Polar Form of Conics

Objectives

1. Analyze the graphs of conic sections in polar form.

Intro

Given a fixed line L, a point F not on L, and a positive number e, a conic section is the set of all points P such that

$$\frac{\text{the distance from } P \text{ to } F}{\text{the distance from } P \text{ to } L} = e$$

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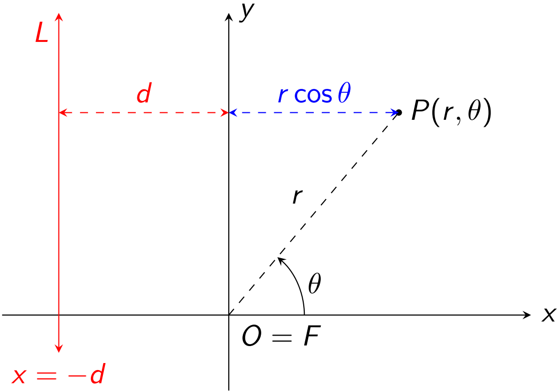
The line L is called the directrix of the conic section, the point F is called a focus of the conic section, and the constant e is called the eccentricity of the conic section.

Eccentricity, Focus, and Directrix Line

The conic section has eccentricity e, a focus F at the origin and directrix line $x = -d$:



General Equation

From which we get

$$e = \frac{\text{the distance from } P \text{ to } F}{\text{the distance from } P \text{ to } L} = \frac{r}{d + r\cos\theta} = e$$

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$$r - er\cos\theta = ed$$

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$$\begin{aligned} r &= ed + er\cos\theta \\ r - er\cos\theta &= ed \\ r(1 - e\cos\theta) &= ed \end{aligned}$$

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$$\begin{aligned} r &= ed + er\cos\theta \\ r - er\cos\theta &= ed \\ r(1 - e\cos\theta) &= ed \\ r &= \frac{ed}{1 - e\cos\theta} \end{aligned}$$

Example 1

Examine the graphs of each of the following for different values of d, but with $e = 1$.

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- y-intercepts at $(0, \pm d)$

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Follow-up to Example 1

Notice each of the previous graphs in Example 1 were parabolas. This is the case when $e = 1$. The directrix lines were either $x = \pm d$ or $y = \pm d$, and the focal diameter is $2d$.

Example 2

Examine the graphs of each of the following for different values of d, but with $0 < e < 1$ and $e > 1$.

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Ellipse (wide)

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Hyperbola (opening left and right)

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Identify the conic for each.

(a) $$r = \frac{4}{1 - \sin\theta}$$

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Goes through $(\pm 4, 0)$

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(b) $r = \dfrac{12}{3 - \cos\theta}$

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Major axis length: $\dfrac{2(1/3)(12)}{1 - (1/3)^2} = 9$

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Minor axis length: $\dfrac{2(1/3)(12)}{\sqrt{1 - (1/3)^2}} = 6\sqrt{3}$

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(c) $r = \dfrac{6}{1 + 2\sin\theta}$

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$e = 2$: Hyperbola (opens up and down)

$2d = 6 \longrightarrow d = 3$

Transverse axis length: $\dfrac{2(2)(3)}{2^2 - 1} = 4$

Conjugate axis length: $\dfrac{2(2)(3)}{\sqrt{2^2 - 1}} = 4\sqrt{3}$

Polar Form of Rotated Conics

For constants $\ell > 0$, $e \geq 0$, and ϕ, the graph of

$$r = \frac{\ell}{1 - e\cos(\theta - \phi)}$$

is a conic section with eccentricity e and one focus at $(0,0)$.

Polar Form of Rotated Conics

If $e = 0$, the graph is a circle centered at $(0, 0)$ with radius ℓ.

Polar Form of Rotated Conics

If $e \neq 0$, the conic has a focus at $(0,0)$ and the directrix contains the point with polar coordinates $(-d, \phi)$ where $d = \dfrac{\ell}{e}$.

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- If $e = 1$, graph is a parabola with focal diameter $2d$.
- If $e > 1$, graph is a hyperbola with transverse axis length $\dfrac{2ed}{e^2 - 1}$ and conjugate axis length $\dfrac{2ed}{\sqrt{e^2 - 1}}$