Math 231A HW 1

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September 29, 2025

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Problem 1

Show that $\mathrm{GL}_n(\mathbb{R})$ has a smooth manifold structure.

Solution: First, we'll view $\mathrm{GL}_n(\mathbb{R}) \subsetneq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ (as $\mathbb{R}$-vector space). Then, if consider the determinant function $\det : M_n(\mathbb{R}) \to \mathbb{R}$, it is precisely a polynomial function in terms of the n^2 entries of the $n \times n$ matrices in $M_n(\mathbb{R})$. Which, since polynomial functions are smooth, then the preimage of any open set under det function is guaranteed to be open also.

Since for all $A \in M_n(\mathbb{R})$, we have $A \in \mathrm{GL}_n(\mathbb{R}) \iff \det(A) \neq 0$, then $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$, which since $\mathbb{R} \setminus \{0\}$ is an open set in $\mathbb{R}$, so is $\mathrm{GL}_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$ under the standard topology of $M_n(\mathbb{R})$ (as $\mathbb{R}^{n^2}$).

Now, since $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ is a finite dimensional $\mathbb{R}$-vector space, then $\mathrm{GL}_n(\mathbb{R})$ as an open subset of it, then for every $A \in \mathrm{GL}_n(\mathbb{R})$ there exists radius $r > 0$, such that $B_r(A) \subseteq \mathrm{GL}_n(\mathbb{R})$ (where the open ball $B_r(A)$ uses Euclidean Norm), then the identity function $\mathrm{id} : B_r(A) \xrightarrow{\sim} B_r(A)$ is precisely the homeomorphism of an open neighborhood of A in $\mathrm{GL}_n(\mathbb{R})$, to an open neighborhood in $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, and all transition maps are simply the composition of restrictions of identity function, hence is smooth. This shows that $\mathrm{GL}_n(\mathbb{R})$ has a smooth manifold structure, and with dimension n^2.

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Problem 2

Show that $\mathrm{SL}_n(\mathbb{R})$ has a smooth manifold structure.

Solution: Our general goal is to prove that the determinant function is a submersion on $\mathrm{GL}_n(\mathbb{R})$ (which contains $\mathrm{SL}_n(\mathbb{R})$), so one can utilize **Proposition 1.18** (also mentioned in **Problem 3**) to prove that $\mathrm{SL}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ forms a smooth submanifold.

First, we can again view $M_n(\mathbb{R}) := \mathbb{R}^{n^2}$ as default, and view $\det : M_n(\mathbb{R}) \to \mathbb{R}$ as a smooth function (or polynomial) with variables being the n^2 entries $(x_{ij})_{1\leq i,j\leq n}$ of the $n \times n$ matrix.

Recall that $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ (because a matrix $A \in \mathrm{GL}_n(\mathbb{R}) \iff \det(A) \neq 0$).

Since det has codomain being $\mathbb{R}$, which the tangent space at any point has dimension 1, hence to prove that det is a submersion on $\mathrm{GL}_n(\mathbb{R})$, it suffices to prove that the differential of det is nonzero at all $A \in \mathrm{GL}_n(\mathbb{R})$.

If consider the $n \times n$ matrix X of indeterminates as follow:

$$X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix} \tag{2.1}$$

Then, for any index $i \in \{1, \dots, n\}$, $\det(X) = \sum_{j=1}^{n} (-1)^{i+j} x_{ij} \cdot \det(M_{ij})$, where M_{ij} is the $(n-1) \times (n-1)$ minor after deleting the i^{th} row and the j^{th} column of X.

(**Rmk:** M_{ij} doesn't contain any x_{ik} since these indeterminates are all on the i^{th} row, hence $\det(M_{ij})$ is independent from all x_{ik}, or $\frac{\partial}{\partial x_{ik}} \det(M_{ij}) = 0$).

Hence, if consider the partial derivative with respect to any x_{ik} of det, we get:

$$\frac{\partial \det(X)}{\partial x_{ik}} = \sum_{j=1}^{n} (-1)^{i+j} \frac{\partial}{\partial x_{ik}} \left(x_{ij} \cdot \det(M_{ij})\right) \tag{2.2}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} \left(\frac{\partial x_{ij}}{\partial x_{ik}} \cdot \det(M_{ij}) + x_{ij} \cdot \frac{\partial}{\partial x_{ik}} \left(\det(M_{ij})\right) \right) \tag{2.3}$$

$$= \sum_{k=1}^{n} (-1)^{i+j} \delta_{jk} \cdot \det(M_{ij}) = (-1)^{i+k} \det(M_{ik}) \tag{2.4}$$

Where δ_{jk} representes the Kronecker Delta. So, since differential of det can be expressed as a $1 \times n^2$ matrix (linear map from $\mathbb{R}^{n^2} \to \mathbb{R}$) that has entries being all $\frac{\partial \det(X)}{\partial x_{ik}}$, to show that differential of det is nonzero for all $A \in \mathrm{GL}_n(\mathbb{R})$, it suffices to show that one of the above partial derivative - which is determinant of an $(n-1) \times (n-1)$ minor of A - is nonzero. Equivalently, it suffices to find an invertible $(n-1) \times (n-1)$ minor of A.

For all $A \in \mathrm{GL}_n(\mathbb{R})$, since it's invertible, the collection of any (nonempty) list of its column vectors is linearly independent. WLOG, if we pick $v_1, \dots, v_{n-1}$ (represents the 1^{st} to $(n-1)^{th}$ column vector of A respectively), then the $n \times (n-1)$ matrix B formed by column vectors $v_1, \dots, v_{n-1} \in \mathbb{R}^n$ has column rank $(n-1)$ (due to the linear independence of $v_1, \dots, v_{n-1}$). Since any matrix satisfies column rank = row rank, it means B has row rank $(n-1)$.

For the n row vectors of B, say $w_1, ..., w_n \in \mathbb{R}^{n-1}$, with the row rank of B being $(n-1)$, then $\text{span}\{w_1, ..., w_n\}$ is an $(n-1)$-dimensional $\mathbb{R}$-subpsace of $\mathbb{R}^{n-1}$, or $\text{span}\{w_1, ..., w_n\} = \mathbb{R}^{n-1}$. Hence, the list $\{w_1, ..., w_n\}$ is a spanning set of $\mathbb{R}^{n-1}$, which can be reduced to a basis of $\mathbb{R}^{n-1}$ (a sublist of $(n-1)$ vectors from $\{w_1, ..., w_n\}$ spanning $\mathbb{R}^{n-1}$).

Therefore, there exists $j \in \{1, ..., n\}$, such that the list of $(n-1)$ row vectors $\{w_1, ..., w_{j-1}, w_{j+1}, ..., w_n\}$ spans $\mathbb{R}^{n-1}$, or it is a basis. Hence, the $(n-1) \times (n-1)$ minor of B (denote as C) by removing the j^{th} row (removing w_j) is in fact invertible (since its rwo vectors form a basis of $\mathbb{R}^{n-1}$). The below is the matrix we choose:

$$A = \begin{pmatrix} | & | & & | & | \\ v_1 & v_2 & ... & v_{n-1} & v_n \\ | & | & & | & | \end{pmatrix}, \quad B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & ... & v_{n-1} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} - & w_1 & - \\ - & w_2 & - \\ & \vdots & \\ - & w_n & - \end{pmatrix} \tag{2.5}$$

$$C = \begin{pmatrix} - & w_1 & - \\ & \vdots & \\ - & w_{j-1} & - \\ - & w_{j+1} & - \\ & \vdots & \\ - & w_n & - \end{pmatrix} \tag{2.6}$$

However, since C can be obtained by removing the n^{th} column of A (to obtain B) then removing the j^{th} row of B, it's the same matrix we obtained by removing the n^{th} column and j^{th} row of A, so $C = M_{jn}$ (the $(n-1) \times (n-1)$ minor of A by removing the j^{th} row and n^{th} column). This shows that A indeed has an invertible $(n-1) \times (n-1)$ minor.

Finally, with the previous derivation, we've shown that the differential of det function at any $A \in \text{GL}_n(\mathbb{R})$ has one of the entries being nonzero (since there is at least one invertible $(n-1) \times (n-1)$ minor of A), hence the differential of det at A is nonzero, showing that det is a submersion at A, or det is a submersion on $\text{GL}_n(\mathbb{R})$.

Then, because $1 \in \det(\text{GL}_n(\mathbb{R})) = \mathbb{R} \setminus \{0\}$ and det is a submersion on $\text{GL}_n(\mathbb{R})$, using **Proposition 1.18**, $\det^{-1}(1) = \text{SL}_n(\mathbb{R})$ is a manifold with smooth structure, and $\dim(\text{SL}_n(\mathbb{R})) = \dim(\text{GL}_n(\mathbb{R})) - \dim(\mathbb{R} \setminus \{0\}) = n^2 - 1$.

(The proof of **Proposition 1.18** will be provided in **Problem 3**).

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Problem 3

Solve **Exercise 1.10** in the lecture notes for smooth manifolds, then prove **Proposition 1.18** (You don't need to do this for C^k / analytic cases).

Exercise 1.10 (Modified):
Let $f_1, ..., f_m$ be functions $\mathbb{R}^n \to \mathbb{R}$ which are smooth. Let $X \subset \mathbb{R}^2$ be the set of points P such that $f_i(P) = 0$ for all i, and $df_i(P)$ are linearly independent. Use the implicit function theorem to show that X is a topological manifold of dimension $n - m$ and equip it with a natural smooth structure.

Proposition: **1.18**

If F is a submersion then for any $Q \in Y$, $F^{-1}(Q)$ is a manifold of dimension $\dim X - \dim Y$.

Solution:

1. **Exercise 1.10:**

Since $f_1, ..., f_m : \mathbb{R}^n \to \mathbb{R}$ are all smooth functions, consider $F : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $F(\overline{x}) = (f_1(\overline{x}), ..., f_m(\overline{x}))$. Since each entry is a smooth function, F itself is smooth.

If considering its differential, we yield the following:

$$dF = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = \begin{pmatrix} - & df_1 & - \\ & \vdots & \\ - & df_m & - \end{pmatrix} \tag{3.1}$$

Since for all points $P \in X$ satisfies $f_i(P) = 0$ for all i, we have $F(P) = (f_1(P), ..., f_m(P)) = \overline{0}$, showing that P is a solution for the smooth function F.

On the other hand, if consider $dF(P)$, since by assumption the row vectors of the differential $df_1(P), ..., df_m(P)$ are all linearly independent, it implies that the row rank of $dF(P)$ is m; and since row rank and column rank are equal, if consider the column vectors $v_1(P), ..., v_n(P) \subseteq \mathbb{R}^m$ of $dF(P)$, $\text{span}\{v_1(P), ..., v_n(P)\} \subseteq \mathbb{R}^m$ is a subspace of dimension = column rank = row rank = m, hence $\text{span}\{v_1(P), ..., v_n(P)\} = \mathbb{R}^m$, showing that it can be reduced to a basis of $\mathbb{R}^m$ (Note: $n \geq m$, since $df_1(P), ..., df_{m(P)} \in \mathbb{R}^n$ are linearly independent).

WLOG, up to coordinate permutation we can assume that $\{v_{n-m+1}(P), ..., v_n(P)\}$ forms a basis of $\mathbb{R}^m$, then the $m \times m$ minor A_y formed by these column vectors is in fact invertible. Together with the $m \times (n-m)$ matrix A_x formed by $\{v_1(P), ..., v_{n-m}(P)\}$, we have the following:

$$dF(P) = \left(\begin{array}{ccc|ccc} | & & | & | & & | \\ v_1(P) & \dots & v_{n-m}(P) & v_{n-m+1}(P) & \dots & v_n(P) \\ | & & | & | & & | \end{array}\right) = (A_x \,|\, A_y) \tag{3.2}$$

Since A_y is invertible, one can apply Implicit Function Theorem. Given $P = (p_1, ..., p_{n-m}, p_{n-m+1}, ..., p_n) \in \mathbb{R}^n$, there exists open neighborhood $U \subseteq \mathbb{R}^{n-m}$ containing $(p_1, ..., p_{n-m})$, open neighborhood $W \subseteq \mathbb{R}^m$ containing $(p_{n-m+1}, ..., p_n)$, and a smooth function $g : U \to W$, such that for all $(x_1, ..., x_{n-m}) \in U$, $g(x_1, ..., x_{n-m}) \in \mathbb{R}^m$ satisfies $F(x_1, ..., x_{n-m}, g(x_1, ..., x_{n-m})) = 0$, and $g(p_1, ..., p_{n-m}) = (p_{n-m+1}, ..., p_n)$. Furthermore, $dg(p_1, ..., p_{n-m}) = -A_y^{-1} A_x$ in matrix form.

Then, if we consider the smooth function $\overline{g} : U \to \mathbb{R}^n$ by $\overline{g}(x_1, ..., x_{n-m}) = (x_1, ..., x_{n-m}, g(x_1, ..., x_{n-m}))$, the image of $\overline{g}$ is solely contained in X. If we consider the projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-m}$ by $\pi(x_1, ..., x_n) = (x_1, ..., x_{n-m})$, if restrict the domain of π onto $\overline{g}(U)$, it satisfies the following for all $(x_1, ..., x_{n-m}) \in U$:

$$\pi(\overline{g}(x_1, ..., x_{n-m})) = \pi(x_1, ..., x_{n-m}, g(x_1, ..., x_{n-m}) = (x_1, ..., x_{n-m})) \tag{3.3}$$

Similarly, for all $(x_1, ..., x_{n-m}, x_{n-m+1}, ..., x_n) \in \overline{g}(U)$, it satisfies:

$$\overline{g}(\pi(x_1, ..., x_{n-m}, x_{n-m+1}, ..., x_n)) = \overline{g}(x_1, ..., x_{n-m}) = (x_1, ..., x_{n-m}, x_{n-m+1}, ..., x_n) \tag{3.4}$$

The above equality holds, since Implicit Function Theorem guarantees every $(x_1, ..., x_{n-m}) \in U$ to pair up with a unique $(x_{n-m+1}, ..., x_n) \in W$, such that $F(x_1, ..., x_{n-m}, x_{n-m+1}, ..., x_n) = 0$, while g is defined to satisfy $g(x_1, ..., x_{n-m}) = (x_{n-m+1}, ..., x_n)$.

Hence, the above two equality shows that the projection π (after restriction, which is still continuous) is an inverse of $\overline{g} : U \to \overline{g}(U)$, hence $\overline{g}$ is a homeomorphism. This shows that $P \in X$ has an open neighborhood $\overline{g}(U) \subseteq X$, where $\overline{g}(U) \cong U$ while $U \subseteq \mathbb{R}^{n-m}$, showing X is a topological manifold with dimension $(n-m)$. Finally, X has a natural smooth structure simply because the map g and its extension $\overline{g}$ can be chosen as smooth functions (while the inverse of $\overline{g}$, projection π is also a smooth function on $\mathbb{R}^n$), hence any transition map would be composition of restrictions of $\overline{g}$ and some other projection π, which is still smooth.

Proposition 1.18:

Given a smooth map $F : X \to Y$ that is a submersion. Let $n := \dim(X)$ and $m := \dim(Y)$. For any $Q \in (Y \cap F(X))$ (can ignore the case $Q \notin F(X)$), since there exists open neighborhood $U \subseteq Y$ containing Q, with local chart $g : U \xrightarrow{\sim} \tilde{U}$, where $\tilde{U} \subseteq \mathbb{R}^m$ is an open set. In particular, one can choose g (up to some translation in $\mathbb{R}^m$) so that $g(Q) = 0$.

Then, for all $P \in F^{-1}(Q)$, let $V \subseteq X$ be an open neighborhood of P, with local chart $h : V \xrightarrow{\sim} \tilde{V}$ where $\tilde{V} \subseteq \mathbb{R}^n$ is open. Then, if consider $W = V \cap F^{-1}(U)$ as an open neighborhood of P (since $P \in F^{-1}(Q) \subseteq F^{-1}(U)$), one has the restriction of h, $h : W \xrightarrow{\sim} h(W)$ be a homeomorphism to $h(W) \subseteq \mathbb{R}^n$ that is open. Hence, if consider the map $g \circ F \circ h^{-1} : h(W) \to U$, for all $x \in W$, we have $h(x) \in h(W)$ satisfies the following:

$$d(g \circ F \circ h^{-1})_{h(x)} = dg_{F \circ h^{-1}(h(x))} \circ dF_{h^{-1}(h(x))} \circ d(h^{-1})_{h(x)} \tag{3.5}$$

Since $g : U \to \tilde{U}$ is a homeomorphism from open subsets of Y (as smooth manifold) to open subsets of $\mathbb{R}^m$, it in fact can be viewed as a smooth function that has diffrential being isomorphism (smooth homeomorphism must have differential being isomorphism of tangent space, since $\mathrm{id} = d(\mathrm{id}_U)_y = d(g^{-1} \circ g)_y = d(g^{-1})_{g(y)} \circ dg_y$, and similar result when interchanging g^{-1} and g, showing dg_y must be an isomorphism at all $y \in U$).

Similar logic can also be applied to h^{-1} (where $d(h^{-1})_z$ is also an isomorphism for all $z \in h(W)$), hence with F being a submersion (or dF_x is surjective for arbitrary $x \in X$), the above $d(g \circ F \circ h^{-1})_{h(x)}$ is also surjective, which $g \circ F \circ h^{-1} : h(W) \to \tilde{U}$ is in fact a submersion from $h(W) \subseteq \mathbb{R}^n$ to $U \subseteq \mathbb{R}^m$, that satisfies $g \circ F \circ h^{-1}(h(P)) = g \circ F(P) = g(Q) = 0$.

Which, we get that $(g \circ F \circ h^{-1})^{-1}(0) = (F \circ h^{-1})^{-1}(g^{-1}(0)) = (F \circ h^{-1})^{-1}(Q) \subseteq h(W)$ is a topological manifold of dimension $\dim(h(W)) - \dim(\tilde{U}) = n - m$, and endows with a smooth manifold structure (since it reduces to the case of **Exercise 1.10**). So, $F^{-1}(Q) \cap W = h\left((F \circ h^{-1})^{-1}(Q)\right)$ also has a smooth manifold structure and with dimension $n - m$.

Since $F^{-1}(Q)$ can be realized as arbitrary union of the above subset $F^{-1}(Q) \cap W$, and the overlapped open regions between two different subsets above still agrees with each other (by restiction of local charts of X onto $F^{-1}(Q)$), hence $F^{-1}(Q)$ can be realized as a smooth manifold with dimension $n - m$, or dimension $\dim(X) - \dim(Y)$.

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Problem 4

Let X be $\{(x, y, t) \in \mathbb{R}^3 \mid x^3 + y^3 - 3txy + 1 = 0\}$, show that X has a smooth manifold structure. Now let Y be the real line with coordinate t, and let $F : X \to Y$ be the map sending (x, y, t) to t. Show that F is not a submersion, but the restriction of F to the open set $t \neq 1$ is a submersion. What does $F^{-1}(1)$ look like?

Solution:

1. Smooth Manifold Structure of X

First, consider the following function $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, t) = x^3 + y^3 - 3txy + 1$, then $X = f^{-1}(0)$ (since X collects precisely the zeros of f). If consider this function's differential, we get $df = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 3x^2-3ty & 3y^2-3tx & -3xy \end{pmatrix}$. Which, for any point $(x, y, t) \in X$, one can consider the case where $x, y \neq 0$, $x = 0$, or $y = 0$ for different usage of Implicit Function Theorem.

(**Note**: We need not consider the case $x, y = 0$, since it yields $f(0, 0, t) = 0^3 + 0^3 - 0 + 1 \neq 0$ when plugging in arbitrary $t \in \mathbb{R}$).

Here, we'll directly construct all the local charts with Implicit Function Theorem (instead of directly applying **Problem 3**) for the easier calculation of differentials on X later on.

- If both $x, y \neq 0$, since $-3xy \neq 0$, then this 1×1 minor of $df(x, y, t)$ is invertible. Hence, one can apply Implicit Function Theorem (smooth version), where there exists open neighborhood $U \subseteq \mathbb{R}^2$ containing (x, y), open neighborhood $W \subseteq \mathbb{R}$ containing t, and a smooth function $\varphi : U \to W$ satisfying $f(x_1, y_1, \varphi(x_1, y_1)) = 0$ for all $(x_1, y_1) \in U$ (or $(x_1, y_1, \varphi(x_1, y_1)) \in X$), and $\varphi(x, y) = t$. Also, the differential at $(x, y) \in U$, $d\varphi(x, y) = -\left(\frac{\partial f}{\partial t}\right)^{-1} \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \frac{1}{3xy} \begin{pmatrix} 3x^2-3ty & 3y^2-3tx \end{pmatrix}$.

  Then, this extends to a smooth function $\overline{\varphi} : U \to X$, satisfying $\overline{\varphi}(x_1, y_1) = (x_1, y_1, \varphi(x_1, y_1))$.

- Else if $x = 0$, then we have $y \neq 0$ (based on the **Note** given above). Hence, $\frac{\partial f}{\partial y} = 3y^2 - 3tx = 3y^2 \neq 0$, so apply Implicit Function Theorem, there exists open neighborhood $U' \subseteq \mathbb{R}^2$ containing (x, t), open neighborhood $W' \subseteq \mathbb{R}$ containing y, and a smooth function $\psi : U' \to W'$ satisfying $f(x_2, \psi(x_2, t_2), t_2) = 0$ (or $(x_2, \psi(x_2, t_2), t_2) \in X$), and $\psi(x, t) = y$. Again, the differential at $(x, t) \in U'$, $d\psi(x, t) = -\left(\frac{\partial f}{\partial y}\right)^{-1} \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial t} \end{pmatrix} = -\frac{1}{3y^2} \begin{pmatrix} 3x^2-3ty & -3xy \end{pmatrix} = \begin{pmatrix} \frac{t}{y} & 0 \end{pmatrix}$ (based on the condition $x = 0, y \neq 0$).

  This again extends to a smooth function $\overline{\psi} : U' \to X$, satisfying $\overline{\psi}(x_2, t_2) = (x_2, \psi(x_2, t_2), t_2)$.

- Else if $y = 0$, we have $x \neq 0$ based on the **Note** again. So, $\frac{\partial f}{\partial x} = 3x^2 - 3ty = 3x^2 \neq 0$, again apply Implicit Function Theorem, there exists open neighborhood $U'' \subseteq \mathbb{R}^2$ containing (y, t), open neighborhood $W'' \subseteq \mathbb{R}$ containing x, and a smooth function $\theta : U'' \to W''$ satisfying $f(\theta(y_3, t_3), y_3, t_3) = 0$ (or $(\theta(y_3, t_3), y_3, t_3) \in X$), and $\theta(y, t) = x$. Similarly, the differential at $(y, t) \in U''$ is provided by $d\theta(y, t) = -\left(\frac{\partial f}{\partial x}\right)^{-1} \begin{pmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial t} \end{pmatrix} = -\frac{1}{3x^2} \begin{pmatrix} 3y^2-3tx & -3xy \end{pmatrix} = \begin{pmatrix} \frac{t}{x} & 0 \end{pmatrix}$ (based on $x \neq 0, y = 0$).

  This also extends to a smooth function $\overline{\theta} : U'' \to X$, satisfying $\overline{\theta}(y_3, t_3) = (\theta(y_3, t_3), y_3, t_3)$.

Notice that for all cases above, $\overline{\varphi}, \overline{\psi}, \overline{\theta}$ are all homeomorphism onto their own image: For definiteness, consider the first case with $\overline{\varphi}$, the projection $\pi : \overline{\varphi}(U) \to U$ by $\pi(\overline{\varphi}(x_1, y_1)) = \pi(x_1, y_1, \varphi(x_1, y_1)) = (x_1, y_1)$ also satisfies $\overline{\varphi}(\pi(x_1, y_1, t_1)) = \overline{\varphi}(x_1, y_1) = (x_1, y_1, \varphi(x_1, y_1)) = (x_1, y_1, t_1)$ (since Implicit Function Theorem guarantees every $(x_1, y_1) \in U$ to pair up with a unique $t_1 \in W$, so that $\varphi(x_1, y_1) = t_1$). Hence, the projection π back to the original coordinate is a continuous inverse of $\overline{\varphi}$, showing it's indeed a homeomorphism to its image.

Similar concept applies to the other two cases (where it projects onto different entries), which shows that all (x, y, t) there exists a neighborhood in X that is homeomorphic to an open subset in $\mathbb{R}^2$ (the explicit maps are provided by $\overline{\varphi}, \overline{\psi}$, or $\overline{\theta}$ depending on the case), showing that X is a topological manifold with dimension 2.

And, the reason it has a smooth structure, is simply because every map mentioned above can be chosen as smooth functions, causing the transition map to also be smooth.

2. Non-Submersion of F:

To show that $F : \mathbb{R}^3 \to \mathbb{R}$ by $F(x, y, t) = t$ (after restricting to X) is not a submersion, we'll consider its behavior at $(1, 1, 1)$: It's clear that $f(1, 1, 1) = 1^3 + 1^3 - 3 \cdot 1 + 1 = 0$, hence $(1, 1, 1) \in X$. Since it satisfies $x, y \neq 0$, then apply the first case, one obtains $\overline{\varphi} : U \to X$ (where $(1, 1) \in U \subseteq \mathbb{R}^2$) with $\overline{\varphi}(x, y) = (x, y, \varphi(x, y))$ and $\overline{\varphi}(1, 1) = (1, 1, 1)$ as the chart. Since $d\varphi(1, 1) = -\frac{1}{3}\begin{pmatrix} 0 & 0 \end{pmatrix}$ (by plugging in $(x, y, t) = (1, 1, 1)$ to the differential calculated beforehand), then $\overline{\varphi}$ has differential at $(1, 1)$ provided as:

$$d\overline{\varphi}(1,1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.1}$$

Hence, if compose this with F (where $dF = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ by calculation), we get:

$$d(F \circ \overline{\varphi})(1,1) = dF(\overline{\varphi}(1,1)) \circ d\overline{\varphi}(1,1) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} \tag{4.2}$$

This shows that the differential of F at $(1, 1, 1)$ (when restricting onto X) in fact is not surjective (since the differential is 0). Hence, F is not a submersion.

3. Submersion of F on $t \neq 1$:

For any $(x, y, t) \in X$ satisfying $t \neq 1$, we'll explicitly calculate the differential:

- For $x, y \neq 0$, the first local chart described by $\overline{\varphi}$ can be used. Since $d\varphi(x, y) = \frac{1}{3xy}\begin{pmatrix} 3x^2 - 3ty & 3y^2 - 3tx \end{pmatrix}$, for $t \neq 1$ we yield $d\varphi(x, y) \neq 0$: Suppose the contrary that $d\varphi(x, y) = 0$, we have $3x^2 - 3ty = 3y^2 - 3tx = 0$, showing that $t = \frac{x^2}{y} = \frac{y^2}{x}$, or $x^3 = y^3$, which implies $x = y$. However, plug back to the given condition, we get $3x^2 - 3ty = 3x^2 - 3tx = 0$, or $x = t$, hence $x = y = t$. But then, we have $f(x, y, t) = x^3 + y^3 - 3txy + 1 = t^3 + t^3 - 3t^3 + 1 = 1 - t^3 = 0$, or $t^3 = 1$, implying $t = 1$, which contradicts the assumption that $t \neq 1$. This shows that $d\varphi(x, y) \neq 0$, hence one of the coordinate is nonzero.

  So, if consider $d\overline{\varphi}(x, y)$ (where $\overline{\varphi}(x, y) = (x, y, \varphi(x, y))$), it's provided as:

  $$d\overline{\varphi}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \end{pmatrix} \tag{4.3}$$

  Where one of the entries in the third row is nonzero. Hence:

  $$d(F \circ \overline{\varphi})(x,y) = dF(\overline{\varphi}(x,y)) \circ d\overline{\varphi}(x,y) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \end{pmatrix} \neq 0 \tag{4.4}$$

With $d(F \circ \overline{\varphi})(x, y) \neq 0$, and the fact that the target space of F (namely $\mathbb{R}$) is a 1-dimensional smooth manifold, then the differential being nonzero implies it's surjective.

- For the case $x = 0$, we can use the local chart in the form of $\overline{\psi}$. Since $d\psi(x, t) = \left(\frac{t}{y} \;\; 0\right)$, we have $\overline{\psi}(x_2, t_2) = (x_2, \psi(x_2, t_2), t_2)$ having the following differential at $(x, t) \in U'$:

$$d\overline{\psi}(x,t) = \begin{pmatrix} 1 & 0 \\ \frac{d\psi}{dx} & \frac{d\psi}{dt} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{t}{y} & 0 \\ 0 & 1 \end{pmatrix} \tag{4.5}$$

  Hence, we get:

$$d\Big(F \circ \overline{\psi}\Big)(x,t) = dF\Big(\overline{\psi}(x,t)\Big) \circ d\overline{\psi}(x,t) = (0 \;\; 0 \;\; 1) \begin{pmatrix} 1 & 0 \\ \frac{t}{y} & 0 \\ 0 & 1 \end{pmatrix} = (0 \;\; 1) \tag{4.6}$$

  Therefore F has nonzero differential at (x, y, t) when considering it as a smooth map on X for $x = 0$, showing the differential of F at (x, y, t) is surjective.

- Finally, for the case $y = 0$, we se the local chart in the form of $\overline{\theta}$. Since $d\theta(y, t) = \left(\frac{t}{x} \;\; 0\right)$, with $\overline{\theta}(y_3, t_3) = (\theta(y_3, t_3), y_3, t_3)$, we have its differential at (y, t) represented as:

$$d\overline{\theta}(y,t) = \begin{pmatrix} \frac{d\theta}{dy} & \frac{d\theta}{dt} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{t}{x} & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.7}$$

  Hence, we get:

$$d\Big(F \circ \overline{\theta}\Big)(y,t) = dF\Big(\overline{\theta}(y,t)\Big) \circ d\overline{\theta}(y,t) = (0 \;\; 0 \;\; 1) \begin{pmatrix} -\frac{t}{x} & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = (0 \;\; 1) \tag{4.8}$$

  This again shows that F has nonzero differential at (x, y, t) when treating it as a smooth map on X for $y = 0$, showing the differential of F at (x, y, t) is again surjective.

Since in all possible cases of $t \neq 1$, F has surjective differential at $(x, y, t) \in X$, then F is indeed a submersion on $t \neq 1$.

Finally, if consider $F^{-1}(1)$, for all $(x, y, 1) \neq (1, 1, 1)$, if $x = 0$ or $y = 0$, then the case of $\overline{\psi}, \overline{\theta}$ from above (in part 3) still applies (since they don't rely on the assumption $t \neq 1$ to derive nontrivial differential for F), so close to $(x, y, 1)$ one can still derive a 1-dimensional manifold structure for $F^{-1}(1)$ (by applying Implicit Function Theorem like **Problem 3**). For the case $x, y \neq 0$, we have $d\overline{\varphi}(x, y) \neq 0$ (since in the above case of part 3, we proved how the differential is 0 implies $x = y = t$, but here one of the $x, y \neq 1$ by assumption, so $d\overline{\varphi}(x, y) \neq 0$), showing if $(x, y, 1) \in F^{-1}(1)$ is different from $(1, 1, 1)$, it locally has a 1-dimensional smooth manifold structure.

And, since at $(1, 1, 1) \in F^{-1}(1)$ has differential of F being 0 indicates that it must be an isolated point in $F^{-1}(1)$ (since if it's not isolated it should be included in one of the neighborhoods of some other points in $F^{-1}(1)$). Hence, $F^{-1}(1)$ is consists of lines (1-dimensional manifold), and the isolated point $(1, 1, 1)$.

5 D

Problem 5

Prove the chain rule stated at the beginning of page 17 of the lecture notes:

Given $F : X \to Y$ and $G : Y \to Z$ that are regular maps between manifolds of same regularity, then:

$$d(G \circ F)_P = dG_{F(P)} \circ dF_P \tag{5.1}$$

Solution: By definition, $d(G \circ F)_P : T_P X \to T_{G \circ F(P)} Z$ is a linear map satisfying $(d(F \circ G)_P \cdot v)(f) = v(f \circ (G \circ F))$ for all derivation $v \in T_P X$, and all class of regular function $f \in O_{G \circ F(P)}$.

Similarly, $dF_P : T_P X \to T_{F(P)} Y$ satisfies $(dF_P \cdot v)(g) = v(g \circ F)$ for all derivation $v \in T_P X$ and all class of regular function $g \in O_{F(P)}$; also, $dG_{F(P)} : T_{F(P)} Y \to T_{G(F(P))} Z$ satisfies $\big(dG_{F(P)} \cdot u\big)(h) = u(h \circ G)$ for all derivation $u \in T_{F(P)} Y$ and class of regular function $h \in O_{G \circ F(P)}$.

By associativity of function composition, given $f \in O_{G \circ F(P)}$ and $v \in T_P X$, the first term $(d(G \circ F)_P \cdot v)(f) = v(f \circ (G \circ F))$ can be rewritten as:

$$v((f \circ G) \circ F) = (dF_P \cdot v)(f \circ G) \tag{5.2}$$

This is due to the fact that $f \circ G$ is a class of regular function in $O_{F(P)}$. Then again, since f is a class of regular function in $O_{G \circ F(P)}$, and $dF_P \cdot v \in T_{F(P)} Y$, using the same argument, we have:

$$(dF_P \cdot v)(f \circ G) = \big(dG_{F(P)} \cdot (dF_P \cdot v)\big)(f) = \big(dG_{F(P)} \circ dF_P(v)\big)(f) \tag{5.3}$$

Hence, we get $(d(G \circ F)_P \cdot v)(f) = \big(dG_{F(P)} \circ dF_P(v)\big)(f)$, showing that $d(G \circ F)_P = dG_{F(P)} \circ dF_P$.