Math CS 121 HW 1

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Question 1. *Chapter 1.2 # 12*
Prove that the event B is impossible if and only if for every event A,

$$A = (B \cap A^c) \cup (B^c \cap A)$$

Proof. $\Longrightarrow$: First, suppose the event B is impossible, which is equivalent to saying $B = \emptyset$. Then, we have $B^c = \Omega$ the whole sample space. For any event A, the following holds:

$$(B \cap A^c) \cup (B^c \cap A) = (\emptyset \cap A^c) \cup (\Omega \cap A) = \emptyset \cup A = A \tag{1}$$

Hence the given equality of event holds.

$\Longleftarrow$: Now, suppose for all event A, $A = (B \cap A^c) \cup (B^c \cap A)$, then in particular it works for setting $A = B$. Which, $B \cap B^c = B^c \cap B = \emptyset$, so we get:

$$B = (B \cap B^c) \cup (B^c \cap B) = \emptyset \cup \emptyset = \emptyset \tag{2}$$

Hence, $B = \emptyset$ is an impossible event.

□

Question 2. *Chapter 1.2 # 16*
Let A and B be two events. Prove the following relations by the elementwise method.

(a) $(A \setminus (A \cap B)) \cup B = A \cup B$

(b) $(A \cup B) \setminus (A \cap B) = (A \cap B^c) \cup (A^c \cap B)$

Proof. (a) $\subseteq$: Suppose $x \in (A \setminus (A \cap B)) \cup B$, either $x \in (A \setminus (A \cap B)) \subseteq A \subseteq A \cup B$, or $x \in B \subseteq A \cup B$, hence $x \in A \cup B$, showing $(A \setminus (A \cap B)) \cup B \subseteq A \cup B$.

$\supseteq$: Suppose $x \in A \cup B$, then either $x \in A$ or $x \in B$. If $x \in B$, then $x \in B \subseteq (A \setminus (A \cap B)) \cup B$; else, if $x \notin B$, it enforces $x \in A$, and shows that $x \notin A \cap B$. Hence, $x \in A \setminus (A \cap B) \subseteq (A \setminus (A \cap B)) \cup B$. With the above two cases, $A \cup B \subseteq (A \setminus (A \cap B)) \cup B$.

(b) $\subseteq$: Suppose $x \in (A \cup B) \setminus (A \cap B)$, it states $x \notin (A \cap B)$, while $x \in A$ or $x \in B$. Suppose $x \in A$, then with $x \notin (A \cap B)$ it concludes $x \notin B$ (or $x \in B^c$), hence $x \in (A \cap B^c) \subseteq (A \cap B^c) \cup (A^c \cap B)$. Else, suppose $x \in B$, then with $x \notin (A \cap B)$ it concludes $x \notin A$ (or $x \in A^c$), hence $x \in (A^c \cap B) \subseteq (A \cap B^c) \cup (A^c \cap B)$.

These two cases conclude that $(A \cup B) \setminus (A \cap B) \subseteq (A \cap B^c) \cup (A^c \cap B)$.

$\supseteq$: Now, suppose $x \in (A \cap B^c) \cup (A^c \cap B)$, then either $x \in A \cap B^c$ (stating $x \in A$ and $x \notin B$), or $x \in A^c \cap B$ (stating $x \notin A$ and $x \in B$).

In the first case $x \in A \subseteq (A \cup B)$, while $x \notin B$ implies $x \notin (A \cap B)$, showing that $x \in (A \cup B) \setminus (A \cap B)$. Similarly, in the second case $x \in B \subseteq (A \cup B)$, while $x \notin A$ implies $x \notin (A \cap B)$, hence again $x \in (A \cup B) \setminus (A \cap B)$.

These two cases conclude that $(A \cap B^c) \cup (A^c \cap B) \subseteq (A \cup B) \setminus (A \cap B)$.

□

Question 3. *Chapter 1.2 # 17*
Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events. rove that for every event B,

(a) $B \cap (\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} (B \cap A_i)$

(b) $B \cup (\bigcap_{i=1}^{\infty} A_i) = \bigcap_{i=1}^{\infty} (B \cup A_i)$.

Proof. (a) $\subseteq$: Suppose $x \in B \cap (\bigcup_{i=1}^{\infty} A_i)$, then $x \in B$ and $x \in \bigcup_{i=1}^{\infty} A_i$, hence there exists $n \in \mathbb{N}$ such that $x \in A_n$. This concludes that $x \in B \cap A_n \subseteq \bigcup_{i=1}^{\infty}(B \cap A_i)$.

Which, it further concludes that $B \cap (\bigcup_{i=1}^{\infty} A_i) \subseteq \bigcup_{i=1}^{\infty}(B \cap A_i)$.

$\supseteq$: Suppose $x \in \bigcup_{i=1}^{\infty}(B \cap A_i)$, then there exists $n \in \mathbb{N}$ such that $x \in B \cap A_n$. In particular $x \in B$, and $x \in A_n \subseteq \bigcup_{i=1}^{\infty} A_i$, showing that $x \in B \cap (\bigcup_{i=1}^{\infty} A_i)$.

This concludes that $\bigcup_{i=1}^{\infty}(B \cap A_i) \subseteq B \cap (\bigcup_{i=1}^{\infty} A_i)$.

(b) $\subseteq$: Suppose $x \in B \cup (\bigcap_{i=1}^{\infty} A_i)$, then either $x \in B$, or $x \in \bigcap_{i=1}^{\infty} A_i$. If $x \in B$, then for all $n \in \mathbb{N}$ it satisfies $x \in B \cup A_n$, hence $x \in \bigcap_{i=1}^{\infty}(B \cup A_i)$; else if $x \in \bigcap_{i=1}^{\infty} A_i$, for eveery $n \in \mathbb{N}$ it satisfies $x \in A_n \subseteq (B \cup A_n)$, hence $x \in \bigcap_{i=1}^{\infty}(B \cup A_i)$.

This concludes that $x \in \bigcap_{i=1}^{\infty}(B \cup A_i)$, or $B \cup (\bigcap_{i=1}^{\infty} A_i) \subseteq \bigcap_{i=1}^{\infty}(B \cup A_i)$.

$\supseteq$: Suppose $x \in \bigcap_{i=1}^{\infty}(B \cup A_i)$, then for each $n \in \mathbb{N}$, one has $x \in B \cup A_n$, showing that $x \in B$ or $x \in A_n$.

If for some $n \in \mathbb{N}$ it satisfies $x \in B$, it's clear that $x \in B \cup (\bigcap_{i=1}^{\infty} A_i)$; else, if $x \notin B$ for all $n \in \mathbb{N}$, then $x \in (B \cup A_n)$ for all $n \in \mathbb{N}$ implies $x \in A_n$ for all $n \in \mathbb{N}$. Hence, $x \in \bigcap_{i=1}^{\infty} A_i \subseteq B \cup (\bigcap_{i=1}^{\infty} A_i)$.

This concludes that $\bigcap_{i=1}^{\infty}(B \cup A_i) \subseteq B \cup (\bigcap_{i=1}^{\infty} A_i)$.

□

Question 4. *Chapter 1.4 # 5*
Suppose that 75% of all investors invest in traditional annuities and 45% of them invest in the stock market. If 85% invest in the stock market and/or traditional annuities, what percentage invest in both?

Proof. Let sample space Ω collects all investors, let event $A \subseteq \Omega$ denotes all investors invest in traditional annuities, and event $B \subseteq \Omega$ denotes all investors invest in the stock market. Which, $A \cup B$ denotes all investors investing in the stock market and/or traditional annuities, while $A \cap B$ denotes all investors investing in both the stock market and traditional annuities.

Based on the description, the provided probability function satisfies $\mathbb{P}(A) = 0.75$, $\mathbb{P}(B) = 0.45$, and $\mathbb{P}(A \cup B) = 0.85$. Then, the following equation hold:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \tag{3}$$

After rearranging, we get $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$. With the above conditions given, we get:

$$\mathbb{P}(A \cap B) = 0.75 + 0.45 - 0.85 = 0.3 \tag{4}$$

□

5,6,7,8,9,10 Not Done

Question 5. *Chapter 1.4 # 15*
Let A, B, and C be three events. Show that exactly two of these events will occur with probability:

$$\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - 3\mathbb{P}(A \cap B \cap C)$$

Proof. □

Question 6. *Chapter 1.5 # 20*
The coefficients of the quadratic equation $x^2 + bx + c = 0$ are determined by tossing a fair die twice (the first outcome is b, the second one is c). Find the probability that the equation has real roots.

Proof. For the equation to have real roots, the discriminat $b^2 - 4ac \geq 0$. In this case $a = 1$, so $b^2 - 4c \geq 0$, or $b^2 \geq 4c$.

Given the fair dice with six sides (from 1 to 6), the probability of getting each number is precisely $\frac{1}{6}$. $\square$

Question 7. *Chapter 1.4 # 25*
A number is selected at random from the set of natural numbers $\{1, 2, ..., 1000\}$*. What is the probability that it is divisible by* 4*but neither by* 5 *nor by* 7*?*

Proof. Let the sample space $\Omega := \{1, 2, ..., 1000\}$. Let A denotes $\square$

Question 8. *Chapter 1.4 # 26*
For a Democratic candidate to win an election, she must win districts I,II, and III. Polls have shown that the probability of winning I and III is 0.55*, losing II but not I is* 0.34*, and losing II and III but not I is* 0.15*. Find the probability that this candidate will win all three districts. (Draw a Venn Diagram).*

Proof. $\square$

Question 9. *Chapter 1.4 # 28*

Proof. □

Question 10. *Chapter 1.7 # 8*

Proof. □