#5

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Given that L is a lower-triangular matrix with diagonal 1s & U is an upper-triangular matrix with nonzero diagonals:

Say a 3×3 dimension:

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

also called "Doolittle algorithm"

we can compute the entries of L & U w/o Gaussian elimination by doing matrix multiplication (row · column) and computing each result with prior information.

i.e. for the 3×3 matrix above:

$$\left.\begin{aligned} u_{11} &= a_{11} \\ u_{12} &= a_{12} \\ u_{13} &= a_{13} \end{aligned}\right\}$$ note: no multiplication / division

$l_{21}u_{11} = a_{21}$

we already have u_{11} so we do 1 division to get l_{21}

$l_{31}u_{11} = a_{31}$ — 1 division to get l_{31}

note: 1 multiplication / division

$l_{21}u_{12} + (1 \cdot u_{22}) = a_{22}$

1 multiplication to compute u_{22}

$l_{21}u_{13} + (1 \cdot u_{23}) = a_{23}$ — also 1 multiplication to get u_{23}

note: 1 multiplication / division

$l_{31}u_{12} + l_{32}u_{22} = a_{32}$

$l_{31}u_{13} + l_{32}u_{23} + (1 \cdot u_{33}) = a_{33}$

note: 2 multiplication / division using same process above

These 9 equations (steps of matrix multiplication) can be used to compute the entries of L & U w/o Gaussian elimination, where both L & U are 3×3 matrices

For the # of multiplication / division to compute each entry, we can see a pattern: (ignore xs they are placeholders)

$$\begin{array}{cc} L & U \\ \begin{bmatrix} x & x & x \\ 1 & x & x \\ 1 & 2 & x \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ x & 1 & 1 \\ x & x & 2 \end{bmatrix} \end{array} = A_{3\times3}$$

If we now do a 4×4:

$$\begin{bmatrix} x & & & \\ 1 & x & & \\ 1 & 2 & x & \\ 1 & 2 & 3 & x \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 1 & 1 & 1 \\ x & x & 2 & 2 \\ x & x & x & 3 \end{bmatrix} = A_{4\times4}$$

☆ These values in the matrices are the # of multiplication / division needed to compute the said entry of L & U.

For 5×5:

$$\begin{bmatrix} x & & & & \\ 1 & x & & & \\ 1 & 2 & x & & \\ 1 & 2 & 3 & x & \\ 1 & 2 & 3 & 4 & x \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ x & 1 & 1 & 1 & 1 \\ x & x & 2 & 2 & 2 \\ x & x & x & 3 & 3 \\ x & x & x & x & 4 \end{bmatrix} = A_{5\times5}$$

and so on following this triangular shaped pattern ...

Comparing the answer with question 3,

A	3x3 matrix	4x4 matrix	5x5 matrix
Gauss	11	26	50
Inverse A	54	128	250
LU w/o Gauss	8	20	40

\# of multiplication/division needed

∴ we can see that computing L & U w/o Gaussian elimination and using the above computational method gives us a much faster compute time.