# Vectors

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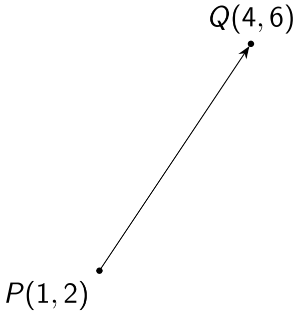
# Intro

A **vector** is a mathematical object with both magnitude (length) and direction.

It is represented geometrically by a directed line segment.

# Intro

The following shows an example of a vector $\vec{v} = \overrightarrow{PQ}$.




$P$ is the initial point and $Q$ is the terminal point.

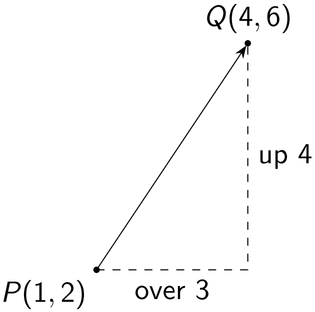
# Magnitude

The magnitude of a vector (denoted $\|\vec{v}\|$ or $|\vec{v}|$) is the distance between $P$ and $Q$.

Think Pythagorean Theorem.

# Component Form

If we connect the point $P$ to point $Q$ we get the following diagram:



# Component Form

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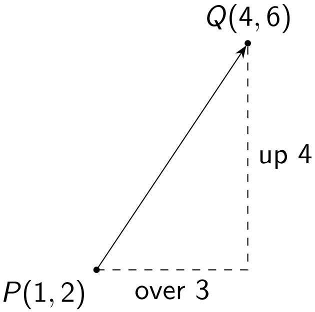


This can be represented by $\vec{v} = \langle 3, 4 \rangle$ or by $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.

# Component Form

To find the component form for $P(x_0, y_0)$ and $Q(x_1, y_1)$, just take the difference in coordinates:

$$\vec{v} = \overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0 \rangle$$

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To find the component form for $P(x_0, y_0)$ and $Q(x_1, y_1)$, just take the difference in coordinates:

$$\vec{v} = \overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0 \rangle$$

or

$$(x_1 - x_0)\mathbf{i} + (y_1 - y_0)\mathbf{j}$$

# Example 1a

Find the component form and exact magnitude of $\overrightarrow{AB}$ for each.

(a) $A(2, -1)$ $\quad B(-5, 3)$

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$$= \langle -5 - 2, 3 - (-1) \rangle$$

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$$= \langle -7, 4 \rangle$$

$$= -7\mathbf{i} + 4\mathbf{j}$$

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$$\|\overrightarrow{AB}\| = \sqrt{7^2 + 4^2}$$

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Find the component form and exact magnitude of $\overrightarrow{AB}$ for each.

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$$\begin{aligned} &= \langle -5 - 2, 3 - (-1) \rangle \\ &= \langle -7, 4 \rangle \\ &= -7\mathbf{i} + 4\mathbf{j} \\ \|\overrightarrow{AB}\| &= \sqrt{7^2 + 4^2} \\ &= \sqrt{49 + 16} \end{aligned}$$

# Example 1a

Find the component form and exact magnitude of $\overrightarrow{AB}$ for each.

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$$
\begin{aligned}
&= \langle -5-2, 3-(-1)\rangle \\
&= \langle -7, 4\rangle \\
&= -7\mathbf{i} + 4\mathbf{j} \\
\|\overrightarrow{AB}\| &= \sqrt{7^2+4^2} \\
&= \sqrt{49+16} \\
&= \sqrt{65}
\end{aligned}
$$

# Example 1b

(b) $A(-3,2)$ $B(0,1)$

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(b) $A(-3,2) \quad B(0,1)$

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$$= 3\mathbf{i} - \mathbf{j}$$

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$$= \langle 0 - (-3), 1 - 2 \rangle$$

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$$\|\overrightarrow{AB}\| = \sqrt{3^2 + 1^2}$$

# Example 1b

(b) $A(-3, 2)$ $B(0, 1)$

$$\begin{aligned} &= \langle 0 - (-3), 1 - 2 \rangle \\ &= \langle 3, -1 \rangle \\ &= 3\mathbf{i} - \mathbf{j} \\ \|\overrightarrow{AB}\| &= \sqrt{3^2 + 1^2} \\ &= \sqrt{10} \end{aligned}$$

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# Vector Addition

We can add vectors by adding the horizontal components, and adding the vertical components.

Think of these as *combining like terms*.

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For instance, when adding $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 1, -2 \rangle$, combine horizontal components and vertical components:

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# Vector Addition

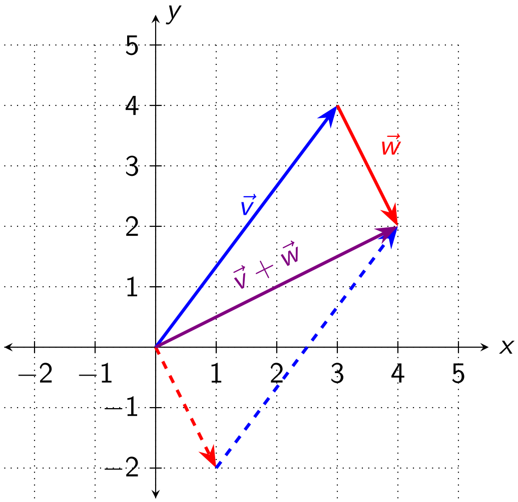
For instance, when adding $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 1, -2 \rangle$, combine horizontal components and vertical components:

$$\vec{v} + \vec{w} = \langle 3 + 1, 4 + (-2) \rangle$$

$$= \langle 4, 2 \rangle$$

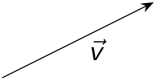
# Vector Addition

Visually, the previous problem is



# Scalar Multiplication

If we multiply our vector by a real number (a scalar), we get a new vector.



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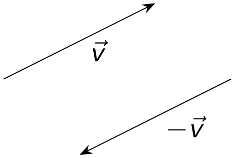

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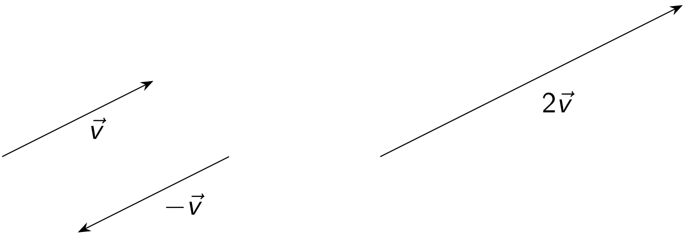

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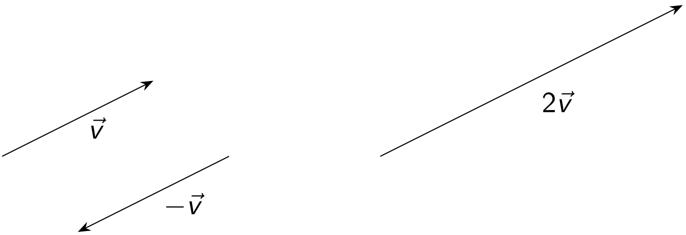
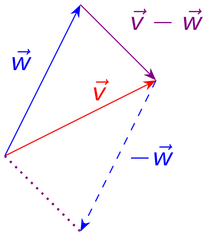



Scalar multiplication "distributes" the scalar to both horizontal and vertical component.

# Vector Subtraction

Vector subtraction $\vec{v} - \vec{w}$ can be thought of as $\vec{v} + (-\vec{w})$ and is illustrated below:



# Example 2a

If $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j}$, and $\vec{w} = \langle 6, -9 \rangle$, find each of the following.

(a) $\mathbf{v} + \mathbf{w}$

# Example 2a

If $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j}$, and $\vec{w} = \langle 6, -9 \rangle$, find each of the following.

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$$\mathbf{v} + \mathbf{w} = \langle 5, 4 \rangle + \langle 6, -9 \rangle$$

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If $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j}$, and $\vec{w} = \langle 6, -9 \rangle$, find each of the following.

(a) $\mathbf{v} + \mathbf{w}$

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle 5, 4 \rangle + \langle 6, -9 \rangle \\ &= \langle 5 + 6, 4 + (-9) \rangle \end{aligned}$$

# Example 2a

If $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j}$, and $\vec{w} = \langle 6, -9 \rangle$, find each of the following.

(a) $\mathbf{v} + \mathbf{w}$

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle 5, 4 \rangle + \langle 6, -9 \rangle \\ &= \langle 5 + 6, 4 + (-9) \rangle \\ &= \langle 11, -5 \rangle \end{aligned}$$

# Example 2b

(b) $\mathbf{v} - \mathbf{w}$

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$$\begin{aligned}\mathbf{v} - \mathbf{w} &= \langle 5, 4\rangle - \langle 6, -9\rangle \\ &= \langle 5 - 6, 4 - (-9)\rangle\end{aligned}$$

# Example 2b

(b) $\mathbf{v} - \mathbf{w}$

$$\begin{aligned}\mathbf{v} - \mathbf{w} &= \langle 5, 4 \rangle - \langle 6, -9 \rangle \\ &= \langle 5 - 6, 4 - (-9) \rangle \\ &= \langle -1, 13 \rangle\end{aligned}$$

# Example 2c

(c) $6\vec{v}$

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$$6\vec{v} = 6\langle 5, 4\rangle$$

# Example 2c

(c) $6\vec{v}$

$$\begin{aligned} 6\vec{v} &= 6\langle 5, 4\rangle \\ &= \langle 30, 24\rangle \end{aligned}$$

# Example 2d

(d) $2\mathbf{v} + 4\mathbf{w}$

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(d) $2\mathbf{v} + 4\mathbf{w}$

$$2\mathbf{v} + 4\mathbf{w} = 2(5\mathbf{i} + 4\mathbf{j}) + 4(6\mathbf{i} - 9\mathbf{j})$$

# Example 2d

(d) $2\mathbf{v} + 4\mathbf{w}$

$$\begin{aligned} 2\mathbf{v} + 4\mathbf{w} &= 2(5\mathbf{i} + 4\mathbf{j}) + 4(6\mathbf{i} - 9\mathbf{j}) \\ &= 10\mathbf{i} + 8\mathbf{j} + 24\mathbf{i} - 36\mathbf{j} \end{aligned}$$

# Example 2d

(d) $2\mathbf{v} + 4\mathbf{w}$

$$\begin{aligned} 2\mathbf{v} + 4\mathbf{w} &= 2(5\mathbf{i} + 4\mathbf{j}) + 4(6\mathbf{i} - 9\mathbf{j}) \\ &= 10\mathbf{i} + 8\mathbf{j} + 24\mathbf{i} - 36\mathbf{j} \\ &= 34\mathbf{i} - 28\mathbf{j} \end{aligned}$$

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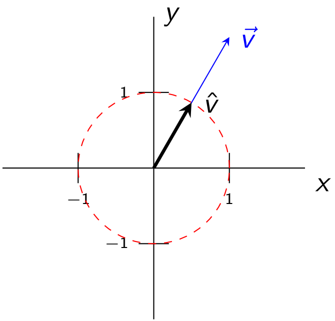
# Unit Vectors

A unit vector, $\hat{v}$, is a vector that has a magnitude of 1.




Notice the unit vector $\hat{v}$ is parallel to $\vec{v}$.

$\mathbf{i}$ and $\mathbf{j}$ are unit vectors with $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$

# Unit Vectors

We get $\hat{v}$ by dividing the magnitude of $\vec{v}$ by itself:

$$\hat{v} = \frac{\vec{v}}{\|v\|}$$

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Since we are dividing the vector (the hypotenuse) by the magnitude, we also divide the $x$ and $y$ components as well:

$$\hat{v} = \left\langle \frac{x}{\|v\|}, \frac{y}{\|v\|} \right\rangle$$

# Example 3a

Find a unit vector for each given vector.

(a) $\mathbf{v} = 5\mathbf{i} - 12\mathbf{j}$

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# Example 3a

Find a unit vector for each given vector.

(a) $\mathbf{v} = 5\mathbf{i} - 12\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{5^2 + 12^2}$$

$$= 13$$

$$\hat{v} = \frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}$$

# Example 3b

(b) $\vec{w} = \langle 4, -3 \rangle$

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$$|\vec{w}| = \sqrt{4^2 + 3^2}$$

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$$= 5$$

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(b) $\vec{w} = \langle 4, -3 \rangle$

$$|\vec{w}| = \sqrt{4^2 + 3^2}$$

$$= 5$$

$$\hat{w} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$$

# Example 3c

(c) $\mathbf{d} = 7\mathbf{i} + \mathbf{j}$

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$$|\mathbf{d}| = \sqrt{7^2 + 1^2}$$

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(c) $\mathbf{d} = 7\mathbf{i} + \mathbf{j}$

$$
\begin{aligned}
|\mathbf{d}| &= \sqrt{7^2 + 1^2} \\
&= \sqrt{50}
\end{aligned}
$$

# Example 3c

(c) $\mathbf{d} = 7\mathbf{i} + \mathbf{j}$

$$
\begin{aligned}
|\mathbf{d}| &= \sqrt{7^2 + 1^2} \\
&= \sqrt{50} \\
&= 5\sqrt{2}
\end{aligned}
$$

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$$\hat{d} = \frac{7}{5\sqrt{2}}\mathbf{i} + \frac{1}{5\sqrt{2}}\mathbf{j}$$

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$$= \frac{7\sqrt{2}}{10}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

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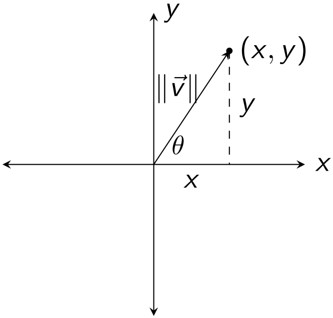
# Magnitude and Direction

If we place a vector $\langle x, y \rangle$ in the coordinate plane, we can put the initial point at the origin and the terminal point at $(x, y)$.



# Magnitude and Direction

From Trig Functions of Any Angle,

$$x = \|\vec{v}\| \cos\theta \text{ and } y = \|\vec{v}\| \sin\theta$$

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. Thus,

$$\langle x, y \rangle = \langle \|\vec{v}\| \cos\theta, \|\vec{v}\| \sin\theta \rangle = \|\vec{v}\| \langle \cos\theta, \sin\theta \rangle$$

# Example 4

Find the horizontal and vertical component form of the vector whose magnitude and direction angle are given by $|u| = 12$ and $\theta = 150^\circ$.

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# Example 4

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$$x = -6\sqrt{3} \qquad y = 6$$

# Example 4

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$$x = 12\left(-\frac{\sqrt{3}}{2}\right) \qquad y = 12\left(\frac{1}{2}\right)$$

$$x = -6\sqrt{3} \qquad y = 6$$

$$\langle -6\sqrt{3}, \quad 6\rangle$$

# Example 5

For $\vec{v} = \langle 3, -3\sqrt{3} \rangle$, find $\|\vec{v}\|$ and $\theta$ $(0 \leq \theta < 2\pi)$ and write in $\|\vec{v}\|\langle \cos\theta, \sin\theta \rangle$ form.

# Example 5

For $\vec{v} = \langle 3, -3\sqrt{3} \rangle$, find $\|\vec{v}\|$ and $\theta$ $(0 \leq \theta < 2\pi)$ and write in $\|\vec{v}\| \langle \cos\theta, \sin\theta \rangle$ form.

$$\|\vec{v}\| = \sqrt{3^2 + (3\sqrt{3})^2}$$

# Example 5

For $\vec{v} = \langle 3, -3\sqrt{3} \rangle$, find $\|\vec{v}\|$ and $\theta$ $(0 \leq \theta < 2\pi)$ and write in $\|\vec{v}\|\langle \cos\theta, \sin\theta \rangle$ form.

$$\|\vec{v}\| = \sqrt{3^2 + (3\sqrt{3})^2}$$

$$= \sqrt{36} = 6$$

# Example 5

For $\vec{v} = \langle 3, -3\sqrt{3} \rangle$, find $\|\vec{v}\|$ and $\theta$ $(0 \le \theta < 2\pi)$ and write in $\|\vec{v}\|\langle \cos\theta, \sin\theta \rangle$ form.

$$\theta' = \tan^{-1}\left|\frac{-3\sqrt{3}}{3}\right|$$

# Example 5

For $\vec{v} = \langle 3, -3\sqrt{3} \rangle$, find $\|\vec{v}\|$ and $\theta$ $(0 \leq \theta < 2\pi)$ and write in $\|\vec{v}\|\langle \cos\theta, \sin\theta \rangle$ form.

$$\theta' = \tan^{-1}\left|\frac{-3\sqrt{3}}{3}\right|$$

$$\theta' = 60^\circ$$

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$$\theta' = \tan^{-1} \left| \frac{-3\sqrt{3}}{3} \right|$$

$$\theta' = 60^\circ$$

$$\theta = 360^\circ - 60^\circ = 300^\circ$$

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$$= \frac{5\pi}{3}$$

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$$= \frac{5\pi}{3}$$

$$6 \left\langle \cos\left(\frac{5\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right) \right\rangle$$