Polynomials and Their Graphs

Polynomial Functions

A polynomial function is a function in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers and $n \geq 1$ is a natural number.

The **domain** of a polynomial function is $(-\infty, \infty)$

Table of Contents

Graphs of Polynomial Functions

Graphs of polynomials are **smooth** and **continuous**.

Graphs of Polynomial Functions

Graphs of polynomials are **smooth** and **continuous**.

By “smooth”, we mean that there are no “corners” or “cusps” (sharp points) on the graph.

Graphs of Polynomial Functions

Graphs of polynomials are **smooth** and **continuous**.

By "smooth", we mean that there are no "corners" or "cusps" (sharp points) on the graph.

By "continuous" we mean that there are no "breaks" or "holes" in the graph (i.e. you can draw it without lifting the pencil off the paper.)

Example 1

Determine if each of the following are polynomials.

(a) $g(x) = \dfrac{4 + x^3}{x}$

Example 1

Determine if each of the following are polynomials.

(a) $\quad g(x) = \dfrac{4 + x^3}{x}$

Not a polynomial. The domain is not $(-\infty, \infty)$

Example 1

Determine if each of the following are polynomials.

(a) $g(x) = \dfrac{4 + x^3}{x}$

Not a polynomial. The domain is not $(-\infty, \infty)$

(b) $p(x) = \dfrac{4x + x^3}{x}$

Example 1

Determine if each of the following are polynomials.

(a) $g(x) = \dfrac{4 + x^3}{x}$

Not a polynomial. The domain is not $(-\infty, \infty)$

(b) $p(x) = \dfrac{4x + x^3}{x}$

Not a polynomial. The domain is not $(-\infty, \infty)$

Example 1

(c) $\quad q(x) = \dfrac{4x + x^3}{x^2 + 4}$

Example 1

(c) $q(x) = \dfrac{4x + x^3}{x^2 + 4}$

$$\frac{4x + x^3}{x^2 + 4} = \frac{x(4 + x^2)}{x^2 + 4}$$

Example 1

(c) $\quad q(x) = \dfrac{4x + x^3}{x^2 + 4}$

$$\frac{4x + x^3}{x^2 + 4} = \frac{x(4 + x^2)}{x^2 + 4}$$

$$= x$$

Example 1

(c) $\quad q(x) = \dfrac{4x + x^3}{x^2 + 4}$

$$\frac{4x + x^3}{x^2 + 4} = \frac{x(4 + x^2)}{x^2 + 4}$$

$$= x$$

Is a polynomial.

Example 1

(c) $\quad q(x) = \dfrac{4x + x^3}{x^2 + 4}$

$$\frac{4x + x^3}{x^2 + 4} = \frac{x(4 + x^2)}{x^2 + 4}$$

$$= x$$

Is a polynomial.

- Original domain is $(-\infty, \infty)$
- Can be written as $q(x) = 1x^1$

Example 1

(d) $f(x) = \sqrt[3]{x}$

Example 1

(d) $f(x) = \sqrt[3]{x}$

Not a polynomial.

Example 1

(d) $f(x) = \sqrt[3]{x}$

Not a polynomial.

$$f(x) = \sqrt[3]{x} = x^{1/3}$$

Example 1

(d) $f(x) = \sqrt[3]{x}$

Not a polynomial.

$$f(x) = \sqrt[3]{x} = x^{1/3}$$

$1/3$ is not a natural number

Example 1

(e) $h(x) = |x|$

Example 1

(e) $h(x) = |x|$

Not a polynomial.

Example 1

(e) $h(x) = |x|$

Not a polynomial.

Could only be written in piecewise form

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 1

(f) $z(x) = 0$

Example 1

(f) $z(x) = 0$

Is a polynomial.

Example 1

(f) $\quad z(x) = 0$

Is a polynomial.

- Can be written as $z(x) = 0x^n + 0x^{n-1} + \cdots + 0$
- Domain is $(-\infty, \infty)$

Table of Contents

Polynomial Vocabulary

For $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$:

- n is the degree of the polynomial.

Polynomial Vocabulary

For $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$:

- n is the degree of the polynomial.
- The term $a_n x^n$ is the leading term of the polynomial.

Polynomial Vocabulary

For $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$:

- n is the degree of the polynomial.
- The term $a_n x^n$ is the leading term of the polynomial.
- a_n is the leading coefficient of the polynomial.

Polynomial Vocabulary

For $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$:

- n is the degree of the polynomial.
- The term $a_n x^n$ is the leading term of the polynomial.
- a_n is the leading coefficient of the polynomial.
- a_0 is the constant term of the polynomial.

Polynomial Vocabulary

For $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$:

- n is the degree of the polynomial.
- The term $a_n x^n$ is the leading term of the polynomial.
- a_n is the leading coefficient of the polynomial.
- a_0 is the constant term of the polynomial.

Polynomial Vocabulary

For $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$:

- n is the degree of the polynomial.
- The term $a_n x^n$ is the leading term of the polynomial.
- a_n is the leading coefficient of the polynomial.
- a_0 is the constant term of the polynomial.

If $f(x) = a_0$, and $a_0 \neq 0$, then f has degree 0.

Polynomial Vocabulary

For $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$:

- n is the degree of the polynomial.
- The term $a_n x^n$ is the leading term of the polynomial.
- a_n is the leading coefficient of the polynomial.
- a_0 is the constant term of the polynomial.

If $f(x) = a_0$, and $a_0 \neq 0$, then f has degree 0.

If $f(x) = 0$, then f has no degree

Example 2

Find the degree, leading term, leading coefficient, and constant term of the following polynomials.

(a) $f(x) = 4x^5 - 3x^2 + 2x - 5$

Example 2

Find the degree, leading term, leading coefficient, and constant term of the following polynomials.

(a) $f(x) = 4x^5 - 3x^2 + 2x - 5$

- Degree is 5

Example 2

Find the degree, leading term, leading coefficient, and constant term of the following polynomials.

(a) $f(x) = 4x^5 - 3x^2 + 2x - 5$

- Degree is 5
- Leading term is $4x^5$

Example 2

Find the degree, leading term, leading coefficient, and constant term of the following polynomials.

(a) $f(x) = 4x^5 - 3x^2 + 2x - 5$

- Degree is 5
- Leading term is $4x^5$
- Leading coefficient is 4

Example 2

Find the degree, leading term, leading coefficient, and constant term of the following polynomials.

(a) $f(x) = 4x^5 - 3x^2 + 2x - 5$

- Degree is 5
- Leading term is $4x^5$
- Leading coefficient is 4
- Constant is -5

Example 2

(b) $g(x) = 12x + x^3$

Example 2

(b) $g(x) = 12x + x^3$

$$g(x) = x^3 + 12x$$

Example 2

(b) $g(x) = 12x + x^3$

$$g(x) = x^3 + 12x$$

- Degree is 3

Example 2

(b) $g(x) = 12x + x^3$

$$g(x) = x^3 + 12x$$

- Degree is 3
- Leading term is x^3

Example 2

(b) $g(x) = 12x + x^3$

$$g(x) = x^3 + 12x$$

- Degree is 3
- Leading term is x^3
- Leading coefficient is 1

Example 2

(b) $g(x) = 12x + x^3$

$$g(x) = x^3 + 12x$$

- Degree is 3
- Leading term is x^3
- Leading coefficient is 1
- Constant is 0

Example 2

(c) $h(x) = \dfrac{4 - x}{5}$

Example 2

(c) $\quad h(x) = \dfrac{4 - x}{5}$

$$h(x) = \frac{4}{5} - \frac{1}{5}x = -\frac{1}{5}x + \frac{4}{5}$$

Example 2

(c) $\quad h(x) = \dfrac{4 - x}{5}$

$$h(x) = \frac{4}{5} - \frac{1}{5}x = -\frac{1}{5}x + \frac{4}{5}$$

- Degree is 1

Example 2

(c) $h(x) = \dfrac{4 - x}{5}$

$$h(x) = \frac{4}{5} - \frac{1}{5}x = -\frac{1}{5}x + \frac{4}{5}$$

- Degree is 1
- Leading term is $-\dfrac{1}{5}x$

Example 2

(c) $h(x) = \dfrac{4 - x}{5}$

$$h(x) = \frac{4}{5} - \frac{1}{5}x = -\frac{1}{5}x + \frac{4}{5}$$

- Degree is 1
- Leading term is $-\dfrac{1}{5}x$
- Leading coefficient is $-\dfrac{1}{5}$

Example 2

(c) $h(x) = \dfrac{4-x}{5}$

$$h(x) = \frac{4}{5} - \frac{1}{5}x = -\frac{1}{5}x + \frac{4}{5}$$

- Degree is 1
- Leading term is $-\dfrac{1}{5}x$
- Leading coefficient is $-\dfrac{1}{5}$
- Constant is $\dfrac{4}{5}$

Example 2

(d) $p(x) = (2x-1)^3(x-2)(3x+2)$

Example 2

(d) $p(x) = (2x-1)^3(x-2)(3x+2)$

$$p(x) = (2x)^3(x)(3x) + \text{ other terms } + (-1)^3(-2)(2)$$

Example 2

(d) $p(x) = (2x-1)^3(x-2)(3x+2)$

$$p(x) = (2x)^3(x)(3x) + \text{ other terms } + (-1)^3(-2)(2)$$

$$p(x) = 8x^3(x)(3x) + \text{ other terms } + (-1)(-2)(2)$$

Example 2

(d) $p(x) = (2x-1)^3(x-2)(3x+2)$

$$p(x) = (2x)^3(x)(3x) + \text{ other terms } + (-1)^3(-2)(2)$$

$$p(x) = 8x^3(x)(3x) + \text{ other terms } + (-1)(-2)(2)$$

$$p(x) = 24x^5 + \text{ other terms } + 4$$

Example 2

(d) $p(x) = (2x-1)^3(x-2)(3x+2)$

$$p(x) = (2x)^3(x)(3x) + \text{ other terms } + (-1)^3(-2)(2)$$

$$p(x) = 8x^3(x)(3x) + \text{ other terms } + (-1)(-2)(2)$$

$$p(x) = 24x^5 + \text{ other terms } + 4$$

- Degree is 5

Example 2

(d) $p(x) = (2x-1)^3(x-2)(3x+2)$

$$p(x) = (2x)^3(x)(3x) + \text{ other terms } + (-1)^3(-2)(2)$$

$$p(x) = 8x^3(x)(3x) + \text{ other terms } + (-1)(-2)(2)$$

$$p(x) = 24x^5 + \text{ other terms } + 4$$

- Degree is 5
- Leading term is $24x^5$

Example 2

(d) $$p(x) = (2x-1)^3(x-2)(3x+2)$$

$$p(x) = (2x)^3(x)(3x) + \text{ other terms } + (-1)^3(-2)(2)$$

$$p(x) = 8x^3(x)(3x) + \text{ other terms } + (-1)(-2)(2)$$

$$p(x) = 24x^5 + \text{ other terms } + 4$$

- Degree is 5
- Leading term is $24x^5$
- Leading coefficient is 24

Example 2

(d) $p(x) = (2x-1)^3(x-2)(3x+2)$

$$p(x) = (2x)^3(x)(3x) + \text{ other terms } + (-1)^3(-2)(2)$$

$$p(x) = 8x^3(x)(3x) + \text{ other terms } + (-1)(-2)(2)$$

$$p(x) = 24x^5 + \text{ other terms } + 4$$

- Degree is 5
- Leading term is $24x^5$
- Leading coefficient is 24
- Constant is 4

Table of Contents

End Behavior

The end behavior of a function refers to what is happening to the output values the further the graph of the function heads left and right.

End Behavior

The end behavior of a function refers to what is happening to the output values the further the graph of the function heads left and right.

In other words, what is happening to your y-coordinates the further x goes to ∞ or to $-\infty$?

End Behavior of Polynomials

INVESTIGATION:

What is the end behavior of

$$f(x) = ax^n$$

when a is a positive real number and n is an odd integer?

End Behavior of Polynomials

INVESTIGATION:

What is the end behavior of

$$f(x) = ax^n$$

when a is a positive real number and n is an odd integer?

As $x \to -\infty$, $f(x) \to -\infty$ and
As $x \to \infty$, $f(x) \to \infty$

End Behavior of Polynomials

INVESTIGATION:

What is the end behavior of

$$f(x) = ax^n$$

when a is a positive real number and n is an odd integer?

As $x \to -\infty$, $f(x) \to -\infty$ and
As $x \to \infty$, $f(x) \to \infty$

We would say that

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty$$

End Behavior of Polynomials

INVESTIGATION:

What is the end behavior of

$$f(x) = ax^n$$

when a is a negative real number and n is an odd integer?

End Behavior of Polynomials

INVESTIGATION:

What is the end behavior of

$$f(x) = ax^n$$

when a is a negative real number and n is an odd integer?

As $x \to -\infty$, $f(x) \to \infty$ and
As $x \to \infty$, $f(x) \to -\infty$

End Behavior of Polynomials

INVESTIGATION:

What is the end behavior of

$$f(x) = ax^n$$

when a is a negative real number and n is an odd integer?

As $x \to -\infty$, $f(x) \to \infty$ and
As $x \to \infty$, $f(x) \to -\infty$

We would say that

$$\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = -\infty$$

End Behavior of Polynomials

Investigation:

What is the end behavior of

$$f(x) = ax^n$$

when a is a positive real number and n is an even integer?

End Behavior of Polynomials

INVESTIGATION:

What is the end behavior of

$$f(x) = ax^n$$

when a is a positive real number and n is an even integer?

As $x \to -\infty$, $f(x) \to \infty$ and
As $x \to \infty$, $f(x) \to \infty$

End Behavior of Polynomials

INVESTIGATION:

What is the end behavior of

$$f(x) = ax^n$$

when a is a positive real number and n is an even integer?

As $x \to -\infty$, $f(x) \to \infty$ and
As $x \to \infty$, $f(x) \to \infty$

We would say that

$$\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty$$

End Behavior of Polynomials

Investigation:

What is the end behavior of

$$f(x) = ax^n$$

when a is a negative real number and n is an even integer?

End Behavior of Polynomials

INVESTIGATION:

What is the end behavior of

$$f(x) = ax^n$$

when a is a negative real number and n is an even integer?

As $x \to -\infty$, $f(x) \to -\infty$ and
As $x \to \infty$, $f(x) \to -\infty$

End Behavior of Polynomials

INVESTIGATION:

What is the end behavior of

$$f(x) = ax^n$$

when a is a negative real number and n is an even integer?

As $x \to -\infty$, $f(x) \to -\infty$ and
As $x \to \infty$, $f(x) \to -\infty$

We would say that

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = -\infty$$

End Behavior Summary

	Odd Degree	**Even Degree**
L.C. is positive	Down and Up	Up and Up
L.C. is negative	Up and Down	Down and Down

Example 3

Determine the end behavior of each of the following.

(a) $f(x) = 4x^5 - 3x^2 + 2x - 5$

Example 3

Determine the end behavior of each of the following.

(a) $f(x) = 4x^5 - 3x^2 + 2x - 5$

Positive lead coefficient, odd degree:

Example 3

Determine the end behavior of each of the following.

(a) $f(x) = 4x^5 - 3x^2 + 2x - 5$

Positive lead coefficient, odd degree:

Down (as we go left) and up (as we go right)

Example 3

Determine the end behavior of each of the following.

(a) $f(x) = 4x^5 - 3x^2 + 2x - 5$

Positive lead coefficient, odd degree:

Down (as we go left) and up (as we go right)

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty$$

Example 3

(b) $g(x) = 12x + x^4$

Example 3

(b) $g(x) = 12x + x^4$

$$g(x) = x^4 + 12x$$

Example 3

(b) $g(x) = 12x + x^4$

$$g(x) = x^4 + 12x$$

Positive lead coefficient, even degree:

Example 3

(b) $g(x) = 12x + x^4$

$$g(x) = x^4 + 12x$$

Positive lead coefficient, even degree:

Up (as we go left) and up (as we go right)

Example 3

(b) $\quad g(x) = 12x + x^4$

$$g(x) = x^4 + 12x$$

Positive lead coefficient, even degree:

Up (as we go left) and up (as we go right)

$$\lim_{x \to -\infty} g(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} g(x) = \infty$$

Example 3

(c) $h(x) = \dfrac{4 - x}{5}$

Example 3

(c) $h(x) = \dfrac{4 - x}{5}$

$$h(x) = -\frac{1}{5}x + \frac{4}{5}$$

Example 3

(c) $h(x) = \dfrac{4 - x}{5}$

$$h(x) = -\frac{1}{5}x + \frac{4}{5}$$

Negative lead coefficient, odd degree:

Example 3

(c) $h(x) = \dfrac{4 - x}{5}$

$$h(x) = -\frac{1}{5}x + \frac{4}{5}$$

Negative lead coefficient, odd degree:

Up (as we go left) and Down (as we go right)

Example 3

(c) $h(x) = \dfrac{4 - x}{5}$

$$h(x) = -\frac{1}{5}x + \frac{4}{5}$$

Negative lead coefficient, odd degree:

Up (as we go left) and Down (as we go right)

$$\lim_{x \to -\infty} h(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} h(x) = -\infty$$

Example 3

(d) $p(x) = 2x^7 - x^8$

Example 3

(d) $p(x) = 2x^7 - x^8$

$$p(x) = -x^8 + 2x^7$$

Example 3

(d) $p(x) = 2x^7 - x^8$

$$p(x) = -x^8 + 2x^7$$

Negative lead coefficient, even degree:

Example 3

(d) $p(x) = 2x^7 - x^8$

$$p(x) = -x^8 + 2x^7$$

Negative lead coefficient, even degree:

Down (as we go left) and Down (as we go right)

Example 3

(d) $p(x) = 2x^7 - x^8$

$$p(x) = -x^8 + 2x^7$$

Negative lead coefficient, even degree:

Down (as we go left) and Down (as we go right)

$$\lim_{x \to -\infty} p(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} p(x) = -\infty$$

Table of Contents

Zeros of Polynomials

The zeros of a polynomial, $f(x)$, have a few aliases:

- x-intercepts
- roots
- solutions to $f(x) = 0$

Example 4

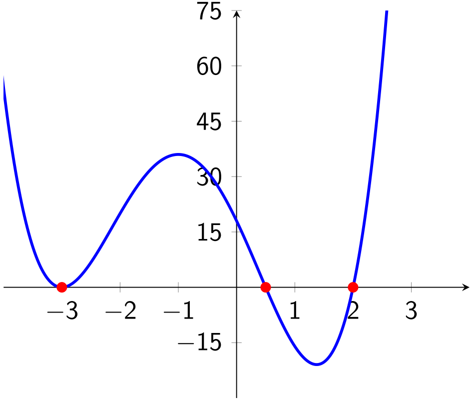
Find the zeros of each.

(a) $f(x) = 2x^4 + 7x^3 - 10x^2 - 33x + 18$

Example 4

Find the zeros of each.

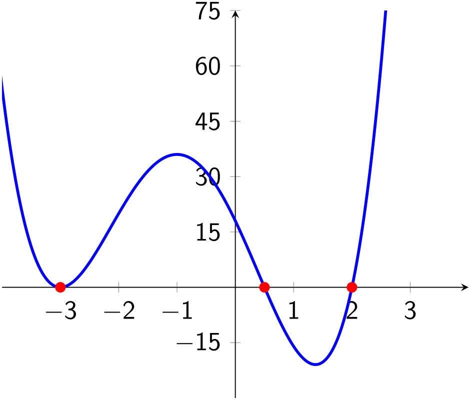
(a) $f(x) = 2x^4 + 7x^3 - 10x^2 - 33x + 18$



Example 4

Find the zeros of each.

(a) $f(x) = 2x^4 + 7x^3 - 10x^2 - 33x + 18$



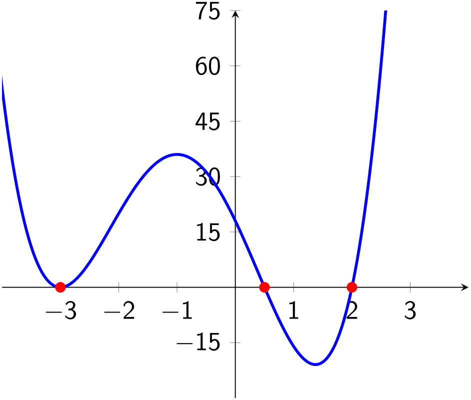
x-intercepts:

$(-3, 0), \left(\frac{1}{2}, 0\right), (2, 0)$

Example 4

Find the zeros of each.

(a) $f(x) = 2x^4 + 7x^3 - 10x^2 - 33x + 18$



x-intercepts:

$(-3, 0), \left(\dfrac{1}{2}, 0\right), (2, 0)$

Zeros:

$x = -3, \dfrac{1}{2}, 2$

Example 4

(b) $g(x) = (x-1)^3(5x+4)(x+7)^4$

Example 4

(b) $g(x) = (x-1)^3(5x+4)(x+7)^4$

$$x - 1 = 0 \qquad 5x + 4 = 0 \qquad x + 7 = 0$$

Example 4

(b) $g(x) = (x-1)^3(5x+4)(x+7)^4$

$$x - 1 = 0 \qquad 5x + 4 = 0 \qquad x + 7 = 0$$

$$x = 1 \qquad x = -\frac{4}{5} \qquad x = -7$$