Math 237A HW 1

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Problem 1

Statement

Solution:

Proof

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Problem 2

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Problem 3

Hartshorne Chapter 1 Exercise 1.1 (a),(b):

(a) Let Y be the plane curve $y = x^2$ (i.e. Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ (or $k[Y]$) is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ (or $k[Z]$) is not isomorphic to a polynomial ring in one variable over k.

Solution:

(a): Let ideal $a = (y - x^2) \subseteq k[x, y]$, then we have $Y = Z(a)$ (the corresponding algebraic set of polynomial $y - x^2$, hence also corresponds to the ideal generated by it). Then, $I(Y) = I(Z(a)) = \sqrt{a}$, so the coordinate ring $k[Y] = k[x, y]/\sqrt{a}$.

However, notice that $y - x^2$ is irreducible in $k[x, y]$: If consider $k[x, y] = (k[x])[y]$ (with base ring $k[x]$), then $y - x^2$ has degree of y being 1, which is irreducible in $(k[x])[y]$. Hence, the ideal $a = (y - x^2)$ is in fact a prime ideal (since the generated element $y - x^2$ is irreducible, and $k[x, y]$ is a UFD), then we get that $\sqrt{a} = a$ (since all prime ideal is its own radical).

Now, to prove that $k[x, y]/\sqrt{a} = k[x, y]/a \cong k[t]$ (where t is an indeterminate), consider a ring homomorphism $\varphi : k[x, y] \to k[t]$ by $\varphi(f(x, y)) = f(t, t^2)$ for all $f(x, y) \in k[x, y]$. This is a well-defined ring homomorphism, since any $f, g \in k[x, y]$ satisfy the following:

$$\varphi(f(x, y) \cdot g(x, y)) = f(t, t^2) \cdot g(t, t^2) = \varphi(f(x, y)) \cdot \varphi(g(x, y))$$

Since for all $f(t) \in k[t]$, consider $f(x) \in k[x] \subseteq k[x, y]$, then $\varphi(f(x)) = f(t)$, showing φ is surjective, hence $k[t] \cong k[x, y]/\ker(\varphi)$.

Now, to show that $\ker(\varphi) = a$, first, for all $f(x, y) \in a$, there exists $g(x, y) \in k[x, y]$ such that $f(x, y) = (y - x^2) \cdot g(x, y)$, hence we have $\varphi(f(x, y)) = \varphi((y - x^2) \cdot g(x, y)) = (t^2 - t^2) \cdot g(t, t^2) = 0$, showing $f(x, y) \in \ker(\varphi)$, which proves $a \subseteq \ker(\varphi)$;

On the other hand, if $f(x, y) \in \ker(\varphi)$, then $\varphi(f(x, y)) = f(t, t^2) = 0$. So, for all $x \in k$, with $y = x^2$ we have $f(x, y) = f(x, x^2) = 0$, hence $f(x, y)$ vanishes for all $(x, y) \in Y$. This shows that $f(x, y) \in I(Y) = \sqrt{a} = a$, hence $\ker(\varphi) \subseteq a$.

As a conclusion, we have $\ker(\varphi) = a$, hence $k[t] \cong k[x, y]/\ker(\varphi) = k[x, y]/a$, while $k[x, y]/a = k[Y]$ the coordinate ring (due to the fact that $a = \sqrt{a}$). Hence, $k[Y] \cong k[t]$ (polynomial ring with single indeterminate).

(b): Given that Z is the plane curve $xy = 1$, then Z is the algebraic set corresponding to the polynomial $xy - 1 \in k[x, y]$. Let ideal $b = (xy - 1)$, we have $Z = Z(a)$ (**Note:** the second Z in $Z(a)$ represents the function of mapping ideal to its algebraic set, not the algebraic set Z itself). Which, we get that $I(Z) = I(Z(b)) = \sqrt{b}$, so the corresponding coordinate ring $k[Z] = k[x, y]/\sqrt{b}$.

Now, again if interpreting $k[x, y] = (k[x])[y]$, since $xy - 1$ is a polynomial with degree of y being 1, it is irreducible in $(k[x])[y]$, hence the ideal $b = (xy - 1)$ is in fact a prime ideal, which implies that $\sqrt{b} = b$. So, the coordinate ring $k[Z] = k[x, y]/\sqrt{b} = k[x, y]/b$.

Finally, we'll show that $k[Z] \ncong k[t]$ the polynomial ring in k with one indeterminate. Suppose the contrary that $k[Z] \cong k[t]$, then there exists a ring isomorphism $\psi : k[Z] = k[x, y]/b \to k[t]$. Then, if consider $\psi(\overline{x}), \psi(\overline{y}) \in k[t]$, since $\overline{x} \cdot \overline{y} = \overline{xy} = 1 \in k[Z]$ (due to the fact that $xy - 1 \equiv 0 \bmod b$, so $\overline{xy - 1} = 0 \in k[Z]$), then we get that $\psi(\overline{x}) \cdot \psi(\overline{y}) = \psi(\overline{xy}) = \psi(1) = 1$, hence both $\psi(\overline{x}), \psi(\overline{y}) \in k[t]$ are invertible. Yet, since group of units $(k[t])^\times = k^\times$, this enforces $\psi(\overline{x}), \psi(\overline{y}) \in k^\times$ (nonzero constant polynomials), but this is a contradiction since ψ is supposed to be surjective, while now $\psi\left(\overline{f(x, y)}\right) = f(\psi(\overline{x}), \psi(\overline{y})) \in k$ for all $\overline{f(x, y)} \in k[Z]$, showing that ψ is not surjective. Hence, we conclude that $k[Z] \ncong k[t]$.

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Problem 4

Hartshorne Chapter 1 Exercise 1.2:

The Twisted Cubic Curve. Let $Y \subseteq \mathbb{A}^3$ be the set $Y = \left\{(t, t^2, t^3) | t \in k\right\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ (or $k[Y]$) is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t, y = t^2, z = t^3$.

Solution: First, we'll show that given any $(x, y, z) \in \mathbb{A}^3$, there exists $t \in k$ such that $(x, y, z) = (t, t^2, t^3) \Longleftrightarrow y = x^2$ and $z = x^3$.

For $\Longrightarrow$, if there exists $t \in k$ such that $(x, y, z) = (t, t^2, t^3)$, it's clear that $y = t^2 = x^2$ and $z = t^3 = x^3$, so the conditions are satisfied. Conversely (for $\Longleftarrow$), if $y = x^2$ and $z = x^3$, choose $t = x \in k$ we have $(x, y, z) = (x, x^2, x^3) = (t, t^2, t^3)$. Hence, the equivalence is shown.

Now, if consider the ideal $I = (y - x^2, z - x^3)$, we claim that I is the ideal corrersponding to Y.

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Problem 5

Hartshorne Chapter 1 Exercise 1.4:

If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Solution: For this we'll prove by contradiction. First, recall the following lemma from point set topology:

Lemma

Given a topological space X, and consider $X \times X$ under the product topology. Then, the diagonal $\Delta = \{(x, x) \in X \times X \mid x \in X\}$ is closed under product topology $\iff X$ is Hausdorff.

Proof:

$\Longrightarrow$: First, suppose $\Delta \subseteq X \times X$ is closed, which means $(X \times X) \setminus \Delta$ is open in $X \times X$ under product topology. Hence, for all $(x, y) \in (X \times X) \setminus \Delta$ (with $x \neq y$), there exists open neighborhood $U_x, U_y \subseteq X$ of x, y respectively, such that $(x, y) \in U_x \times U_y \subseteq (X \times X) \setminus \Delta$. Then, for all $z \in U_x$ and $w \in U_y$, since $(z, w) \in U_x \times U_y \subseteq (X \times X) \setminus \Delta$, we have $z \neq w$, hence $U_x \cap U_y = \emptyset$. Which, with $x, y \in X$ being arbitrary, $x \neq y$, $U_x \ni x$ and $U_y \ni y$ are open neighborhoods that're disjoint, hence X is Hausdorff.

$\Longleftarrow$: Suppose X is Hausdorff, then for all $(x, y) \in (X \times X) \setminus \Delta$ (where $x \neq y$), there exists open neighborhoods $U_x, U_y \subseteq X$ of x, y respectively, such that $U_x \cap U_y = \emptyset$. Hence, for all $(z, w) \in U_x \times U_y$, with $z \in U_x$ and $w \in U_y$, the two sets being disjoint implies $z \neq w$, hence $(z, w) \in (X \times X) \setminus \Delta$. Hence, $(x, y) \in U_x \times U_y \subseteq (X \times X) \setminus \Delta$, showing that $(X \times X) \setminus \Delta$ is open in $X \times X$ under product topology, hence $\Delta \subseteq X \times X$ is closed under product topology. $\square$

With this lemma in mind, suppose the contrary that the Zariski Topology on $\mathbb{A}^2$ is the same as the product topology of $\mathbb{A}^1 \times \mathbb{A}^1$ (with $\mathbb{A}^1$ equipped with its own Zariski Topology). Then, by the lemma above, the diagonal $\Delta = \{(x, x) \in \mathbb{A}^2 \mid x \in k\} \subseteq \mathbb{A}^2$ is closed in $\mathbb{A}^2 \iff \mathbb{A}^1$ is Hausdorff.

Now, notice that with the polynomial $y - x \in k[x, y]$, the corresponding algebraic set $Z(y - x) = \Delta$ (since $(x, y) \in \mathbb{A}^2$ satisfies $y - x = 0$ iff $y = x$ iff $(x, y) \in \Delta$). Hence, Δ itself is closed in $\mathbb{A}^2$ under Zariski Topology, so based on our assumption (together with the lemma), $\mathbb{A}^1$ is Hausdorff.

However, $\mathbb{A}^1$ has Zariski Topology being the same as Finite Complement Topology (here assume that k is algebraically closed, hence k is infinite):

Since $k[x]$ is a PID (given that k is a field), then for any nonempty and proper algebraic set $Y \subseteq \mathbb{A}^1 = k$, its corresponding ideal $I(Y) = (f(x))$ for some $f(x) \in k[x]$, hence $t \in Y$ iff $f(t) = 0$, or t is a zero of $f(x)$. Since $f(x)$ only has finitely many roots, it follows that Y is finite.

Conversely, given any nonempty finite subset $X \subsetneq \mathbb{A}^1$, let $f(x) := \prod_{a \in X}(x - a)$, we have X being the algebraic set corresponding to $f(x)$ (since $a \in X$ iff $f(a) = 0$). Hence, the closed set in $\mathbb{A}^1$ under Zariski Topology (beside $\mathbb{A}^1$ and $\emptyset$) are all finite subsets of $\mathbb{A}^1$, showing that all open sets in $\mathbb{A}^1$ (besides $\emptyset$ and $\mathbb{A}^1$ itself) are precisely the subsets with their complements being finite, hence the Zariski Topology on $\mathbb{A}^1$ is equivalent to the Finite Complement Topology.

Then, given $\mathbb{A}^1 = k$ is infinite, the Finite Complement Topology on $\mathbb{A}^1$ is not Hausdorff: Suppose the contrary that it is Hausdorff, then for any $x, y \in \mathbb{A}^1$ with $x \neq y$, there exists open neighborhoods $U_x, U_y \subseteq \mathbb{A}^1$ containing x, y respectively, such that $U_x \cap U_y = \emptyset$. However, it implies that $U_y \subseteq \mathbb{A}^1 \setminus U_x$, while $\mathbb{A}^1 \setminus U_x$ is finite, hence U_y is finite. Yet, this implies that $\mathbb{A}^1 \setminus U_y$ is infinite (since $\mathbb{A}^1$ is infinite, while U_y is finite), which reaches a contradiction. So, $\mathbb{A}^1$ cannot be Hausdorff.

However, this contradicts one of the previous conclusion that $\mathbb{A}^1$ is Hausdorff. Hence, the initial assumption must be false, showing that Zariski Topology on $\mathbb{A}^2$ is not the same as product topology of $\mathbb{A}^1 \times \mathbb{A}^1$ (with $\mathbb{A}^1$ equipped with its own Zariski Topology).

(I think Here we can conclude that $\mathbb{A}^2$ has Zariski Topology being the same as product topology of $\mathbb{A}^1 \times \mathbb{A}^1$ iff the base field k is finite, since this is the only case where the Finite Complement Topology, i.e. the Zariski Topology on $\mathbb{A}^1$, is Hausdorff).

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Problem 6

Hartshorne Chapter 1 Exercise 1.5:

Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$ for some n if and only if B is a finitely generated k-algebra with no nilpotent elements.

Solution:

$\Longrightarrow$: Suppose B is a k-algebra (here B can be assumed as a commutative algebra) that is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$ for some n.

Then, there exists an algebraic set $Y \subseteq \mathbb{A}^n$, such that $B \cong k[Y]$, where let $J = I(Y) \subseteq k[x_1, ..., x_n]$ the corresponding ideal (which J is a radical), we have $k[Y] = k[x_1, ..., x_n]/J$. This shows that B is a finitely generated k-algebra (since it's isomorphic to a quotient of the polynomial ring $k[x_1, ..., x_n]$), and also B has no nilpotent elements (since J is a radical ideal, so for all $f \in k[x_1, ..., x_n]$, if the quotient $\overline{f} \in k[Y]$ satisfies $\overline{f}^k = 0$ for some $k \in \mathbb{N}$, then $f^k \in J$, hence $f \in J$ since J is a radical, or $\overline{f} = 0$). This proves the forward implication.

$\Longleftarrow$: Now, suppose B is a finitely generated k-algebra with no nilpotent elements, then there exists $a_1, ..., a_n \in B$, such that for every elemnt $b \in B$, there exists a polynomial $f \in k[x_1, ..., x_n]$ such that $b = f(a_1, ..., a_n)$. Hence, if consider the ring homomorphism $\varphi : k[x_1, ..., x_n] \to B$ by $\varphi(f) = f(a_1, ..., a_n)$, the ring homomorphism φ is surjective, showing that $B \cong k[x_1, ..., x_n]/\ker(\varphi)$.

Also, the assumption that B has no nilpotent elements implies that $\ker(\varphi) \subseteq k[x_1, ..., x_n]$ is a radical (since for all $f \in k[x_1, ..., x_n]$, if $f^k \in \ker(\varphi)$ for some $k \in \mathbb{N}$, we have $\varphi(f)^k = \varphi(f^k) = 0$, showing that $\varphi(f) \in B$ is nilpotent, or $\varphi(f) = 0$. Hence $f \in \ker(\varphi)$, therefore $\sqrt{\ker(\varphi)} = \ker(\varphi)$).

Then, if we take $Y = Z(\ker(\varphi)) \subseteq \mathbb{A}^n$ as the algebraic set, since $\ker(\varphi) = I(Y) = I(Z(\ker(\varphi)))$ (due to $\ker(\varphi)$ being a radical), we have the coordinate ring $k[Y] = k[x_1, ..., x_n]/\ker(\varphi)$, hence $B \cong k[x_1, ..., x_n]/\ker(\varphi) = k[Y]$, so B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$ (for some $n \in \mathbb{N}$). This proves the converse.