Homework 6

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Question #1

a) If λ is an eigenvalue of A, then by definition, $A\vec{v} = \lambda\vec{v}$ where the corresponding non-zero vector $\vec{v} \neq 0$ is called the eigenvector of A.

Now, $B = cA + dI$ then:

$$B\vec{v} = (cA + dI)\vec{v} = c\underbrace{A\vec{v}}_{A\vec{v} = \lambda\vec{v}} + d\vec{v} = c\lambda\vec{v} + d\vec{v} = (c\lambda + d)\vec{v}$$

$\therefore$ $c\lambda + d$ is an eigenvalue of $B = cA + dI$.

b) By mathematical induction:

Consider the base case where $k = 1$:

$$A^1\vec{v} = \lambda^1\vec{v} \iff A\vec{v} = \lambda\vec{v}$$

which holds true per definition.

Now, assume that for some positive integer $k = n$ where $A^n\vec{v} = \lambda^n\vec{v}$.

We want to prove that for $k = n+1$ then $A^{n+1}\vec{v} = \lambda^{n+1}\vec{v}$.

$$A^{n+1}\vec{v} = A^n(A\vec{v}) \iff A^{n+1}\vec{v} = A(A^n\vec{v})$$

Using the assumption $A^n\vec{v} = \lambda^n\vec{v}$:

$$A^{n+1}\vec{v} = A(\lambda^n\vec{v})$$

$$A^{n+1}\vec{v} = \lambda^n(A\vec{v}) \quad \text{substitute base case}$$

$$A^{n+1}\vec{v} = \lambda^n(\lambda\vec{v})$$

$$A^{n+1}\vec{v} = \lambda^{n+1}\vec{v}$$

Since we have proven the base case & show that if the statement holds for $k = n$, it also holds for $k = n+1$.

$\therefore$ By induction: $A^k\vec{v} = \lambda^k\vec{v}$ for all positive k. $\square$

c) By definition, we know that if λ is an eigenvalue of A, then $A\vec{v} = \lambda\vec{v}$ where the corresponding non-zero vector $\vec{v}$ is the eigenvector of A.

This means that for $\lambda = 0$, we have $A\vec{v} = 0\vec{v} = \vec{0}$.

If A is singular (non-invertible), then A must have a $\det(A) = 0$ & a non trivial kernel, $\ker(A) \neq \{\vec{0}\}$. As such, there exists an non-zero vector $\vec{v} \in \ker(A)$. Subsequently, the subspace consisting of eigenvector (eigenspace) with $\lambda = 0$ is precisely $\ker(A)$ with the same dimension since kernel is the number of linearly independent solutions to $A\vec{v} = 0$ that spans the subspace.

Conversely, A must be singular for $\lambda = 0$ to be its eigenvalue. If A is nonsingular (invertible), then A has a trivial kernel where there exists no non-zero vector $\vec{v}$ such that $A\vec{v} = \vec{0}$.

d) Let A be an $n \times n$ matrix where each entry is equal to 1:

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

Observation: $\text{rank}(A) = 1$ b/c all columns are multiple of each other & A has a nontrivial kernel. As per part c, A is clearly singular, thus $\lambda = 0$ is an eigenvector of A.

Using the rank-nullity theorem, we get nullity of A as $n - \text{rank}(A) = n-1$ which is equivalent to saying $\dim(\ker(A)) = n-1$. From part c, we found that the dimension of eigenspace corresponding to the eigenvalue $\lambda = 0$ is the nullity of A with multiplicity $n-1$. This means there exists one other non-zero eigenvalue.

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Now, to find the non-zero eigenvalue & its vector:
Since the sum of each row of A is n as shown:

$$A\vec{v} = \begin{bmatrix} \sum_{i=1}^{n} v_i \\ \sum_{i=1}^{n} v_i \\ \vdots \\ \sum_{i=1}^{n} v_i \end{bmatrix}$$

we can take $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ an column vector of all 1's then we have:

$$A\vec{v} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} n \\ n \\ \vdots \\ n \end{bmatrix} = n \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \lambda \vec{v}$$

$\therefore$ $\lambda = n$ is an eigenvalue of A, with its corresponding eigenvector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

Lastly, as already stated, $\lambda = 0$ is also an eigenvalue of A, and its corresponding eigenvector is any non-zero $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \vec{0}$.

e) By definition, we know that $A\vec{v} = \lambda\vec{v}$.

NOW, if A is a nonsingular matrix, then applying A^{-1} to both sides of $A\vec{v} = \lambda\vec{v}$ gives:

$$A^{-1}A\vec{v} = A^{-1}\lambda\vec{v} \implies \vec{v} = A^{-1}\lambda\vec{v}$$

Dividing both sides by λ:

$$\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v} \iff A^{-1}\vec{v} = \lambda^{-1}\vec{v}$$

$\therefore$ λ^{-1} is an eigenvalue of A^{-1} with $\lambda \neq 0$ as A is invertible.

Question #2

a) Since u is a unit vector, we know that $\|u\| = 1$ & $u^T u = 1$.

We also know that $A = uu^T$ which is constructed in such a way that the columns of A are spanned by $\vec{u}$ $\therefore$ rank$(A) = 1$.

This problem is similar to part 1c & part 1d, where we concluded that such a singular rank$(A)=1$ matrix contains an eigenvalue of $\lambda = 0$ with multiplicity $n-1$ & one other non-zero eigenvalue.

To find the non-zero eigenvalue, we can let $\vec{v} = \vec{u}$, then:

$$A\vec{v} = \lambda\vec{v} \Rightarrow (\underbrace{\vec{u}\vec{u}^T)\vec{u}}_{1} = \lambda\vec{u} \Rightarrow \vec{u} = \lambda\vec{u}$$

which is equivalent to saying $A\vec{u} = (1)\vec{u}$ as $A\vec{u} = \lambda\vec{u}$.

$\therefore$ $\lambda = 1$ is the eigenvalue of A with the corresponding eigenvector $\vec{u}$.

Similarly, for the eigenvalue of $\lambda = 0$, its corresponding eigenvector is any non-zero $\vec{v} \in \mathbb{R}$ such that $A\vec{v} = \vec{0}$

b) Using the eigenvalue equation of $A\vec{v} = \lambda\vec{v}$, we can substitute $A = H = (I - 2\vec{u}\vec{u}^T)$:

$$(I - 2\vec{u}\vec{u}^T)\vec{v} = \lambda\vec{v}$$

$\Rightarrow$ $\vec{v} - 2(uu^T\vec{v}) = \lambda\vec{v}$ then rearrange

$\Rightarrow$ $\vec{v} - \lambda\vec{v} = 2(\vec{u}\vec{u}^T\vec{v})$

$\Rightarrow$ $(1 - \lambda)\vec{v} = 2(\vec{u}^T\vec{v})\vec{u}$

Here, we can consider 2 cases: $\vec{v}$ is parallel to $\vec{u}$ & $\vec{v}$ is orthogonal to $\vec{u}$.

Case 1 ($\vec{v}$ is parallel to $\vec{u}$):

Let $\vec{v} = \vec{u}$:

$$(1-\lambda)\vec{u} = 2(\underbrace{\vec{u}^T\vec{u}}_{1})\vec{u} \Rightarrow (1-\lambda)\vec{u} = 2\vec{u} \Rightarrow 1-\lambda = 2 \Rightarrow \lambda = -1$$

$\therefore$ $A\vec{v} = H\vec{v} = -\vec{u}$ where $\vec{u}$ is the corresponding eigenvector to $\lambda = -1$.

Case 2 ($\vec{v}$ is orthogonal to $\vec{u}$):

If $\vec{v}$ is orthogonal to $\vec{u}$, then $\vec{u}^T\vec{v} = 0$.

$$(1-\lambda)\vec{v} = 2(\underbrace{\vec{u}^T\vec{v}}_{0})\vec{u} \Rightarrow (1-\lambda)\vec{v} = 2(0)\vec{u} \Rightarrow (1-\lambda)\vec{v} = 0 \Rightarrow 1-\lambda = 0$$

$\therefore$ $\lambda = 1$ where its corresponding eigenvectors are all $\vec{v}$ orthogonal to $\vec{u}$.

c) The eigenvalue equation respective to P is : $P\vec{v} = \lambda\vec{v}$

Now, applying $P^2 = P$ to the equation :

$$P^2\vec{v} = P\vec{v}$$

Substituting $P\vec{v} = \lambda\vec{v}$:

$$P(\lambda\vec{v}) = \lambda\vec{v}$$

$$\lambda P\vec{v} = \lambda\vec{v}$$

$$\lambda(\lambda\vec{v}) = \lambda\vec{v}$$

$$\lambda^2\vec{v} = \lambda\vec{v}$$

$$\lambda^2\vec{v} - \lambda\vec{v} = 0$$

$$\lambda^2 - \lambda = 0 \iff \lambda(\lambda - 1) = 0$$

$\therefore$ $\lambda = 0$ & $\lambda = 1$ where $\lambda = 0$ corresponding eigenvectors is orthogonal of the subspace & $\lambda = 1$ is the corresponding to eigenvectors in the subspace where P projects.

Question #3

Finding the eigenvalues of the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{bmatrix} = -\lambda\det\begin{bmatrix} -\lambda & a \\ -a & -\lambda \end{bmatrix} - c\begin{bmatrix} -c & a \\ b & -\lambda \end{bmatrix} - b\begin{bmatrix} -c & -\lambda \\ b & -a \end{bmatrix}$$

$$= -\lambda(\lambda^2 + a^2) - c(c\lambda - ab) - b(ca + b\lambda)$$

$$= -\lambda^3 - a^2\lambda - c^2\lambda + \cancel{abc} - \cancel{abc} - b^2\lambda$$

$$= -\lambda^3 - a^2\lambda - c^2\lambda - b^2\lambda$$

$$= -\lambda(\lambda^2 - a^2 - c^2 - b^2) = 0$$

Eigenvalues: $\lambda_1 = 0$ & $\lambda^2 - a^2 - c^2 - b^2 = 0$

$$\lambda^2 = a^2 + b^2 + c^2$$

$$\lambda_{2,3} = \pm i\sqrt{a^2 + b^2 + c^2}$$

A is diagonalizable as indicated by the eigenvalues. Since there are 3 distinct eigenvalues, there is 3 linearly independent eigenvector forming a basis of $\mathbb{R}^3$

$\therefore$ A is diagonalizable.

Question #4

$\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = -2$ with corresponding eigenvectors $\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}$

$S = \begin{bmatrix}-1 & 2 & 0\\ 1 & -1 & 1\\ 0 & 1 & 3\end{bmatrix}$ ← augmented eigenvectors

$\Lambda = \begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -2\end{bmatrix}$ ← diagonal matrix where diagonal entries are the eigenvalues

computing S^{-1}:

$$S^{-1} = \left[\begin{array}{ccc|ccc}-1 & 2 & 0 & 1 & 0 & 0\\ 1 & -1 & 1 & 0 & 1 & 0\\ 0 & 1 & 3 & 0 & 0 & 1\end{array}\right] \text{swap} \Rightarrow \left[\begin{array}{ccc|ccc}1 & -1 & 1 & 0 & 1 & 0\\ -1 & 2 & 0 & 1 & 0 & 0\\ 0 & 1 & 3 & 0 & 0 & 1\end{array}\right] R_2 + R_1 \Rightarrow \left[\begin{array}{ccc|ccc}1 & -1 & 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 1 & 1 & 0\\ 0 & 1 & 3 & 0 & 0 & 1\end{array}\right] R_3 - R_2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc}1 & -1 & 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 1 & 1 & 0\\ 0 & 0 & 2 & -1 & -1 & 1\end{array}\right] \frac{R_3}{2} \Rightarrow \left[\begin{array}{ccc|ccc}1 & -1 & 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 1 & 1 & 0\\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{array}\right] R_2 - R_3 \Rightarrow \left[\begin{array}{ccc|ccc}1 & -1 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & \frac{3}{2} & \frac{3}{2} & -\frac{1}{2}\\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{array}\right] R_1 + R_2 - R_3$$

$$\Rightarrow \begin{bmatrix}2 & 3 & -1\\ \frac{3}{2} & \frac{3}{2} & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

Now computing $A = S \Lambda S^{-1}$:

$$A = \begin{bmatrix}-1 & 2 & 0\\ 1 & -1 & 1\\ 0 & 1 & 3\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}2 & 3 & -1\\ \frac{3}{2} & \frac{3}{2} & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$= \begin{bmatrix}0 & 4 & 0\\ 0 & -2 & -2\\ 0 & 2 & -6\end{bmatrix}\begin{bmatrix}2 & 3 & -1\\ \frac{3}{2} & \frac{3}{2} & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{bmatrix} = \boxed{\begin{bmatrix}6 & 6 & -2\\ -2 & -2 & 0\\ 6 & 6 & -4\end{bmatrix}}$$

Question #5

a) Since A is a real & symmetric matrix, we can use $\vec{v} = e_i$ where e_i is the standard basis vector:

$$\vec{v} = e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{ith position}$$

NOW, if we substitute $\vec{v} = e_i$ in $\vec{v}^T A \vec{v}$, we will acquire $e_i^T A e_i$ which gives the (i,i) entry of A. To save on extensive notation, e_i^T summarize to taking the i-th row of A, then e_i takes the column of A, giving us the diagonal entries of A.

By the definition of positive semidefiniteness:

$\vec{v}^T A \vec{v} \geq 0$ for all $\vec{v}$, then:

$$e_i^T A e_i = A_{ii} \geq 0$$

$\therefore$ All diagonal entries of a positive semidefinite matrix are nonnegative.

b) Since A is a real symmetric matrix, there exists an orthogonal matrix Q where its columns form an orthonormal basis of $\mathbb{R}^n$ which are eigenvectors of A. There also exists a diagonal matrix Λ such that $A = Q \Lambda Q^T$. We know that $\vec{v}^T A \vec{v} \geq 0$, then taking the spectral decomposition:

$$\vec{v}^T A \vec{v} = \vec{v}^T (Q \Lambda Q^T) \vec{v} = (\vec{v}^T Q) \Lambda (Q^T \vec{v}) \geq 0$$

we can let the change of basis be:

$$y = Q^T \vec{v} \quad \& \quad y^T = v^T Q$$

then substituting:

$$\vec{v}^T (Q \Lambda Q^T) \vec{v} \Leftrightarrow y^T \Lambda y \geq 0$$

Since Λ is the diagonal matrix with its diagonal entries being the eigenvalues & $y = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_n \end{bmatrix}$ being the orthonormal basis, we can express the equation as the following summation: $y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2 \geq 0$

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Now, since $\vec{v}^T A \vec{v} = y^T \Lambda y \geq 0$ when all $\lambda_i \geq 0$, then subsequently $\vec{v}^T A \vec{v} \geq 0$ for all $\vec{v} \in \mathbb{R}^n$ so A is a positive semi definite matrix.

c) Given that A is a positive semidefinite matrix for all $\vec{v} \in \mathbb{R}^n$: $\vec{v}^T A \vec{v} \geq 0$.

Proof by contradiction:

Assume that A has a <u>negative eigenvalue</u> $\lambda < 0$ with a corresponding non-negative eigen vector.

Now, computing $\vec{v}^T A \vec{v}$ with the eigenvalue equation $A\vec{v} = \lambda \vec{v}$:

$$\vec{v}^T A \vec{v} = \vec{v}^T (\lambda \vec{v}) = \lambda (\vec{v}^T \vec{v}) \geq 0$$

Here $\vec{v}^T \vec{v} = \|\vec{v}\|^2 \geq 0$, the magnitude, where $\vec{v}$ is a non-zero eigenvector, then by the assumption that $\lambda < 0$,

$$\vec{v}^T A \vec{v} = \lambda \|\vec{v}\|^2 < 0$$

This contradicts the definition of positive semi-definiteness, which requires $\vec{v}^T A \vec{v} \geq 0$ for all $\vec{v}$, but we got $\vec{v}^T A \vec{v} < 0$.

This means that the eigenvalues of a positive semi definite matrix A must be nonnegative.

<u>Question #6</u>

In MATLAB file.