

# Math CS 121 HW 1

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## Question 1. Chapter 1.2 # 12

Prove that the event  $B$  is impossible if and only if for every event  $A$ ,

$$A = (B \cap A^c) \cup (B^c \cap A)$$

*Proof.*  $\Rightarrow$  : First, suppose the event  $B$  is impossible, which is equivalent to saying  $B = \emptyset$ . Then, we have  $B^c = \Omega$  the whole sample space. For any event  $A$ , the following holds:

$$(B \cap A^c) \cup (B^c \cap A) = (\emptyset \cap A^c) \cup (\Omega \cap A) = \emptyset \cup A = A \quad (1)$$

Hence the given equality of event holds.

$\Leftarrow$  : Now, suppose for all event  $A$ ,  $A = (B \cap A^c) \cup (B^c \cap A)$ , then in particular it works for setting  $A = B$ . Which,  $B \cap B^c = B^c \cap B = \emptyset$ , so we get:

$$B = (B \cap B^c) \cup (B^c \cap B) = \emptyset \cup \emptyset = \emptyset \quad (2)$$

Hence,  $B = \emptyset$  is an impossible event.

□

## Question 2. Chapter 1.2 # 16

Let  $A$  and  $B$  be two events. Prove the following relations by the elementwise method.

(a)  $(A \setminus (A \cap B)) \cup B = A \cup B$

(b)  $(A \cup B) \setminus (A \cap B) = (A \cap B^c) \cup (A^c \cap B)$

*Proof.* (a)  $\subseteq$ : Suppose  $x \in (A \setminus (A \cap B)) \cup B$ , either  $x \in (A \setminus (A \cap B)) \subseteq A \subseteq A \cup B$ , or  $x \in B \subseteq A \cup B$ , hence  $x \in A \cup B$ , showing  $(A \setminus (A \cap B)) \cup B \subseteq A \cup B$ .

$\supseteq$ : Suppose  $x \in A \cup B$ , then either  $x \in A$  or  $x \in B$ . If  $x \in B$ , then  $x \in B \subseteq (A \setminus (A \cap B)) \cup B$ ; else, if  $x \notin B$ , it enforces  $x \in A$ , and shows that  $x \notin A \cap B$ . Hence,  $x \in A \setminus (A \cap B) \subseteq (A \setminus (A \cap B)) \cup B$ . With the above two cases,  $A \cup B \subseteq (A \setminus (A \cap B)) \cup B$ .

(b)  $\subseteq$ : Suppose  $x \in (A \cup B) \setminus (A \cap B)$ , it states  $x \notin (A \cap B)$ , while  $x \in A$  or  $x \in B$ . Suppose  $x \in A$ , then with  $x \notin (A \cap B)$  it concludes  $x \notin B$  (or  $x \in B^c$ ), hence  $x \in (A \cap B^c) \subseteq (A \cap B^c) \cup (A^c \cap B)$ . Else, suppose  $x \in B$ , then with  $x \notin (A \cap B)$  it concludes  $x \notin A$  (or  $x \in A^c$ ), hence  $x \in (A^c \cap B) \subseteq (A \cap B^c) \cup (A^c \cap B)$ .

These two cases conclude that  $(A \cup B) \setminus (A \cap B) \subseteq (A \cap B^c) \cup (A^c \cap B)$ .

$\supseteq$ : Now, suppose  $x \in (A \cap B^c) \cup (A^c \cap B)$ , then either  $x \in A \cap B^c$  (stating  $x \in A$  and  $x \notin B$ ), or  $x \in A^c \cap B$  (stating  $x \notin A$  and  $x \in B$ ).

In the first case  $x \in A \subseteq (A \cup B)$ , while  $x \notin B$  implies  $x \notin (A \cap B)$ , showing that  $x \in (A \cup B) \setminus (A \cap B)$ . Similarly, in the second case  $x \in B \subseteq (A \cup B)$ , while  $x \notin A$  implies  $x \notin (A \cap B)$ , hence again  $x \in (A \cup B) \setminus (A \cap B)$ .

These two cases conclude that  $(A \cap B^c) \cup (A^c \cap B) \subseteq (A \cup B) \setminus (A \cap B)$ .

□

**Question 3.** Chapter 1.2 # 17

Let  $\{A_n\}_{n=1}^{\infty}$  be a sequence of events. prove that for every event  $B$ ,

$$(a) \quad B \cap \left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} (B \cap A_i)$$

$$(b) \quad B \cup \left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} (B \cup A_i).$$

*Proof.* (a)  $\subseteq$ : Suppose  $x \in B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)$ , then  $x \in B$  and  $x \in \bigcup_{i=1}^{\infty} A_i$ , hence there exists  $n \in \mathbb{N}$  such that  $x \in A_n$ . This concludes that  $x \in B \cap A_n \subseteq \bigcup_{i=1}^{\infty} (B \cap A_i)$ .

Which, it further concludes that  $B \cap \left(\bigcup_{i=1}^{\infty} A_i\right) \subseteq \bigcup_{i=1}^{\infty} (B \cap A_i)$ .

$\supseteq$ : Suppose  $x \in \bigcup_{i=1}^{\infty} (B \cap A_i)$ , then there exists  $n \in \mathbb{N}$  such that  $x \in B \cap A_n$ . In particular  $x \in B$ , and  $x \in A_n \subseteq \bigcup_{i=1}^{\infty} A_i$ , showing that  $x \in B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)$ .

This concludes that  $\bigcup_{i=1}^{\infty} (B \cap A_i) \subseteq B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)$ .

(b)  $\subseteq$ : Suppose  $x \in B \cup \left(\bigcap_{i=1}^{\infty} A_i\right)$ , then either  $x \in B$ , or  $x \in \bigcap_{i=1}^{\infty} A_i$ . If  $x \in B$ , then for all  $n \in \mathbb{N}$  it satisfies  $x \in B \cup A_n$ , hence  $x \in \bigcap_{i=1}^{\infty} (B \cup A_i)$ ; else if  $x \in \bigcap_{i=1}^{\infty} A_i$ , for every  $n \in \mathbb{N}$  it satisfies  $x \in A_n \subseteq (B \cup A_n)$ , hence  $x \in \bigcap_{i=1}^{\infty} (B \cup A_i)$ .

This concludes that  $x \in \bigcap_{i=1}^{\infty} (B \cup A_i)$ , or  $B \cup \left(\bigcap_{i=1}^{\infty} A_i\right) \subseteq \bigcap_{i=1}^{\infty} (B \cup A_i)$ .

$\supseteq$ : Suppose  $x \in \bigcap_{i=1}^{\infty} (B \cup A_i)$ , then for each  $n \in \mathbb{N}$ , one has  $x \in B \cup A_n$ , showing that  $x \in B$  or  $x \in A_n$ .

If for some  $n \in \mathbb{N}$  it satisfies  $x \in B$ , it's clear that  $x \in B \cup \left(\bigcap_{i=1}^{\infty} A_i\right)$ ; else, if  $x \notin B$  for all  $n \in \mathbb{N}$ , then  $x \in (B \cup A_n)$  for all  $n \in \mathbb{N}$  implies  $x \in A_n$  for all  $n \in \mathbb{N}$ . Hence,  $x \in \bigcap_{i=1}^{\infty} A_i \subseteq B \cup \left(\bigcap_{i=1}^{\infty} A_i\right)$ .

This concludes that  $\bigcap_{i=1}^{\infty} (B \cup A_i) \subseteq B \cup \left(\bigcap_{i=1}^{\infty} A_i\right)$ .

□

**Question 4.** Chapter 1.4 # 5

Suppose that 75% of all investors invest in traditional annuities and 45% of them invest in the stock market. If 85% invest in the stock market and/or traditional annuities, what percentage invest in both?

*Proof.* Let sample space  $\Omega$  collect all investors, let event  $A \subseteq \Omega$  denotes all investors invest in traditional annuities, and event  $B \subseteq \Omega$  denotes all investors invest in the stock market. Which,  $A \cup B$  denotes all investors investing in the stock market and/or traditional annuities, while  $A \cap B$  denotes all investors investing in both the stock market and traditional annuities.

Based on the description, the provided probability function satisfies  $\mathbb{P}(A) = 0.75$ ,  $\mathbb{P}(B) = 0.45$ , and  $\mathbb{P}(A \cup B) = 0.85$ . Then, the following equation hold:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \quad (3)$$

After rearranging, we get  $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$ . With the above conditions given, we get:

$$\mathbb{P}(A \cap B) = 0.75 + 0.45 - 0.85 = 0.3 \quad (4)$$

□

## 5,6,7,8,9,10 Not Done

**Question 5.** Chapter 1.4 # 15

Let  $A, B$ , and  $C$  be three events. Show that exactly two of these events will occur with probability:

$$\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - 3\mathbb{P}(A \cap B \cap C)$$

*Proof.*

□

**Question 6.** Chapter 1.5 # 20

The coefficients of the quadratic equation  $x^2 + bx + c = 0$  are determined by tossing a fair die twice (the first outcome is  $b$ , the second one is  $c$ ). Find the probability that the equation has real roots.

*Proof.* For the equation to have real roots, the discriminant  $b^2 - 4ac \geq 0$ . In this case  $a = 1$ , so  $b^2 - 4c \geq 0$ , or  $b^2 \geq 4c$ .

□

**Question 7.** Chapter 1.4 # 25

A number is selected at random from the set of natural numbers  $\{1, 2, \dots, 1000\}$ . What is the probability that it is divisible by 4 but neither by 5 nor by 7?

*Proof.* Let the sample space  $\Omega := \{1, 2, \dots, 1000\}$ . Let  $A$  denotes

□

**Question 8.** Chapter 1.4 # 26

*For a Democratic candidate to win an election, she must win districts I, II, and III. Polls have shown that the probability of winning I and III is 0.55, losing II but not I is 0.34, and losing II and III but not I is 0.15. Find the probability that this candidate will win all three districts. (Draw a Venn Diagram).*

*Proof.*

□

**Question 9.** *Chapter 1.4 # 28*

*Proof.*



**Question 10.** *Chapter 1.7 # 8*

*Proof.*

