Rational Functions and Their Graphs

What is a Rational Function?

A rational function is a function in the form

$$\frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

Objectives

1. Determine the end behavior of a rational function
2. Determine the equations of vertical asymptotes and coordinates of holes of a rational function

End Behavior

Recall from Polynomial Functions that end behavior refers to the graph's behavior as x approaches ∞ and also $-\infty$.

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In other words, what are the output values approaching as our input values get larger (in either direction)?

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- The function will look like a horizontal line.
- The function will not look like a horizontal line.

Example 1

Determine the end behavior of the function

$$f(x) = \frac{2x}{x^2 - 3}$$

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x	$f(x)$
1,000	0.002
10,000	0.0002
100,000	0.00002
1,000,000	0.000002

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x	$f(x)$
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As you zoom out of the function, the graph more and more resembles the graph of $y = 0$.

As far as notation goes, we would say

$$\lim_{x \to -\infty} f(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} f(x) = 0$$

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This will happen if either

- Degree of numerator $<$ Degree of denominator (horizontal asymptote will be $y = 0$)
- Degree of numerator $=$ Degree of denominator (horizontal asymptote will be ratio of leading coefficients)

Oblique Asymptotes

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You will need to use polynomial division to find the equation of an oblique asymptote.

Example 2

Determine the end behavior of each, then find the equation of the asymptote.

(a) $f(x) = \dfrac{5x}{x^2 + 1}$

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- Horizontal asymptote
- $y = 0$

Example 2

Note: If you used polynomial division, you would get

$$f(x) = {\color{red}0} + \frac{5x}{x^2+1}$$

and as $x \to \pm\infty$,

$f(x) \to {\color{red}0} + 0 = 0$

Example 2

(b) $$g(x) = \frac{x^2 - 4}{x + 1}$$

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$$(x^2 + 0x - 4) \div (x + 1)$$

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 & \color{red}{x} & \\ \hline
x & x^2 & -x \\ \hline
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Slant asymptote: $y = x - 1$

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(c) $$h(x) = \frac{6x^3 - 3x + 1}{5 - 2x^3}$$

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- Ratio of leading coefficients $= \frac{6}{-2} = -3$
- $y = -3$

Example 2 $h(x) = \frac{6x^3-3x+1}{5-2x^3}$

Note: If you perform polynomial division, you get

$$h(x) = -3 + \frac{-3x+16}{5-2x^3}$$

And as $x \to \pm\infty$,

$h(x) \to -3 + 0 = -3$

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Domain of Rational Functions

Recall that the domain of a rational function is all real numbers *****EXCEPT***** values that make the denominator equal 0.

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At each of the values that cause the denominator to $= 0$, there will be **only one of two things there**:

- A vertical asymptote
- Or a hole in the graph

Vertical Asymptotes

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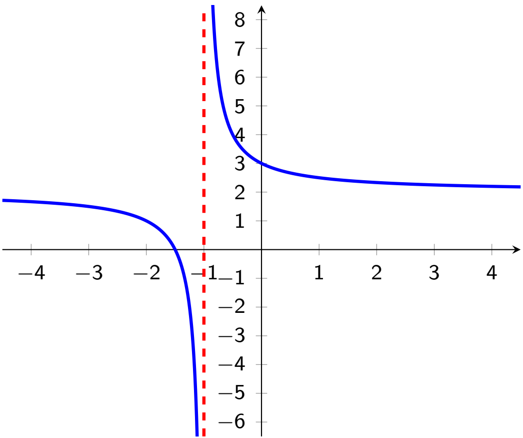
A vertical asymptote is a vertical line that the function will approach, but never intersect.

As the graph of the function gets closer to a vertical asymptote, the graph will either skyrocket up towards ∞ or plummet downward towards $-\infty$.

Vertical Asymptotes



Holes in Rational Functions

Most graphing technology will not show holes in graphs upon sight.

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1. Factor both numerator and denominator ****completely**** and then **simplify that**.

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   - Otherwise, there is a hole in the graph there.

To get the y-coordinate of a hole in the graph, plug in the value of x into your simplified expression.

Example 3

Determine the domain of each. Then determine the equations of vertical asymptotes and/or coordinates of any holes.

(a) $f(x) = \dfrac{2x}{x^2 - 9}$

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$$\begin{aligned} x + 3 &\neq 0 \\ x &\neq -3 \end{aligned} \qquad\qquad \begin{aligned} x - 3 &\neq 0 \\ x &\neq 3 \end{aligned}$$

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Domain: $x \neq -3, 3$

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There are no holes in the graph

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(b) $$g(x) = \frac{x^2 - 6x - 7}{x^2 + 5x + 4}$$

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Vertical asymptote at $x = -4$

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$$y\text{-coordinate: } \frac{-1-7}{-1+4} = -\frac{8}{3}$$

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Hole at $\left(-1, -\dfrac{8}{3}\right)$

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Domain: $\quad x \neq -3, 3$

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$$y\text{-coordinate: } \frac{3+2}{3+3} = \frac{5}{6}$$

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Domain: $x \neq -3, 3$

Vertical asymptote at $x = -3$

There is a hole in the graph at $x = 3$

$$y\text{-coordinate: } \frac{3+2}{3+3} = \frac{5}{6}$$

Hole at $\left(3, \frac{5}{6}\right)$

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Since $x^2 + 9 \neq 0$ for any real number x, domain is All Real Numbers

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Since domain is all reals, there are no vertical asymptotes or holes

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Vertical asymptote at $x = -2$

There is no hole in the graph