Journal of Economic Dynamics & Control 37 (2013) 1972–1981

Contents lists available at SciVerse ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Dynamic pairs trading using the stochastic control approach

Agnès Tourin [a,*], Raphael Yan [b]

[a] New York University Polytechnic Institute, Department of Finance and Risk Engineering, Room RH519, Six MetroTech Center, Brooklyn, NY 11201, United States
[b] BlackRock, Inc., 55 East 52nd Street, New York, NY 10055, United States

ARTICLE INFO

Article history:
Received 22 March 2011
Accepted 29 January 2013
Available online 20 May 2013

JEL classification:
91G80

Keywords:
Optimal stochastic control
Pairs trading
Co-integration
Hamilton Jacobi Bellman equation
Merton problem

ABSTRACT

We propose a model for analyzing dynamic pairs trading strategies using the stochastic control approach. The model is explored in an optimal portfolio setting, where the portfolio consists of a bank account and two co-integrated stocks and the objective is to maximize for a fixed time horizon, the expected terminal utility of wealth. For the exponential utility function, we reduce the problem to a linear parabolic partial differential equation which can be solved in closed form. In particular, we exhibit the optimal positions in the two stocks.

1. Introduction

We develop an optimal stochastic control model for analyzing dynamic pairs trading strategies. For an introduction to pairs trading, we recommend the paper by Elliott et al. (2005) and the references therein. In this paper, we consider a portfolio composed of a risk-free asset and two co-integrated and correlated stocks. For a given maturity $T > 0$, the goal is to determine the trading policies that maximize the expected utility of terminal wealth.

This work was inspired by an earlier work of Mudchanatongsuk et al. (2008) who apply the optimal stochastic control approach to a simplified model of optimal pairs trading. In their model, they only allow positions that are short one stock and long the other, in equal dollar amounts. We use a similar co-integration model but we relax the above constraint and allow strategies with arbitrary amounts in each stock.

The model for the co-integrated stocks is taken from Duan and Pliska (2004) who obtain it as the diffusion limit of a discrete time co-integration model. While their focus is to value options on co-integrated assets, our goal is to compute optimal allocations directly on the co-integrated assets. Consequently, we work with the historical probability measure, not a risk-neutral measure. The parameters in our co-integration model can be estimated as in Duan and Pliska (2004), by using a two-step (Engle and Granger, 1987) method coupled with a Dicker–Fulley test. It is worth noting that, alternately, a filtering method could be used as in Elliott et al. (2005).

For the exponential utility function and for a zero risk-free interest rate, we are able to reduce the problem to a one-dimensional linear parabolic partial differential equation (PDE in short); we compute explicitly the optimal strategies and

* Corresponding author. Tel.: +1 646 240 2608.
E-mail address: atourin@poly.edu (A. Tourin).

0165-1889/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.jedc.2013.05.010

the value function which turns out to be smooth. As in the standard Merton problem, the amounts invested in each stock are inversely proportional to the risk aversion coefficient. From a practitioner's point of view, the investor must pick a particular value for the risk aversion coefficient. For instance, cutting in half the risk aversion level is equivalent to investing twice as much capital in each stock. Besides, at the optimum, the amounts invested in each stock do not depend on the values of the individual stocks, but rather on a mean reverting process which is a linear combination of the stock prices.

There is a similar paper by Benth and Karlsen (2005) who consider a Merton investment problem with a mean reverting asset price. They derive an explicit solution for the value function and the optimal trading strategies by reducing the model to a one-dimensional linear parabolic PDE. Although our application is different than theirs, our calculations and proofs are similar. Besides, we refer to Fleming and Soner (1993) for an overview of the theory of stochastic optimal control. Finally, we want to point out the paper by Gatev et al. (2006) which contains an extensive empirical study on pairs trading.

In the next section, we describe our optimal stochastic control model. In Section 3, we compute in closed form the solution of the stochastic control problem by deriving formally the associated Hamilton–Jacobi–Bellman (HJB in short) equation and solving it explicitly. We provide a verification result in Section 4 which holds under two conditions on the parameters in the model. In Section 5, we add correlations to our model and we calculate the corresponding solution. In the last section, in order to illustrate our results, we produce an example with historical minute-by-minute stock data.

2. Formulation of the optimal stochastic control model

We essentially use for the co-integrated stocks the model derived in Duan and Pliska (2004) as the diffusion limit of a discrete-time model, in the case when there are only two assets. We fix a time horizon $T>0$. We denote by S_t^1 and S_t^2 the co-integrated stock prices for $t\in[0,T]$. S_t^1, S_t^2 satisfy the stochastic differential equations

$$d\log S_t^1=\left(\mu_1-\frac{\sigma_1^2}{2}+\delta z_t\right)dt+\sigma_1\,dB_t^1$$
$$d\log S_t^2=\left(\mu_2-\frac{\sigma_2^2}{2}\right)dt+\sigma_2\,dB_t^2$$

where (B_t^1,B_t^2) is a two-dimensional Brownian motion defined on a filtered probability space $(\Omega,\mathcal{F}_t,IP)$, the underlying filtration is $\mathcal{F}_t=\sigma((B_s^1,B_s^2):0\le s\le t)$ and the co-integrating vector z_t is defined by

$$z_t=a+\log S_t^1+\beta\log S_t^2. \tag{1}$$

The main differences with Duan and Pliska (2004) are the following: first of all, we work under the historical probability measure instead of the risk-neutral probability measure; secondly, for a sake of simplicity, the co-integration term z_t appears only in the drift of the first stock instead of affecting the drifts of the two stocks as in Duan and Pliska (2004). However, our results can be easily generalized to the fully symmetric case. Thirdly, we omit the deterministic linear trend in our definition of z_t. This only affects the estimation of the coefficients in the dynamics of z_t and our results are readily applicable in the case when a linear deterministic trend is present.

Furthermore, as in Duan and Pliska (2004), we can see that z_t is mean-reverting

$$\begin{aligned}dz_t&=\left(\mu_1-\frac{\sigma_1^2}{2}+\delta z_t\right)dt+\sigma_1\,dB_t^1+\beta\left(\mu_2-\frac{\sigma_2^2}{2}\right)dt+\beta\sigma_2\,dB_t^2\\&=\left(\mu_1-\frac{\sigma_1^2}{2}+\beta\mu_2-\beta\frac{\sigma_2^2}{2}+\delta z_t\right)dt+\sigma_1 dB_t^1+\beta\sigma_2\,dB_t^2\\&=\alpha(\eta-z_t)\,dt+\sigma_\beta\,dB_t\end{aligned}$$

where $\alpha=-\delta$ is the speed of mean reversion, $\sigma_\beta=\sqrt{\sigma_1^2+\beta^2\sigma_2^2}$, $B_t=(\sigma_1/\sigma_\beta)B_t^1+\beta(\sigma_2/\sigma_\beta)B_t^2$ is a Brownian motion adapted to $\mathcal{F}_t$ and

$$\eta=-\frac{1}{\delta}\left(\mu_1-\frac{\sigma_1^2}{2}+\beta\left(\mu_2-\frac{\sigma_2^2}{2}\right)\right)$$

is the equilibrium level.

We also assume that there is a risk free asset such as a money market account. In this paper, we set the risk-free interest rate to 0 because this allows us to derive a closed form solution by factoring out the wealth variable.

Next, we introduce the variable W_s representing the value of the investor's portfolio at time s. Note that we do not require the wealth W_s to be non-negative. The investor starts at time t with an initial wealth $W_t=w$, in the money market account. Then she invests, at every time $s\in[t,T]$, in both the risk-free money market account and the two stocks. We denote by π_s^1 and π_s^2 the number of shares held respectively in the first and second stocks at time s. We only consider self-financing strategies and hence, the evolution of the wealth variable is given by

$$dW_s=\pi_s^1\,dS_s^1+\pi_s^2\,dS_s^2.$$

The value of the portfolio W_s satisfies the stochastic differential equation

$$dW_s = \pi_s^1(\mu_1+\delta z_s)S_s^1\,ds + \pi_s^2\mu_2 S_s^2\,ds + \pi_s^1\sigma_1 S_s^1\,dB_s^1 + \pi_s^2\sigma_2 S_s^2\,dB_s^2.$$

Finally, the dynamics of the state variables W_s, S_s^1, S_s^2 are given by the following controlled system of stochastic differential equations

$$dW_s = \pi_s^1(\mu_1+\delta z_s)S_s^1\,ds + \pi_s^2\mu_2 S_s^2\,ds + \pi_s^1\sigma_1 S_s^1\,dB_s^1 + \pi_s^2\sigma_2 S_s^2\,dB_s^2, \tag{2}$$

$$dS_s^1 = (\mu_1+\delta z_s)S_s^1\,dt + \sigma_1 S_s^1\,dB_s^1, \tag{3}$$

$$dS_s^2 = \mu_2 S_s^2\,ds + \sigma_2 S_s^2\,dB_s^2, \tag{4}$$

$$W_t = w, S_t^1 = s_1, S_t^2 = s_2, \tag{5}$$

where z_t is defined in (1).

We defined without ambiguity the evolution of all the state variables. However, the above system may not have a unique solution for every control pair (π^1,π^2).

A pair of controls (π^1,π^2) is said to be admissible if π^1,π^2 are real-valued, progressively measurable, are such that, (2)–(5) define a unique solution (W_s, S_s^1, S_s^2) for every time $s\in[0,T]$ and (π^1,π^2,S^1,S^2) satisfy the integrability condition

$$\mathbb{E}\int_t^T (\pi_s^1 S_s^1)^2 + (\pi_s^2 S_s^2)^2\,ds < +\infty. \tag{6}$$

We denote the set of admissible controls at the initial time of investment t, by $\mathcal{A}_t$. Next, we define the value function $u(t,w,s_1,s_2)$ of the following backward dynamic optimization problem: the investor seeks an admissible strategy (π_s^1,π_s^2) for every $s\in[t,T)$ that maximizes the utility he derives from wealth at time T, i.e.

$$u(t,w,s_1,s_2) = \sup_{(\pi^1,\pi^2)\in\mathcal{A}_t} \mathbb{E}[U(W_T^{t,w,s_1,s_2,(\pi^1,\pi^2)})], \tag{7}$$

where $W_T^{t,w,s_1,s_2,(\pi^1,\pi^2)}$ denotes the solution of (2) at time T, corresponding to the control pair (π^1,π^2), the stock prices S^1,S^2 whose dynamics are respectively defined by (3) and (4) and for the initial conditions (5).

Furthermore, in this paper, we only treat the case of the exponential utility function, i.e.

$$U(w) = -e^{-\gamma w},$$

where $\gamma>0$ denotes the constant risk aversion coefficient.

One could alternately use the power function $U(w)=(1/\gamma)c^\gamma$ as in Benth and Karlsen (2005). However, for this model, we are unable to carry out the calculations explicitly and derive a solution in closed form. As we will see in the next section, the exponential utility function allows us to factor out the wealth variable and reduce the problem to a two-dimensional PDE. We simply cannot make the same ansatz for the power function.

3. The Hamilton–Jacobi–Bellman equation and its solution

Since we do not know a priori the regularity of the value function of the stochastic control problem, we only proceed formally with the hope of obtaining a smooth candidate solution for the stochastic control problem. We will verify later, in Section 4, that our candidate solution coincides with the solution of the optimal stochastic control.

We expect the value function u defined above to satisfy the following HJB partial differential equation:

$$\begin{aligned}-u_t - \sup_{\pi_1,\pi_2}[&(\pi_1(\mu_1+\delta z)s_1 + \pi_2\mu_2 s_2)u_w + (\mu_1+\delta z)s_1 u_{s_1}\\ &+ \mu_2 s_2 u_{s_2} + \pi_1\sigma_1^2 s_1^2 u_{ws_1} + \pi_2\sigma_2^2 s_2^2 u_{ws_2} + \tfrac{1}{2}(\pi_1^2\sigma_1^2 s_1^2 + \pi_2^2\sigma_2^2 s_2^2)u_{ww} + \tfrac{1}{2}\sigma_1^2 s_1^2 u_{s_1s_1} + \tfrac{1}{2}\sigma_2^2 s_2^2 u_{s_2s_2}] = 0,\end{aligned} \tag{8}$$

for all $0\le t<T, w, 0\le s_1, 0\le s_2$, together with the final condition

$$u(T,w,s_1,s_2) = U(w) = -\exp(-\gamma w). \tag{9}$$

We introduce the following notations that will be needed later in Section 4. We define the linear operator

$$\begin{aligned}\mathcal{L}^{(\pi_1,\pi_2)}u(t,w,s_1,s_2) &= (\pi_1(\mu_1+\delta z)s_1 + \pi_2\mu_2 s_2)u_w\\ &+ (\mu_1+\delta z)s_1 u_{s_1} + \mu_2 s_2 u_{s_2} + \pi_1\sigma_1^2 s_1^2 u_{ws_1} + \pi_2\sigma_2^2 s_2^2 u_{ws_2}\\ &+ \tfrac{1}{2}(\pi_1^2\sigma_1^2 s_1^2 + \pi_2^2\sigma_2^2 s_2^2)u_{ww} + \tfrac{1}{2}\sigma_1^2 s_1^2 u_{s_1s_1} + \tfrac{1}{2}\sigma_2^2 s_2^2 u_{s_2s_2}.\end{aligned}$$

We also define the matrix

$$A = \begin{bmatrix} p_{ww} & p_{ws_1} & p_{ws_2} \\ p_{s_1w} & p_{s_1s_1} & p_{s_1s_2} \\ p_{s_2w} & p_{s_2s_1} & p_{s_2s_2} \end{bmatrix}$$

and the Hamiltonian

$$\begin{aligned} H(t,w,s_1,s_2,p_w,p_{s_1},p_{s_2},A) = \sup_{\pi_1,\pi_2}\{&(\pi_1(\mu_1+\delta z)s_1+\pi_2\mu_2 s_2)p_w \\ &+(\mu_1+\delta z)s_1 p_{s_1}+\mu_2 s_2 p_{s_2}+\pi_1\sigma_1^2 s_1^2 p_{ws_1}+\pi_2\sigma_2^2 s_2^2 p_{ws_2} \\ &+\tfrac{1}{2}(\pi_1^2\sigma_1^2 s_1^2+\pi_2^2\sigma_2^2 s_2^2)p_{ww}+\tfrac{1}{2}\sigma_1^2 s_1^2 p_{s_1s_1}+\tfrac{1}{2}\sigma_2^2 s_2^2 p_{s_2s_2}\}. \end{aligned}$$

We then proceed by making two classic changes of variable: we first apply the standard logarithmic transformation and secondly, we reduce the number of dimensions in the HJB equation by factoring out the wealth variable. To this end, we let $s_1 = e^x, s_2 = e^y$ and $u(t,w,s_1,s_2) = -e^{-\gamma w} g(t,x,y)$. Then g solves the transformed HJB equation

$$\begin{aligned} -g_t + \sup_{\pi_1,\pi_2}[&(\pi_1(\mu_1+\delta z)s_1+\pi_2\mu_2 s_2)\gamma g - (\mu_1+\delta z)g_x \\ &-\mu_2 g_y+\pi_1\sigma_1^2 s_1\gamma g_x+\pi_2\sigma_2^2 s_2\gamma g_y \\ &-\tfrac{1}{2}(\pi_1)^2\sigma_1^2 s_1^2\gamma^2 g-\tfrac{1}{2}(\pi_2)^2\sigma_2^2 s_2^2\gamma^2 g \\ &-\tfrac{1}{2}\sigma_1^2(g_{xx}-g_x)-\tfrac{1}{2}\sigma_2^2(g_{yy}-g_y)] = 0, \end{aligned} \tag{10}$$

subject to

$$g(T,x,y) = 1. \tag{11}$$

We notice that the maximization over π_1, π_2 in (10) is achieved at

$$\pi_1^* = \frac{(\mu_1+\delta z)g+\sigma_1^2 g_x}{\sigma_1^2 s_1\gamma g}, \tag{12}$$

$$\pi_2^* = \frac{\mu_2 g+\sigma_2^2 g_y}{\sigma_2^2 s_2\gamma g}. \tag{13}$$

Plugging the formulae (12) and (13) into the PDE (10), we obtain

$$g_t = \frac{1}{2}\left(\frac{(\mu_1+\delta z)^2}{\sigma_1^2}+\frac{\mu_2^2}{\sigma_2^2}\right)g+\frac{\sigma_1^2}{2}g_x+\frac{\sigma_2^2}{2}g_y+\frac{1}{2}\left(\frac{\sigma_1^2 g_x^2}{g}+\frac{\sigma_2^2 g_y^2}{g}\right)-\frac{1}{2}\left(\sigma_1^2 g_{xx}+\sigma_2^2 g_{yy}\right) \tag{14}$$

Note that we can recover the HJB equation for the standard Merton problem (see Merton, 1971) in two dimensions by setting $\delta = 0$ in our model; in this well known case, the value function is independent of the stocks and the dollar amounts invested in each stock are constant. We recall below the closed form formulae for the value function and the dollar amounts corresponding to the particular case $\delta = 0$

$$g(t,x,y) = \exp\left(-\frac{\mu_1^2}{2\sigma_1^2}(T-t)-\frac{\mu_2^2}{2\sigma_2^2}(T-t)\right), \quad s_1\pi_1^*(t,x,y) = \frac{\mu_1}{\sigma_1^2\gamma}, \quad s_2\pi_2^*(t,x,y) = \frac{\mu_2}{\sigma_2^2\gamma}.$$

In the co-integrated case, the solution is no longer independent of the stocks. It essentially depends on the value of the co-integration process z, rather than on the individual stocks. We can therefore reduce (14) to a one-dimensional equation in the variable

$$X = \mu_1+\delta z = \mu_1+\delta(a+x+\beta y).$$

Furthermore, we combine this change of variable with a logarithmic transformation of the value function, in order to get rid of the nonlinearity. Indeed, simple calculations show that the function $\Phi(t,X) = -\log(g(t,x,y))$ solves the linear parabolic PDE

$$\Phi_t = -\frac{1}{2}\left(\frac{X^2}{\sigma_1^2}+\frac{\mu_2^2}{\sigma_2^2}\right)+\frac{1}{2}(\sigma_1^2+\beta\sigma_2^2)(\delta\Phi_X)-\frac{1}{2}(\sigma_1^2+\beta^2\sigma_2^2)(\delta^2\Phi_{XX}) \tag{15}$$

for any real number X and time $0 \le t < T$ and satisfies the terminal condition

$$\Phi(T,X) = 0. \tag{16}$$

It is easy to see that $\Phi(t,X) = a(t)X^2+b(t)X+c(t)$ is an explicit solution of the linear PDE (15) and (16), where the coefficients a, b, c are given by

$$a(t) = \frac{1}{2}\frac{(T-t)}{\sigma_1^2} \tag{17}$$

$$b(t) = -\frac{1}{4}\frac{(\sigma_1^2+\beta\sigma_2^2)\delta}{\sigma_1^2}(T-t)^2 \tag{18}$$

$$c(t) = \frac{1}{2}\frac{\mu_2^2}{\sigma_2^2}(T-t) + \frac{1}{4}\frac{(\sigma_1^2+\beta^2\sigma_2^2)\delta^2}{\sigma_1^2}(T-t)^2 + \frac{1}{24}\frac{(\sigma_1^2+\beta\sigma_2^2)^2\delta^2}{\sigma_1^2}(T-t)^3. \tag{19}$$

We also compute the optimal policies

$$\pi_1^* s_1 = \frac{X}{\sigma_1^2\gamma} + \frac{\delta(-2a(t)X - b(t))}{\gamma},$$
$$\pi_2^* s_2 = \frac{\mu_2}{\gamma\sigma_2^2} + \frac{\delta\beta(-2a(t)X - b(t)}{\gamma}.$$

After substituting for $a(t), b(t)$ their expressions in (17) and (18), we obtain

$$\pi_1^* s_1 = \frac{(\mu_1+\delta z)}{\gamma\sigma_1^2} - \delta\frac{(\mu_1+\delta z)}{\gamma\sigma_1^2}(T-t) + \frac{1}{4}\frac{\delta^2(\sigma_1^2+\beta\sigma_2^2)}{\gamma\sigma_1^2}(T-t)^2, \tag{20}$$

$$\pi_2^* s_2 = \frac{\mu_2}{\gamma\sigma_2^2} - \beta\delta\frac{(\mu_1+\delta z)}{\gamma\sigma_1^2}(T-t) + \frac{1}{4}\beta\frac{\delta^2(\sigma_1^2+\beta\sigma_2^2)}{\gamma\sigma_1^2}(T-t)^2. \tag{21}$$

4. Verification

We must verify that the smooth candidate solution we derived in the previous section is indeed the value function of the stochastic control problem. This can be achieved by proving a *verification* result, which connects the HJB equation to the optimal control problem. In this section, we first state below the *verification* theorem that we need. We omit its proof since it is standard (for instance see Fleming and Soner, 1993). The next step then consists in checking that the assumptions of the *verification theorem* are satisfied and we show that this is indeed the case, under certain conditions on the parameters in the model.

First of all, we recall that u, defined in (7), denotes the value function of the optimal stochastic control problem and we state the *verification theorem*.

Theorem 1 (*Verification*). *Let $v \in C^{1,2}([0,T), \mathbb{R}\times(0,+\infty)^2) \cap C([0,T)\times\mathbb{R}\times[0,+\infty)^2)$. Assume that $v(\tau, W_\tau, S_\tau^1, S_\tau^2)$ is a nonpositive and uniformly integrable function for every stopping-time $\tau \in [t,T]$ for the process (W_u, S_u^1, S_u^2) starting at (w, s_1, s_2) at time t and for every admissible control pair (π_1, π_2) such that*

$$0 = v_t(t,w,s_1,s_2) + \mathcal{L}^{(\pi_1,\pi_2)}v(t,w,s_1,s_2).$$

(i) *Suppose that $v(t,w,\dot{}) \geq U(w)$ and*
$$-v_t(t,w,s_1,s_2) - H(t,w,s_1,s_2,Dv,D^2v) \geq 0$$
on $[0,T)\times\mathbb{R}\times(0,+\infty)^2$. Then $v \geq u$ on $[0,T]\times\mathbb{R}\times[0,+\infty)^2$.

(ii) *Assume further that $v(T,w,.) = U(w)$ and there exists a maximizer $(\hat{\pi_1}, \hat{\pi_2})$ of $(\pi_1,\pi_2) \to \mathcal{L}^{(\pi_1,\pi_2)}v(t,w,s_1,s_2)$ such that*
- $0 = v_t(t,w,s_1,s_2) + \mathcal{L}^{(\hat{\pi_1},\hat{\pi_2})}v(t,w,s_1,s_2)$.
- *The system of stochastic differential equations (2)–(4) for the pair of controls $(\hat{\pi_1}, \hat{\pi_2})$ and each initial data $(W_t, S_t^1, S_t^2) = (w, s_1, s_2)$ defines a unique solution.*
- *The processes $\mu_s^1 = \hat{\pi_1}(s, W_s, S_s^1, S_s^2)$, $\mu_s^2 = \hat{\pi_2}(s, W_s, S_s^1, S_s^2)$ are well-defined control processes in $\mathcal{A}$.*

Then, $v = u$ and (μ_s^1, μ_s^2) is an optimal Markov control process.

Next, we must check that the candidate solution satisfies the conditions required in Theorem 1. First of all, it is easy to show that the optimal control pair $(\hat{\pi_1}, \hat{\pi_2})$ given by the formula (20) and (21) is a Markovian control and is admissible. In particular, the integrability condition (6) holds. Furthermore, one can write explicitly the unique solution W_t of (2) corresponding to the optimal control, in terms of the process z_t and the parameters in the model.

Finally, we have to verify that the uniform integrability condition holds. The proof of this fact almost follows along the lines of Benth and Karlsen (2005). More precisely, we apply the Lemma in Benth and Karlsen (2005) proving the exponential integrability of the square of an Ornstein–Uhlenbeck process and use it to derive some sufficient conditions on the parameters of our model, under which the uniform integrability condition holds. First, we recall the lemma in Benth and Karlsen (2005).

Lemma 1 (Benth and Karlsen, 2005). *If λ is a constant such that*

$$\lambda < \frac{|\delta|}{2\sigma_\beta^2(T-t)},$$

then

$$\mathbb{E}\left[\exp\left\{\lambda \int_t^T (z_u^{t,z})^2\, du\right\}\right] < \infty.$$

We then state our main result, which is an application of Theorem 1. It provides an explicit solution for the optimal stochastic control under some conditions on the parameters.

Theorem 2 (*Main result*). *If*

$$|\delta|T < \frac{1}{4}\left(\frac{\sigma_1^4}{\sigma_2^2\sigma_\beta^2\beta^2}\right)^{1/3}$$

and

$$|\delta|T(1-\delta T)^2 < \frac{1}{64}\frac{\sigma_1^2}{\sigma_\beta^2},$$

the value function of the optimal stochastic problem is given by

$$u(t,w,s_1,s_2) = -\exp(-\gamma w)\exp(-a(t)X^2 - b(t)X - c(t))$$

where

$$X = \mu_1 + \delta z = \mu_1 + \delta(a + \log s_1 + \beta \log s_2)$$

and the coefficients $a(\cdot), b(\cdot), c(\cdot)$ are given by (17)–(19).

Furthermore, the optimal control pair is given by (20) and (21).

Since the proof of Theorem 2 uses exactly the same arguments as in Benth and Karlsen (2005), we do not present it here. We can see that the conditions on the parameters translate into a limit on the time-horizon. Beyond a certain time horizon, we cannot guarantee that the solution we computed is the unique solution of the optimal stochastic problem.

5. A more general model with correlations

It is also quite straightforward to incorporate correlations between the stocks into the above model and we present briefly this extension in this section. The dynamics will read in this case

$$dW_t = \pi_1\, dS_t^1 + \pi_2\, dS_t^2$$
$$dS_t^1 = (\mu_1 + \delta z_t)S_t^1\, dt + \sigma_1 S_t^1\, dB_t^1$$
$$dS_t^2 = \mu_2 S_t^2\, dt + \sigma_2 S_t^2(\rho\, dB_t^1 + \sqrt{1-\rho^2}dB_t^2)$$

where $0 < \rho < 1$ denotes the correlation coefficient and the co-integrating vector z_t is still defined by

$$z_t = a + \log S_t^1 + \beta \log S_t^2.$$

Substituting, we find that the wealth satisfies the stochastic differential equation

$$dW_t = \pi_1(\mu_1 + \delta z_t)S_t^1\, dt + \pi_2\mu_2 S_t^2\, dt + \pi_1\sigma_1 S_t^1\, dB_t^1 + \pi_2\sigma_2 S_t^2(\rho\, dB_t^1 + \sqrt{1-\rho^2}\, dB_t^2).$$

In this case z_t satisfies the SDE

$$\begin{aligned} dz_t &= \left(\mu_1 - \frac{\sigma_1^2}{2} + \delta z_t\right)dt + \sigma_1\, dB_t^1 + \beta\left(\mu_2 - \frac{\sigma_2^2}{2}\right)dt + \beta\sigma_2\, dB_t^2 \\ &= \left(\mu_1 - \frac{\sigma_1^2}{2} + \beta\mu_2 - \beta\frac{\sigma_2^2}{2} + \delta z_t\right)dt + \sigma_1\, dB_t^1 \\ &\quad + \beta\sigma_2\left(\rho\, dB_t^1 + \sqrt{1-\rho^2}\, dB_t^2\right) \\ &= \alpha(\eta - z_t)dt + \sigma_\beta\, dB_t \end{aligned}$$

where $\alpha=-\delta$ is the speed of mean reversion, $\sigma_\beta=\sqrt{\sigma_1^2+\beta^2\sigma_2^2+2\beta\sigma_1\sigma_2\rho}$, $B_t=((\sigma_1+\beta\sigma_2\rho)/\sigma_\beta)B_t^1+\beta(\sigma_2\sqrt{1-\rho^2}/\sigma_\beta)B_t^2$ is a Brownian motion adapted to $\mathcal{F}_t$ and

$$\eta=-\frac{1}{\delta}\left(\mu_1-\frac{\sigma_1^2}{2}+\beta\left(\mu_2-\frac{\sigma_2^2}{2}\right)\right)$$

is the equilibrium level. The value function of this stochastic control problem is defined, as earlier, in (7).

For a sake of simplicity, we postpone the calculations of the explicit solution to Appendix A and we just summarize our findings in the following theorem.

Theorem 3. *If*

$$|\delta|T\left(\frac{\rho\sigma_1}{\sigma_2\beta}+\delta T\right)^2<\frac{1}{64}\left(\frac{\sigma_1^4}{\sigma_2^2\sigma_\beta^2\beta^2}\right)(1-\rho^2) \tag{22}$$

and

$$|\delta|T\left(1-\delta T\frac{1}{1-\rho^2}(1+\rho\beta\sigma_2/\sigma_1)\right)^2<\frac{1}{64}\frac{\sigma_1^2}{\sigma_\beta^2}, \tag{23}$$

the value function of the optimal stochastic problem with the correlation coefficient ρ is given by

$$u(t,w,s_1,s_2)=-\exp(-\gamma w)\exp(-a(t)X^2-b(t)X-c(t))$$

where

$$X=\mu_1+\delta z=\mu_1+\delta(a+\log s_1+\beta\log s_2)$$

and the coefficients $a(\cdot),b(\cdot),c(\cdot)$ are given by (24)–(26) below

$$a(t)=\frac{1}{2}\frac{(T-t)}{(1-\rho^2)\sigma_1^2} \tag{24}$$

$$b(t)=-\frac{1}{4}\frac{(T-t)^2}{(1-\rho^2)\sigma_1^2}(\sigma_1^2+\beta\sigma_2^2)\delta-\frac{\rho}{(1-\rho^2)}\frac{\mu_2(T-t)}{\sigma_1\sigma_2} \tag{25}$$

$$c(t)=\frac{1}{2}\left(\frac{(T-t)\mu_2^2}{(1-\rho^2)\sigma_2^2}+\frac{1}{4}\frac{(\sigma_1^2+\beta^2\sigma_2^2+2\sigma_1\sigma_2\beta\rho)\delta^2}{(1-\rho^2)\sigma_1^2}(T-t)^2\right)+\frac{1}{4}\frac{\delta\rho}{(1-\rho^2)}\frac{\mu_2(\sigma_1^2+\beta\sigma_2^2)(T-t)^2}{\sigma_1\sigma_2}+\frac{1}{24}\frac{(\sigma_1^2+\beta\sigma_2^2)^2\delta^2}{(1-\rho^2)\sigma_1^2}(T-t)^3. \tag{26}$$

Furthermore, the optimal control pair is given by (27) and (28)

$$\pi_1^*s_1=\frac{\mu_1+\delta z}{\gamma(1-\rho^2)\sigma_1^2}-\frac{\rho\mu_2}{\gamma(1-\rho^2)\sigma_1\sigma_2}+\frac{\delta}{\gamma(1-\rho^2)}\left(-\frac{(\mu_1+\delta z)}{\sigma_1^2}+\frac{\rho\mu_2}{\sigma_1\sigma_2}\right)(T-t)+\frac{1}{4}\frac{\delta^2(\sigma_1^2+\beta\sigma_2^2)}{\gamma(1-\rho^2)\sigma_1^2}(T-t)^2 \tag{27}$$

$$\pi_2^*s_2=\frac{\mu_2}{\gamma(1-\rho^2)\sigma_2^2}-\frac{\rho(\mu_1+\delta z)}{\gamma(1-\rho^2)\sigma_1\sigma_2}+\frac{\delta\beta}{\gamma(1-\rho^2)}\left(-\frac{(\mu_1+\delta z)}{\sigma_1^2}+\frac{\rho\mu_2}{\sigma_1\sigma_2}\right)(T-t)+\frac{1}{4}\frac{\delta^2\beta(\sigma_1^2+\beta\sigma_2^2)}{\gamma(1-\rho^2)\sigma_1^2}(T-t)^2. \tag{28}$$

6. Example

We provide an example to illustrate our results. We wish to emphasize that we are not conducting a comprehensive study here, on whether stocks are co-integrated. Furthermore, we have not tested our strategy's performance in real-time. This would require implementing an algorithm that sequentially detects co-integrated assets, estimates the parameters in the model and finally applies the computed optimal strategies. Here, we are assuming that the whole data set is available prior to the beginning of the trading day and we calibrate the model based on this data set. We then apply the optimal strategy, starting at the beginning of the trading day. This exercise, that we present in this section, is merely to illustrate our method with a concrete real-life example.

We simply browsed a number of arbitrary data sets and we picked one among those. We collected on October 17, 2011, minute-by-minute data, on two stocks traded on the New York Stock Exchange, Goldman Sachs Group, Incorporated, with ticker symbols GS, and J.P. Morgan Chase and Company, with ticker symbol JPM. This gave us a two-dimensional time series with 390 data points. As in Duan and Pliska (2004), we followed the standard two-step Engle–Granger methodology coupled with a Dickey–Fuller test (Engle and Granger, 1987), to test for co-integration and estimate the parameters in the co-integration model. We also performed in addition a Phillips and Ouliaris (1990) test whose outcome further confirmed that our time series was co-integrated. We refer to the book by Enders (2004) for a general presentation of co-integration tests.

More precisely, we first ran a regression as in Eq. (7) in Duan and Pliska (2004), but without a time trend variable, which seems reasonable for an intra-day data set. The Augmented-Dickey–Fuller (up to 16 lags) test suggested that $\log(S^1)$ and

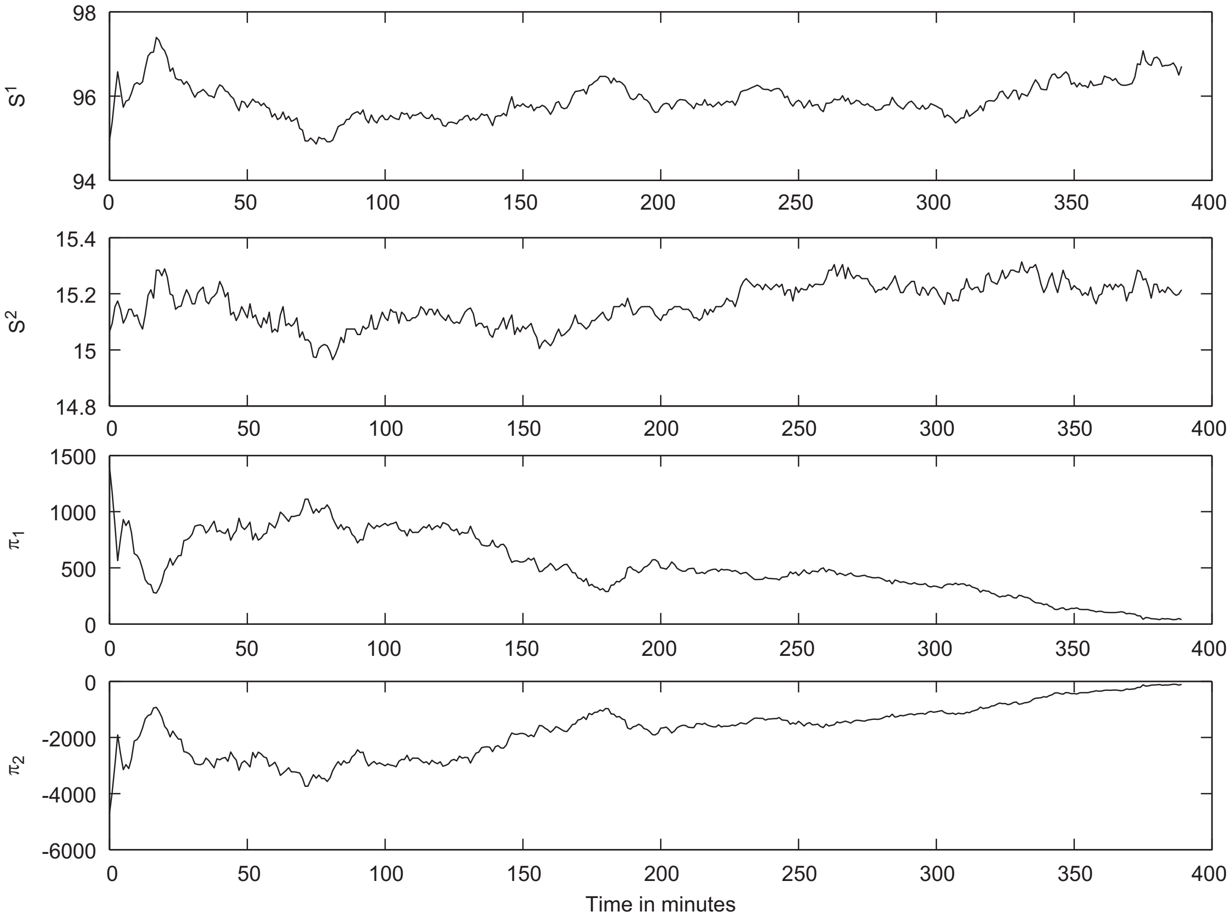

Fig. 1. Stocks and optimal policies.

$\log(S^2)$ were co-integrated at the 5% level and the variance-ratio Phillips–Ouliaris test statistic suggested that after de-trending with constant and linear trend, the data were co-integrated at the 5% level. We obtained the following annualized parameters:

$a = -3.105148$, $\beta = -0.5363258$, $\delta = -3561.194$, $\eta = 9.298768e\text{–}04$, $\sigma_\beta = 0.3085159$, $\sigma_1 = 0.3828025$, $\sigma_2 = 0.5076713$, $\rho = 0.61181479$, $\mu_1 = 4.662472$, $\mu_2 = 2.623764$.

For this set of parameters, we estimated the maximal time horizon T satisfying the set of conditions (22) and (23) and we found about 38 s. For the purpose of illustrating, we chose not to comply with the 38-s bound and plotted our stock prices and the optimal policies for a whole trading day.

More precisely, we show the stock prices S^1, S^2, as well as the optimal policies π_1, π_2 in Fig. 1. We then present in Fig. 2, the ratio $|\pi_1/\pi_2|$ and the cumulative profit and loss function.

As expected, in a pairs trading setting, the controls are opposite in sign. For this data set and for a risk tolerance $\gamma = 0.1$, a significant profit is instantly realized. The profit then fluctuates throughout the day but remains strongly positive and by the end of the day, it is approximately $1348. Of course, this figure does not take into account the cost of borrowing or transaction costs which are both assumed to be 0 in this model. We also notice that the positions, which are large during the first half of the day, are both progressively unwound in the second half, ending close to 0 by the end of the trading day.

7. Conclusion

In this paper, we propose a dynamic model for pairs trading based on the theory of optimal stochastic control and we briefly illustrate the applicability of our method with minute-by-minute historical stock data.

In our model, the two stock processes are co-integrated, correlated, and have constant volatility. We also assume that the risk free interest rate is zero and we ignore the costs associated with trading. The simplicity of the present formulation enables a feasible implementation of parameter calibration and the derivation of analytical formulae for the optimal trading strategies.

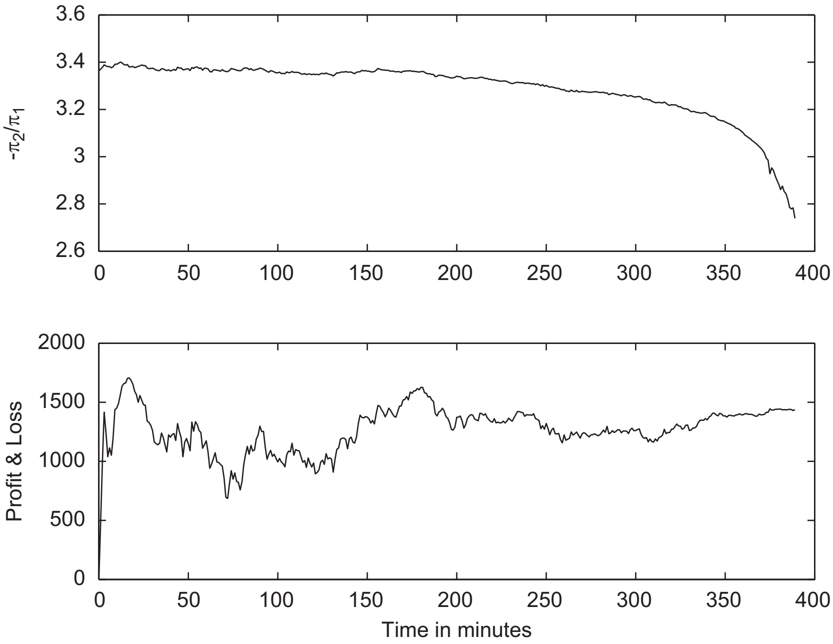

Fig. 2. The optimal ratio and the profit and loss processes.

Finally, we acknowledge that we have not addressed in this work the question of detecting two instruments whose market price tends to evolve in tandem, although this is undoubtedly a fundamental issue.

Acknowledgments

Agnès Tourin was partially supported by an immersion fellowship from the Fields Institute in Toronto (Canada) from January to June 2010. We also thank the anonymous referees for their thorough reviews and their thoughtful comments and suggestions.

Appendix A. Computation of the optimal solution when the stocks are correlated

Next, we expect the value function $u(t,w,s_1,s_2)$ to solve the HJB equation

$$\begin{aligned}
&u_t + \sup_{\pi_1,\pi_2}[(\pi_1(\mu_1+\delta z)s_1+\pi_2\mu_2 s_2)u_w+(\mu_1+\delta z)s_1u_1+\mu_2 s_2 u_2\\
&\quad+\pi_1\sigma_1^2 s_1^2 u_{ws_1}+\pi_2\rho\sigma_1\sigma_2 s_1 s_2 v_{s_1w}+\pi_2\sigma_2^2 s_2^2 u_{ws_2}\\
&\quad+\pi_1\rho\sigma_1\sigma_2 s_1 s_2 v_{s_2w}+\tfrac{1}{2}(\pi_1^2\sigma_1^2 s_1^2+\pi_2^2\sigma_2^2 s_2^2+\rho\pi_1\pi_2\sigma_1\sigma_2 s_1 s_2)u_{ww}\\
&\quad+\tfrac{1}{2}\,\sigma_1^2 s_1^2 u_{s_1s_1}+\tfrac{1}{2}\sigma_2^2 s_2^2 u_{s_2s_2}+\rho\sigma_1\sigma_2 s_1 s_2 v_{s_1s_2}]=0
\end{aligned}$$

The function $g(t,x,y)$ satisfies the HJB equation

$$\begin{aligned}
&-g_t+\sup_{\pi_1,\pi_2}[(\pi_1(\mu_1+\delta z)s_1+\pi_2\mu_2 s_2)\gamma g-(\mu_1+\delta z)g_x-\mu_2 g_y\\
&\quad+\pi_1\sigma_1^2 s_1\gamma g_x+\pi_2\gamma\rho\sigma_1\sigma_2 s_2 g_x+\pi_2\sigma_2^2 s_2\gamma g_y+\pi_1\gamma\rho\sigma_1\sigma_2 s_1 g_y\\
&\quad-\tfrac{1}{2}\,(\pi_1)^2\sigma_1^2 s_1^2\gamma^2 g-\tfrac{1}{2}(\pi_2)^2\sigma_2^2 s_2^2\gamma^2 g-\gamma^2\pi_1\pi_2\rho\sigma_1\sigma_2 s_1 s_2 g\\
&\quad-\tfrac{1}{2}\,\sigma_1^2(g_{xx}-g_x)-\tfrac{1}{2}\sigma_2^2(g_{yy}-g_y)-\rho\sigma_1\sigma_2 g_{xy}]=0
\end{aligned}\tag{A.1}$$

subject to

$$g(T,x,y)=1. \tag{A.2}$$

The optimal controls are

$$\pi_1^*=\frac{(\mu_1+\delta z)}{\gamma(1-\rho^2)\sigma_1^2 s_1}+\frac{g_x}{\gamma g s_1}-\rho\frac{\mu_2}{\gamma(1-\rho^2)\sigma_1\sigma_2 s_1}, \tag{A.3}$$

$$\pi_2^* = \frac{\mu_2}{\gamma(1-\rho^2)\sigma_2^2 s_2} + \frac{g_y}{\gamma g s_2} - \rho\frac{(\mu_1+\delta z)}{\gamma(1-\rho^2)\sigma_1\sigma_2 s_2}. \tag{A.4}$$

After substituting the controls into the HJB equation, we obtain the PDE

$$g_t = \left(\frac{1}{2}\frac{(\mu_1+\delta z)^2}{(1-\rho^2)\sigma_1^2} + \frac{1}{2}\frac{\mu_2^2}{(1-\rho^2)\sigma_2^2} - \frac{\rho\mu_2(\mu_1+\delta z)}{(1-\rho^2)\sigma_1\sigma_2}\right)g + \frac{\sigma_1^2}{2}g_x + \frac{\sigma_2^2}{2}g_y + \frac{1}{2}\sigma_1^2\frac{g_x^2}{g} + \frac{1}{2}\sigma_2^2\frac{g_y^2}{g} + \rho\sigma_1\sigma_2\frac{g_x g_y}{g}$$
$$-\frac{1}{2}\sigma_2^2\sigma_1^2 g_{xx} - \frac{1}{2}\sigma_2^2 g_{yy} - \rho\sigma_1\sigma_2 g_{xy}.$$

Replacing the variables (x,y) by the single variable $X = \mu_1 + \delta z$ and by using the exponential change of variable $g = \exp(-\Phi)$, we reduce the problem to the linear parabolic PDE

$$\Phi_t = -\frac{1}{1-\rho^2}\left(\frac{1}{2}\left(\frac{X^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2}\right) - \rho\frac{\mu_2 X}{\sigma_1\sigma_2}\right) + \frac{1}{2}(\sigma_1^2 + \beta\sigma_2^2)(\delta\Phi_X) - \frac{1}{2}(\sigma_1^2 + \beta^2\sigma_2^2 + 2\sigma_1\sigma_2\beta\rho)(\delta^2\Phi_{XX}) \tag{A.5}$$

for any real number X and time $0 \le t < T$ and is subject to the terminal condition

$$\Phi(T,X) = 0. \tag{A.6}$$

The above linear PDE has the explicit solution $\phi(t,X) = a(t)X^2 + b(t)X + c(t)$ where

$$a(t) = \frac{1}{2}\frac{(T-t)}{(1-\rho^2)\sigma_1^2}, \tag{A.7}$$

$$b(t) = -\frac{1}{4}\frac{(T-t)^2}{(1-\rho^2)\sigma_1^2}(\sigma_1^2 + \beta\sigma_2^2)\delta - \frac{\rho}{(1-\rho^2)}\frac{\mu_2(T-t)}{\sigma_1\sigma_2}, \tag{A.8}$$

$$c(t) = \frac{1}{2}\left(\frac{(T-t)\mu_2^2}{(1-\rho^2)\sigma_2^2} + \frac{1}{4}\frac{(\sigma_1^2 + \beta^2\sigma_2^2 + 2\sigma_1\sigma_2\beta\rho)\delta^2}{(1-\rho^2)\sigma_1^2}(T-t)^2\right) + \frac{1}{4}\frac{\delta\rho}{(1-\rho^2)}\frac{\mu_2(\sigma_1^2+\beta\sigma_2^2)(T-t)^2}{\sigma_1\sigma_2} + \frac{1}{24}\frac{(\sigma_1^2+\beta\sigma_2^2)^2\delta^2}{(1-\rho^2)\sigma_1^2}(T-t)^3. \tag{A.9}$$

Finally, we substitute the values of g, g_x and g_y into the formulae (A.3 and A.4), in order to obtain the optimal policies

$$\pi_1^* s_1 = \frac{\mu_1+\delta z}{\gamma(1-\rho^2)\sigma_1^2} + \frac{\delta(-2a(t)(\mu_1+\delta z) - b(t))}{\gamma} - \frac{\rho\mu_2}{\gamma(1-\rho^2)\sigma_1\sigma_2},$$
$$\pi_2^* s_2 = \frac{\mu_2}{\gamma(1-\rho^2)\sigma_2^2} + \frac{\delta\beta(-2a(t)(\mu_1+\delta z) - b(t))}{\gamma} - \frac{\rho(\mu_1+\delta z)}{\gamma(1-\rho^2)\sigma_1\sigma_2}.$$

References

Benth, F.E., Karlsen, K.H., 2005. A note on Merton's portfolio selection problem for the Schwartz mean-reversion model. Stochastic Analysis and Applications 23, 687–704.

Duan, J.-C., Pliska, S., 2004. Option valuation with co-integrated asset prices. Journal of Economic Dynamics and Control 28, 727–754.

Elliott, R.J., Hoek, J.V.D., Malcolm, W.P., 2005. Pairs trading. Quantitative Finance 5, 271–276.

Enders, W., 2004. Applied Econometric Time Series. Wiley Series in Probability and Statistics, Wiley.

Engle, R.F., Granger, C.W.J., 1987. Co-integration and error correction: representation, estimation and testing. Econometrica 55, 251–276.

Fleming, W.H., Soner, H.M., 1993. Controlled Markov Processes and Viscosity Solutions. Springer-Verlag.

Gatev, E., Goetzmann, W.N., Rouwenhorst, K.G., 2006. Pairs trading: performance of a relative value arbitrage rule. The Review of Financial Studies 19, 797–827.

Merton, R., 1971. Optimum consumption and portfolio rules in a continuous time model. Journal of Economic Theory 3, 373–413.

Mudchanatongsuk, S., Primbs, J.A., Wong, W., 2008. Optimal pairs trading: a stochastic control approach. In: Proceedings of the American Control Conference, pp. 1035–1039.

Phillips, P.C.B., Ouliaris, S., 1990. Asymptotic properties of residual based tests for cointegration. Econometrica 58, 165–193.