# Math CS 5 HW 1

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**Question 1.** *The following poset is a Heyting algebra:* $\{\perp \leq \mu \leq \top\}$. *Fill in tables for the Heyting algebra operations.*

*Proof.* Given the basicrules of Heyting algebra (as an easier place of reference for myself):

- $\perp \leq x \leq T$ for all $x \in \mathcal{H}$.
- $x \leq y \wedge z$ iff $x \leq y$ and $x \leq z$.
- $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$.
- $x \leq y \to z$ iff $x \wedge y \leq z$.

With these operations in mind, here are the tables for its Heyting Algebra Operations:

| $\wedge$ | $\top$ | $\mu$ | $\perp$ |
|---|---|---|---|
| $\top$ | $\top$ | $\mu$ | $\perp$ |
| $\mu$ | $\mu$ | $\mu$ | $\perp$ |
| $\perp$ | $\perp$ | $\perp$ | $\perp$ |

| $\vee$ | $\top$ | $\mu$ | $\perp$ |
|---|---|---|---|
| $\top$ | $\top$ | $\top$ | $\top$ |
| $\mu$ | $\top$ | $\mu$ | $\mu$ |
| $\perp$ | $\top$ | $\mu$ | $\perp$ |

| $\to$ | $\top$ | $\mu$ | $\perp$ |
|---|---|---|---|
| $\top$ | $\top$ | $\mu$ | $\perp$ |
| $\mu$ | $\top$ | $\top$ | $\perp$ |
| $\perp$ | $\top$ | $\top$ | $\top$ |

$\square$

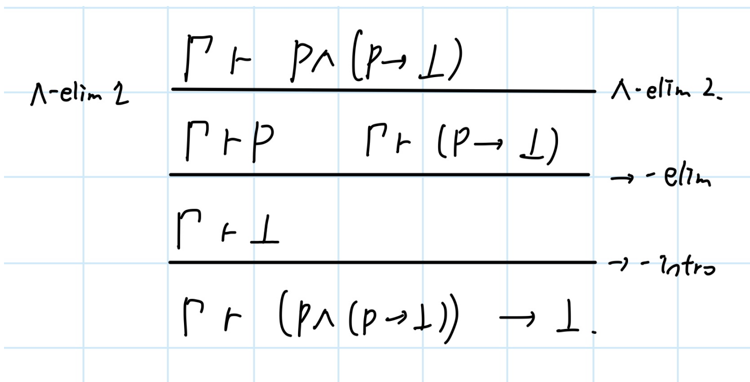
**Question 2.** *A notable absence in our list of logical connectives is negation. Luckily, we can define it in terms of existing connectives: $\neg A := A \to \bot$. Let $\mathcal{A} = \{P\}$. The "law of non-contradiction" is the proposition $\neg(P \wedge \neg P)$. A closely related "law" is the "law of excluded middle," which is the proposition $P \vee \neg P$.*

*(a) Construct a proof tree of $\vdash \neg(P \wedge \neg P)$. (By definition, $(P \wedge (P \to \bot)) \to \bot$).*

*(b) Prove that there does not exist a derivation of $\vdash P \vee \neq P$ (Tip: consider the Heyting algebra in the previous problem).*

*Proof.*

(a) Let $\Gamma = \{P \wedge (P \to \bot)\}$ as a premise. We get the following proof tree:




Where the last $\to$-intro is also based on the premis (where we form $\Gamma, P \wedge (P \to \bot) \vdash \bot$).

(Yeah I failed using the package, I'll try to learn it by the next time).

(b) Suppose the contrary that there exists a derivation of $\Gamma \vdash P \vee \neg P$ (given some premise $\Gamma$), if consider the Heyting algebra from the previous problem, define the model $\varphi : \mathcal{A} \to \mathcal{H}$ by $\varphi(P) = \bot$. Then, by Soundness Theorem, we derived:

$$[\![\Gamma]\!]_\varphi \leq [\![P \vee \neg P]\!]_\varphi = [\![P]\!]_\varphi \vee [\![P \to \bot]\!]_\varphi = [\![P]\!]_\varphi \vee ([\![P]\!]_\varphi \to [\![\bot]\!]_\varphi) = \bot \vee (\bot \to \bot)$$

Using the previous diagram, we get $[\![\Gamma]\!]_\varphi \leq \bot \vee \bot = \bot$, so $[\![\Gamma]\!]_\varphi = \bot$ based on the inequality. But this becomes an inconsistency (where the premise outputs false). So, this concludes that the derivation of $\Gamma \vdash P \vee \neg P$ doesn't exist.

$\square$