

**Module 3:****Relations and Functions:**

- ▲ Cartesian Products and Relations,
- ▲ Functions – Plain and One-to-One, Onto Functions
- ▲ Stirling Numbers of the Second Kind, Special Functions,
- ▲ The Pigeon-hole Principle,
- ▲ Function Composition and Inverse Functions.

**Relations****Introduction**

**Product set:** If A and B are any 2 non-empty sets then the product set of A and B are the Cartesian product or product of A and B.

$$A \times B = \{(a, b) / (a \in A, b \in B)\}$$

$$A \times B \neq B \times A$$

Example: (a) Let,  $A = \{1, 2, 3\}$   $B = \{a, b\}$

$$\text{Then, } A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\} \quad A \times B \neq B \times A$$

$$(b) \text{ Let, } A = \{1, 2\} \quad B = \{a, b\} \quad C = \{x, y\}$$

$$B \times C = \{(a, x), (a, y), (b, x), (b, y)\}$$

$$A \times (B \times C) = \{(1, (a, x)), (1, (a, y)), (1, (b, x)), (1, (b, y)), \\ (2, (a, x)), (2, (a, y)), (2, (b, x)), (2, (b, y))\}$$

$$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$$

$$(A \times B) \times C = \{((1, a), x), ((1, a), y), ((1, b), x), ((1, b), y), \\ ((2, a), x), ((2, a), y), ((2, b), x), ((2, b), y)\}$$

\*Remarks:

$$a. A \times (B \times C) = (A \times B) \times C$$

$$b. A \times A = A^2$$

c. If  $R$  is the set of all real numbers then  $R \times R = R^2$ , set of all points in plane.

d.  $(a, b) = (c, d)$  if  $a = c$  and  $b = d$

**Partition set:** Let ' $A$ ' be a non-empty set. A partition of ' $A$ ' or quotient set of ' $A$ ' is a collection  $P$  of subsets of

' $A$ ' such that.

(a) Every element of  $A$  belongs to some set in  $P$

(b) If  $A_1$  and  $A_2$  are any two distinct members of  $P$ , then  $A_1 \cap A_2 = \phi$ .

(c) The members of  $P$  are called 'blocks' or 'cells'.

Example:

Let,

$A = \{1, 2, 3, 4, 5\}$  then,

$P_1 = \{\{1, 2, 3\}, \{4\}, \{5\}\}$

$P_2 = \{\{1, 5\}, \{4, 3\}, \{2\}\}$

$P_3 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$

**Relations:** Let  $A$  and  $B$  be any two non-empty sets. A relation  $R$  from a set  $A$  to the set  $B$  is a subset of  $A \times B$ .

If  $(a, b) \in R$  then we write  $a R b$ , otherwise we write  $a \not R b$  (ie.  $a$  not related to  $b$ ).

Example:

Let,

$A = \{1, 2, 3, 4, 5\}$ , Let  $R$  be a relation on  $A$  defined as  $a R b$  if  $a < b$ .  $R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

$$\Rightarrow R \subseteq A \times A.$$

Domain of  $R$ :  $\text{Dom}(R) = \{1, 2, 3, 4\} \subseteq A$

Range of  $R$ :  $\text{Ran}(R) = \{2, 3, 4, 5\} \subseteq B$

$\text{Dom}(R) = \{x \in A / x R y \text{ for some } y \in A\}$

$\text{Ran}(R) = \{y \in B / x R y \text{ for some } x \in A\}$

**R - Relative set:** If  $R$  is a relation from  $A$  to  $B$  and if  $x \in A$  then the  $R$  relative set of  $x$  is defined as

$$R(x) = \{y \in B / x R y\}$$

If  $A_1 \subseteq A$  then the  $R$  relative set of  $A_1$  is defined as,

$$R(A_1) = \{y \in B / x R y \text{ for some } x \in A_1\}$$

$$= \bigcup_{x \in A_1} R(x)$$

Example:

Let,

$$A = \{a, b, c, d\}$$

$$R = \{(a, a), (a, b), (b, c), (c, a), (c, b), (d, a)\}$$

$$R(a) = \{a, b\}$$

$$R(b) = \{c\}$$

$$R(c) = \{a, b\}$$

$$R(d) = \{a\}$$

Let,

$$A_1 = \{a, c\} \text{ be a subset of } A,$$

$$\text{Then, } R(A_1) = R(a) \cup R(c)$$

$$= \{a, b\} \cup \{a, b\}$$

$$= \{a, b\}$$

**Matrix of a relation / Relation Matrix:** Let  $A = \{a_1, a_2, a_3, \dots, a_m\}$  and  $B = \{b_1, b_2, b_3, \dots, b_n\}$  be any two finite sets.

Let  $R$  be relation from  $A$  to  $B$  then the matrix of the relation  $R$  is defined as the  $m \times n$  matrix,

$$M_R = [M_{ij}]$$

Where  $M_{ij} = 1$ , if  $(a_i, b_j) \in R$

$$= 0, \text{ if } (a_i, b_j) \notin R$$

Example:

(a) Let,

$$A = \{1, 2, 3\} \text{ and } B = \{x, 4\}$$

$$R = \{(1, x), (1, 4), (2, 4), (3, x)\}$$

Thus,  $M_r =$




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(b) Given  $M_R =$  . Find Relation R.

?

Define set,

$A = \{a_1, a_2, a_3\}$

and  $B = \{b_1,$

$b_2, b_3, b_4\}$

$R = \{(a_1, b_2) (a_1, b_4) (a_2, b_2) (a_2, b_3) (a_3, b_1) (a_3, b_3)\}$

**Digraph of a relation:** Let A be a finite set and R be a relation on A. Then R can be represented pictorially as follows,

(a) Draw a small circle for each element of A and label the circles with the corresponding element of A. These circles are called "Vertices".

(b) Draw an arrow from  $a_i$  to  $a_j$  if  $a_i R a_j$ . These arrows are called "edges".

(c) The resulting picture representing the relation R is called the "directed graph of R" or "digraph of R".

Example:

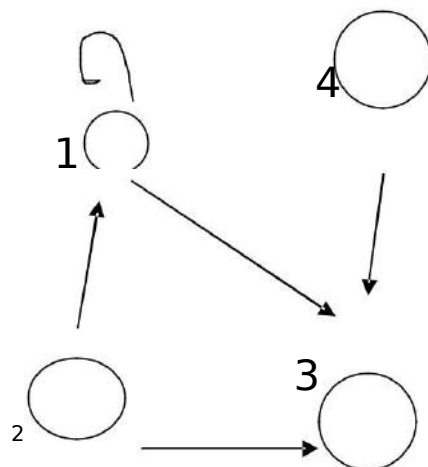
(a) Let, A be equal to the set

$A = \{1, 2, 3, 4\}$

$R = \{(1, 1), (1, 3), (2, 1), (2, 3), (3, 2), (4, 3)\}$





Diagram:

The "indegree" of  $a \in A$  is the number of elements  $b \in A$  such that  $b R a$ . The "outdegree" of  $a \in A$  is the number of elements  $b \in A$  such that  $a R b$

Elements	Indegree	Outdegree
1	2	2
2	1	2
3	3	1
4	0	1

(b) If  $A = \{1, 2, 3, 4\}$  and  $B = \{1, 4, 6, 8, 9\}$  and  $R: A \rightarrow B$  defined by

$a R b$  if  $b = a^2$ . Find the domain, Range, and  $M_R$

$$A = \{1, 2, 3, 4\} \quad B = \{1, 4, 6, 8, 9\}$$

$$R = \{(x, y) / x \in A, y \in B \text{ and } y = x^2\}$$

$$R = \{(1, 1), (2, 4), (3, 9)\}$$

Domain:  $\text{Dom}(R) = \{1, 2, 3\}$



Range:  $\text{Ran}(R) = \{1, 4, 9\}$

$$M_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

### Properties of a relation:

1. **Reflexive:** Let  $R$  be a relation on a set  $A$ .

The " $R$  is reflexive" if  $(a, a) \in R \forall a \in A$  or  $a R a, \forall a \in A$ .

Example:  $A = \{1, 2, 3\}$

$$R = \{(1, 1), (2, 2), (1, 2), (3, 2), (3, 3)\}$$

Therefore,  $R$  is reflexive.

A relation  $R$  on a set  $A$  is "non-reflexive" if ' $a$ ' is not related to ' $a$ ' for some  $a \in A$  or  $(a, a) \notin R$ .

a)  $\nexists a \in A$  such that  $a R a$

$$A = \{1, 2, 3\}$$

$$R = \{(1, 1), (2, 1), (3, 2), (3, 3)\}$$

$\Rightarrow (2, 2) \notin R$  Therefore,  $R$  is *not-reflexive*.

2. **Irreflexive:** A relation  $R$  on a set  $A$  is irreflexive if  $a \not R a, \forall a \in A$ .

Example:  $R = \{(1, 2), (2, 1), (3, 2), (3, 1)\}$

$(1, 1), (2, 2), (3, 3) \notin R$  hence  $R$  is irreflexive.



A relation  $R$  on a set  $A$  is “not irreflexive” if ‘ $a$ ’ is not Relation to ‘ $a$ ’ for some  $a \in A$ .

Example:  $R = \{(1, 1) (1, 2) (2, 1) (3, 2) (3, 1)\}$

$(1, 1) \in R$  hence  $R$  is “not irreflexive”.

3. **Symmetric Relation:** Let  $R$  be a relation on a set  $A$ , then  $R$  is “symmetric” if whenever  $a R b$ , then  $b R a$ ;  $\forall a \in A, b \in A$ .

Example: Let  $A = \{1, 2, 3\}$  and  $R = \{(1, 1) (1, 2) (2, 1) (3, 2) (2, 3)\}$

Therefore,  $R$  is symmetric.

A relation  $R$  on a set  $A$  is said to be "not symmetric" if  $a R b$  and  $b R a$  for some  $a, b \in A$ .  
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Example:  $A = \{1, 2, 3\}$  and  $R = \{(1, 2) (3, 2) (1, 3) (2, 1) (2, 3)\}$

Therefore,  $R$  is not symmetric.

4. **Asymmetric:** Let  $R$  be a relation on a set  $A$  then  $R$  is “Asymmetric”, if whenever  $a R b$  then  $b R a$ ,  $\forall a, b \in A$ .

$$R = \{(1, 2), (1, 3) (3, 2)\}$$

Therefore,  $R$  is asymmetric.

A relation  $R$  on a set  $A$  is said to be "not Asymmetric" if  $a R b$  and  $b R a$  for some  $a, b \in A$   $R = \{(1, 1) (1, 2) (1, 3) (3, 2)\}$

$R$  is not asymmetric.

5. **Anti-symmetric:** Let  $R$  be a relation on a set  $A$ , then  $R$  is anti-symmetric if whenever  $a R b$  and  $b R a$  then  $a = b$  (for some  $a, b \in A$ )

Example: Let,  $A = \{1, 2, 3\}$  and  $R = \{(1, 1), (1, 2), (3, 2)\}$

$R$  is anti-symmetric  $\in 1R1$  and  $1 = 1$ .



Example:  
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$$R = \{(1, 2) (2, 1)\}$$

$1R2, 2R1$  but  $2 \neq 1$  hence  $R$  is not anti symmetric.

**6. Transitive Property:** Let  $R$  be a relation on a set  $A$ , then  $R$  is transitive if whenever  $a R b$  and  $b R c$ , then  $a R c \forall a, b, c \in A$ .

Example: Let,  $A = \{1, 2, 3\}$  and  $R = \{(1, 1), (1, 3), (2, 3), (3, 1) (2, 1), (3, 3)\}$  (all should satisfy)

**Equivalence relation:** A Relation  $R$  is said to be an equivalence relation if it is, Reflexive (b) Symmetric and

(c) Transitive.

Therefore,  $R$  is an equivalence Relation.

**Symmetric:** Let  $a R b$

$$\Rightarrow 1R2$$

2 is not Related to 1 and also b is not

Related to a Hence,  $R$  is not symmetric

**Transitive:** Let  $a R b$  and  $b R c$

$\Rightarrow 1R2$  and  $2R3$  but, 1 is not Related to 3 and also a is not Related to c

Hence,  $R$  is not transitive.

Therefore,  $R$  is not an equivalence Relation.

$$b. R = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$$

**Reflexive:**  $a R a \forall a \in A$

$$\Rightarrow 1 R 1, 2 R 2, 3 R 3 \quad \text{not true,}$$

Hence, R is not reflexive

**Symmetric:** Let  $a R b$

$$\Rightarrow 1 R 3$$

$$\Rightarrow 3 R 1$$

$$\Rightarrow b R a$$

Hence, R is symmetric.

**Transitive:** Let  $a R b$  and  $b R c$

$$\Rightarrow 1 R 2 \text{ and } 2 R 3$$

$$\Rightarrow 1 R 3$$

$$\Rightarrow a R c$$

Hence, R is transitive

Therefore, R is not an equivalence Relation.

$$c. A = \{1, 2, 3\}$$

$$R = A \times A = \{(1, 1)(1, 2)(1, 3)(2, 1)(2, 2)(2, 3)(3, 1)(3, 2)(3, 3)\}$$

It is reflexive, symmetric and transitive and hence R is an equivalence

Relation.

**Theorem:** "Let R be an equivalence relation on a set A, and P be the collection of all distinct R - relative set of A. Then P is a partition of A, and R is the equivalence relation"





determined by P"

OR

"Show that an equivalence relation R in a set S which is non-empty, determine a partition of S"

*Proof:* Given,  $P = \{R(a) \mid \forall a \in A\}$

We know that  $\forall a \in A$ , we have,  $a R a$

$$\Rightarrow (a, a) \in R$$

$$\Rightarrow a \in R(a)$$

Therefore, for every element of A belongs to one of the sets of P.

If  $R(a)$  and  $R(b)$  are 2 distinct relative sets  $R(a) \cap R(b) = \Phi$

If possible, let  $x \in R(a) \cap R(b)$

$$\Rightarrow x \in R(a) \text{ and } x \in R(b)$$

$$\Rightarrow a R x \text{ and } b R x$$

This partition determines the relation R in the sense that  $a R b$  if a and b belong to the same block of the partition.

Hence proved.....

**\*NOTE:** The partition of a set A determined by an equivalence relation R is called the partition induced by R and is denoted by  $A/R$ .

### **Manipulation of relations:**

1. **Complement:** Let R be a relation from A to B. The complement of R is a relation

defined as  $a R b$  if  $a R^{\sim} b$ , where  $R^{\sim}$  is the complement of  $R$ .

$$\Rightarrow (a, b) R^{\sim} \text{ if } (a, b) R^{\sim}$$

2. **Union:** Let  $R$  and  $S$  be 2 relations from  $A$  to  $B$ . The union  $R \cup S$  is a relation from  $A$  to  $B$  defined as,

$$a (R \cup S) b \text{ if either } a R b \text{ or } a S b$$

That is  $(a, b) \in R \cup S$  if either  $(a, b) \in R$  or  $(a, b) \in S$ .

3. **Intersection:** Let  $R$  and  $S$  be relations from  $A$  to  $B$ . The intersection  $R \cap S$  is a relation from  $A$  to  $B$  defined as,

$$a (R \cap S) b \text{ if } a R b \text{ and } a S b$$

That is  $(a, b) \in R \cap S$  if  $(a, b) \in R$  and  $(a, b) \in S$ .

4. **Inverse:** Let  $R$  be a relation from  $A$  to  $B$ . The inverse  $R^{-1}$  is a relation from  $B$  to  $A$  defined as,  $a R b$  if  $b R^{-1} a$

$$\text{i.e., } (a, b) \in R \text{ if } (b, a) \in R^{-1}$$

**Composition of relations:** Let  $R$  and  $S$  be relations from  $A$  to  $B$  and  $B$  to  $C$  respectively. The composition of  $R$  and  $S$  is the

relation  $S \circ R$  from  $A$  to  $C$  defined as,

$$a(S \circ R) c \text{ if there-exist } b \in B/a R b \text{ and } b S c.$$

$$R^2 = R \circ R = \{(a, a), (a, c), (a, b), (b, a), (b, c), (b, b), (c, a), (c, b), (c, c)\}$$

$$S^2 = S \circ S = \{(a, a), (b, b), (b, c), (b, a), (c, a), (c, c)\}$$

**Reflexive closure:** Let  $R$  be a relation on a set  $A$ . Suppose  $R$  lacks a particular property, the smallest relation that contain  $R$  and which, processes the desired property is called the closure of  $R$  with respective a property in question.

Given a relation  $R$  on a set  $A$  the relation  $R_1 = (A \cup R)$  is the "reflexive closure of  $R$ ".



Example:

$$A = \{1, 2, 3\}$$

$R = \{(1, 1)(1, 2)(2, 1)(1, 3)(3, 2)\}$  find the reflexive closure of  $R$ .

Solution: We know that,  $R$  is not reflexive because  $(2, 2) \notin R$  and  $(3, 3) \notin R$ .

$$\text{Now, } A = \{(1, 1) (2, 2) (3, 3)\}$$

Therefore,  $R_1 = R \cup A = \{(1, 1) (1, 2) (2, 1) (2, 2) (1, 3) (3, 2) (3, 3)\}$

$R_1$  is the reflexive closure of  $R$ .

**Symmetric closure :** If  $R$  is not symmetric then there exists  $(x, y) \in R$  such that  $(x, y) \in R$ , but  $(y, x) \notin R$ . To make  $R$  symmetric we need to add the ordered pairs of  $R^{-1}$ .

$R_1 = R \cup R^{-1}$  is the "symmetric closure of  $R$ ".

$$A = \{1, 2, 3\}$$

$R = \{(1, 1) (1, 2) (2, 1) (1, 3) (3, 2)\}$  find the symmetric closure of  $R$ .

Solution: We know that,  $R$  is not symmetric because  $(1, 3) \in R$  but  $(3, 1) \notin R$  and  $(3, 2) \in R$  but  $(2, 3) \notin R$ .

*Example:*  $R^{-1} = \{(1, 1) (2, 1) (1, 2) (3, 1) (2, 3)\}$

Therefore,  $R_1 = R \cup R^{-1} = \{(1, 1) (1, 2) (2, 1) (1, 3) (3, 1) (3, 2) (2, 3)\}$

$R_1$  is called the symmetric closure of  $R$ .

**Transitive closure:** Let  $R$  be a relation on a set  $A$  the smallest transitive relation containing  $R$  is called the "Transitive closure of  $R$ ".

## **Functions**

### **Introduction**

A person counting students present in a class assigns a number to each student under consideration. In this case a correspondence between two sets is established: between students and whole numbers. Such correspondence is called functions. Functions



are central to the study of physics and enumeration, but they occur in many other situations as well. For instance, the correspondence between the data stored in computer memory and the standard symbols  $a, b, c, \dots, z, 0, 1, \dots, 9, ?, !, +, \dots$  into strings of 0's and 1's for digital processing and the subsequent decoding of the strings obtained: these are functions. Thus, to understand the general use of functions, we must study their properties in the general terms of set theory, which is what we will do in this chapter.

**Definition:** Let  $A$  and  $B$  be two sets. A function  $f$  from  $A$  to  $B$  is a rule that assigned to each element  $x$  in  $A$  exactly one element  $y$  in  $B$ . It is denoted by  $f: A \rightarrow B$

**Note:**

1. The set  $A$  is called domain of  $f$ .
2. The set  $B$  is called codomain of  $f$ .

**Value of  $f$ :** If  $x$  is an element of  $A$  and  $y$  is an element of  $B$  assigned to  $x$ , written  $y = f(x)$  and call function value of  $f$  at  $x$ . The element  $y$  is called the image of  $x$  under  $f$ .

Example:  $A = \{1, 2, 3, 4\}$  and  $B = \{a, b, c, d\}$

$R = \{(1, a), (2, b), (3, c),$

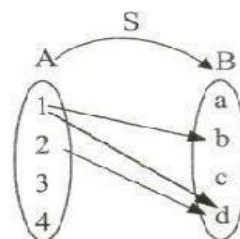
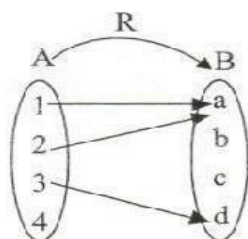
$\{4, d\}\}$   $S = \{(1, b), (1, d),$

$(2, d)\}$

Therefore,  $R$  is a function and  $S$  is not a function. Since the element 1 has two images

$b$  and  $d$ ,  $S$  is not a function.

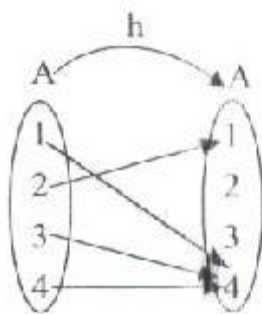
Example: Let  $A = \{1, 2, 3, 4\}$  determine whether or not the following relations on  $A$  are







functions.



$$1. f = \{(2, 3), (1, 4), (2, 1), (3, 1), (4, 4)\}$$

(Since element 2 has 2 images 3 and 1,  $f$  is not a function.)

$$2. g = \{(3, 1), (4, 2),$$

$(1, 1)\}$   $g$  is a

function

$$3. h = \{(2, 1), (3, 4), (1, 4), (2, 1),$$

$(4, 4)\}$   $h$  is a function

4. Let  $A = \{0, \pm 1, \pm 2, 3\}$ . Consider the function  $F: A \rightarrow \mathbb{R}$ , where  $\mathbb{R}$  is the set of all real numbers, defined by  $f(x) = x^3 - 2x^2 + 3x + 1$  for  $x \in A$ . Find the range of  $f$ .

$$f(0) = 1$$

$$f(1) = 1 - 2 + 3 + 1 = 3$$

$$f(-1) = -1 - 2 - 3 + 1 = -5$$

$$f(2) = 8 - 8 - 6 + 1 = -5$$

$$f(-2) = -8 - 8 - 6 + 1 = -21$$

$$f(3) = 27 - 18 + 9 + 1 = 19$$

$$\blacklozenge \text{ Range} = \{1, 3, -5, -21, 19\}$$



5. If  $A = \{0, \pm 1, \pm 2\}$  and  $f: A \rightarrow \mathbb{R}$  is defined by  $f(x) = x^2 - x + 1$ ,

Find the range.  $f(0) = 1$

$$f(1) = 1 -$$

$$1 + 1 = 1 \quad f(-1)$$

$$= 1 + 1 + 1 = 3 \quad f$$

$$(2) = 4 -$$

$$2 + 1 = 3 \quad f(-2)$$

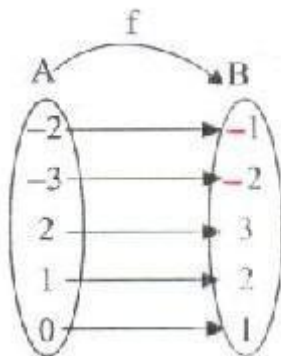
$$= 4 + 2 + 1 = 7$$

$$\text{Range} = \{1, 3, 7\}$$

### Types of functions:

#### 1. Everywhere defined -2

A function  $f: A \rightarrow B$  is everywhere defined if domain of  $f$  equal to  $A$  (dom  $f = A$ )

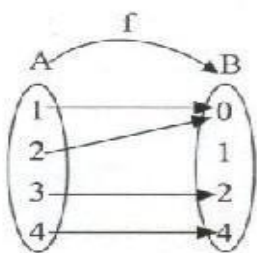


A function  $f: A \rightarrow B$  is onto or surjection if Range of  $f = B$ . In other words, a function  $f$  is surjection or onto if for any value  $y$  in  $B$ , there is at least one element  $x$  in  $A$  for which  $f(x) = y$ .

#### 3. Many to one function

A function  $F$  is said to be a many-to-one function if  $a \neq b$ ,  $f(a) = f(b)$ , where  $(a, b) \in A$ .

Example:



Here,  $1, 2 \in A$  but  $f(1) = f(2) = 0$ , where  $1, 2 \in A$

#### 4. One-to-one function or injection

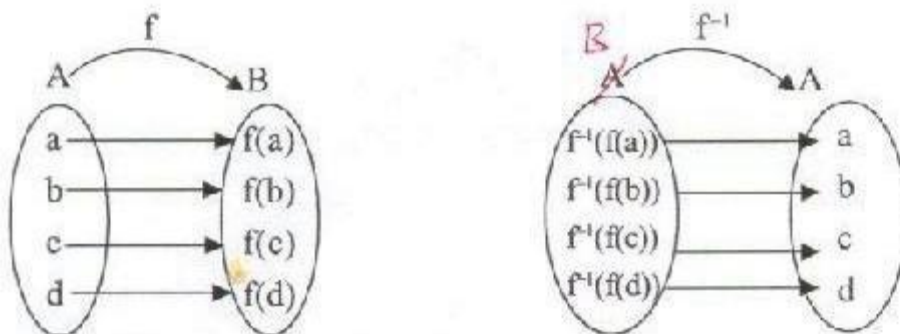
A function  $f: A \rightarrow B$  is one-to-one or injection if  $f(a) = f(b)$  then  $a = b$ ,

where  $a, b \in A$ . In other words if  $a \neq b$  then  $f(a) \neq f(b)$ .

#### 5. Bijection function

A function  $f: A \rightarrow B$  is Bijection if it is both onto and one-to-one.

#### 6. Invertible function



A function  $f: A \rightarrow B$  is said to be an invertible function if its inverse relation,  $f^{-1}$  is a function from  $B \rightarrow A$ .

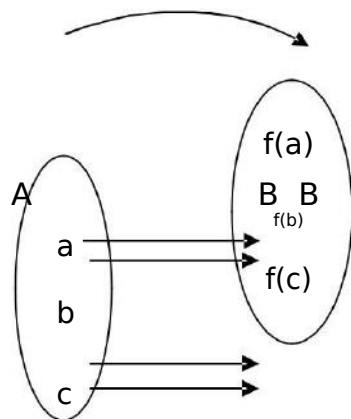
If  $f: A \rightarrow B$  is Bijection, then  $f^{-1}: B \rightarrow A$  exists,  $f$  is said to be invertible.

Example:  $Y = f(x) = x+1$

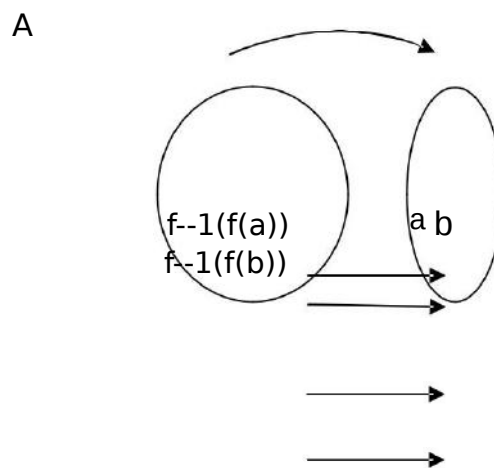
Here ,  $\text{dom } f = A$

## 2. Onto or surjection function

Example:  $f$



$f^{-1}$



$$f^{-1} : B \rightarrow A$$

$$B = \{b_1, b_2, b_3\} \quad C = \{c_1, c_2\} \quad D = \{d_1, d_2, d_3, d_4\}$$

Let  $f_1: A \rightarrow B$ ,  $f_2: A \rightarrow D$ ,  $f_3: B \rightarrow C$ ,  $f_4: D \rightarrow B$  be functions defined as follows,

$$1. f_1 = \{(a_1, b_2) (a_2, b_3) (a_3, b_1)\}$$

$$2. f_2 = \{(a_1, d_2) (a_2, d_1) (a_3, d_4)\}$$

$$3. f_3 = \{(b_1, c_2) (b_2, c_2) (b_3, c_1)\}$$

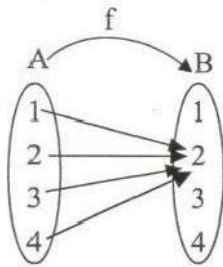
$$4. f_4 = \{(d_1, b_1) (d_2, b_2) (d_3, b_1)\}$$

### Identity function

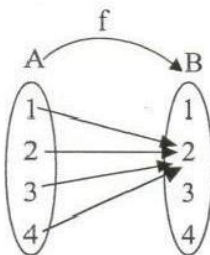
A function  $f: A \rightarrow A$  such that  $f(a) = a$ , 'if  $a \in A$  is called the identity function or identity mapping on  $A$ .  $\text{Dom}(f) = \text{Ran}(f) = A$

**Constant function**

A function  $f: A \rightarrow B$  such that  $f(a) = c$ ,  $\forall a \in \text{dom}(f)$  where  $c$  is a fixed element of  $B$ , is called a constant function.

**Into function**

A function  $f: A \rightarrow B$  is said to be an into function if there exist some  $b$  in  $B$  which is not the image of any  $a$  in  $A$  under  $f$ .



3 is not the image of any element.

**One-to-one correspondence**

If  $f: A \rightarrow B$  is everywhere defined and is Bijective, then corresponding to every  $a \in A$  there is a unique  $b \in B$  such that  $b = f(a)$  and corresponding to every  $b \in B$  there is a unique  $a \in A$  such that  $f(a) = b$ . For this reason a everywhere defined bijection function from  $A \rightarrow B$  is called as one-one correspondence from  $A \rightarrow B$ .

**Composition of function**

Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be any 2 functions, and then the composition of  $f$  and  $g$  is a function  $g \circ f: A \rightarrow C$  defined as,  $g \circ f(a) = g[f(a)]$  ( $a \in \text{dom}(f)$ ).





**Inverse function**

Consider a function  $f: A \rightarrow B$ . Then  $f$  is a relation from  $A$  to  $B$  with  $\text{Dom}(f) \subseteq A$  and  $\text{Ran}(f) \subseteq B$ . Its inverse,  $f^{-1}$ , is a relation from  $B$  to  $A$  which is such that if whenever  $(a, b) \in f$  then  $(b, a) \in f^{-1}$ .

Also,  $\text{Dom}(f^{-1}) = \text{Ran}(f)$   
 $\text{Ran}(f^{-1}) = \text{Dom}(f)$  and

$$(f^{-1})^{-1} = f$$

**Definition**

A function  $f: A \rightarrow B$  is invertible if its inverse relation  $f^{-1}$  is a function from  $B$  to  $A$ . Then,  $f^{-1}$  is called the inverse function of  $f$ .

Ex: let  $A = \{a, b, c, d\}$  and  $B = \{e, f, g, h\}$  and  $f: A \rightarrow B$  be a function defined by

$$f(a) = e, f(b) = f, f(c) = h, f(d) = g$$

Then, as a relation from  $A$  to  $B$ ,  $f$  reads

$$f = \{(a, e), (b, f), (c, h), (d, g)\}$$

And  $f^{-1}$  is a relation from  $B$  to  $A$ , given by

$$f^{-1} = \{(e, a), (f, b), (h, c), (g, d)\}$$

Now,  $\text{Dom}(f^{-1}) = \{e, f, g, h\} = \text{Ran}(f)$  and

$$\text{Ran}(f^{-1}) = \{a, b, c, d\} = A = \text{Dom}(f)$$

$$\text{Also, } (f^{-1})^{-1} = f$$

Although  $f^{-1}$  is a relation from  $B$  to  $A$ , it is not function from  $B$  to  $A$ , because  $e$  is related to two elements 'a' and 'b' under  $f^{-1}$ .

Let  $A = \{1, 2, 3, 4\}$  and  $B = \{5, 6, 7, 8\}$  and the function  $f: A \rightarrow B$  defined by

$$f(1) = 6, f(2) = 8, f(3) = 5, f(4) = 7$$

$$\text{Then, } f = \{(1, 6), (2, 8), (3, 5), (4, 7)\}$$

$$\diamond f^{-1} = \{(6, 1), (8, 2), (5, 3), (7, 4)\}$$

In this case,  $f^{-1}$  is not only a relation from  $B$  to  $A$  but a function as well.

## Characteristic function Introduction

Characteristic function is a special type of function. It is very useful in the field of computer science. Through this function one can tell whether an element present in the set or not. If the function has the value 1 then the particular element belongs to the set and if it has value 0 then the element is not present in the set.

Definition

Associated with the subset  $A$  of  $\diamond$  we can define a characteristic function of  $A$  over  $\diamond$  as  $f: \diamond \rightarrow \{0, 1\}$  where

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

## Properties of the characteristics function

$$f_{A \cap B}(x) = (f_A \wedge f_B)(x)$$

$$1. \quad f_{A \cap B}(x) = f_A(x) \wedge f_B(x)$$

**Proof:**

$$i. \text{ if } x \in A \cap B \text{ then } x \in A \text{ and } x \in B$$

$$(f_A \wedge f_B)(x) = 1$$

$$\diamond f_A(x) = 1 \quad \text{and} \quad f_B(x) = 1$$

$$\diamond (f_A \wedge f_B)(x) = 1 = f_{A \cap B}(x)$$

$$ii. \text{ if } x \notin A \cap B \text{ then } (f_A \wedge f_B)(x) = 0. \text{ but if } x \notin A \cap B \text{ then } x \notin A \text{ and } x \notin B$$

$$ii. \text{ if } f_A(x) = 0 \text{ or } f_B(x) = 0$$

$$(f_A \wedge f_B)(x) = 0$$

$$\diamond f_{A \cap B}(x) = 0$$



$$\diamond f(x)_{A \cap B} = 0 = f(x) \cdot f_A(x) \cdot f_B(x)$$

$\diamond$  From case 1 and 2

$$\begin{aligned} (A \cup B)_B &= f \\ f(x) \cdot f_A(x) &= f_B(x) \\ f_{A \cup B}(x) &= f(x) - f_B(x) \cdot f_A(x) \\ &= f(x) + f_A(x) \cdot f_B(x) \end{aligned}$$

**Proof:**

i. Let  $x \in A \cup B$  then  $f_{A \cup B}(x) = 1$ . But if  $x \notin A \cup B$  then there are three cases

**case1:** let  $x \in A$  but not in  $B$  then  $f_A(x) = 1$  and  $f_B(x) = 0 \Rightarrow f(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$

$$= 1 + 0 - 1 \cdot 0 = 1$$

$f_B(x)$  [Because

$$1 + 0 + 0]$$

**case2:**

let  $x \in B$  but not in  $A$

$$\text{Then } f_B(x) = 1 \quad f_A(x) =$$

$$\text{and } f_A(x) = 0$$

$$(f_{A \cup B})(x) = 1 = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$$

$$\diamond f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$$

[Because  $0 + 1 - 0$ ]

**case3:** let  $x \in A$  and  $x \in B$

$$\text{Then } f_A(x) = 1 \quad f_B(x) =$$

$$\text{and } f_B(x) = 1$$

$$(f_{A \cup B})(x) = 1 = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$$

$$\diamond f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$$

[Because  $1 + 1 - 1$ ]

$$f(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$$

AUB

Let  $x \notin A \cup B$  then  $f_{A \cup B}(x) = 0$

$x \in A \cup B$

then  $x \in A$  and  $x \in B$  then

$$f_A(x) = 0 \text{ and } f_B(x) = 0$$

$$\Rightarrow f_{A \cup B}(x) = 0 = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$$

[because  $0+0-1$ ]

$\Rightarrow$  From case i and ii.

$$\Rightarrow f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$$

A symmetric difference is associative on sets

To prove  $(A \oplus B) \oplus C = A \oplus (B \oplus C)$  we have to prove

$$f_{(A \oplus B) \oplus C}(x) = f_{A \oplus (B \oplus C)}(x) \quad \forall x$$

$$\text{LHS} = f_{(A \oplus B) \oplus C}$$

$$= f_{(D \oplus C)} \text{ where } D = A \oplus B$$

$$= f_D + f_C - 2 f_D f_C$$

$$= f_C + f_D (1 - 2 f_C)$$

$$= f_D + f_{A \oplus B} (1 - 2 f_C)$$

$$= f_C + (f_A + f_B - 2 f_A f_B) (1 - 2 f_C)$$

$$= f_C + f_A + f_B - 2 f_A f_B - 2 f_A f_C - 2 f_B f_C + 4 f_A f_B f_C$$

$$= f_A + (f_B + f_C - 2 f_B f_C) - 2 f_A (f_B + f_C - 2 f_B f_C)$$

$$= f_A + f_B + f_C - 2 f_B f_C (1 - 2 f_A)$$

$$= f_A + f_{B \oplus C} (1 - 2 f_A)$$

$$= f_A + f_{B \oplus C} - 2 f_A f_{B \oplus C}$$

$$= f_{A \oplus (B \oplus C)}$$

$$= \text{RHS}$$

$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$



## Permutation function

A permutation on 'A' is a bijection of 'A' onto itself. Let 'A' = {a<sub>1</sub>, a<sub>2</sub>, a<sub>3</sub>, ----- a<sub>n</sub>}. Where A is a finite set, if P is a permutation on A then P can be represented as ,

$$P = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ P(a_1) & P(a_2) & P(a_3) & \dots & P(a_n) \end{pmatrix}$$

This is called as two line notation of a permutation of A.

**NOTE:** (a) if  $|A| = n$ , then there  $n!$  Permutation on A

(b) The composition of two permutations is again a permutation called Product of permutation.

### Cycle

Consider a permutation P of a set  $A = \{a_1, a_2, a_3, \dots, a_n\}$

In this permutation suppose r elements of A say  $\{b_1, b_2, b_3, \dots, b_r\}$  are such that  $P(b_1) = b_2$ ,  $P(b_2) = b_3$ , .....  $P(b_{r-1}) = b_r$ ,  $P(b_r) = b_1$ , and the remaining elements of A are images of themselves, Then P is called a cycle of length 'r', and is denoted by  $(b_1, b_2, b_3, \dots, b_r)$ .

Example 1:

$$A = \{1, 2, 3, 4\} \text{ and } P(A) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

$$P(1) = 3 ; P(3) = 4 ; P(4) = 1$$

◆ (1, 3, 4) forms a cycle of length 3.

◆ In P the elements (1, 3, 4) forms a cycle and '2' remains unchanged.





◆ P is a cycle of

length 3. Example 2:

$A = \{1, 2, 3, 4, 5, 6\}$   
and  $P =$

$$\begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ & 3 & 2 & 4 & 6 & 5 & 1 \end{pmatrix}$$

$P(1) = 3; P(3) = 4; P(4)$   
 $= 6; P(6) = 1$

◆ (1, 3, 4, 6) forms a cycle (2 and 5 remain unchanged)

◆ P is a cycle of length 4.

### T ransposition

A cycle of length 2 is called a “transposition” if  $A = \{a_1, a_2, a_3, \dots, a_n\}$  then  $P = (a_i, a_j), i \neq j$  is a transposition of A.

Example:

$A = \{1, 2, 3, 4, 5, 6\}$  compute

1.  $(4, 1, 3, 5) \circ (5, 6, 3)$  and

2.  $(5, 6, 3) \circ (4, 1, 3, 5)$

$$P_1 = \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ & 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix}$$

$$P_2 = (5, 6, 3) = \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ & 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix}$$



$$P_1 \circ P_2 =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix}$$

$$2. P_2 \circ P_1 = (5, 6, 3) \circ (4, 1, 3, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix}$$

### Even and odd permutations

Example:

$A = \{1, 2, 3, 4, 5, 6, 7, 8\}$  find whether the following permutation are even or odd 1.  $P =$

$$P = (1, 3) \circ (1, 8) \circ (1, 6) \circ (1, 4)$$

◆  $P$  is an even permutation.

a.  $P = (4, 8) \circ (3, 5, 2, 1) \circ (2, 4, 7, 1)$

$$P = (4, 8) \circ (3, 1) \circ (3, 5) \circ (2, 1) \circ (2, 7) \circ (2, 4)$$

◆  $P$  is an odd permutation because  $P$  is expressed as a composition of odd number of transpositions.

**Note:** Product of even-even permutation is even

Product of even-odd permutation is odd

Product of odd-odd permutation is odd

Even permutation cannot be expressed in terms of odd

Odd permutation cannot be expressed in terms of even.

## Hashing function Introduction

Suppose we want to store the mail address of all voters of a large city in  $n$  number of files, numbered from 0 to  $n-1$ , in such a way that the file containing the address any chosen voter can be located almost instantly. The following is one way of doing this task. First, to each voter let us assign a unique positive integer as an identification number. Next, to each identification number, let us assign a unique positive integer called a key. The keys can be such that two identification numbers can have the same key but two different keys are not assigned to the same identification number.

Therefore the number of identification number will be equal to the number of voters, but the number of keys can be less than the no. of identification number.

### Definition

Let  $A$  denote the set of all keys and  $B = \{0, 1, 2, \dots, (n-1)\}$ , the set of all files.

Consider an everywhere defined function  $h_n$ ;  $h_n : A \rightarrow B$  specified by  $h_n$

$(a) = r$ , where  $r$  is the remainder,  $r = a/n$  and  $a \in A$ . This function determines a unique  $r$ . for any specified  $a \in A$ , this  $r$  will be one and only one of the numbers from 0 to  $n-1$ , (both inclusive).

The function  $h_n$  is called hashing function. For this function a set of all keys is domain.

**NOTE:** The key need not be different from the identification number. If the keys are identical with the identification number, then the domain of the hashing function is the set of all identification number.

