Trig Equations

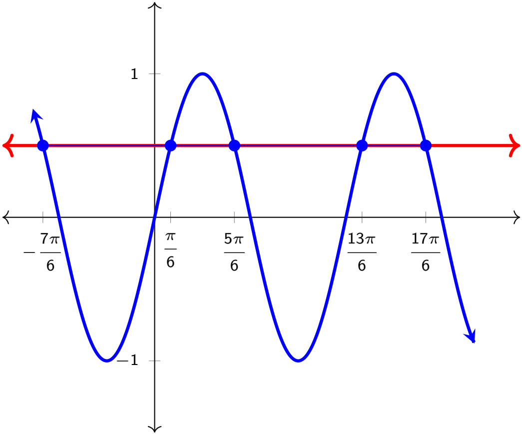
Objectives

1. Solve trigonometric equations

Trig Equations

A trigonometric equation is one that contains a trig function with variable, such as $\sin x = \frac{1}{2}$.

Trig Equations



General Form of Solutions

The general form of this solution is

$$x = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad x = \frac{5\pi}{6} + 2\pi n$$

where 2π is the period of the sine function.

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where 2π is the period of the sine function.

Since there are an infinite number of solutions, we will usually confine our answers to be between 0 and 2π.

How to Solve a Trig Equation

1. Get the trig function by itself, if possible.
2. Solve for the variable using inverse trig.

Example 1

Solve each of the following in the interval $[0, 2\pi)$.

(a) $\cos(2x) = -\dfrac{\sqrt{3}}{2}$

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$$x = 75^\circ + 180n \qquad\qquad x = 105^\circ + 180n$$

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(a) $\cos(2x) = -\dfrac{\sqrt{3}}{2}$

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$$x = 75^\circ,\ 255^\circ \qquad\qquad x = 105^\circ,\ 285^\circ$$

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$$x = 75^\circ,\ 255^\circ \qquad\qquad x = 105^\circ,\ 285^\circ$$

$$x = \left\{ \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12} \right\}$$

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$$\frac{1}{3}x - 180^\circ = 45^\circ + 360n \qquad \frac{1}{3}x - 180^\circ = 135^\circ + 360n$$

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$$\frac{1}{3}x - 180^\circ = 45^\circ + 360n \qquad \frac{1}{3}x - 180^\circ = 135^\circ + 360n$$

$$\frac{1}{3}x = 225^\circ + 360n \qquad \frac{1}{3}x = 315^\circ + 360n$$

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$$x = 675^\circ + 1080n \qquad x = 945^\circ + 1080n$$

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No angles between 0 and 2π

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$$x = 30^\circ + 60n \qquad\qquad x = 90^\circ + 60n$$

$$x = 30^\circ, 90^\circ, 150^\circ, 210^\circ, 270^\circ, 330^\circ \qquad\qquad x = 90^\circ, 150^\circ, \ldots$$

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$$x = 30^\circ, 90^\circ, 150^\circ, 210^\circ, 270^\circ, 330^\circ \qquad\qquad x = 90^\circ, 150^\circ, \ldots$$

$$x = \left\{ \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6} \right\}$$

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$$x = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$$

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$$x = 2\tan^{-1}(-3) + 360^\circ$$

$$x = 2\tan^{-1}(-3) + 2\pi$$

Using Algebraic Techniques and Trig Identities

The following examples make use of trig identities and algebraic techniques to solve the equations.

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Solve each in the interval $[0, 2\pi)$

(a) $3\sin^3 x = \sin^2 x$

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$$\sin x = 0 \qquad \sin x = \frac{1}{3}$$

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$$\sin^2 x = 0 \qquad 3\sin x - 1 = 0$$

$$\sin x = 0 \qquad \sin x = \frac{1}{3}$$

$$x = 0, 180^\circ \qquad x \approx 19.471^\circ, 160.529^\circ$$

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$$\sin x = 0 \qquad \sin x = \frac{1}{3}$$

$$x = 0, 180^\circ \qquad x \approx 19.471^\circ, 160.529^\circ$$

$$x = \left\{0,\ \text{arcsin}\left(\frac{1}{3}\right), \pi - \text{arcsin}\left(\frac{1}{3}\right), \pi\right\}$$

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$$\tan x = 2 \qquad\qquad \tan x = -1$$

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$$\begin{aligned} \tan x - 2 &= 0 \\ \tan x &= 2 \\ x &\approx 63.5^\circ, 243.5^\circ \end{aligned}$$

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$$x = \left\{ \arctan(2),\ \frac{3\pi}{4},\ \pi + \arctan(2),\ \frac{7\pi}{4} \right\}$$

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