Double and Half-Angle Identities

Objectives

1. Solve problems using double-angle identities
2. Solve problems using power-reduction identities
3. Solve problems using half-angle identities

Double-Angle Identities

The double-angle identities are an extension of the angle sum identities.

In this case, the angles would be equal.

Double-Angle Identities

The double-angle identities are an extension of the angle sum identities.

In this case, the angles would be equal.

$$\sin(2A) = \sin(A + A)$$

Double-Angle Identities

The double-angle identities are an extension of the angle sum identities.

In this case, the angles would be equal.

$$\begin{aligned}\sin(2A) &= \sin(A + A) \\ &= \sin A \cdot \cos A + \sin A \cdot \cos A\end{aligned}$$

Double-Angle Identities

The double-angle identities are an extension of the angle sum identities.

In this case, the angles would be equal.

$$\begin{aligned}\sin(2A) &= \sin(A + A) \\ &= \sin A \cdot \cos A + \sin A \cdot \cos A \\ &= 2 \sin A \cos A\end{aligned}$$

Double-Angle Identities

$$\sin(2A) = 2\sin A\cos A$$

$$\cos(2A) = \cos^2 A - \sin^2 A$$

$$\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

Alternate Forms of $\cos(2A)$

$\cos(2A)$ has two alternate forms.

Alternate form # 1:

Alternate Forms of $\cos(2A)$

$\cos(2A)$ has two alternate forms.

Alternate form # 1:

$$\cos(2A) = \cos^2 A - \sin^2 A$$

Alternate Forms of $\cos(2A)$

$\cos(2A)$ has two alternate forms.

Alternate form # 1:

$$\begin{aligned}\cos(2A) &= \cos^2 A - \sin^2 A \\ &= 1 - \sin^2 A - \sin^2 A\end{aligned}$$

Alternate Forms of $\cos(2A)$

$\cos(2A)$ has two alternate forms.

Alternate form # 1:

$$\begin{aligned}\cos(2A) &= \cos^2 A - \sin^2 A \\ &= 1 - \sin^2 A - \sin^2 A \\ &= 1 - 2\sin^2 A\end{aligned}$$

Alternate Forms of $\cos(2A)$

Alternate form # 2:

Alternate Forms of $\cos(2A)$

Alternate form # 2:

$$\cos(2A) = \cos^2 A - \sin^2 A$$

Alternate Forms of $\cos(2A)$

Alternate form # 2:

$$\cos(2A) = \cos^2 A - \sin^2 A$$

$$= \cos^2 A - (1 - \cos^2 A)$$

Alternate Forms of $\cos(2A)$

Alternate form # 2:

$$\begin{aligned}\cos(2A) &= \cos^2 A - \sin^2 A \\ &= \cos^2 A - (1 - \cos^2 A) \\ &= \cos^2 A - 1 + \cos^2 A\end{aligned}$$

Alternate Forms of $\cos(2A)$

Alternate form # 2:

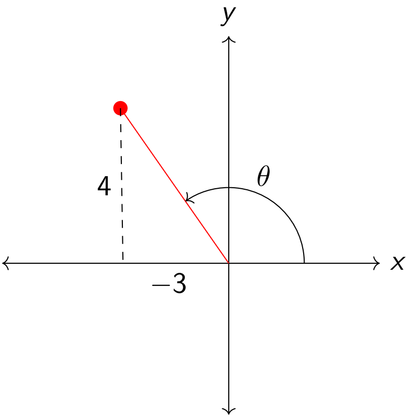
$$\begin{aligned}
\cos(2A) &= \cos^2 A - \sin^2 A \\
&= \cos^2 A - (1 - \cos^2 A) \\
&= \cos^2 A - 1 + \cos^2 A \\
&= 2\cos^2 A - 1
\end{aligned}$$

Example 1

Suppose $P(-3, 4)$ lies on the terminal side of θ when θ is in standard position.

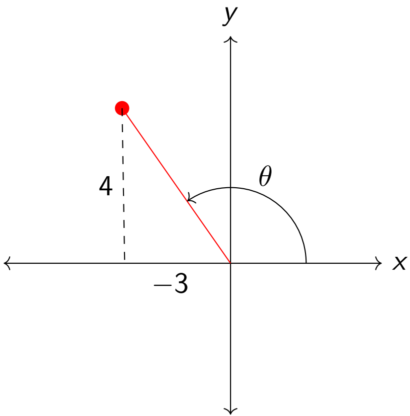
Example 1

Suppose $P(-3,4)$ lies on the terminal side of θ when θ is in standard position.



Example 1

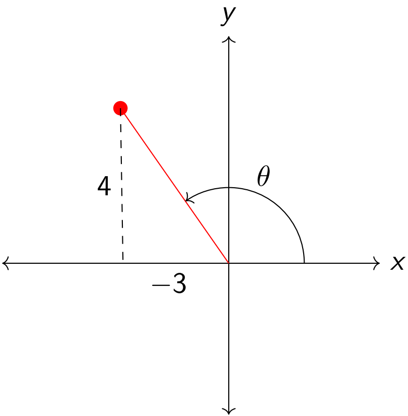
Suppose $P(-3, 4)$ lies on the terminal side of θ when θ is in standard position.



$$x^2 + y^2 = r^2$$

Example 1

Suppose $P(-3,4)$ lies on the terminal side of θ when θ is in standard position.

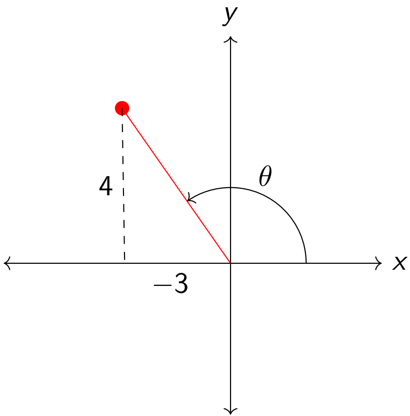


$$x^2 + y^2 = r^2$$

$$3^2 + 4^2 = r^2$$

Example 1

Suppose $P(-3, 4)$ lies on the terminal side of θ when θ is in standard position.



$$x^2 + y^2 = r^2$$

$$3^2 + 4^2 = r^2$$

$$r = 5$$

Example 1 $x = -3, y = 4, r = 5$

(a) Find $\cos(2\theta)$

Example 1 $x = -3, y = 4, r = 5$

(a) Find $\cos(2\theta)$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Example 1 $\quad x = -3, y = 4, r = 5$

(a) Find $\cos(2\theta)$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \left(\frac{-3}{5}\right)^2 - \left(\frac{4}{5}\right)^2\end{aligned}$$

Example 1 $x = -3, y = 4, r = 5$

(a) Find $\cos(2\theta)$

$$\begin{aligned}
\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\
&= \left(\frac{-3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 \\
&= \frac{9}{25} - \frac{16}{25}
\end{aligned}$$

Example 1 $x = -3, y = 4, r = 5$

(a) Find $\cos(2\theta)$

$$\begin{aligned}
\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\
&= \left(\frac{-3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 \\
&= \frac{9}{25} - \frac{16}{25} \\
&= \frac{-7}{25}
\end{aligned}$$

Example 1 $x = -3, y = 4, r = 5$

(b) Find $\sin(2\theta)$

Example 1 $x = -3, y = 4, r = 5$

(b) Find $\sin(2\theta)$

$$\sin(2\theta) = 2 \sin\theta \cos\theta$$

Example 1 $x = -3, y = 4, r = 5$

(b) Find $\sin(2\theta)$

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{4}{5}\right)\left(\frac{-3}{5}\right) \end{aligned}$$

Example 1 $x = -3, y = 4, r = 5$

(b) Find $\sin(2\theta)$

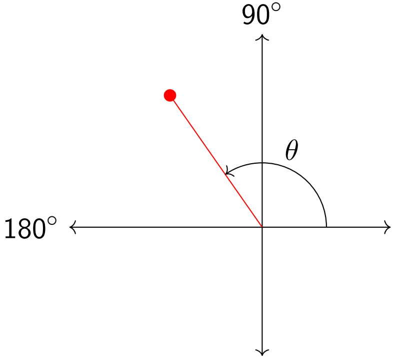
$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{4}{5}\right)\left(\frac{-3}{5}\right) \\ &= \frac{-24}{25}\end{aligned}$$

Example 1

(c) Which quadrant does 2θ lie in?

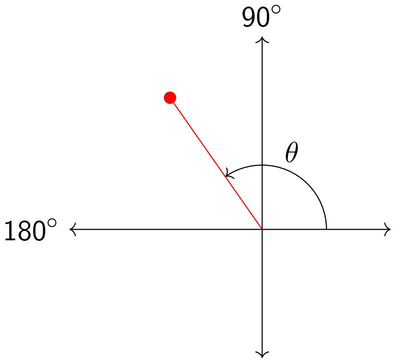
Example 1

(c) Which quadrant does 2θ lie in?



Example 1

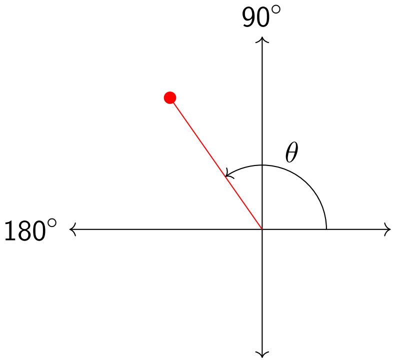
(c) Which quadrant does 2θ lie in?



$$90^\circ < \theta < 180^\circ$$

Example 1

(c) Which quadrant does 2θ lie in?

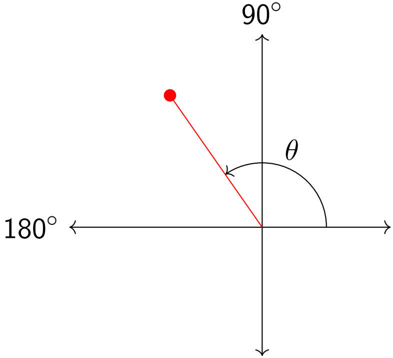


$$90^\circ < \theta < 180^\circ$$

$$180^\circ < 2\theta < 360^\circ$$

Example 1

(c) Which quadrant does 2θ lie in?



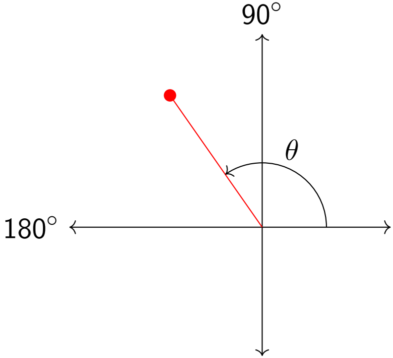
$90^\circ < \theta < 180^\circ$

$180^\circ < 2\theta < 360^\circ$

θ in quadrant 3 or 4

Example 1

(c) Which quadrant does 2θ lie in?



$90^\circ < \theta < 180^\circ$

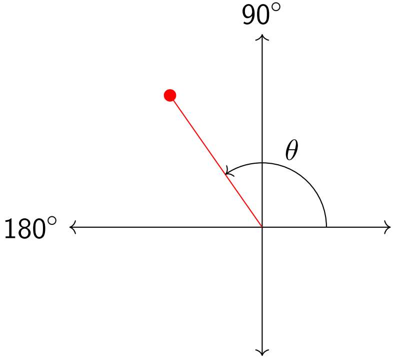
$180^\circ < 2\theta < 360^\circ$

θ in quadrant 3 or 4

Example 1a, $\cos(2\theta) = -$

Example 1

(c) Which quadrant does 2θ lie in?



$90^\circ < \theta < 180^\circ$

$180^\circ < 2\theta < 360^\circ$

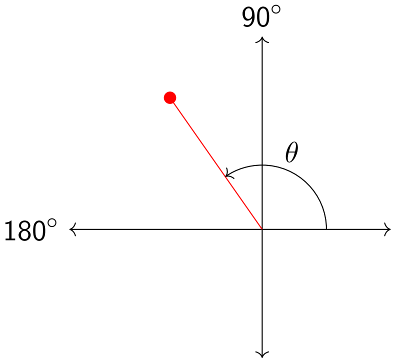
θ in quadrant 3 or 4

Example 1a, $\cos(2\theta) = -$

Example 1b, $\sin(2\theta) = -$

Example 1

(c) Which quadrant does 2θ lie in?



$90^\circ < \theta < 180^\circ$

$180^\circ < 2\theta < 360^\circ$

θ in quadrant 3 or 4

Example 1a, $\cos(2\theta) = -$

Example 1b, $\sin(2\theta) = -$

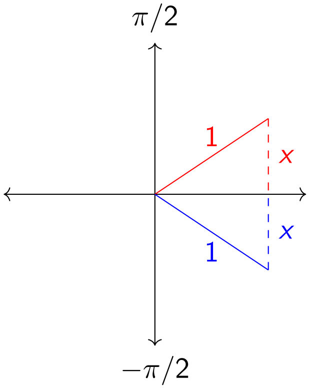
both negative in Quadrant 3

Example 2

If $\sin\theta = x$ for $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

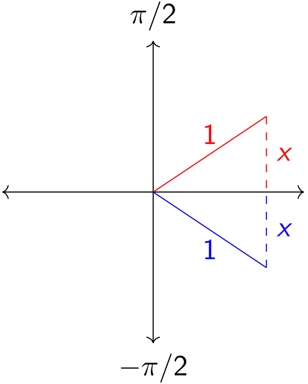
Example 2

If $\sin\theta = x$ for $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.



Example 2

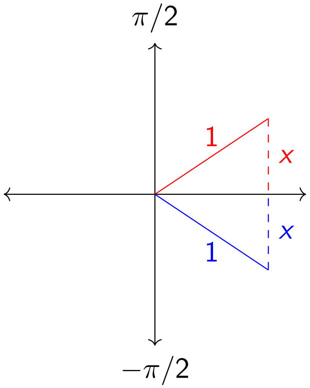
If $\sin\theta = x$ for $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.



$$\sin\theta = x$$

Example 2

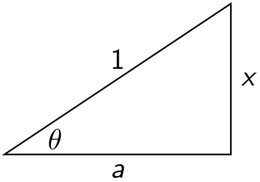
If $\sin\theta = x$ for $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.



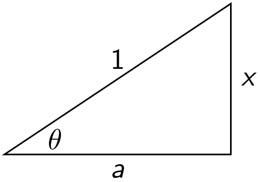
$$\sin\theta = x$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Example 2

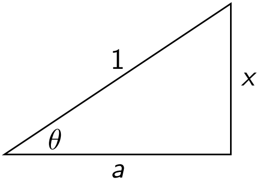


Example 2



$$\cos\theta = \frac{a}{1} = a$$

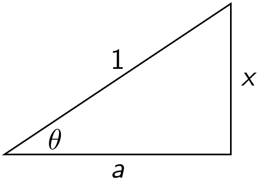
Example 2



$$\cos\theta = \frac{a}{1} = a$$

$$a^2 + x^2 = 1^2$$

Example 2

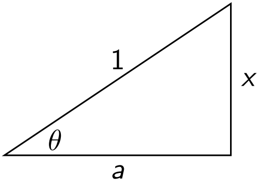


$$\cos\theta = \frac{a}{1} = a$$

$$a^2 + x^2 = 1^2$$

$$a^2 = 1 - x^2$$

Example 2



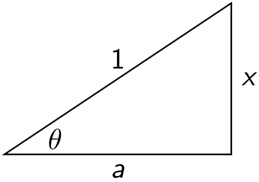
$$\cos\theta = \frac{a}{1} = a$$

$$a^2 + x^2 = 1^2$$

$$a^2 = 1 - x^2$$

$$a = \pm\sqrt{1 - x^2}$$

Example 2



$$\cos\theta = \frac{a}{1} = a$$

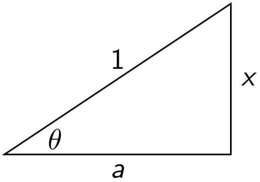
$$a^2 + x^2 = 1^2$$

$$a^2 = 1 - x^2$$

$$a = \pm\sqrt{1 - x^2}$$

$$a = \sqrt{1 - x^2}$$

Example 2



$$\cos\theta = \frac{a}{1} = a$$

$$a^2 + x^2 = 1^2$$

$$a^2 = 1 - x^2$$

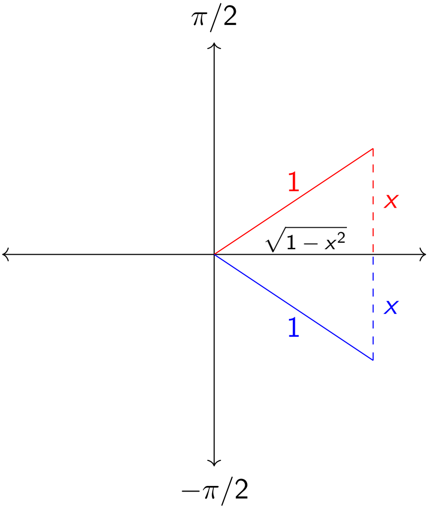
$$a = \pm\sqrt{1 - x^2}$$

$$a = \sqrt{1 - x^2}$$

$$\cos\theta = \sqrt{1 - x^2}$$

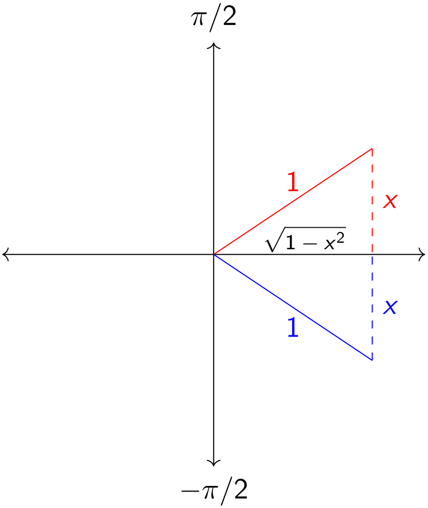
Example 2

$\sin\theta = x \quad \cos\theta = \sqrt{1-x^2}$



Example 2

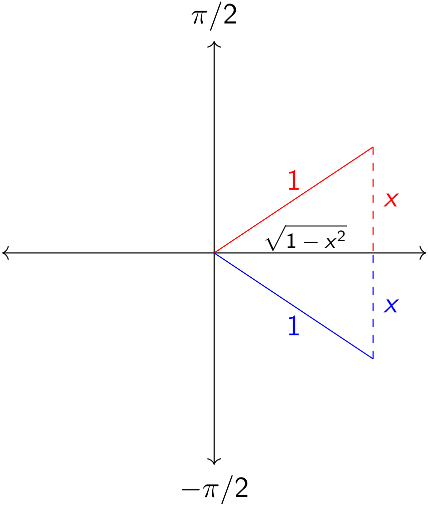
$\sin\theta = x \quad \cos\theta = \sqrt{1-x^2}$



$\sin(2\theta) = 2\sin\theta\cos\theta$

Example 2

$\sin\theta = x \quad \cos\theta = \sqrt{1-x^2}$



$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$= 2x\sqrt{1-x^2}$$

Example 3

Verify the identity

$$\sin(2\theta) = \frac{2\tan\theta}{1+\tan^2\theta}$$

Example 3

Verify the identity

$$\sin(2\theta) = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$2\sin\theta\cos\theta = \frac{2\left(\frac{\sin\theta}{\cos\theta}\right)}{1+\frac{\sin^2\theta}{\cos^2\theta}}$$

Example 3

Verify the identity

$$\sin(2\theta) = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$2\sin\theta\cos\theta = \frac{2\left(\frac{\sin\theta}{\cos\theta}\right)}{1+\frac{\sin^2\theta}{\cos^2\theta}}\left(\frac{\cos^2\theta}{\cos^2\theta}\right)$$

Example 3

Verify the identity

$$\sin(2\theta) = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$2\sin\theta\cos\theta = \frac{2\left(\frac{\sin\theta}{\cos\theta}\right)}{1+\frac{\sin^2\theta}{\cos^2\theta}}\left(\frac{\cos^2\theta}{\cos^2\theta}\right)$$

$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta+\sin^2\theta}$$

Example 3

Verify the identity

$$\sin(2\theta) = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$2\sin\theta\cos\theta = \frac{2\left(\frac{\sin\theta}{\cos\theta}\right)}{1+\frac{\sin^2\theta}{\cos^2\theta}}\left(\frac{\cos^2\theta}{\cos^2\theta}\right)$$

$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta+\sin^2\theta}$$

$$= 2\sin\theta\cos\theta$$

Example 4

Express $\cos(3\theta)$ as a polynomial in terms of $\cos\theta$.

Example 4

Express $\cos(3\theta)$ as a polynomial in terms of $\cos\theta$.

$$\cos(3\theta) = \cos(2\theta + \theta)$$

Example 4

Express $\cos(3\theta)$ as a polynomial in terms of $\cos\theta$.

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta + \theta) \\ &= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta\end{aligned}$$

Example 4

Express $\cos(3\theta)$ as a polynomial in terms of $\cos\theta$.

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta + \theta) \\ &= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta\end{aligned}$$

Example 4

Express $\cos(3\theta)$ as a polynomial in terms of $\cos\theta$.

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta + \theta) \\
&= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta
\end{aligned}$$

Example 4

Express $\cos(3\theta)$ as a polynomial in terms of $\cos\theta$.

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta + \theta) \\
&= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta)
\end{aligned}$$

Example 4

Express $\cos(3\theta)$ as a polynomial in terms of $\cos\theta$.

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta + \theta) \\
&= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta) \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta
\end{aligned}$$

Example 4

Express $\cos(3\theta)$ as a polynomial in terms of $\cos\theta$.

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta + \theta) \\
&= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta) \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta
\end{aligned}$$

Objectives

1. Solve problems using double-angle identities
2. Solve problems using power-reduction identities
3. Solve problems using half-angle identities

Power-Reduction Identities

The power-reduction identities can be derived from the alternate equations for $\cos(2A)$.

Power-Reduction Identities

The power-reduction identities can be derived from the alternate equations for $\cos(2A)$.

For $\cos(2A) = 2\cos^2 A - 1$, if we solve for $\cos^2 A$, we get

$$\cos^2 A = \frac{1 + \cos(2A)}{2}$$

Power-Reduction Identities

The power-reduction identities can be derived from the alternate equations for $\cos(2A)$.

For $\cos(2A) = 2\cos^2 A - 1$, if we solve for $\cos^2 A$, we get

$$\cos^2 A = \frac{1 + \cos(2A)}{2}$$

And for $\cos(2A) = 1 - 2\sin^2 A$, solving for $\cos^2 A$ gives us

$$\sin^2 A = \frac{1 - \cos(2A)}{2}$$

Example 5

Rewrite $\sin^2\theta\cos^2\theta$ as a sum and difference of cosines to the first power.

Example 5

Rewrite $\sin^2\theta\cos^2\theta$ as a sum and difference of cosines to the first power.

$$(\sin^2\theta)(\cos^2\theta) = \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right)$$

Example 5

Rewrite $\sin^2\theta\cos^2\theta$ as a sum and difference of cosines to the first power.

$$(\sin^2\theta)(\cos^2\theta) = \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right)$$

$$= \frac{1-\cos^2(2\theta)}{4}$$

Example 5

Rewrite $\sin^2\theta\cos^2\theta$ as a sum and difference of cosines to the first power.

$$(\sin^2\theta)(\cos^2\theta) = \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right)$$

$$= \frac{1-\cos^2(2\theta)}{4}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Example 5

Rewrite $\sin^2\theta\cos^2\theta$ as a sum and difference of cosines to the first power.

$$(\sin^2\theta)(\cos^2\theta) = \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right)$$

$$= \frac{1-\cos^2(2\theta)}{4}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\cos^2(2\theta) = \frac{1+\cos(2(2\theta))}{2}$$

Example 5

Rewrite $\sin^2\theta\cos^2\theta$ as a sum and difference of cosines to the first power.

$$(\sin^2\theta)(\cos^2\theta) = \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right)$$

$$= \frac{1-\cos^2(2\theta)}{4}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\cos^2(2\theta) = \frac{1+\cos(2(2\theta))}{2}$$

$$\cos^2(2\theta) = \frac{1+\cos(4\theta)}{2}$$

Example 5

$$= \frac{1 - \frac{1 + \cos(4\theta)}{2}}{4}$$

Example 5

$$= \frac{1 - \frac{1+\cos(4\theta)}{2}}{4}\left(\frac{2}{2}\right)$$

Example 5

$$= \frac{1 - \frac{1+\cos(4\theta)}{2}}{4}\left(\frac{2}{2}\right)$$

$$= \frac{2 - (1+\cos(4\theta))}{8}$$

Example 5

$$= \frac{1 - \frac{1+\cos(4\theta)}{2}}{4}\left(\frac{2}{2}\right)$$

$$= \frac{2 - (1+\cos(4\theta))}{8}$$

$$= \frac{2 - 1 - \cos(4\theta)}{8}$$

Example 5

$$= \frac{1 - \frac{1+\cos(4\theta)}{2}}{4}\left(\frac{2}{2}\right)$$

$$= \frac{2 - (1+\cos(4\theta))}{8}$$

$$= \frac{2 - 1 - \cos(4\theta)}{8}$$

$$= \frac{1 - \cos(4\theta)}{8}$$

Objectives

1. Solve problems using double-angle identities
2. Solve problems using power-reduction identities
3. Solve problems using half-angle identities

Half-Angle Identities

The Half-Angle Identities can be found by evaluating the Power-Reducing Identities for $\frac{\theta}{2}$ instead of θ, and then taking the square root of both sides.

Half-Angle Identities

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Half-Angle Identities

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\left(2\cdot\frac{\theta}{2}\right)}{2}$$

Half-Angle Identities

$$\cos^2 \theta = \frac{1+\cos(2\theta)}{2}$$

$$\cos^2 \left(\frac{\theta}{2}\right) = \frac{1+\cos\left(2\cdot\frac{\theta}{2}\right)}{2}$$

$$\cos^2 \left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$$

Half-Angle Identities

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\left(2\cdot\frac{\theta}{2}\right)}{2}$$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Half-Angle Identities

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

Example 6

Find the exact value of $\cos 112.5^\circ$.

Example 6

Find the exact value of $\cos 112.5^\circ$.

$$\cos 112.5^\circ = \cos\left(\frac{225^\circ}{2}\right)$$

Example 6

Find the exact value of $\cos 112.5^\circ$.

$$\cos 112.5^\circ = \cos\left(\frac{225^\circ}{2}\right)$$

$$= -\sqrt{\frac{1+\cos(225^\circ)}{2}}$$

Example 6

Find the exact value of $\cos 112.5^\circ$.

$$\cos 112.5^\circ = \cos\left(\frac{225^\circ}{2}\right)$$

$$= -\sqrt{\frac{1+\cos(225^\circ)}{2}}$$

$$= -\sqrt{\frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}}$$

Example 6

Find the exact value of $\cos 112.5^\circ$.

$$
\begin{aligned}
\cos 112.5^\circ &= \cos\left(\frac{225^\circ}{2}\right) \\
&= -\sqrt{\frac{1+\cos(225^\circ)}{2}} \\
&= -\sqrt{\frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}\cdot\frac{2}{2}}
\end{aligned}
$$

Example 6

Find the exact value of $\cos 112.5^\circ$.

$$\cos 112.5^\circ = \cos\left(\frac{225^\circ}{2}\right)$$

$$= -\sqrt{\frac{1+\cos(225^\circ)}{2}}$$

$$= -\sqrt{\frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}\cdot\frac{2}{2}}$$

$$= -\sqrt{\frac{2-\sqrt{2}}{4}}$$

Example 6

$$= \frac{-\sqrt{2-\sqrt{2}}}{\sqrt{4}}$$

Example 6

$$= \frac{-\sqrt{2-\sqrt{2}}}{\sqrt{4}}$$

$$= \frac{-\sqrt{2-\sqrt{2}}}{2}$$

Example 7

Suppose $-\pi \leq \theta \leq 0$ with $\cos\theta = -\dfrac{3}{5}$. Find $\sin\left(\dfrac{\theta}{2}\right)$.

Example 7

Suppose $-\pi \leq \theta \leq 0$ with $\cos\theta = -\dfrac{3}{5}$. Find $\sin\left(\dfrac{\theta}{2}\right)$.

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Example 7

Suppose $-\pi \leq \theta \leq 0$ with $\cos\theta = -\dfrac{3}{5}$. Find $\sin\left(\dfrac{\theta}{2}\right)$.

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$= \pm\sqrt{\frac{1-\left(\frac{-3}{5}\right)}{2}}$$

Example 7

Suppose $-\pi \leq \theta \leq 0$ with $\cos\theta = -\frac{3}{5}$. Find $\sin\left(\frac{\theta}{2}\right)$.

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$= \pm\sqrt{\frac{1-\left(\frac{-3}{5}\right)}{2}\cdot\frac{5}{5}}$$

Example 7

Suppose $-\pi \leq \theta \leq 0$ with $\cos\theta = -\frac{3}{5}$. Find $\sin\left(\frac{\theta}{2}\right)$.

$$\begin{aligned}
\sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\
&= \pm\sqrt{\frac{1-\left(\frac{-3}{5}\right)}{2}\cdot\frac{5}{5}} \\
&= \pm\sqrt{\frac{5+3}{10}}
\end{aligned}$$

Example 7

Suppose $-\pi \leq \theta \leq 0$ with $\cos\theta = -\dfrac{3}{5}$. Find $\sin\left(\dfrac{\theta}{2}\right)$.

$$
\begin{aligned}
\sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\
&= \pm\sqrt{\frac{1-\left(\frac{-3}{5}\right)}{2}\cdot\frac{5}{5}} \\
&= \pm\sqrt{\frac{5+3}{10}} \\
&= \pm\sqrt{\frac{4}{5}}
\end{aligned}
$$

Example 7

$$\pm\sqrt{\frac{4}{5}} = \pm\frac{2\sqrt{5}}{5}$$

$$\text{If} -\pi \leq \theta \leq 0 \longrightarrow -\frac{\pi}{2} \leq \frac{\theta}{2} \leq 0$$

Example 7

$$\pm\sqrt{\frac{4}{5}} = \pm\frac{2\sqrt{5}}{5}$$

$$\text{If} -\pi \leq \theta \leq 0 \longrightarrow -\frac{\pi}{2} \leq \frac{\theta}{2} \leq 0$$

$\frac{\theta}{2}$ is in quadrant IV, where sine is negative.

Example 7

$$\pm\sqrt{\frac{4}{5}} = \pm\frac{2\sqrt{5}}{5}$$

$$\text{If} -\pi \leq \theta \leq 0 \longrightarrow -\frac{\pi}{2} \leq \frac{\theta}{2} \leq 0$$

$\frac{\theta}{2}$ is in quadrant IV, where sine is negative.

$$-\frac{2\sqrt{5}}{5}$$

Example 8

Using $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$ derive the identity

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta}$$

Example 8

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

Example 8

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}}$$

Example 8

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}}$$

$$= \pm\sqrt{\frac{1-\cos^2\theta}{(1+\cos\theta)^2}}$$

Example 8

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}}$$

$$= \pm\sqrt{\frac{1-\cos^2\theta}{(1+\cos\theta)^2}}$$

$$= \pm\sqrt{\frac{\sin^2\theta}{(1+\cos\theta)^2}}$$

Example 8

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}}$$

$$= \pm\sqrt{\frac{1-\cos^2\theta}{(1+\cos\theta)^2}}$$

$$= \pm\sqrt{\frac{\sin^2\theta}{(1+\cos\theta)^2}}$$

$$= \frac{\sin\theta}{1+\cos\theta}$$