Math 425 Homework	**Applied & Comput. Lin. Algebra**	**Fall 2024**

Homework VI

For this homework, include all code and computations in a `MATLAB` file named `math425hw6.m`. You will need to submit this file along with a document containing your answers which do not involve `MATLAB`. **Do not submit a zipped (compressed) folder**.

1.a) Suppose λ is an eigenvalue of A. Show that $c\lambda + d$ is an eigenvalue of $B = cA + dI$ for scalars c and d.
1.b) Prove that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k for any positive integere k.
1.c) Show that $\lambda = 0$ is an eigenvalue of A if and only if A is singular. Conclude that the dimension of the subspace consisting of eigenvectors with eigenvalue $\lambda = 0$ is the dimension of the kernel of A.
1.d) Use part **c)** to compute all eigenvalues and eigenvectors of the $n \times n$ matrix A where each entry of A is equal to one.
1.e) Let A be a nonsingular matrix and let λ be an eigenvalue of A. Prove that λ^{-1} is an eigenvalue of A^{-1}.

2.a) Let $u \in \mathbb{R}^n$ be a unit vector (in the standard Euclidean inner product). Compute the eigenvectors and eigenvalues of $A = uu^T$. [Hint: what is the rank of A ?]
2.b) Compute the eigenvalues of the Householder matrix $H = I - 2uu^T$.
2.c) Compute the eigenvalues of a matrix P such that $P^2 = P$ (such as a projection matrix).

3. Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$. Is A diagonalizable ? Justify your answer.

4. Construct a real matrix with eigenvalues $0, 2, -2$ and the corresponding eigenvectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$.

5. A real symmetrix $n \times n$ matrix A is called positive semidefinite if for all $v \in \mathbb{R}^n$, $v^T A v \geq 0$.
a) Show that all diagonal entries of a positive semidefinite matrix are nonnegative [Hint: use $v = e_i$].
b) Prove that if all eigenvalues of a symmetric matrix A are nonnegative then A is a positive semidefinite matrix [Hint: use the spectral decomposition of A].
c) Conversely, prove that the eigenvalues of positive semidefinite matrix are nonnegative [Hint: to prove by contradiction assume the matrix has a negative eigenvalue with corresponding eigenvector v. Now compute $v^T A v$, and again use the spectral decomposition].

6. Write a `MATLAB` script that will compute the eigenvalues and the corresponding set of orthonormal eigenvectors of a symmetric matrix using the algorithm we learned in class based on the QR-factorization. Your algorithm should terminate when all off-diagonal entries of $A^{(k)}$ are smaller than $\varepsilon = 10^{-6}$.