Parametric Equations

Objectives

1. Sketch a parametric curve.

2. Rewrite an equation by eliminating the parameter.

Intro

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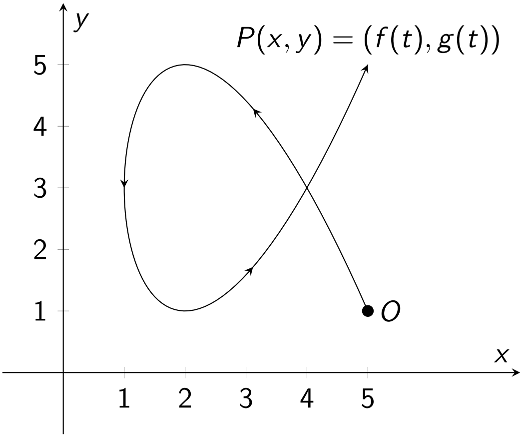
In this section, we will look at parametric functions: ones in which x and y are defined by a parameter, such as t.

For instance, the plot below could show the path a bug might take (starting at O) while walking on a table:



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Note: The curve itself is a set of points and is devoid of any orientation.

Example 1

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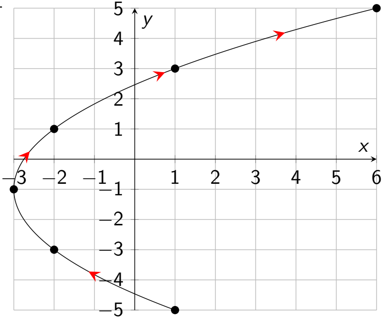

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Eliminating the Parameter

We can eliminate the parameter t by solving one of the equations for t and substituting it into the other.

Example 2

Eliminate the parameter in Example 1 and write the equation using only x and y.

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\begin{aligned}
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x + 3 &= \frac{(y+1)^2}{4} \\
4(x+3) &= (y+1)^2
\end{aligned}
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\frac{y+1}{2} &\geq -2
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$$\frac{y+1}{2} \geq -2$$

$$y \geq -5$$

Example 2

$$4(x+3) = (y+1)^2, \quad y \geq -5$$

Example 3a

Sketch each of the following curves.

(a) $$\begin{cases} x &= t^3 \\ y &= 2t^2 \end{cases} \quad \text{for } -1 \leq t \leq 1$$

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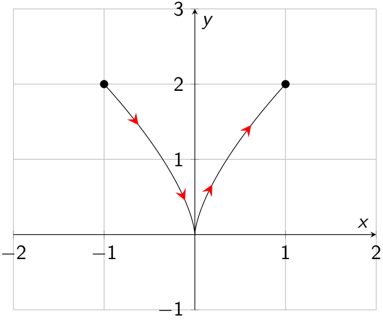

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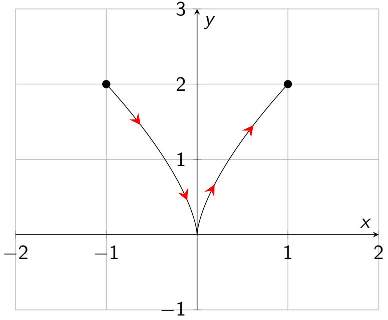

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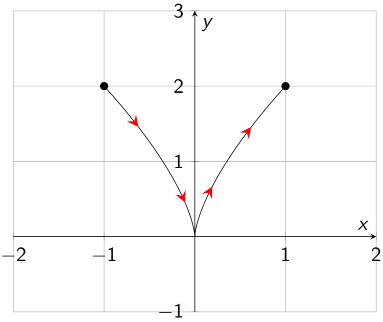

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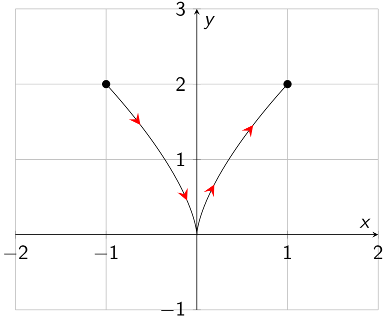

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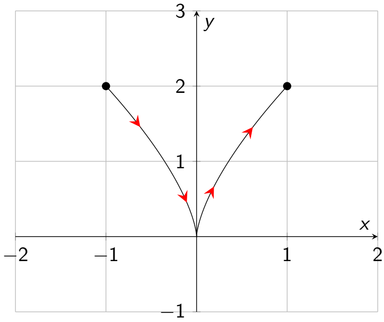

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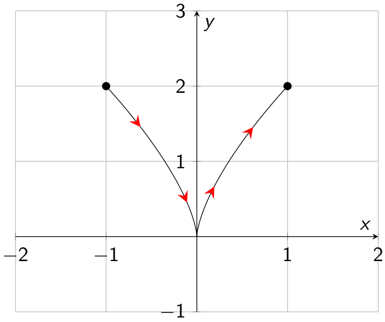

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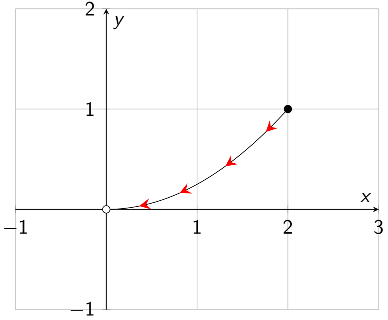

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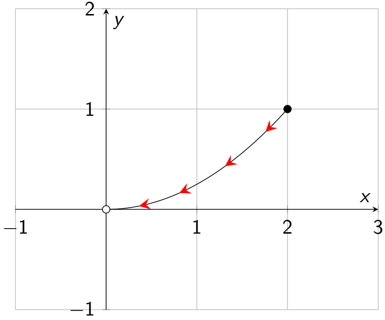

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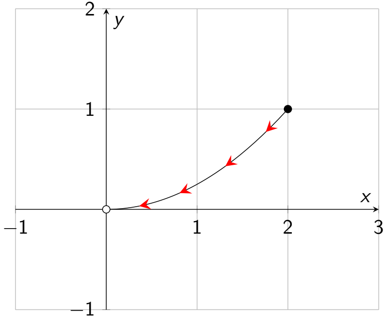

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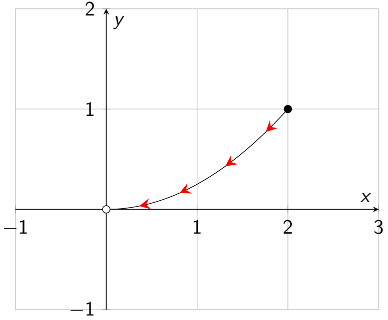

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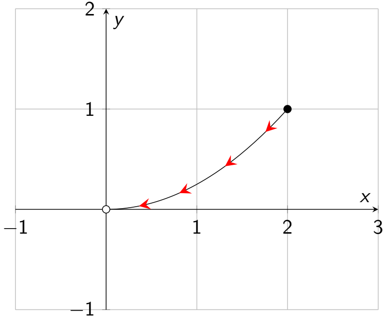

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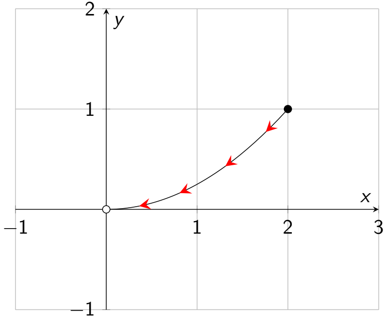

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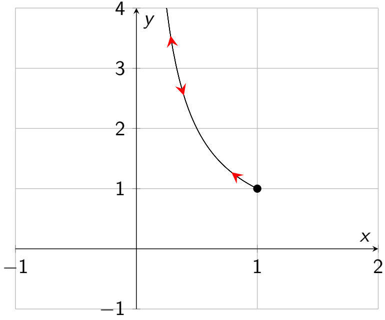

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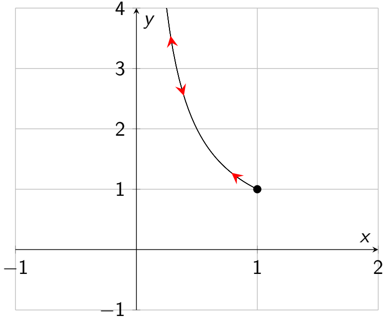

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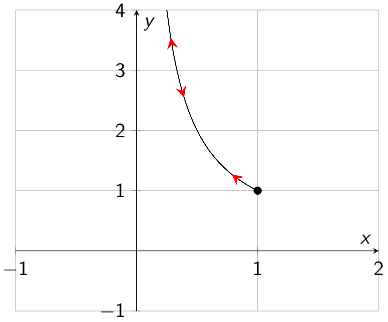

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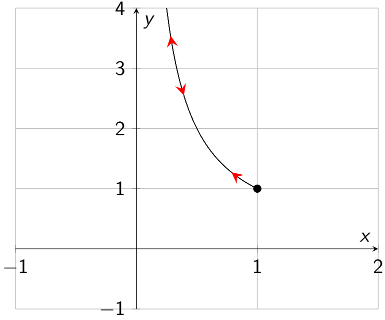

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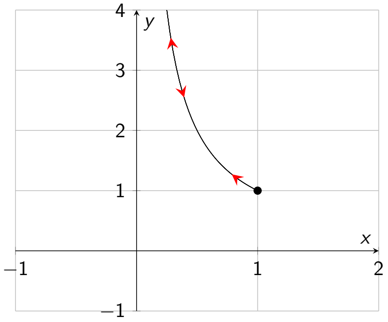

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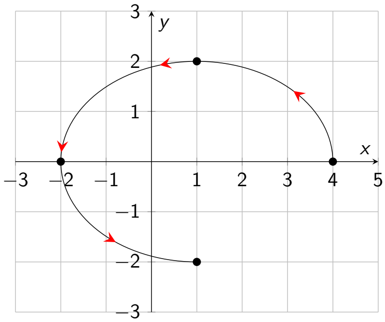

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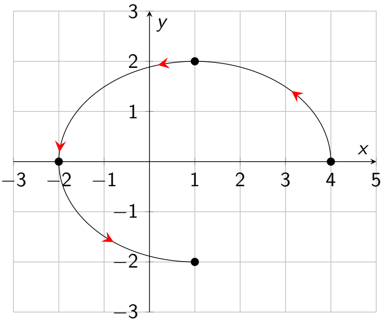

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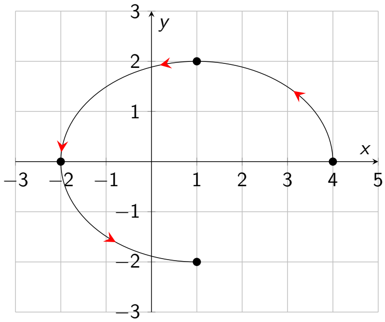

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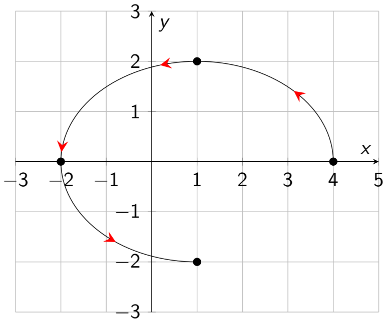

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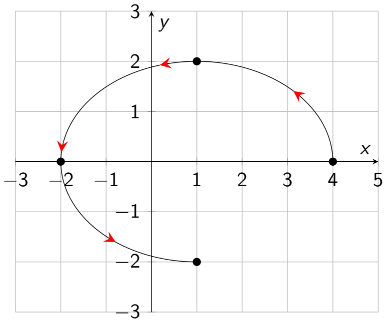

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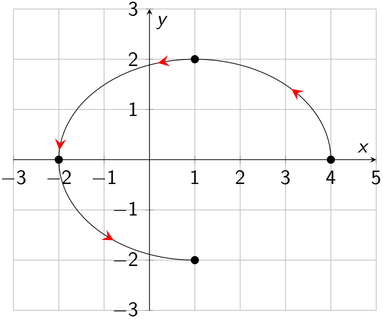

$$\cos^2 t + \sin^2 t = 1$$

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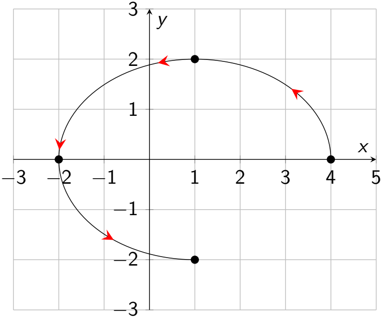

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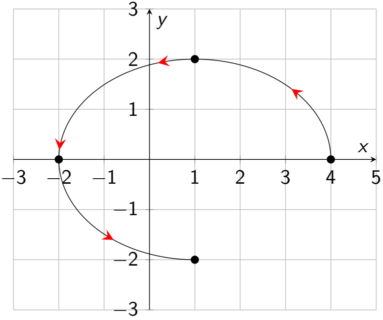

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{(x-1)^2}{9} + \frac{y^2}{4} = 1$$

Parametrizations of Common Curves

- For $y = f(x)$, as x runs through some interval I, let $x = t$ and $y = f(t)$ and let t run through I.

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- For a directed line segment with initial point (x_0, y_0) and terminal point (x_1, y_1) let $x = x_0 + (x_1 - x_0)t$ and let $y = y_0 + (y_1 - y_0)t$ for $0 \leq t \leq 1$.

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- For an ellipse in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, let $x = h + a\cos t$ and $y = k + a\sin t$ for $0 \leq t < 2\pi$.

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Find a parametrization for each of the following.

(a) $y = x^2$ from $x = -3$ to $x = 2$

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$$x = t \quad \text{and} \quad y = t^2 \quad \text{for} \ -3 \leq t \leq 2$$

(b) $y = f^{-1}(x)$ where $f(x) = x^5 + 2x + 1$

$$y = x^5 + 2x + 1$$

$$x = y^5 + 2y + 1$$

Example 4

Find a parametrization for each of the following.

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$$y = x^5 + 2x + 1$$

$$x = y^5 + 2y + 1$$

$$y = t \qquad x = t^5 + 2t + 1 \quad \text{for} \ -\infty < t < \infty$$

Example 4c

(c) The line segment which starts at $(2, -3)$ and ends at $(1, 5)$

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$$x_1 - x_0 = 1 - 2$$

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$$\begin{aligned} x_1 - x_0 &= 1 - 2 \\ &= -1 \end{aligned}$$

Example 4c

(c) The line segment which starts at $(2, -3)$ and ends at $(1, 5)$

$$\begin{aligned} x_1 - x_0 &= 1 - 2 \\ &= -1 \\ x &= x_0 + (x_1 - x_0)t \end{aligned}$$

Example 4c

(c) The line segment which starts at $(2, -3)$ and ends at $(1, 5)$

$$\begin{aligned} x_1 - x_0 &= 1 - 2 \\ &= -1 \\ x &= x_0 + (x_1 - x_0)t \\ x &= 2 + (-1)t \end{aligned}$$

Example 4c

(c) The line segment which starts at $(2, -3)$ and ends at $(1, 5)$

$$\begin{aligned} x_1 - x_0 &= 1 - 2 \\ &= -1 \\ x &= x_0 + (x_1 - x_0)t \\ x &= 2 + (-1)t \\ x &= 2 - t \\ y_1 - y_0 &= 5 - (-3) \end{aligned}$$

Example 4c

(c) The line segment which starts at $(2, -3)$ and ends at $(1, 5)$

$$
\begin{aligned}
x_1 - x_0 &= 1 - 2 \\
&= -1 \\
x &= x_0 + (x_1 - x_0)t \\
x &= 2 + (-1)t \\
x &= 2 - t \\
y_1 - y_0 &= 5 - (-3) \\
&= 8
\end{aligned}
$$

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x_1 - x_0 &= 1 - 2 \\
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x &= x_0 + (x_1 - x_0)t \\
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x &= 2 - t \\
y_1 - y_0 &= 5 - (-3) \\
&= 8 \\
y &= y_0 + (y_1 - y_0)t \\
y &= -3 + 8t \\
&\text{for } 0 \le t \le 1
\end{aligned}
$$

Example 4

(d) The circle $x^2 + 2x + y^2 - 4y = 4$

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$$x = -1 + 3\cos t \qquad y = 2 + 3\sin t \quad \text{for } 0 \le t < 2\pi$$

Example 4

(e) The left half of the ellipse $\dfrac{x^2}{4} + \frac{y^2}{9} = 1$

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$$\text{for } \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$$