Statistics 101C - Week 2 - Thursday

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Logistic Regression and Logistic Loss function

- Logistic Loss function:

$$L(f(\boldsymbol{x}_i), y_i) = \log\Big(1 + \exp(-f(\boldsymbol{x}_i)y_i)\Big)$$

- The expected logistic loss (Logistic Risk):

$$\begin{aligned} R_{\log}(f) =& \mathbb{E}_{\boldsymbol{X},Y}\left(L(f(\boldsymbol{X}), Y)\right) \\ =& \mathbb{E}_{\boldsymbol{X}}\Big[\mathbb{P}(Y=1|\boldsymbol{X})\log\Big(1+\exp(-f(\boldsymbol{X}))\Big) + \mathbb{P}(Y=-1|\boldsymbol{X})\log\Big(1+\exp(f(\boldsymbol{X}))\Big) \end{aligned}$$

- The optimal function minimizing $R_{\log}(f)$ is defined as

$$f^*_{log}(\boldsymbol{X}) = \log\left(\frac{\mathbb{P}(Y=1|\boldsymbol{X})}{1-\mathbb{P}(Y=1|\boldsymbol{X})}\right)$$

Logistic Loss for classification

- We can construct a classifier as $\text{sign}(f^*_{log}(\boldsymbol{x})) \in \{-1, 1\}$. This classifier is identical to the Bayes classifier.

$$\text{sign}(f^*_{log}(\boldsymbol{x})) = \text{sign}(\eta(\boldsymbol{x}) - 1/2)$$

Logistic Loss for classification

- Suppose we have a dataset $D = \{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$, and we employ the logistic loss for classification. Then the training error in terms of logistic loss can be written as

$$R_n(f) = \frac{1}{n}\sum_{i=1}^{n} \log\left(1 + \exp(-f(\boldsymbol{x}_i)y_i)\right)$$

where $y_i \in \{-1, 1\}$.

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  where $y_i \in \{-1, 1\}$.
- Here we do not make any assumption of f.
- **Question**: What if we assume that $f(\boldsymbol{x}_i) = \beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i$

If $f(\boldsymbol{x}_i) = \beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i$

- $R_n(f)$ can be further written as

$$
\begin{aligned}
R_n(f) =& \frac{1}{n} \sum_{i=1}^{n} \log\Big(1 + \exp(-f(\boldsymbol{x}_i) y_i)\Big) \\
=& \frac{1}{n} \sum_{i=1}^{n} \log\Big(1 + \exp\big(-(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i) y_i\big)\Big)
\end{aligned}
$$

- We suppose that $y_i = 1 \Leftrightarrow \widetilde{y}_i = 1$ and $y_i = -1 \Leftrightarrow \widetilde{y}_i = 0$

$$
\begin{aligned}
R_n(f) = \frac{1}{n} \sum_{i=1}^{n} \Big[& \widetilde{y}_i \log\Big(1 + \exp\big(-(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i)\big)\Big) \\
& + (1 - \widetilde{y}_i) \log\Big(1 + \exp\big(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i\big)\Big)\Big]
\end{aligned}
$$

If $f(\boldsymbol{x}_i) = \beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i$

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& + (1 - \widetilde{y}_i) \log\Big(1 + \exp\big(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i\big)\Big)\Big] \\
= \frac{1}{n}\sum_{i=1}^{n} \Big[& \widetilde{y}_i \log\Big(\exp\big(-(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i)\big)\Big) \\
& + \log\Big(1 + \exp\big(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i\big)\Big)\Big]
\end{aligned}
$$

Logistic Regression

- **Logistic regression** estimates the conditional probability probability:

$$\mathbb{P}\left(\widetilde{Y} = 1 \middle| \boldsymbol{X}\right)$$

- In logistic regression, it is assumed that

$$\mathbb{P}\left(\widetilde{Y} = 1 \middle| \boldsymbol{X} = \boldsymbol{x}\right) = \frac{\exp(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x})}{1 + \exp(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x})},$$
$$\mathbb{P}\left(\widetilde{Y} = 0 \middle| \boldsymbol{X} = \boldsymbol{x}\right) = \frac{1}{1 + \exp(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x})},$$

where

- $\boldsymbol{x} = (x_1, \ldots, x_p)^T$ is a p-dimensional predictor
- β_0 and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)$ are unknown parameters
- $\boldsymbol{\beta}^T \boldsymbol{x} = \sum_{l=1}^{p} \beta_i x_i$

Estimation in Logistic Regression

- Suppose a dataset in logistic regression is $\{(\boldsymbol{x}_i, \widetilde{y}_i)\}_{i=1}^{n}$, here $\widetilde{y}_i \in \{0, 1\}$.
- Likelihood function $L(\beta_0, \boldsymbol{\beta})$:

$$L(\beta_0, \boldsymbol{\beta}) = \prod_{i=1}^{n} \left(\mathbb{P}\left(\widetilde{Y} = 1 \middle| \boldsymbol{X} = \boldsymbol{x}\right)\right)^{\widetilde{y}_i} \left(\mathbb{P}\left(\widetilde{Y} = 0 \middle| \boldsymbol{X} = \boldsymbol{x}\right)\right)^{1-\widetilde{y}_i}$$

Negative Log-likelihood

Negative Log-likelihood:

$$
\begin{aligned}
& -\log L(\beta_0, \boldsymbol{\beta}) \\
= & -\sum_{i=1}^{n} \left[\widetilde{y}_i \log\left(\frac{\exp(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i)}{1 + \exp(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i)} \right) + (1 - \widetilde{y}_i) \log\left(\frac{1}{1 + \exp(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i)} \right) \right] \\
= & \sum_{i=1}^{n} \left[\widetilde{y}_i \log\Big(\exp(-\beta_0 - \boldsymbol{\beta}^T \boldsymbol{x}_i) \Big) + \log\Big(1 + \exp(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i) \Big) \right]
\end{aligned}
$$

If we take the expectation with respect to y_i, we have

$$
\begin{aligned}
& \mathbb{P}(\widetilde{y}_i = 1 | \boldsymbol{x}_i) \log(1 + \exp(-\beta_0 - \boldsymbol{\beta}^T \boldsymbol{x}_i)) \\
+ & \mathbb{P}(\widetilde{y}_i = 0 | \boldsymbol{x}_i) \log(1 + \exp(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}_i))
\end{aligned}
$$

Conclusion

- Logistic Regression is an special example of using **Logistic loss** for classification.
- Logistic Regression assumes that the $\mathbb{P}(\widetilde{Y} = 1|\boldsymbol{X} = \boldsymbol{x})$ as

$$\eta(\boldsymbol{x}) = \mathbb{P}(\widetilde{Y} = 1|\boldsymbol{X} = \boldsymbol{x}) = \frac{\exp(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x})}{1 + \exp(\beta_0 + \boldsymbol{\beta}^T \boldsymbol{x})}.$$

- The optimal function minimize the logistic risk is $f^*(\boldsymbol{x}) = \log(\frac{\eta(\boldsymbol{x})}{1-\eta(\boldsymbol{x})}) = \beta_0 + \boldsymbol{\beta}^T \boldsymbol{x}$.

Relationship: Logistic Loss and Logistic Regression

Training Error (0-1): $\frac{1}{n}\sum_{i=1}^{n} I(f(x_i) \neq y_i)$

$\Downarrow$

Training Error (Logistic): $\frac{1}{n}\sum_{i=1}^{n} L(f(x_i), y_i)$

$\Downarrow$

Training Error (Logistic): $\frac{1}{n}\sum_{i=1}^{n} L(f(x_i), y_i)$ $\Leftarrow$ Consider: $f(x_i)=\beta_0 + \beta^T x$

Logistic Regression

Contents

Classification

- A typical dataset in classification $\mathcal{D} = \{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$.
  - $\boldsymbol{x}_i$: the covariate vector of i-th instance
  - $y_i \in \{0, 1\}$: binary label of i-th instance

- Bayes classifier f^*:

$$f^*(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \mathbb{P}(Y = 1|\boldsymbol{X} = \boldsymbol{x}) > 1/2 \\ 0 & \text{if } \mathbb{P}(Y = 1|\boldsymbol{X} = \boldsymbol{x}) < 1/2 \end{cases}$$

- Minimal risk $R(f^*)$:

$$R(f^*) = \mathbb{E}\Big[f^*(\boldsymbol{X}) \neq Y\Big] = \mathbb{E}\Big[\min(\eta(\boldsymbol{X}), 1 - \eta(\boldsymbol{X}))\Big],$$

where $\eta(\boldsymbol{x}) = \mathbb{P}(Y = 1|\boldsymbol{X} = \boldsymbol{x})$.

How can we construct classifier?

- **Discriminative models**
  - Discriminative modeling studies the $P(Y|\boldsymbol{X})$
  - Examples: Logistic regression (LR)
- **Generative models**
  - Generative models studies the joint probability distribution $\mathbb{P}(\boldsymbol{X}, Y)$
  - Examples: linear discriminant analysis and quadratic discriminant analysis

Basics of Generative models

- LDA and QDA are **generative models**, we need to consider the structure of $\mathbb{P}(\boldsymbol{X}, Y)$

$$\mathbb{P}(\boldsymbol{X}, Y) = \mathbb{P}(\boldsymbol{X}|Y)\mathbb{P}(Y)$$
$$\mathbb{P}(\boldsymbol{X}, Y) = \mathbb{P}(Y|\boldsymbol{X})\mathbb{P}(\boldsymbol{X})$$

An alternative look

Let $k \in \{0, 1\}$. We can develop an alternative formulation of $\mathbb{P}(Y = k|\boldsymbol{X} = \boldsymbol{x})$ from the definition of conditional probability.

$$\begin{aligned}\mathbb{P}(Y = k|\boldsymbol{X} = \boldsymbol{x}) = \frac{\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}, Y = k)}{\mathbb{P}(\boldsymbol{X} = \boldsymbol{x})} &= \frac{\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}|Y = k) \cdot \mathbb{P}(Y = k)}{\mathbb{P}(\boldsymbol{X} = \boldsymbol{x})} \\ &= \frac{\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}|Y = k) \cdot \mathbb{P}(Y = k)}{\sum_{k=0}^{1} \mathbb{P}(\boldsymbol{X} = \boldsymbol{x}|Y = k) \cdot \mathbb{P}(Y = k)}\end{aligned}$$

- $\mathbb{P}(\boldsymbol{X} = \boldsymbol{x})$ the marginal distribution
- $\mathbb{P}(Y = k|\boldsymbol{X} = \boldsymbol{x})$: given $\boldsymbol{X} = \boldsymbol{x}$ the probability that outcome $Y = k$.

Banknote Dataset

conterfeit	Length	Left	Right	Bottom	Top	Diagonal
0	214.70000	129.70000	129.30000	8.60000	9.60000	141.60000
0	215.40000	130.00000	129.90000	8.50000	9.70000	141.40000
0	214.90000	129.40000	129.50000	8.20000	9.90000	141.50000
0	214.50000	129.50000	129.30000	7.40000	10.70000	141.50000
0	214.70000	129.60000	129.50000	8.30000	10.00000	142.00000
0	215.60000	129.90000	129.90000	9.00000	9.50000	141.70000
0	215.00000	130.40000	130.30000	9.10000	10.20000	141.10000
0	214.40000	129.70000	129.50000	8.00000	10.30000	141.20000
0	215.10000	130.00000	129.80000	9.10000	10.20000	141.50000
0	214.70000	130.00000	129.40000	7.80000	10.00000	141.20000
1	214.40000	130.10000	130.30000	9.70000	11.70000	139.80000
1	214.90000	130.50000	130.20000	11.00000	11.50000	139.50000
1	214.90000	130.30000	130.10000	8.70000	11.70000	140.20000
1	215.00000	130.40000	130.60000	9.90000	10.90000	140.30000
1	214.70000	130.20000	130.30000	11.80000	10.90000	139.70000
1	215.00000	130.20000	130.20000	10.60000	10.70000	139.90000
1	215.30000	130.30000	130.10000	9.30000	12.10000	140.20000

Discriminant Analysis (DA)

Discriminant Analysis models $\mathbb{P}(Y|\boldsymbol{X})$ as follows:

- Step 1: Make assumptions on data structure

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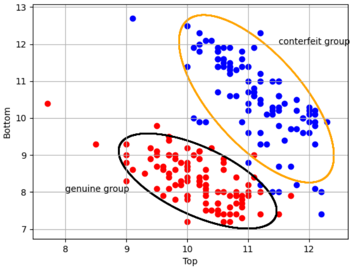

Figure: Black ellipsoid: covariance structure of genuine group. Green ellipsoid: covariance structure of the counterfeit group

Discriminant Analysis (DA)

- $\mathbb{P}(\boldsymbol{X} = \boldsymbol{x} | Y = k)$ is a multivariate normal distribution with mean $\boldsymbol{\mu}_k$ and covariance matrix $\boldsymbol{\Sigma}_k$.

$$\mathbb{P}(\boldsymbol{X} = \boldsymbol{x} | Y = k) = \frac{1}{(2\pi)^{p/2} |\boldsymbol{\Sigma}_k|^{1/2}} \exp\Big(-\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu}_k)^T \boldsymbol{\Sigma}_k^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_k) \Big),$$

where

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}, \boldsymbol{\mu}_k = \begin{pmatrix} \mu_{1,k} \\ \mu_{2,k} \\ \vdots \\ \mu_{p,k} \end{pmatrix}, \boldsymbol{\Sigma}_k = \begin{pmatrix} \sigma^2_{1,1,k} & \sigma^2_{1,2,k} & \cdots & \sigma^2_{2,2,k} \\ \sigma^2_{2,1,k} & \sigma^2_{2,2,k} & \cdots & \sigma^2_{2,p,k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2_{p,1,k} & \sigma^2_{p,2,k} & \cdots & \sigma^2_{p,p,k} \end{pmatrix}.$$

Discriminant Analysis (DA)

- Step 2: We use the Bayes' theorem to compute $\mathbb{P}(Y = k|\boldsymbol{X} = \boldsymbol{x}), k = 0, 1.$

$$\mathbb{P}(Y = k|\boldsymbol{X} = \boldsymbol{x}) = \frac{\pi_k \mathbb{P}(\boldsymbol{X} = \boldsymbol{x}|Y = k)}{\pi_1 \mathbb{P}(\boldsymbol{X} = \boldsymbol{x}|Y = 1) + \pi_0 \mathbb{P}(\boldsymbol{X} = \boldsymbol{x}|Y = 0)}$$

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Question: What is the difference between Linear and Quadratic discriminant analyses?

Discriminant Analysis (DA)

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Question: What is the difference between Linear and Quadratic discriminant analyses?

- **Linear** Discriminant Analysis (LDA) assumes that the classes have a common covariance matrix. In other words, that is $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}_1$
- **Quadratic** Discriminant Analysis (QDA) does not assumes this. So, we have a covariance matrix $\boldsymbol{\Sigma}_0$ for class 0 and $\boldsymbol{\Sigma}_1$ for class 1.

Linear Discriminant Analysis (LDA)

Three Assumptions in LDA

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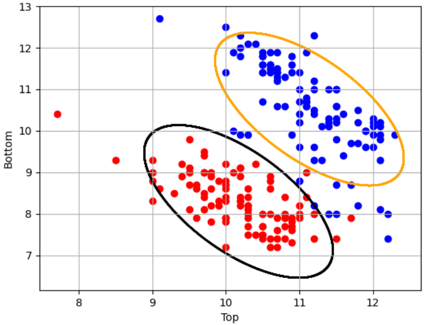
1. Multivariate normal distribution for each group, that $\mathbb{P}(\boldsymbol{X} = \boldsymbol{x} | Y = k)$ is multivariate normal
2. They have different mean vectors
3. **Same covariance matrices**



Use LDA for classification

We make predictions using LDA as follows:

$$f_{LDA}(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \frac{\pi_1 \mathbb{P}(\boldsymbol{X}=\boldsymbol{x}|Y=1)}{\pi_1 \mathbb{P}(\boldsymbol{X}=\boldsymbol{x}|Y=1)+\pi_0 \mathbb{P}(\boldsymbol{X}=\boldsymbol{x}|Y=0)} > 0.5 \\ 0, & \text{if } \frac{\pi_1 \mathbb{P}(\boldsymbol{X}=\boldsymbol{x}|Y=1)}{\pi_1 \mathbb{P}(\boldsymbol{X}=\boldsymbol{x}|Y=1)+\pi_0 \mathbb{P}(\boldsymbol{X}=\boldsymbol{x}|Y=0)} \leq 0.5 \end{cases}$$

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Conclusions we can make

1. Similar to the Bayes classifier, we classify to the most probable class using the posterior probability

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Conclusions we can make

1. Similar to the Bayes classifier, we classify to the most probable class using the posterior probability
2. The decision boundary can be easily derived as

$$\frac{\pi_1 \mathbb{P}(\boldsymbol{X} = \boldsymbol{x}|Y = 1)}{\pi_1 \mathbb{P}(\boldsymbol{X} = \boldsymbol{x}|Y = 1) + \pi_0 \mathbb{P}(\boldsymbol{X} = \boldsymbol{x}|Y = 0)} = 1/2$$

$$\Leftrightarrow \log \frac{\pi_1}{\pi_0} + \log \frac{\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}|Y = 1)}{\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}|Y = 0)} = 0.$$

Decision boundary in LDA

A closer look at the decision boundary.

$$\log \frac{\pi_1}{\pi_0} + \log \frac{\mathbb{P}(\boldsymbol{X} = \boldsymbol{x} | Y = 1)}{\mathbb{P}(\boldsymbol{X} = \boldsymbol{x} | Y = 0)} = 0$$

$$\Updownarrow$$

$$\log \frac{\pi_1}{\pi_0} + \boldsymbol{x}^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0) - \frac{1}{2} \boldsymbol{\mu}_1^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_1 + \frac{1}{2} \boldsymbol{\mu}_0^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_0 = 0$$

$$\Updownarrow$$

$$\log \frac{\pi_1}{\pi_0} + \boldsymbol{x}^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0) - \frac{1}{2} (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0) = 0.$$

Decision boundary in LDA

A closer look at the decision boundary.

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$$\log\frac{\pi_1}{\pi_0} + \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) - \frac{1}{2}(\boldsymbol{\mu_1}+\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu_1}-\boldsymbol{\mu}_0) = 0.$$

The decision boundary can be written as (a linear equation)

$$\boldsymbol{x}^T C_1(\boldsymbol{\mu}_0,\boldsymbol{\mu}_1,\boldsymbol{\Sigma}) + C_2(\boldsymbol{\mu}_0,\boldsymbol{\mu}_1,\boldsymbol{\Sigma}) = 0,$$

where $C_1(\boldsymbol{\mu}_0,\boldsymbol{\mu}_1,\boldsymbol{\Sigma}) = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)$ and
$C_2(\boldsymbol{\mu}_0,\boldsymbol{\mu}_1,\boldsymbol{\Sigma}) = \log\frac{\pi_1}{\pi_0} - \frac{1}{2}(\boldsymbol{\mu_1}+\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu_1}-\boldsymbol{\mu}_0)$.

Parameter estimation in LDA

Thanks to the formulation of LDA, we can easily estimate its parameters.

- The prior probability π_0 and π_1.

$$\widehat{\pi}_0 = \frac{n_0}{n_0 + n_1} \text{ and } \widehat{\pi}_1 = \frac{n_1}{n_0 + n_1},$$

  where n_k is the number of observations in the training data set that belong to class.

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- The means are estimated as

$$\widehat{\boldsymbol{\mu}}_k = \frac{1}{n_k} \sum_{i:y_i=k} \boldsymbol{x}_i, k = 0, 1$$

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- The means are estimated as

$$\widehat{\boldsymbol{\mu}}_k = \frac{1}{n_k} \sum_{i:y_i=k} \boldsymbol{x}_i, k = 0, 1$$

- The covariance matrices are estimated as

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{n-2} \sum_{k=0}^{1} \sum_{i:y_i=k} (\boldsymbol{x}_i - \widehat{\boldsymbol{\mu}}_k)(\boldsymbol{x}_i - \widehat{\boldsymbol{\mu}}_k)^T$$

Quadratic Discriminant Analysis (QDA)

Three Assumptions in QDA

1. Multivariate normal distribution for each group, that $\mathbb{P}(\boldsymbol{X} = \boldsymbol{x} | Y = k)$ is multivariate normal
2. They have different mean vectors
3. **Different covariance matrices**

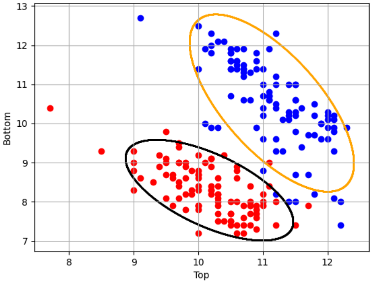

Figure: Different covariance structures

Decision boundary in QDA

We follow a similar analysis of QDA as with LDA. After some algebra, we arrive to the following (interesting) equation:

$$\log \frac{\pi_1}{\pi_0} - \frac{1}{2}\boldsymbol{x}^T(\boldsymbol{\Sigma}_1^{-1} - \boldsymbol{\Sigma}_0^{-1})\boldsymbol{x} + \boldsymbol{x}^T(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0) + \cdots = 0$$

Conclusion

- The decision boundary in QDA is a quadratic function

LDA vs QDA

The difference between LDA and QDA can be summarized as

- LDA is simpler than QDA. (LDA is a special case of QDA)
- QDA needs to estimate more parameters. One covariance matrix for each class.
- LDA is much less flexible than QDA, but this also means that it has low variance
- If the assumptions of LDA do not hold, then it can lead to poor estimates and so, a high bias.

Exercise: Prediction of counterfeit banknotes

conterfeit	Length	Left	Right	Bottom	Top	Diagonal
0	214.70000	129.70000	129.30000	8.60000	9.60000	141.60000
0	215.40000	130.00000	129.90000	8.50000	9.70000	141.40000
0	214.90000	129.40000	129.50000	8.20000	9.90000	141.50000
0	214.50000	129.50000	129.30000	7.40000	10.70000	141.50000
0	214.70000	129.60000	129.50000	8.30000	10.00000	142.00000
0	215.60000	129.90000	129.90000	9.00000	9.50000	141.70000
0	215.00000	130.40000	130.30000	9.10000	10.20000	141.10000
0	214.40000	129.70000	129.50000	8.00000	10.30000	141.20000
0	215.10000	130.00000	129.80000	9.10000	10.20000	141.50000
0	214.70000	130.00000	129.40000	7.80000	10.00000	141.20000
1	214.40000	130.10000	130.30000	9.70000	11.70000	139.80000
1	214.90000	130.50000	130.20000	11.00000	11.50000	139.50000
1	214.90000	130.30000	130.10000	8.70000	11.70000	140.20000
1	215.00000	130.40000	130.60000	9.90000	10.90000	140.30000
1	214.70000	130.20000	130.30000	11.80000	10.90000	139.70000
1	215.00000	130.20000	130.20000	10.60000	10.70000	139.90000
1	215.30000	130.30000	130.10000	9.30000	12.10000	140.20000

- Length: length of banknote (mm)
- Left: length of left edge (mm)
- Right: length of right edge (mm)
- Top: distance from the image to top edge
- Bottom: distance from image to bottom
- Diagonal: length of diagonal (mm)
- counterfeit: 1 means counterfeit and 0 means genuine

Exercise: Prediction of counterfeit banknotes using R

- Step 1: Loading the dataset and split the dataset into training set and testing set:

```
library(mclust)
# Load the data set.
data(banknote)
banknote$Status<-factor(banknote$Status,levels=c("genuine", "counterfeit"))
# Split into training and test data.
set.seed(123) # Set seed to reproduce results.
i <- 1:dim(banknote)[1]
# Generate a random sample.
i.train <- sample(i, 130, replace = F) # 130 samples are used for training
bn.train <- banknote[i.train,] # training dataset
bn.test <- banknote[-i.train,] # testing dataset
```

- Step 2: Implement LDA and make prediction by LDA

```
library(MASS)
lda.mod <- lda(Status~Length + Right + Left + Top, data = bn.train) # Fit a LD
pred.lda.test <- predict(lda.mod,bn.test[,-1])

table('Reference' = bn.test[,1], "Predicted" =
       pred.lda.test$class)
```

Exercise: Prediction of counterfeit banknotes using R

- Result:

```
> table('Reference' = bn.test[,1], "Predicted" =
+         pred.lda.test$class)
             Predicted
Reference     genuine counterfeit
  genuine          30           3
  counterfeit       4          33
```

- Conclusion: The prediction accuracy of LDA is $(30+33)/70=0.9$.

Exercise: Prediction of counterfeit banknotes using R

- Implementation of QDA

```
qda.mod <- qda(Status~Length + Right + Left + Top, data = bn.train)
pred.qda.test <- predict(qda.mod, bn.test[,-1])
table('Reference' = bn.test[,1],
      "Predicted" = pred.qda.test$class)
```

- Result:

```
> table('Reference' = bn.test[,1],
+       "Predicted" = pred.qda.test$class)
             Predicted
Reference     genuine counterfeit
  genuine          29           4
  counterfeit       3          34
```

- Conclusion: The prediction accuracy of LDA is $(29+34)/70=0.9$. No improvement is observed.

Some questions

- Can you finish an implementation of LDA and QDA in R or Python?
- Can you summarize the difference between Logistic regression, LDA, and QDA?
- What is the difference between generative model and discriminative model?
- Is K-nearest neighbor classifier a generative model or a discriminative model?