Limits and Asymptotes

Summary

1. Infinite limits are often described by vertical and horizontal asymptotes.
2. A function never crosses a vertical asymptote but its behavior is warped around it.
3. Horizontal asymptotes can help determine end behavior of a function.

Infinite Limts and Asymptotes

Example 1. For $f(x) = \dfrac{1}{x}$, estimate $\lim_{x \to 0} f(x)$

Example 2. For $g(x) = \dfrac{1}{x^2}$, estimate $\lim_{x \to 0} g(x)$

A vertical line is a **vertical asymptote** if <u>any</u> are true:

- If as $x \to a^-$, $\lim\limits_{x \to a^-} = -\infty$ or ∞
- If as $x \to a^+$, $\lim\limits_{x \to a^+} = -\infty$ or ∞

Example 3. If $f(x) = \dfrac{x-1}{x^2-1}$

(a) State the domain of $f(x)$

(b) Determine $\lim\limits_{x \to 1} f(x)$ *algebraically*.

(c) Determine $\lim\limits_{x \to -1} f(x)$.

Example 4. The cost, $C(x)$, in thousands of dollars of removing $x\%$ of a city's pollutants discharged into a lake is given by

$$C(x) = \frac{113x}{100 - x}$$

(a) Determine the reasonable domain for C.

(b) Evaluate and interpret $C(50)$

(c) Determine and interpret $\lim_{x \to 100^-} C(x)$

Limits at Infinity and Horizontal Asymptotes

What happens as $x \to -\infty$ or $x \to \infty$?

Example 5. Consider the doubling function $f(x) = 2^x$.

(a) What is $\lim_{x \to \infty}$?

(b) What is $\lim_{x \to -\infty}$?

In the previous example, the line $y = 0$ is a **horizontal asymptote** of $f(x) = 2^x$.

A **horizontal asymptote** of a function $f(x)$ is a horizontal line with equation $y = L$ where $\lim_{x \to \pm\infty} f(x) = L$.

Horizontal asymptotes are used to determine *end behavior.*

Example 6. Pharmacological studies have determined that the amount of medication present in the body is a function of the amount given and how much time has elapsed since taking the medicine.

For a certain medication, the amount present in milliliters, $A(t)$, can be approximated by the function

$$A(t) = 3e^{-0.123t}$$

where t is the number of hours since taking the medication.

(a) Determine and interpret $A(0)$.

(b) Determine and interpret $\lim_{t \to \infty} A(t)$.

Example 7. Evaluate each of the following.

(a) $\lim_{x\to\infty} \frac{1}{x}$

(b) $\lim_{x\to-\infty} \frac{1}{x}$

(c) $\lim_{x\to\infty} \frac{1}{x^{5/3}}$

(d) $\lim_{x\to-\infty} \frac{1}{x^{5/3}}$

(e) $\lim_{x\to\infty} \frac{1}{x^{1/2}}$

(f) $\lim_{x\to-\infty} \frac{1}{x^{1/2}}$

Special Limits at Infinity

- For $n > 0$, $\lim_{x\to\infty} \frac{1}{x^n} = 0$
- For $n > 0$, $\lim_{x\to-\infty} \frac{1}{x^n} = 0$ **provided x^n is a real number when $x < 0$**

Note: All limit properties from the last section are true for limits at infinity.

Example 8. For $f(x) = \dfrac{x^2+1}{2x^2-1}$, determine $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

Example 9. The total cost (in dollars) to produce x units of a certain product is given by $C(x) = 22,500 + 7.35x$. The **average cost**, AC, is given by

$$AC(x) = \frac{C(x)}{x} = \frac{22,500 + 7.35x}{x}$$

Find and interpret $\lim\limits_{x\to\infty} AC(x)$