Matrix Multiplication

Summary

1. Matrix multiplication creates new basis vectors $\hat{\imath}$ and $\hat{\jmath}$.
2. An $m \times n$ matrix can be multiplied by an $n \times r$ matrix to make an $m \times r$ matrix.
3. Matrix multiplication transforms the coordinate plane itself as a series of compositions of functions.

Matrix Times a Vector

Matrix multiplication does not work the way you might think. For instance, if

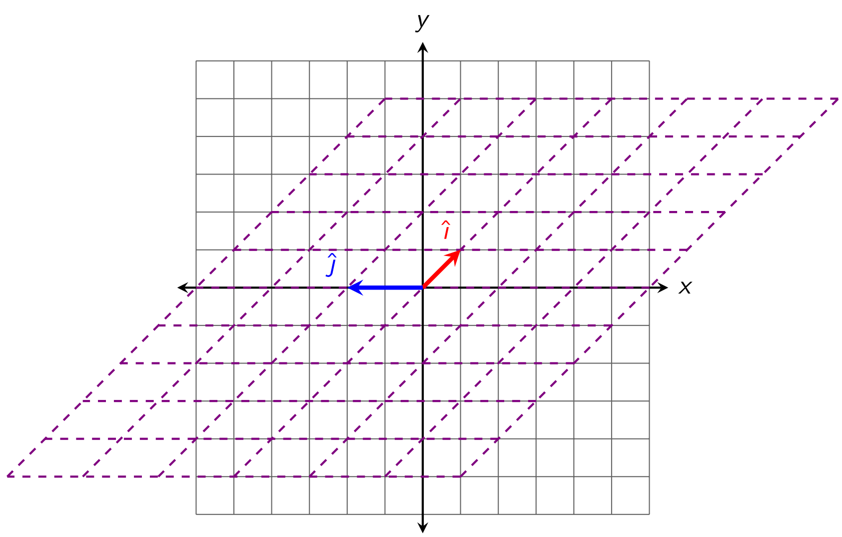
$$A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$$

Then AB is not $\begin{bmatrix} -1 & -2 \\ 2 & 0 \end{bmatrix}$.

In other words, we don't multiply corresponding elements.

In order to even multiply a matrix or a vector by a vector or another matrix, **the number of columns in the first matrix must equal the number of rows in the second.**

Multiplying by $\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$ sends $\hat{\imath}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\hat{\jmath}$ to $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$



Notice we get a new coordinate plane (in purple).

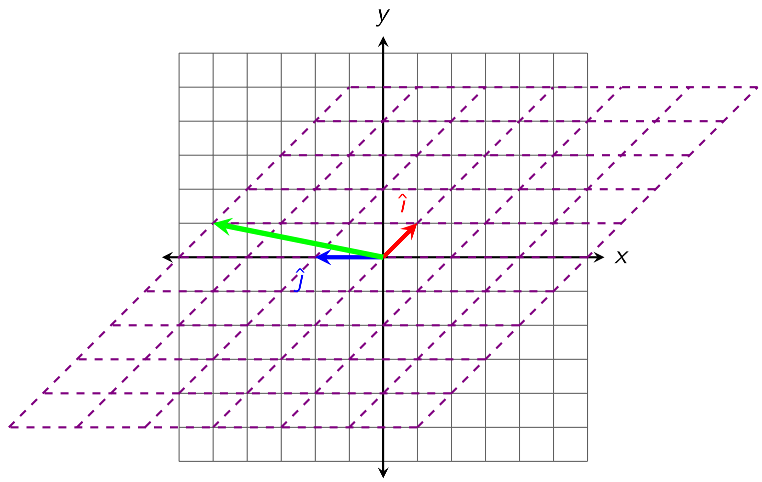
Along the purple coordinate plane, $\hat{\imath}$ now points along the new "positive x-axis."

And $\hat{\jmath}$ now points along the new "positive y-axis."

Multiplying by a matrix transforms $\hat{\imath}$ to the first column and $\hat{\jmath}$ to the second column.

The following creates a new vector along our new coordinate plane (in purple):

$$\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



$$\begin{aligned} \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} &= 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -6 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -5 \\ 1 \end{bmatrix} \end{aligned}$$

Example 1. Multiply each.

(a) $\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} -5 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

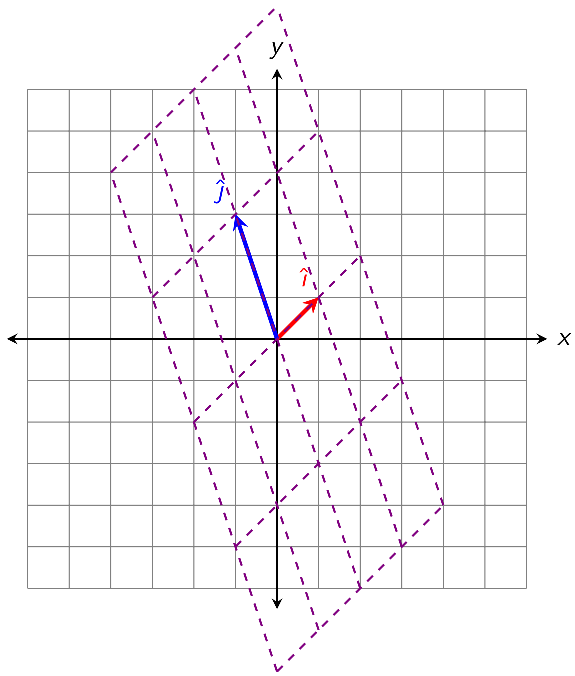
Matrix Times a Matrix

Multiplying matrices combines matrix-vector multiplication with composition of functions; in which we evaluate the composition from the "inside" to the "outside" (or from right-to-left).

For instance, in the product

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

We first look at the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ which sends $\hat{\imath}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\hat{\jmath}$ to $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.



We can then use matrix-vector multiplication to find where $\hat{\imath}$ and $\hat{\jmath}$ will end up:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

Where $\hat{\imath}$ ends up

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Where $\hat{\jmath}$ ends up

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Example 2. Multiply each of the following.

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(c) Does the order in which we multiply matrices matter?

When Matrix Multiplication May Fail

Sometimes we run out of components when multiplying a matrix times a vector.

This will happen if the number of columns in the first matrix do not equal the number of rows in the second matrix.

Otherwise, we will end up with a matrix where

$$\begin{aligned} \text{number of rows} &= \text{number of rows of 1st matrix} \\ \text{number of columns} &= \text{number of columns of 2nd matrix} \end{aligned}$$

Example 3. Multiply each (if possible).

(a) $\begin{bmatrix} 2 & 4 \\ -2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -3 & 6 \\ 0 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 6 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -2 & 3 \\ 5 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 4 \end{bmatrix}$