Limits

Summary

1. The limit of a function as x approaches a value is only concerned what happens as x *gets closer* to that value, not what happens *at* that value.

2. Limits can be found numerically, graphically, or algebraically.

Limit Notation

Consider $f(x) = \dfrac{x^2 - 4}{x - 2}$.

- $f(2)$ is not defined (division by 0)

- What is the output behavior as x *gets closer* to 2?

We can fill out a table of values less than 2 and greater than 2:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

To indicate the limit L of a function $f(x)$ as x approaches the value of a, we use the notation

$$\lim_{x \to a} f(x) = L$$

In other words, as x gets closer to the x-coordinate a,
the y-values get closer to the y-coordinate L.

- Sometimes you can just plug the value of a into the function.
- Sometimes you can't (e.g. might get division by 0).

*** IMPORTANT ***

Limits only look at what happens **as you get closer to the value of a**

They **are not concerned with** what the value of the function is *at* that value of a.

Example 1. Find each limit.

(a) $\lim_{x \to 7} \dfrac{x^2 - 6x - 7}{x - 7}$

(b) $\lim_{x \to 2} (3x + 5)$

Left-Hand and Right-Hand Limits

In the previous examples, we found limits by evaluating values less than the value of a and also values greater than a.

These are called **left-hand limits** and **right-hand limits**, respectively.

Left-Hand Limit

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

	Values of x approach 7 from the left ($x < 7$)			
x	6.99	6.999	6.9999	7
$f(x)$	**7.99**	**7.999**	**7.9999**	Undefined

Right-Hand Limit

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

		Values of x approach 7 from the right ($x > 7$)		
x	7	7.0001	7.001	7.01
$f(x)$	Undefined	**8.001**	**8.01**	**8.01**

Left-Hand Limit: $\lim\limits_{x \to a^-} f(x)$

Right-Hand Limit: $\lim\limits_{x \to a^+} f(x)$

Two-Sided Limit

$$\lim_{x \to a} f(x) = L$$

$$\Updownarrow \text{Implies}$$

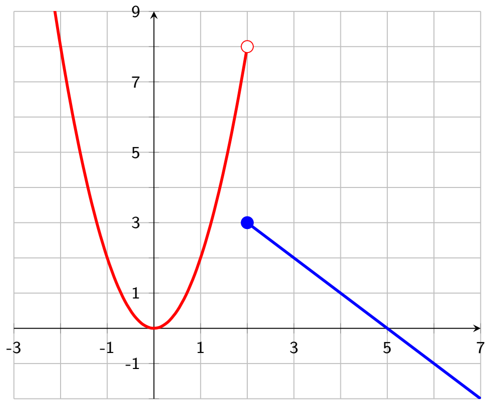
$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Limits Using a Graph

For $f(x)$ as x approaches a:

- Examine the graph to see if left-hand limit exists.
  - Won't if there is a vertical asymptote at $x = a$
- Examine the graph to see if right-hand limit exists.
  - Ditto from above
- If the 2 one-sided limits exist and are equal, there is a "limit."
- If there is a point at $x = a$, then $f(a)$ is the value of the function at $x = a$.

Example 2. Use the graph of $f(x)$ to find each.



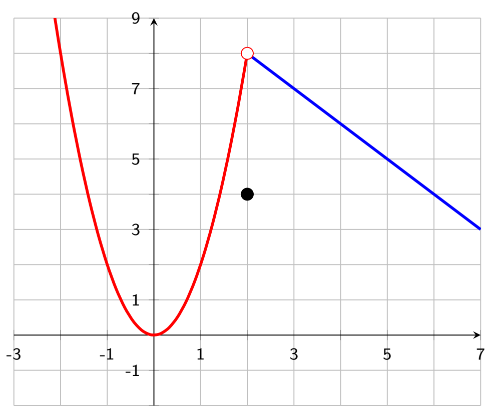
(a) $\lim\limits_{x \to 2^-} f(x)$

(b) $\lim\limits_{x \to 2^+} f(x)$

(c) $\lim\limits_{x \to 2} f(x)$

(d) $f(2)$

Example 3. Use the graph of $f(x)$ to find each.



(a) $\lim\limits_{x \to 2^-} f(x)$

(b) $\lim\limits_{x \to 2^+} f(x)$

(c) $\lim\limits_{x \to 2} f(x)$

(d) $f(2)$

Algebraically Determining Limits

Limit Rules:

Let a, c, and n be real numbers.

1. $\lim_{x\to a} c = c$ (limit of a constant is that constant)
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} [c \cdot f(x)] = c \cdot \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
5. $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
6. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
7. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$

Example 4. Determine each of the following limits.

(a) $\lim_{x\to 2} 7$

(b) $\lim_{x\to 1} \left(2x^2 - 3x + 5\right)$

(c) $\lim_{x\to 3} \sqrt{2x-1}$

You can also use techniques like factoring before evaluating limits.

Example 5. Find each limit algebraically.

(a) $\lim_{x \to 7} \dfrac{x^2 - 6x - 7}{x - 7}$

(b) $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

When two variables are involved, keep in mind **which variable** is changing in the limit.

Example 6. Find each limit.

(a) $\lim_{h \to 0} (3x + 2h)$

(b) $\lim_{h \to 0} \dfrac{5xh + 2h^2}{h}$

Example 7. For $f(x) = x^2$, compute $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$