

Generalized G-inflation: Inflation with the most general second-order field equations

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We study generalized Galileons as a framework to develop the most general single-field inflation models ever, Generalized G-inflation, containing yet further generalization of G-inflation, as well as previous examples such as k-inflation, extended inflation, and new Higgs inflation as special cases. We investigate the background and perturbation evolution in this model, calculating the most general quadratic actions for tensor and scalar cosmological perturbations to give the stability criteria and the power spectra of primordial fluctuations. It is pointed out in the Appendix that the Horndeski theory and the generalized Galileons are equivalent. In particular, even the non-minimal coupling to the Gauss-Bonnet term is included in the generalized Galileons in a non-trivial manner.

§1. Introduction

Scalar fields play important roles in cosmology. On the one hand, inflation in the early Universe is now becoming a part of standard cosmology that is driven by a scalar field called the inflaton.¹⁾⁻³⁾ The conventional inflaton action consists of a canonical kinetic term and a sufficiently flat potential.³⁾ [See Ref. 4) for the latest review.] Non-canonical kinetic terms⁵⁾ also arise naturally in some particle physics models of inflation such as Dirac-Born-Infeld inflation.⁶⁾

On the other hand, it is strongly suggested that the present Universe is dominated by mysterious dark energy, and its identity might be a dynamical scalar field.⁷⁾ In relation to the present accelerated expansion, modified gravity theories have been studied extensively, and in such theories, an extra gravitational degree of freedom can often be equivalently described by a scalar field coupled non-minimally to gravity or matter. In the decoupling limit of the Dvali-Gabadadze-Porrati brane model,⁸⁾ the scalar field has a non-linear derivative self-interaction,⁹⁾ which was later generalized to Galileons¹⁰⁾ with a number of applications to various contexts in cosmology.¹¹⁾⁻²³⁾ Thus, in recent years, there have been growing interests in scalar field theories beyond the canonical one.

The most attractive feature of higher derivative theories possessing the Galilean invariance $\partial_\mu\phi \rightarrow \partial_\mu\phi + b_\mu$ is that field equations derived from such a theory contain derivatives only up to second order,¹⁰⁾ so that it can easily avoid ghosts. Unfortunately, however, this desired feature ceases to exist once the background spacetime

is curved.²⁴⁾

To preserve the second-order nature of field equations, the “covariantization” of the Galileon has been proposed by Deffayet et al.²⁴⁾ where the theory is no longer Galilean invariant. This line of analysis has been further pursued recently to yield a more generic class of higher-derivative theories that result in second-order field equations.²⁵⁾ In this theory, the Galilean invariance is absent even in the flat spacetime limit.

The purpose of this paper is to provide a comprehensive and thorough study of the most general non-canonical and non-minimally coupled single-field inflation models yielding second-order field equations making use of Ref. 25), which is the most general extension of the Galileons but is no longer based on a symmetry argument. It would be nice if one could develop a new class of viable inflation models fully respecting the Galileon symmetry, $\phi \rightarrow \phi + b_\mu x^\mu + c$, as was attempted in Refs. 21), 22), but such a symmetry must be actually broken to construct a phenomenologically viable inflation model, namely, to terminate inflation and reheat the Universe. Thus, our strategy here is more similar to the G-inflation model¹⁴⁾ and is indeed the most general extension of it. Hence, one may call the model presented here as *Generalized G-inflation* or *G²-inflation*.

Special cases of the generalized Galileons can be derived from a relativistic probe brane embedded in a five-dimensional bulk,^{26)–29)} and hence, they are possibly related to fundamental theory and particle physics. The scalar field theories we are going to study thus include not only all the previous examples considered in the context of single-field inflation, but also recent developments and their further generalization. We clarify the generic behavior of the inflationary background and investigate the nature of primordial tensor and scalar perturbations at linear order. Given a specific model, our formulas are helpful to determine the evolution of cosmological perturbations and its observational consequences.

This paper is organized as follows. In the next section, we define the scalar-field theories that we consider. We then provide the background cosmological equations in §3, and explore two possible inflationary mechanisms. In §4, cosmological perturbations are considered and the quadratic actions for tensor and scalar perturbations are computed, which are used to give stability conditions and evaluate the primordial power spectra. Our conclusion is drawn in §5.

§2. Generalized Higher-order Galileons and Kinetic Gravity Braiding

Galileons¹⁰⁾ and their covariant extension²⁴⁾ have been further generalized recently to yield the most general scalar field theories having second-order field equations.²⁵⁾ The first two terms of the generalized Lagrangian corresponding to $(\partial\phi)^2$ and $(\partial\phi)^2\Box\phi$ in the original theory are given by^{13),14)}

$$\mathcal{L}_2 = K(\phi, X), \quad (2.1)$$

$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \quad (2.2)$$

where K and G_3 are generic functions of ϕ and $X := -\partial_\mu\phi\partial^\mu\phi/2$. Similarly, higher-order Galileons can be generalized to give²⁵⁾

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4X} \left[(\square\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2 \right], \quad (2.3)$$

$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6} \left[(\square\phi)^3 - 3(\square\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3 \right], \quad (2.4)$$

where R is the Ricci tensor, $G_{\mu\nu}$ is the Einstein tensor, $(\nabla_\mu\nabla_\nu\phi)^2 = \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi$, $(\nabla_\mu\nabla_\nu\phi)^3 = \nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\lambda\phi\nabla_\lambda\nabla^\mu\phi$, and $G_{iX} = \partial G_i/\partial X$. Setting $G_3 = X$, $G_4 = X^2$, and $G_5 = X^2$, the above Lagrangians reproduce the covariant Galileons introduced in Ref. 24). The non-minimal couplings to gravity in \mathcal{L}_4 and \mathcal{L}_5 are necessary to eliminate higher derivatives that would otherwise appear in the field equations. Note that we do not need a separate gravitational Lagrangian other than \mathcal{L}_4 ; for $G_4 = M_{\text{Pl}}^2/2$, \mathcal{L}_4 reduces to the Einstein-Hilbert term. We also obtain a non-minimal coupling of the form $f(\phi)R$ from \mathcal{L}_4 by taking $G_4 = f(\phi)$. The non-standard kinetic term $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ that is considered, such as in Ref. 30), turns out to be a special case $G_5 \propto \phi$ of \mathcal{L}_5 after integration by parts. Equation (24) of Ref. 31), which is obtained from a Kaluza-Klein compactification of higher-dimensional Lovelock gravity, turns out to be equivalent to \mathcal{L}_5 with $G_5 = -3X/2$.

We thus consider a gravity + scalar system described by the action

$$S = \sum_{i=2}^5 \int d^4x \sqrt{-g} \mathcal{L}_i, \quad (2.5)$$

which is the most general single scalar theory resulting in equations of motion containing derivatives up to second order. This action contains only four independent arbitrary functions of ϕ and X . This theory represents a general class of single-field inflation, including models that have not been studied so far, as well as almost all the previously known models such as potential-driven slow-roll inflation,³⁾ k-inflation,⁵⁾ extended inflation,³²⁾ and even new Higgs inflation³⁰⁾ as special cases.*)

Note in passing that

$$\begin{aligned} XR + (\square\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2 &= G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi + \text{total derivative} \\ &= -\phi G_{\mu\nu}\nabla^\mu\nabla^\nu\phi + \text{total derivative}, \end{aligned}$$

which implies that \mathcal{L}_4 with $G_4 = X$ and \mathcal{L}_5 with $G_5 = -\phi$ are equivalent. Similarly, \mathcal{L}_3 with $G_3 = f(\phi)$ and \mathcal{L}_2 with $K = -2Xf_\phi$ coincide up to total derivative. These facts can be used to check the calculations.

§3. Background equations

Let us derive the equations of motion describing the background evolution from (2.5). The easiest way is to substitute $\phi = \phi(t)$ and the metric $ds^2 = -N^2(t)dt^2 +$

*⁾ Even the curvature-square inflation model as well as more general $f(R)$ inflation,^{2),33)} which do not contain any scalar field and result in fourth-order equations of motion, can be recast in the present form by defining a new field as $\phi = df/dR$.

$a^2(t)d\mathbf{x}^2$ to the action. Variation with respect to $N(t)$ gives the constraint equation corresponding to the Friedmann equation, which can be written as

$$\sum_{i=2}^5 \mathcal{E}_i = 0, \quad (3.1)$$

where

$$\mathcal{E}_2 = 2XK_X - K, \quad (3.2)$$

$$\mathcal{E}_3 = 6X\dot{\phi}HG_{3X} - 2XG_{3\phi}, \quad (3.3)$$

$$\mathcal{E}_4 = -6H^2G_4 + 24H^2X(G_{4X} + XG_{4XX}) - 12HX\dot{\phi}G_{4\phi X} - 6H\dot{\phi}G_{4\phi}, \quad (3.4)$$

$$\mathcal{E}_5 = 2H^3X\dot{\phi}(5G_{5X} + 2XG_{5XX}) - 6H^2X(3G_{5\phi} + 2XG_{5\phi X}). \quad (3.5)$$

The above quantities contain derivatives of the metric and the scalar field up to first order.

Variation with respect to $a(t)$ yields the evolution equation,

$$\sum_{i=2}^5 \mathcal{P}_i = 0, \quad (3.6)$$

where

$$\mathcal{P}_2 = K, \quad (3.7)$$

$$\mathcal{P}_3 = -2X\left(G_{3\phi} + \ddot{\phi}G_{3X}\right), \quad (3.8)$$

$$\begin{aligned} \mathcal{P}_4 = & 2\left(3H^2 + 2\dot{H}\right)G_4 - 12H^2XG_{4X} - 4H\dot{X}G_{4X} - 8\dot{H}XG_{4X} - 8HX\dot{X}G_{4XX} \\ & + 2\left(\ddot{\phi} + 2H\dot{\phi}\right)G_{4\phi} + 4XG_{4\phi\phi} + 4X\left(\ddot{\phi} - 2H\dot{\phi}\right)G_{4\phi X}, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \mathcal{P}_5 = & -2X\left(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi}\right)G_{5X} - 4H^2X^2\ddot{\phi}G_{5XX} \\ & + 4HX\left(\dot{X} - HX\right)G_{5\phi X} + 2[2(HX)\dot{+} 3H^2X]G_{5\phi} + 4HX\dot{\phi}G_{5\phi\phi}. \end{aligned} \quad (3.10)$$

The background quantities \mathcal{E}_i and \mathcal{P}_i are defined in an analogous way in which the energy density and the isotropic pressure of a usual scalar field are defined. In the present case, however, the distinction between the gravitational and scalar-field portions of the Lagrangian is ambiguous, and hence, in that sense, the gravitational contribution is included in the above expressions. Indeed, one can see that, for $G_4 = M_{\text{Pl}}^2/2$, \mathcal{E}_4 and \mathcal{P}_4 reduce to the “minus” of the Einstein tensor G_μ^ν : $\mathcal{E}_4 = -3M_{\text{Pl}}^2H^2$ and $\mathcal{P}_4 = M_{\text{Pl}}^2(3H^2 + 2\dot{H})$.

Variation with respect to $\phi(t)$ gives the scalar-field equation of motion,

$$\frac{1}{a^3} \frac{d}{dt} (a^3 J) = P_\phi, \quad (3.11)$$

where

$$\begin{aligned} J = & \dot{\phi}K_X + 6HXG_{3X} - 2\dot{\phi}G_{3\phi} + 6H^2\dot{\phi}(G_{4X} + 2XG_{4XX}) - 12HXG_{4\phi X} \\ & + 2H^3X(3G_{5X} + 2XG_{5XX}) - 6H^2\dot{\phi}(G_{5\phi} + XG_{5\phi X}), \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} P_\phi = & K_\phi - 2X \left(G_{3\phi\phi} + \ddot{\phi}G_{3\phi X} \right) + 6 \left(2H^2 + \dot{H} \right) G_{4\phi} + 6H \left(\dot{X} + 2HX \right) G_{4\phi X} \\ & - 6H^2 X G_{5\phi\phi} + 2H^3 X \dot{\phi} G_{5\phi X}. \end{aligned} \quad (3.13)$$

Not all of the equations (3.1), (3.6), and (3.11) are mutually independent: Eq. (3.11) [(3.6)] can be derived from Eqs. (3.6) [(3.11)] and (3.1).

Below, we present two extreme cases of inflation, one driven purely kinetically, which is an extension of the original G-inflation¹⁴⁾ corresponding to the case with $G_4 = M_{\text{Pl}}^2/2$ and $G_5 = 0$, the other driven by a scalar potential as an extension of Higgs G-inflation.¹⁵⁾

3.1. Kinetically driven G-inflation

Let us start with a shift-symmetric model, $\phi \rightarrow \phi + c$. This in particular implies that ϕ does not have any potential. In this case, the field equations are

$$\sum_{i=2}^5 \mathcal{E}_i = \dot{\phi}J - K - 6H^2(G_4 - 2XG_{4X}) + 4H^3X\dot{\phi}G_{5X} = 0, \quad (3.14)$$

$$\begin{aligned} \sum_{i=2}^5 (\mathcal{E}_i + \mathcal{P}_i) = & \dot{\phi}J - 2X\ddot{\phi}G_{3X} + 2\frac{d}{dt} \left[2H(G_4 - 2XG_{4X}) - H^2X\dot{\phi}G_{5X} \right] \\ = & 0, \end{aligned} \quad (3.15)$$

$$\frac{d}{dt}(a^3J) = 0. \quad (3.16)$$

From Eq. (3.16), one immediately finds that $J \propto a^{-3} \rightarrow 0$. Shift-symmetric models thus have an attractor, $J = 0$, along which $H = \text{const}$, $\dot{\phi} = \text{const}$, satisfying

$$\begin{aligned} \dot{\phi}K_X + 6HXG_{3X} + 6H^2\dot{\phi}(G_{4X} + 2XG_{4XX}) \\ + 2H^3X(3G_{5X} + 2XG_{5XX}) = 0, \end{aligned} \quad (3.17)$$

$$K + 6H^2(G_4 - 2XG_{4X}) - 4H^3X\dot{\phi}G_{5X} = 0. \quad (3.18)$$

Provided that Eqs. (3.17) and (3.18) have a non-trivial root, $H \neq 0$, $\dot{\phi} \neq 0$, we obtain inflation driven by ϕ 's kinetic energy. This is the generalization of kinetically driven G-inflation¹⁴⁾ and kinetic gravity braiding.¹³⁾

Although the shift-symmetric Lagrangian can nicely accommodate a de Sitter solution as an attractor, the shift symmetry must be broken in some region in the field space to end inflation and to reheat the Universe. Following the arguments in Refs.,^{5),14)} a graceful exit from kinetically driven inflation is possible with gravitational reheating.³⁴⁾ However, a detailed analysis of the reheating stage after kinetically driven inflation is beyond the scope of the present paper.

3.2. Potential-driven slow-roll G-inflation

Suppose that the functions in the Lagrangian can be expanded in terms of X as

$$K(\phi, X) = -V(\phi) + \mathcal{K}(\phi)X + \dots, \quad (3.19)$$

$$G_i(\phi, X) = g_i(\phi) + h_i(\phi)X + \dots, \quad (3.20)$$

and consider the case in which the inflaton field value $\phi(t)$ changes very slowly. In this case, the potential term manifestly breaks the shift symmetry, and thereby, the model is capable of a graceful exit from inflation.³⁵⁾ Neglecting all the terms multiplied by $\dot{\phi}$ in the gravitational field equations, we obtain

$$\sum_{i=2}^5 \mathcal{P}_i \simeq - \sum_{i=2}^5 \mathcal{E}_i \simeq -V(\phi) + 6g_4(\phi)H^2, \quad (3\cdot21)$$

where we have assumed

$$|\dot{H}| \ll H^2 \text{ and } |\ddot{\phi}| \ll |H\dot{\phi}|. \quad (3\cdot22)$$

We may thus have slow-roll inflation with

$$H^2 \simeq \frac{V}{6g_4}. \quad (3\cdot23)$$

During slow-roll, we approximate

$$|\dot{J}| \ll |HJ|, \quad |\dot{g}_i| \ll |Hg_i|, \quad |\dot{h}_i| \ll |Hh_i|. \quad (3\cdot24)$$

Under the above approximation, we have the slow-roll equation of motion for ϕ ,

$$3HJ \simeq -V_\phi + 12H^2g_{4\phi}, \quad (3\cdot25)$$

with

$$J \simeq \mathcal{K}\dot{\phi} - 2g_{3\phi}\dot{\phi} + 6 \left(Hh_3X + H^2h_4\dot{\phi} - H^2g_{5\phi}\dot{\phi} + H^3h_5X \right). \quad (3\cdot26)$$

Which term is dominant in Eq. (3·26) depends on the magnitude of the coefficients $h_i(\phi)$ of X . Note here that we can set $g_3 = 0$ and $g_5 = 0$ without loss of generality, because $g_{3\phi}$ can be absorbed into the redefinition of \mathcal{K} and $g_{5\phi}$ into h_4 , that is, $\mathcal{K} - 2g_{3\phi} \rightarrow \mathcal{K}$, $h_4 - g_{5\phi} \rightarrow h_4$.

Equations (3·23) and (3·26) imply that the dominant contribution to the inflationary Hubble parameter is the potential V in \mathcal{L}_2 , while any of the terms in Eq. (3·26) can participate to determine the actual dynamics of the scalar field. Therefore, the slow-roll parameters expressed in terms of the potential may look very different from the standard ones in general. This is the generalization of the Higgs G-inflation¹⁵⁾ (see also Ref. 36)).

§4. Quadratic actions for tensor and scalar perturbations

In this section, our goal is to compute quadratic actions for tensor and scalar cosmological perturbations in Generalized G-inflation. We use the unitary gauge in which $\phi = \phi(t)$ and begin with writing the perturbed metric as

$$ds^2 = -N^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (4\cdot1)$$

where

$$N = 1 + \alpha, \quad N_i = \partial_i \beta, \quad \gamma_{ij} = a^2(t) e^{2\zeta} \left(\delta_{ij} + h_{ij} + \frac{1}{2} h_{ik} h_{kj} \right). \quad (4.2)$$

Here, α , β , and ζ are scalar perturbations and h_{ij} is a tensor perturbation satisfying $h_{ii} = 0 = h_{ij,j}$. With the above definition of the perturbed metric, $\sqrt{-g}$ does not contain h_{ij} up to second order, and the coefficients of ζ^2 and $\alpha\zeta$ vanish, thanks to the background equations.

4.1. Tensor perturbations

The quadratic action for the tensor perturbations is found to be

$$S_T^{(2)} = \frac{1}{8} \int dt d^3x a^3 \left[\mathcal{G}_T \dot{h}_{ij}^2 - \frac{\mathcal{F}_T}{a^2} (\vec{\nabla} h_{ij})^2 \right], \quad (4.3)$$

where

$$\mathcal{F}_T := 2 \left[G_4 - X \left(\ddot{\phi} G_{5X} + G_{5\phi} \right) \right], \quad (4.4)$$

$$\mathcal{G}_T := 2 \left[G_4 - 2X G_{4X} - X \left(H \dot{\phi} G_{5X} - G_{5\phi} \right) \right]. \quad (4.5)$$

One may notice that \mathcal{G}_T can also be expressed as

$$\mathcal{G}_T = \frac{1}{2} \sum_{i=2}^5 \frac{\partial \mathcal{P}_i}{\partial \dot{H}}. \quad (4.6)$$

The squared sound speed is given by

$$c_T^2 = \frac{\mathcal{F}_T}{\mathcal{G}_T}. \quad (4.7)$$

One sees from the action (4.3) that ghost and gradient instabilities are avoided provided that

$$\mathcal{F}_T > 0, \quad \mathcal{G}_T > 0. \quad (4.8)$$

Note that c_T^2 is not necessarily unity in general cases, contrary to the standard inflation models.

To canonically normalize the tensor perturbation, we define

$$dy_T := \frac{c_T}{a} dt, \quad z_T := \frac{a}{2} (\mathcal{F}_T \mathcal{G}_T)^{1/4}, \quad v_{ij} := z_T h_{ij}, \quad (4.9)$$

and then the quadratic action is written as

$$S_T^{(2)} = \frac{1}{2} \int dy_T d^3x \left[(v'_{ij})^2 - (\vec{\nabla} v_{ij})^2 + \frac{z''_T}{z_T} v_{ij}^2 \right], \quad (4.10)$$

where a prime denotes differentiation with respect to y_T . In terms of the Fourier wavenumber k , sound horizon crossing occurs when $k^2 = z''_T/z_T \sim 1/y_T^2$ for each mode.

On superhorizon scales, the two independent solutions to the perturbation equation that follows from the action (4.10) are

$$v_{ij} \propto z_T \quad \text{and} \quad z_T \int \frac{dy_T}{z_T^2}. \quad (4.11)$$

In terms of the original variables, the two independent solutions on superhorizon scales are given by

$$h_{ij} = \text{const} \quad \text{and} \quad \int^t \frac{dt'}{a^3 \mathcal{G}_T}. \quad (4.12)$$

The second solution corresponds to a decaying mode.

To evaluate the primordial power spectrum, let us assume that $\epsilon := -\dot{H}/H^2 \simeq \text{const}$,

$$f_T := \frac{\dot{\mathcal{F}}_T}{H\mathcal{F}_T} \simeq \text{const} \quad \text{and} \quad g_T := \frac{\dot{\mathcal{G}}_T}{H\mathcal{G}_T} \simeq \text{const}. \quad (4.13)$$

We also define the variation parameter of the sound velocity of tensor perturbations as

$$s_T := \frac{\dot{c}_T}{Hc_T} = \frac{1}{2}(f_T - g_T). \quad (4.14)$$

Clearly, only two of the three parameters are independent. We additionally impose conditions

$$1 - \epsilon - f_T/2 + g_T/2 > 0, \quad (4.15)$$

$$3 - \epsilon + g_T > 0. \quad (4.16)$$

The former (equivalent to $1 > \epsilon + s_T$) guarantees that the time coordinate y_T runs from $-\infty$ to 0 as the Universe expands. The latter implies that the second solution in (4.12) indeed decays. We see that z_T can be written as

$$z_T = \frac{\mathcal{F}_{T*}^{3/4}}{2\mathcal{G}_{T*}^{1/4}} \frac{1}{H_*(-y_{T*})} \frac{[(-y_T)/(-y_{T*})]^{1/2-\nu_T}}{1 - \epsilon - f_T/2 + g_T/2}, \quad (4.17)$$

where the quantities with * are those evaluated at some reference time $y_T = y_{T*}$. The normalized mode solution to the perturbation equation is given in terms of the Hankel function:

$$v_{ij} = \frac{\sqrt{\pi}}{2} \sqrt{-y} H_{\nu_T}^{(1)}(-ky_T) e_{ij}, \quad (4.18)$$

where

$$\nu_T := \frac{3 - \epsilon + g_T}{2 - 2\epsilon - f_T + g_T}, \quad (4.19)$$

and e_{ij} is a polarization tensor. Notice that the conditions (4.15) and (4.16) guarantee the positivity of ν_T . On superhorizon scales, $-ky_T \ll 1$, we obtain

$$k^{3/2} h_{ij} \approx 2^{\nu_T - 2} \frac{\Gamma(\nu_T)}{\Gamma(3/2)} \frac{(-y_T)^{1/2 - \nu_T}}{z_T} k^{3/2 - \nu_T} e_{ij}. \quad (4.20)$$

Thus, we find the power spectrum of the primordial tensor perturbation:

$$\mathcal{P}_T = 8\gamma_T \left. \frac{\mathcal{G}_T^{1/2}}{\mathcal{F}_T^{3/2}} \frac{H^2}{4\pi^2} \right|_{-ky_T=1}, \quad (4.21)$$

where $\gamma_T = 2^{2\nu_T - 3} |\Gamma(\nu_T)/\Gamma(3/2)|^2 (1 - \epsilon - f_T/2 + g_T/2)$. The tensor spectral tilt is given by

$$n_T = 3 - 2\nu_T. \quad (4.22)$$

Contrary to the predictions of the conventional inflation models, the blue spectrum $n_T > 0$ can be obtained if the following condition is satisfied,

$$4\epsilon + 3f_T - g_T < 0. \quad (4.23)$$

This condition is easily compatible with the conditions (4.15) and (4.16). Thus, positive and large g_T compared with ϵ and f_T can lead to a blue spectrum of tensor perturbations. In deriving the above formulas, we only assumed that ϵ , f_T , and g_T are constant. These parameters may not necessarily be very small as long as the inequalities (4.15) and (4.16) are satisfied under a sensible background solution.

4.2. Scalar perturbations

We now focus on scalar fluctuations putting $h_{ij} = 0$. Plugging the perturbed metric into the action and expanding it to second order, we obtain

$$S_S^{(2)} = \int dt d^3x a^3 \left[-3\mathcal{G}_T \dot{\zeta}^2 + \frac{\mathcal{F}_T}{a^2} (\vec{\nabla} \zeta)^2 + \Sigma \alpha^2 - 2\Theta \alpha \frac{\vec{\nabla}^2}{a^2} \beta + 2\mathcal{G}_T \dot{\zeta} \frac{\vec{\nabla}^2}{a^2} \beta + 6\Theta \alpha \dot{\zeta} - 2\mathcal{G}_T \alpha \frac{\vec{\nabla}^2}{a^2} \zeta \right], \quad (4.24)$$

where

$$\begin{aligned} \Sigma := & X K_X + 2X^2 K_{XX} + 12H\dot{\phi}XG_{3X} \\ & + 6H\dot{\phi}X^2 G_{3XX} - 2XG_{3\phi} - 2X^2 G_{3\phi X} - 6H^2 G_4 \\ & + 6 \left[H^2 (7XG_{4X} + 16X^2 G_{4XX} + 4X^3 G_{4XXX}) \right. \\ & \left. - H\dot{\phi} (G_{4\phi} + 5XG_{4\phi X} + 2X^2 G_{4\phi XX}) \right] \\ & + 30H^3 \dot{\phi}XG_{5X} + 26H^3 \dot{\phi}X^2 G_{5XX} \\ & + 4H^3 \dot{\phi}X^3 G_{5XXX} - 6H^2 X (6G_{5\phi} + 9XG_{5\phi X} + 2X^2 G_{5\phi XX}), \end{aligned} \quad (4.25)$$

$$\begin{aligned} \Theta := & -\dot{\phi}XG_{3X} + 2HG_4 - 8HXG_{4X} - 8HX^2 G_{4XX} + \dot{\phi}G_{4\phi} + 2X\dot{\phi}G_{4\phi X} \\ & - H^2 \dot{\phi} (5XG_{5X} + 2X^2 G_{5XX}) + 2HX (3G_{5\phi} + 2XG_{5\phi X}). \end{aligned} \quad (4.26)$$

It is interesting to see that even in the most generic case, some of the coefficients are given by \mathcal{F}_T and \mathcal{G}_T , *i.e.*, the functions characterizing the tensor perturbation, and only two new functions show up in the scalar quadratic action. Note that the following relations hold:

$$\Sigma = X \sum_{i=2}^5 \frac{\partial \mathcal{E}_i}{\partial X} + \frac{1}{2} H \sum_{i=2}^5 \frac{\partial \mathcal{E}_i}{\partial H}, \quad (4.27)$$

$$\Theta = -\frac{1}{6} \sum_{i=2}^5 \frac{\partial \mathcal{E}_i}{\partial H}, \quad (4.28)$$

which compactify the above lengthy expressions.

Varying the action (4.24) with respect to α and β , we obtain the constraint equations

$$\Sigma \alpha - \Theta \frac{\vec{\nabla}^2}{a^2} \beta + 3\Theta \dot{\zeta} - \mathcal{G}_T \frac{\vec{\nabla}^2}{a^2} \zeta = 0, \quad (4.29)$$

$$\Theta \alpha - \mathcal{G}_T \dot{\zeta} = 0. \quad (4.30)$$

Using the constraint equations, we eliminate α and β from the action (4.24) and finally arrive at

$$S_S^{(2)} = \int dt d^3x a^3 \left[\mathcal{G}_S \dot{\zeta}^2 - \frac{\mathcal{F}_S}{a^2} (\vec{\nabla} \zeta)^2 \right], \quad (4.31)$$

where

$$\mathcal{F}_S := \frac{1}{a} \frac{d}{dt} \left(\frac{a}{\Theta} \mathcal{G}_T^2 \right) - \mathcal{F}_T, \quad (4.32)$$

$$\mathcal{G}_S := \frac{\Sigma}{\Theta^2} \mathcal{G}_T^2 + 3\mathcal{G}_T. \quad (4.33)$$

The analysis of the curvature perturbation hereafter is completely parallel to that of the tensor perturbation. The squared sound speed is given by $c_S^2 = \mathcal{F}_S/\mathcal{G}_S$, and ghost and gradient instabilities are avoided as long as

$$\mathcal{F}_S > 0 \quad \mathcal{G}_S > 0. \quad (4.34)$$

In the case of k-inflation where $G_3 = 0 = G_5$ and $G_4 = M_{\text{Pl}}^2/2$, we have $\mathcal{F}_S = M_{\text{Pl}}^2 \epsilon$. This implies that the interesting regime $\dot{H} > 0$ is prohibited by the stability requirement in k-inflation.³⁷⁾ However, the sign of \dot{H} and the stability criteria are not correlated in more general situations. This point was already clear in G-inflation and kinetic gravity braiding for which $G_3 \neq 0$.^{13), 14)} Stable cosmological solutions with $\dot{H} > 0$ offer a radical and very interesting scenario of the earliest Universe.^{19), 38)}

The stability conditions (4.34) for the scalar perturbation as well as (4.8) for the tensor perturbation have been derived in the case of the covariant Galileon, for which $K = -c_2 X$, $G_3 = -c_3 X/M^3$, $G_4 = M_{\text{Pl}}^2/2 - c_4 X^2/M^6$, and $G_5 = 3c_5 X^2/M^9$.¹²⁾ It can be verified that our general formulas correctly reproduce the result of.¹²⁾

Using the new variables

$$dy_S := \frac{c_S}{a} dt, \quad z_S := \sqrt{2}a (\mathcal{F}_S \mathcal{G}_S)^{1/4}, \quad u := z_S \zeta, \quad (4.35)$$

the curvature perturbation is canonically normalized and the action is now given by

$$S_S^{(2)} = \frac{1}{2} \int dy_S d^3x \left[(u')^2 - (\vec{\nabla} u)^2 + \frac{z_S''}{z_S} u^2 \right], \quad (4.36)$$

where a prime denotes differentiation with respect to y_S . Each perturbation mode exits the sound horizon when $k^2 = z_S''/z_S \sim 1/y_S^2$, where k is the Fourier wavenumber.

The two independent solutions on superhorizon scales are

$$\zeta = \text{const} \quad \text{and} \quad \int^t \frac{dt'}{a^3 \mathcal{G}_S}. \quad (4.37)$$

During inflation, it may be assumed that \mathcal{G}_S is slowly varying. In this case, the second solution decays rapidly. Note, however, that the non-trivial dynamics of the scalar field can induce a temporal rapid evolution of \mathcal{G}_S , which would affect the superhorizon behavior of the curvature perturbation through the contamination of the second mode in the same way as in Ref. 39). Given the specific background dynamics, one can evaluate such an effect using our general formulas.

Closely following the procedure we did in the case of the tensor perturbation, we now evaluate the power spectrum of the primordial curvature perturbation. To do so, we assume that $\epsilon \simeq \text{const}$,^{*)}

$$f_S := \frac{\dot{\mathcal{F}}_S}{H\mathcal{F}_S} \simeq \text{const}, \quad g_S := \frac{\dot{\mathcal{G}}_S}{H\mathcal{G}_S} \simeq \text{const}, \quad (4.38)$$

and then define

$$\nu_S := \frac{3 - \epsilon + g_S}{2 - 2\epsilon - f_S + g_S}. \quad (4.39)$$

The power spectrum is given by

$$\mathcal{P}_\zeta = \frac{\gamma_S}{2} \left. \frac{\mathcal{G}_S^{1/2}}{\mathcal{F}_S^{3/2}} \frac{H^2}{4\pi^2} \right|_{-ky_S=1}, \quad (4.40)$$

where $\gamma_S = 2^{2\nu_S-3} |\Gamma(\nu_S)/\Gamma(3/2)|^2 (1 - \epsilon - f_S/2 + g_S/2)$. The spectral index is

$$n_s - 1 = 3 - 2\nu_S. \quad (4.41)$$

An exactly scale-invariant spectrum is obtained if

$$\epsilon + \frac{3}{4}f_S - \frac{1}{4}g_S = 0. \quad (4.42)$$

^{*)} By defining the variation parameter of the sound velocity of scalar perturbations as $s_S := \dot{c}_S/(Hc_S) = (f_S - g_S)/2$, the formulae with f_S and/or g_S can be rewritten in terms of c_S .

Here again, ϵ , f_S , and g_S are not necessarily very small (as long as $n_s - 1 \simeq 0$).

Taking now the limit $\epsilon, f_T, g_T, f_S, g_S \ll 1$, the tensor-to-scalar ratio is given by

$$r = 16 \left(\frac{\mathcal{F}_S}{\mathcal{F}_T} \right)^{3/2} \left(\frac{\mathcal{G}_S}{\mathcal{G}_T} \right)^{-1/2} = 16 \frac{\mathcal{F}_S}{\mathcal{F}_T} \frac{c_S}{c_T}. \quad (4.43)$$

Note that even in the de Sitter limit where $\epsilon, f_T, g_T, f_S, g_S \rightarrow 0$, the scalar perturbation can be produced in general, $r \neq 0$.

In the case of potential-driven slow-roll inflation in §3.2, we have, to leading order in slow-roll,

$$\mathcal{F}_S \simeq \frac{X}{H^2} (\mathcal{K} + 6H^2 h_4) + \frac{4\dot{\phi}X}{H} (h_3 + H^2 h_5), \quad (4.44)$$

$$\mathcal{G}_S \simeq \frac{X}{H^2} (\mathcal{K} + 6H^2 h_4) + \frac{6\dot{\phi}X}{H} (h_3 + H^2 h_5), \quad (4.45)$$

and $\mathcal{F}_T \simeq \mathcal{G}_T \simeq 2g_4$, where we used the slow-roll equation $2g_4\epsilon + \dot{g}_4/H \simeq \dot{\phi}J/2H^2$. In this case, we have $c_T^2 \simeq 1$ and $n_T \simeq -(2\epsilon + g_T)$ with $f_T \simeq g_T \simeq \dot{g}_4/(Hg_4)$. If the \mathcal{K} or h_4 term dominates in J , we have $c_S^2 \simeq 1$ and $\mathcal{F}_S \simeq \mathcal{G}_S \simeq J\dot{\phi}/(2H^2) \simeq g_4(2\epsilon + g_T)$, which yields the standard consistency relation:

$$r \simeq -8n_T. \quad (4.46)$$

On the other hand, if the h_3 or h_5 term dominates, then we have $c_S^2 \simeq 2/3$, $\mathcal{F}_S \simeq 2J\dot{\phi}/(3H^2) \simeq (4g_4/3)(2\epsilon + g_T)$, and $\mathcal{G}_S \simeq J\dot{\phi}/H^2 \simeq 2g_4(2\epsilon + g_T)$, which yields a new consistency relation:¹⁵⁾

$$r \simeq -\frac{32\sqrt{6}}{9}n_T. \quad (4.47)$$

Thus, we can discriminate which term dominates in the dynamics using the consistency relations.

§5. Summary

In this paper, generic inflation models driven by a single scalar field have been studied. Our gravity + scalar-field system is described by the generalized Galileons, which do not give rise to higher derivatives in the field equations despite the non-minimal coupling, *e.g.*, of the form $G(\phi, X)R$. The class of inflation models is the most general ever proposed in the context of single-field inflation.

We have seen that if the Lagrangian has a shift symmetry, $\phi \rightarrow \phi + c$, de Sitter attractors are present and, hence, inflation can be driven by ϕ 's kinetic energy. Reheating after kinetically driven inflation is possible by breaking the shift symmetry, but the way to break it depends on the explicit construction of the originally shift-symmetric Lagrangian itself.

We have also derived slow-roll equations of motion for potential-driven inflation, in which the scalar-field dynamics is modified by the higher-order Galileon terms.

We have determined the most generic quadratic actions for tensor and scalar cosmological perturbations. Using them, we have presented the stability criteria for

both types of perturbations. The primordial power spectra have also been computed. Note that, since the propagation speeds of the two types of fluctuations can be different, we must evaluate the power spectra for the same comoving wavenumber at different epochs, which may have some consequence.⁴⁰⁾

It would be interesting to extend the present linear analysis of the curvature perturbation to non-linear order along the line of Refs. 23), 41). In relation to cosmological perturbations beyond linear order, it would be important to evaluate primordial non-Gaussianities generated from Generalized G -inflation.

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Appendix A

— The Horndeski action, generalized Galileons, and non-minimal coupling to the Gauss-Bonnet term —

In 1974, Horndeski presented the most general action (in four dimensions) constructed from the metric $g_{\mu\nu}$, the scalar field ϕ , and their derivatives $\partial g_{\mu\nu}, \partial^2 g_{\mu\nu}, \partial^3 g_{\mu\nu}, \dots, \partial\phi, \partial^2\phi, \partial^3\phi, \dots$,⁴²⁾ still having second-order field equations. The Horndeski theory has been revisited recently by the authors of Ref. 43). In this appendix, we point out that the Horndeski theory and the generalized Galileons are equivalent.

In terms of the notation of Ref. 43) but using $X = -\partial_\mu\phi\partial^\mu\phi/2$ rather than $\rho = \partial_\mu\phi\partial^\mu\phi$, the Lagrangian of the Horndeski theory is given by

$$\begin{aligned} \mathcal{L}_H = & \delta_{\mu\nu\sigma}^{\alpha\beta\gamma} \left[\kappa_1 \nabla^\mu \nabla_\alpha \phi R_{\beta\gamma}^{\nu\sigma} + \frac{2}{3} \kappa_{1X} \nabla^\mu \nabla_\alpha \phi \nabla^\nu \nabla_\beta \phi \nabla^\sigma \nabla_\gamma \phi + \kappa_3 \nabla_\alpha \phi \nabla^\mu \phi R_{\beta\gamma}^{\nu\sigma} \right. \\ & + 2\kappa_{3X} \nabla_\alpha \phi \nabla^\mu \phi \nabla^\nu \nabla_\beta \phi \nabla^\sigma \nabla_\gamma \phi \Big] + \delta_{\mu\nu}^{\alpha\beta} \left[(F + 2W) R_{\alpha\beta}^{\mu\nu} + 2F_X \nabla^\mu \nabla_\alpha \phi \nabla^\nu \nabla_\beta \phi \right. \\ & \left. \left. + 2\kappa_8 \nabla_\alpha \phi \nabla^\mu \phi \nabla^\nu \nabla_\beta \phi \right] - 6(F_\phi + 2W_\phi - X\kappa_8) \square\phi + \kappa_9, \end{aligned} \quad (\text{A.1})$$

where $\delta_{\mu_1\mu_2\dots\mu_n}^{\alpha_1\alpha_2\dots\alpha_n} = n! \delta_{\mu_1}^{[\alpha_1} \delta_{\mu_2}^{\alpha_2} \dots \delta_{\mu_n]}^{\alpha_n]}$, and $\kappa_1, \kappa_3, \kappa_8$, and κ_9 are arbitrary functions of ϕ and X . We also have two functions $F = F(\phi, X)$ and $W = W(\phi)$, and the former is constrained so that $F_X = 2(\kappa_3 + 2X\kappa_{3X} - \kappa_{1\phi})$, while the latter can be absorbed into a redefinition of the former. We are therefore left with four arbitrary functions of ϕ and X , in accordance with the generalized Galileon.

The above Lagrangian can be mapped to that of the generalized Galileon by identifying

$$K = \kappa_9 + 4X \int^X dX' (\kappa_{8\phi} - 2\kappa_{3\phi\phi}), \quad (\text{A.2})$$

$$G_3 = 6F_\phi - 2X\kappa_8 - 8X\kappa_{3\phi} + 2 \int^X dX' (\kappa_8 - 2\kappa_{3\phi}), \quad (\text{A.3})$$

$$G_4 = 2F - 4X\kappa_3, \quad (\text{A.4})$$

$$G_5 = -4\kappa_1, \quad (\text{A.5})$$

where we redefined F so that $F + 2W \rightarrow F$. Now we see that the two theories are in fact equivalent. In deriving the Lagrangian (A·1), Horndeski started from the assumptions that are weaker than those made by Deffayet et al.,²⁵⁾ although the latter worked in arbitrary dimensions.

Since the generalized Galileon is the most general theory in four dimensions composed of $g_{\mu\nu}$, ϕ , and their derivatives, which gives the second-order field equations, it must reproduce the non-minimal coupling to the Gauss-Bonnet term,⁴⁴⁾

$$\xi(\phi) (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}), \quad (\text{A}\cdot6)$$

which seems non-trivial at first glance. One can show that, by taking

$$K = 8\xi^{(4)}X^2(3 - \ln X), \quad (\text{A}\cdot7)$$

$$G_3 = 4\xi^{(3)}X(7 - 3\ln X), \quad (\text{A}\cdot8)$$

$$G_4 = 4\xi^{(2)}X(2 - \ln X), \quad (\text{A}\cdot9)$$

$$G_5 = -4\xi^{(1)}\ln X, \quad (\text{A}\cdot10)$$

or, equivalently, $\kappa_1 = \xi^{(1)}\ln X$, $\kappa_3 = \xi^{(2)}\ln X$, $\kappa_8 = 0$, and $\kappa_9 = 16\xi^{(4)}X^2$, where $\xi^{(n)} := \partial^n\xi/\partial\phi^n$, the generalized Galileon indeed reproduces the non-minimal coupling of the form (A·6). Probably the shortest way to confirm this fact is to substitute κ_i to the field equations presented in Ref. 42) and to compare them with those obtained from (A·6).

Similarly to $f(R)$ gravity, the gravitational theory described by

$$\mathcal{L} = \frac{R}{2} + f(\mathcal{G}), \quad \mathcal{G} := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \quad (\text{A}\cdot11)$$

where $f(\mathcal{G})$ is an arbitrary function of the Gauss-Bonnet term, contains an extra scalar degree of freedom, and hence, (A·11) must be recast in the Lagrangian of the generalized Galileon. Noting that the Lagrangian (A·11) can be equivalently written as

$$\mathcal{L} = \frac{R}{2} + f(\phi) + f_\phi(\mathcal{G} - \phi), \quad (\text{A}\cdot12)$$

and the non-minimal coupling $f_\phi\mathcal{G}$ is reproduced by Eqs. (A·7)–(A·10), it is now straightforward to translate (A·11) to the generalized Galileon.

It is easy to see explicitly in the cosmological equations of motion that the contribution from the non-minimal coupling (A·6) can indeed be reproduced from the non-trivial functions (A·7)–(A·10). Both the generalized Galileon with (A·7)–(A·10) and the Lagrangian (A·6) give the following identical contributions to the background and perturbation equations: for the background gravitational field equations,

$$\mathcal{E} \supset -24H^3\dot{\xi}, \quad (\text{A}\cdot13)$$

$$\mathcal{P} \supset 8 \left[H^2\ddot{\xi} + 2H \left(H^2 + \dot{H} \right) \dot{\xi} \right], \quad (\text{A}\cdot14)$$

for the background equation of motion for ϕ ,

$$P_\phi \supset 24H^2 \left(\dot{H} + H^2 \right) \xi_\phi, \quad (\text{A}\cdot15)$$

and for the quadratic actions of the tensor and scalar perturbations,

$$\mathcal{F}_T \supset 8\ddot{\xi}, \quad \mathcal{G}_T \supset 8H\dot{\xi}, \quad \Sigma \supset -48H^3\dot{\xi}, \quad \Theta \supset 12H^2\dot{\xi}. \quad (\text{A.16})$$

Appendix B — Field equations —

In this Appendix, we present both gravitational- and scalar-field equations derived from the action (2.5) for completeness.*⁴⁵⁾ Varying the action, we obtain

$$\delta \left(\sqrt{-g} \sum_{i=2}^5 \mathcal{L}_i \right) = \sqrt{-g} \left[\sum_{i=2}^5 \mathcal{G}_{\mu\nu}^i \delta g^{\mu\nu} + \sum_{i=2}^5 (P_\phi^i - \nabla^\mu J_\mu^i) \delta \phi \right] + \text{total derivative,} \quad (\text{B.1})$$

where

$$\mathcal{G}_{\mu\nu}^2 = -\frac{1}{2} K_X \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} K g_{\mu\nu}, \quad (\text{B.2})$$

$$\mathcal{G}_{\mu\nu}^3 = \frac{1}{2} G_{3X} \square \phi \nabla_\mu \phi \nabla_\nu \phi + \nabla_{(\mu} G_3 \nabla_{\nu)} \phi - \frac{1}{2} g_{\mu\nu} \nabla_\lambda G_3 \nabla^\lambda \phi, \quad (\text{B.3})$$

$$\begin{aligned} \mathcal{G}_{\mu\nu}^4 = & G_4 G_{\mu\nu} - \frac{1}{2} G_{4X} R \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} G_{4XX} [(\square \phi)^2 - (\nabla_\alpha \nabla_\beta \phi)^2] \nabla_\mu \phi \nabla_\nu \phi \\ & - G_{4X} \square \phi \nabla_\mu \nabla_\nu \phi + G_{4X} \nabla_\lambda \nabla_\mu \phi \nabla^\lambda \nabla_\nu \phi + 2 \nabla_\lambda G_{4X} \nabla^\lambda \nabla_{(\mu} \phi \nabla_{\nu)} \phi \\ & - \nabla_\lambda G_{4X} \nabla^\lambda \phi \nabla_\mu \nabla_\nu \phi + g_{\mu\nu} (G_{4\phi} \square \phi - 2X G_{4\phi\phi}) \\ & + g_{\mu\nu} \left\{ -2 G_{4\phi X} \nabla_\alpha \nabla_\beta \phi \nabla^\alpha \phi \nabla^\beta \phi + G_{4XX} \nabla_\alpha \nabla_\lambda \phi \nabla_\beta \nabla^\lambda \phi \nabla^\alpha \phi \nabla^\beta \phi \right. \\ & \left. + \frac{1}{2} G_{4X} [(\square \phi)^2 - (\nabla_\alpha \nabla_\beta \phi)^2] \right\} + 2 \left[G_{4X} R_{\lambda(\mu} \nabla_{\nu)} \phi \nabla^\lambda \phi \right. \\ & \left. - \nabla_{(\mu} G_{4X} \nabla_{\nu)} \phi \square \phi \right] - g_{\mu\nu} \left[G_{4X} R^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi - \nabla_\lambda G_{4X} \nabla^\lambda \phi \square \phi \right] \\ & + G_{4X} R_{\mu\alpha\nu\beta} \nabla^\alpha \phi \nabla^\beta \phi - G_{4\phi} \nabla_\mu \nabla_\nu \phi - G_{4\phi\phi} \nabla_\mu \phi \nabla_\nu \phi \\ & + 2 G_{4\phi X} \nabla^\lambda \phi \nabla_\lambda \nabla_{(\mu} \phi \nabla_{\nu)} \phi - G_{4XX} \nabla^\alpha \phi \nabla_\alpha \nabla_\mu \phi \nabla^\beta \phi \nabla_\beta \nabla_\nu \phi, \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} \mathcal{G}_{\mu\nu}^5 = & G_{5X} R_{\alpha\beta} \nabla^\alpha \phi \nabla^\beta \nabla_{(\mu} \phi \nabla_{\nu)} \phi - G_{5X} R_{\alpha(\mu} \nabla_{\nu)} \phi \nabla^\alpha \phi \square \phi \\ & - \frac{1}{2} G_{5X} R_{\alpha\beta} \nabla^\alpha \phi \nabla^\beta \phi \nabla_\mu \nabla_\nu \phi - \frac{1}{2} G_{5X} R_{\mu\alpha\nu\beta} \nabla^\alpha \phi \nabla^\beta \phi \square \phi \\ & + G_{5X} R_{\alpha\lambda\beta(\mu} \nabla_{\nu)} \phi \nabla^\lambda \phi \nabla^\alpha \nabla^\beta \phi + G_{5X} R_{\alpha\lambda\beta(\mu} \nabla_{\nu)} \nabla^\lambda \phi \nabla^\alpha \phi \nabla^\beta \phi \\ & - \frac{1}{2} \nabla_{(\mu} [G_{5X} \nabla^\alpha \phi] \nabla_\alpha \nabla_{\nu)} \phi \square \phi + \frac{1}{2} \nabla_{(\mu} [G_{5\phi} \nabla_{\nu)} \phi] \square \phi \\ & - \nabla_\lambda [G_{5\phi} \nabla_{(\mu} \phi] \nabla_{\nu)} \nabla^\lambda \phi \\ & + \frac{1}{2} [\nabla_\lambda (G_{5\phi} \nabla^\lambda \phi) - \nabla_\alpha (G_{5X} \nabla_\beta \phi) \nabla^\alpha \nabla^\beta \phi] \nabla_\mu \nabla_\nu \phi \\ & + \nabla^\alpha G_5 \nabla^\beta \phi R_{\alpha(\mu\nu)\beta} - \nabla_{(\mu} G_5 G_{\nu)\lambda} \nabla^\lambda \phi \\ & + \frac{1}{2} \nabla_{(\mu} G_{5X} \nabla_{\nu)} \phi [(\square \phi)^2 - (\nabla_\alpha \nabla_\beta \phi)^2] \end{aligned}$$

*⁴⁵⁾ In the course of the preparation of this manuscript, we became aware that Gao has also calculated gravitational field equations in the present model.⁴⁵⁾ After some iterations, his result has converged with ours and we are in full agreement.

$$\begin{aligned}
& -\nabla^\lambda G_5 R_{\lambda(\mu} \nabla_{\nu)} \phi + \nabla_\alpha [G_{5X} \nabla_\beta \phi] \nabla^\alpha \nabla_{(\mu} \phi \nabla^{\beta} \nabla_{\nu)} \phi \\
& -\nabla_\beta G_{5X} [\square \phi \nabla^\beta \nabla_{(\mu} \phi - \nabla^\alpha \nabla^\beta \phi \nabla_\alpha \nabla_{(\mu} \phi] \nabla_{\nu)} \phi \\
& + \frac{1}{2} \nabla^\alpha \phi \nabla_\alpha G_{5X} [\square \phi \nabla_\mu \nabla_\nu \phi - \nabla_\beta \nabla_\mu \phi \nabla^{\beta} \nabla_\nu \phi] \\
& - \frac{1}{2} G_{5X} G_{\alpha\beta} \nabla^\alpha \nabla^\beta \phi \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} G_{5X} \square \phi \nabla_\alpha \nabla_\mu \phi \nabla^\alpha \nabla_\nu \phi \\
& + \frac{1}{2} G_{5X} (\square \phi)^2 \nabla_\mu \nabla_\nu \phi + \frac{1}{12} G_{5XX} [(\square \phi)^3 - 3 \square \phi (\nabla_\alpha \nabla_\beta \phi)^2 \\
& + 2 (\nabla_\alpha \nabla_\beta \phi)^3] \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} \nabla_\lambda G_5 G_{\mu\nu} \nabla^\lambda \phi \\
& + g_{\mu\nu} \left\{ -\frac{1}{6} G_{5X} [(\square \phi)^3 - 3 \square \phi (\nabla_\alpha \nabla_\beta \phi)^2 + 2 (\nabla_\alpha \nabla_\beta \phi)^3] + \nabla_\alpha G_5 R^{\alpha\beta} \nabla_\beta \phi \right. \\
& - \frac{1}{2} \nabla_\alpha (G_{5\phi} \nabla^\alpha \phi) \square \phi + \frac{1}{2} \nabla_\alpha (G_{5\phi} \nabla_\beta \phi) \nabla^\alpha \nabla^\beta \phi - \frac{1}{2} \nabla_\alpha G_{5X} \nabla^\alpha X \square \phi \\
& + \frac{1}{2} \nabla_\alpha G_{5X} \nabla_\beta X \nabla^\alpha \nabla^\beta \phi - \frac{1}{4} \nabla^\lambda G_{5X} \nabla_\lambda \phi [(\square \phi)^2 - (\nabla_\alpha \nabla_\beta \phi)^2] \\
& \left. + \frac{1}{2} G_{5X} R_{\alpha\beta} \nabla^\alpha \phi \nabla^\beta \phi \square \phi - \frac{1}{2} G_{5X} R_{\alpha\lambda\beta\rho} \nabla^\alpha \nabla^\beta \phi \nabla^\lambda \phi \nabla^\rho \phi \right\}, \tag{B.5}
\end{aligned}$$

$$P_\phi^2 = K_\phi, \tag{B.6}$$

$$P_\phi^3 = \nabla_\mu G_{3\phi} \nabla^\mu \phi, \tag{B.7}$$

$$P_\phi^4 = G_{4\phi} R + G_{4\phi X} [(\square \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2], \tag{B.8}$$

$$P_\phi^5 = -\nabla_\mu G_{5\phi} G^{\mu\nu} \nabla_\nu \phi - \frac{1}{6} G_{5\phi X} [(\square \phi)^3 - 3 \square \phi (\nabla_\mu \nabla_\nu \phi)^2 + 2 (\nabla_\mu \nabla_\nu \phi)^3], \tag{B.9}$$

and

$$J_\mu^2 = -\mathcal{L}_{2X} \nabla_\mu \phi, \tag{B.10}$$

$$J_\mu^3 = -\mathcal{L}_{3X} \nabla_\mu \phi + G_{3X} \nabla_\mu X + 2 G_{3\phi} \nabla_\mu \phi, \tag{B.11}$$

$$\begin{aligned}
J_\mu^4 &= -\mathcal{L}_{4X} \nabla_\mu \phi + 2 G_{4X} R_{\mu\nu} \nabla^\nu \phi - 2 G_{4XX} (\square \phi \nabla_\mu X - \nabla^\nu X \nabla_\mu \nabla_\nu \phi) \\
&\quad - 2 G_{4\phi X} (\square \phi \nabla_\mu \phi + \nabla_\mu X), \tag{B.12}
\end{aligned}$$

$$\begin{aligned}
J_\mu^5 &= -\mathcal{L}_{5X} \nabla_\mu \phi - 2 G_{5\phi} G_{\mu\nu} \nabla^\nu \phi \\
&\quad - G_{5X} \left[G_{\mu\nu} \nabla^\nu X + R_{\mu\nu} \square \phi \nabla^\nu \phi - R_{\nu\lambda} \nabla^\nu \phi \nabla^\lambda \nabla_\mu \phi - R_{\alpha\mu\beta\nu} \nabla^\nu \phi \nabla^\alpha \nabla^\beta \phi \right] \\
&\quad + G_{5XX} \left\{ \frac{1}{2} \nabla_\mu X [(\square \phi)^2 - (\nabla_\alpha \nabla_\beta \phi)^2] - \nabla_\nu X (\square \phi \nabla_\mu \nabla^\nu \phi - \nabla_\alpha \nabla_\mu \phi \nabla^\alpha \nabla^\nu \phi) \right\} \\
&\quad + G_{5\phi X} \left\{ \frac{1}{2} \nabla_\mu \phi [(\square \phi)^2 - (\nabla_\alpha \nabla_\beta \phi)^2] + \square \phi \nabla_\mu X - \nabla^\nu X \nabla_\nu \nabla_\mu \phi \right\}. \tag{B.13}
\end{aligned}$$

The gravitational- and scalar-field equations are thus given by

$$\sum_{i=2}^5 \mathcal{G}_{\mu\nu}^i = 0, \quad \nabla^\mu \left(\sum_{i=2}^5 J_\mu^i \right) = \sum_{i=2}^5 P_\phi^i. \tag{B.14}$$

One might worry that $\nabla^\mu J_\mu^i$ gives rise to higher derivatives as J_μ^i apparently contains second-order derivatives. However, this is not the case because the commutations of

higher derivatives can be replaced by the curvature tensors and thus are canceled. For instance, one has $\nabla_\mu (\square\phi\nabla^\mu + \nabla^\mu X) = (\square\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2 - R_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi$.

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