Homework 3

Bryan Lee

October 4, 2024

1. a) Let $\mathbf{v} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^n$. Show that the $m \times n$ matrix $\mathbf{vw}^T$ has a rank equal to 1. Assuming that v and w cannot be zero vectors.

$$vw^T = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \begin{pmatrix} w_1 & w_2 & \dots & w_n \end{pmatrix}$$

$$= \begin{pmatrix} v_1w_1 & v_1w_2 & \dots & v_1w_n \\ v_2w_1 & v_2w_2 & \dots & v_2w_n \\ \vdots & \vdots & \ddots & \vdots \\ v_mw_1 & v_mw_2 & \dots & v_mw_n \end{pmatrix}$$

From doing the dot product of vw^T, we acquire a matrix where each row is a scale multiple of the vector v. Specifically, the i-th row of the matrix is v_iw^T, where v_i is the i-th element of the vector v. Since all rows are multiples of the same vector v, this means that the rows are linearly dependent and, therefore, multiples of each other. Now, if we proceed with reduced row echelon reduction by applying the following row operation for all $i = 2, 3, ..., m$:

$$\frac{-v_i}{v_1} R_1 + R_i \to R_i$$

We will get the following resulting matrix:

$$\begin{pmatrix} v_1w_1 & v_1w_2 & \dots & v_1w_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Thus, we can see that the mxn matrix vw^T has a rank equal to 1.

b) Conversely, show that if A is an $m \times n$ matrix with rank(A) $= 1$, then $A = \mathbf{v}\mathbf{w}^T$ for some $\mathbf{v} \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$

Let A be an $m \times n$ matrix:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

Now, given that A is an $R^{m\times n}$ and that rank(A) $= 1$, it means that all rows of A are multiple of each other. To be more precise, the rows of A can be expressed as a scalar multiple of a non-zero row. For example, taking the first row of A (denoted by A_1), you can express the rows of the matrix A as:

$$A_i = v_i A_1$$

Where each subsequent row A_i there exists a scalar $v_i \in \mathbb{R}$.

Ultimately, you will get the following matrix:

$$A = \begin{pmatrix} v_1 a_{1,1} & v_1 a_{1,2} & \cdots & v_1 a_{1,n} \\ v_2 a_{1,1} & v_2 a_{1,2} & \cdots & v_2 a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ v_m a_{1,1} & v_m a_{1,2} & \cdots & v_m a_{1,n} \end{pmatrix}$$

Thus, to construct the matrix A, we take the <u>outer product</u> of the vectors v and w, where $w^T = (a_{1,1}, a_{1,2}, ..., a_{1,n})$. This gives us:

$$A = vw^T = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \end{pmatrix}$$

Therefore, if A is an $m \times n$ matrix with rank(A) $= 1$, then $A = vw^T$ for some $\mathbf{v} \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$.