Limits and Algebra

Intro

The graphs of $f(x) = \dfrac{x^2 - 6x - 7}{x - 7}$ and $g(x) = x + 1$ are <u>not the same</u>.

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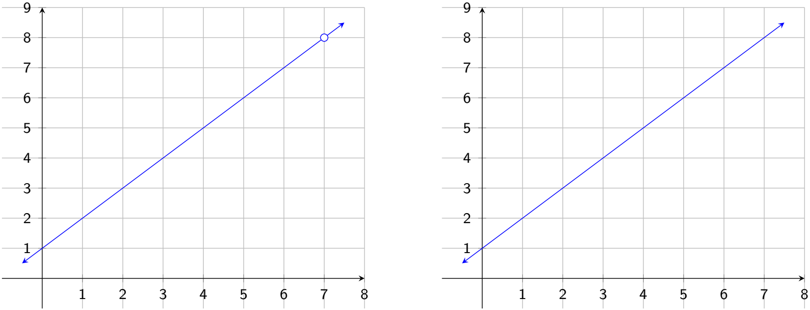

$$f(x) = \frac{x^2 - 6x - 7}{x - 7} \qquad g(x) = x + 1$$

Objectives

1. Find Limits via Factoring
2. Limits with Complex Fractions
3. Limits with Radicals

Algebraic Limits

Some limits that can't be evaluated directly can be evaluated after cancelling out common factors.

Algebraic Limits

Some limits that can't be evaluated directly can be evaluated after cancelling out common factors.

This is called removable discontinuity.

Example 1

(a) Evaluate $\lim\limits_{x \to -3} \dfrac{x^2 + 4x + 3}{x + 3}$

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$$\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3} = \lim_{x \to -3} \frac{(x+3)(x+1)}{x+3}$$

Example 1

(a) Evaluate $\lim_{x \to -3} \dfrac{x^2 + 4x + 3}{x + 3}$

$$\begin{aligned} \lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3} &= \lim_{x \to -3} \frac{(x+3)(x+1)}{x+3} \\ &= \lim_{x \to -3} \frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}} \end{aligned}$$

Example 1

(a) Evaluate $\lim_{x \to -3} \dfrac{x^2 + 4x + 3}{x + 3}$

$$\begin{aligned}
\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3} &= \lim_{x \to -3} \frac{(x+3)(x+1)}{x+3} \\
&= \lim_{x \to -3} \frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}} \\
&= \lim_{x \to -3} (x+1)
\end{aligned}$$

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$$
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\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3} &= \lim_{x \to -3} \frac{(x+3)(x+1)}{x+3} \\
&= \lim_{x \to -3} \frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}} \\
&= \lim_{x \to -3} (x+1) \\
&= -3 + 1
\end{aligned}
$$

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\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3} &= \lim_{x \to -3} \frac{(x+3)(x+1)}{x+3} \\
&= \lim_{x \to -3} \frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}} \\
&= \lim_{x \to -3} (x+1) \\
&= -3 + 1 \\
&= -2
\end{aligned}$$

Example 1

(b) Evaluate $\lim_{x \to -2} \dfrac{x+2}{x^2+7x+10}$

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$$\lim_{x \to -2} \frac{x+2}{x^2+7x+10} = \lim_{x \to -2} \frac{x+2}{(x+2)(x+5)}$$

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$$\lim_{x \to -2} \frac{x+2}{x^2+7x+10} = \lim_{x \to -2} \frac{x+2}{(x+2)(x+5)}$$

$$= \lim_{x \to -2} \frac{\cancel{x+2}}{\cancel{(x+2)}(x+5)}$$

Example 1

(b) Evaluate $\lim_{x \to -2} \dfrac{x+2}{x^2+7x+10}$

$$\begin{aligned}
\lim_{x \to -2} \frac{x+2}{x^2+7x+10} &= \lim_{x \to -2} \frac{x+2}{(x+2)(x+5)} \\
&= \lim_{x \to -2} \frac{\cancel{x+2}}{\cancel{(x+2)}(x+5)} \\
&= \lim_{x \to -2} \frac{1}{x+5}
\end{aligned}$$

Example 1

(b) Evaluate $\lim\limits_{x \to -2} \dfrac{x+2}{x^2+7x+10}$

$$\begin{aligned}
\lim_{x \to -2} \frac{x+2}{x^2+7x+10} &= \lim_{x \to -2} \frac{x+2}{(x+2)(x+5)} \\
&= \lim_{x \to -2} \frac{\cancel{x+2}}{\cancel{(x+2)}(x+5)} \\
&= \lim_{x \to -2} \frac{1}{x+5} \\
&= \frac{1}{-2+5}
\end{aligned}$$

Example 1

(b) Evaluate $\lim_{x \to -2} \frac{x+2}{x^2+7x+10}$

$$\begin{aligned}
\lim_{x \to -2} \frac{x+2}{x^2+7x+10} &= \lim_{x \to -2} \frac{x+2}{(x+2)(x+5)} \\
&= \lim_{x \to -2} \frac{\cancel{x+2}}{\cancel{(x+2)}(x+5)} \\
&= \lim_{x \to -2} \frac{1}{x+5} \\
&= \frac{1}{-2+5} \\
&= \frac{1}{3}
\end{aligned}$$

Objectives

Complex Fractions

Simplify the complex fraction by multiplying every term by the least common tiny denominator.

Example 2

Evaluate each.

(a) $$\lim_{x \to -5} \left(\frac{\frac{1}{x} + \frac{1}{5}}{x + 5} \right)$$

Example 2

Evaluate each.

(a) $\displaystyle\lim_{x\to-5}\left(\frac{\frac{1}{x}+\frac{1}{5}}{x+5}\right)$

$$\lim_{x\to-5}\left(\frac{\frac{1}{x}+\frac{1}{5}}{x+5}\right)=\lim_{x\to-5}\left(\frac{\frac{1}{x}+\frac{1}{5}}{x+5}\right)\left(\frac{5x}{5x}\right)$$

Example 2

Evaluate each.

(a) $\lim_{x \to -5} \left(\dfrac{\frac{1}{x} + \frac{1}{5}}{x+5} \right)$

$$\lim_{x \to -5} \left(\frac{\frac{1}{x} + \frac{1}{5}}{x+5} \right) = \lim_{x \to -5} \left(\frac{\frac{1}{x} + \frac{1}{5}}{x+5} \right) \left(\frac{5x}{5x} \right)$$

$$= \lim_{x \to -5} \frac{5+x}{5x(x+5)}$$

Example 2

$$= \lim_{x \to -5} \frac{\cancel{5+x}}{5x\cancel{(x+5)}}$$

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$$= \frac{1}{5(-5)}$$

Example 2

$$= \lim_{x \to -5} \frac{\cancel{5+x}}{5x\cancel{(x+5)}}$$

$$= \lim_{x \to -5} \frac{1}{5x}$$

$$= \frac{1}{5(-5)}$$

$$= -\frac{1}{25}$$

Example 2

(b) $\displaystyle \lim_{x \to 3} \left(\frac{\frac{1}{3} - \frac{1}{x}}{3 - x} \right)$

Example 2

(b) $\lim_{x\to 3}\left(\frac{\frac{1}{3}-\frac{1}{x}}{3-x}\right)$

$$\lim_{x\to 3}\left(\frac{\frac{1}{3}-\frac{1}{x}}{3-x}\right)=\lim_{x\to 3}\left(\frac{\frac{1}{3}-\frac{1}{x}}{3-x}\right)\left(\frac{3x}{3x}\right)$$

Example 2

(b) $\lim_{x \to 3} \left(\dfrac{\frac{1}{3} - \frac{1}{x}}{3 - x} \right)$

$$\lim_{x \to 3} \left(\frac{\frac{1}{3} - \frac{1}{x}}{3 - x} \right) = \lim_{x \to 3} \left(\frac{\frac{1}{3} - \frac{1}{x}}{3 - x} \right) \left(\frac{3x}{3x} \right)$$

$$= \lim_{x \to 3} \frac{3 - x}{3x(x - 3)}$$

Example 2

$$= \lim_{x \to 3} \frac{\cancel{3 - x}}{3x\cancel{(x - 3)}}$$

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$$= \lim_{x \to 3} \frac{\cancel{3 - x}}{3x\cancel{(x-3)}}$$

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$$= \lim_{x \to 3} \frac{-1}{3x}$$

$$= \frac{-1}{3(3)}$$

Example 2

$$= \lim_{x \to 3} \frac{\cancel{3 - x}}{3x\cancel{(x - 3)}}$$

$$= \lim_{x \to 3} \frac{-1}{3x}$$

$$= \frac{-1}{3(3)}$$

$$= \frac{-1}{9}$$

Objectives

Radicals

When working with radicals, multiply by the conjugate.

Expression	**Conjugate**
$a + \sqrt{b}$	$a - \sqrt{b}$
$a - \sqrt{b}$	$a + \sqrt{b}$

Example 3

(a) $\lim_{x \to 0} \left(\frac{\sqrt{25 - x} - 5}{x} \right)$

Example 3

(a) $\displaystyle \lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right)$

$$\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right)=\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right)\left(\frac{\sqrt{25-x}+5}{\sqrt{25-x}+5}\right)$$

Example 3

(a) $\displaystyle \lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right)$

$$\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right) = \lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right)\left(\frac{\sqrt{25-x}+5}{\sqrt{25-x}+5}\right)$$

$$= \lim_{x\to 0}\frac{25-x-25}{x\left(\sqrt{25-x}+5\right)}$$

Example 3

(a) $\lim_{x \to 0} \left(\frac{\sqrt{25-x}-5}{x} \right)$

$$\lim_{x \to 0} \left(\frac{\sqrt{25-x}-5}{x} \right) = \lim_{x \to 0} \left(\frac{\sqrt{25-x}-5}{x} \right) \left(\frac{\sqrt{25-x}+5}{\sqrt{25-x}+5} \right)$$

$$= \lim_{x \to 0} \frac{25-x-25}{x\left(\sqrt{25-x}+5\right)}$$

$$= \lim_{x \to 0} \frac{-x}{x\left(\sqrt{25-x}+5\right)}$$

Example 3

$$= \lim_{x \to 0} \frac{-1}{\sqrt{25 - x} + 5}$$

Example 3

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Example 3

$$= \lim_{x \to 0} \frac{-1}{\sqrt{25-x}+5}$$

$$= \frac{-1}{\sqrt{25-0}+5}$$

$$= \frac{-1}{10}$$

Example 3

(b) $$\lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right)$$

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(b) $\lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right)$

$$\lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right)=\lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right)\left(\frac{\sqrt{16+h}+4}{\sqrt{16+h}+4}\right)$$

Example 3

(b) $\lim_{h \to 0} \left(\frac{\sqrt{16+h} - 4}{h} \right)$

$$\lim_{h \to 0} \left(\frac{\sqrt{16+h} - 4}{h} \right) = \lim_{h \to 0} \left(\frac{\sqrt{16+h} - 4}{h} \right) \left(\frac{\sqrt{16+h} + 4}{\sqrt{16+h} + 4} \right)$$

$$= \lim_{h \to 0} \frac{16 + h - 16}{h\left(\sqrt{16+h} + 4\right)}$$

Example 3

(b) $\lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right)$

$$\begin{aligned}\lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right) &= \lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\right)\left(\frac{\sqrt{16+h}+4}{\sqrt{16+h}+4}\right)\\ &= \lim_{h\to 0}\frac{16+h-16}{h\left(\sqrt{16+h}+4\right)}\\ &= \lim_{h\to 0}\frac{h}{h\left(\sqrt{16+h}+4\right)}\end{aligned}$$

Example 3

$$= \lim_{h \to 0} \frac{1}{\sqrt{16+h}+4}$$

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$$= \lim_{h \to 0} \frac{1}{\sqrt{16+h}+4}$$

$$= \frac{1}{\sqrt{16+0}+4}$$

Example 3

$$= \lim_{h \to 0} \frac{1}{\sqrt{16+h}+4}$$

$$= \frac{1}{\sqrt{16+0}+4}$$

$$= \frac{1}{8}$$

Example 3

(c) $\lim_{x \to 4} \dfrac{4 - x}{\sqrt{x} - 2}$

Example 3

(c) $\lim_{x\to 4} \dfrac{4-x}{\sqrt{x}-2}$

$$\lim_{x\to 4} \frac{4-x}{\sqrt{x}-2} = \lim_{x\to 4} \frac{4-x}{\sqrt{x}-2}\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right)$$

Example 3

(c) $\lim_{x \to 4} \dfrac{4 - x}{\sqrt{x} - 2}$

$$\begin{aligned}\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} &= \lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} \left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2} \right) \\ &= \lim_{x \to 4} \frac{(4 - x)(\sqrt{x} + 2)}{x - 4}\end{aligned}$$

Example 3

(c) $\displaystyle\lim_{x\to 4} \frac{4-x}{\sqrt{x}-2}$

$$\begin{aligned}
\lim_{x\to 4} \frac{4-x}{\sqrt{x}-2} &= \lim_{x\to 4} \frac{4-x}{\sqrt{x}-2}\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right) \\
&= \lim_{x\to 4} \frac{(4-x)(\sqrt{x}+2)}{x-4} \\
&= \lim_{x\to 4} -1(\sqrt{x}+2)
\end{aligned}$$

Example 3

(c) $\lim_{x\to 4} \dfrac{4-x}{\sqrt{x}-2}$

$$\begin{aligned}
\lim_{x\to 4} \frac{4-x}{\sqrt{x}-2} &= \lim_{x\to 4} \frac{4-x}{\sqrt{x}-2}\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right) \\
&= \lim_{x\to 4} \frac{(4-x)(\sqrt{x}+2)}{x-4} \\
&= \lim_{x\to 4} -1(\sqrt{x}+2) \\
&= -1(\sqrt{4}+2)
\end{aligned}$$

Example 3

(c) $\lim_{x \to 4} \dfrac{4 - x}{\sqrt{x} - 2}$

$$\begin{aligned}
\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} &= \lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} \left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2} \right) \\
&= \lim_{x \to 4} \frac{(4 - x)(\sqrt{x} + 2)}{x - 4} \\
&= \lim_{x \to 4} -1(\sqrt{x} + 2) \\
&= -1(\sqrt{4} + 2) \\
&= -1(2 + 2) = -4
\end{aligned}$$

Example 3

(d) $\lim_{x \to 3} \dfrac{x-3}{\sqrt{x}-\sqrt{3}}$

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$$\lim_{x \to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} = \lim_{x \to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} \left(\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} \right)$$

Example 3

(d) $\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}$

$$\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} = \lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} \left(\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}\right)$$

$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{x-3}$$

Example 3

(d) $\displaystyle\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}$

$$\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} = \lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}\left(\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}\right)$$

$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{x-3}$$

$$= \lim_{x\to 3} (\sqrt{x}+\sqrt{3})$$

Example 3

(d) $\displaystyle\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}$

$$\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} = \lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} \left(\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}\right)$$

$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{x-3}$$

$$= \lim_{x\to 3} (\sqrt{x}+\sqrt{3})$$

$$= \sqrt{3}+\sqrt{3}$$

Example 3

(d) $\lim_{x \to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}$

$$
\begin{aligned}
\lim_{x \to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} &= \lim_{x \to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} \left(\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} \right) \\
&= \lim_{x \to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{x-3} \\
&= \lim_{x \to 3} (\sqrt{x}+\sqrt{3}) \\
&= \sqrt{3}+\sqrt{3} \\
&= 2\sqrt{3}
\end{aligned}
$$