Properties of Logarithms

Objectives

1. Use properties of logarithms to expand logarithmic expressions.

2. Use properties of logarithms to condense an expression into a single logarithm

3. Rewrite a logarithmic expression using the Change of Base Rules

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	**Exponents**	**Logarithms**

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	Exponents	Logarithms
Product		

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	Exponents	Logarithms
Product	$b^x \cdot b^y = b^{x+y}$	

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	Exponents	Logarithms
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	Exponents	Logarithms
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$
Quotient		

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	**Exponents**	**Logarithms**
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$
Quotient	$\frac{b^x}{b^y} = b^{x-y}$	

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	Exponents	Logarithms
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$
Quotient	$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	**Exponents**	**Logarithms**
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$
Quotient	$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$
Power		

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	Exponents	Logarithms
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$
Quotient	$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$
Power	$(b^x)^y = b^{xy}$	

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	**Exponents**	**Logarithms**
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$
Quotient	$\frac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
Power	$(b^x)^y = b^{xy}$	$\log_b(x)^y = y \cdot \log_b(x)$

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	**Exponents**	**Logarithms**
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$
Quotient	$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$
Power	$(b^x)^y = b^{xy}$	$\log_b(x)^y = y \cdot \log_b(x)$
Equality		

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	Exponents	Logarithms
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$
Quotient	$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$
Power	$(b^x)^y = b^{xy}$	$\log_b(x)^y = y \cdot \log_b(x)$
Equality	$b^x = b^y \iff x = y$	

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	**Exponents**	**Logarithms**
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$
Quotient	$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$
Power	$(b^x)^y = b^{xy}$	$\log_b(x)^y = y \cdot \log_b(x)$
Equality	$b^x = b^y \iff x = y$	
	x and y are real	

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	Exponents	Logarithms
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$
Quotient	$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$
Power	$(b^x)^y = b^{xy}$	$\log_b(x)^y = y \cdot \log_b(x)$
Equality	$b^x = b^y \iff x = y$	$\log_b(x) = \log_b(y) \iff x = y$
	x and y are real	

Exponent and Logarithm Properties

Exponents and logarithms have similar properties.

Property	**Exponents**	**Logarithms**
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x) + \log_b(y) = \log_b(xy)$
Quotient	$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$
Power	$(b^x)^y = b^{xy}$	$\log_b(x)^y = y \cdot \log_b(x)$
Equality	$b^x = b^y \iff x = y$	$\log_b(x) = \log_b(y) \iff x = y$
	x and y are real	$x > 0,\ y > 0$

Example 1

Expand each of the following and simplify numerical values when possible. Assume all quantities represent positive real numbers.

(a) $\log_2 \left(\dfrac{8}{x} \right)$

Example 1

Expand each of the following and simplify numerical values when possible. Assume all quantities represent positive real numbers.

(a) $\log_2\left(\frac{8}{x}\right)$

$$\log_2\left(\frac{8}{x}\right) = \log_2(8) - \log_2(x) \qquad \text{Quotient Prop.}$$

Example 1

Expand each of the following and simplify numerical values when possible. Assume all quantities represent positive real numbers.

(a) $\log_2\left(\frac{8}{x}\right)$

$$\log_2\left(\frac{8}{x}\right) = \log_2(8) - \log_2(x) \qquad \text{Quotient Prop.}$$

$$= 3 - \log_2(x) \qquad \log_2 8 = 3$$

Example 1

(b) $\log_{0.1}\left(10x^2\right)$

Example 1

(b) $\log_{0.1}\left(10x^2\right)$

$$\log_{0.1}\left(10x^2\right) = \log_{0.1}(10) + \log_{0.1}\left(x^2\right) \qquad \text{Product Prop.}$$

Example 1

(b) $\log_{0.1}\left(10x^2\right)$

$$\begin{aligned} \log_{0.1}\left(10x^2\right) &= \log_{0.1}(10) + \log_{0.1}\left(x^2\right) && \text{Product Prop.} \\ &= \log_{0.1}(10) + 2\log_{0.1}(x) && \text{Power Prop.} \end{aligned}$$

Example 1

(b) $\log_{0.1}\left(10x^2\right)$

$$\begin{aligned} \log_{0.1}\left(10x^2\right) &= \log_{0.1}(10) + \log_{0.1}\left(x^2\right) && \text{Product Prop.} \\ &= \log_{0.1}(10) + 2\log_{0.1}(x) && \text{Power Prop.} \\ &= -1 + 2\log_{0.1}(x) && \log_{0.1}(10) = -1 \end{aligned}$$

Example 1

(c) $\ln\left(\dfrac{3}{ex}\right)^2$

Example 1

(c) $\ln\left(\frac{3}{ex}\right)^2$

$$\ln\left(\frac{3}{ex}\right)^2 = 2\ln\left(\frac{3}{ex}\right) \qquad \text{Power Prop.}$$

Example 1

(c) $\ln\left(\frac{3}{ex}\right)^2$

$$\ln\left(\frac{3}{ex}\right)^2 = 2\ln\left(\frac{3}{ex}\right) \qquad \text{Power Prop.}$$

$$= 2\left(\ln(3) - \ln(ex)\right) \qquad \text{Quotient Prop.}$$

Example 1

(c) $\ln\left(\frac{3}{ex}\right)^2$

$$\ln\left(\frac{3}{ex}\right)^2 = 2\ln\left(\frac{3}{ex}\right) \qquad \text{Power Prop.}$$

$$= 2\left(\ln(3) - \ln(ex)\right) \qquad \text{Quotient Prop.}$$

$$= 2\left(\ln(3) - (\ln(e) + \ln(x)\right) \qquad \text{Product Prop.}$$

Example 1

(c) $\ln\left(\frac{3}{ex}\right)^2$

$$\begin{aligned}
\ln\left(\frac{3}{ex}\right)^2 &= 2\ln\left(\frac{3}{ex}\right) && \text{Power Prop.}\\
&= 2\left(\ln(3) - \ln(ex)\right) && \text{Quotient Prop.}\\
&= 2\left(\ln(3) - (\ln(e) + \ln(x))\right) && \text{Product Prop.}\\
&= 2\left(\ln(3) - \ln(e) - \ln(x)\right) && \text{Distribute the negative}
\end{aligned}$$

Example 1

(c) $\ln\left(\frac{3}{ex}\right)^2$

$$\begin{aligned}
\ln\left(\frac{3}{ex}\right)^2 &= 2\ln\left(\frac{3}{ex}\right) && \text{Power Prop.}\\
&= 2\left(\ln(3) - \ln(ex)\right) && \text{Quotient Prop.}\\
&= 2\left(\ln(3) - (\ln(e) + \ln(x)\right)) && \text{Product Prop.}\\
&= 2\left(\ln(3) - \ln(e) - \ln(x)\right) && \text{Distribute the negative}\\
&= 2\left(\ln(3) - 1 - \ln(x)\right) && \ln(e) = 1
\end{aligned}$$

Example 1

(c) $\ln\left(\dfrac{3}{ex}\right)^2$

$$\begin{aligned}
\ln\left(\frac{3}{ex}\right)^2 &= 2\ln\left(\frac{3}{ex}\right) && \text{Power Prop.}\\
&= 2\left(\ln(3) - \ln(ex)\right) && \text{Quotient Prop.}\\
&= 2\left(\ln(3) - (\ln(e) + \ln(x))\right) && \text{Product Prop.}\\
&= 2\left(\ln(3) - \ln(e) - \ln(x)\right) && \text{Distribute the negative}\\
&= 2\left(\ln(3) - 1 - \ln(x)\right) && \ln(e) = 1\\
&= 2\ln(3) - 2 - 2\ln(x) && \text{Distribute the 2}
\end{aligned}$$

Example 1

(d) $\log_{117}\left(x^2 - 4\right)$

Example 1

(d) $\log_{117}\left(x^2 - 4\right)$

$$\log_{117}(x^2 - 4) = \log_{117}\left((x+2)(x-2)\right) \qquad \text{Factor } x^2 - 4$$

Example 1

(d) $\log_{117}\left(x^2 - 4\right)$

$$\begin{aligned} \log_{117}(x^2 - 4) &= \log_{117}\left((x+2)(x-2)\right) && \text{Factor } x^2 - 4 \\ &= \log_{117}(x+2) + \log_{117}(x-2) && \text{Product Prop.} \end{aligned}$$

Example 1

(e) $\log\left(\sqrt[3]{\dfrac{100x^2}{yz^5}}\right)$

Example 1

(e) $\log\left(\sqrt[3]{\dfrac{100x^2}{yz^5}}\right)$

$$\log\left(\sqrt[3]{\frac{100x^2}{yz^5}}\right) = \log\left(\frac{100x^2}{yz^5}\right)^{1/3} \qquad \sqrt[3]{a} = a^{1/3}$$

Example 1

(e) $\log\left(\sqrt[3]{\dfrac{100x^2}{yz^5}}\right)$

$$\log\left(\sqrt[3]{\frac{100x^2}{yz^5}}\right) = \log\left(\frac{100x^2}{yz^5}\right)^{1/3} \qquad \sqrt[3]{a} = a^{1/3}$$

$$= \frac{1}{3}\log\left(\frac{100x^2}{yz^5}\right) \qquad \text{Power Prop.}$$

Example 1

(e) $\log\left(\sqrt[3]{\frac{100x^2}{yz^5}}\right)$

$$\log\left(\sqrt[3]{\frac{100x^2}{yz^5}}\right) = \log\left(\frac{100x^2}{yz^5}\right)^{1/3} \qquad \sqrt[3]{a} = a^{1/3}$$

$$= \frac{1}{3}\log\left(\frac{100x^2}{yz^5}\right) \qquad \text{Power Prop.}$$

$$= \frac{1}{3}\left(\log\left(100x^2\right) - \log\left(yz^5\right)\right) \qquad \text{Quotient Prop.}$$

Example 1

$\frac{1}{3}\left(\log\left(100x^2\right) - \log\left(yz^5\right)\right)$

Example 1

$$\frac{1}{3}\left(\log\left(100x^2\right) - \log\left(yz^5\right)\right)$$

$$= \frac{1}{3}\left(\log(100) + \log(x^2) - \left(\log(y) + \log(z^5)\right)\right) \qquad \text{Product Prop.}$$

Example 1

$$\frac{1}{3}\left(\log\left(100x^2\right) - \log\left(yz^5\right)\right)$$

$$= \frac{1}{3}\left(\log(100) + \log(x^2) - \left(\log(y) + \log(z^5)\right)\right) \qquad \text{Product Prop.}$$

$$= \frac{1}{3}\left(\log(100) + \log(x^2) - \log(y) - \log(z^5)\right) \qquad \text{Distribute the negative}$$

Example 1

$$\frac{1}{3}\left(\log\left(100x^2\right) - \log\left(yz^5\right)\right)$$

$$= \frac{1}{3}\left(\log(100) + \log(x^2) - \left(\log(y) + \log(z^5)\right)\right) \qquad \text{Product Prop.}$$

$$= \frac{1}{3}\left(\log(100) + \log(x^2) - \log(y) - \log(z^5)\right) \qquad \text{Distribute the negative}$$

$$= \frac{1}{3}\left(2 + \log(x^2) - \log(y) - \log(z^5)\right) \qquad \log(100) = 2$$

Example 1

$$\frac{1}{3}\left(\log\left(100x^2\right) - \log\left(yz^5\right)\right)$$

$$= \frac{1}{3}\left(\log(100) + \log(x^2) - \left(\log(y) + \log(z^5)\right)\right) \qquad \text{Product Prop.}$$

$$= \frac{1}{3}\left(\log(100) + \log(x^2) - \log(y) - \log(z^5)\right) \qquad \text{Distribute the negative}$$

$$= \frac{1}{3}\left(2 + \log(x^2) - \log(y) - \log(z^5)\right) \qquad \log(100) = 2$$

$$= \frac{1}{3}\left(2 + 2\log(x) - \log(y) - 5\log(z)\right) \qquad \text{Power Prop.}$$

Example 1

$$\frac{1}{3}\left(\log\left(100x^2\right) - \log\left(yz^5\right)\right)$$

$$= \frac{1}{3}\left(\log(100) + \log(x^2) - \left(\log(y) + \log(z^5)\right)\right) \qquad \text{Product Prop.}$$

$$= \frac{1}{3}\left(\log(100) + \log(x^2) - \log(y) - \log(z^5)\right) \qquad \text{Distribute the negative}$$

$$= \frac{1}{3}\left(2 + \log(x^2) - \log(y) - \log(z^5)\right) \qquad \log(100) = 2$$

$$= \frac{1}{3}\left(2 + 2\log(x) - \log(y) - 5\log(z)\right) \qquad \text{Power Prop.}$$

$$= \frac{2}{3} + \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) \qquad \text{Distribute the } \frac{1}{3}$$

Objectives

1. Use properties of logarithms to expand logarithmic expressions.
2. Use properties of logarithms to condense an expression into a single logarithm
3. Rewrite a logarithmic expression using the Change of Base Rules

Condensing Logarithmic Expressions

This is just working backwards from what we did in Example 1.

This will come in handy when we solve logarithmic equations that have more than one logarithm.

Example 2

Use the properties of logarithms to write the following as a single logarithm.

(a) $\log_3(x-1) - \log_3(x+1)$

Example 2

Use the properties of logarithms to write the following as a single logarithm.

(a) $\log_3(x-1) - \log_3(x+1)$

$$\log_3(x-1) - \log_3(x+1) = \log_3\left(\frac{x-1}{x+1}\right) \quad \text{Quotient Prop.}$$

Example 2

(b) $\log(x) + 2\log(y) - \log(z)$

Example 2

(b) $\log(x) + 2\log(y) - \log(z)$

$$\log(x) + 2\log(y) - \log(z) = \log(x) + \log(y^2) - \log(z) \qquad \text{Power Prop.}$$

Example 2

(b) $\log(x) + 2\log(y) - \log(z)$

$$\begin{aligned}
\log(x) + 2\log(y) - \log(z) &= \log(x) + \log(y^2) - \log(z) && \text{Power Prop.} \\
&= \log(xy^2) - \log(z) && \text{Product Prop.}
\end{aligned}$$

Example 2

(b) $\log(x) + 2\log(y) - \log(z)$

$$\begin{aligned}
\log(x) + 2\log(y) - \log(z) &= \log(x) + \log(y^2) - \log(z) && \text{Power Prop.} \\
&= \log(xy^2) - \log(z) && \text{Product Prop.} \\
&= \log\left(\frac{xy^2}{z}\right) && \text{Quotient Prop.}
\end{aligned}$$

Example 2

(c) $4\log_2(x) + 3$

Example 2

(c) $4\log_2(x)+3$

$$4\log_2(x)+3=\log_2(x^4)+3 \qquad \text{Power Prop.}$$

Example 2

(c) $4\log_2(x) + 3$

$$4\log_2(x) + 3 = \log_2(x^4) + 3 \qquad \text{Power Prop.}$$

$$= \log_2(x^4) + \log_2(2^3) \qquad \log_2(2^3) = 3$$

Example 2

(c) $4\log_2(x) + 3$

$$\begin{aligned} 4\log_2(x) + 3 &= \log_2(x^4) + 3 && \text{Power Prop.} \\ &= \log_2(x^4) + \log_2(2^3) && \log_2(2^3) = 3 \\ &= \log_2(x^4) + \log_2(8) && 2^3 = 8 \end{aligned}$$

Example 2

(c) $4\log_2(x) + 3$

$$\begin{aligned} 4\log_2(x) + 3 &= \log_2(x^4) + 3 && \text{Power Prop.} \\ &= \log_2(x^4) + \log_2(2^3) && \log_2(2^3) = 3 \\ &= \log_2(x^4) + \log_2(8) && 2^3 = 8 \\ &= \log_2(8x^4) && \text{Product Prop.} \end{aligned}$$

Example 2

(d) $\quad -\ln(x) - \dfrac{1}{2}$

Example 2

(d) $-\ln(x) - \dfrac{1}{2}$

$-\ln(x) - \frac{1}{2} = \ln\left(x^{-1}\right) - \frac{1}{2}$ Power Prop.

Example 2

(d) $\quad -\ln(x) - \dfrac{1}{2}$

$$-\ln(x) - \tfrac{1}{2} = \ln\left(x^{-1}\right) - \tfrac{1}{2} \qquad \text{Power Prop.}$$

$$= \ln\left(x^{-1}\right) - \ln\left(e^{1/2}\right) \qquad \tfrac{1}{2} = \ln(e^{1/2})$$

Example 2

(d) $-\ln(x) - \dfrac{1}{2}$

$$-\ln(x) - \tfrac{1}{2} = \ln\left(x^{-1}\right) - \tfrac{1}{2} \qquad \text{Power Prop.}$$

$$= \ln\left(x^{-1}\right) - \ln\left(e^{1/2}\right) \qquad \tfrac{1}{2} = \ln(e^{1/2})$$

$$= \ln\left(x^{-1}\right) - \ln\left(\sqrt{e}\right) \qquad e^{1/2} = \sqrt{e}$$

Example 2

(d) $-\ln(x) - \dfrac{1}{2}$

$$
\begin{aligned}
-\ln(x) - \tfrac{1}{2} &= \ln\left(x^{-1}\right) - \tfrac{1}{2} && \text{Power Prop.} \\
&= \ln\left(x^{-1}\right) - \ln\left(e^{1/2}\right) && \tfrac{1}{2} = \ln(e^{1/2}) \\
&= \ln\left(x^{-1}\right) - \ln\left(\sqrt{e}\right) && e^{1/2} = \sqrt{e} \\
&= \ln\left(\frac{x^{-1}}{\sqrt{e}}\right) && \text{Quotient Prop.}
\end{aligned}
$$

Example 2

(d) $-\ln(x) - \dfrac{1}{2}$

$$
\begin{aligned}
-\ln(x) - \tfrac{1}{2} &= \ln\left(x^{-1}\right) - \tfrac{1}{2} && \text{Power Prop.} \\
&= \ln\left(x^{-1}\right) - \ln\left(e^{1/2}\right) && \tfrac{1}{2} = \ln(e^{1/2}) \\
&= \ln\left(x^{-1}\right) - \ln\left(\sqrt{e}\right) && e^{1/2} = \sqrt{e} \\
&= \ln\left(\frac{x^{-1}}{\sqrt{e}}\right) && \text{Quotient Prop.} \\
&= \ln\left(\frac{1}{x\sqrt{e}}\right) && x^{-1} = \tfrac{1}{x}
\end{aligned}
$$

Objectives

1. Use properties of logarithms to expand logarithmic expressions.
2. Use properties of logarithms to condense an expression into a single logarithm
3. Rewrite a logarithmic expression using the Change of Base Rules

Change of Base Rules

Let $a, b > 0,\ a, b \neq 1.$

Change of Base Rules

Let $a, b > 0,\ a, b \neq 1$.

- $a^x = b^{x \log_b(a)}$ for all real numbers x.

Change of Base Rules

Let $a, b > 0,\ a, b \neq 1$.

- $a^x = b^{x \log_b(a)}$ for all real numbers x.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$

Example 3

Write an equivalent expression for each using base e (natural logarithms).

(a) $\log_7(2)$

Example 3

Write an equivalent expression for each using base e (natural logarithms).

(a) $\log_7(2) = \dfrac{\ln(2)}{\ln(7)}$

Example 3

Write an equivalent expression for each using base e (natural logarithms).

(a) $\log_7(2) = \dfrac{\ln(2)}{\ln(7)}$

(b) $\log(5)$

Example 3

Write an equivalent expression for each using base e (natural logarithms).

(a) $\log_7(2) = \dfrac{\ln(2)}{\ln(7)}$

(b) $\log(5) = \dfrac{\ln(5)}{\ln(10)}$

Example 3

Write an equivalent expression for each using base e (natural logarithms).

(a) $\log_7(2) = \dfrac{\ln(2)}{\ln(7)}$

(b) $\log(5) = \dfrac{\ln(5)}{\ln(10)}$

(c) $\log(x)$

Example 3

Write an equivalent expression for each using base e (natural logarithms).

(a) $\log_7(2) = \dfrac{\ln(2)}{\ln(7)}$

(b) $\log(5) = \dfrac{\ln(5)}{\ln(10)}$

(c) $\log(x) = \dfrac{\ln(x)}{\ln(10)}$