Systems of Equations

Objectives

1. Write a system of equations using matrices
2. Solve a system of equations using inverse matrices
3. Solve applications of systems of equations

Matrix Multiplication

Previously, we looked at the method for multiplying matrices:

$$\begin{bmatrix} 7 & 2 & -1 \\ 0 & 5 & 4 \\ -3 & 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

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In this section, we will use matrix multiplication to solve a system of equations.

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EXAMPLES

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$$\begin{aligned} 2x + 5y &= 8 \\ -7x - 3y &= 9 \end{aligned}$$

$$\begin{aligned} 4x - 9y + 2z &= 10 \\ -x + z &= 15 \\ 3x + 10y - 12z &= 0 \end{aligned}$$

Writing a System of Equations Using Matrices

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$$\begin{aligned} 2x + 5y &= 8 \\ -7x - 3y &= 9 \end{aligned}$$

becomes

$$\begin{bmatrix} 2 & 5 \\ -7 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

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$$\begin{aligned} 4x - 9y + 2z &= 10 \\ -x + z &= 15 \\ 3x + 10y - 12z &= 0 \end{aligned}$$

becomes

$$\begin{bmatrix} 4 & -9 & 2 \\ -1 & 0 & 1 \\ 3 & 10 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 0 \end{bmatrix}$$

Example 1

Write each of the following systems of equations using matrices.

(a)

$$\begin{aligned} 4y + z &= -1 \\ -4x - 5y + 5z &= 0 \\ 6x + 5y &= 25 \end{aligned}$$

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\begin{aligned}
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Example 1

(b)

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Solving Equations Using Inverse Operations

Long ago, you learned that to solve something like

$$5x = 10$$

you divide (the inverse operation of multiplication) both sides by 5.

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We don't "divide" matrices in this sense, but we do need to use an inverse operation to solve the previous examples.

Identity Matrices

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$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{etc.}$$

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$$X = A^{-1} \cdot B$$

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and that if $AX = B$ where A is the matrix of coefficients and B are the constants on the right side, then

$$X = A^{-1} \cdot B$$

Note: We will not discuss the techniques of how to actually find the inverse of a matrix without a calculator.

Example 2

Solve each of the following using matrices.

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$$\begin{aligned} 4y + z &= -1 \\ -4x - 5y + 5z &= 0 \\ 6x + 5y &= 25 \end{aligned}$$

$$\begin{bmatrix} 0 & 4 & 1 \\ -4 & -5 & 5 \\ 6 & 5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 25 \end{bmatrix}$$

Example 2

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B$$

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

$$x = 5, \quad y = -1, \quad z = 3$$

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(b)

$$\begin{aligned} x - y &= -9 \\ -3x - 4y &= -8 \end{aligned}$$

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$$\begin{aligned} x - y &= -9 \\ -3x - 4y &= -8 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9 \\ -8 \end{bmatrix}$$

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$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

$$x = -4, \quad y = 5$$

Example 2

(c)

$$\begin{aligned} x - 3y - 5z &= -18 \\ -2x + 3y + 5z &= 23 \\ -x + 3y + 6z &= 17 \end{aligned}$$

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$$\begin{bmatrix} 1 & -3 & -5 \\ -2 & 3 & 5 \\ -1 & 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18 \\ 23 \\ 17 \end{bmatrix}$$

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$$x = -5, \quad y = 6, \quad z = -1$$

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Applied Systems of Equations

Setting up many applied systems of equations problems boils down to

$$\text{unit rate} \times \text{amount} = \text{total amount}$$

For instance, \$2.00 per gallon times 5 gallons of gas costs a total amount of \$10.

Example 3

(a) How many mL of a solution containing 10% pure hydrochloric acid must be mixed with a solution containing 15% hydrochloric acid to produce 30 mL of a solution that is 11% hydrochloric acid?

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Example 3

(a) How many mL of a solution containing 10% pure hydrochloric acid must be mixed with a solution containing 15% hydrochloric acid to produce 30 mL of a solution that is 11% hydrochloric acid?

x = mL of the 10% acid solution
y = mL of the 15% acid solution

$$\begin{aligned} x + y &= 30 && \text{volume of total liquid} \\ 0.10x + 0.15y &= 30(0.11) && \text{volume of total hydrocholic acid} \end{aligned}$$

Example 3

$$\begin{bmatrix} 1 & 1 \\ 0.1 & 0.15 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 3.3 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 \\ 0.1 & 0.15 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 3.3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 24 \\ 6 \end{bmatrix}$$

We need to mix 24 mL of a 10% hydrochloric acid solution with 6 mL of a 15% hydrochloric acid solution.

Example 3

(b) A coffee shop wants to sell 8 pound bags of a coffee blend for \$28.64. They will do this by blending coffee that costs \$2.25 per pound with coffee that costs \$4.00 per pound. How much of each type of coffee should they use in the blend?

Example 3

(b) A coffee shop wants to sell 8 pound bags of a coffee blend for \$28.64. They will do this by blending coffee that costs \$2.25 per pound with coffee that costs \$4.00 per pound. How much of each type of coffee should they use in the blend?

$x =$ number of pounds of cheaper coffee (\$2.25 per pound)
$y =$ number of pounds of more expensive coffee (\$4.00 per pound)

Example 3

(b) A coffee shop wants to sell 8 pound bags of a coffee blend for \$28.64. They will do this by blending coffee that costs \$2.25 per pound with coffee that costs \$4.00 per pound. How much of each type of coffee should they use in the blend?

x = number of pounds of cheaper coffee (\$2.25 per pound)
y = number of pounds of more expensive coffee (\$4.00 per pound)

$$x + y = 8 \qquad \text{weight of each bag}$$
$$2.25x + 4.00y = 28.64 \qquad \text{total cost}$$

Example 3

$$\begin{bmatrix} 1 & 1 \\ 2.25 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 28.64 \end{bmatrix}$$

Example 3

$$\begin{bmatrix} 1 & 1 \\ 2.25 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 28.64 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.92 \\ 6.08 \end{bmatrix}$$

They will need 1.92 pounds of the cheaper coffee and 6.08 pounds of the more expensive coffee.