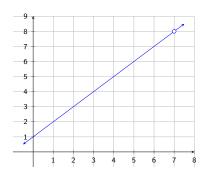
# Limits and Algebra

#### Intro

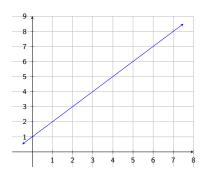
The graphs of  $f(x) = \frac{x^2 - 6x - 7}{x - 7}$  and g(x) = x + 1 are not the same.

#### Intro

The graphs of 
$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$
 and  $g(x) = x + 1$  are not the same.



$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$



$$g(x) = x + 1$$

# Objectives

1 Find Limits via Factoring

2 Limits with Complex Fractions

3 Limits with Radicals

4 Limits with Absolute Value

#### Algebraic Limits

Some limits that can't be evaluated directly can be evaluated after cancelling out common factors.

#### Algebraic Limits

Some limits that can't be evaluated directly can be evaluated after cancelling out common factors.

This is called removable discontinuity.

(a) Evaluate 
$$\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3}$$

$$\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3} = \lim_{x \to -3} \frac{(x + 3)(x + 1)}{x + 3}$$

$$\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3} = \lim_{x \to -3} \frac{(x + 3)(x + 1)}{x + 3}$$
$$= \lim_{x \to -3} \frac{(x + 3)(x + 1)}{(x + 3)}$$

$$\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3} = \lim_{x \to -3} \frac{(x + 3)(x + 1)}{x + 3}$$

$$= \lim_{x \to -3} \frac{\cancel{(x + 3)}(x + 1)}{\cancel{(x + 3)}}$$

$$= \lim_{x \to -3} (x + 1)$$

$$\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3} = \lim_{x \to -3} \frac{(x + 3)(x + 1)}{x + 3}$$

$$= \lim_{x \to -3} \frac{\cancel{(x + 3)}(x + 1)}{\cancel{(x + 3)}}$$

$$= \lim_{x \to -3} (x + 1)$$

$$= -3 + 1$$

$$\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3} = \lim_{x \to -3} \frac{(x + 3)(x + 1)}{x + 3}$$

$$= \lim_{x \to -3} \frac{\cancel{(x + 3)}(x + 1)}{\cancel{(x + 3)}}$$

$$= \lim_{x \to -3} (x + 1)$$

$$= -3 + 1$$

$$= -2$$

$$\lim_{x \to -2} \frac{x+2}{x^2 + 7x + 10} = \lim_{x \to -2} \frac{x+2}{(x+2)(x+5)}$$

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$$\lim_{x \to -2} \frac{x+2}{x^2 + 7x + 10} = \lim_{x \to -2} \frac{x+2}{(x+2)(x+5)}$$

$$= \lim_{x \to -2} \frac{\cancel{x+2}}{\cancel{(x+2)}(x+5)}$$

$$= \lim_{x \to -2} \frac{1}{x+5}$$

$$\lim_{x \to -2} \frac{x+2}{x^2 + 7x + 10} = \lim_{x \to -2} \frac{x+2}{(x+2)(x+5)}$$

$$= \lim_{x \to -2} \frac{\cancel{x+2}}{\cancel{(x+2)}(x+5)}$$

$$= \lim_{x \to -2} \frac{1}{x+5}$$

$$= \frac{1}{-2+5}$$

$$\lim_{x \to -2} \frac{x+2}{x^2 + 7x + 10} = \lim_{x \to -2} \frac{x+2}{(x+2)(x+5)}$$

$$= \lim_{x \to -2} \frac{\cancel{x+2}}{\cancel{(x+2)}(x+5)}$$

$$= \lim_{x \to -2} \frac{1}{\cancel{x+5}}$$

$$= \frac{1}{-2+5}$$

$$= \frac{1}{3}$$

# Objectives

Find Limits via Factoring

2 Limits with Complex Fractions

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#### Complex Fractions

Simplify the complex fraction by multiplying every term by the least common tiny denominator.

Evaluate each.

(a) 
$$\lim_{x \to -5} \left( \frac{\frac{1}{x} + \frac{1}{5}}{x + 5} \right)$$

Evaluate each.

(a) 
$$\lim_{x \to -5} \left( \frac{\frac{1}{x} + \frac{1}{5}}{x + 5} \right)$$

$$\lim_{x \to -5} \left( \frac{\frac{1}{x} + \frac{1}{5}}{x+5} \right) = \lim_{x \to -5} \left( \frac{\frac{1}{x} + \frac{1}{5}}{x+5} \right) \left( \frac{5x}{5x} \right)$$

Evaluate each.

(a) 
$$\lim_{x \to -5} \left( \frac{\frac{1}{x} + \frac{1}{5}}{x + 5} \right)$$

$$\lim_{x \to -5} \left( \frac{\frac{1}{x} + \frac{1}{5}}{x+5} \right) = \lim_{x \to -5} \left( \frac{\frac{1}{x} + \frac{1}{5}}{x+5} \right) \left( \frac{5x}{5x} \right)$$
$$= \lim_{x \to -5} \frac{5+x}{5x(x+5)}$$

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$$= \lim_{x \to -5} \frac{5+x}{5x(x+5)}$$
$$= \lim_{x \to -5} \frac{1}{5x}$$

$$= \lim_{x \to -5} \frac{\cancel{5} + \cancel{x}}{5\cancel{x}\cancel{(x+5)}}$$

$$= \lim_{x \to -5} \frac{1}{5\cancel{x}}$$

$$= \frac{1}{5(-5)}$$

$$= \lim_{x \to -5} \frac{5 + x}{5x(x + 5)}$$

$$= \lim_{x \to -5} \frac{1}{5x}$$

$$= \frac{1}{5(-5)}$$

$$= -\frac{1}{25}$$

(b) 
$$\lim_{x \to 3} \left( \frac{\frac{1}{3} - \frac{1}{x}}{3 - x} \right)$$

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$$\lim_{x \to 3} \left( \frac{\frac{1}{3} - \frac{1}{x}}{3 - x} \right) = \lim_{x \to 3} \left( \frac{\frac{1}{3} - \frac{1}{x}}{3 - x} \right) \left( \frac{3x}{3x} \right)$$

(b) 
$$\lim_{x \to 3} \left( \frac{\frac{1}{3} - \frac{1}{x}}{3 - x} \right)$$

$$\lim_{x \to 3} \left( \frac{\frac{1}{3} - \frac{1}{x}}{3 - x} \right) = \lim_{x \to 3} \left( \frac{\frac{1}{3} - \frac{1}{x}}{3 - x} \right) \left( \frac{3x}{3x} \right)$$
$$= \lim_{x \to 3} \frac{3 - x}{3x(x - 3)}$$

$$= \lim_{x \to 3} \frac{3 - x}{3x(x - 3)}$$

$$= \lim_{x \to 3} \frac{3 - x}{3x(x - 3)}$$
$$= \lim_{x \to 3} \frac{-1}{3x}$$

$$= \lim_{x \to 3} \frac{3 - x}{3x(x - 3)}$$

$$= \lim_{x \to 3} \frac{-1}{3x}$$

$$= \frac{-1}{3(3)}$$

$$= \lim_{x \to 3} \frac{3 - x}{3x(x - 3)}$$

$$= \lim_{x \to 3} \frac{-1}{3x}$$

$$= \frac{-1}{3(3)}$$

$$= \frac{-1}{9}$$

# Objectives

Find Limits via Factoring

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#### Radicals

When working with radicals, multiply by the conjugate.

#### Expression Conjugate

$$a + \sqrt{b}$$
  $a - \sqrt{b}$   $a + \sqrt{b}$ 

$$a - \sqrt{b}$$
  $a + \sqrt{b}$ 

(a) 
$$\lim_{x\to 0} \left( \frac{\sqrt{25-x}-5}{x} \right)$$

(a) 
$$\lim_{x \to 0} \left( \frac{\sqrt{25 - x} - 5}{x} \right)$$
  
 $\lim_{x \to 0} \left( \frac{\sqrt{25 - x} - 5}{x} \right) = \lim_{x \to 0} \left( \frac{\sqrt{25 - x} - 5}{x} \right) \left( \frac{\sqrt{25 - x} + 5}{\sqrt{25 - x} + 5} \right)$ 

(a) 
$$\lim_{x \to 0} \left( \frac{\sqrt{25 - x} - 5}{x} \right)$$
  
 $\lim_{x \to 0} \left( \frac{\sqrt{25 - x} - 5}{x} \right) = \lim_{x \to 0} \left( \frac{\sqrt{25 - x} - 5}{x} \right) \left( \frac{\sqrt{25 - x} + 5}{\sqrt{25 - x} + 5} \right)$   
 $= \lim_{x \to 0} \frac{25 - x - 25}{x \left( \sqrt{25 - x} + 5 \right)}$ 

(a) 
$$\lim_{x \to 0} \left( \frac{\sqrt{25 - x} - 5}{x} \right)$$

$$\lim_{x \to 0} \left( \frac{\sqrt{25 - x} - 5}{x} \right) = \lim_{x \to 0} \left( \frac{\sqrt{25 - x} - 5}{x} \right) \left( \frac{\sqrt{25 - x} + 5}{\sqrt{25 - x} + 5} \right)$$

$$= \lim_{x \to 0} \frac{25 - x - 25}{x \left( \sqrt{25 - x} + 5 \right)}$$

$$= \lim_{x \to 0} \frac{-x}{x \left( \sqrt{25 - x} + 5 \right)}$$

$$=\lim_{x\to 0}\frac{-1}{\sqrt{25-x}+5}$$

$$= \lim_{x \to 0} \frac{-1}{\sqrt{25 - x} + 5}$$
$$= \frac{-1}{\sqrt{25 - 0} + 5}$$

$$= \lim_{x \to 0} \frac{-1}{\sqrt{25 - x} + 5}$$

$$= \frac{-1}{\sqrt{25 - 0} + 5}$$

$$= \frac{-1}{10}$$

(b) 
$$\lim_{h\to 0} \left(\frac{\sqrt{16+h}-4}{h}\right)$$

(b) 
$$\lim_{h \to 0} \left( \frac{\sqrt{16+h} - 4}{h} \right)$$
  $\lim_{h \to 0} \left( \frac{\sqrt{16+h} - 4}{h} \right) = \lim_{h \to 0} \left( \frac{\sqrt{16+h} - 4}{h} \right) \left( \frac{\sqrt{16+h} + 4}{\sqrt{16+h} + 4} \right)$ 

(b) 
$$\lim_{h \to 0} \left( \frac{\sqrt{16+h} - 4}{h} \right)$$

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$$= \lim_{h \to 0} \frac{16+h-16}{h(\sqrt{16+h} + 4)}$$

(b) 
$$\lim_{h \to 0} \left( \frac{\sqrt{16 + h} - 4}{h} \right)$$

$$\lim_{h \to 0} \left( \frac{\sqrt{16 + h} - 4}{h} \right) = \lim_{h \to 0} \left( \frac{\sqrt{16 + h} - 4}{h} \right) \left( \frac{\sqrt{16 + h} + 4}{\sqrt{16 + h} + 4} \right)$$

$$= \lim_{h \to 0} \frac{16 + h - 16}{h \left( \sqrt{16 + h} + 4 \right)}$$

$$= \lim_{h \to 0} \frac{h}{h \left( \sqrt{16 + h} + 4 \right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{16+h}+4}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{16 + h} + 4}$$
$$= \frac{1}{\sqrt{16 + 0} + 4}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{16 + h} + 4}$$

$$= \frac{1}{\sqrt{16 + 0} + 4}$$

$$= \frac{1}{8}$$

(c) 
$$\lim_{x \to 4} \frac{4-x}{\sqrt{x}-2}$$

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$$\lim_{x \to 4} \frac{4-x}{\sqrt{x}-2}$$

$$\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} \left( \frac{\sqrt{x} + 2}{\sqrt{x} + 2} \right)$$

(c) 
$$\lim_{x \to 4} \frac{4-x}{\sqrt{x}-2}$$

$$\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} \left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2}\right)$$
$$= \lim_{x \to 4} \frac{(4 - x)(\sqrt{x} + 2)}{x - 4}$$

(c) 
$$\lim_{x\to 4} \frac{4-x}{\sqrt{x}-2}$$

$$\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} \left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2}\right)$$

$$= \lim_{x \to 4} \frac{(4 - x)(\sqrt{x} + 2)}{x - 4}$$

$$= \lim_{x \to 4} -1(\sqrt{x} + 2)$$

(c) 
$$\lim_{x\to 4}\frac{4-x}{\sqrt{x}-2}$$

$$\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} \left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2}\right)$$

$$= \lim_{x \to 4} \frac{(4 - x)(\sqrt{x} + 2)}{x - 4}$$

$$= \lim_{x \to 4} -1(\sqrt{x} + 2)$$

$$= -1(\sqrt{4} + 2)$$

(c) 
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$$\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} \left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2}\right)$$

$$= \lim_{x \to 4} \frac{(4 - x)(\sqrt{x} + 2)}{x - 4}$$

$$= \lim_{x \to 4} -1(\sqrt{x} + 2)$$

$$= -1(\sqrt{4} + 2)$$

$$= -1(2 + 2) = -4$$

(d) 
$$\lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}}$$

(d) 
$$\lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}}$$

$$\lim_{x \to 3} \frac{x - 3}{\sqrt{x} - \sqrt{3}} = \lim_{x \to 3} \frac{x - 3}{\sqrt{x} - \sqrt{3}} \left( \frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}} \right)$$

(d) 
$$\lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}}$$

$$\lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}} = \lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}} \left(\frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}}\right)$$

$$= \lim_{x \to 3} \frac{(x-3)(\sqrt{x} + \sqrt{3})}{x-3}$$

(d) 
$$\lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}}$$

$$\lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}} = \lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}} \left(\frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}}\right)$$

$$= \lim_{x \to 3} \frac{(x-3)(\sqrt{x} + \sqrt{3})}{x-3}$$

$$= \lim_{x \to 3} (\sqrt{x} + \sqrt{3})$$

(d) 
$$\lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}}$$

$$\lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}} = \lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}} \left(\frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}}\right)$$

$$= \lim_{x \to 3} \frac{(x-3)(\sqrt{x} + \sqrt{3})}{x-3}$$

$$= \lim_{x \to 3} (\sqrt{x} + \sqrt{3})$$

$$= \sqrt{3} + \sqrt{3}$$

(d) 
$$\lim_{x \to 3} \frac{x - 3}{\sqrt{x} - \sqrt{3}}$$

$$\lim_{x \to 3} \frac{x - 3}{\sqrt{x} - \sqrt{3}} = \lim_{x \to 3} \frac{x - 3}{\sqrt{x} - \sqrt{3}} \left(\frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}}\right)$$

$$= \lim_{x \to 3} \frac{(x - 3)(\sqrt{x} + \sqrt{3})}{x - 3}$$

$$= \lim_{x \to 3} (\sqrt{x} + \sqrt{3})$$

$$= \sqrt{3} + \sqrt{3}$$

$$= 2\sqrt{3}$$

### Objectives

Find Limits via Factoring

2 Limits with Complex Fractions

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4 Limits with Absolute Value

#### Limits and Absolute Value

With absolute value, it might be better to use a table or a graph to find the limit.

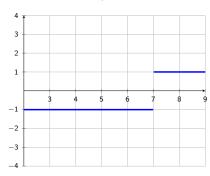
#### Limits and Absolute Value

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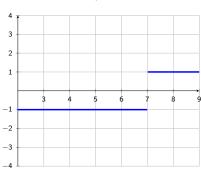
Also, look at the left-hand and right-hand limits.

(a) 
$$\lim_{x \to 7} \frac{|x-7|}{x-7}$$

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$$\lim_{x \to 7} \frac{|x-7|}{x-7}$$

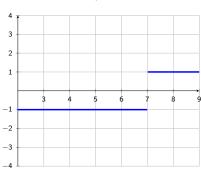


(a) 
$$\lim_{x \to 7} \frac{|x-7|}{x-7}$$



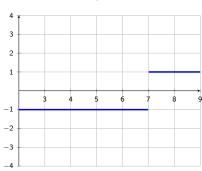
$$\lim_{x \to 7^-} \frac{|x-7|}{x-7}$$

(a) 
$$\lim_{x \to 7} \frac{|x-7|}{x-7}$$



$$\lim_{x \to 7^{-}} \frac{|x - 7|}{x - 7} = -1$$

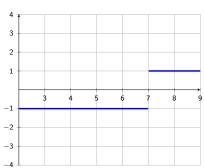
(a) 
$$\lim_{x \to 7} \frac{|x-7|}{x-7}$$



$$\lim_{x \to 7^{-}} \frac{|x - 7|}{x - 7} = -1$$

$$\lim_{x \to 7^+} \frac{|x-7|}{x-7}$$

(a) 
$$\lim_{x \to 7} \frac{|x-7|}{x-7}$$

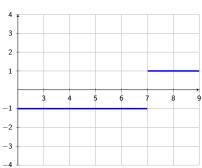


$$\lim_{x \to 7^{-}} \frac{|x - 7|}{x - 7} = -1$$

$$\lim_{x \to 7^+} \frac{|x - 7|}{x - 7} = 1$$

Find the limit for each.

(a) 
$$\lim_{x \to 7} \frac{|x-7|}{x-7}$$

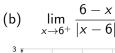


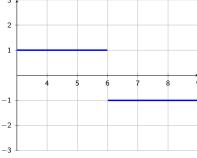
$$\lim_{x \to 7^{-}} \frac{|x - 7|}{x - 7} = -1$$

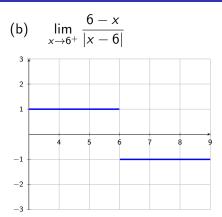
$$\lim_{x \to 7^+} \frac{|x - 7|}{x - 7} = 1$$

Does Not Exist

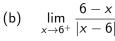
(b) 
$$\lim_{x \to 6^+} \frac{6-x}{|x-6|}$$

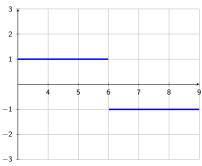






$$\lim_{x \to 6^+} \frac{6-x}{|x-6|}$$





$$\lim_{x \to 6^+} \frac{6 - x}{|x - 6|} = -1$$