# Instituto Tecnológico de Estudios Superiores de Occidente

## Materia: Finanzas Cuantitativas

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**Tarea 6: Modelado de Derivados**

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Fecha:
14 de junio de 2021

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# 1. Using Taylor Series solve in detail:

- $a = 0$
- $b = 0$
- $k = 2$

$f(x,y) = e^{xy} + x^2y^2 + 5$

**Starting from Taylor Series for two variables:**

$$f(x,y) = f(a,b) + \frac{\partial f(a,b)}{\partial x}(x-a) + \frac{\partial f(a,b)}{\partial y}(y-b) + \frac{1}{2!}\Bigg[\frac{\partial^2 f(a,b)}{\partial x^2}(x-a)^2 + \frac{\partial^2 f(a,b)}{\partial y^2}(y-b)^2 + \frac{\partial^2 f(a,b)}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 f(a,b)}{\partial y \partial x}(y-b)(x-a)\Bigg]$$

**The first step is to evaluate a and b in the function:**

$f(a,b) = e^{(0)(0)} + (0)^2(0)^2 + 5$
$f(a,b) = 1 + 0 + 5 = 6$

**Then, we calculate every partial derivative:**

$\frac{\partial f(x,y)}{\partial x} = ye^{xy} + 2xy^2$
$\frac{\partial f(a,b)}{\partial x} = 0e^{(0)(0)} + 2(0)(0)^2 = 0$

$\frac{\partial f(x,y)}{\partial y} = xe^{xy} + 2yx^2$
$\frac{\partial f(a,b)}{\partial y} = 0e^{(0)(0)} + 2(0)(0)^2 = 0$

$\frac{\partial^2 f(x,y)}{\partial x^2} = y^2e^{xy} + 2y^2$
$\frac{\partial^2 f(a,b)}{\partial x^2} = 0^2e^{(0)(0)} + 2(0)^2 = 0$

$\frac{\partial^2 f(x,y)}{\partial x \partial y} = e^{xy} + xye^{xy} + 4xy$
$\frac{\partial^2 f(a,b)}{\partial x \partial y} = e^{(0)(0)} + (0)(0)e^{(0)(0)} + 4(0)(0) = 1$

$\frac{\partial^2 f(x,y)}{\partial y \partial x} = e^{xy} + xye^{xy} + 4xy$
$\frac{\partial^2 f(a,b)}{\partial x \partial y} = e^{(0)(0)} + (0)(0)e^{(0)(0)} + 4(0)(0) = 1$

**And now we can replace in the formula and simplify:**

$f(x,y) = 6 + 0(x-0) + 0(y-0) + \frac{1}{2}\Big[0(x-0)^2 + 0(y-0)^2 + 1(x-0)(y-0) + 1(y-0)(x-0)\Big]$
$f(x,y) = 6 + \frac{1}{2}\Big[xy + xy\Big]$

$f(x,y) = 6 + \frac{1}{2}\Big[2xy\Big]$
$f(x,y) = 6 + xy$

## 2. Let's consider certain diffusion process $dX_t = X_t \sigma dW_t$ find differential form of a function $f(X_{(t+\Delta t),t+\Delta t}) = df_{(t+\Delta t)} := df_t$

**Using Taylor Series solution for two variables:**

$f(x,y) = f(a,b) + \frac{\partial f(a,b)}{\partial x}(x-a) + \frac{\partial f(a,b)}{\partial y}(y-b) + \frac{1}{2!}\Bigg[\frac{\partial^2 f(a,b)}{\partial x^2}(x-a)^2 + \frac{\partial^2 f(a,b)}{\partial y^2}(y-b)^2 + \frac{\partial^2 f(a,b)}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 f(a,b)}{\partial y \partial x}(y-b)(x-a)\Bigg]$

Let´s define some variables that will help us:

- $x = X_{(t+\Delta t)}$
- $y = t + \Delta t$
- $a = X_t$
- $b = t$
- $k = 2$

Rewriting our expression of $f(X_t, t)$ in Taylor Series form:

$f(X_t,t) = f(X_t,t) + \frac{\partial f(X_t,t)}{\partial X_t}(X_{(t+\Delta t)} - X_t) + \frac{\partial f(X_t,t)}{\partial t}(t+\Delta t - t) + \frac{1}{2!}\Bigg[\frac{\partial^2 f(X_t,t)}{\partial {X_t}^2}(X_{t+\Delta t} - X_t)^2 + \frac{\partial^2 f(X_t,t)}{\partial t^2}(t+\Delta t - t)^2 + \frac{\partial^2 f(X_t,t)}{\partial X_t \partial t}(X_{(t+\Delta t)} - X_t)(t+\Delta t - t) + \frac{\partial^2 f(X_t,t)}{\partial t \partial X_t}(t+\Delta t - t)(X_{(t+\Delta t)} - X_t)\Bigg]$

When $\Delta t \to 0$ we can express the subtraction $(X_t - X_t)$ as a differential of $X_t$ $(\Delta X_t)$. A similar case with $(t - t)$ that will become $(\Delta t)$.

$f(X_t,t) = f(X_t,t) + \frac{\partial f(X_t,t)}{\partial X_t}(\Delta X_t) + \frac{\partial f(X_t,t)}{\partial t}(\Delta t) + \frac{1}{2!}\Bigg[\frac{\partial^2 f(X_t,t)}{\partial {X_t}^2}(\Delta X_t)^2 + \frac{\partial^2 f(X_t,t)}{\partial t^2}(\Delta t)^2 + \frac{\partial^2 f(X_t,t)}{\partial X_t \partial t}(\Delta X_t)(\Delta t) + \frac{\partial^2 f(X_t,t)}{\partial t \partial X_t}(\Delta t)(\Delta X_t)\Bigg]$

By making $\Delta X_t \to 0$ and $\Delta t \to 0$, we have the differential form for our expression.

$df(X_t,t) = \frac{\partial f(X_t,t)}{\partial X_t}(dX_t) + \frac{\partial f(X_t,t)}{\partial t}(dt) + \frac{1}{2!}\Bigg[\frac{\partial^2 f(X_t,t)}{\partial {X_t}^2}(dX_t)^2 + \frac{\partial^2 f(X_t,t)}{\partial t^2}(dt)^2 + \frac{\partial^2 f(X_t,t)}{\partial X_t \partial t}(dX_t)(dt) + \frac{\partial^2 f(X_t,t)}{\partial t \partial X_t}(dt)(dX_t)\Bigg]$

Let´s focus now on $(dX_t)^2$. By definition we know $dX_t = X_t \sigma dW_t$.

Replacing in our differential form:

$df(X_t,t) = \frac{\partial f(X_t,t)}{\partial X_t}(dX_t) + \frac{\partial f(X_t,t)}{\partial t}(dt) + \frac{1}{2}\Bigg[\frac{\partial^2 f(X_t,t)}{\partial {X_t}^2}(X_t \sigma dW_t)^2 + \frac{\partial^2 f(X_t,t)}{\partial t^2}(dt)^2 + \frac{\partial^2 f(X_t,t)}{\partial X_t \partial t}(X_t \sigma dW_t)(dt) + \frac{\partial^2 f(X_t,t)}{\partial t \partial X_t}(dt)(X_t \sigma dW_t)\Bigg]$

Applying Itô's Lemma:

| | dt | $dW_t$ |
|---|---|---|
| dt | 0 | 0 |
| $dW_t$ | 0 | dt |

$$df(X_t,t) = \frac{\partial f(X_t,t)}{\partial X_t}(dX_t) + \frac{\partial f(X_t,t)}{\partial t}(dt) + \frac{1}{2}\left[\frac{\partial^2 f(X_t,t)}{\partial {X_t}^2}(X_t^2 \sigma^2 dt)\right]$$

Grouping by common factor:

$$df(X_t,t) = \frac{\partial f(X_t,t)}{\partial X_t}(dX_t) + \left[\frac{\partial f(X_t,t)}{\partial t}(dt) + \frac{1}{2}\frac{\partial^2 f(X_t,t)}{\partial {X_t}^2}(X_t^2 \sigma^2 dt)\right]$$

This way, we have our final differential expression:

$$df(X_t,t) = \frac{\partial f(X_t,t)}{\partial X_t}(dX_t) + \left[\frac{\partial f(X_t,t)}{\partial t} + \frac{1}{2}\left(\frac{\partial^2 f(X_t,t)}{\partial {X_t}^2} X_t^2 \sigma^2\right)\right] dt$$

# 3. Let's consider a certain diffusion process $dX_t = \mu S_t^3 + \sigma^2 dW_t + \frac{1}{2}dt$ find a differential form of a function: $f(X_{(t+\Delta t),t+\Delta t}) = df_{(t+\Delta t)} := df_t$

**Using Taylor Series solution for two variables:**

$$f(x,y) = f(a,b) + \frac{\partial f(a,b)}{\partial x}(x-a) + \frac{\partial f(a,b)}{\partial y}(y-b) + \frac{1}{2!}\left[\frac{\partial^2 f(a,b)}{\partial x^2}(x-a)^2 + \frac{\partial^2 f(a,b)}{\partial y^2}(y-b)^2 + \frac{\partial^2 f(a,b)}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 f(a,b)}{\partial y \partial x}(y-b)(x-a)\right]$$

Let's define:

- $f(X_{(t+\Delta t)}, t+\Delta t) = f(Xt, t)$
- $x = X_t$
- $y = t$
- $a = X_t$
- $b = t$
- $k = 2$

$$f(X_t,t) = f(X_t,t) + \frac{\partial f(X_t,t)}{\partial X_t}(X_t - X_t) + \frac{\partial f(X_t,t)}{\partial t}(t-t) + \frac{1}{2!}\left[\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(X_t - X_t)^2 + \frac{\partial^2 f(X_t,t)}{\partial t^2}(t-t)^2 + \frac{\partial^2 f(X_t,t)}{\partial X_t t}(X_t - X_t)(t-t) + \frac{\partial^2 f(X_t,t)}{\partial t X_t}(t-t)(X_t - X_t)\right]$$

$$f(X_t,t) = f(X_t,t) + \frac{\partial f(X_t,t)}{\partial X_t}(\Delta X_t) + \frac{\partial f(X_t,t)}{\partial t}(\Delta t) + \frac{1}{2!}\left[\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(\Delta X_t)^2 + \frac{\partial^2 f(X_t,t)}{\partial t^2}(\Delta t)^2 + \frac{\partial^2 f(X_t,t)}{\partial X_t t}(\Delta X_t)(\Delta t) + \frac{\partial^2 f(X_t,t)}{\partial t X_t}(\Delta t)(\Delta X_t)\right]$$

Differential form:

As $\Delta t \to 0$
and $\Delta X_t \to 0$

$$df_t = \frac{\partial f(X_t,t)}{\partial X_t}(dX_t) + \frac{\partial f(X_t,t)}{\partial t}(dt) + \frac{1}{2}\left[\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(dX_t)^2 + \frac{\partial^2 f(X_t,t)}{\partial t^2}(dt)^2 + \frac{\partial^2 f(X_t,t)}{\partial X_t t}(dX_t)(dt) + \frac{\partial^2 f(X_t,t)}{\partial t X_t}(dt)(dX_t)\right]$$

Applying It$\hat{o}$'s Lemma:

| | dt | $dW_t$ |
|---|---|---|
| dt | 0 | 0 |
| $dW_t$ | 0 | dt |

$$df_t = \frac{\partial f(X_t,t)}{\partial X_t}(dX_t) + \frac{\partial f(X_t,t)}{\partial t}(dt) +$$
$$\frac{1}{2}\left[\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(dX_t)^2 + \frac{\partial^2 f(X_t,t)}{\partial X_t t}(dX_t)(dt) + \frac{\partial^2 f(X_t,t)}{\partial t X_t}(dt)(dX_t)\right]$$

Let's expand the expression in terms of $dX_t$

$$df_t = \frac{\partial f(X_t,t)}{\partial X_t}(\mu S_t^3 + \sigma^2 dW_t + \tfrac{1}{2}dt) + \frac{\partial f(X_t,t)}{\partial t}(dt) +$$
$$\frac{1}{2}\left[\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(\mu S_t^3 + \sigma^2 dW_t + \tfrac{1}{2}dt)^2 + \frac{\partial^2 f(X_t,t)}{\partial X_t t}(\mu S_t^3 + \sigma^2 dW_t + \tfrac{1}{2}dt)(dt) +\right.$$
$$\left.\frac{\partial^2 f(X_t,t)}{\partial t X_t}(dt)(\mu S_t^3 + \sigma^2 dW_t + \tfrac{1}{2}dt)\right]$$

Solving the trinomial squared:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$
$$(\mu S_t^3 + \sigma^2 dW_t + \tfrac{1}{2}dt)^2 = \mu^2 S_t^6 + \sigma^4 dW_t^2 + \tfrac{1}{4}dt^2 + 2(\mu S_t^3)(\sigma^2 dW_t) + 2(\mu S_t^3)(\tfrac{1}{2}dt) + 2(\sigma^2 dW_t)(\tfrac{1}{2}dt)$$

Applying It$\hat{o}$'s Lemma to the trinomial squared.

$$(\mu S_t^3 + \sigma^2 dW_t + \tfrac{1}{2}dt)^2 = \mu^2 S_t^6 + \sigma^4 dt + 2(\mu S_t^3)(\sigma^2 dW_t) + 2(\mu S_t^3)(\tfrac{1}{2}dt)$$

We expand the series in terms of the trinomial squared.

$$df_t = \frac{\partial f(X_t,t)}{\partial X_t}(\mu S_t^3 + \sigma^2 dW_t + \tfrac{1}{2}dt) + \frac{\partial f(X_t,t)}{\partial t}(dt) +$$
$$\frac{1}{2}\left\{\frac{\partial^2 f(X_t,t)}{\partial X_t^2}\left[(\mu^2 S_t^6 + \sigma^4 dt + 2(\mu S_t^3)(\sigma^2 dW_t) + 2(\mu S_t^3)(\tfrac{1}{2}dt)\right] +\right.$$
$$\left.\frac{\partial^2 f(X_t,t)}{\partial X_t t}(\mu S_t^3 dt + \sigma^2 dW_t dt + \tfrac{1}{2}dt^2) + \frac{\partial^2 f(X_t,t)}{\partial t X_t}(\mu S_t^3 dt + \sigma^2 dW_t dt + \tfrac{1}{2}dt^2)\right\}$$

We applied It$\hat{o}$'s Lemma to the last two partial derivatives.

$$df_t = \frac{\partial f(X_t,t)}{\partial X_t}(\mu S_t^3 + \sigma^2 dW_t + \tfrac{1}{2}dt) + \frac{\partial f(X_t,t)}{\partial t}(dt) +$$
$$\frac{1}{2}\left\{\frac{\partial^2 f(X_t,t)}{\partial X_t^2}\left[(\mu^2 S_t^6 + \sigma^4 dt + 2(\mu S_t^3)(\sigma^2 dW_t) + 2(\mu S_t^3)(\tfrac{1}{2}dt)\right] +\right.$$
$$\left.\frac{\partial^2 f(X_t,t)}{\partial X_t t}(\mu S_t^3 dt) + \frac{\partial^2 f(X_t,t)}{\partial t X_t}(\mu S_t^3 dt)\right\}$$

Simplifying:

$$df_t = \frac{\partial f(X_t,t)}{\partial X_t}(dX_t) + \frac{\partial f(X_t,t)}{\partial t}(dt) +$$
$$\frac{1}{2}\frac{\partial^2 f(X_t,t)}{\partial X_t^2}\Big(\mu^2 S_t^6 + \sigma^4 dt + 2\mu S_t^3 \sigma^2 dW_t + \mu S_t^3 dt\Big) +$$
$$\frac{1}{2}\mu S_t^3 dt\bigg(\frac{\partial^2 f(X_t,t)}{\partial X_t t} + \frac{\partial^2 f(X_t,t)}{\partial t X_t}\bigg)$$

$$df_t = \frac{\partial f(X_t,t)}{\partial X_t}(dX_t) + \frac{\partial f(X_t,t)}{\partial t}(dt) +$$
$$\frac{1}{2}\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(\mu^2 S_t^6) + \frac{1}{2}\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(\sigma^4 dt) + \frac{1}{2}\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(2\mu S_t^3 \sigma^2 dW_t) + \frac{1}{2}\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(\mu S_t^3 dt) +$$
$$\frac{1}{2}\mu S_t^3 dt\bigg(\frac{\partial^2 f(X_t,t)}{\partial X_t t} + \frac{\partial^2 f(X_t,t)}{\partial t X_t}\bigg)$$

Finally we got:

$$df_t = \frac{\partial f(X_t,t)}{\partial X_t} dX_t + \left[\frac{\partial f(X_t,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(\sigma^4) + \frac{1}{2}\mu S_t^3\bigg(\frac{\partial^2 f(X_t,t)}{\partial X_t t} + \frac{\partial^2 f(X_t,t)}{\partial t X_t} + \frac{\partial^2 f(X_t,t)}{\partial X_t^2}\bigg)\right] dt +$$
$$\frac{1}{2}\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(\mu^2 S_t^6) + \frac{\partial^2 f(X_t,t)}{\partial X_t^2}(\mu S_t^3 \sigma^2 dW_t)$$