MATH425 - Homework 1

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Exercise 2

In this exercise, we will use the following matrix to illustrate how to compute the LU decomposition and how to use it:

$$A = \begin{pmatrix} -8 & -2 & 3 & 1 \\ 1 & -2 & 0 & 2 \\ -4 & -1 & 3 & 2 \\ 4 & 1 & -1 & -1 \end{pmatrix}$$

By Gaussian elimination, one can reduce the above A to an upper triangular matrix by using row operations of the type $cR_i + R_j \to R_j$.
a) State the first-row operation you would do in order to reduce A to an upper triangular matrix and apply this row operation to obtain the matrix A_1.

$$A = \begin{pmatrix} -8 & -2 & 3 & 1 \\ 1 & -2 & 0 & 2 \\ -4 & -1 & 3 & 2 \\ 4 & 1 & -1 & -1 \end{pmatrix} \frac{1}{8}R_1 + R_2 \to R_2 \quad \Longrightarrow \quad A_1 = \begin{pmatrix} -8 & -2 & 3 & 1 \\ 0 & -\frac{9}{4} & \frac{3}{8} & \frac{17}{8} \\ -4 & -1 & 3 & 2 \\ 4 & 1 & -1 & -1 \end{pmatrix}$$

b) Find the corresponding elementary matrix E_1. Then compute E_1A. What do you observe?

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{8} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \& \quad E_1A = \begin{pmatrix} -8 & -2 & 3 & 1 \\ 0 & -\frac{9}{4} & \frac{3}{8} & \frac{17}{8} \\ -4 & -1 & 3 & 2 \\ 4 & 1 & -1 & -1 \end{pmatrix}$$

I observe that the matrix multiplication of E_1A yields the A_1 matrix. It also turns the $a_{2,1}$ element into 0, the start of the triangle of 0s.

c) State and apply the next row operation that needs to be applied to A_1 to obtain A_2.

$$A_1 = \begin{pmatrix} -8 & -2 & 3 & 1 \\ 0 & -\frac{9}{4} & \frac{3}{8} & \frac{17}{8} \\ -4 & -1 & 3 & 2 \\ 4 & 1 & -1 & -1 \end{pmatrix} \quad -\frac{1}{2}R_1 + R_3 \to R_3 \quad \Longrightarrow \quad A_2 = \begin{pmatrix} -8 & -2 & 3 & 1 \\ 0 & -\frac{9}{4} & \frac{3}{8} & \frac{17}{8} \\ 0 & 0 & \frac{3}{2} & \frac{3}{2} \\ 4 & 1 & -1 & -1 \end{pmatrix}$$

d) Construct the corresponding elementary matrix E_2. Compute $E_2 A_1$.

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \& \quad E_2 A_1 = \begin{pmatrix} -8 & -2 & 3 & 1 \\ 0 & -\frac{9}{4} & \frac{3}{8} & \frac{17}{8} \\ 0 & 0 & \frac{3}{2} & \frac{3}{2} \\ 4 & 1 & -1 & -1 \end{pmatrix}$$

e) Now continue to construct E_3 and E_4. Verify $E_4 E_3 E_2 E_1 A = U$ where U is an upper triangular matrix.

$$A_2 = \begin{pmatrix} -8 & -2 & 3 & 1 \\ 0 & -\frac{9}{4} & \frac{3}{8} & \frac{17}{8} \\ 0 & 0 & \frac{3}{2} & \frac{3}{2} \\ 4 & 1 & -1 & -1 \end{pmatrix} \quad \frac{1}{2}R_1 + R_4 \to R_4 \quad \Longrightarrow \quad A_3 = \begin{pmatrix} -8 & -2 & 3 & 1 \\ 0 & -\frac{9}{4} & \frac{3}{8} & \frac{17}{8} \\ 0 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad -\frac{1}{3}R_3 + R_4 \to R_4$$

$$\Longrightarrow \quad A_4 = E_4 E_3 E_2 E_1 A = U = \begin{pmatrix} -8 & -2 & 3 & 1 \\ 0 & -\frac{9}{4} & \frac{3}{8} & \frac{17}{8} \\ 0 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus,

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix} \quad \& \quad E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & 1 \end{pmatrix}$$

f) Compute the inverses of $E_1, \ldots, E_4$. State the pattern you see.

$$E_1^{-1} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{8} & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad -\frac{1}{8}R_1 + R_2 \to R_2 \quad \Longrightarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{8} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad \Longrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{8} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Repeated process for E2, E3, E4, thus:

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \& \quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix} \quad \& \quad E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 1 \end{pmatrix}$$

Observation: The inverses swap the sign of the constant in the row operation $cR_i + R_j \to R_j$, where each row operation is represented by a E_i matrix.

g) Argue that $A = LU$ where $L = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$. What kind of matrix is L?
We have obtained an LU factorization of A. To check your work, use
`>> [L, U] = lu(A)`
command in MATLAB to get the LU factorization of A.

Since $E_1, ..., E_4$ are matrices that represent the respective row operations performed on the matrix A to obtain the upper-triangular matrix U, taking the inverse of $E_1, ..., E_4$, representing L, will undo these operations, restoring the upper-triangular matrix to its original form of A. Additionally, L is a product of inverses of elementary matrices, and elementary matrices themselves are invertible; thus, L is invertible and a lower-triangular matrix.

h) In Matlab file.

Exercise 3

Suppose in the linear system $Ax = b$ the matrix A is an $n\ timesn$ regular matrix.
a) Count the number of multiplication/division operations needed to find the solution x by Gaussian elimination and backward substitution. The answer should be a formula involving a polynomial in n. What is the degree and the leading coefficient of this polynomial?

For Gaussian elimination, taking the summation of the for-loops and computing them:

$$\begin{aligned}
T(n) &= \sum_{j=1}^{n} \sum_{i=j+1}^{n} \sum_{x=j}^{n+1} 1 \\
&= \sum_{j=1}^{n} \sum_{i=j+1}^{n} ((n+1)+1-j) = \sum_{j=1}^{n} \sum_{i=j+1}^{n} (n+2-j) \\
&= \sum_{j=1}^{n} (n+1-(j+1)(n+2-j) = \sum_{j=1}^{n} ((n-j)^2 + 2(n-j)) \\
&= \sum_{j=1}^{n} (n-j)^2 + 2\sum_{j=1}^{n} (n-j)
\end{aligned}$$

Let $k = (n-j),$ then

$$\begin{aligned}
&= \sum_{k=0}^{n-1} k^2 + 2\sum_{k=0}^{n-1} k \\
&= \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6} + 2(\frac{(n-1)((n-1)+1)}{2}) \\
&= \frac{(n-1)(n)(2n-1)}{6} + n^2 - n \\
&= \frac{2n^3 + 3n^2 - 5n}{6}
\end{aligned}$$

Now for Backward substitution:

$$
\begin{aligned}
T(n) &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} 1 \\
&= \sum_{i=1}^{n} (n + 1 - (i + 1)) = \sum_{i=1}^{n} (n - i) \\
&= \sum_{i=1}^{n} n - \sum_{i=1}^{n} i \\
&= n^2 - \frac{n(n+1)}{2} \\
&= \frac{n^2 - n}{2}
\end{aligned}
$$

Merging Gaussian elimination and backward substitution:

$$
T(n) = \frac{2n^3 + 3n^2 - 5n}{6} + \frac{n^2 - n}{2} = \frac{n^3}{3} + n^2 - \frac{4n}{3}
$$

The degree of the polynomial is: 3
The coefficient of the polynomial is: $\frac{1}{3}$

b) Now count the number of multiplications/divisions to compute A^{-1} and to find x by $x = A^{-1}b$. What is the degree and the leading coefficient of the resulting polynomial in n in your formula?

Compute number of multiplications/divisions to inverse A by taking the summation of the for-loops and computing them:

$$
\begin{aligned}
T(n) &= \sum_{i=1}^{n}(\sum_{x=1}^{2n} 1 + (\sum_{j=1}^{n-1}\sum_{x=1}^{2n} 1)) \\
&= \sum_{i=1}^{n}((2n+1-1) + (\sum_{j=1}^{n-1}(2n+1-1))) \\
&= \sum_{i=1}^{n}(2n + (\sum_{j=1}^{n-1} 2n)) \\
&= \sum_{i=1}^{n}(2n + (2n(n-1+1-1)) = \sum_{i=1}^{n}(2n + (2n^2 - 2n)) \\
&= \sum_{i=1}^{n}(2n^2) = n(2n^2) = 2n^3
\end{aligned}
$$

Now, to solve for x, it is simply multiplying A inverse with b, which is a double for-loop:

$$T(n) = \sum_{n}^{i=1}\sum_{n}^{j=1} 1 = n^2$$

Combining the two operations gives us:

$$T(n) = \sum_{n}^{i=1}\sum_{n}^{j=1} 1 = n^2$$

And merging the two together yields:

$$T(n) = 2n^3 + n^2$$

Where the degree of the polynomial is 3 and the coefficient is 2.

c) Based on your answers above, which method will you use for large n? Explain.

Since both methods have a polynomial degree of 3, I have to look at the coefficient. Gaussian elimination and backward substitution have a smaller coefficient of $\frac{1}{3}$ while finding the inverse of A has a coefficient of 2. In conclusion, the Gaussian elimination and backward substitution would be a more viable method for large n, as it will be about six times faster than finding the inverse of A.