Module 1: Set Theory:

- Sets and Subsets,
- Set Operations and the Laws of Set Theory,
- Counting and Venn Diagrams,
- A First Word on Probability,
- Countable and
- Uncountable Sets

Fundamentals of Logic:

- Basic Connectives and Truth Tables,
- Logic Equivalence –The Laws of Logic,
- Logical Implication – Rules of Inference.

Set Theory:

Sets and Subsets:

A set is a collection of objects, called elements of the set. A set can be represented by listing its elements between braces: A = {1, 2, 3, 4, 5}. The symbol e is (or belongs to) a set. For instance 3 e A. Its negation is represented by /e, e.g. 7 /e A. If the set Is finite, its number of elements is represented |A|, e.g. if A = {1, 2, 3, 4, 5} then | A| = 5

1. N = {0, 1, 2, 3, ·· · } = the set of natural numbers.
2. Z = {··· , -3, -2, -1, 0, 1, 2, 3, ··· } = the set of integers.
3. Q = the set of rational numbers.
4. R = the set of real numbers.
5. C = the set of complex numbers.

If S is one of those sets then we also use the following notations :

1. S^+ = set of positive elements in S, for instance

   Z^+ = {1, 2, 3, ··· } = the set of positive integers.

2. S^- = set of negative elements in S, for instance

   Z^- = {-1, -2, -3, ··· } = the set of negative integers.

3. S^* = set of elements in S excluding zero, for instance R^* = the set of non zero real numbers.

Set-builder notation: An alternative way to define a set, called set-builder notation, is by stating a property (predicate) P (x) verified by exactly its elements, for instance A = {x e Z | 1 ≤ x ≤ 5} = "set of integers x such that 1 ≤ x ≤ 5"— i.e.: A = {1, 2, 3, 4, 5}. In general: A = {x e U | p(x)}, where U is the universe of discourse in which the predicate P (x) must be interpreted, or A = {x | P (x)} if the universe of discourse for P (x) is understood. In set the term universal is often used in

implicitly . theory set
place of “universe of discourse” for a given
predicate.

Principle of Extension: Two sets are equal only if if and they have the same elements, i.e.: A = B � ∀x (x e A ↔ x e B) .

Subset: We say that A is a subset of set B, or A is contained in B, and we represent it “A ⊆ B”, if all elements of A are in B, e.g., if A = {a, b, c} and B = {a, b, c, d, e} then A ⊆ B.

Proper subset: A is a proper subset of B, represented “A ⊂ B”, if A ⊆ B A = B, i.e., there is some element in B which is not in A.

Empty Set: A set with no elements is called empty set (or null set, or void set), and is represented by ∅ or {}.

Note that nothing prevents a set from possibly being an element of another set (which is not the same as being a subset!). For i n stance if A = {1, a, {3, t}, { 1, 2, 3}} and B= { 3, t}, then obvious ly B is an element of A, i.e., B e A.

Power Set: The collection of all subsets of a set A is called the power set of A, and is represented P(A). For instance, if A = {1, 2, 3}, then

P(A) = {∅, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, A} .

MultCSEts: Two ordinary sets are identical if they have the same elements, so for instance, {a, a, b} and {a, b} are the same set because they have exactly the same elements, namely a and b. However, in some applications it might be useful to allow repeated elements in a set. In that case we use multCSEts, which are mathematical entities similar to sets, but with possibly repeated elements. So, as multCSEts, {a, a, b} and {a, b} would be consi dered different, since in the first one the element a occurs twice and in the second one it occurs only once.

S et Oper atio ns:

1. Intersection : The common elements of two sets:

   A ∩ B = {x | (x e A) ∧ (x e B)} .

   If A ∩ B = ∅, the sets are said to be disjoint.

2. Union : The set of elements that belong to either of two sets:

   A ∪ B = {x | (x e A) ∨ (x e B)} .

3. Complement : The set of elements (in the universal set) that do not belong to a given set:

A = {x e U | ¬x /e A} .

4. Difference or Relative Complement : The set of elements that belong to a set but not to another:

A - B = {x | (x e A) ∧ (x /e B)} = A ∩ B .

5. Symmetric Difference : Given two sets, their symmetric differ- ence is the set of elements that belong to either one or the other set but not both.

$$A \oplus B = \{x \mid (x \in A) \oplus (x \in B)\} .$$

It can be expressed also in the following way:

$$A \oplus B = A \cup B - A \cap B = (A - B) \cup (B - A) .$$

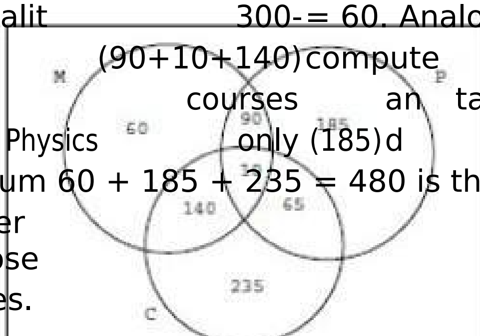
Counting with Venn Diagra ms:

A Venn diagram with n sets intersecting in the most general way divides the plane into 2^n regions. If we have information about the number of elements of some portions of the diagram, then we can find the number of elements in each of the regions and use that information for obtaining the number of elements in other portions of the plane.

Example : Let M , P and C be the sets of students taking Mathematics courses, Physics courses and Computer Science courses respec- tively in a university. Assume |M| = 300, |P | = 350, |C| = 450,
|M � P | = 100, |M � C| = 150, |P � C| = 75, |M � P � C| = 10. How many students are taking exactly one of those courses?

We see that |(M �P)-(M �P �C)| = 100-10 = 90, |(M �C)-(M � P � C)| = 150 - 10 = 140 and |(P � C) - (M � P � C)| = 75 - 10 = 65.

Then the region corresponding to students taking Mathematics courses only has cardinality 300-(90+10+140) = 60. Analogously we compute the number of students taking Physics courses only (185) and taking Computer Science courses only (235). The sum 60 + 185 + 235 = 480 is the number of students taking exactly one of those courses.



Venn Diagrams:

Venn diagrams are graphic representa- tions of sets as enclosed areas in the plane. For instance, in figure 2.1, the rectangle represents the universal set (the set of all elements con- sidered in a given problem) and the shaded region represents a set A. The other figures represent various set operations.

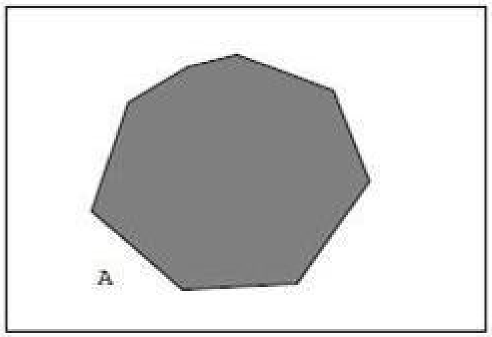


FIGURE 2.1. Venn Diagram.

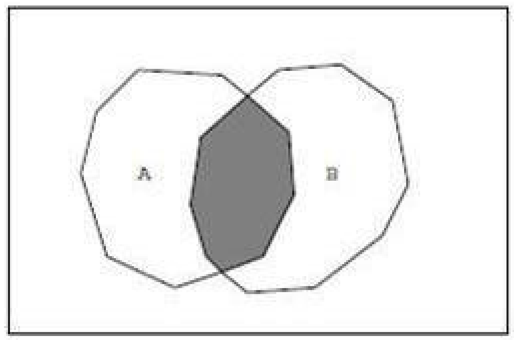


FIGURE 2.2. Intersection $A \cap B$.

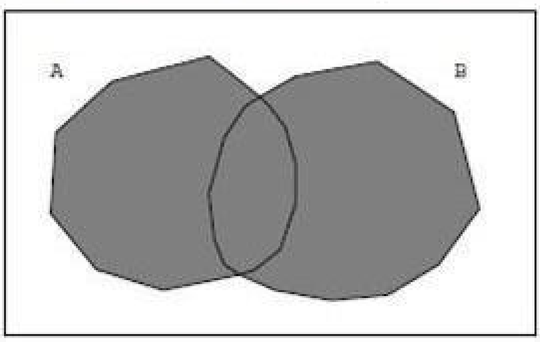


FIGURE 2.3. Union $A \cup B$.

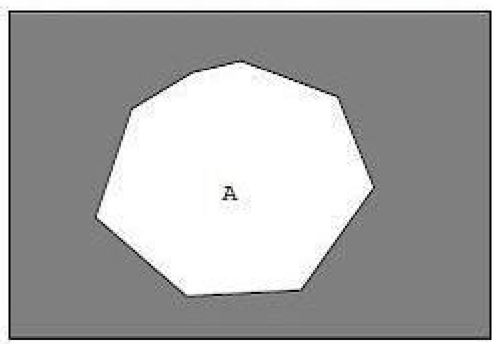


FIGURE 2.4. Complement $\overline{A}$.

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many students are taking exactly one of those courses? (fig. 2.7)

We see that |(M �P)-(M �P �C)| = 100-10 = 90, |(M �C)-(M � P � C)| = 150 - 10 = 140 and |(P � C) - (M � P � C)| = 75 - 10 = 65.

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1. *Associative Laws:*
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$
2. *Commutative Laws:*
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
3. *Distributive Laws:*
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
4. *Identity Laws:*
$$A \cup \emptyset = A$$
$$A \cap \mathfrak{U} = A$$
5. *Complement Laws:*
$$A \cup \overline{A} = \mathfrak{U}$$
$$A \cap \overline{A} = \emptyset$$
6. *Idempotent Laws:*
$$A \cup A = A$$
$$A \cap A = A$$
7. *Bound Laws:*
$$A \cup \mathfrak{U} = \mathfrak{U}$$
$$A \cap \emptyset = \emptyset$$
8. *Absorption Laws:*
$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$
9. *Involution Law:*
$$\overline{\overline{A}} = A$$

Gene ral ized Un ion and Inters ec ti on: Given a collec- tion of sets A_1 , A_2, . . . , A_N , their union is defined as the set of elements that belong to at least one of the sets (here n represents an integer in the range from 1 to N):

Analogously, their intersection is the set of elements that belong to all the sets simultaneously:

$$\bigcap_{n=1}^{N} A_n = A_1 \cap A_2 \cap \cdots \cap A_N = \{x \mid \forall n\, (x \in A_n)\}\,.$$

These definitions can be applied to infinite collections of sets as well. For instance assume that S_m = {kn | k = 2, 3, 4, . . . } = set of multiples of n greater than n. Then

$$\bigcup_{n=2}^{\infty} S_n = S_2 \cup S_3 \cup S_4 \cup \cdots = \{4, 6, 8, 9, 10, 12, 14, 15, \dots\}$$
$$= \text{set of composite positive integers}\,.$$

Partitions: A partition of a set X is a collection S of non overlapping non empty subsets of X whose union is the whole X . For instance a partition of X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} could be S = {{1, 2, 4, 8}, {3, 6}, {5, 7, 9, 10}} . Given a partition S of a set X , every element of X belongs to exactly one member of S.

Example : The division of the integers Z into even and odd numbers is a partition: S = {E, O}, where E = { 2n | n e Z}, O = { 2n + 1 | n e Z}.

Example : The divisions of Z in negative integers, positive integers and zero is a partition: S = {Z^+ , Z^-, {0 }}.

Ordered Pairs, Cartesian Product:

Any ordinary pair {a, b} is a set with two elements. In a set the order of the elements is irrelevant, so {a, b} = {b, a}. If the order of the elements is relevant, then we use a different object called ordered pair, represented (a, b). Now (a, b) = (b, a) (unless a = b). In general (a, b) = ($a^!$, $b^!$) iff a = $a^!$ and b = $b^!$.

Given two sets A, B, their Cartesian product $A \times B$ is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$:

$A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}$.

Analogously we can define triples or 3-tuples (a, b, c), 4-tuples (a, b, c, d),

. . . , n-tuples $(a_1, a_2, \ldots, a_m)$, and the corresponding 3-fold, 4-fold,. . . , n-fold Cartesian products:

$A_1 \times A_2 \times \cdots \times A_m =$

$\{(a_1, a_2, \ldots, a_m) \mid (a_1 \in A_1) \wedge (a_2 \in A_2) \wedge \cdots \wedge (a_m \in A_m)\}$.

$= A \times A \times A$, etc. In

If all the sets in a Cartesian product are the same, then we can use an exponent: $A^2 = A \times A$, A^3

(m times) m

$= A \times A \times \cdots \times A$.

A First Word on Probability:

Introduction: Assume that we perform an experiment such as tossing a coin or rolling a die. The set of possible outcomes is called the sample space of the experiment. An event is a subset of the sample space. For instance, if we toss a coin three times, the sample space is

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

The event "at least two heads in a row" would be the subset

$E = \{HHH, HHT, THH\}$.

If all possible outcomes of an experiment have the same likelihood of occurrence, then the probability of an event $A \subset S$ is given by Laplace's rule:

For instance, the probability of getting at least two heads in a row in the above experiment is 3/8.

Example: Assume that a die is loaded so that the probability of obtaining n point is proportional to n. Find the probability of getting an odd number when rolling that die.

Answer: First we must find the probability function $P(n)$ ($n = 1, 2, \ldots, 6$). We are told that $P(n)$ is proportional to n, hence $P(n) = kn$. Since $P(S) = 1$ we have $P(1)+P(2)+\cdots P(6) = 1$, i.e., $k\cdot 1+k\cdot 2+\cdots+k\cdot 6 = 21k = 1$, so $k = 1/21$ and $P(n) = n/21$. Next we want to find the probability of $E = \{2,4,6\}$, i.e. $P(E) = P(2)+P(4)+P(6) = \frac{2}{21}+\frac{4}{21}+\frac{6}{21} = \boxed{\frac{12}{21}}$.

Then: **Properties of probability:** Let P be a probability func- tion on a sample space S.

1. For every event $E \subseteq S$,
$$0 \leq P(E) \leq 1\,.$$
2. $P(\emptyset) = 0$, $P(S) = 1$.
3. For every event $E \subseteq S$, if $\overline{E} =$ is the complement of E ("not E") then
$$P(\overline{E}) = 1 - P(E)\,.$$
4. If $E_1, E_2 \subseteq S$ are two events, then
$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)\,.$$
In particular, if $E_1 \cap E_2 = \emptyset$ (E_1 and E_2 are *mutually exclusive*, i.e., they cannot happen at the same time) then
$$P(E_1 \cup E_2) = P(E_1) + P(E_2)\,.$$

THE CONCEPT OF PROBALITY:

Pr(A)=|A| / |S| where |A| is an event and |S| is sample space

Pr(A)=|A| / |S|=(|S|-|A|)/|S|= 1- (|A|/|S|)= 1-Pr(A).

Pr(A)=0 if and only if Pr(A)=1 and Pr(A)=1 if and only if Pr(A)=0

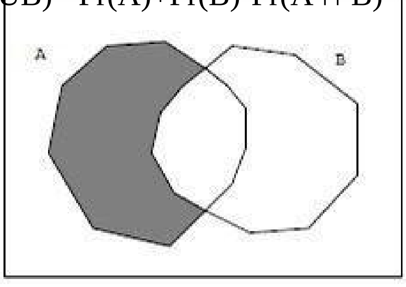
ADDITION THEROM:

Suppose A and B are 2 events is a sample space S then A UB is an event in S consisting of outcomes that are in A or B or both and A ∩ B is an event is S consisting of outcomes thata recommon to A and B. accordingly by the principle of addition we have |AUB|=|A|+|B|-|A ∩B| and formula 1 gives

P r(AUB)=|AUB|/|S|=(|A|+|B|-|A ∩B|)/|S|

= |A|/|S| + |B|/|S| - |A ∩ B| / |S|

P r(AUB) =Pr(A)+Pr(B)-Pr(A ∩ B)



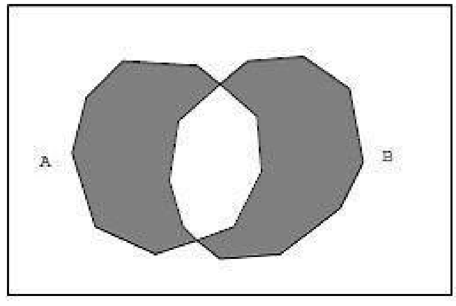


FIGURE 2.5. Difference $A - B$. FIGURE 2.6. Symmetric Difference $A \oplus B$.

MUTUALY EXCLUSIVE EVENTS:

Two events A and B in a sample space are said to be mutual exclusive if $A \cap B = \emptyset$ then $Pr(A \cap B)=0$ and the addition theorem reduces to $P\,r(AUB)= Pr(A)+Pr(B)$

If A1, A2…….An are mutualy exclusive events, then Pr(A1UA2U…….UAn)= Pr(A1)+Pr(A2)+….+Pr(An)

COND ITIONAL PROBABILITY:

If E is an event in a finite sample S with Pr(E)>0 then the probability that an event A in S occurs when E has already occurred is called the probability of A relative to E or the conditional p robability of A , given E

This p robability, denoted by Pr(A|E) is defined by Pr(A|

E)=|A∩ E|/ |E| from this |A∩ E|=|E| . Pr(A|E) Pr(A∩ E)= |

A∩ E|/ S=|=|E|/|S| . Pr(A|E)=Pr(E) . Pr(A|E)

Example : Find the probability of obtaining a sum of 10 after rolling two fair dice. Find the probability of that event if we know that at least one of the dice shows 5 points.

Answer : We call E — "obtaining sum 10" and F — "at least one of the dice shows 5 points". The number of possible outcomes is 6 × 6 — 36. The event "obtaining a sum 10" is E — {(4, 6), (5, 5), (6, 4)}, so|E| — 3. Hence the probability is P (E) — |E|/|S| — 3/36 — 1/12.Now, if we know that at least one of the dice shows 5 points then the sample space shrinks to

F — {(1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 6)} ,

so |F | — 11, and the ways to obtain a sum 10 are E n F — {(5, 5)}, |E n F | — 1, so the probability is P (E | F) — P (E n F)/P (F) — 1/11.

MUTUALLY INDEPENDENT EVENTS:

The event A and E in a sample space S are said to be mutually independent if the probability of the occurrence of A is independent of the probability of the occurrence of E, So that Pr(A)=Pr(A|E). For such events $Pr(A \cap E)=Pr(A).Pr(E)$

This is known as the product rule or the multiplication theorem for mutually independent events .

A gen eralization of expression is if A1,A2,A3………..An are mutually in dependent events in a sample space S then

$Pr(A1\cap A2\cap \ldots\ldots\ldots\ldots\cap An)=Pr(A1).Pr(A2)\ldots\ldots\ldots..Pr(An)$

Example : Assume that the probability that a shooter hits a target is p — 0.7, and that hitting the target in different shots are independent events. Find:

1. The probability that the shooter does not hit the target in one shot.
2. The probability that the shooter does not hit the target three times in a row.

3. The probability that the shooter hits the target at least once after shooting three times.

Answer :

1. P (not hitting the target in one shot) — 1 — 0.7 — 0.3.

2. P (not hitting the target three times in a row) — 0.3^3 — 0.027.
3. P (hitting the target at least once in three shots) — 1—0.027 — 0.973.

COUNTABLE AND UNCOUNTABLE SETS

A set A is said to be the c ountable if A is a finite set. A set which is not countable is called an uncountable set.

THE ADDITION PRINCIPLE:

- |AUB|=|A|+|B|-|A∩ B| is the addition principle rule or the principle of inclusion – exclusion.
- |A-B|=|A|-|A∩ B|
- |A ∩ B|=|U|-|A|-|B| +|A∩ B|
- |AUBUC|=|A|+|B|+|C|-|A ∩B|-|B ∩ C|-|A ∩ C|+|A ∩ B ∩ C| is extended addition principle
- NOTE: |A ∩ B ∩ C|=|AUBUC|
  =|U|-| AUBUC|
  = |U|-|A|-|B| -|C|+|B ∩C|+|A ∩B|+|A ∩C|- |A ∩B ∩C| |
  A-B-C|=|A|-|A ∩ B|-|A ∩ C|+|A ∩ B ∩ C|

Fundamentals of Logic:

Intr oduction:

Propositi ons:

A proposition is a declarative sentence that is either true or false (but not both). For instance, the following are propositions: "Paris is in France" (true), "London is in Denmark" (false), "2 < 4" (true), "4 = 7 (false)". However the following are not propositions: "what is your name?" (this is a question), "do your homework" (this is a command), "this sentence is false" (neither true nor false), "x is an even number" (it depends on what x represents), "So crates" (it is not even a sentence). The truth or falsehood of a proposition is called its truth value.

Basic Connectives and Truth Tables:

Connectives are used for making compound propositions. The main ones are the following (p and q represent given propositions):

Name	Represented	Meaning
Negation	ed	"not p"
Conjunction	¬p	"p and q"
Disjunction	p ∧ q	"p or q (or both)"
Exclusive Or		"either p or q, but not both"
Implication	p ∨q	"if p then q"
Biconditional		"p if and only if q"
	p ⊕ q	

The truth value of a compound proposition depends only on the value of its components. Writing F for "false" and T for "true", we can summarize the meaning of the connectives in the following way:

p	q	¬p	p ∧ q	p ∨ q	p ⊕ q	p →p	↔ q
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

Note that ∨ represents a non-exclusive or, i.e., p ∨ q is true when any of p, q is true and also when both are true. On the other hand ⊕ represents an exclusive or, i.e., p ⊕ q is true only when exactly one of p and q is true.

Tautology, Contradiction, Contingency:

1. A proposition is said to be a tautology if its truth value is T for any assignment of truth values to its compon ents. Example : The proposition p ∨ ¬p is a tautology.

2. A proposition is said to be a contradiction if its truth value is F for any assignment of truth values to its components. Example : The proposition p ∧ ¬p is a c ontradiction.

3. A proposition that is neither a tautology nor a contradiction is called a contingency

Conditional Propositions: A proposition of the form "if p then q" or "p implies q", represented "p → q" is called a conditio nal proposition. For instance: "if John is from Chicago then John is from Illinois". The proposition p is called hypothesis or antecedent, and the proposition q is the conclusion or consequent.

Note that p → q is true always except when p is true and q is false. So, the following sentences are true: “if 2 < 4 then Paris is in France” (true → true), “if London is in Denmark t hen 2 < 4” (false → true),

“if 4 = 7 then London is in Denmark” (false → false). However the following one is false: “if 2 < 4 then London is in Denmark” (true → false).

In might seem strange that “p → q” is considered true when p is false, regardless of the truth value of q. This will become clearer when we study predicates such as “if x is a multiple of 4 then x is a multiple of 2”. That implication is obviously true, although for the particular
case x = 3 it becomes “if 3 is a multiple of 4 then 3 is a multiple of 2”.

The proposition p ↔ q, read “p if and only if q”, is called bicon- ditional. It is true precCSEly when p and q have the same truth value, i.e., they are both true or both false.

~~**Logic al Equival ence:** N~~ote that the compound proposi- tions p → q a nd ¬p ∨ q have the same truth values:

p	q	¬p	¬p ∨ q	p	→
T	T	F	T	T	
T	F	F	F	F	
F	T	T	T	T	
F	F	T	T	T	

When two compound propositions have the same truth values no matter what truth value their constituent propositions have, they are called logically equivalent. For

inst an ce p → q and ¬p ∨ q are logically equivalent, and we write it:

p → q ≡ ¬p ∨q

Note that that two propositions A and B are logically equivalent precCSEly when A ↔ B is a tautology.

Example : De Morgan’s Laws for Logic. The following propositions are logically equivalent:

¬(p ∨ q) ≡ ¬p ∧ ¬q
¬(p ∧ q) ≡ ¬p ∨ ¬q

p	q	¬p	¬q	p ∨ q	¬(p ∨ q)	¬p ∧ ¬q	p ∧ q	¬(p ∧ q)	¬p ∨ ¬q	
T	T	F	F	T	F	F	T	F	F	
T	F	F	T	T	F	F	F	T	T	
F	T	T	F	T	F	F	F	T	T	
F	F	T	T	F	T	T	F	T	T	

Example : The following propositions are logically e quivalent:

p ↔ q ≡ (p → q) ∧ (q → p)

Again, this can be checked with the truth tables:

p	q	p	→ q	→ (p	→ q)∧(qp ↔ q	
T	T	T	T	T	T	
T	F	F	T	F	F	
F	T	T	F	F	F	
F	F	T	T	T	T	

ExercCSE : Check the following logical equivalences:

¬(p → q) ≡ p ∧ ¬q
p → q ≡ ¬q → ¬p

¬(p ↔ q) ≡ p ⊕ q

Converse, C ontrapo sitive: The converse of a conditional proposition p → q is the proposition q → p. As we have seen, the bi- conditional proposition is equivalent to the co njunction of a con ditional propo sition an its converse.

p ↔ q ≡ (p → q) ∧ (q → p)

So, for instance, saying that "John is married if and only if he has a spouse" is the same as saying “if John is married then he has a spouse” and “if he has a spouse then he is married”.

Note that the converse is not equivalent to the given conditional proposition, for instance “if John is from Chicago then John is from Illinois” is true, but the converse “if John is from Illinois then John is from Chicago” may be false.
The contrapositive of a conditional proposition $p \rightarrow q$ is the propo- sition $\neg q \rightarrow \neg p$. They are logically equivalent. For instance the con- trapositive of “if John is from Chicago then John is from Illinois” is “if John is not from Illinois then John is not from Chicago”.

LOGICAL CONNECTIVES: New propositions are obtained with the aid of word or phrases like “not”,”and”,”if….then”,and “if and only if”. Such words or phrases are called logical connectives. The new propositions obtained by the use of connectives are called compound propositions. The original propositions from which a compound proposition is obtained are called the components or the primitives of the compound proposition. Propositions which do not contain any logical connective are called simple propositions

NE GATION: A Proposition obtained by inserting the word “not” at an appropriate place in a given proposition is called the negation of the given proposition. The negation of a proposition p is denoted by ~p(read “not p”)
Ex: p: 3 is a prime number
~p: 3 is not a prime number
Truth Table:

p	~p
0	1
1	0

CONJUNCTION:

A compound proposition obtained by combining two given propositions by inserting the word “and” in between them is called the conjunction of the given proposition.The conjunction of two proposition p and q is denoted by p^q(read “p and q”).

- The conjunction p^q is true only when p is true and q is true; in all other cases it is false.
- Ex: p:√2 is an irational number q: 9 is a prime number
  p^q: √2 is an i rational number and 9 is a prime number
- Truth table:

p	q	p^q
0	0	0
0	1	0
1	0	0
1	1	1

DISJUNCTION:

A compound proposition obtained by combining two given propositions by inserting the word “or” in between them is called the disjunction of the given proposition.The di sjunction of two proposition p and q is denoted by p�q(read “p or q”).

- The di sjunction p�q is false only when p is false and q is false ; in all other cases it is true.
- Ex: p:√2 is an irational number q: 9 is a prime number

  p�q : √2 is an irational number or 9 is a prime number Truth table:

p	q	p�q
0	0	0
0	1	1
1	0	1
1	1	1

EXCLUSIVE DISJUNCTION:

- The compound proposition “p or q” to be true only when either p is true or q is true but not both. The exclusive or is denoted by symbol v. __

- Ex: p:√2 is an ir rational number q: 2+3=5

Pvq: Either √2 is an i rrational number or 2+3=5 but not both.

- Truth Table:

p	q	pvq
0	0	0
0	1	1
1	0	1
1	1	0

COND ITIONAL(or IMP LICATION):

- A compound proposition obtained by combining two given propositions by using the words “if” and “then” at appropriate places is called a conditional or an implication..

Given two propositions p and q, we can form the conditionals “if p, then q” and “if q, then p:. The conditional “if p, then q”is denoted by p→q and the conditional “if q, then p” is denoted by q →p.

- The conditional p→q is false only when p is true and q is false ;in all other cases it

is true.

- Ex: p: 2 is a prime number q: 3 is a prime number

p→q: If 2 is a prime number then 3 is a prime number; it is true

- Truth Table:

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

BICONDITIONAL:

- Let p and q be two propositions,then the conjunction of the conditionals p→q and q→p is called bi- conditional of p and q. It is denoted by p↔q.

- p↔q is same as (p→q)∧(q→p). As such p↔q is read as " if p then q and if q then p".

- Ex: p: 2 is a prime number q: 3 is a prime number p↔q are true.

Truth Table:

p	q	p→q	q→p	p↔q
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

COMBINED TRUTH TABLE

P	q	~p	p^q	p∨q	p⊻q	p→q	p↔q
0	0	1	0	0	0	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	1	0	0

1 1 0 1 1 0 1 1

TAUTOLOGIES; CONTRADICTIONS:

A compound proposition which is always true regardless of the truth values of its components is called a tautology.

A compound proposition which is always false regardless of the truth values of its components is called a cont radiction or an absurdity.

A compound proposition that can be true or false (depending upon the truth values of its components) is called a contingency I.e contingency is a compound proposition which is neither a tautology nor a contradiction.

LOGICAL EQUIVALENCE

- Two propositions ‘u’ and ‘v’ are said to be logically equivalent whenever u and v have the same truth value, or equivalently .

- Then we write u�v. Here the symbol �stands for “logically equivalent to”.

- When the propositions u and v are not logically eq uivalent we write u�v.

LAWS OF LOGIC:

To denote a tautology and To denotes a contradiction.

- Law of Double negation: For any proposition p,(~~p)�p
- Idempotent laws: For any propositions p, 1) (p�p) �p 2) (p�p) �p
- Identity laws: For any proposition p, 1)(p�Fo) �p 2)(p�To) �p
- Inverse laws: For any proposition p, 1) (p � �p) �To 2)(p�~p)�Fo
- Commutative Laws: For any proposition p and q, 1)(p�q)�(q�p) 2)(p�q)�(q�p)
- Domination Laws: For any proposition p, 1) (p�To) �To 2) (p�Fo) �Fo
- Absorption Laws: For any proposition p and q,1) [p� (p�q)] �p 2)[p� (p�q)] �p
- De-Morgan Laws: For any proposition p and q, 1)~ (p�q)��p��q

Associative Laws : For any proposition p ,q and r, 1) p � (q � r) �(p �q) �r 2)

Distributive Laws: For any proposition p ,q and r, 1) p � (q � r) � (p �q) � (p �r)

2) p�(q�r)� (p�q) � (p�r)

- Law for the negation of a conditional : Given a conditional p→q, its negation is obtained by using the following law: �(p→q)�[p�(�q)]

TRANSITIVE AND SUBSTITUTION RULES If u,v,w are propositions such that u�v and v �w, then u �w. (this is transitive rule)

- Suppose that a compound proposition u is a tautology and p is a component of u, we replace each occurrence of p in u by a proposition q, then the resulting compound proposition v is also a tautology(This is called a substitution rule).

- Suppose that u is a compound proposition which contains a proposition p. Let q be a proposition such that q �p , suppose we replace one or more occurrences of p by q and obtain a compound proposition v. Then u �v (This is also substitution rule)

APPLICATION TO SWITCHING NETWORKS

- If a switch p is open, we assign the symbol o to it and if p is closed we assign the symbol 1 to it.

- Ex: current flows from the terminal A to the terminal B if the switch is closed i.e if p is assigned the symbol 1. This network is represented by the s ymbol p

A P B

Ex: parallel network consists of 2 switches p and q in which the current flows from the terminal A to the terminal B , if p or q or both are closed i.e if p or q (or both) are assigned the symbol 1. This network is represent by p�q

Ex: Series network consists of 2 switches p and q in which the current flows from the terminal A to the terminal B if both of p and q are closed; that is if both p and q are assigned the symbol 1. This network is repr esented by p�q

DUALITY:

Suppose u is a compound proposition that contains the connectives � and �. Suppose we replace each occurrence of � and � in u by � and � re spectively.

Also if u contains To and Fo as components, suppose we replace each occurrence of To and Fo by Fo and To respectively, then the resulting compound proposition is called the dual of u and is denoted by u^d.

Ex: u: p �(q � �r) � (s � To) u^d: p � (q � � r) � (s � Fo)

NOTE:

- $(u^d)^d$ �u. The dual of the dual of u is logically equ ivalent to u.
- For any two propositions u and v if u �v, then u^d �v^d . This is known as the pr inciple of duality.

The connectives NAND and NOR

(p ↑q)= �(p � q) � � p � � q

(p ↓q)= �(p � q) � � p � � q

CONVERSE,INVERSE AND CONTRAPOSITIVE

Consider a conditional (p→q) , Then :

1) q→p is called the converse of p→q

2) �p→�q is called the inverse of p→q

3) �q→�p is called the cont rapositive of p→q

RULES OF INFERENCE:

There exist rules of logic which can be employed for establishing the validity of a rguments . These rules are called the Rules of Inference.

1) Rule of conjunctive simpli fication: This rule states that for any two propositions p

and q if p�q is true, then p is true i.e (p�q)�p.

2) Rule of Disjunctive amplification: This rule states that for any two proposition p and q if p is true then p�q is true i.e p�(p �q)

3) 3) Rule of Syllogism: This rule states that for any three propositions p,q r if p→q is true and q→r is true then p→r is true. i.e {(p→q)�(q→)} �(p →r) In tabular form:

p→q q→r � (p →r)

4) 4) Modus pones(Rule of Detachment): This rule states that if p is true and p→q is true, then q is true, ie {p �(p→q)} �q. Tabular form

p p→q � q

5) Modus Tollens: This rule states that if p→q is true and q is false, then p is false. {(p→q)��q}�q Tabular form: p→q

�q ��p

6) Rule of Disjunctive Syllogism: This rule states that if p�q is true and p is false, then q is true i.e. {(p�q)��p}�q Tabular Form p�q

�p � q

QUANTIFIERS:

1. The words “ALL”,”EVERY”,”SOME”,”THERE EXISTS” are called quantifiers in the proposition

2. The symbol � is used to denote the phrases “FOR ALL”,”FOR EVE RY”,”FOR EACH” and “FOR ANY”.this is called as universal quantifier.

3. � is used to denote the phrases “FOR SOME”and “THERE EXISTS”and “FOR ATLEAST ONE”.this s ymbol is called existential quantifier.

A proposition involving the universal or the existential quantifier is called a quantified statement

LOGICAL EQUIVALENCE:

1. � x,[p(x)�q(x)]�(�x p(x))�(�x,q(x))

2. � x, [p(x)�q(x)]�(�x p(x)) �(�x,q(x))

3. � x, [p(x)→q(x)] ��x ,[�p(x)�q(x)]

<u>RULE FOR NEGATION OF A QUANTIFIED STATEMENT</u>:

�{�x,p(x)}≡�x{�p(x)} �{�x,p(x)}≡�x{�p(x)}

<u>RULES OF INTERFERENCE:</u>

1. RULE OF UNIVERSAL SPECIFICATION

2. RULE OF UNIVERSAL GENERALIZATION

If an open statement p(x) is proved to be true for any (arbitrary)x chosen from a set S, then the quantified statement �x€S, p(x) is true.

<u>**ME THODS OF PROOF AND DIS PROOF:**</u>

1.DIRECT PROOF:

The direct method of proving a conditional p→q has the following lines of argument:

a) hypothesis : First assume that p is true

b) Analysis: starting with the hypothesis and employing the rules /laws of logic and other known facts , infer that q is true

c) Conclusion:p→q is true.

2. INDIRECT PROOF:

Condition p→q and its contrapositive �q→�p are logically equivalent. On basis of this proof, we infer that the conditional p→q is true. This method of proving a conditional is called an indirect method of proof.

3 .PROOF BY CONTRADICTION

The indirect method of proof is equivalent to what is known as the proof by contradiction.
The lines of argument in this method of proof of the statement $p \to q$ are as follows:

1) Hypothesis: Assume that $p \to q$ is false i.e assume that p is true and q is false.

2)Analysis: starting with the hypothesis that q is false and employing the rules of logic and other known facts , infer that p is false. This contradicts the assumption that p is true

3)Conculsion: because of the contradiction arrived in the analysis , we infer that $p \to q$ is true

4 .PROOF BY E XHAUSTION:

"�x €*S*,p(x)" is true if p(x)is true for every (each) x in S.If S consists of only a limited number of elements , we can prove that the statement "�x €*S*,p(x)" is true by considering p(a) for each a in S and verifying that p(a) is true .such a method of prove is called method of exhaustion.

5 .PROOF OF EXISTENCE:

"�x €*S*,p(x)" is true if any one element a € S such that p(a) is true is exhibited. Hence , the best way of proving a proposition of the form "�x €*S*,p(x)" is to exhibit the existence of one a€S such that p(a) is true. This method of proof is called proof of existence.

6.DI SPROOF BY CONTRADICTION :

Suppose we wish to disprove a conditional $p \to q$. for this propose we start with the hypothesis that p is true and q is true, and end up with a contradiction. In view of the contradiction , we conclude that the conditional $p \to q$ is false.this method of disproving $p \to q$ is called DISPROOF BY CONTRADICTION