Limits and Algebra

Intro

The graphs of $f(x) = \dfrac{x^2 - 6x - 7}{x - 7}$ and $g(x) = x + 1$ are not the same.

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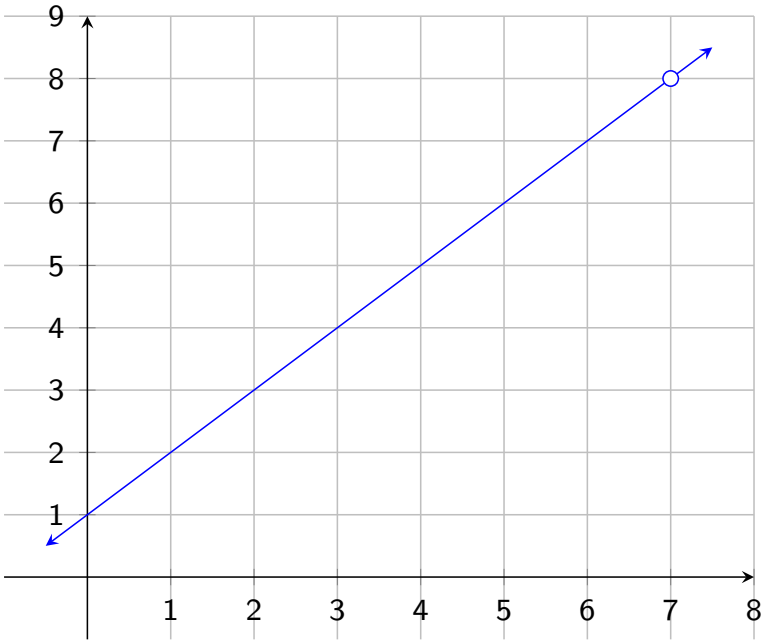

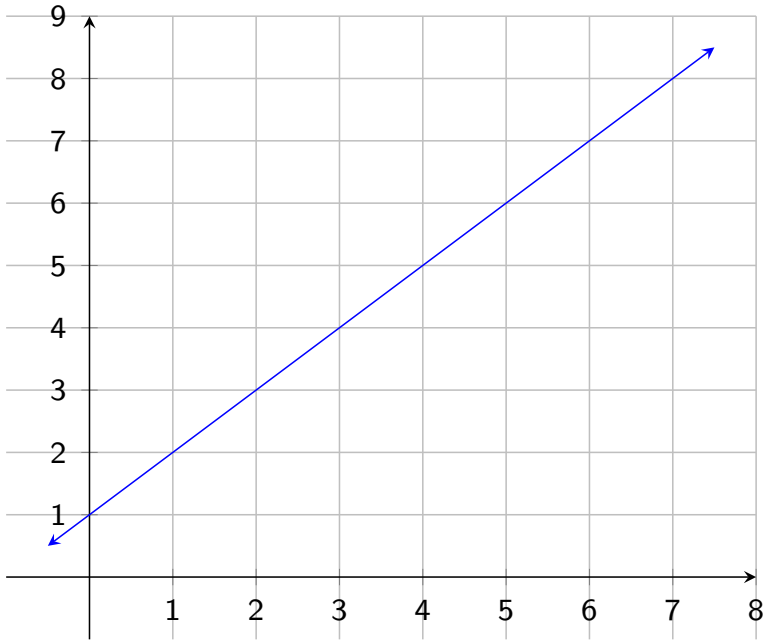

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

$$g(x) = x + 1$$

Objectives

1. Find Limits via Factoring
2. Limits with Complex Fractions
3. Limits with Radicals
4. Limits with Absolute Value

Algebraic Limits

Some limits that can't be evaluated directly can be evaluated after cancelling out common factors.

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Some limits that can't be evaluated directly can be evaluated after cancelling out common factors.

This is called removable discontinuity.

Example 1

(a) Evaluate $\lim_{x \to -3} \dfrac{x^2 + 4x + 3}{x + 3}$

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$$= -2$$

Example 1

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$$= \frac{1}{3}$$

Objectives

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Complex Fractions

Simplify the complex fraction by multiplying every term by the least common tiny denominator.

Example 2

Evaluate each.

(a) $\lim_{x \to -5} \left(\dfrac{\frac{1}{x} + \frac{1}{5}}{x + 5} \right)$

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$$= \lim_{x \to -5} \frac{5 + x}{5x(x + 5)}$$

Example 2

$$= \lim_{x \to -5} \frac{\cancel{5 + x}}{5x\cancel{(x + 5)}}$$

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$$= \frac{1}{5(-5)}$$

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$$= \lim_{x \to -5} \frac{1}{5x}$$

$$= \frac{1}{5(-5)}$$

$$= -\frac{1}{25}$$

Example 2

(b) $\lim_{x \to 3} \left(\dfrac{\frac{1}{3} - \frac{1}{x}}{3 - x} \right)$

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$$= \lim_{x \to 3} \frac{x - 3}{3x(3 - x)}$$

Example 2

$$= \lim_{x \to 3} \frac{\cancel{x - 3}}{3x\cancel{(3 - x)}}$$

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$$= \frac{-1}{3(3)}$$

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$$= \lim_{x \to 3} \frac{\cancel{x - 3}}{3x\cancel{(3 - x)}}$$

$$= \lim_{x \to 3} \frac{-1}{3x}$$

$$= \frac{-1}{3(3)}$$

$$= \frac{-1}{9}$$

Objectives

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Radicals

When working with radicals, multiply by the conjugate.

Expression	**Conjugate**
$a + \sqrt{b}$	$a - \sqrt{b}$
$a - \sqrt{b}$	$a + \sqrt{b}$

Example 3

(a) $\lim_{x \to 0} \left(\frac{\sqrt{25 - x} - 5}{x} \right)$

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$$= \lim_{x \to 0} \frac{25 - x - 25}{x \left(\sqrt{25 - x} + 5 \right)}$$

Example 3

(a) $\lim_{x \to 0} \left(\frac{\sqrt{25 - x} - 5}{x} \right)$

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$$= \lim_{x \to 0} \frac{25 - x - 25}{x \left(\sqrt{25 - x} + 5 \right)}$$

$$= \lim_{x \to 0} \frac{-x}{x \left(\sqrt{25 - x} + 5 \right)}$$

Example 3

$$= \lim_{x \to 0} \frac{-1}{\sqrt{25 - x} + 5}$$

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$$= \frac{-1}{10}$$

Example 3

(b) $\lim_{h \to 0} \left(\frac{\sqrt{16+h} - 4}{h} \right)$

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Example 3

(b) $\lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right)$

$$\lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right)=\lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right)\left(\frac{\sqrt{16+h}+4}{\sqrt{16+h}+4}\right)$$

$$=\lim_{h\to 0}\frac{16+h-16}{h\left(\sqrt{16+h}+4\right)}$$

Example 3

(b) $\lim_{h\to 0}\left(\dfrac{\sqrt{16+h}-4}{h}\right)$

$$\lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right) = \lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right)\left(\frac{\sqrt{16+h}+4}{\sqrt{16+h}+4}\right)$$

$$= \lim_{h\to 0}\frac{16+h-16}{h\left(\sqrt{16+h}+4\right)}$$

$$= \lim_{h\to 0}\frac{h}{h\left(\sqrt{16+h}+4\right)}$$

Example 3

$$= \lim_{h \to 0} \frac{1}{\sqrt{16+h}+4}$$

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$$= \frac{1}{8}$$

Example 3

(c) $\lim_{x \to 4} \dfrac{4 - x}{\sqrt{x} - 2}$

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$$\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} \left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2} \right)$$

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$$= \lim_{x \to 4} \frac{(4 - x)(\sqrt{x} + 2)}{x - 4}$$

Example 3

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$$= \lim_{x \to 4} \frac{(4 - x)(\sqrt{x} + 2)}{x - 4}$$

$$= \lim_{x \to 4} -1(\sqrt{x} + 2)$$

Example 3

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$$\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} \left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2} \right)$$

$$= \lim_{x \to 4} \frac{(4 - x)(\sqrt{x} + 2)}{x - 4}$$

$$= \lim_{x \to 4} -1(\sqrt{x} + 2)$$

$$= -1(\sqrt{4} + 2)$$

Example 3

(c) $\lim_{x\to 4} \dfrac{4-x}{\sqrt{x}-2}$

$$\lim_{x\to 4} \frac{4-x}{\sqrt{x}-2} = \lim_{x\to 4} \frac{4-x}{\sqrt{x}-2}\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right)$$

$$= \lim_{x\to 4} \frac{(4-x)(\sqrt{x}+2)}{x-4}$$

$$= \lim_{x\to 4} -1(\sqrt{x}+2)$$

$$= -1(\sqrt{4}+2)$$

$$= -1(2+2) = -4$$

Example 3

(d) $\lim_{x \to 3} \dfrac{x-3}{\sqrt{x}-\sqrt{3}}$

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$$= \lim_{x \to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{x-3}$$

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$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{x-3}$$

$$= \lim_{x\to 3} (\sqrt{x}+\sqrt{3})$$

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$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{x-3}$$

$$= \lim_{x\to 3} (\sqrt{x}+\sqrt{3})$$

$$= \sqrt{3}+\sqrt{3}$$

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$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{x-3}$$

$$= \lim_{x\to 3} (\sqrt{x}+\sqrt{3})$$

$$= \sqrt{3}+\sqrt{3}$$

$$= 2\sqrt{3}$$

Objectives

1. Find Limits via Factoring
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Limits and Absolute Value

With absolute value, it might be better to use a table or a graph to find the limit.

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Also, look at the left-hand and right-hand limits.

Example 4

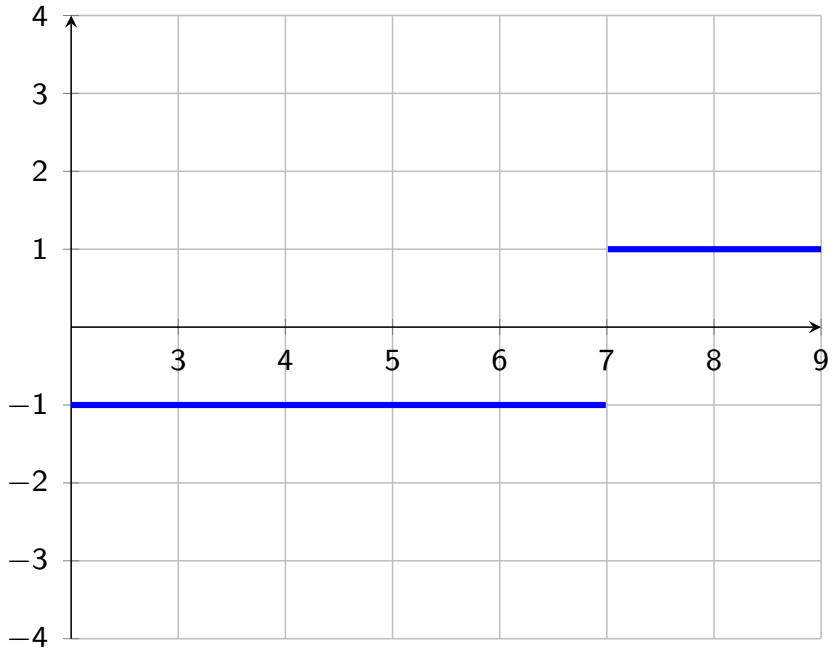
Find the limit for each.

(a) $\lim_{x \to 7} \dfrac{|x - 7|}{x - 7}$

Example 4

Find the limit for each.

(a) $\lim_{x \to 7} \dfrac{|x-7|}{x-7}$



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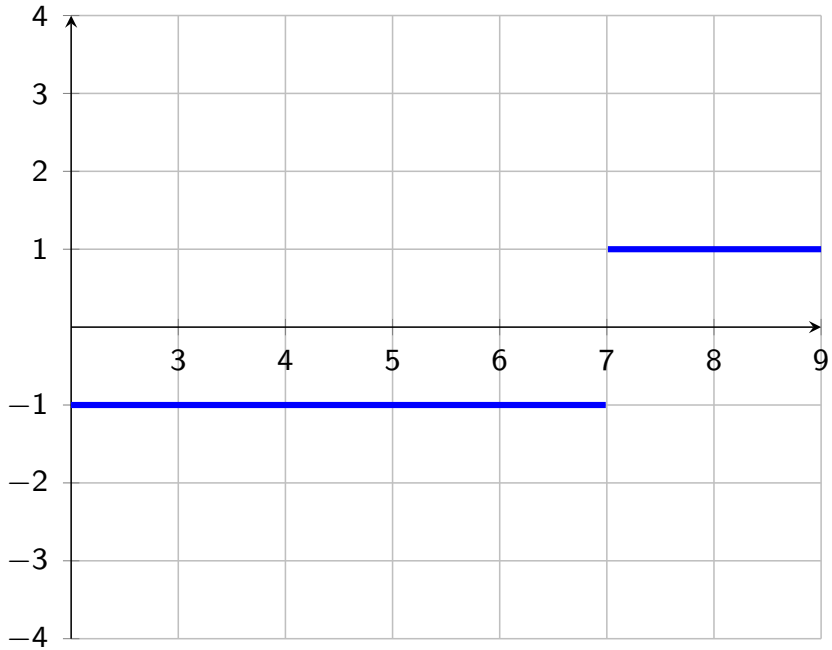
Find the limit for each.

(a) $\lim_{x \to 7} \dfrac{|x-7|}{x-7}$

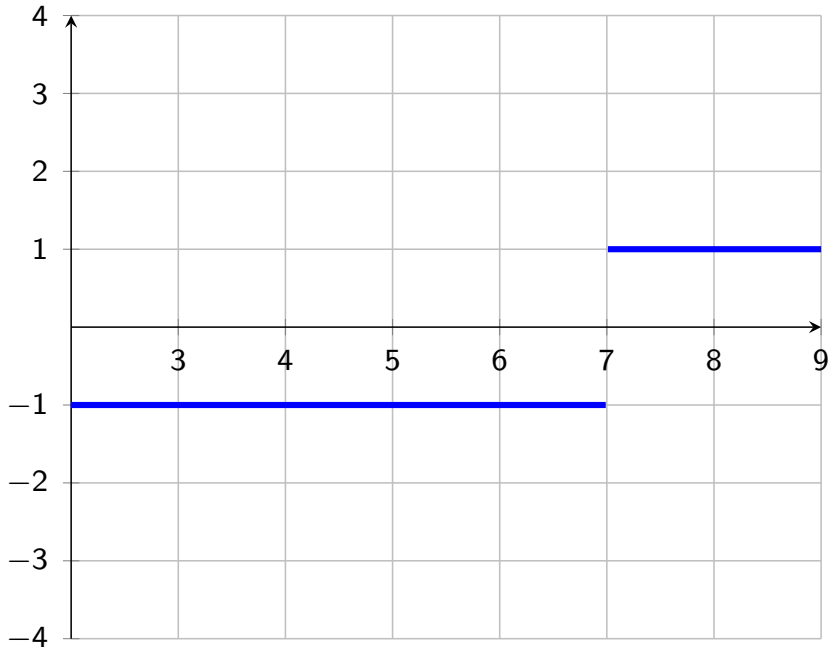


$\lim_{x \to 7^-} \dfrac{|x-7|}{x-7}$

Example 4

Find the limit for each.

(a) $\lim_{x \to 7} \dfrac{|x-7|}{x-7}$

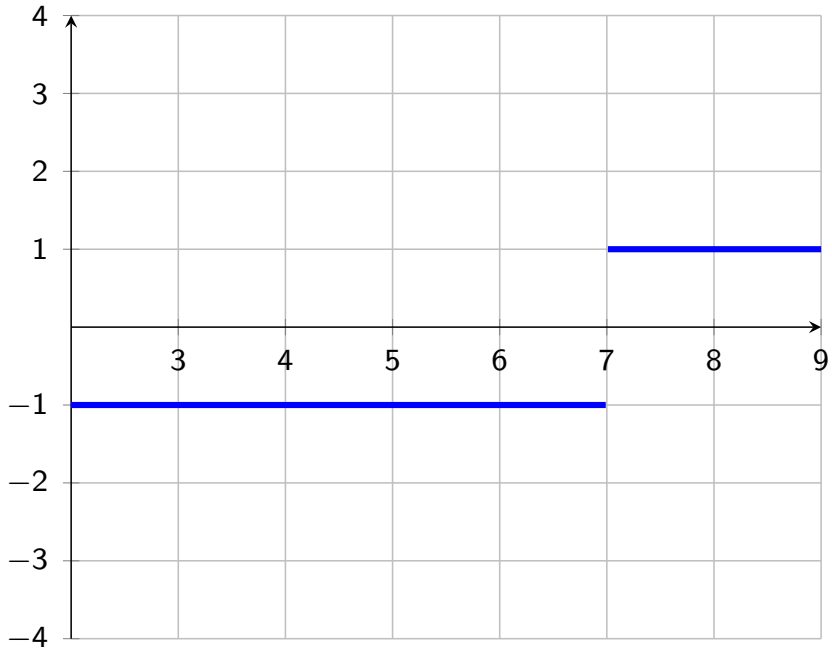


$$\lim_{x \to 7^-} \frac{|x-7|}{x-7} = -1$$

Example 4

Find the limit for each.

(a) $\lim_{x \to 7} \dfrac{|x-7|}{x-7}$



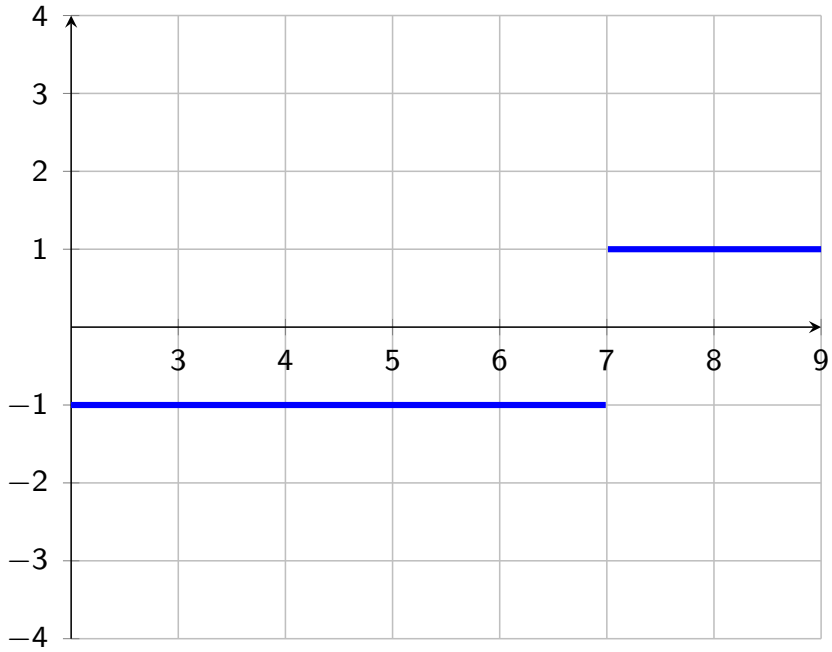
$$\lim_{x \to 7^-} \frac{|x-7|}{x-7} = -1$$

$$\lim_{x \to 7^+} \frac{|x-7|}{x-7}$$

Example 4

Find the limit for each.

(a) $\lim_{x \to 7} \dfrac{|x-7|}{x-7}$



$$\lim_{x \to 7^-} \frac{|x-7|}{x-7} = -1$$

$$\lim_{x \to 7^+} \frac{|x-7|}{x-7} = 1$$

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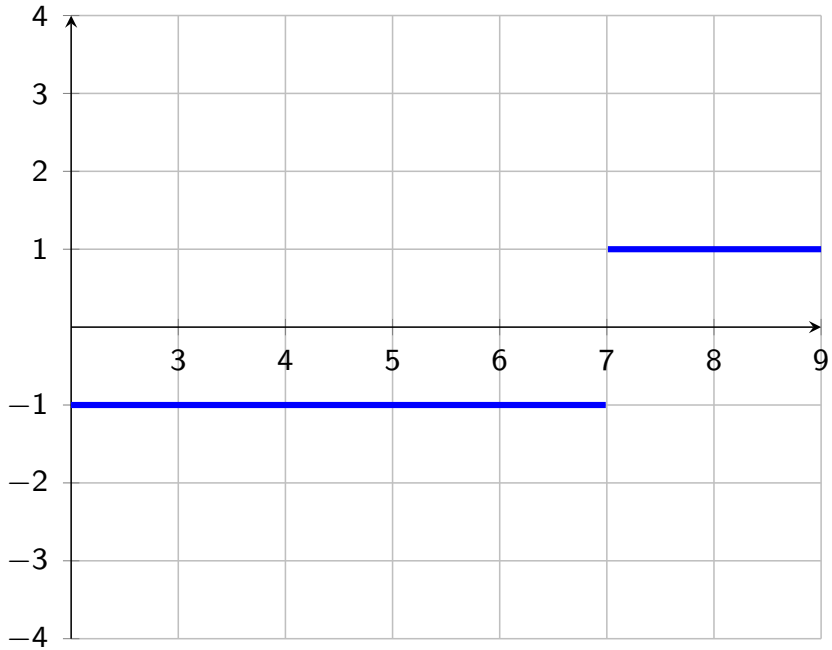

$$\lim_{x \to 7^-} \frac{|x-7|}{x-7} = -1$$

$$\lim_{x \to 7^+} \frac{|x-7|}{x-7} = 1$$

Does Not Exist

Example 4

(b) $\lim_{x \to 6^+} \dfrac{6 - x}{|x - 6|}$

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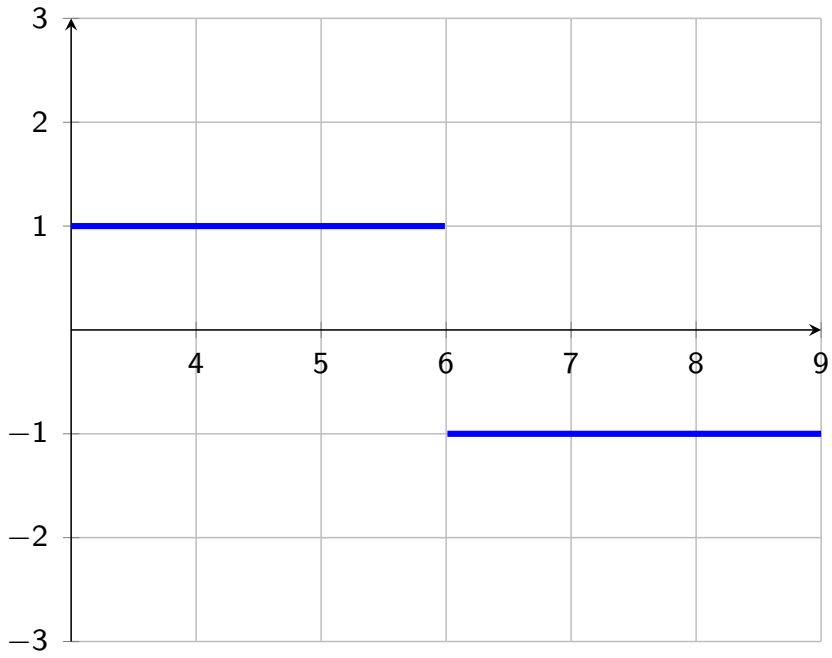
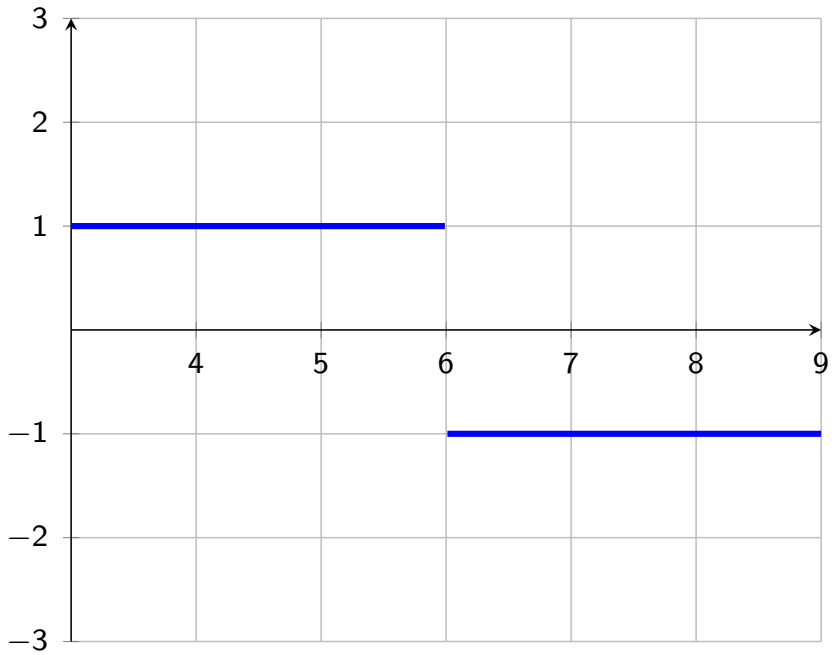


Example 4

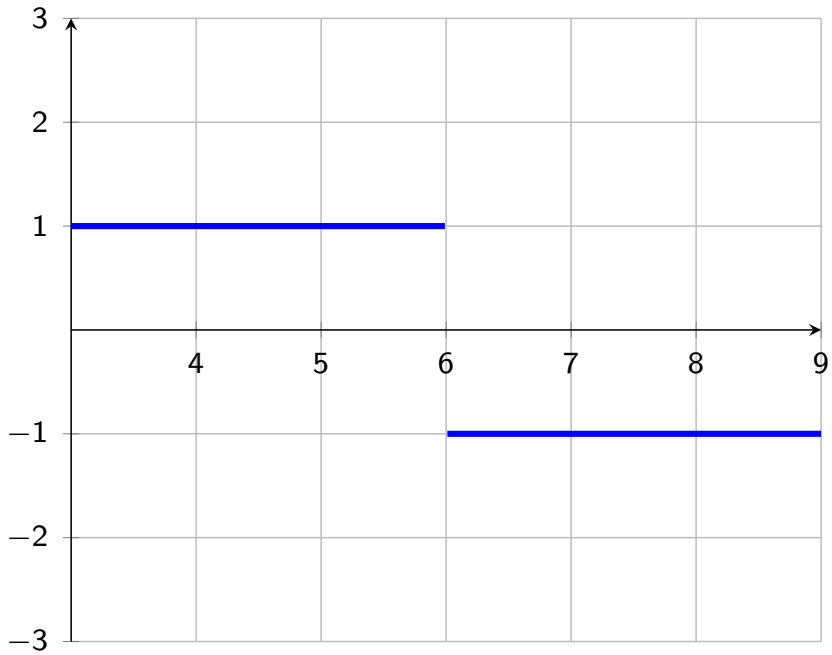
(b) $\lim_{x \to 6^+} \dfrac{6 - x}{|x - 6|}$



$$\lim_{x \to 6^+} \frac{6 - x}{|x - 6|}$$

Example 4

(b) $\lim_{x \to 6^+} \dfrac{6 - x}{|x - 6|}$



$$\lim_{x \to 6^+} \frac{6 - x}{|x - 6|} = -1$$