Verifying Trig Identities

Objectives

1. Verify trigonometric identities

Identities

Recall that an identity is an equation that is always true.

To verify an identity, we will work with the left and/or right sides of the equation given to show that the equation is true for all values of the variable.

In this section, the identities will involve trig functions.

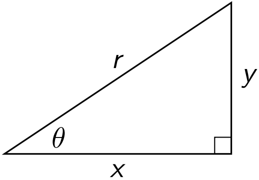
Quotient Identities

The Quotient Identities are

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Example 1

Use the triangle to verify $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$



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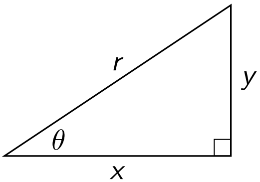

$$\frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)}$$

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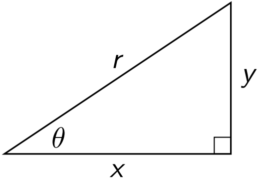

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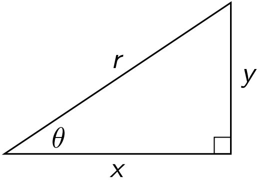

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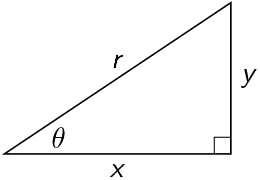

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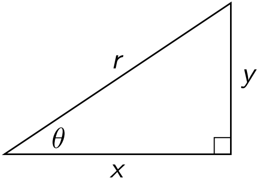
Example 2

Verify $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$



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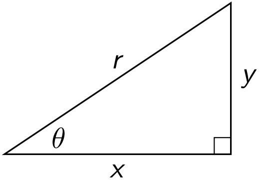

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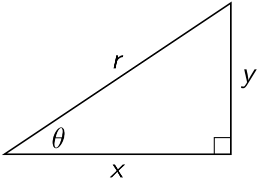

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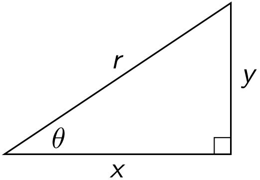

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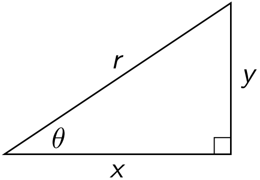

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$$= \frac{x}{y}$$

$$= \cot\theta$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

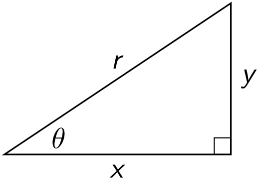
Pythagorean Identities

These are some of the most important identities in this course.

They are based on the Pythagorean Theorem.

Example 3

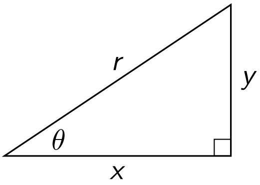
Verify $\cos^2\theta + \sin^2\theta = 1$



Example 3

Verify $\cos^2\theta + \sin^2\theta = 1$

$$x^2 + y^2 = r^2$$



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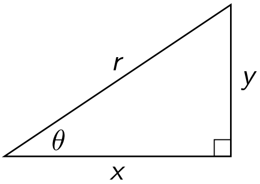

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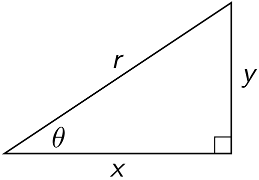

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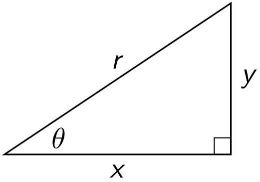

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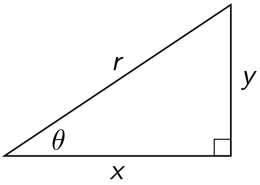

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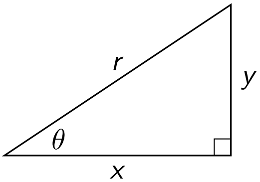
Example 4

Verify $1 + \tan^2\theta = \sec^2\theta$



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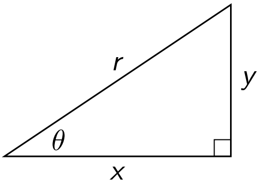

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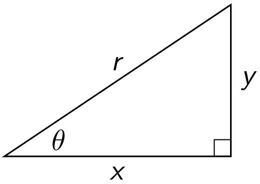

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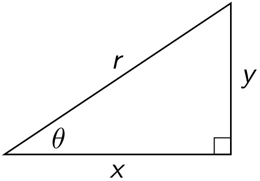

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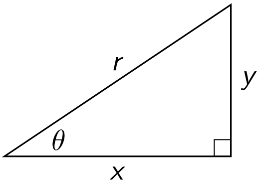

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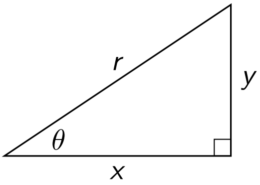

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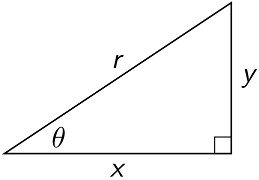
$$1 + \tan^2\theta = \sec^2\theta$$

Example 5

Verify $\cot^2\theta + 1 = \csc^2\theta$



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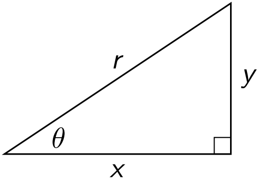

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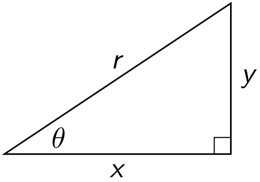

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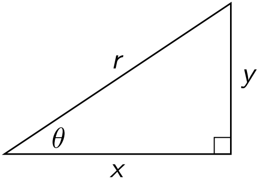

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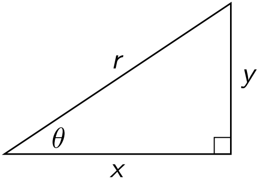

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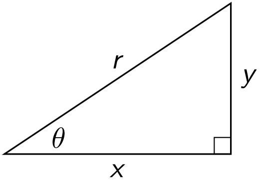

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$$(\cot\theta)^2 + 1 = (\csc\theta)^2$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Alternate Forms of Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1$$

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$$\cos^2\theta + \sin^2\theta = 1$$

- $1 - \sin^2\theta = \cos^2\theta$

Alternate Forms of Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1$$

- $1 - \sin^2\theta = \cos^2\theta$
  - $(1 - \sin\theta)(1 + \sin\theta) = \cos^2\theta$

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- $\csc^2\theta - \cot^2\theta = 1$
    - $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$

Alternate Forms of Pythagorean Identities

$$1 + \cot^2\theta = \csc^2\theta$$

- $\csc^2\theta - \cot^2\theta = 1$
    - $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$
- $\csc^{\theta} - 1 = \cot^2\theta$

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- $\csc^2\theta - \cot^2\theta = 1$
  - $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$
- $\csc^\theta - 1 = \cot^2\theta$
  - $(\csc\theta + 1)(\csc\theta - 1) = \cot^2\theta$

Strategies For Verifying Trig Identities

- Rewrite other functions in terms of sine and/or cosine.
- Work on one or both sides of the equation at the same time.
- Try working on the more complicated side of the identity.
- Use Reciprocal and Quotient Identities to write complex fractions that you can then simplify.
- Obtain common denominators before adding rational expressions.
- Try Pythagorean Identities, especially if you find trig functions raised to a power.

Example 6

Verify each.

(a) $\tan\theta = \sin\theta \cdot \sec\theta$

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Example 6

(b) $(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$

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$$\sec\theta(\sec\theta + \tan\theta) - \tan\theta(\sec\theta + \tan\theta) = 1$$

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(b) $(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$

$$\sec\theta(\sec\theta + \tan\theta) - \tan\theta(\sec\theta + \tan\theta) = 1$$

$$\sec^2\theta + \sec\theta\tan\theta - \sec\theta\tan\theta - \tan^2\theta = 1$$

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(b) $\quad (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$

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$$\sec^2\theta + \sec\theta\tan\theta - \sec\theta\tan\theta - \tan^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$1 = 1$$

Example 6

(c) $$\frac{\sec\theta}{1-\tan\theta} = \frac{1}{\cos\theta - \sin\theta}$$

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$$\frac{\frac{1}{\cos\theta}}{1-\frac{\sin\theta}{\cos\theta}} = \frac{1}{\cos\theta - \sin\theta}$$

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$$\frac{\frac{1}{\cos\theta}}{1-\frac{\sin\theta}{\cos\theta}} \left(\frac{\cos\theta}{\cos\theta}\right)$$

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$$\frac{\frac{1}{\cos\theta}}{1-\frac{\sin\theta}{\cos\theta}}\left(\frac{\cos\theta}{\cos\theta}\right)$$

$$\frac{1}{\cos\theta - \sin\theta} = \frac{1}{\cos\theta - \sin\theta}$$

Example 6

(d) $$6\sec\theta\tan\theta = \frac{3}{1-\sin\theta} - \frac{3}{1+\sin\theta}$$

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$$6\sec\theta\tan\theta = 6\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right)$$

Example 6

(d) $\quad 6\sec\theta\tan\theta = \dfrac{3}{1-\sin\theta} - \dfrac{3}{1+\sin\theta}$

$$6\sec\theta\tan\theta = 6\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$= \frac{6\sin\theta}{\cos^2\theta}$$

Example 6 $\quad 6\sec\theta\tan\theta = \dfrac{3}{1-\sin\theta} - \dfrac{3}{1+\sin\theta}$

$$\frac{3}{1-\sin\theta} - \frac{3}{1+\sin\theta} =$$

Example 6 $\quad 6\sec\theta\tan\theta = \dfrac{3}{1-\sin\theta} - \dfrac{3}{1+\sin\theta}$

$$\frac{3}{1-\sin\theta} - \frac{3}{1+\sin\theta} = \frac{3}{1-\sin\theta}\left(\frac{1+\sin\theta}{1+\sin\theta}\right) - \frac{3}{1+\sin\theta}\left(\frac{1-\sin\theta}{1-\sin\theta}\right)$$

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$$= \frac{6\sin\theta}{\cos^2\theta}$$

Example 6

(e) $$\frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$$ (Half-Angle Tangent Identity)

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