Inverse Functions

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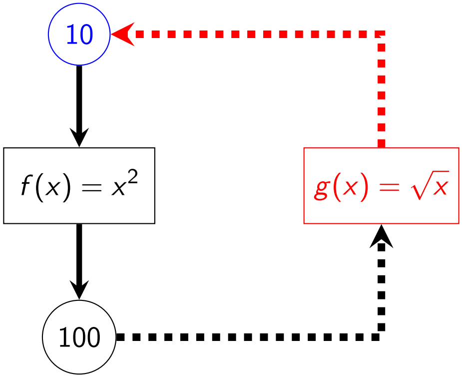
If we give a function input it gives us output in return.

If we want our original input back, we can give our output to the inverse function and get our original input back.

Visual Interpretation

Original Function $f(x) = x^2$ Inverse Function: $g(x) = \sqrt{x}$



Mathematical Definition of Inverse Functions

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- $(g \circ f)(x) = x$ for all x in the domain of f
- $(f \circ g)(x) = x$ for all x in the domain of g

Objectives

1. Determine if a relation or function has an inverse
2. Find the inverse of a function
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Horizontal Line Test

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For a function to have an inverse that is also a function, it must pass a similar test known as the **Horizontal Line Test**.

Horizontal Line Test

If each horizontal line intersects the graph **at most once**, then the function has an inverse.

Note: A function that passes the Horizontal Line Test is also said to be invertible or one-to-one.

Example 1

Determine if the following relations or functions have an inverse.

(a) $F = \{(-1, 1), (0, 2), (2, 1)\}$

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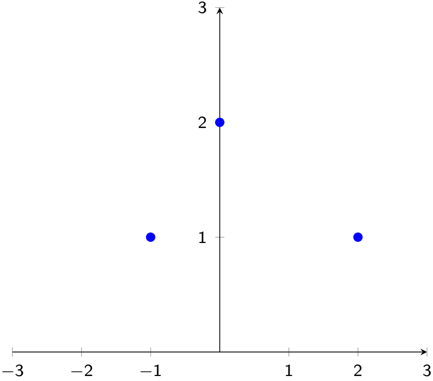

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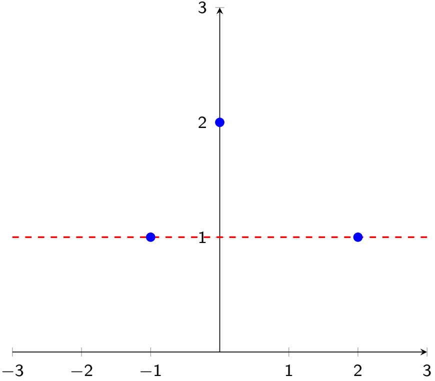

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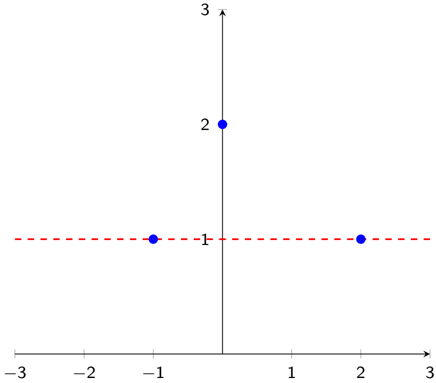

Does not have an inverse.

Example 1

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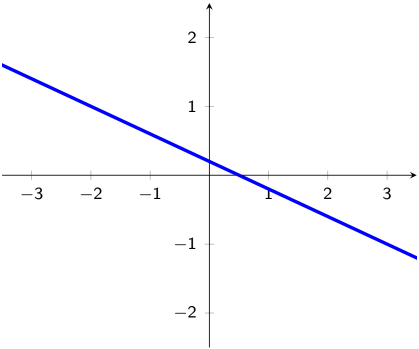

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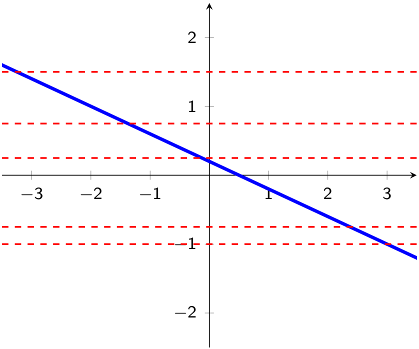

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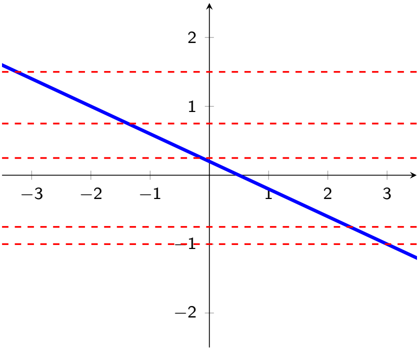

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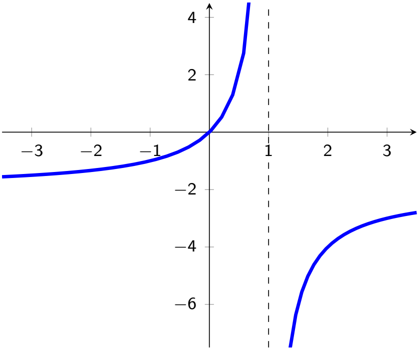

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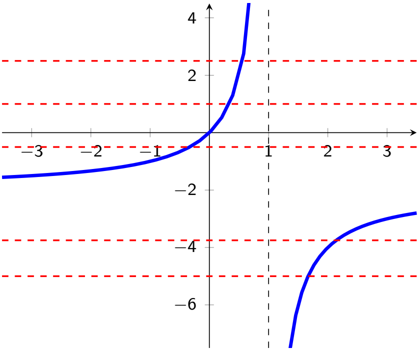

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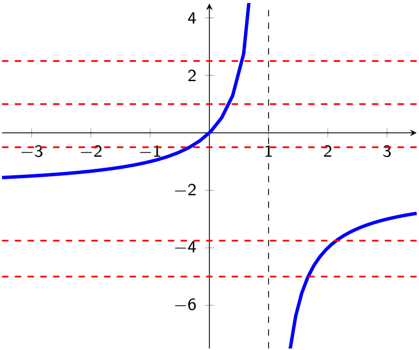

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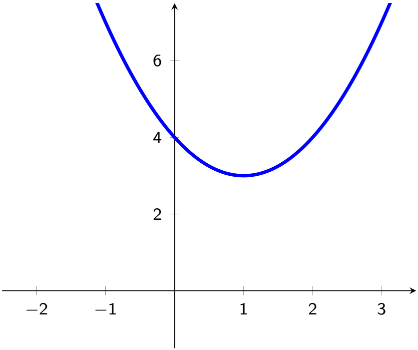

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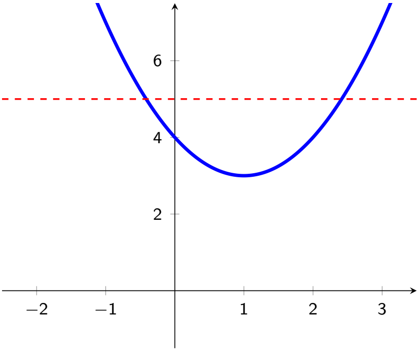

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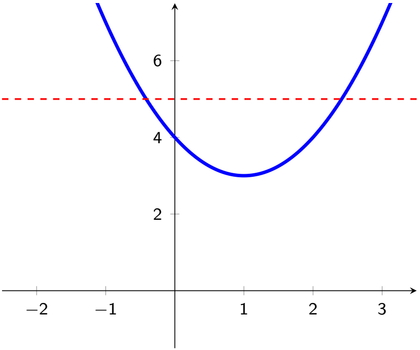

Does not have an inverse.

Objectives

1. Determine if a relation or function has an inverse
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Inverse Function Notation

For a given function $f(x)$, the notation for the inverse of $f(x)$ is

$$f^{-1}(x)$$

How to Find the Inverse of a Function

1. Write as $y = f(x)$

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1. Write as $y = f(x)$
2. Switch the x and y
3. Solve for y and write using inverse notation $f^{-1}(x)$.

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Find the inverse of each.

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$$-2y = 5x - 1$$

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y &= \frac{1-2x}{5} \\
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5x &= 1-2y \\
-2y &= 5x-1 \\
y &= \frac{5x-1}{-2}
\end{aligned}
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Example 2

$$f^{-1}(x) = \frac{-5x+1}{2}$$

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$$x = y(2+x)$$

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$$(x+1)^2 - 2 = y$$

$$h^{-1}(x) = (x+1)^2 - 2$$

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Domain and Range of Inverse Functions

In the previous example, we found the inverse of a function by switching the x and the y in the equation.

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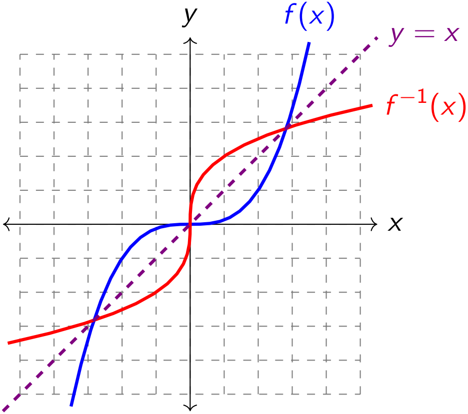

Domain and Range Restrictions

Sometimes you have to restrict the domain of the function so that it is a reflection of its inverse across the line $y = x$

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Domain of $f^{-1}(x)$: $(-\infty, \infty)$

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$$\text{Domain of } f^{-1}(x): \quad (-\infty, \infty)$$
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Domain of $h^{-1}(x)$: $[-1, \infty)$

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	Domain	Range
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$$\text{Domain of } h^{-1}(x): \quad [-1, \infty)$$
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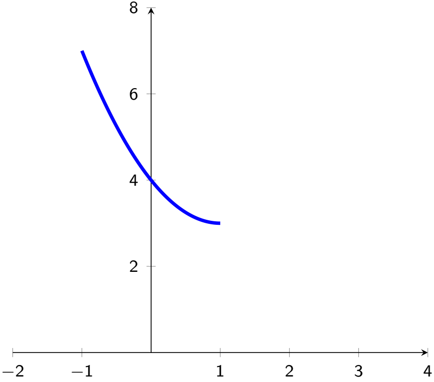

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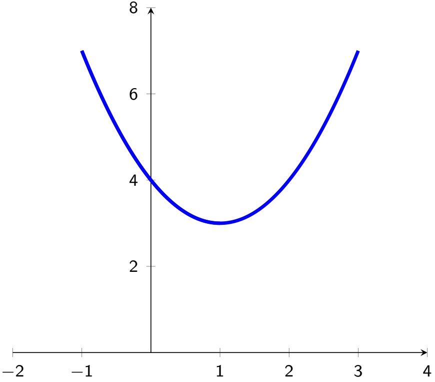

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$$x - 3 = (y - 1)^2$$

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$$
\begin{aligned}
y &= x^2 - 2x + 4 \\
x &= y^2 - 2y + 4 \\
x - 3 &= (y - 1)^2 \\
y - 1 &= \pm\sqrt{x - 3}
\end{aligned}
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$$\begin{aligned} y &= x^2 - 2x + 4 \\ x &= y^2 - 2y + 4 \\ x - 3 &= (y - 1)^2 \\ y - 1 &= \pm\sqrt{x - 3} \\ y &= 1 \pm \sqrt{x - 3} \\ j^{-1}(x) &= 1 \pm \sqrt{x - 3} \\ j^{-1}(x) &= 1 - \sqrt{x - 3} \end{aligned}$$

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(d) $j(x) = x^2 - 2x + 4 \quad x \leq 1 \qquad j^{-1}(x) = 1 - \sqrt{x - 3}$

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	Domain	**Range**
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$j^{-1}(x)$	$x \geq 3$	$y \leq 1$

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Domain of $j^{-1}(x)$: $[3, \infty)$

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Range of $j^{-1}(x)$: $\quad (-\infty, 1]$