## Introduction to Cryptography

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# Cryptography

- = "Secret writing" throughout most of its history.
  - More precisely, "writing" with a hidden meaning as opposed to steganography where the existence of the "writing" itself is hidden.
- The idea is to make a message unintelligble except to the intended receiver.

- ▶ Up until the 1970's, the typical pattern was ([1])
  - Somebody creates a cipher.
  - ▶ They claim (or assume) the cipher is unbreakable.
  - Their enemy breaks the cipher using cryptanalysis.

#### Two Periods: BDH and ADH

- ▶ BDH Before Diffie-Hellman < 1976
- ► ADH After Diffie-Hellman > 1976

#### Two big changes in 1976

- Selection of Data Encryption Standard (DES) block cipher.
- Public-key cryptography Diffie-Hellman.

## BDH: Symmetric Cryptography

- A symmetric cipher uses the same key for encryption and decryption.
- ► Two main types:
  - Stream cipher.
  - Block cipher.
- ▶ Prior to 1970's, most ciphers were stream ciphers.
- ► A symmetric cipher consists of three algorithms G, E and D:
  - G generates a secret key k.
  - ▶ E takes key k and plaintext m and ouputs a ciphertext c i.e. c = E(k, m).
  - ▶ D takes a key k and a ciphertext c and ouputs a plaintext m i.e. m = D(k, c).

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- H gives the average number of bits of information contained in some message, which we call the amount of entropy.

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  - ► Then the unicity distance is the length of ciphertext U such that  $k \cdot \frac{s}{p} = 1$ .



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  - ▶ The value D = n H(M) is the redundancy of the plaintext.

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where H(M|C) the *conditional entropy* of M given a ciphertext from C.

▶ Put another way, for all plaintexts  $m \in M$  and all ciphertexts  $c \in C$ , we have

$$\Pr(x) = \Pr(m \mid c)$$

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- Example: using addition modulo 2 (XOR) when encrypting a binary message.



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  - Can be achieved via the technique of permutation (aka transposition).

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  - The unicity distance is  $U = H(K)/(n H(M)) = \log_2 26!/(\log_2 26 1.5) \approx 28.$

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- ▶ In English, the most frequently occurring letters are (in order) E, T, A, O, I, N, S, H, R, D, L, U...
- So given a ciphertext generated from an English plaintext, the most frequently occurring character likely corresponds to E etc.

# Polyalphabetic Substitution Cipher

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- ▶ The main idea is to change the substitution alphabet with each plaintext character, so the first letter is encrypted according to one alphabet, the second according to a different alphabet and so on (note the alphabets may repeat after a certain period).

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- ► Encrypting the text "HELLO" with keyword "BLAZE" yields the ciphertext "IPLKS".

### Viginère Cipher - Tabula Recta

ABCDEFGHIJKLMNOPQRSTUVWXYZ F G H I J K L M N O P Q R S T U V W X Y Z B B C D E F G H I J K L M N O P Q R S T U V W X Y Z A CCDEFGHIJKLMNOPQRSTUVW MNOPQRSTUVW LMNOPQRSTUVWXYZA F F G H I J K L M N O P Q R S T U V W X Y Z A B C D E I K L M N O P Q R S T U V W X Y Z A B C D E F LMNOPQRSTUVWXYZABC KLMNOPQRSTUVWXYZABCD LMNOPQRSTUVWXYZABCDEF K K L M N O P O R S T U V W X Y Z A B C D E F LLMNOPQRSTUVWXYZABCDE MMNOPORSTUVWXYZABCDEFGH NNOPQRSTUVWXYZABCDEFGHI OOPQRSTUVWXYZABCDE PPQRSTUVWXYZABCDEFGHI QQRSTUVWXYZABCDEFGHIJK RRSTUVWXYZABCDEFGHIJK SSTUVWXYZABCDEFGHI TTUVWXYZABCDEFGHIIKLMNO UUVWXYZABCDEFGH V V W X Y Z A B C D E F G H I J K L M N O P Q R S T U WWXYZABCDEFGHIJKLMNOPQRS XXYZABCDEFGHIJKLMNOPQRSTUVW Y Y Z A B C D E F G H I I K L M N O P Q R S T U V W X ZZABCDEFGHI IKLMNOPORSTUVWXY

### Transposition

- Idea: rearrange the plaintext (change the order) to produce the ciphertext.
- ► The positions of the plaintext characters are shifted.
- Also known as permutation.
- Examples:
  - Rail Fence
  - Route Cipher
  - Columnar Transposition

## Example: Columnar Transposition

- Write plaintext along rows whose length is determined by the key
- Example (here X denotes a null character):

```
T H E R E M U S T B E S O M E K I N D O F W A Y O U T O F H E R E X X
```

- ▶ Suppose the key specifies the row length as 5 and the order of columns to write out as 4, 2, 5, 1, 3.
- Then we get the ciphertext by writing out the columns in the specified order:
  - we obtain: RTMDYFXHUSTWTREBEOOHXTMEKFUEESONAO



## Example: Columnar Transposition (Cont'd)

- The key could be alternatively given as a keyword such as TOWER
  - ▶ the length of the keyword represents the row length.
  - the alphabetical order of the letters in the keyword gives the order of the columns to be written out.

### References



Barak, B.:

Cos 433: Cryptography.

www.cs.princeton.edu/courses/archive/fall07/cos433/.../lec1-intro.ppt (2007)