Homework 2

Bryan Lee

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3. Let A be an $n \times n$ symmetric matrix. Describe a strategy of *symmetric pivoting* so that after the Gaussian elimination, the matrix A is reduced to a diagonal matrix D. Make sure to argue that after each symmetric pivoting, the resulting matrix is still symmetric. [Hint: this will require both row and column operations]

Using Theorem 1.34 from the textbook (page 45-46):
A symmetric matrix A is regular if and only if it can be factored as

$$A = LDL^T$$

where L is a lower uni-triangular matrix, and D is a diagonal matrix with nonzero diagonal entries.

Theorem 1.29 also proves that we can factor

$$A = LDV$$

where transposing both sides of the equation, noting that V^T is a lower uni-triangular matrix, and L^T is an upper uni-triangular matrix. Then, if A is symmetric, then

$$LDV = A = A^T = V^T DL^T$$

Uniqueness of the LDV factorization implies that

$$L = V^T \quad \text{and} \quad V = L^T$$

There is more in-depth proofing in the textbook, but I want to get to the primary address of the problem.

Now let's assume a 3x3 matrix A:

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ 4 & 2 & 6 \end{pmatrix}$$

I) We want to find the upper-triangular matrix U:

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ 4 & 2 & 6 \end{pmatrix} \; -\frac{3}{2}R_1 + R_2 \to R_2 \quad \Longrightarrow \quad \begin{pmatrix} 2 & 3 & 4 \\ 0 & -\frac{7}{2} & -4 \\ 4 & 2 & 6 \end{pmatrix} - 2R_1 + R_3 \to R_3$$

$$\Longrightarrow \begin{pmatrix} 2 & 3 & 4 \\ 0 & -\frac{7}{2} & -4 \\ 0 & 0 & \frac{18}{7} \end{pmatrix} = U$$

II) Now, we want to acquire the diagonal matrix D:

$$D = \begin{pmatrix} 2 & 3 & 4 \\ 0 & -\frac{7}{2} & -4 \\ 0 & 0 & \frac{18}{7} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 \\ 0 & 0 & \frac{18}{7} \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{2} & 2 \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & 1 \end{pmatrix} = DV$$

We can see that the L matrix corresponds to the LU-factorization of A and $V = L^T$,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 2 & \frac{8}{7} & 1 \end{pmatrix}$$

But why is the values inverted? i.e. $-\frac{3}{2} \to \frac{3}{2}$. Since we are trying to prove $A = LDL^T$, we are seeking the inverse of L to reacquire A rather than finding upper-triangular matrix U.

III) Now, since $A = LDL^T$, we can verify that it is indeed true.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 2 & \frac{8}{7} & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 \\ 0 & 0 & \frac{18}{7} \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{2} & 2 \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ 4 & 2 & 6 \end{pmatrix}$$

Reference: Textbook example 1.35 page 46

IV) But what does this mean in terms of finding diagonal matrix D while arguing that after each symmetric pivoting, the resulting matrix will remain symmetric?

Let's apply the first-row operation (the $\frac{3}{2}$ from the L matrix) to begin finding our diagonal matrix D. This will turn $\frac{3}{2}$ into $-\frac{3}{2}$, effectively inverting L and L^T to find D, corresponding to the equation: $L^{-1}A(L^{-1})^T = D$

KEY NOTE: The pivot value chosen will just be the diagonal values of A undergoing the row operations.

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ 4 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 4 \\ 0 & -\frac{7}{2} & -4 \\ 4 & -4 & 6 \end{pmatrix}$$

We can see that the matrix is indeed still symmetric after the 1st row and column pivoting.

Now repeat for the next row and column operation:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 0 & -\frac{7}{2} & -4 \\ 4 & -4 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{7}{2} & -4 \\ 0 & -4 & 2 \end{pmatrix}$$

The resulting matrix is still symmetric.

Now, for the last operation:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{8}{7} & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{7}{2} & -4 \\ 0 & -4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{8}{7} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 \\ 0 & 0 & \frac{18}{7} \end{pmatrix} = D$$

The resulting matrix is still symmetric and is now the corresponding D matrix.

Therefore, we can repeat the same strategy of symmetric pivoting for smaller/larger matrices in a more general form, using Theorem 1.34 as its foundation. After every row and column pivoting, the matrix remains symmetrical, and eventually, we acquire the diagonal matrix D.

In a sense, for every row operation we apply to A, we will have to apply the transpose of said row operation (column operation) to keep the matrix symmetrical.