Math 231A HW 3

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Problem 1

Etingof Problem Set 2.11:

Show that for $n \geq 1$, we have $\pi_0(\mathrm{SU}(n+1)) = \pi_0(\mathrm{SU}(n))$, $\pi_0(U(n+1)) = \pi_0(U(n))$ and deduce from it that groups $U(n), \mathrm{SU}(n)$ are connected for all n. Similarly, show that for $n \geq 2$, we have $\pi_1(\mathrm{SU}(n+1)) = \pi_1(\mathrm{SU}(n))$, $\pi_1(U(n+1)) = \pi_1(U(n))$ and deduce from it that for $n \geq 2$, $\mathrm{SU}(n)$ is simply-connected and $\pi_1(U(n)) = \mathbb{Z}$.

Solution:

1. Statements about π_0, the Connected Components:

We'll approach the first part by induction. For $n = 1$, we have $U(1) = S^1$ and $\mathrm{SU}(1) = \{1\}$, which are both connected (these will be treated as base case).

Now, suppose for given $n \geq 1$, suppose $U(n)$ (respectively, $\mathrm{SU}(n)$) is connected (which $\pi_0(U(n)) = \pi_0(\mathrm{SU}(n)) = \{*\}$). Consider $U(n+1)$ (and $\mathrm{SU}(n)$ respectively): It has a natural action on $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ while preserving the norm, which descends to a natural action on $S^{2n+1} := \{v \in \mathbb{R}^{2n+2} \mid \|v\| = 1\}$.

Notice that for both $U(n)$ and $\mathrm{SU}(n)$ such action on S^{2n+1} is transitive, since let $\{e_1, ..., e_{n+1}\}$ be the elementary basis of $\mathbb{C}^{n+1}$, for any $u_1 \in S^{2n+1}$ one can extend this to an orthonormal basis, say $\{u_1, ..., u_{n+1}\}$, then the matrix $\begin{pmatrix} | & & | \\ u_1 & \cdots & u_{n+1} \\ | & & | \end{pmatrix}$ is unitary, and sends e_1 to u_1, showing that $U(n)$ has transitive action on S^{2n+1}; and, for $\mathrm{SU}(n+1)$ it's also transitive, since one can always modify the list $\{u_1, ..., u_{n+1}\}$ (by multiplying one some scalar $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ to one of the $u_2, ..., u_{n+1}$) to get determinant 1, hence one can talk about their stabilizers for $e_1 \in S^{2n+1}$ for instance.

If fix $e_1 \in S^{2n+1}$, then $G_{e_1} \subset U(n+1)$ must fixe e_1, while send $\mathrm{span}\{e_1\}^\perp = \mathrm{span}\{e_2, ..., e_{n+1}\}$ back to itself (due to the unitarity of the matrices). Hence, any $O \in G_{e_1}$ must have $O(e_1) = e_1$, and $O(e_i) \in \mathrm{span}\{e_2, ..., e_{n+1}\}$ for all index $i > 1$, which has the following form under standard basis:

$$\mathcal{M}(O) = \begin{pmatrix} 1 & 0 \\ 0 & O^{(n)} \end{pmatrix} \tag{1.1}$$

Where here $O^{(n)} \in U(n)$ (since O itself restricts to a unitary operator on $\mathrm{span}\{e_2, ..., e_{n+1}\}$). Hence, by taking the map $O \mapsto O^{(n)}$, it defines an injective map (in fact a group homomorphism because $O^{(n)}$ is a block matrix on the diagonal) of $G_{e_1} \hookrightarrow U(n)$. This map is also surjective since every $O^{(n)} \in U(n)$ has $O := \begin{pmatrix} 1 & 0 \\ 0 & O^{(n)} \end{pmatrix}$ be a unitary operator on $\mathbb{C}^{n+1}$. Hence, $G_{e_1} \cong U(n)$.

As a consequence, since the action on S^{2n+1} is transitive, by the theorem related to the homogeneous space we have $S^{2n+1} = \mathrm{Orb}(e_1) \cong U(n+1)/G_{e_1} \cong U(n+1)/U(n)$. Hence, by assumption since $U(n)$ is connected, while S^{2n+1} is simply-connected, we have $\pi_0(U(n+1)) \cong \pi_0(U(n+

$1)/U(n)) = \pi_0(S^{2n+1}) = \{*\}$ (from a set-theoretic point of view, due to **Corollary 4.2** in Etingof's notes), hence $\pi_0(U(n+1))$ is trivial, which $U(n+1)$ is connected.

Then, recall that $\mathrm{SU}(n+1)$ also acts transitively on S^{2n+1}, and its stabilizer for e_1 is $\mathrm{SU}(n+1) \cap G_{e_1}$, which from the formual in (1.1) it's every matrix of the form $O = \begin{pmatrix} 1 & 0 \\ 0 & O^{(n)} \end{pmatrix}$ with $\det(O) = \det(O^{(n)}) = 1$, hence we have $\mathrm{SU}(n+1) \cap G_{e_1} \cong \mathrm{SU}(n)$ by definition. So by similar logic, $\pi_0(\mathrm{SU}(n+1)) \cong \pi_0(\mathrm{SU}(n+1)/\mathrm{SU}(n)) = \pi_0(S^{2n+1}) = \{*\}$ (as sets), again showing that $\mathrm{SU}(n+1)$ is also connected.

2. Statements about π_1, the Fundamental Groups:
Since we've proven by induction that $U(n)$ and $\mathrm{SU}(n)$ are all connected for $n \geq 1$, hence one can identify its fundamental group. Again, by applying **Corollary 4.2** in Etingof's Notes, using the same (transitive) action defined in the previous part, we have the following exact sequences for all $n \geq 1$:

$$\pi_2(U(n+1)/U(n)) \to \pi_1(U(n)) \to \pi_1(U(n+1)) \to \pi_1(U(n+1)/U(n)) \to \{1\} \tag{1.2}$$

$$\pi_2(\mathrm{SU}(n+1)/\mathrm{SU}(n)) \to \pi_1(\mathrm{SU}(n)) \to \pi_1(\mathrm{SU}(n+1)) \to \pi_1(\mathrm{SU}(n+1)/\mathrm{SU}(n)) \to \{1\} \tag{1.3}$$

Then, because $U(n+1)/U(n) \cong \mathrm{SU}(n+1)/\mathrm{SU}(n) \cong S^{2n+1}$ by the previous construction of homogeneous spaces, then because $\pi_2(S^{2n+1})$ and $\pi_1(S^{2n+1})$ are both trivial groups, the above exact sequences reduce to the following:

$$\pi_1(U(n)) \xrightarrow{\sim} \pi_1(U(n+1)), \quad \pi_1(\mathrm{SU}(n)) \xrightarrow{\sim} \pi_1(\mathrm{SU}(n+1)) \tag{1.4}$$

Hence, one can identify $\pi_1(U(n)) = \pi_1(U(n+1))$ and $\pi_1(\mathrm{SU}(n)) = \pi_1(\mathrm{SU}(n+1))$.

Finally, because $U(1) = S^1$ (which has $\pi_1(S^1) = \mathbb{Z}$) and $\mathrm{SU}(1) = \{1\}$ (which has $\pi_1(\{1\}) = \{1\}$), then we conclude that $\pi_1(U(n)) = \mathbb{Z}$ and $\pi_1(\mathrm{SU}(n)) = \{1\}$ for all $n \geq 1$.

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Problem 2

Etingof Problem Set 2.13:

Using Gram-Schmidt orthogonalization process, show that $\mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ is diffeomorphic to the space of upper-triangular matrices with positive entries on the diagonal. Deduce from this that $\mathrm{GL}(n, \mathbb{R})$ is homotopic (as a topological space) to $O(n, \mathbb{R})$.

Solution:

1. Diffeomorphism between $\mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ and $\mathrm{UT}_+(n, \mathbb{R})$:
Here we'll utilize QR-Factorization: Given $A \in \mathrm{GL}(n, \mathbb{K})$ where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, then there exists unique matrices Q and R (with Q being unitary and R being upper-triangular with positive entries on the diagonal), such that $A = QR$.

For the case $\mathbb{K} = \mathbb{R}$, since the unitary matrices are precisely collected by $O(n, \mathbb{R})$, then with $\mathrm{UT}_+(n, \mathbb{R})$ denoting groups of upper triangular matrices with positive enries on the diagonal, if identify $\mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ with the right cosets (instead of the left cosets), then for all matrix $A = QR \in \mathrm{GL}(n, \mathbb{R})$ (using QR-Factorization), we have $\overline{A} = \overline{R}$. THis shows the set function $\mathrm{UT}_+(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ by $R \mapsto \overline{R}$ is surjective.

Also, notice that $O(n, \mathbb{R}) \cap \mathrm{UT}_+(n, \mathbb{R}) = \{\mathrm{id}_n\}$: Given any $T \in O(n, \mathbb{R}) \cap \mathrm{UT}_+(n, \mathbb{R})$, we have its first column being normalized (and only with entry $T_{11} > 0$), hence we must have $T_{11} = 1$, while any other $T_{1k} = 0$ (by the claim that T is upper-triangular). Inductively, for the i^{th} column vector of T (with $i < k$), if given $T_{i1}, ..., T_{i(i-1)}, T_{i(i+1)}, ..., T_{in}$ are all 0 and $T_{ii} = 1$, then by orthonormality if taking the inner product of the k^{th} column vector with the first $(k-1)$ of them, we must yield $T_{k1} = ... = T_{k(k-1)} = 1$, while $T_{k(k+1)} = ... = T_{kn} = 0$ by the fact that T is upper-triangular. Finally, because T_{kk} must have norm 1 and being positive, then $T_{kk} = 1$. Hence, all the diagonals are 1 (by induction) while non-diagonal emtries are all 0, showing $T = \mathrm{id}_n$.

Hence, given any $R, R' \in \mathrm{UT}_+(n, \mathbb{R})$, if $\overline{R} = \overline{R'} \in \mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$, one must have $R(R')^{-1} \in O(n, \mathbb{R})$ (since they have the same right coset), showing that $R(R')^{-1} = \mathrm{id}_n$, or $R = R'$. Therefore as set function, $\mathrm{UT}_+(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ by $R \mapsto \overline{R}$ is also injective. Hence, since it is bijective, there's no ambiguity identifying the representatives of each coset in $\mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ using $\mathrm{UT}_+(n, \mathbb{R})$.

Since for all right coset $\overline{A}$ in $\mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ there exists a unique $R \in \mathrm{UT}_+(n, \mathbb{R})$ such that $\overline{R} = \overline{A}$. Then, define a right action of $\mathrm{UT}_+(n, \mathbb{R}) \curvearrowright \mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ by $\overline{R} \cdot T := \overline{RT}$. It is transitive since any coset $\overline{R} = \overline{\mathrm{id}_n} \cdot R$. Also, notice that the stabilizer for any element is trivial: Given any $R \in \mathrm{UT}_+(n, \mathbb{R})$, suppose $R' \in \mathrm{UT}_+(n, \mathbb{R})$ satisfies $\overline{R} \cdot R' = \overline{RR'} = \overline{R}$, then since $RR' \in \mathrm{UT}_+(n, \mathbb{R})$, so the one-to-one correspondance between $\mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ and $\mathrm{UT}_+(n, \mathbb{R})$ indicates that $R = RR'$, hence $R' = \mathrm{id}_n$.

Hence, if consider $\mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ as a manifold and fix $\overline{\mathrm{id}_n}$ as an element, since $\mathrm{Orb}\left(\overline{\mathrm{id}_n}\right) = \mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ and $G_{\overline{\mathrm{id}_n}} = \{\mathrm{id}_n\}$ under the action of $\mathrm{UT}_+(n, \mathbb{R})$, then we have $\mathrm{UT}_+(n, \mathbb{R}) \cong \mathrm{UT}_+(n, \mathbb{R})/\{\mathrm{id}_n\} \cong \mathrm{Orb}\left(\overline{\mathrm{id}_n}\right) = \mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ (by **Corollary 4.13** in Etingof's notes).

(**Rmk:** Here the reason why the action is smooth, is because the action corresponds to a natural right action on $\mathrm{GL}(n, \mathbb{R})$, say $a : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{UT}_+(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ by $a(A, R) = AR$. Then, the action on $\mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ can be obtained through $\pi \circ a$, where $\pi : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ is the projection).

2. Homotopy between $\mathrm{GL}(n, \mathbb{R})$ and $O(n, \mathbb{R})$:

Here, notice that $\mathrm{UT}_+(n, \mathbb{R}) \cong (0, \infty)^n \times \mathbb{R}^{\frac{n(n-1)}{2}}$ (since the n diagonal entries are positive, while the $\frac{n(n-1)}{2}$ strict upper-triangular entries can be arbitrary real numbers), then this space is simply-connected (since $\pi_1((0, \infty)) \cong \pi_1((\mathbb{R})) = \{e\}$, so $\pi_1(\mathrm{UT}_+(n, \mathbb{R})) \cong \prod_{i=1}^n \pi_1((0, \infty)) \times \prod_{j=1}^{\frac{n(n-1)}{2}} \pi_1(\mathbb{R}) = \{e\}$). So, that means the space is contractable, hence there exists smooth homotopy $H : \mathrm{UT}_+(n, \mathbb{R}) \times [0, 1] \to \mathrm{UT}_+(n, \mathbb{R})$, such that $H(R, 1) = \mathrm{id}_n$ for all $R \in \mathrm{UT}_+(n, \mathbb{R})$.

Then, one can basically pull back the homotopy to the homotopy on $\mathrm{GL}(n, \mathbb{R})$: Define $H' : \mathrm{GL}(n, \mathbb{R}) \times [0, 1] \to \mathrm{GL}(n, \mathbb{R})$ by $H'(A, t) = Q_A \cdot H(R_A, t)$, where $A = Q_A R_A$ by QR-Factorization, so $R_A \in \mathrm{UT}_+(n, \mathbb{R})$. Notice that such H' in fact is well-defined by the uniqueness of QR-Factorization, and it's a smooth map due to the fact that the map $O(n, \mathbb{R}) \times \mathrm{UT}_+(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ by $(Q, R) \mapsto QR$ is a smooth surjective map (not a group homomorphism though) that's a locally trivial fibration (or one can choose small neighborhoods of $Q \in O(n, \mathbb{R})$ so that such map is diffeomorphic to its own image). Hence, since the map $((Q, R), t) \mapsto (Q, H(R, t))$ defines a smooth homotopy on $O(n, \mathbb{R}) \times \mathrm{UT}_+(n, \mathbb{R})$ that descends to $H' : \mathrm{GL}(n, \mathbb{R}) \times [0, 1] \to .\ \mathrm{GL}(n, \mathbb{R})$ (by the local diffeoorphism onto $\mathrm{GL}(n, \mathbb{R})$), then H' can be identified as a smooth map. Hence, one has $H'(A, 1) = Q_A \cdot H(R_A, 1) = Q_A \cdot \mathrm{id}_n = Q_A \in O(n, \mathbb{R})$ for all $A \in \mathrm{GL}(n, \mathbb{R})$, showing that $\mathrm{GL}(n, \mathbb{R})$ in fact is homotopic to $O(n, \mathbb{R})$.

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Problem 3

Etingof Problem Set 2.14:

Let L_n be the set of all Lagrangian subspaces in $\mathbb{R}^{2n}$ with the standard symplectic form w. (A subspace V is Lagrangian if $\dim(V) = n$ and $w(x, y) = 0$ for any $x, y \in V$). Show that the group $\mathrm{Sp}(n, \mathbb{R})$ acts transitively on L_n and use it to define on L_n a structure of a smooth manifold and find its dimension.

Solution:

1. Symplectic Action on Space of Lagrangian:

First, we'll fix $L^{(0)} := \mathrm{span}\{e_1, ..., e_n\}$ (given that $\mathbb{R}^{2n}$ has standard basis $\{e_1, ..., e_n, f_1, ..., f_n\}$). Notice that $L^{(0)}$ is an n-dimensional subspace, and it's in fact Lagrangian: Notice that with symplectic form w, its matrix under standard basis is $\mathcal{M}(w) = J = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}$, hence since for all $e_i, e_j \in \{e_1, ..., e_n\}$ we have $J(e_j) = -f_j$, then one has $w(e_i, e_j) = e_i \cdot J(e_j) = e_i \cdot (-f_j) = 0$ (using standard Euclidean inner product). Hence, since all the pairs of basis elements of $L^{(0)}$ (when fixing a basis) yields 0 under symplectic form, every $x, y \in L^{(0)}$ satisfies $w(x, y) = 0$ (by inductively breaking down the bilinear form into sums of the form applies to each pair of he basis). Hence, $L^{(0)}$ is a Lagrangian subspace.

To prove that $\mathrm{Sp}(n, \mathbb{R})$ acts transitively on L_n (all Lagrangian Subspace), it suffices to prove that one can send $L^{(0)}$ to any Lagrangian Subspace, say $L \in L_n$. We'll first establish some tools:

***Lemma:* 1**

Given any Lagrangian Subspace $L \in L_n$, with $J := \mathcal{M}(w)$ (the matrix of symplectic form), then $JL := \{J(v) | \ v \in L\}$ is also Lagrangian.

Proof: This can be seen by the following equation:

$$J^T J J = \begin{pmatrix} 0 & -\mathrm{id}_n \\ \mathrm{id}_n & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix} = J \tag{3.1}$$

Hence, for all $x, y \in L$, it satisfies $w(Jx, Jy) = (Jx)^T J(Jy) = x^T J^T J J y = x^T J y = w(x, y) = 0$, hence JL is also Lagrangian (since J is invertible, JL is also n-dimensional). $\square$

***Lemma:* 2**

Given any Lagrangian $L \in L_n$, $L \oplus JL = \mathbb{R}^{2n}$ and $JL = L^{\perp}$.

Proof: For any $x = (a_1, ..., a_n, b_1, ..., b_n) \in L \cap JL$, we have the following:

$$Jx = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \\ -a_1 \\ \vdots \\ -a_n \end{pmatrix} \in JL \tag{3.2}$$

$$J^{-1}x = \begin{pmatrix} 0 & -\operatorname{id}_n \\ \operatorname{id}_n & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} -b_1 \\ \vdots \\ -b_n \\ a_1 \\ \vdots \\ a_n \end{pmatrix} \in L \tag{3.3}$$

Hence, we also have $Jx \in L \cap JL$. So, if we consider $w(x, Jx) = 0$, it is expressed as follow:

$$0 = w(x, Jx) = (a_1 \; \dots \; b_n) \begin{pmatrix} 0 & \operatorname{id}_n \\ -\operatorname{id}_n & 0 \end{pmatrix}^2 \begin{pmatrix} a_1 \\ \vdots \\ b_n \end{pmatrix} \tag{3.4}$$

$$= (a_1 \; \dots \; b_n) \begin{pmatrix} -\operatorname{id}_n & 0 \\ 0 & -\operatorname{id}_n \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ b_n \end{pmatrix} = -x \cdot x \tag{3.5}$$

(Here it's using $J^2 = -\operatorname{id}_{2n}$).

Hence, $x \cdot x = 0$ under Euclidean inner product, showing that $x = 0$. This concludes that $L \cap JL = \{0\}$, hence $L \oplus JL$ forms a direct sum; however, since both L, JL are n-dimensional subspaces, $L \oplus JL$ is $2n$-dimensional, so $L \oplus JL = \mathbb{R}^{2n}$.

Also, let $a, b, c, d \in \mathbb{R}^n$ satisfying $(a, b), (c, d) \in L$, then the matrix tells that $w((a, b), (c, d)) = (a, b) \cdot (d, -c) = a \cdot d - b \cdot c = 0$. So, we get that $(a, b) \cdot J((c, d)) = w((a, b), (c, d)) = 0$. Hence, any $(a, b) \in L$ and $J((c, d)) \in JL$ are orthogonal to each other, showing that $JL \subseteq L^\perp$; and with the two subspace having the same dimension, $JL = L^\perp$. $\square$

Which by the proof in **Lemma 2** in particular, let $\{u_1, \dots, u_n\}$ denotes an orthonormal basis of L, then $\{J(u_1), \dots, J(u_n)\}$ actually forms an orthonormal basis (since J swaps entry only when fixing standard basis, which preserves the norm; while $J^T = J^{-1}$, hence $(Ju_i) \cdot (Ju_j) = (u_i^T J^T)(Ju_j) = u_i^T u_j = u_i \cdot u_j = 0$ for all $i \neq j$). Hence, $\{u_1, \dots, u_n, Ju_1, \dots, Ju_n\}$ in fact forms an orthonormal basis of $\mathbb{R}^{2n}$ (since $u_1, .., u_n \in L$, while $Ju_1, \dots, Ju_n \in JL = L^\perp$ by **Lemma 2**).

Now, consider the matrix $T \in \mathrm{GL}(n, \mathbb{R})$ satisfying $T(e_i) = u_i$ and $T(f_i) = -Ju_i$, we claim that $T \in \mathrm{Sp}(n, \mathbb{R})$: To prove that T preserves symplectic form, it suffices to show it preserves the form's relation on the standard basis (since the rest follows by decomposing the form to the evaluation on pairs of basis elements, using linearity of T and bilinearity of w). Given any e_i, e_j we have $w(e_i, e_j) = 0$, which $w\big(T(e_i), T(e_j)\big) = w(u_i, u_j) = 0$ also (since u_i, u_j are chosen to be basis elements of Lagrangian L). Similarly, any f_i, f_j satisfies $w\big(T(f_i), T(f_j)\big) = w\big(-Ju_i, -Ju_j\big) = 0$ (since they're basis elements of $JL = L^\perp$, the complement Lagrangian or L). Also, given any e_i, f_i we have $w(T(e_i), T(f_i)) = w(u_i, -Ju_i) = u_i^T J(-Ju_i) = u_i \cdot u_i = 1 = e_i \cdot e_i = e_i^T J(-Je_i) = w(e_i, f_i)$ (since $J(e_i) = -f_i$, and u_i is chosen to be normalized). Finally, we also have e_i, f_j satisfies $w\big(T(e_i), T(f_j)\big) = w\big(u_i, -Ju_j\big) = u_i^T J\big(-Ju_j\big) = u_i^T u_j = u_i \cdot u_j = 0 = e_i \cdot e_j = e_i^T J\big(-Je_j\big) = w\big(e_i, f_j\big)$ (give $i \neq j$). Hence, T indeed preserves the symplectic form between any two pairs of the standard basis elements, hence $T \in \mathrm{Sp}(n, \mathbb{R})$.

Finally, since $L = \mathrm{span}\{u_1, .., u_n\} = \mathrm{span}\{T(e_1), \dots, T(e_n)\}$, then we have $T\big(L^{(0)}\big) = L$ as Lagrangian subspace, which shows that the natural action $\mathrm{Sp}(n, \mathbb{R}) \curvearrowright L_n$ by $T \cdot L = TL$ is transitive (Note: if $T \in \mathrm{Sp}(n, \mathbb{R})$, then for all $x, y \in L$, we have $w(T(x), T(y)) = w(x, y) = 0$, henc TL is also Lagrangian).

2. Dimension of Space of Lagrangian:

Here, we'll fix $L^{(0)} = \mathrm{span}\{e_1, \dots, e_n\}$ and compute its stabilizer (Note: we also have $JL^{(0)} = \mathrm{span}\{f_1, \dots, f_n\}$, with $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ being standard basis on $\mathbb{R}^{2n}$).

Given any $T \in G_{L^{(0)}}$, since $T\left(L^{(0)}\right) = L^{(0)}$, then with $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ for some $A, B, C, D \in M_{n(\mathbb{R})}$, we must have $C = 0$ (since $T(e_i) \in \operatorname{span}\{e_1, ..., e_n\}$). Then using the properties of symplectic matrix, we have:

$$A^T C = C^T A = 0, \quad B^T D = D^T B, \quad A^T D - C^T B = A^T D = \mathrm{id}_n \tag{3.6}$$

Hence, fixing any $A \in \mathrm{GL}(n, \mathbb{R})$, we automatically has $D = \left(A^T\right)^{-1} = \left(A^{-1}\right)^T$ and $B^T = D^T B D^{-1} = A^{-1} B A^T$.

Notice that the correspondance for B above actually is encoding $\frac{n(n-1)}{2}$ equations (since T must preserve Lagrangian, then in particular it also preserves $T\left(JL^{(0)}\right)$ as Lagrangian, where $JL^{(0)} = \operatorname{span}\{f_1, ..., f_n\}$), hence the last n column vectors of T (which is formed by entries of B, D) must pairwise satisfy symplectic form being 0, showing that $B \in M_n(\mathbb{R})$ (with n^2 entries) actually has a solution space of $n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ dimension (Note: The collection of B with the above properties form a subspace in $M_n(\mathbb{R})$ of $\frac{n(n-1)}{2}$ non-degenerate linear equations).

Hence, $G_{L^{(0)}}$ can be identified using $A \in \mathrm{GL}(n, \mathbb{R})$ and its corresponding solution space for B (denoted as $V_A \subseteq M_n(\mathbb{R})$), which $\dim\left(G_{e_1}\right) = \dim(\mathrm{GL}(n, \mathbb{R})) + \dim(V_A) = n^2 + \frac{n(n+1)}{2}$.

Finally, since $L_n \cong \mathrm{Sp}(n, \mathbb{R}) / G_{L^{(0)}}$ based on the group action, then since $\mathrm{Sp}(n, \mathbb{R})$ has dimension $2n^2 + n$, we have the following:

$$\dim(L_n) = \dim(\mathrm{Sp}(n, \mathbb{R})) - \dim(G_{L^{(0)}}) = 2n^2 + n - n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2} \tag{3.7}$$

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Problem 4

Etingof Problem Set 2.16:

(1) Show that $\mathrm{Sp}(1) \cong \mathrm{SU}(2) \cong S^3$.

(2) Using the previous exercise (2.15), show that we have a natural transitive action of $\mathrm{Sp}(n)$ on the sphere S^{4n-1} and a stabilizer of a point is isomorphic to $\mathrm{Sp}(n-1)$.

(3) Deduce that $\pi_1(\mathrm{Sp}(n+1)) = \pi_1(\mathrm{Sp}(n))$, $\pi_0(\mathrm{Sp}(n+1)) = \pi_0(\mathrm{Sp}(n))$.

Solution:

(1) Given that $\mathrm{Sp}(1) := \mathrm{Sp}(1,\mathbb{C}) \cap \mathrm{SU}(2)$, it suffices to prove that $\mathrm{SU}(2) \subseteq \mathrm{Sp}(1,\mathbb{C})$. Recall that any $A \in \mathrm{SU}(2)$ can be expressed as $A = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ where $|a|^2 + |b|^2 = 1$. Then, consider the symplectic matrix, we get:

$$A^T J A = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.1}$$

Hence, $A \in \mathrm{Sp}(1,\mathbb{C})$, showing that $\mathrm{SU}(2) \subseteq \mathrm{Sp}(1,\mathbb{C})$, or $\mathrm{SU}(1) = \mathrm{Sp}(1,\mathbb{C}) \cap \mathrm{SU}(2) = \mathrm{SU}(2)$. As a consequence, $\mathrm{Sp}(1) = \mathrm{SU}(2) \cong S^3$.

(2) Notice that S^{4n-1} can be identified as all normal vectors in $\mathbb{R}^{4n}$, which can be reclassified as $\mathbb{H}^n$. Hence, since $U(n,\mathbb{H})$ has a natural transitive action on S^{4n-1}, while $U(n,\mathbb{H}) \cong \mathrm{Sp}(n) := \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{SU}(2n)$, there is a natural transitive action of $\mathrm{Sp}(n)$ on S^{4n-1}.

Then, let $\mathbb{R}^{4n}$ be identified as $\mathbb{C}^{2n}$ with standard basis $\{e_1, ..., e_n, f_1, ..., f_n\}$. If we consider the action of $\mathrm{Sp}(n)$ on S^{4n-1} and consider the stabilizer of e_1 (denote as G_{e_1}).

For $n = 1$ one has the stabilizer being $A = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \in G_{e_1} \subseteq \mathrm{Sp}(1) = \mathrm{SU}(2)$, which $d = \overline{1} = 1$ and $b = -\overline{0} = 0$, so $A = \mathrm{id}_2$.

Now, for $n \geq 2$, suppose $A \in G_{e_1}$, then express in matrix form (using standard basis) the first column vector is $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Notice for the specific case f_1, because $w(e_1, f_1) = 1$ (Since $J(f_1) = e_1$), while symplectic matrix preserves it, then we have $w(A(e_1), A(f_1)) = w(e_1, A(f_1))$. Let $A(f_1) = a_{(n+1)1}e_1 + ... + a_n((n+1)n)e_n + a_{(n+1)(n+1)}f_1 + ... + a_{(n+1)(2n)}f_n$, we have $w(e_1, A(f_1)) = a_{(n+1)(n+1)}$ (since symplectic form J swaps the first n entries with the last n entries, while providing a negative sign for the moved first n entries), hence $a((n+1)(n+1)) = 1$. Also, recall that $\mathrm{Sp}(n) \subseteq \mathrm{SU}(2n)$, then every column vector must itself be normalized. So, with the $(n+1)^{\text{th}}$ column vector (i.e. $A(f_1)$) having the $(n+1)^{\text{th}}$ entry being 1, the other entry is enforced to be 0. Hence, we get that $A(f_1) = f_1$ by appealing to the matrix form of A. Which, there exists $A', B', C', D' \in M_{(n-1)}(\mathbb{C})$ satisfying the following:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A' & 0 & B' \\ 0 & 0 & 1 & 0 \\ 0 & C' & 0 & D' \end{pmatrix} \tag{4.2}$$

Since the original column vectors in A (involving A', B', C', D') forms an orthonormal list, after truncating the two entries involving 0, they still form an orthonormal list, so we have $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in U(2n-2)$. Then, by the claim that it is special unitary, we have $\det(A) = 1 \cdot$

$\det\begin{pmatrix} A' & 0 & B' \\ 0 & 1 & 0 \\ C' & 0 & D' \end{pmatrix} = 1 \cdot \det\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = 1$, hence we also have $A \in \mathrm{SU}(2n-2)$. Finally, since A is symplectic, it satisfies the following three relations:

$$\begin{pmatrix} 0 & 0 \\ 0 & (A')^T C \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & C' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & C' \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix} = \left((C')^T A\right) \tag{4.3}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & (B')^T D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & B' \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & D' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D' \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & B' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (D')^T B \end{pmatrix} \tag{4.4}$$

$$\mathrm{id}_n = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & D' \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & C' \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & B' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (A')^T D' - (C')^T B' \end{pmatrix} \tag{4.5}$$

So, we get $(A')^T C' = (C')^T A'$, $(B')^T D' = (D')^T B'$, and $(A')^T D' - (C')^T B' = \mathrm{id}_{n-1}$, showing that $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \mathrm{Sp}(n-1, \mathbb{C})$. Hence, $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \mathrm{Sp}(n-1) = \mathrm{Sp}(n-1, \mathbb{C}) \cap \mathrm{SU}(2n-2)$.

Hence, this creates an injection $G_{e_1} \hookrightarrow \mathrm{Sp}(n-1)$ by mapping $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A' & 0 & B' \\ 0 & 0 & 1 & 0 \\ 0 & C' & 0 & D' \end{pmatrix} \mapsto \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ (and it's a group homomorphism also). Also, every $T' \in \mathrm{Sp}(n-1)$ after decomposing into four pieces of $(n-1) \times (n-1)$ block matrices (say $T' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$) and plug into the desired positions shown above (say $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A' & 0 & B' \\ 0 & 0 & 1 & 0 \\ 0 & C' & 0 & D' \end{pmatrix}$), we also recover something contained in G_{e_1}, which creates a mutual inverse. Hence, $G_{e_1} \cong \mathrm{Sp}(n-1)$ as groups, and we get the following:

$$S^{4n-1} = \mathrm{Orb}(e_1) \cong \mathrm{Sp}(n)/G_{e_1} \cong \mathrm{Sp}(n)/\mathrm{Sp}(n-1) \tag{4.6}$$

Where the middle part is appealing to homogeneous space theory.

(3) From the previous part we've deduce that as homogeneous space, $S^{4n-1} \cong \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$. For $\mathrm{Sp}(1) = \mathrm{SU}(2) \cong S^3$ (by (1)), we have $\mathrm{Sp}(1)$ being simply-connected. Now, by induction suppose $\mathrm{Sp}(n)$ is connected ($\pi_0(\mathrm{Sp}(n)) = \{*\}$), then using **Corollary 4.2** in Etingof's Notes, we have $\pi_0(\mathrm{Sp}(n+1)) = \pi_0(\mathrm{Sp}(n+1)/\mathrm{Sp}(n)) = \pi_0(S^{4n+3}) = \{*\}$, hence $\mathrm{Sp}(n+1)$ is also connected.

Then, also by induction assume that $\mathrm{Sp}(n)$ is simlpy-connected, using the connectedness and **Corollary 4.2** again, we have the following long exact sequence:

$$\pi_2(\mathrm{Sp}(n+1)/\mathrm{Sp}(n)) \to \pi_1(\mathrm{Sp}(n)) \to \pi_1(\mathrm{Sp}(n+1)) \to \pi_1(\mathrm{Sp}(n+1)/\mathrm{Sp}(n)) \to \{e\} \tag{4.7}$$

Since $\mathrm{Sp}(n+1)/\mathrm{Sp}(n) \cong S^{4n+3}$, we have $\pi_2(S^{4n+3}) = \pi_1(S^{4n+3}) = \{e\}$, hence the above exact sequence reduces to the following:

$$\{e\} \to \pi_1(\mathrm{Sp}(n)) \to \pi_1(\mathrm{Sp}(n+1)) \to \{e\} \tag{4.8}$$

This shows that $\pi_1(\mathrm{Sp}(n)) \cong \pi_1(\mathrm{Sp}(n+1))$. As a consequence, $\mathrm{Sp}(n)$ is simply-connected implies $\mathrm{Sp}(n+1)$ is also simply-connected.