notation, the ODE for $B(\tau)$ can be re-expressed in the following form:

$$\frac{dB}{d\tau} = c_2 B(\tau) + c_1 A(\tau) B(\tau) + c_4 A(\tau) + c_5 \tag{32}$$

where the two additional constants are defined as follows:

$$c_4 = \frac{2\kappa\overline{S}}{\gamma} \tag{33}$$

$$c_5 = -\left(\frac{\kappa + r}{\sigma^2}\right)\left(\frac{1-\gamma}{\gamma}\right)\kappa\overline{S} \tag{34}$$

In the special corner case when $\overline{S}$ we have $c_4 = c_5 = 0$, which leads to a significant simplification of the solution, as will be shown below. As with $A(\tau)$, the solution to the ODE for $B(\tau)$ will take on different forms depending on the location of the model in the parameter space. The solution of the above ODE can be shown to be given by:

$$B(\tau) = \begin{cases} \frac{c_4 \phi_1(\tau)}{c_1 \sqrt{-\Delta}} + \frac{4\phi_2(\tau)}{\sqrt{-\Delta}} \left(c_5 - \frac{c_4}{c_1}\right) \cos\left(\frac{\sqrt{-\Delta}}{2}\tau - \tan^{-1}\left(\frac{c_2}{\sqrt{-\Delta}}\right)\right) & \gamma \in (0, \gamma_0) \\ \left(\frac{c_1 c_5 (c_2\tau - 2) - c_2^2 c_4}{2c_1(c_2\tau - 1)}\right)\tau & \gamma = \gamma_0 \\ \frac{4\left(c_2c_5 - c_3c_4 + (c_3c_4 - c_2c_5)\cosh\left(\frac{\sqrt{\Delta}}{2}\tau\right)\right) + 2c_5\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}}{2}\tau\right)}{\Delta\cosh\left(\frac{\sqrt{\Delta}}{2}\tau\right) - 2c_2\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}}{2}\tau\right)} & \gamma \in (\gamma_0, \infty) \end{cases} \tag{35}$$

where $\phi_1(\tau)$ and $\phi_2(\tau)$ are given by:

$$\begin{aligned} \phi_1(\tau) &= \sqrt{-\Delta}\left(\cos\left(\frac{\sqrt{-\Delta}}{2}\tau\right) - 1\right) + 2c_2 \sin\left(\frac{\sqrt{-\Delta}}{2}\tau\right) \\ \phi_2(\tau) &= \tanh^{-1}\left(\tan\left(\frac{1}{4}\left(\sqrt{-\Delta}\tau - 2\tan^{-1}\left(\frac{2c_2}{\sqrt{-\Delta}}\right)\right)\right)\right) + \tanh^{-1}\left(\tan\left(\frac{1}{2}\tan^{-1}\left(\frac{2c_2}{\sqrt{-\Delta}}\right)\right)\right) \end{aligned}$$

¿From the above solution it is immediate that $B(\tau) = 0$, for all values of τ, when $c_4 = c_5 = 0$, which is the case when $\overline{S} = 0$. Furthermore, in this special case, the form of the ODE for $C(\tau)$ is particularly straightforward and a closed form solution for $C(\tau)$ is readily attainable.

A.2.3 Solving for $C(\tau)$

Having solved the ODEs for $A(\tau)$ and $B(\tau)$ allows us to proceed to the last ODE - for $C(\tau)$. This last ODE is conceptually trivial to solve as it involves only one univariate integration. The complexity of the integrand, however, is quite significant and involves cross-products and powers of hyperbolic trigonometric functions. Although analytical solutions are available for the integrals of all of the functions appearing linearly ($A(\tau)$ and $B(\tau)$), the quadratic term, $B(\tau)^2$, does not have a known closed form solution (Gradshteyn et al (2000)). This prevents us from being able to derive an analytical expression for $C(\tau)$ in the general case, when $\overline{S} \neq 0$. However, since the function $C(\tau)$ only appears in the value function - and not in the policy function - it only impacts welfare computations. Therefore the lack of an analytical solution for $C(\tau)$ does not present an obstacle in computing the comparative statics of the optimal dynamic policy. Ultimately, a variety of numerical methods are available for the computation of the integral yielding $C(\tau)$ (e.g. Gaussian quadrature (Judd (1998))), but we do not pursue these computations here. We do note however that closed-form solutions are readily available in the special case when $\overline{S} = 0$.

In the special case when $\overline{S} = 0$ we have shown that $B(\tau) = 0$ for all τ. This result makes solving the last ODE considerably easier than in the general case with $\overline{S} \neq 0$, due to the disappearance of all terms involving $B(\tau)$. Consequently, in order to recover $C(\tau)$ all that is necessary is a single integration of an affine transformation of $A(\tau)$. Performing this integration and imposing that $C(0) = 0$, leads us to the following solution:

$$C(\tau) = \begin{cases} \frac{1}{2}\left((\kappa + 3r(1-\gamma))\tau + \gamma \log\left(\frac{\sqrt{\frac{-\Delta}{c_1 c_3}}}{2\cos\left(-\frac{\sqrt{-\Delta}}{2}\tau - \tan^{-1}\left(\frac{2c_2}{\sqrt{-\Delta}}\right)\right)}\right)\right) & \gamma \in (0, \gamma_0) \\ \frac{1}{2}\left((\kappa + 3r(1-\gamma))\tau + \gamma \log\left(\frac{1}{1 - c_2\tau}\right)\right) & \gamma = \gamma_0 \\ \frac{1}{2}\left((\kappa + 3r(1-\gamma))\tau + \gamma \log\left(\frac{\sqrt{\frac{\Delta}{c_1 c_3}}}{2\sinh\left(\frac{\sqrt{\Delta}}{2}\tau - \coth^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right)}\right)\right) & \gamma \in (\gamma_0, 1) \\ \frac{1}{2}\left((\kappa + 3r(1-\gamma))\tau + \gamma \log\left(\frac{\sqrt{\frac{\Delta}{-c_1 c_3}}}{2\cosh\left(\frac{\sqrt{\Delta}}{2}\tau - \tanh^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right)}\right)\right) & \gamma \in (1, \infty) \end{cases} \tag{36}$$

Although $C(\tau)$ is irrelevant for the determination of the optimal policy, the availability of a closed form solution allows us to perform a welfare analysis - in the special case of $\overline{S} = 0$ - without resorting to numerical simulations.

B O-U Spread with Utility of Intermediate Consumption

Let S denote the spread and let B denote the price of the riskless bond. Again suppose that the asset prices satisfy the following SDEs:

$$dS = \kappa(\overline{S} - S)dt + \sigma dZ \tag{37}$$

$$dB = rBdt \tag{38}$$

That is, the price of the risky asset follows an OU process with a long run mean of $\overline{S}$. The equation describing wealth evolution now includes a term for consumption flow C:

$$\begin{aligned} dW &= N\, dS + (W - NS)r\, dt - Cdt \\ &= N\left(\kappa(\overline{S} - S) - rS\right)\, dt + (rW - C)\, dt + N\sigma dZ \end{aligned} \tag{39}$$

where N denotes the number of spread units held by the investor.

Suppose that the agent's objective is to maximize the utility of the discounted stream of consumption over a finite horizon and the agent's preferences are described by Epstein-Zin recursive utility. The normalized aggregator of current consumption and the continuation utility is, $f(C,J)$, and takes the form:

$$f(C_t, J_t) = \frac{\beta}{1-(1/\psi)}(1-\gamma)\cdot J(W_t,S_t)\cdot\left[\left(\frac{C_t}{((1-\gamma)J(W_t,S_t))^{1/(1-\gamma)}}\right)^{1-1/\psi}-1\right] \tag{40}$$

Here $\beta > 0$ is the rate of time preference, $\gamma > 0$ is the coefficient of relative risk aversion, and $\psi > 0$ is the elasticity of intertemporal substitution. For $\psi = 1$, we have

$$f(C_t, J_t) = \beta(1-\gamma)\cdot J(W_t,S_t)\cdot\left[\log C_t - \frac{1}{1-\gamma}\log((1-\gamma)J(W_t,S_t))\right] \tag{41}$$

If we let T denote the finite horizon over which the agent is optimizing, his objective function becomes:

$$V(W_t,S_t) = \max_{N(\cdot),\,C(\cdot)} E_t\left[\int_t^T f(C_s,J_s)ds\right] \tag{42}$$

The value function for the problem will additionally have to satisfy the usual boundary condition at time T:

$$V(W_T,S_T) = \frac{W_T^{1-\gamma}}{1-\gamma} \tag{43}$$

Since the Bellman principle of optimality applies to recursive utility without modification (see Duffie and Epstein (1992)) we have:

$$0 = \max_{N(\cdot),\,C(\cdot)} f(C_t,V_t)dt + E_t\left[dV_t\right] \tag{44}$$

By Ito's Lemma we know the value function satisfies the following stochastic differential equation:

$$\begin{aligned} dV &= V_S\,dS + V_W\,dW - \frac{\partial V}{\partial\tau}\,dt + \frac{1}{2}V_{SS}\,dS^2 + \frac{1}{2}V_{WW}\,dW^2 + V_{SW}\,dWdS \\ &= V_S\,\left(\kappa(\overline{S}-S)\,dt + \sigma\,dZ\right) + V_W\left(N\left(\kappa(\overline{S}-S) - rS\right)\,dt + (rW - C_t)\,dt + N\sigma dZ\right) \\ &\quad -\frac{\partial V}{\partial\tau}dt + \frac{1}{2}V_{SS}\sigma^2 dt + \frac{1}{2}V_{WW}N^2\sigma^2\,dt + V_{SW}N\sigma^2\,dt \end{aligned}$$

Substituting this result into the Bellman equation and imposing the no arbitrage condition we arrive at the following formulation for the problem:

$$\begin{aligned} 0 &= \max_{N(\cdot),\,C(\cdot)} f(C_t,V_t) + V_S\,\kappa(\overline{S}-S)V_S + \left(N\left(\kappa(\overline{S}-S)-rS\right) + rW - C\right)V_W \\ &\quad -\frac{\partial V}{\partial\tau} + \frac{1}{2}V_{SS}\sigma^2 + \frac{1}{2}V_{WW}N^2\sigma^2 + V_{SW}N\sigma^2 \end{aligned} \tag{45}$$

Using the first order conditions for optimal consumption and investment in the spread asset we arrive at the following policy functions for consumption (defined implicitly by the envelope condition) and investment:

$$f_C(C_t,V_t) = V_W \tag{46}$$

$$N = -\left(\frac{V_W}{\sigma^2 V_{WW}}\right)\left(\kappa(\overline{S}-S) - rS\right) - \frac{V_{SW}}{V_{WW}} \tag{47}$$

Substituting the optimal policy functions into the Bellman equation yields a PDE for the value function. We guess and verify the form of the solution for all cases of interest.

B.1 Case 1: $\gamma > 1$, $\psi = 1$

We postulate and verify the following form of the value function:

$$V(S,W,\tau) = \frac{W^{1-\gamma}}{1-\gamma}\exp\left(A(\tau)\cdot S^2 + B(\tau)\cdot S + C(\tau)\right) \tag{48}$$

where $A(\tau)$, $B(\tau)$, and $C(\tau)$ are functions that satisfy the boundary constraints:

$$A(0) = B(0) = C(0) = 0 \tag{49}$$

The optimal policy functions are given by:

$$C = \beta(1-\gamma)\frac{V}{V_W} \tag{50}$$

$$N = -\frac{V_W}{V_{WW}}\left(\frac{\kappa(\overline{S}-S)-rS}{\sigma^2}\right)-\frac{V_{SW}}{V_{WW}} \tag{51}$$

Substituting in the postulated value function yields:

$$C = \beta W \tag{52}$$

$$N = \left(\frac{\kappa(\overline{S}-S)-rS}{\gamma\sigma^2}+\frac{2A(\tau)S+B(\tau)}{\gamma}\right)W \tag{53}$$

Now we turn to determining the coefficient functions $A(\tau)$, $B(\tau)$, and $C(\tau)$.

After substituting in the proposed value function along with the associated policy functions into the Bellman equation we arrive at a second order polynomial in S_t. Since the Bellman equation must hold for all values of the state variable we know that each of the terms must be identically equal to zero. This leads to a system of three ordinary differential equations for the $A(\tau)$, $B(\tau)$, and $C(\tau)$:

$$A'(\tau) = \frac{2\sigma^2}{\gamma}\cdot A(\tau)^2-\left(\beta+\frac{2\kappa}{\gamma}+2r\left(\frac{1-\gamma}{\gamma}\right)\right)\cdot A(\tau)+\frac{1}{2}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\kappa+r}{\sigma}\right)^2 \tag{54}$$

$$B'(\tau) = -\left(\beta+\frac{\kappa}{\gamma}+r\left(\frac{1-\gamma}{\gamma}\right)-\frac{2\sigma^2}{\gamma}\cdot A(\tau)\right)B(\tau)+\left(\frac{2}{\gamma}\cdot A(\tau)-\left(\frac{\kappa+r}{\sigma^2}\right)\left(\frac{1-\gamma}{\gamma}\right)\right)\kappa\overline{S} \tag{55}$$

$$\begin{aligned} C'(\tau) = {} & \sigma^2\cdot A(\tau)+\frac{\sigma^2}{2\gamma}\cdot B(\tau)^2+\frac{\kappa\overline{S}}{\gamma}\cdot B(\tau)-\beta\cdot C(\tau)+(1-\gamma)r+\left(\frac{1}{2\sigma^2}\right)\left(\frac{1-\gamma}{\gamma}\right)(\kappa\overline{S})^2 \\ & +\{1-\beta(1-\gamma)\} \end{aligned} \tag{56}$$

B.1.1 Solving for $\mathbf{A(\tau)}$

The ordinary differential equation for $A(\tau)$ belongs to the class of Riccati equations and the form of the solution will depend on the discriminant of the quadratic form appearing on the right-hand side. In particular, in order to minimize on notation let:

$$c_1 = \frac{2\sigma^2}{\gamma} \tag{57}$$

$$c_2 = \frac{\gamma(2r-\beta)-2(\kappa+r)}{2\gamma} \tag{58}$$

$$c_3 = \frac{(\kappa+r)^2(1-\gamma)}{2\gamma\sigma^2} \tag{59}$$

The ODE can then we re-expressed as:

$$A'(\tau) = c_1A(\tau)^2+2c_2A(\tau)+c_3$$

The discriminant of the quadratic equation is then:

$$\begin{aligned}\Delta &= 4\left(c_2^2-c_1c_3\right)\\ &= \frac{1}{\gamma}\left((2\kappa+\beta)^2+(\gamma-1)(-2r+\beta)^2\right)\end{aligned} \tag{60}$$

The discriminant Δ is always positive for $\gamma \geq 1$. The discriminant is also always positive if $\kappa > r-\beta$. If $\kappa < r-\beta$, then the discriminant is positive as long as

$$\gamma > \frac{4(\kappa+r)(r-\beta-\kappa)}{(2r-\beta)^2} \equiv \gamma_0$$

As in (31), we write the solution in terms of cases:

$$A(\tau) = \begin{cases} -\frac{c_2}{c_1}+\frac{\sqrt{-\Delta}}{2c_1}\tan\left(\frac{\sqrt{-\Delta}}{2}\tau+\tan^{-1}\left(\frac{2c_2}{\sqrt{-\Delta}}\right)\right) & \gamma\in(0,\gamma_0)\\ -\frac{c_2}{c_1}\left(1+\frac{1}{c_2\tau-1}\right) & \gamma=\gamma_0\\ -\frac{c_2}{c_1}+\frac{\sqrt{\Delta}}{2c_1}\coth\left(-\frac{\sqrt{\Delta}}{2}\tau+\coth^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right) & \gamma\in(\gamma_0,1)\\ -\frac{c_2}{c_1}+\frac{\sqrt{\Delta}}{2c_1}\tanh\left(-\frac{\sqrt{\Delta}}{2}\tau+\tanh^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right) & \gamma\in(1,\infty)\end{cases} \tag{61}$$

Note that $\gamma_0 > 0$ need not exist, i.e. it may be that γ_0 as defined above is negative, in which case only the tanh and coth solutions apply.

B.1.2 Solving for B(τ)

We can rewrite the ODE for $B(\tau)$ as

$$B'(\tau) = c_1 A(\tau)B(\tau) + c_4 A(\tau) + c_5 B(\tau) + c_6$$

where

$$\begin{aligned} c_4 &\equiv \frac{2\kappa\bar{S}}{\gamma} \\ c_5 &\equiv \frac{-(\kappa + r(1-\gamma) + \beta\gamma)}{2\gamma} \\ c_6 &\equiv \frac{\kappa(\kappa+r)(\gamma-1)\bar{S}}{\gamma\sigma^2} \end{aligned}$$

Unlike in equation (32), the coefficient on $B(\tau)$ is not the same as c_2. The symmetry is broken by the term in β, which is not present in the case of no intermediate consumption. In the special case that $\bar{S} = 0$ (zero mean spread), note that $c_4 = c_6 = 0$. When $c_4 = c_6 = 0$, the solution $B(\tau) = 0$ immediately obtains from the boundary condition $B(0) = 0$.

The general case-by-case solutions are given below:

Case I: $\gamma \in (0, \gamma_0)$

$$\begin{aligned} B(\tau) = \; & e^{-\tau c_2}\left(e^{\tau c_2}\sqrt{\frac{c_1 c_3}{c_1 c_3 - c_2^2}}c_4 c_5\left(c_2 - \sqrt{c_1 c_3 - c_2^2}\tan\left(\sqrt{c_1 c_3 - c_2^2}\tau + \tan^{-1}\left(\frac{c_2}{\sqrt{c_1 c_3 - c_2^2}}\right)\right)\right)\right. \\ & + c_1\left(c_6\left(c_2\left(e^{\tau c_2}\sqrt{\frac{c_1 c_3}{c_1 c_3 - c_2^2}} - 2e^{\tau c_5}\sec\left(\sqrt{c_1 c_3 - c_2^2}\tau + \tan^{-1}\left(\frac{c_2}{\sqrt{c_1 c_3 - c_2^2}}\right)\right)\right)\right.\right. \\ & + \left(e^{\tau c_5}\sec\left(\sqrt{c_1 c_3 - c_2^2}\tau + \tan^{-1}\left(\frac{c_2}{\sqrt{c_1 c_3 - c_2^2}}\right)\right) - e^{\tau c_2}\sqrt{\frac{c_1 c_3}{c_1 c_3 - c_2^2}}\right)c_5 \\ & \left. + e^{\tau c_2}\sqrt{\frac{c_1 c_3}{c_1 c_3 - c_2^2}}\sqrt{c_1 c_3 - c_2^2}\tan\left(\sqrt{c_1 c_3 - c_2^2}\tau + \tan^{-1}\left(\frac{c_2}{\sqrt{c_1 c_3 - c_2^2}}\right)\right)\right) \\ & \left.\left. - c_3\left(e^{\tau c_2}\sqrt{\frac{c_1 c_3}{c_1 c_3 - c_2^2}} - e^{\tau c_5}\sec\left(\sqrt{c_1 c_3 - c_2^2}\tau + \tan^{-1}\left(\frac{c_2}{\sqrt{c_1 c_3 - c_2^2}}\right)\right)\right)c_4\right)\right) \\ & \div\left(c_1\sqrt{\frac{c_1 c_3}{c_1 c_3 - c_2^2}}\left(c_1 c_3 + c_5\left(c_5 - 2c_2\right)\right)\right) \end{aligned}$$

Case II: $\gamma = \gamma_0$

$$\begin{aligned} B(\tau) = \; & \Big\{ -e^{\tau c_2}\tau c_4 c_2^3 + \left(c_4\left(e^{\tau c_2}(\tau c_5 + 1) - e^{\tau c_5}\right) + e^{\tau c_2}\tau c_1 c_6\right)c_2^2 - c_1\left(e^{\tau c_2}\tau c_5 + 2(e^{\tau c_2} - e^{\tau c_5})\right)c_6 c_2 \\ & + \left(e^{\tau c_2} - e^{\tau c_5}\right)c_1 c_5 c_6 \Big\} \div \left\{c_1 e^{\tau c_2}\left(\tau c_2 - 1\right)\left(c_2 - c_5\right)^2\right\} \end{aligned}$$

Case III: $\gamma \in (\gamma_0, 1)$

$$\begin{aligned} B(\tau) = \; & e^{-\tau c_2}\left(e^{\tau c_2}\sqrt{\frac{c_1 c_3}{c_2^2}}c_6 c_2^2 - \left(e^{\tau c_2}c_3\sqrt{\frac{c_1 c_3}{c_2^2}}c_4 + \left(\sqrt{c_2^2 - c_1 c_3}\left(2e^{\tau c_5}\operatorname{csch}\left(\coth^{-1}\left(\frac{c_2}{\sqrt{c_2^2 - c_1 c_3}}\right)\right.\right.\right.\right.\right. \\ & \left.\left. - \tau\sqrt{c_2^2 - c_1 c_3}\right) - e^{\tau c_2}\coth\left(\coth^{-1}\left(\frac{c_2}{\sqrt{c_2^2 - c_1 c_3}}\right) - \tau\sqrt{c_2^2 - c_1 c_3}\right)\sqrt{\frac{c_1 c_3}{c_2^2}}\right) + e^{\tau c_2}\sqrt{\frac{c_1 c_3}{c_2^2}}c_5\Big)c_6\Big)c_2 \\ & + \operatorname{csch}\left(\coth^{-1}\left(\frac{c_2}{\sqrt{c_2^2 - c_1 c_3}}\right) - \tau\sqrt{c_2^2 - c_1 c_3}\right)\left(c_3 c_4\left(e^{\tau c_5}\sqrt{c_2^2 - c_1 c_3} - e^{\tau c_2}\sinh\left(\tau\sqrt{c_2^2 - c_1 c_3}\right)c_5\right)\right. \\ & \left.\left. + e^{\tau c_5}\sqrt{c_2^2 - c_1 c_3}c_5 c_6\right)\right) \div \left\{c_2\sqrt{\frac{c_1 c_3}{c_2^2}}\left(c_1 c_3 + c_5\left(c_5 - 2c_2\right)\right)\right\} \end{aligned}$$

Case IV: $\gamma > 1$

$$\begin{aligned} B(\tau) &= e^{-\tau c_2}\left(2e^{\tau c_2}\sqrt{-\frac{c_1c_3}{\Delta}}c_4c_5\left(c_2+\frac{1}{2}\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\tau}{2}-\tanh^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right)\right)\right. \\ &\quad +c_1\left(c_6\left(c_2\left(2e^{\tau c_2}\sqrt{-\frac{c_1c_3}{\Delta}}-2e^{\tau c_5}\operatorname{sech}\left(\frac{\sqrt{\Delta}\tau}{2}-\tanh^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right)\right)\right.\right. \\ &\quad +\left(e^{\tau c_5}\operatorname{sech}\left(\frac{\sqrt{\Delta}\tau}{2}-\tanh^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right)-2e^{\tau c_2}\sqrt{-\frac{c_1c_3}{\Delta}}\right)c_5 \\ &\quad \left.\left. -e^{\tau c_2}\sqrt{\Delta}\sqrt{-\frac{c_1c_3}{\Delta}}\tanh\left(\frac{\sqrt{\Delta}\tau}{2}-\tanh^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right)\right)\right) \\ &\quad \left.\left. -c_3\left(2e^{\tau c_2}\sqrt{-\frac{c_1c_3}{\Delta}}-e^{\tau c_5}\operatorname{sech}\left(\frac{\sqrt{\Delta}\tau}{2}-\tanh^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right)\right)c_4\right)\right) \\ &\quad \div\left\{2c_1\sqrt{-\frac{c_1c_3}{\Delta}}\left(c_1c_3+c_5\left(c_5-2c_2\right)\right)\right\} \end{aligned}$$

This last case will be the most important for our analysis.

B.1.3 Solving for C(τ)

The complexity of the ODE defining $C(\tau)$ prevents us from being able to obtain a closed form solution. Fortunately, however the coefficient function $C(\tau)$ does not influence the policy function.

B.2 Case 2: $\gamma = \psi = 1$

The special case when $\psi = 1/\gamma$ is identical to standard CRRA preferences. In particular, $\psi = \frac{1}{\gamma} = 1$ reduces to standard log utility. Unlike the problem in section A, the investor has intermediate consumption, and solves the objective function:

$$V(W,S,t) = \max_{C,N} \mathbf{E}_t\left[\int_0^{T-t} e^{-\beta s}\beta\log C_{t+s}\,ds + e^{-\beta(T-t)}\log W_T\right] \tag{62}$$

For investors with arbitrary weight on intermediate consumption relative to the final bequest function, the consumption-wealth ratio will be a deterministic function of time. We choose to weight intermediate consumption with β in order to maintain consistency with the normalization implicit in the generalized Epstein-Zin aggregator and to generate a constant consumption-wealth ratio. As usual, the optimized value function must satisfy the Bellman equation:

$$E[dV] = \beta V - \beta\log C \tag{63}$$

We postulate and verify that the value function takes the form

$$V(W,S,\tau) = \log W + e^{-\beta\tau}\left(A(\tau)S^2 + B(\tau)S + C(\tau)\right) \tag{64}$$

The Bellman equation simplifies to

$$\begin{aligned} \beta\log W &= \beta\log C + e^{-\beta\tau}\left(2A(\tau)S + B(\tau)\right)\left(\kappa(\overline{S}-S)\right) + \left(\frac{N}{W}\left(\kappa(\overline{S}-S)-rS\right) + \left(r-\frac{C}{W}\right)\right) \\ &\quad +\sigma^2 e^{-\beta\tau}A(\tau) - \frac{\sigma^2}{2W^2}N^2 - e^{-\beta\tau}\left(A'(\tau)S^2 + B'(\tau)S + C'(\tau)\right) \end{aligned}$$

The first-order conditions are

$$C = \beta W \tag{65}$$

$$N = \left(\frac{\kappa(\overline{S}-S)-rS}{\sigma^2}\right)W \tag{66}$$

Requiring that the coefficients on all powers of S vanish identically after substitution of the optimal policies yields three ordinary differential equations for $A(\tau)$, $B(\tau)$, and $C(\tau)$, giving the solutions

$$\begin{aligned}
A(\tau) &= \frac{e^{-2\kappa\tau}\left(-1+e^{(\beta+2\kappa)\tau}\right)(r+\kappa)^2}{2(\beta+2\kappa)\sigma^2} \\
B(\tau) &= -\frac{e^{-2\kappa\tau}(r+\kappa)\left(\left(-1+e^{(\beta+2\kappa)\tau}\right)\kappa(\beta+\kappa)+r\left(\left(-1+e^{\kappa\tau}\right)\beta+\left(-1+2e^{\kappa\tau}-e^{(\beta+2\kappa)\tau}\right)\kappa\right)\right)\mu}{\left(\beta^2+3\kappa\beta+2\kappa^2\right)\sigma^2} \\
C(\tau) &= \Big\{e^{-2\kappa\tau}\left(\left(4e^{\kappa\tau}\beta\kappa(\beta+2\kappa)\mu^2+\beta(\beta+\kappa)\left(\sigma^2-2\kappa\mu^2\right)\right.\right. \\
&\quad +e^{2\kappa\tau}\left(2e^{\beta\tau}\kappa\left(2\kappa^2\mu^2+(\beta+\kappa)\sigma^2\right)-(\beta+\kappa)(\beta+2\kappa)\left(2\kappa\mu^2+\sigma^2\right)\right)\Big)r^2 \\
&\quad +2\kappa\left(\left(2\left(-1+e^{\kappa\tau}\right)\kappa\mu^2+\left(e^{2\kappa\tau}\left(-3+2e^{\beta\tau}\right)+1\right)\sigma^2\right)\beta^2\right. \\
&\quad +\kappa\left(\left(e^{2\kappa\tau}\left(-9+8e^{\beta\tau}\right)+1\right)\sigma^2-2\left(1-2e^{\kappa\tau}+e^{(\beta+2\kappa)\tau}\right)\kappa\mu^2\right)\beta+6e^{2\kappa\tau}\left(-1+e^{\beta\tau}\right)\kappa^2\sigma^2\Big)r \\
&\quad +\kappa(\beta+\kappa)\left(-4e^{2\kappa\tau}\left(-1+e^{\beta\tau}\right)\beta^2\sigma^2+2e^{2\kappa\tau}\left(-1+e^{\beta\tau}\right)\kappa^2\sigma^2\right. \\
&\quad +\beta\kappa\left(2\left(-1+e^{(\beta+2\kappa)\tau}\right)\kappa\mu^2+\left(e^{2\kappa\tau}\left(7-8e^{\beta\tau}\right)+1\right)\sigma^2\right)\Big) \\
&\quad +4e^{2\kappa\tau}\left(-1+e^{\beta\tau}\right)\beta\kappa(\beta+\kappa)(\beta+2\kappa)\sigma^2\log(\beta)\Big)\Big\} \div \left\{4\beta\kappa(\beta+\kappa)(\beta+2\kappa)\sigma^2\right\}
\end{aligned}$$

C Fund Flows

In this section we derive the optimal policy function for a arbitrageur who acts as a delegated manager and is subject to withdrawals of funds. Letting $\tilde{N}=\tilde{N}(W,S,\tau)$ denote the optimal policy function for an investor facing fund flows (to be specified below) the performance of the fund, Π, is given by:

$$d\Pi = \tilde{N}dS + (W-\tilde{N}S)rdt$$

and the evolution of the total wealth under management is determined by the combined effect of performance and fund flows, F:

$$dW = d\Pi + dF$$

We assume that fund flows can be decomposed into two parts - a component proportional to performance (i.e. flows due to performance chasing) and a random component, uncorrelated with the performance of the fund. The coefficient of proportionality for fund flows is f and will generally be assumed to be positive. We therefore have:

$$dF = fd\Pi + \sigma_f W dZ_f \tag{67}$$

where $E[dZ_f dZ]=0$. The idiosyncratic component of fund flows is assumed to be proportional to the fund size and - as a fraction of total wealth - has a standard deviation of σ_f. Outside of these additional assumptions we maintain the structure of our general model and assume the delegated manager is interested in maximizing the utility of his terminal wealth. The value function for the problem is therefore defined as:

$$V(W(\tau),S(\tau),\tau) = \max_{\tilde{N}(\cdot)} E_t\left[e^{-\beta\tau}\frac{W_T^{1-\gamma}}{1-\gamma}\,\middle|\,\tilde{N}(\cdot)\right] \tag{68}$$

where we have defined $\tau = T-t$. By Ito's Lemma, we get:

$$\begin{aligned}
dV &= V_S\,dS + V_W\,dW - \frac{\partial V}{\partial\tau}\,dt + \frac{1}{2}V_{SS}\,dS^2 + \frac{1}{2}V_{WW}\,dW^2 + V_{SW}\,dWdS \\
&= V_S\left(\kappa(\overline{S}-S)dt+\sigma dZ\right) + (1+f)V_W\left(\left(r(W-\tilde{N}S)+\kappa(\overline{S}-S)\tilde{N}\right)dt+\sigma\tilde{N}dZ\right)+ \\
&\quad +\sigma_f W V_W dZ_f - \frac{\partial V}{\partial\tau}dt + \frac{1}{2}V_{SS}\cdot\sigma^2 dt + \frac{1}{2}V_{WW}\cdot(1+f)^2\sigma^2\tilde{N}^2 dt + \frac{1}{2}\sigma_f^2 W^2 dt+ \\
&\quad +V_{SW}\cdot(1+f)\sigma^2\tilde{N}dt
\end{aligned}$$

Bellman's principle of optimality requires, $E[dV]=\beta V$:

$$\begin{aligned}
\beta V &= V_S k(\overline{S}-S) + (1+f)V_W\left(r(W-\tilde{N}S)+\kappa(\overline{S}-S)\tilde{N}\right) - \frac{\partial V}{\partial\tau} \\
&\quad + \frac{1}{2}V_{SS}\cdot\sigma^2 + \frac{1}{2}V_{WW}\cdot(1+f)^2\sigma^2\tilde{N}^2 + \frac{1}{2}V_{WW}\cdot\sigma_f^2 W^2 + V_{SW}\cdot(1+f)\sigma^2\tilde{N}
\end{aligned} \tag{69}$$

The first order condition for the optimal number of shares of the arbitrage yields the optimal portfolio allocation:

$$\tilde{N} = \frac{1}{1+f}\left[-\left(\frac{V_W}{\sigma^2 V_{WW}}\right)\left(\kappa(\overline{S}-S) - rS\right) - \frac{V_{SW}}{V_{WW}}\right] \tag{70}$$

In other words, under the specified model of fund flows, the optimal portfolio allocation in the presence of the flows is exactly $\frac{1}{1+f}$ as large as in the absence of fund flows. Proceeding as before, we postulate and verify that the value function takes on the following forms depending on the agent's coefficient of relative risk aversion:

$$V(S,W,\tau) = \begin{cases} e^{-\beta\tau}\log W + e^{-\beta\tau}\left(A(\tau)\cdot S^2 + B(\tau)\cdot S + C(\tau)\right) & \gamma = 1 \\ e^{-\beta\tau}\frac{W^{1-\gamma}}{1-\gamma}\exp\left(A(\tau)\cdot S^2 + B(\tau)\cdot S + C(\tau)\right) & \gamma \neq 1 \end{cases} \tag{71}$$

where $A(\tau)$, $B(\tau)$, and $C(\tau)$ are functions that satisfy the boundary constraints:

$$A(0) = B(0) = C(0) = 0 \tag{72}$$

Under this value function the optimal portfolio allocation is given by:

$$\tilde{N} = \begin{cases} \left(\frac{\kappa(\overline{S}-S)-rS}{\sigma^2}\right)\frac{W}{1+f} & \gamma = 1 \\ \left(\frac{\kappa(\overline{S}-S)-rS}{\gamma\sigma^2} + \frac{2A(\tau)S+B(\tau)}{\gamma}\right)\frac{W}{1+f} & \gamma \neq 1 \end{cases} \tag{73}$$

For $\gamma = 1$, it is immediately apparent that the addition of fund flows only affects the portfolio rule by introducing a coefficient of proportionality $\frac{1}{1+f}$. The component of fund flows that is orthogonal to fund performance has **no** effect on the optimal portfolio strategy. For $\gamma > 1$, it will also be the case that fund flows affect the portfolio rule only by introducing a coefficient of proportionality. To show that this is the case, we will have to prove that the coefficient functions $A(\tau)$ and $B(\tau)$ do not depend on f and σ_f, i.e. that they are identical to case of zero fund flows.

The Bellman equation for the problem with fund flows is very similar to the model that we have already solved. Making use of the particular structure of $\tilde{N}$ in relation to the previously optimal value of N we observe that the Bellman equation can be re-written as:

$$\begin{aligned}\beta V &= V_S\kappa(\overline{S}-S) + V_W\left(r((1+f)W - NS) + \kappa(\overline{S}-S)N\right) - \frac{\partial V}{\partial \tau} + \\ &\quad + \frac{1}{2}V_{SS}\cdot\sigma^2 + \frac{1}{2}V_{WW}\cdot\left(\sigma^2 N^2 + \sigma_f^2 W^2\right) + V_{SW}\cdot\sigma^2 N \end{aligned} \tag{74}$$

The new terms are a $1+f$ multiplied by $rV_W W$ and σ_f^2 multiplied by $V_{WW}W^2$. Since both of these terms are of zeroth order in S for the postulated value function, the introduction of fund flows will not affect the ordinary differential equation for $A(\tau)$ (this is given by the S^2 coefficient) nor will it affect the equation for $B(\tau)$ (this is given by the S coefficient). Since f and σ_f do not enter the equations that determine $A(\tau)$ and $B(\tau)$, it is immediately apparent that these functions are unchanged from the no fund flows case. Hence, fund flows can affect the value function only through $C(\tau)$, which does not enter the optimal portfolio rule.

We proceed as in the previous section, by solving the Bellman equation separately for the case of the log utility investor, and the general case of $\gamma \neq 1$.

C.1 Case 1: Log-utility Investor ($\gamma = 1$)

Substituting the proposed portfolio rule into the no-arbitrage condition, along with the relevant derivatives of the value function yields the following condition:

$$\begin{aligned} 0 &= S^2\left(\frac{(\kappa+r)^2}{2\sigma^2} - 2\kappa A(\tau) - A'(\tau)\right) + S\left(-\kappa B(\tau) - \frac{\kappa(\kappa+r)\overline{S}}{\sigma^2} + 2\kappa\overline{S}A(\tau) - B'(\tau)\right) + \\ &\quad + r(1+f) - \frac{\sigma_f^2}{2} + \sigma^2 A(\tau) + \frac{1}{2}\kappa\overline{S}\left(2B(\tau) + \frac{\kappa\overline{S}}{\sigma^2}\right) - C'(\tau) \end{aligned} \tag{75}$$

We can note immediately that the addition of fund flows only affects the ODE for $C(\tau)$, leaving the ODEs for $A(\tau)$ and $B(\tau)$ unaffected. Therefore the coefficient functions, $A(\tau)$ and $B(\tau)$, will be identical to the ones derived in Appendix A for the case without fund flows. The presence of fund flows exerts an effect on investor welfare only by modifying the $C(\tau)$ function. In particular, solving the system of ODEs and applying the boundary conditions we find that: Solving the ODEs consecutively, and applying the boundary condition yields:

$$
\begin{aligned}
A(\tau) &= \frac{(\kappa+r)^2\left(1-e^{-2\kappa\tau}\right)}{4\kappa\sigma^2} && (76)\\
B(\tau) &= -\frac{(\kappa+r)(\kappa-r+e^{-\kappa\tau}(\kappa+r))}{2\kappa\sigma^2}\left(1-e^{-\kappa\tau}\right)\overline{S} && (77)\\
C(\tau) &= -e^{-\kappa\tau}\left(\frac{\kappa+r}{2\kappa}\right)^2\sinh(\kappa\tau)+\left(rf-\frac{\sigma_f^2}{2}+\frac{\kappa^2+6\kappa r+r^2}{4\kappa}+\frac{1}{2}\left(\frac{r\overline{S}}{\sigma}\right)^2\right)\tau \\
&\quad+\frac{(\kappa+r)(\kappa-3r+e^{-\kappa\tau}(\kappa+r))}{2\kappa\sigma^2}\left(1-e^{-\kappa\tau}\right)\overline{S}^2 && (78)
\end{aligned}
$$

C.2 Case 2: General CRRA Investor ($\gamma \neq 1$)

We leave the derivation of the policy function coefficients in the $\gamma \neq 1$ case as an exercise for the reader. As before, the presence of fund flows only affects the solution of the ODE for $C(\tau)$. The ODEs for $A(\tau)$ and $B(\tau)$ are left unchanged relative to the solutions provided in Appendix A.

D O-U Log Price Model

Whereas the main model of the paper is that of a mean-reverting price spread, the case of mean-reverting log prices is also of interest, as it imposes the restriction that prices remain positive. In this, second variant of the model the investable universe is comprised of two assets - a riskless bond and a risky asset with a mean-reverting log price, whose prices at time t are B_t and P_t, respectively. The log price of the risky asset is assumed to have constant volatility (σ) and its is assumed to mean-revert around a long-run mean of $\log\overline{P}$, with mean-reversion parameter κ. A similar modelling assumption can be found in Balvers, et. al. (2000), and is substantially different from the assumption of mean reversion in returns in Kim and Omberg (1996). The price of the bond follows the standard accumulation equation, with an instantaneous riskless rate of r. Mathematically, the asset prices satisfy the following SDEs:

$$
\begin{aligned}
d(\log P) &= \kappa(\log\overline{P}-\log P)dt+\sigma dZ && (79)\\
dB &= rBdt && (80)
\end{aligned}
$$

In order to arrive at an expression for the instantaneous return on the mean-reverting asset, we apply Ito's lemma to the expression for the evolution of the log price, to arrive at:

$$
\frac{dP}{P}=\left(\kappa(\log\overline{P}-\log P)+\frac{\sigma^2}{2}\right)dt+\sigma dZ \tag{81}
$$

The instantaneous return on the mean-reverting asset has constant volatility and inherits the time-varying drift of the log price process.

In order to simplify the ensuing calculations it is useful to define the following spread variable and its associated stochastic differential equation:

$$
\begin{aligned}
S &= \log\overline{P}-\log P && (82)\\
dS &= -d\log P=-\kappa S dt-\sigma dZ && (83)
\end{aligned}
$$

Using the variable, S, which will become one of the state-variables of the dynamic programming problem, we can re-express the instantaneous return process for the risky asset more succinctly as:

$$
\frac{dP}{P}=\left(\kappa S+\frac{\sigma^2}{2}\right)dt+\sigma dZ \tag{84}
$$

Consequently, if there is no intermediate consumption and the agent invests a fraction α of his portfolio in the mean-reverting asset, the return on wealth is:

$$
\begin{aligned}
\frac{dW}{W} &= \alpha\frac{dP}{P}+(1-\alpha)\frac{dB}{B}\\
&= \left(\alpha\left(\kappa S+\frac{\sigma^2}{2}\right)+(1-\alpha)r\right)dt+\alpha\sigma dZ && (85)
\end{aligned}
$$

which constitutes the relevant dynamic budget constraint for the optimization problem of the finite-horizon CRRA investor. With intermediate-consumption - as in the case of the Epstein-Zin investor - the dynamic budget is modified to include a consumption term:

$$\begin{aligned} \frac{dW}{W} &= \alpha \frac{dP}{P} + (1-\alpha)\frac{dB}{B} - \frac{C}{W}dt \\ &= \left(\alpha\left(\kappa S + \frac{\sigma^2}{2}\right) + (1-\alpha)r - \frac{C}{W}\right)dt + \alpha\sigma dZ \end{aligned} \tag{86}$$

D.1 Power Utility with No Consumption and a Finite Horizon

First, we derive the value function in the case where an investor has power utility over terminal wealth at a finite horizon and has no intermediate consumption. We can think of this as approximating the value function of an investment manager whose returns will be evaluated at some terminal date. The investor's objective function is defined as in (4).

The evolution of the value function is given by:

$$\begin{aligned} dV &= V_S\ dS + V_W\ dW - \frac{\partial V}{\partial \tau}\ dt + \frac{1}{2}V_{SS}\ dS^2 + \frac{1}{2}V_{WW}\ dW^2 + V_{SW}\ dW dS \\ &= V_S\left(-\kappa S dt - \sigma dZ\right) + V_W W\left(\left(\alpha\left(\kappa S + \frac{\sigma^2}{2}\right) + (1-\alpha)r\right)dt + \alpha\sigma dZ\right) - \\ &\quad - \frac{\partial V}{\partial \tau}dt + \frac{1}{2}V_{SS}\sigma^2 dt + \frac{1}{2}V_{WW}\cdot\alpha^2\sigma^2 W^2 dt - V_{SW}\cdot\alpha W\sigma^2 dt \end{aligned}$$

Imposing the no arbitrage condition we obtain:

$$\beta V = -\kappa S V_S + \left(\alpha\left(\kappa S + \frac{\sigma^2}{2} - r\right) + r\right)W V_W - \frac{\partial V}{\partial \tau} + \frac{\sigma^2}{2}V_{SS} + \frac{\alpha^2\sigma^2 W^2}{2}V_{WW} - \alpha W\sigma^2 V_{SW} \tag{87}$$

The optimal portfolio rule therefore must satisfy:

$$\alpha = -\left(\frac{V_W}{W V_{WW}}\right)\left(\frac{\kappa S + \frac{\sigma^2}{2} - r}{\sigma^2}\right) + \frac{V_{SW}}{W V_{WW}} \tag{88}$$

Using the method of separation of variables we again postulate a value function of the form:

$$V(S,W,\tau) = \begin{cases} e^{-\beta\tau}\log W + e^{-\beta\tau}\left(A(\tau)\cdot S^2 + B(\tau)\cdot S + C(\tau)\right) & \gamma = 1 \\ e^{-\beta\tau}\frac{W^{1-\gamma}}{1-\gamma}\exp\left(A(\tau)\cdot S^2 + B(\tau)\cdot S + C(\tau)\right) & \gamma \neq 1 \end{cases} \tag{89}$$

where $A(\tau)$, $B(\tau)$, and $C(\tau)$ are functions that satisfy the boundary constraints:

$$A(0) = B(0) = C(0) = 0 \tag{90}$$

This value function implies the following optimal portfolio rule:

$$\alpha = \begin{cases} \frac{\kappa S + \frac{\sigma^2}{2} - r}{\sigma^2} & \gamma = 1 \\ \frac{\kappa S + \frac{\sigma^2}{2} - r}{\gamma\sigma^2} - \frac{2A(\tau)S + B(\tau)}{\gamma} & \gamma \neq 1 \end{cases} \tag{91}$$

This form of the solution is intuitive, since $kS + \frac{\sigma^2}{2} - r$ is the instantaneous expected return on the asset with the mean-reverting log price, the log utility investor holds the myopic portfolio weight of the instantaneous expected return divided by the variance. For $\gamma \neq 1$, the investor's optimal allocation includes the myopic component as well as an intertemporal hedging component governed by $A(\tau)$ and $B(\tau)$. Since $A(\tau)$ and $B(\tau)$ go to zero as $\tau \to 0$, the portfolio weights become more and more myopic as the horizon draws nearer.

D.1.1 Case 1: Log-utility Investor ($\gamma = 1$)

Substituting the policy function into the optimality condition and setting the coefficients on powers of S equal to zero yields the following systems of ODEs:

$$A'(\tau) = -2\kappa\cdot A(\tau) + \frac{\kappa^2}{2\sigma^2} \tag{92}$$

$$B'(\tau) = -\kappa\cdot B(\tau) + \frac{\kappa(\sigma^2 - 2r)}{2\sigma^2} \tag{93}$$

$$C'(\tau) = \sigma^2\cdot A(\tau) + \frac{1}{2}\left(\frac{2r + \sigma^2}{2\sigma}\right)^2 \tag{94}$$

which can be solved in closed form , subject to the boundary condition $A(0) = B(0) = C(0) = 0$, to yield:

$$A(\tau) = \frac{\kappa(1 - e^{-2\kappa\tau})}{4\sigma^2} \tag{95}$$

$$B(\tau) = \frac{(1 - e^{-\kappa\tau})(\sigma^2 - 2r)}{2\sigma^2} \tag{96}$$

$$C(\tau) = -\frac{1}{8}\left(1 - e^{-2\kappa\tau} - \left(2\kappa + \left(\frac{2r + \sigma^2}{\sigma}\right)^2\right)\tau\right) \tag{97}$$

D.1.2 Case 2: General CRRA Investor ($\gamma \neq 1$)

Substituting the policy function into the optimality condition and setting the coefficients on powers of S equal to zero yields the following systems of ODEs:

$$A'(\tau) = \frac{2\sigma^2}{\gamma} \cdot A(\tau)^2 - \frac{2\kappa}{\gamma} A(\tau) + \frac{(1-\gamma)\kappa^2}{2\gamma\sigma^2} \tag{98}$$

$$B'(\tau) = -\frac{\kappa}{\gamma} \cdot B(\tau) + \frac{2\sigma^2}{\gamma} \cdot A(\tau)B(\tau) + \left(1 - \frac{1}{\gamma}\right)(2r - \sigma^2) \cdot A(\tau) + \frac{\kappa}{2\sigma^2}\left(1 - \frac{1}{\gamma}\right)(2r - \sigma^2) \tag{99}$$

$$C'(\tau) = \sigma^2 \cdot A(\tau) + \frac{(1-\gamma)(2r - \sigma^2)}{2\gamma} \cdot B(\tau) + \frac{\sigma^2}{2\gamma} \cdot B(\tau)^2 + \frac{(1-\gamma)(4r^2 + 4r\sigma^2(2\gamma - 1) + \sigma^4)}{8\gamma\sigma^2} \tag{100}$$

The discriminant of the Riccati equation is strictly positive and equal to $\Delta = (2c_1)^2 - 4c_1c_3 = \frac{4\kappa^2}{\gamma}$, which implies that the solutions to the first two ODEs will take the following form (see Appendix A):

$$A(\tau) = -\frac{c_2}{c_1} + \frac{\sqrt{\Delta}}{2c_1} \tanh\left(-\frac{\sqrt{\Delta}}{2}\tau + \tanh^{-1}\left(\frac{2c_2}{\sqrt{\Delta}}\right)\right)$$

$$B(\tau) = \frac{4\left(c_2c_5 - c_3c_4 + (c_3c_4 - c_2c_5)\cosh\left(\frac{\sqrt{\Delta}}{2}\tau\right)\right) + 2c_5\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}}{2}\tau\right)}{\Delta\cosh\left(\frac{\sqrt{\Delta}}{2}\tau\right) - 2c_2\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}}{2}\tau\right)}$$

where:

$$c_1 = \frac{2\sigma^2}{\gamma}$$

$$c_2 = \frac{\kappa}{\gamma}$$

$$c_3 = \frac{(1-\gamma)\kappa^2}{2\gamma\sigma^2}$$

$$c_4 = \left(1 - \frac{1}{\gamma}\right)(2r - \sigma^2)$$

$$c_5 = \frac{\kappa}{2\sigma^2}\left(1 - \frac{1}{\gamma}\right)(2r - \sigma^2)$$

D.2 Epstein-Zin Utility with Consumption and a Finite Horizon

As before, we assume that asset price dynamics are given by (79) and (80), and that the spread is defined by (82). In contrast to the previous subsection, we now suppose that the agent's objective is to maximize the utility of the discounted stream of consumption over a finite horizon and the agent's preferences are described by Epstein-Zin recursive utility. Epstein-Zin utility is a generalized utility function that, unlike power utility, does not force the elasticity of intertemporal substitution to be fully determined by the coefficient of relative risk aversion.

In our framework, we will assume that the elasticity of intertemporal substitution equals one. Like a log-utility investor in a CRRA framework, the investor chooses to consume a constant fraction β of total wealth. The natural interpretation is that of a hedge fund investor who cares both about terminal wealth in some risk-averse way and consumes a constant fraction, β, of assets under management (the management fee).

Proceeding as in Appendix B, the normalized aggregator of current consumption and the continuation utility, $f(C_t, V_t)$, for the case of a unit elasticity of substitution is given by:

$$f(C_t, V_t) = \beta(1-\gamma) \cdot V(W_t, S_t) \cdot \left[\log C_t - \frac{1}{1-\gamma}\log((1-\gamma)V(W_t, S_t))\right] \tag{101}$$

If we let T denote the finite horizon over which the agent is optimizing, his objective function becomes:

$$V(W_t, S_t) = \max_{\alpha(\cdot), C(\cdot)} E_t \left[\int_t^T f(C_s, V_s) ds \right] \tag{102}$$

The value function for the problem will additionally have to satisfy the usual boundary condition at time T:

$$V(W_T, S_T) = f(C_T, V_T) \tag{103}$$

Since the Bellman principle of optimality applies to recursive utility without modification (see Duffie and Epstein (1992)) we have:

$$0 = \max_{\alpha_t, C_t} f(C_t, V_t)dt + E_t [dV] \tag{104}$$

Differentiating the optimality condition we find that the optimal policy rules for an investor maximizing (102) are given by:

$$C_t = \beta W \tag{105}$$

$$\alpha_t = \frac{\kappa S + \frac{\sigma^2}{2} - r}{\gamma \sigma^2} - \frac{2A(\tau)S + B(\tau)}{\gamma} \tag{106}$$

As is standard in Epstein-Zin models with $\psi = 1$, the investor consumes a constant fraction of wealth given by her discount rate. To derivation of the functional forms of the coefficient functions we proceed as in the previous sections and is left as an exercise for the reader.

E Estimating the Parameters of an OU Process

While in general, continuous-time processes require sophisticated techniques to estimate (e.g. Elerian, Chib and Shephard, 2001), the OU process is solvable in closed form, and so can be estimated much more simply. Consider a price process given by

$$dX = \kappa(\mu - X)dt + \sigma dZ \tag{107}$$

If observations are spaced Δt apart, then the solution is given by

$$X(t_i) = \mu + e^{-\kappa \Delta t}(X(t_{i-1}) - \mu) + \sigma e^{-\kappa \Delta t} \int_{t_{i-1}}^{t_i} e^{\kappa s} dZ(s) \tag{108}$$

Hence,

$$\begin{aligned} \frac{X(t_i) - \mu - e^{-\kappa \Delta t}(X(t_{i-1}) - \mu)}{\sigma e^{-\kappa \Delta t}} &= \int_{t_{i-1}}^{t_i} e^{\kappa s} dZ(s) \\ &\sim \sqrt{\int_{t_{i-1}}^{t_i} e^{2\kappa s} ds} \cdot \mathcal{N}(0,1) \\ &= \sqrt{\frac{e^{2\kappa \Delta t} - 1}{2\kappa}} \cdot \mathcal{N}(0,1). \end{aligned} \tag{109}$$

Rearranging, we have

$$\left(\frac{X(t_i) - \mu - e^{-\kappa \Delta t}(X(t_{i-1}) - \mu)}{\sigma e^{-\kappa \Delta t}} \right) \cdot \sqrt{\frac{2\kappa}{e^{2\kappa \Delta t} - 1}} \sim \mathcal{N}(0,1) \tag{110}$$

Note that the errors on the right-hand side are i.i.d., since Brownian jumps over non-overlapping time intervals are independent. This allows us to use OLS regression of $X(t_i)$ on $X(t_{i-1})$ and a constant to estimate the parameters of the OU process. In particular, we get

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} X(t_i) \tag{111}$$

$$\hat{\kappa} = -\frac{1}{\Delta t} \log \left(\frac{\sum_{i=2}^{N} (X(t_i) - \hat{\mu})(X(t_{i-1}) - \hat{\mu})}{\sum_{i=2}^{N} (X(t_i) - \hat{\mu})^2} \right) \tag{112}$$

$$\hat{\sigma} = \sqrt{\frac{2\hat{\kappa}}{e^{2\hat{\kappa}\Delta t} - 1} \cdot \frac{1}{N-2} \sum_{i=1}^{N} \left(\frac{X(t_i) - \hat{\mu} - e^{-\hat{\kappa}\Delta t}(X(t_{i-1}) - \hat{\mu})}{e^{-\hat{\kappa}\Delta t}} \right)^2} \tag{113}$$

The standard errors of the estimators can be constructed using the delta method. To test whether our parameters are significant against the null of a random walk, we use Monte Carlo methods, simulating random walks where the price increments are Gaussian and have mean and standard deviation equal to that estimated for the actual data.