Math 425 **Applied & Comput. Lin. Algebra** **Fall 2024**
Homework

Homework VII

For this homework, include all code and computations in a `MATLAB` file named `math425hw7.m`. You will need to submit this file along with a document containing your answers which do not involve `MATLAB`. **Do not submit a zipped (compressed) folder**.

1. Let A be a nonsingular $n \times n$ matrix with real entries and $b \in \mathbb{R}^n$. Explain carefully how you can use the SVD of A to solve the system of linear equations $Ax = b$. Why is A being nonsingular important ?

2. Let A be a nonsingular $n \times n$ matrix with real entries. How are the singular values of A and the singular values of A^{-1} related? Justify.

3. Let A be an $m \times n$ matrix with real entries a_{ij}. We will denote $\sqrt{\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}^2}$ by $||A||$.
a) Let B be a $p \times r$ matrix and C be a $r \times p$ matrix. Prove that $\text{trace}(BC) = \text{trace}(CB)$.
b) Show that $||A||^2 = \text{trace}(AA^T) = \text{trace}(A^TA)$.
c) Let U be an $m \times m$ orthogonal matrix. Prove that $||UA|| = ||A||$.
d) Now let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ be the singular values of A. Show that $||A|| = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$.

4.a) Let A be an $m \times n$ matrix with real entries and let $A = P\Sigma Q^T$ be its singular value decomposition. Let $p_1, p_2, \ldots, p_r$ be the columns of P and let $q_1, q_2, \ldots, q_r$ be the columns of Q. Show that $A = \sigma_1 p_1 q_1^T + \cdots + \sigma_r p_r q_r^T$.
4.b) Now let $A_k = P_k \Sigma_k Q_k^T$ be the truncated SVD as we have done in the class. Show that $||A - A_k|| = \sqrt{\sigma_{k+1} + \cdots + \sigma_r}$. [Hint: obtain the SVD of $A - A_k$ by using **4.a)**, then use **3.d)**]

5.a) Upload an image into your `MATLAB` directory. Using `imread` and `im2gray` (if necessary) and `im2double` store the image in a matrix A.
5.b) Compute the SVD of A using `MATLAB`, and using various truncated matrices A_k of rank k determine a small k for which the image generated from A_k is a good approximation of the image generated by from A. [`imshow` displays the image]
5.c) Pay attention to the singular values of A. By looking at them could you have predicted a good value of k ? Elaborate.