Homework 4

Bryan Lee

October 10, 2024

1. Recall that an $n \times n$ matrix Q is orthogonal if the columns of Q form an orthonormal basis of $\mathbb{R}^n$. This is equivalent to $Q^TQ = QQ^T = I_n$.
a) Show that the product of two $n \times n$ orthogonal matrices is an orthogonal matrix.

Let $Q_1 = Q_1Q_2$, then we have

$$Q^T = (Q_1Q_2)^T = Q_2^TQ_1^T$$

Now, to verify orthogonality (multiplying both sides with Q):

$$Q^TQ = (Q_2^TQ_1^T)(Q_1Q_2)$$
$$Q^TQ = Q_2^T(Q_1^TQ_1)Q_2$$
$$Q^TQ = Q_2^TI_nQ_2$$
$$Q^TQ = I_n$$

Thus, the product of the two nxn orthogonal matrices is an orthogonal matrix because $Q^TQ = I_n$ where Q is the product of the two nxn orthogonal matrices.

b) Prove that if Q is an orthogonal matrix, so is Q^T. Deduce that the rows of an orthogonal matrix also form an orthonormal basis.

Given that $Q^TQ = QQ^T = I_n$, then

$$(Q^TQ)^T = I_n$$

$$(Q^T)^TQ^T = I_n \Leftrightarrow QQ^T = I_n$$

This shows that if Q is an orthogonal matrix, so is Q^T. We know that if Q is orthogonal, then its columns form an orthonormal basis for $\mathbb{R}^n$. Now, since we have just proven that Q^T is also orthogonal, we can consider the columns of Q^T. The columns of Q^T are exactly the rows of Q, due to transpose swapping the rows and columns. Thus, the rows of Q also form an orthonormal basis of $\mathbb{R}^n$.

c) Show that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{pmatrix}$ is an orthogonal matrix for any $0 \leq \theta < 2\pi$. What does this matrix do?

$$Let\ Q = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

Now, to show that it is orthogonal:

$$i)\ Q^T = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$$

$$ii)\ Q^T Q = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} =$$

$$\begin{pmatrix} cos\theta cos\theta + sin\theta sin\theta & cos\theta(-sin\theta) + sin\theta cos\theta \\ (-sin\theta)cos\theta + cos\theta sin\theta & (-sin\theta)(-sin\theta) + cos\theta cos\theta \end{pmatrix} =$$

$$\begin{pmatrix} cos^2\theta + sin^2\theta & 0 \\ 0 & sin^2\theta + cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_n$$

Thus, $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{pmatrix}$ is an orthogonal matrix for any $0 \leq \theta < 2\pi$ as $Q^T Q = I_n$. The matrix represents a counterclockwise rotation by angle θ.

d) Prove that if Q is an $n \times n$ orthogonal matrix then $||Q\mathbf{x}|| = ||\mathbf{x}||$ for any $\mathbf{x} \in \mathbb{R}^n$.

The norm of a vector $x \in \mathbb{R}^n$ is defined as:

$$||x|| = \sqrt{x^T x}$$

$$||Qx|| = \sqrt{(Qx)^T (Qx)}$$

Note: $(Qx)^T \Leftrightarrow x^T Q^T$

$$||Qx|| = \sqrt{x^T Q^T Q x}$$

$$||Qx|| = \sqrt{x^T x} = ||x||$$

Thus, if Q is an nxn orthogonal matrix then $||Qx|| = ||x||$ for any $x \in \mathbb{R}^n$.