Speed Counting

Objectives

1. Use the Fundamental Counting Rule
2. Understand factorial notation
3. Find permutations of objects
4. Find combinations of objects
5. Find probabilities using counting techniques

Example 1

For a special at a restaurant, you can choose between 3 appetizers, 4 entrees, and 2 desserts. If you select one item from each category (appetizer, entree, and dessert), how many different meals can you create?

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With appetizer A: ADH, ADI, AEH, AEI, AFH, AFI, AGH, AGI

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With appetizer A: ADH, ADI, AEH, AEI, AFH, AFI, AGH, AGI
With appetizer B: BDH, BDI, BEH, BEI, BFH, BFI, BGH, BGI

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With appetizer C: CDH, CDI, CEH, CEI, CFH, CFI, CGH, CGI

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With appetizer C: CDH, CDI, CEH, CEI, CFH, CFI, CGH, CGI

For a total of 24 possible different meals.

Fundamental Counting Rule

If event A can occur in a different ways and event B can occur in b different ways, then the total number of ways both events can occur is ab ways.

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This can be generalized to multiple events, such as those in example 1: $3 \times 4 \times 2 = 24$

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9	8	7	...	3		

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9	8	7	...	3	2	

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9	8	7	...	3	2	1

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$$9 \times 8 \times 7 \times \cdots \times 2 \times 1$$

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$$9 \times 8 \times 7 \times \cdots \times 2 \times 1 = 362,880 \text{ unique lineups}$$

Factorial Notation

Rather than write out all the numbers from 9 to 1 and then multiplying them, mathematicians created **factorial notation** to expedite the process.

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In general, for a positive integer n,

$$n! = n(n-1)(n-2) \cdot \cdots \cdot 3(2)(1)$$

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$$n! = n(n-1)(n-2) \cdot \cdots \cdot 3(2)(1)$$

with $0! = 1$

Factorial Growth

Factorial values grow very quickly:

$$
\begin{aligned}
2! &= 2(1) && = 2 \\
3! &= 3(2)(1) && = 6 \\
4! &= 4(3)(2)(1) && = 24 \\
5! &= 5(4)(3)(2)(1) && = 120 \\
6! &= 6(5)(4)(3)(2)(1) && = 720 \\
7! &= 7(6)(5)(4)(3)(2)(1) && = 5,040
\end{aligned}
$$

Example 3

How many ways are there to arrange 5 books on a shelf?

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$$5! = 120 \text{ different arrangements}$$

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Example 4

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Using the Fundamental Counting Rule:

$$5 \times 4 \times 3 = 60 \text{ different ways}$$

Takeaways from Example 4

We had more contestants available to win prizes than we had prizes available. We could have had an equal number of contestants and prizes, but we can't have more prizes available than contestants.

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So is there an easy way to do this if that's the case?

Yes, and that is where **permutations** come into play.

Example 5

How many ways are there to award gold, silver, and bronze medals to 8 contestants?

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Gold	Silver	Bronze

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Gold	Silver	Bronze
8	7	

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Gold	Silver	Bronze
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Using the Fundamental Counting Rule:

$$8 \times 7 \times 6 = 336 \text{ different ways}$$

Example 5

$$8 \times 7 \times 6 \times \cdots \times 2 \times 1$$

Example 5

$$\frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$$

Example 5

$$8 \times 7 \times 6 = 336$$

Permutations

If there are n items available and we take r at a time, then the total number of permutations is given by

$$_nP_r = \frac{n!}{(n-r)!}$$

with $n \geq r$

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With permutations, the order in which an item is selected matters.

Knowing to Use Permutations

Permutations

- Offering various prizes

Knowing to Use Permutations

Permutations

- Offering various prizes
- Running a race

Knowing to Use Permutations

Permutations

- Offering various prizes
- Running a race
- Assigning officer positions

Knowing to Use Permutations

Permutations

- Offering various prizes
- Running a race
- Assigning officer positions
- Combination locks and passwords

Example 6

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Combinations

With permutations, order selection mattered, so

ABC, ACB, BAC, BCA, CAB, and CBA

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ABC, ACB, BAC, BCA, CAB, and CBA

were all different.

With combinations, selection order does not matter, so there is no distinction among the orderings above. So,

ABC, ACB, BAC, BCA, CAB, and CBA

are all the same.

Combinations

Notice that there are 6, or 3!, arrangements of the letters A, B, and C.

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This can help us develop the formula for finding the number of combinations of n items taken r at a time.

Example 7

Five people are competing for three equal prizes. How many ways can the prizes be awarded?

If order mattered, there would be ${}_5P_3 = 60$ different possibilities:

$$\begin{array}{ccccccc} \text{ABC} & \text{ABD} & \text{ABE} & \text{ACB} & \text{ACD} & \text{ACE} & \dots \\ \text{BAC} & \text{BAD} & \text{BAE} & \text{BCA} & \text{BCD} & \text{BCE} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{array}$$

Example 7

Since ABC is the same as ACB, BAC, BCA, CAB, and CBA in the eyes of combinations, we can divide our permutation result of 60 by 3! to get 10 combinations:

ABC	ABD	ABE	ACD	ACE
ADE	BCD	BCE	BDE	CDE

Combinations

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Combinations

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- Committees

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A committee of 5 is to be formed from a pool of 12 potential candidates. How many committees are possible?

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$$_{12}C_5 = \frac{12!}{5!(12-5)!}$$

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A committee of 5 is to be formed from a pool of 12 potential candidates. How many committees are possible?

We have $n = 12$ candidates to choose from
We are selecting $r = 5$ at a time

$$_{12}C_5 = \frac{12!}{5!(12-5)!}$$

$$= 792$$

Example 9

A committee of 5 is to be formed from a pool of 12 potential candidates. The committee is to be made up of 3 managers and 2 accountants. If there are 8 managers and 4 accountants available, how many committees can be formed?

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Thus, we must blend the Fundamental Counting Rule with combinations.

Example 9

Select 3 managers from 8	$\times$	Select 2 accountants from 4
	$\times$	
	$\times$	

Example 9

Select 3 managers from 8	$\times$	Select 2 accountants from 4
${}_8C_3$	$\times$	
	$\times$	

Example 9

Select 3 managers from 8	$\times$	Select 2 accountants from 4
${}_8C_3$	$\times$	${}_4C_2$
	$\times$	

Example 9

Select 3 managers from 8	$\times$	Select 2 accountants from 4
${}_8C_3$	$\times$	${}_4C_2$
56	$\times$	

Example 9

Select 3 managers from 8	$\times$	Select 2 accountants from 4
${}_8C_3$	$\times$	${}_4C_2$
56	$\times$	6

Example 9

Select 3 managers from 8	$\times$	Select 2 accountants from 4
${}_8C_3$	$\times$	${}_4C_2$
56	$\times$	6

There are 336 ways to do this.

Example 10

Starting at point A and only moving right or up, how many paths are there to get to point B?

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