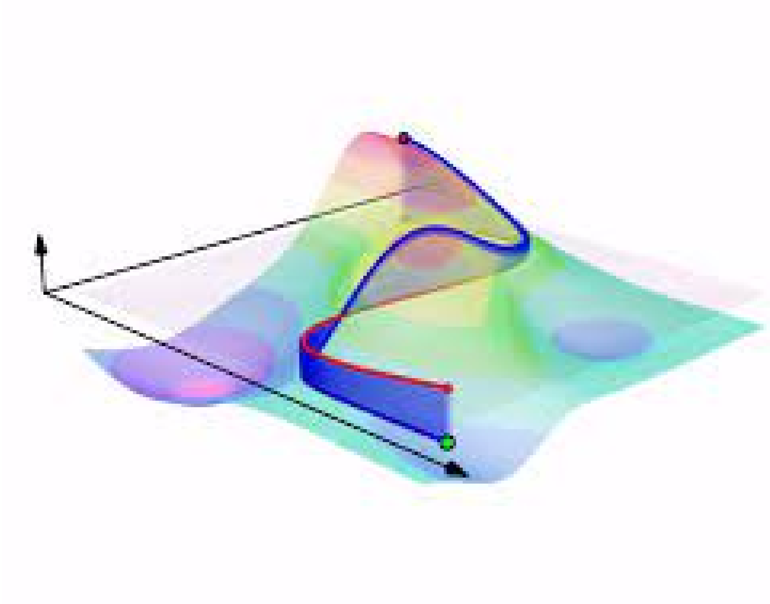
Universidad Autonoma de Aguascalientes
LICENCIATURA EN MATEMATICAS APLICADAS

MATERIA: Calculo Vectorial Integral

Docente: Jorge G. Macias Díaz

FECHA DE CREACION: 12 de agosto de 2023



BRYAN RICARDO BARBOSA OLVERA
12 de agosto de 2023

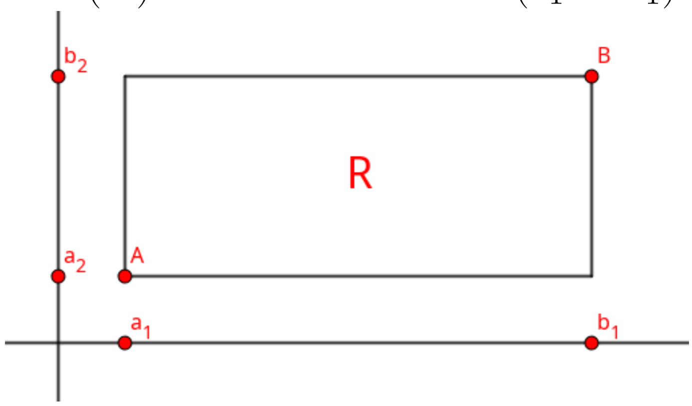
Profesor: Jorge G. Macias Díaz
Edificio: 117
Oficina: 6:30 - 3:30
Email: Jemacias@correo.uaa.mx
Email Secundario: jorge.maciasdiaz@edu.uaa.mx
Tel: +4494527006

1. INTEGRALES DOBLES

Def. Un rectangulo en $\mathbb{R}^2$ es

$$R = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$$

$$\text{Definimos } A(R) = \text{"Area de } R\text{"} = (b_1 - a_1)(b_2 - a_2)$$



Sean

$$\bar{a} = (a_1, a_2)$$
$$\bar{b} = (b_1, b_2)$$

Importante: Haaser usa la notacion $R = [\bar{a}, \bar{b}]$
Obs. Si $\mathcal{R} = \{R \subseteq \mathbb{R}^2; R \text{ es rectángulo } \}$ ent
$\mathcal{A} : \mathcal{R} \to \mathbb{R}^+ \cup \{0\}$ es una funcion.
Def. Sea $R = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$ una particion de R es

$$\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \text{ con}$$

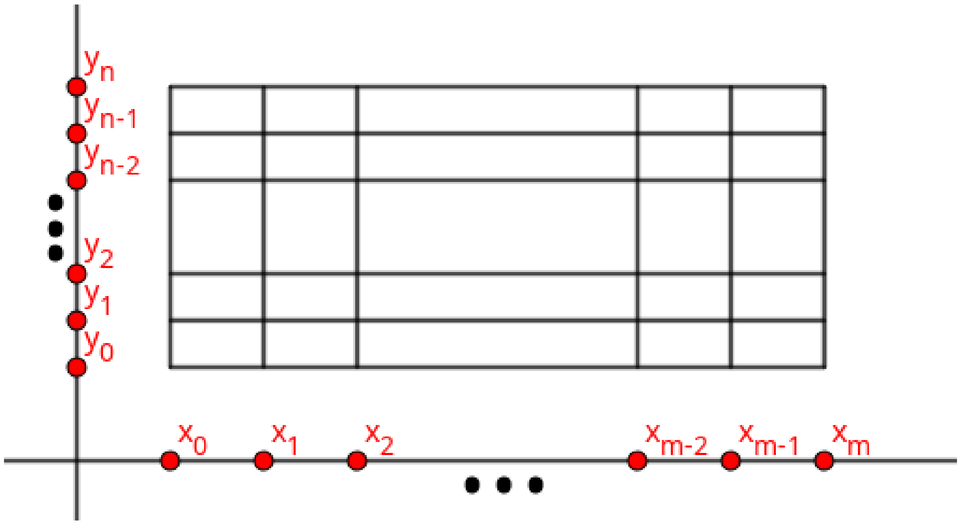
$\mathcal{P}$ es particicon de $[a_1, b_1]$
$\mathcal{P}$ es particicon de $[a_2, b_2]$

Concretamente, Si

$$\mathcal{P}_1 : a_1 = x_0 < x_1 < \cdots < x_m = b_1$$

$$\mathcal{P}_2 : a_2 = y_0 < y_1 < \cdots < y_n = b_2$$

$$\mathcal{P} = \{(x_i, y_j) \in \mathbb{R}^2 | i = 0, 1, \cdots, m; j = 0, 1, \cdots, n\}$$



Definimos $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \; \forall \; i = 1, \cdots, m \; \forall \; j = 1, \cdots, n$

Obs. $R = \cup_{i=1}^{m} \cup_{j=1}^{n} R_{ij}$

Def. Definimos la norma de $\mathcal{P}$ como

$$|\mathcal{P}| = \max\{|\mathcal{P}_1, \mathcal{P}_2|\}$$

$$|\mathcal{P}_1| = \max\{x_i - x_{i-1} | i = 1, \cdots, m\}$$

$$|\mathcal{P}_2| = \max\{y_j - y_{j-1} | j = 1, \cdots, n\}$$

Obs Con la notacion de la definicion anterior

$$\mathcal{A}(R) = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathcal{A}(R_{ij})$$

Dem.

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (\mathcal{A}(R_{ij})) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i - x_{i-1})(y_j - y_{j-1}) =$$

$$= \sum_{i=1}^{m} (x_i - x_{i-1}) \sum_{j=1}^{n} (y_j - y_{j-1}) = \sum_{i=1}^{m} (x_i - x_{i-1})(b_2 - a_2) =$$

$$= (b_2 - a_2) \sum_{i=1}^{m} (x_i - x_{i-1}) = (b_2 - a_2)(b_1 - a_1) = \mathcal{A}(R)$$

$\blacksquare$

Def. Definimos

$$\wp(R) = \{\mathcal{P} | \mathcal{P} \text{ particion de } R\}$$

Def. Sea R un rectangulo $R = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$.
Sean $\mathcal{P}, \mathcal{P}' \subset \wp(R)$
Diga

$$\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \wedge \mathcal{P}' = \mathcal{P}'_1 \times \mathcal{P}'_2$$

decimos $\mathcal{P}'$ es refinamiento de $\mathcal{P}$
Si $\mathcal{P}_1 \subseteq \mathcal{P}'_1 \wedge \mathcal{P}_2 \subseteq \mathcal{P}'_2$
Notacion: $\mathcal{P} \subseteq \mathcal{P}'$
Obs. En lo que sigue:

$$R = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2 \wedge f : R \to \mathbb{R} \quad \text{es acotada}$$

Mas aún, $m, M \in \mathbb{R} \ni$

$$\forall\ x \in \mathbb{R}\ : m \leq f(x) \leq M$$

Def. Sea $\mathcal{P} = \mathcal{P}_\infty \times \mathcal{P}_\in \in \wp(\mathbb{R})$
Diga

$$\mathcal{P}_1 : a_1 = x_0 < x_1 < \cdots < x_m = b_1$$

$$\mathcal{P}_2 : a_2 = y_0 < y_1 < \cdots < y_m = b_2$$

Definimos

$$L(f, \mathcal{P}) = \text{"Suma inferior de } f \text{ con respecto a } \mathcal{P}\text{"}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij}(f)\, \mathcal{A}(R_{ij})$$

con $m_{ij}(f) = \inf\{f(x) | x \in R_{ij}\}$

Definimos

$$U\left(f,\mathcal{P}\right) = \text{"Suma superior de } f \text{ con respecto a } \mathcal{P}\text{"}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\left(f\right)\mathcal{A}\left(R_{ij}\right)$$

con $M_{ij}\left(f\right) = \sup\{f\left(x\right)|x \in R_{ij}\}$

Lemma. $\forall\ \mathcal{P} \in \wp\left(R\right)$:

$$m\mathcal{A}\left(R\right) \leq L\left(f,\mathcal{P}\right) \leq U\left(f,\mathcal{P}\right) \leq M\mathcal{A}$$

Dem.

$$m\mathcal{A}\left(R\right) = m\sum_{i=1}^{m}\sum_{j=1}^{n}\mathcal{A}\left(R_{ij}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} m\mathcal{A}\left(R_{ij}\right)$$
$$\leq \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\left(f\right)\mathcal{A}\left(R_{ij}\right) = L\left(f,\mathcal{P}\right)$$
$$\leq \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\left(f\right)\mathcal{A}\left(R_{ij}\right) = U\left(f,\mathcal{P}\right)$$
$$\leq \sum_{i=1}^{m}\sum_{j=1}^{n} M\mathcal{A}\left(R_{ij}\right) = M\sum_{i=1}^{m}\sum_{j=1}^{n}\mathcal{A}\left(R_{ij}\right) = M\mathcal{A}\left(R\right)$$
$$\therefore m\mathcal{A}\left(R\right) \leq L\left(f,\mathcal{P}\right) \leq U\left(f,\mathcal{P}\right) \leq M\mathcal{A}\left(R\right)$$

■

Def. Definimos

$$\mathscr{L}\left(f\right) = \text{"Conjunto de sumas superiores de } f\text{"}$$
$$= \{L\left(f,\mathcal{P}\right)|\mathcal{P} \in \wp\left(R\right)\}$$
$$\mathscr{U}\left(f\right) = \text{"Conjunto de sumas inferiores de } f\text{"}$$
$$= \{U\left(f,\mathcal{P}\right)|\mathcal{P} \in \wp\left(R\right)\}$$

Obs. Del lema anterior.

$$\forall L(f,\mathcal{P}) \in \mathscr{L}(f) : L(f,\mathcal{P}) \subseteq M\mathcal{A}(R)$$

Asi, $\emptyset \neq \mathscr{L}(f)$ acotado sumeriormente $\Rightarrow\ \exists \sup \mathscr{L}(f)$
Similarmente $\exists$ $\inf \mathscr{U}(f)$
Def. Definimos

$$\overline{\int_R} f = \text{"Integral inferior de } f \text{ sobre } R\text{"} = \sup \mathscr{L}(f)$$
$$\underline{\int_R} f = \text{"Integral superior de } f \text{ sobre } R\text{"} = \inf \mathscr{U}(f)$$

Lemma Sean $\mathcal{P}, \mathcal{P}' \in \wp(R) \ni \mathcal{P} \subseteq \mathcal{P}'$. Entonces

$$m\mathcal{A}(R) \leq L(f,\mathcal{P}) \leq L(f,\mathcal{P}') \leq U(f,\mathcal{P}') \leq U(f,\mathcal{P}) \leq M\mathcal{A}(R)$$

Dem.
Sean

$$\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \in \wp(R)$$
$$\mathcal{P}' = \mathcal{P}_1' \times \mathcal{P}_2' \in \wp(R)$$

Caso1. $\mathcal{P}_1 = \mathcal{P}_1'$. Sea
$n = \text{card}\ \{\mathcal{P}_2' \backslash \mathcal{P}_2\}\ \in\ \mathbb{N} \cup\ \{0\}$ Usaremos inducción sobre n
Si $n = 0$, Ent. $\mathcal{P}_2' = \mathcal{P}_2$
$\mathcal{P}' = \mathcal{P}$, Ent.

$$L(f,\mathcal{P}) = L(f,\mathcal{P}') \leq U(f,\mathcal{P}') = U(f,\mathcal{P})$$

PasoGeneral : Supongamos la conclusion valida para algun
$n \in \mathbb{N} \cup \{0\}$
veremos que es valido para $n+1$. Sea $\mathcal{P}_2'$ refinamiento de $\mathcal{P}_2\ \ni$
$n + 1 =\ \text{card}\ \ (\mathcal{P}_2' \backslash \mathcal{P}_2)$

Definimos $\mathcal{P}_2 \subseteq \mathcal{P}_2'' \subseteq \mathcal{P}_2'$ con
card $(\mathcal{P}_2'' \backslash \mathcal{P}_2) = n \wedge$ card $(\mathcal{P}_2' \backslash \mathcal{P}_2'') = 1$
por hipótesis de inducción:

$$L(f, \mathcal{P}) \leq L(f, \mathcal{P}'') \leq U(f, \mathcal{P}'') \leq U(f, \mathcal{P})$$

donde $\mathcal{P}'' = \mathcal{P}_2 \times \mathcal{P}_2''$
Digo

$$\mathcal{P}_2'' : a_2 = y_0 < y_1 < \cdots < y_{j_0} < y_{j_0+1} < \cdots < y_p = b_2$$
$$\mathcal{P}_2' : a_2 = y_0 < y_1 < \cdots < y_{j_0} < y^* < y_{j_0+1} < \cdots < y_p = b_2$$
$$\mathcal{P}_1 : a_1 = x_0 < x_1 < x_m = b_1$$

Obs.

$$L(f, \mathcal{P}) = \sum_{i=1}^{m} \sum_{j=1}^{p} \left(\inf_{[x_{i-1}, x_i] \times [y_{j-1}, y_j]} \right) \cdot \mathcal{A}\left([x_{i-1}, x_i] \times [y_{j-1}, y_j] \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1, j \neq j_0+1}^{p} \left(\inf_{[x_{i-1}, x_i] \times [y_{j-1}, y_j]} \right) \cdot \mathcal{A}\left([x_{i-1}, x_i] \times [y_{j-1}, y_j] \right)$$
$$+ \sum_{i=1}^{m} \left(\inf_{[x_{i-1}, x_i] \times [y_{j_0}, y_{j_0+1}]} \right) \cdot \mathcal{A}\left([x_{i-1}, x_i] \times [y_{j_0}, y_{j_0+1}] \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1, j \neq j_0+1}^{p} \left(\inf_{[x_{i-1}, x_i] \times [y_{j-1}, y_j]} \right) \cdot \mathcal{A}\left([x_{i-1}, x_i] \times [y_{j-1}, y_j] \right)$$
$$+ \sum_{i=1}^{m} \left(\inf_{[x_{i-1}, x_i] \times [y_{j_0}, y^*]} \right) \cdot \mathcal{A}\left([x_{i-1}, x_i] \times [y_{j_0}, y^*] \right)$$
$$+ \sum_{i=1}^{m} \left(\inf_{[x_{i-1}, x_i] \times [y^*, y_{j_0+1}]} \right) \cdot \mathcal{A}\left([x_{i-1}, x_i] \times [y^*, y_{j_0+1}] \right)$$
$$= L(f, \mathcal{P}') \leq U(f, \mathcal{P}') \leq U(f, \mathcal{P}') \leq U(f, \mathcal{P}'')$$

(El caso 1 se sigue por inducción)
Caso2 : $\mathcal{P}_2 = \mathcal{P}_2' \wedge \mathcal{P}_1 \subseteq \mathcal{P}_1'$ similar al caso 1

Caso3 : $\mathcal{P}_1 \subseteq \mathcal{P}_1' \ \wedge \ \mathcal{P}_2 \subseteq \mathcal{P}_2'$

$$\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$$
$$\mathcal{P}' = \mathcal{P}_1 \times \mathcal{P}_2'$$

Definimos $\mathcal{P}'' = \mathcal{P}_1 \times \mathcal{P}_2'$ de los casos anteriores

$$L\left(f, \mathcal{P}\right) \leq L\left(f, \mathcal{P}''\right) \leq U\left(f, \mathcal{P}'\right) \leq U\left(f, \mathcal{P}''\right) \leq U\left(f, \mathcal{P}\right)$$

■

9/8/2023