Finding Limits: Numerically and Graphically

Objectives

1. Understand Limit Notation
2. Find a a Limit Using a Table
3. Find a Limit Using a Graph

Intro

As we increase the term numbers of a sequence such as

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

notice how the values of the terms get closer and closer to 0.

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$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

notice how the values of the terms get closer and closer to 0.

It may or may not equal 0 at some point, but we would say **the limit of the sequence is 0.**

Limit Notation

Like sequences, functions can also have limits.

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To indicate the limit L of a function $f(x)$ as x approaches the value of a, we use the notation

$$\lim_{x \to a} f(x) = L$$

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To indicate the limit L of a function $f(x)$ as x approaches the value of a, we use the notation

$$\lim_{x \to a} f(x) = L$$

This is read

The limit of $f(x)$ as x approaches a is L.

Limit Notation

In other words, as x gets closer to the x-coordinate a, the y-values get closer to the y-coordinate L.

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Sometimes you can just plug the value of a into the function. Other times you can't (e.g. might get division by 0).

*** IMPORTANT ***

Limits only look at what happens **as you get closer to the value of a**

They are not concerned with what the value of the function is *at* that value of a.

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Example 1

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$\boldsymbol{x}$	$\boldsymbol{f(x)}$
6.99	7.99

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$\boldsymbol{x}$	$\boldsymbol{f(x)}$
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6.999	7.999

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x	$f(x)$
6.99	7.99
6.999	7.999
6.9999	7.9999

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$\boldsymbol{x}$	$\boldsymbol{f(x)}$
6.99	7.99
6.999	7.999
6.9999	7.9999
$\downarrow$	$\downarrow$

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$$\begin{array}{cc}
\boldsymbol{x} & \boldsymbol{f(x)} \\
\hline
6.99 & 7.99 \\
6.999 & 7.999 \\
6.9999 & 7.9999 \\
\downarrow & \downarrow \\
7 &
\end{array}$$

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$\boldsymbol{x}$	$\boldsymbol{f(x)}$
6.99	7.99
6.999	7.999
6.9999	7.9999
$\downarrow$	$\downarrow$
7	
7.01	8.01

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$\boldsymbol{x}$	$\boldsymbol{f(x)}$
6.99	7.99
6.999	7.999
6.9999	7.9999
$\downarrow$	$\downarrow$
7	
7.001	8.001
7.01	8.01

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x	$f(x)$
6.99	7.99
6.999	7.999
6.9999	7.9999
$\downarrow$	$\downarrow$
7	
$\uparrow$	$\uparrow$
7.0001	8.0001
7.001	8.001
7.01	8.01

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6.99	7.99
6.999	7.999
6.9999	7.9999
$\downarrow$	$\downarrow$
7	
$\uparrow$	$\uparrow$
7.0001	8.0001
7.001	8.001
7.01	8.01

$$\lim_{x \to 7} f(x) = 8$$

Example 2

For $f(x) = 3x + 5$, find $\lim_{x \to 2} f(x)$

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$\boldsymbol{x}$	$\boldsymbol{f(x)}$
1.99	10.97

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x	$f(x)$
1.99	10.97
1.999	10.997

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$\boldsymbol{x}$	$\boldsymbol{f(x)}$
1.99	10.97
1.999	10.997
1.9999	10.9997

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x	$f(x)$
1.99	10.97
1.999	10.997
1.9999	10.9997
$\downarrow$	$\downarrow$
2	

Example 2

For $f(x) = 3x + 5$, find $\lim_{x \to 2} f(x)$

x	$f(x)$
1.99	10.97
1.999	10.997
1.9999	10.9997
$\downarrow$	$\downarrow$
2	
2.01	11.03

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For $f(x) = 3x + 5$, find $\lim_{x \to 2} f(x)$

x	$f(x)$
1.99	10.97
1.999	10.997
1.9999	10.9997
$\downarrow$	$\downarrow$
2	
2.001	11.003
2.01	11.03

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For $f(x) = 3x + 5$, find $\lim_{x \to 2} f(x)$

x	$f(x)$
1.99	10.97
1.999	10.997
1.9999	10.9997
$\downarrow$	$\downarrow$
2	
2.0001	11.0003
2.001	11.003
2.01	11.03

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For $f(x) = 3x + 5$, find $\lim_{x \to 2} f(x)$

x	$f(x)$
1.99	10.97
1.999	10.997
1.9999	10.9997
$\downarrow$	$\downarrow$
2	
$\uparrow$	$\uparrow$
2.0001	11.0003
2.001	11.003
2.01	11.03

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For $f(x) = 3x + 5$, find $\lim_{x \to 2} f(x)$

x	$f(x)$
1.99	10.97
1.999	10.997
1.9999	10.9997
$\downarrow$	$\downarrow$
2	
$\uparrow$	$\uparrow$
2.0001	11.0003
2.001	11.003
2.01	11.03

$$\lim_{x \to 2} f(x) = 11$$

Example 3

Find $\lim_{x \to 5} \left(\dfrac{x^3 - 125}{x - 5} \right)$

x	$f(x)$

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$\boldsymbol{x}$	$\boldsymbol{f(x)}$
4.99	74.8501

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4.99	74.8501
4.999	74.985001

Example 3

Find $\lim_{x \to 5} \left(\frac{x^3 - 125}{x - 5} \right)$

x	$f(x)$
4.99	74.8501
4.999	74.985001
4.9999	74.9985

Example 3

Find $\lim_{x \to 5} \left(\dfrac{x^3 - 125}{x - 5} \right)$

$\boldsymbol{x}$	$\boldsymbol{f(x)}$
4.99	74.8501
4.999	74.985001
4.9999	74.9985
$\downarrow$	$\downarrow$

Example 3

Find $\lim_{x \to 5} \left(\frac{x^3 - 125}{x - 5} \right)$

x	$f(x)$
4.99	74.8501
4.999	74.985001
4.9999	74.9985
$\downarrow$	$\downarrow$
5	

Example 3

Find $\lim_{x \to 5} \left(\dfrac{x^3 - 125}{x - 5} \right)$

$\boldsymbol{x}$	$\boldsymbol{f(x)}$
4.99	74.8501
4.999	74.985001
4.9999	74.9985
$\downarrow$	$\downarrow$
5	
5.001	75.0015

Example 3

Find $\lim_{x \to 5} \left(\frac{x^3 - 125}{x - 5} \right)$

x	$f(x)$
4.99	74.8501
4.999	74.985001
4.9999	74.9985
$\downarrow$	$\downarrow$
5	
5.000001	75.000015
5.001	75.0015

Example 3

Find $\lim_{x \to 5} \left(\dfrac{x^3 - 125}{x - 5} \right)$

x	$f(x)$
4.99	74.8501
4.999	74.985001
4.9999	74.9985
$\downarrow$	$\downarrow$
5	
$\uparrow$	$\uparrow$
5.00000001	75.000001
5.000001	75.000015
5.001	75.0015

Example 3

Find $\lim_{x \to 5} \left(\dfrac{x^3 - 125}{x - 5} \right)$

x	$f(x)$
4.99	74.8501
4.999	74.985001
4.9999	74.9985
$\downarrow$	$\downarrow$
5	
$\uparrow$	$\uparrow$
5.00000001	75.000001
5.000001	75.000015
5.001	75.0015

$$\lim_{x \to 5} \frac{x^3 - 125}{x - 5} = 75$$

Example 4

Find the limit. *Note:* x is in radians.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

$\boldsymbol{x}$	$\boldsymbol{f(x)}$

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x	$f(x)$
-0.001	0.99999983

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x	$f(x)$
-0.001	0.99999983
-0.0001	1

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$\boldsymbol{x}$	$\boldsymbol{f(x)}$
-0.001	0.99999983
-0.0001	1
-0.00000001	1

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$$\lim_{x \to 0} \frac{\sin x}{x}$$

x	$f(x)$
-0.001	0.99999983
-0.0001	1
-0.00000001	1
$\downarrow$	$\downarrow$

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Find the limit. *Note:* x is in radians.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

x	$f(x)$
-0.001	0.99999983
-0.0001	1
-0.00000001	1
$\downarrow$	$\downarrow$
0	

Example 4

Find the limit. *Note:* x is in radians.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

x	$f(x)$
-0.001	0.99999983
-0.0001	1
-0.00000001	1
$\downarrow$	$\downarrow$
0	
0.001	0.99999983

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Find the limit. *Note:* x is in radians.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

$\boldsymbol{x}$	$\boldsymbol{f(x)}$
-0.001	0.99999983
-0.0001	1
-0.00000001	1
$\downarrow$	$\downarrow$
0	
0.00001	1
0.001	0.99999983

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Find the limit. *Note:* x is in radians.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

x	$f(x)$
-0.001	0.99999983
-0.0001	1
-0.00000001	1
$\downarrow$	$\downarrow$
0	
$\uparrow$	$\uparrow$
0.000000001	1
0.00001	1
0.001	0.99999983

Example 4

Find the limit. *Note:* x is in radians.

$$\lim_{x\to 0} \frac{\sin x}{x}$$

$\boldsymbol{x}$	$\boldsymbol{f(x)}$
-0.001	0.99999983
-0.0001	1
-0.00000001	1
$\downarrow$	$\downarrow$
0	
$\uparrow$	$\uparrow$
0.000000001	1
0.00001	1
0.001	0.99999983

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Left-Hand and Right-Hand Limits

In the previous examples, we found limits by evaluating values less than the value of a and also values greater than a.

These are called left-hand limits and right-hand limits, respectively.

Left-Hand Limit

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

	Values of x approach 7 from the left ($x < 7$)			
x	6.99	6.999	6.9999	7
$f(x)$	7.99	7.999	7.9999	Undefined

Values of the output approach the limit, 8

Right-Hand Limit

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

		Values of x approach 7 from the right ($x < 7$)		
x	7	7.0001	7.001	7.01
$f(x)$	Undefined	8.001	8.01	8.01

$\longleftarrow$ Values of the output approach the limit, 8

Notation

Left-Hand Limit: $\lim_{x \to a^-} f(x)$

Notation

Left-Hand Limit: $\lim_{x \to a^-} f(x)$

Right-Hand Limit: $\lim_{x \to a^+} f(x)$

Two-Sided Limit

$$\lim_{x \to a} f(x) = L$$

$$\Big\updownarrow \text{Implies}$$

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

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Finding a limit using a graph

For $f(x)$ as x approaches a:

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For $f(x)$ as x approaches a:

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  - Won't if there is a vertical asymptote at $x = a$

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For $f(x)$ as x approaches a:

- Examine the graph to see if left-hand limit exists.
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- Examine the graph to see if right-hand limit exists.

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For $f(x)$ as x approaches a:

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- Examine the graph to see if right-hand limit exists.
  - Ditto from above

Finding a limit using a graph

For $f(x)$ as x approaches a:

- Examine the graph to see if left-hand limit exists.
  - Won't if there is a vertical asymptote at $x = a$
- Examine the graph to see if right-hand limit exists.
  - Ditto from above
- If the 2 one-sided limits exist and are equal, there is a "limit."

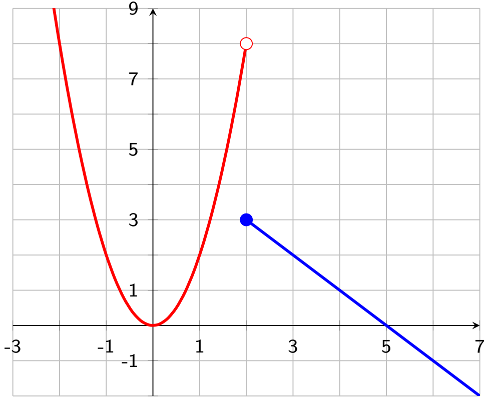
Finding a limit using a graph

For $f(x)$ as x approaches a:

- Examine the graph to see if left-hand limit exists.
  - Won't if there is a vertical asymptote at $x = a$
- Examine the graph to see if right-hand limit exists.
  - Ditto from above
- If the 2 one-sided limits exist and are equal, there is a "limit."
- If there is a point at $x = a$, then $f(a)$ is the value of the function at $x = a$.

Example 5

Use the graph of $f(x)$ to find each.



(a) $\lim_{x \to 2^-} f(x)$

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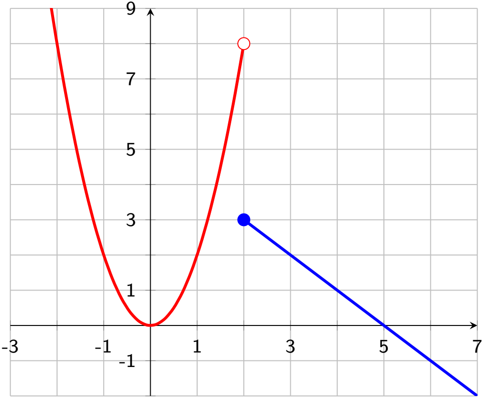

(a) $\lim_{x \to 2^-} f(x) = 8$

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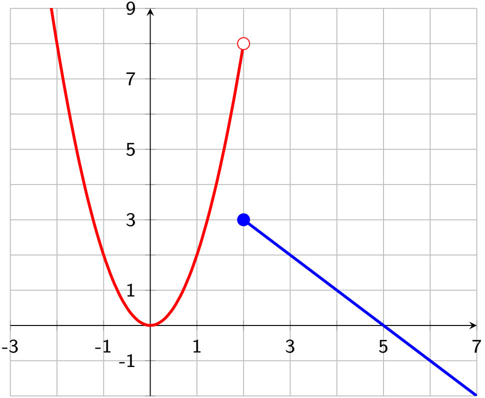

$(a) \quad \lim_{x \to 2^-} f(x) = 8$

$(b) \quad \lim_{x \to 2^+} f(x)$

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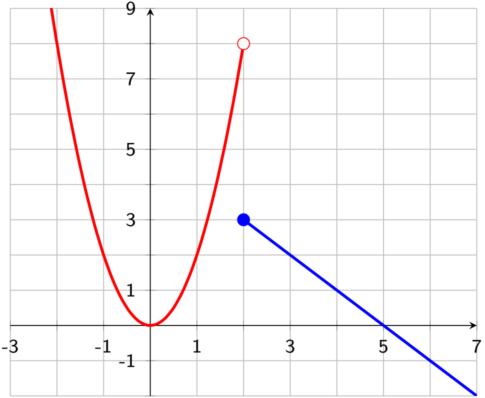

(a) $\lim_{x \to 2^-} f(x) = 8$

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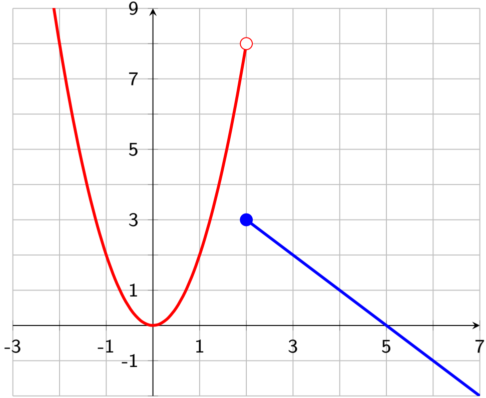

$(a)\quad \lim_{x \to 2^-} f(x) = 8$

$(b)\quad \lim_{x \to 2^+} f(x) = 3$

$(c)\quad \lim_{x \to 2} f(x)$

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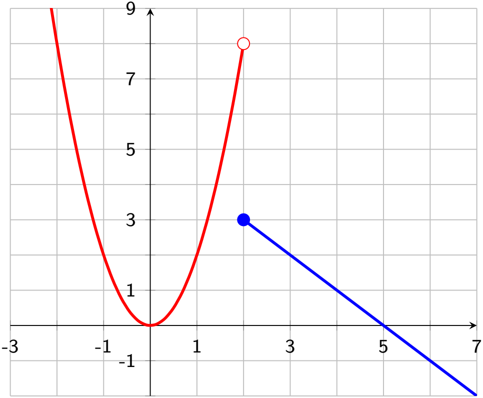

(a) $\lim_{x \to 2^-} f(x) = 8$

(b) $\lim_{x \to 2^+} f(x) = 3$

(c) $\lim_{x \to 2} f(x) = \text{DNE}$

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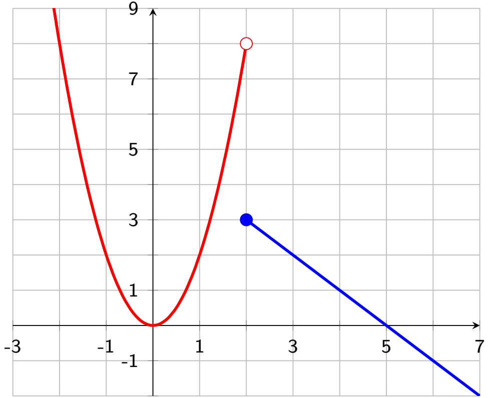

(a) $\lim_{x \to 2^-} f(x) = 8$

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(c) $\lim_{x \to 2} f(x) = \text{DNE}$

(d) $f(2)$

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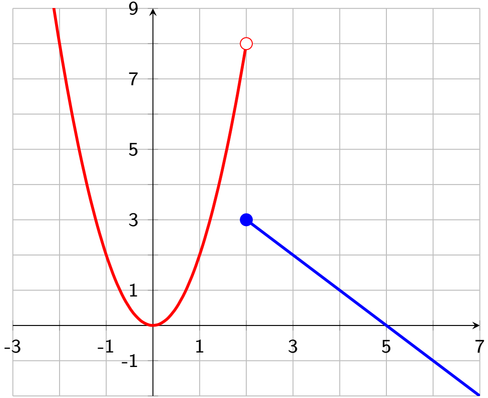


$(a)\quad \lim_{x\to 2^-} f(x) = 8$

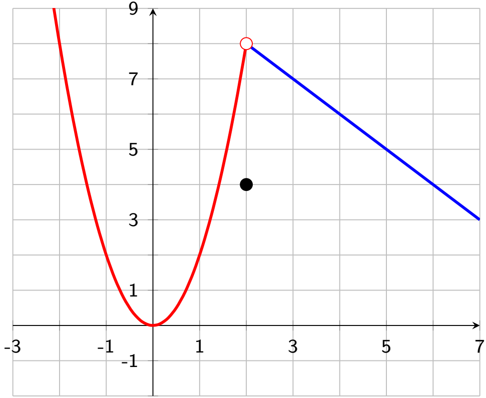
$(b)\quad \lim_{x\to 2^+} f(x) = 3$

$(c)\quad \lim_{x\to 2} f(x) = \text{DNE}$

$(d)\quad f(2) = 3$

Example 6

Use the graph of $f(x)$ to find each.



(a) $\lim_{x\to 2^-} f(x)$

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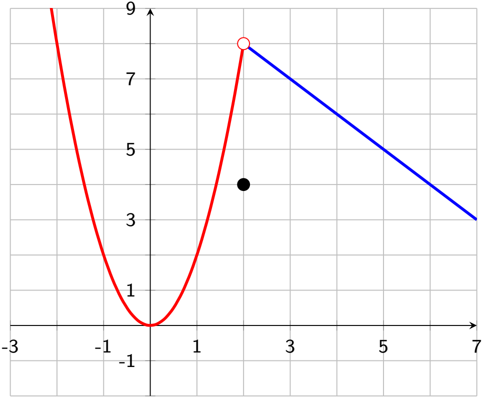

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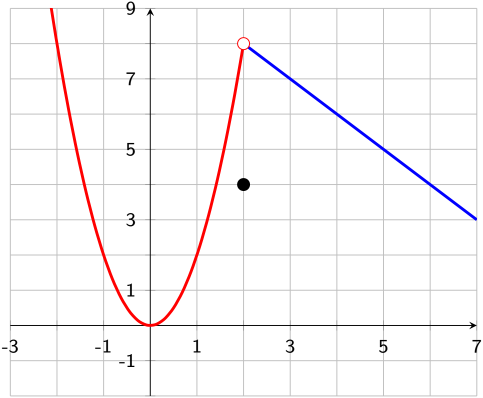
Example 6

Use the graph of $f(x)$ to find each.



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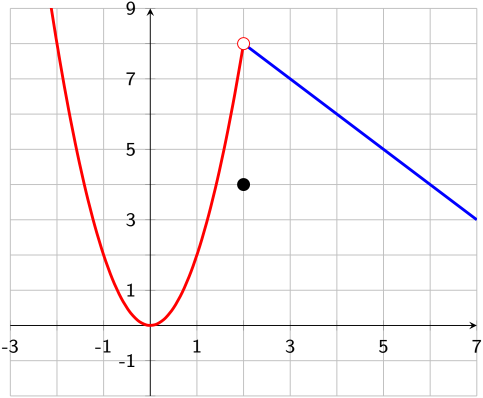

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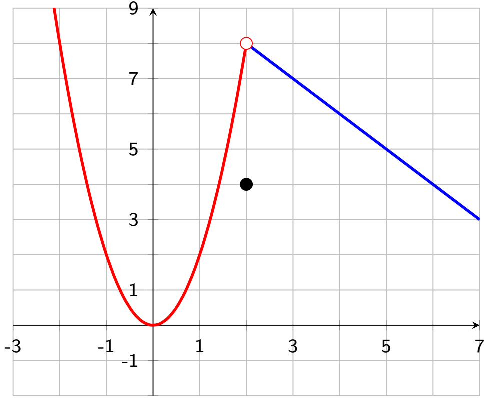

$(a) \quad \lim_{x\to 2^-} f(x) = 8$

$(b) \quad \lim_{x\to 2^+} f(x) = 8$

$(c) \quad \lim_{x\to 2} f(x)$

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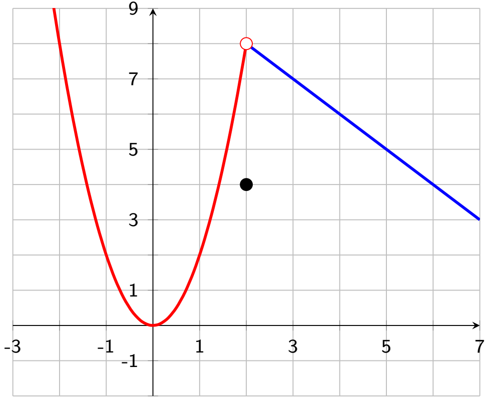

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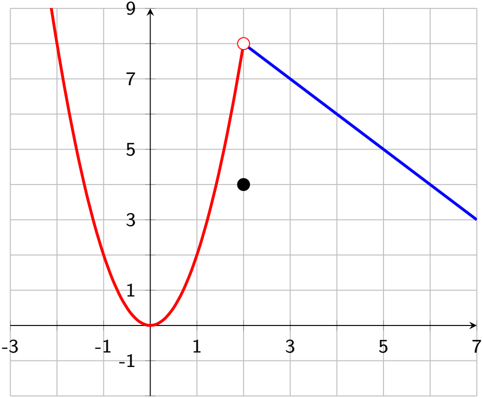

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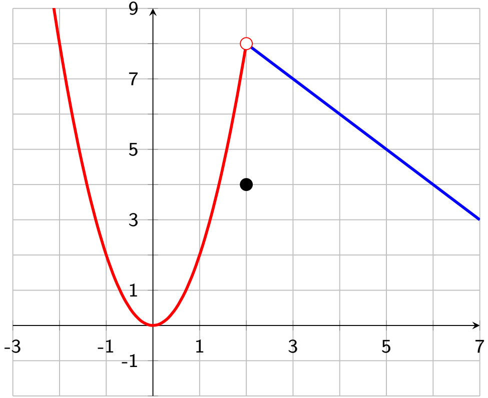
$(d)\quad f(2)$

Example 6

Use the graph of $f(x)$ to find each.



$(a)\quad \lim_{x\to 2^-} f(x) = 8$

$(b)\quad \lim_{x\to 2^+} f(x) = 8$

$(c)\quad \lim_{x\to 2} f(x) = 8$

$(d)\quad f(2) = 4$