Math 425	**Applied & Comput. Lin. Algebra**	**Fall 2024**
Homework		

Homework IV

For this homework, include all code and computations in a `MATLAB` file named `math425hw4.m`. You will need to submit this file along with a document containing your answers which do not involve `MATLAB`. Do not submit a zipped (compressed) folder.

1. Recall that an $n \times n$ matrix Q is orthogonal if the columns of Q form an orthonormal basis of $\mathbb{R}^n$. This is equivalent to $Q^T Q = QQ^T = I_n$.
a) Show that the product of two $n \times n$ orthogonal matrices is an orthogonal matrix.
b) Prove that if Q is an orthogonal matrix, so is Q^T. Deduce that the rows of an orthogonal matrix also form an orthonormal basis.
c) Show that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is an orthogonal matrix for any $0 \le \theta < 2\pi$. What does this matrix do?
d) Prove that if Q is an $n \times n$ orthogonal matrix then $||Q\mathbf{x}|| = ||\mathbf{x}||$ for any $\mathbf{x} \in \mathbb{R}^n$.

2. Let $H_n = Q_n R_n$ be the QR factorization of the $n \times n$ Hilbert matrix (see Homework 1).
a) Find Q_n and R_n for $n = 5, 10, 20$, using `MATLAB`'s command `[Q,R] = qr(A)`.
b) Let $\mathbf{x}^* \in \mathbb{R}^n$ be the vector with ith entry $\mathbf{x}^*{}_i = (-1)^i \frac{i}{i+1}$. For the values n as in part **a)** compute $\mathbf{b}^* = H_n \mathbf{x}^*$ and then solve the system $H_n \mathbf{x} = \mathbf{b}^*$ using first Gaussian elimination and then QR factorization.
c) Compare the results to the correct solution $\mathbf{x}^*$ and discuss the pros and cons of each method.

3. Create a function called `myHouseholder` which takes as input two non-zero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$. Your function should first normalize these vectors to vectors $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$, and then compute the Householder matrix H so that $H\hat{\mathbf{v}} = \hat{\mathbf{w}}$ and $H\hat{\mathbf{w}} = \hat{\mathbf{v}}$. Test your code on three randomly generated vectors in $\mathbb{R}^4$.