Math 231A HW 2

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Problem 1

Etingof Problem Sets 2.2:

(1) Show that every discrete normal subgroup of a connected Lie group is central (hint: consider the map $G \to N : g \mapsto ghg^{-1}$ where h is fixed element in N).

(2) By applying part (1) to kernel of the map $\tilde{G} \to G$, show that for any connected Lie group G, the fundamental group $\pi_1(G)$ is commutative.

Solution: We'll let $m : G \times G \to G$ denotes the group operation, while $\iota : G \rightsquigarrow G$ denotes the inversion map of the Lie group G. We'll also use $\mathbb{1}$ to denote the identity of G (to prevent confusion with $1 \in I = [0,1]$).

(1): Recall that on a Lie group, inversion and multiplication are both regular maps. Now, fix an element $h \in N$, if consider the map $\Delta : G \to G \times G$ by $\Delta(g) = (g, g)$, and consider the right multiplication of h and inversion map, $R_h, \iota : G \rightsquigarrow G$ which collects into a regular product map $(R_h, \iota) : G \times G \to G \times G$ (by $(R_h, \iota)(k, l) = (kh, l^{-1})$). Then, consider the map $f = m \circ (R_h, \iota) \circ \Delta : G \to G$, we get:

$$\forall g \in G, \quad f(g) = m \circ (R_h, \iota) \circ \Delta(g) = m \circ (R_h, \iota)(g, g) = m(gh, g^{-1}) = ghg^{-1} \tag{1.1}$$

Which, this is the desired map we're working with, and it is a regular map. In particular, it's continuous.

Then, since G is a connected Lie group, one have $f(G) \subseteq G$ being connected; then, because N is a normal subgroup, for any $g \in G$, $f(g) = ghg^{-1} \in N$, so $f(G) \subseteq N$; finally, since N is discrete, then for any $h \in N$, there exists open neighborhood $U \ni h$ (where $U \subseteq G$) such that $U \cap N = \{h\}$, such that its closure $\overline{U} \cap N = \{h\}$ also (since G is a manifold, choose a local chart with domain containing h then one can use the discreteness in $\mathbb{R}^n$ or $\mathbb{C}^n$ to construct such open neighborhood).

As a consequence, f is restricted to be a constant map onto h, since if h is non-constant, then $f(G) \nsubseteq \{h\}$, hence using the open neighborhood $U \ni h$ above, take $V = G \setminus \overline{U}$, one have $V \cap U = \emptyset$, while $f(G) \in V \sqcup U$ (since $\overline{U}$ only contains h while $h \in U$ the interior of $\overline{U}$, then since $\overline{U} = U^\circ \sqcup \partial U$, there's no point of $f(G)$ containing on the boundary ∂U, and $\partial U = G \setminus (V \sqcup U)$ by some point set topology). Then, $V \cap f(G)$ and $U \cap f(G) = \{h\}$ forms a separation of $f(G)$ under its subspace topology, while $V \cap f(G)$ is not empty based on the assumption that $f(G)$ is non-constant (i.e. there exists $h' \neq h$, where $h' \in f(G)$; and since $h' \in V$, one has $f(G) \cap V \neq \emptyset$). Yet, this contradicts the statement that $f(G)$ is connected. So, we must have $f(G)$ being constant, or $f(G) = \{h\}$.

Hence, all $g \in G$ satisfies $f(g) = ghg^{-1} = h$, or $gh = hg$, showing that $h \in Z(G)$ (the center of G). And, with $h \in N$ chosen arbitrarily, N is central.

(2): Given the covering map $p : \tilde{G} \to G$ that also serves as a group homomorphism. First, say we fix the base point of G at identity $\mathbb{1}$, and the base point of $\tilde{G}$ at $[c_{\mathbb{1}}]$, the path class of constant map $c_{\mathbb{1}} : I \to G$ by $c_1(s) = 1$ (Note: the path class $[c_{\mathbb{1}}]$ also serves as the identity of $\tilde{G}$, since for all other path classes $[g]$, its representative g satisfies $g \cdot c_1(s) = g(s) \cdot c_1(s) = g(s) \cdot 1 = g(s) = c_1 \cdot g(s)$, so $[g] \cdot [c_1] = [g \cdot c_1] = [g] = [c_1 \cdot g] = [c_1] \cdot [g]$).

If we consider any path class $[g] \in \ker(p)$, one has $p([g]) = g(1) = \mathbb{1}$. Since $g(0) = \mathbb{1}$ also (by definition, the construction of $\tilde{G}$ is all path classes with starting point at base point $\mathbb{1}$ of G), then g is in fact a loop, hence $[g] \in \pi_1(G) := \pi_1(G, \mathbb{1})$

This shows that $\ker(p) \subseteq \pi_1(G)$, and $\pi_1(G) \subseteq \ker(p)$ simply because for any loop $[h] \in \pi_1(G) = \pi_1(G, \mathbb{1})$, if identify $[h] \in \tilde{G}$, $p([h]) = h(1) = \mathbb{1}$, hence $[h] \in \ker(p)$. So, $\ker(p) = \pi_1(G)$ **as a set**.

Finally, to prove that the multiplication on $\tilde{G}$ restricted to $\ker(p)$ is compatible with the concatination operation on $\pi_1(G)$, choose any path classes $[f], [g] \in \pi_1(G)$. If we define the map $H : I \times I \to G$ by $H(s,t) = f(s) \cdot g(t)$ (where the $\cdot$ is the group operation of G).

Then, consider the diagonal path $H_\Delta : I \to G$ by $H_{\Delta(s)} = H(s,s) = f(s) \cdot g(s) = (f \cdot g)(s)$, and the following path $h : I \to G$:

$$h(s) := \begin{cases} H(2s, 0) & s \leq \frac{1}{2} \\ H(1, 2s-1) & s > \frac{1}{2} \end{cases} \tag{1.2}$$

Which, h is tracing out the interval $[0,1] \times \{0\}$, then continue with the interval $\{1\} \times [0,1]$ in $I \times I$, which is continuous.

Similarly, H_Δ is tracing out the diagonal within $I \times I$, then since $I \times I$ is itself convex, and $H_{\Delta(0)} = H(0,0) = h(0)$ while $H_{\Delta(1)} = H(1,1) = h(1)$, then H_Δ and h are in fact path homotopic (since looking at their parametrization within $I \times I$, they're convex hence homotopic, or can cf. **Square Lemma** in **Introduction to Topological Manifold** by Lee). So, $[h] = [H_\Delta]$ as paths.

Finally, notice that H_Δ is a path characterized by $f \cdot g$ (the multiplication defined for $\tilde{G}$), while h is a concatination of f and g (since $H(t,0) = f(t) \cdot g(0) = f(t) \cdot \mathbb{1} = f(t)$, while $H(0,t) = f(0) \cdot g(t) = \mathbb{1} \cdot g(t) = g(t)$). Hence, H_Δ and h being path homotopic, implies that the $[f] \cdot_{\pi_1(G)} [g] = [f] \cdot_{\tilde{G}} [g]$ (concatination in the fundamental group is the same as group operation in $\tilde{G}$). Therefore, $\ker(p) = \pi_1(G)$ not only as sets, but as groups (since their group structure are th same from a topological perspective).

Which, with $\ker(p) = \pi_1(G)$ being discrete due to the fact that p is a covering map, hence there exists evenly covered neighborhood $U \subseteq G$ of $\mathbb{1}$, where $p^{-1}(U) = \bigsqcup_{i \in I} \tilde{U}_i$ (with each $\tilde{U}_i \cong U$ via restriction of p), and each component in $\ker(p)$ must be containd in exactly one $\tilde{U}_i$, while each $\tilde{U}_i$ can contain only one candidate from $\ker(p)$. So, these open subsets $\tilde{U}_i$ forms a separation of $\ker(p)$, showing it's discrete.

Then, based on the fact that $\ker(p) \trianglelefteq \tilde{G}$ and its discrete, statement in (1) implies that it's central. Hence, $\pi_1(G)$ is central.

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Problem 2

Etingof Problem Sets 2.4:

Let $\mathcal{F}_n(\mathbb{C})$ be the set of all flags in $\mathbb{C}^n$. Show that

$$\mathcal{F}_n(\mathbb{C}) = \mathrm{GL}(n,\mathbb{C})/B(n,\mathbb{C}) = U(n)/T(n) \tag{2.1}$$

Where $B(n,\mathbb{C})$ is the group of invertible complex upper triangular matrices, and $T(n)$ is the group of diagonal unitary matrices (which is easily shown to be the n-dimensional torus $(\mathbb{R}/\mathbb{Z})^n$). Deduce from this that $\mathcal{F}_{n(\mathbb{C})}$ is a compact complex manifold and find its dimension over $\mathbb{C}$.

Solution: As a premise, we'll identify $\mathrm{GL}(n,\mathbb{C})$ as a collection of linear operators on $\mathbb{C}^n$ with standard basis $\{e_1,...,e_n\}$ being the ordered basis. Let $V_i \coloneqq \mathrm{span}\{e_1,...,e_i\}$, we'll denote the flag $\{0\} = V_0 \subsetneq V_1 \subsetneq ... \subsetneq V_n = \mathbb{C}^n$ as $\mathcal{F}_0$, called the *Standard Flag.*

Equality between Sets:

Notice that $\mathrm{GL}(n,\mathbb{C})$ has a natural action on $\mathcal{F}_n(\mathbb{C})$: For each $A \in \mathrm{GL}(n,\mathbb{C})$ (viewed as an operator on $\mathbb{C}$), any subspace has its dimension being preserved (i.e. given $V \subseteq \mathbb{C}^n$ with $\dim(V) = k$, $A(V) \subseteq \mathbb{C}^n$ also has $\dim(A(V)) = k$), and it preserves subspace inclusion (so $U \subseteq V \implies A(U) \subseteq A(V)$). Hence, if $U_0 \subsetneq U_1 \subsetneq ... \subsetneq U_n$ is a flag, so is $A(U_0) \subsetneq A(U_1) \subsetneq ... \subsetneq A(U_n)$; so, it makes sense to define $\mu : \mathrm{GL}(n,\mathbb{C}) \times \mathcal{F}_n(\mathbb{C}) \to \mathcal{F}_n(\mathbb{C})$ by $\mu(A,\mathcal{F}) = A(\mathcal{F})$ (where given $\mathcal{F}$ as $U_0 \subsetneq U_1 \subsetneq ... \subsetneq U_n$, $A(\mathcal{F})$ denotes $A(U_0) \subsetneq A(U_1) \subsetneq ... \subsetneq A(U_n)$). It's clear that $\mu(B,\mu(A,\mathcal{F})) = \mu(BA,\mathcal{F})$ for all $A, B \in \mathrm{GL}(n,\mathbb{C})$ and $\mathcal{F} \in \mathcal{F}_n(\mathbb{C})$ by composition of operators, while the identity matrix satisfies $\mu(\mathrm{id},\mathcal{F}) = \mathcal{F}$, hence μ in fact forms a left action.

Now, we claim that $\mathrm{GL}(n,\mathbb{C})$ can be partitioned through its action on $\mathcal{F}_0$, which can be seen through the following statements:

1. $B(n,\mathbb{C})$ is the stabilizer of $\mathcal{F}_0$:

We'll claim that $A \in \mathrm{GL}(n,\mathbb{C})$ satisfies $\mu(A,\mathcal{F}_0) = A(\mathcal{F}_0) = \mathcal{F}_0$ iff $A \in B(n,\mathbb{C})$:

If $A(\mathcal{F}_0) = \mathcal{F}_0$, it satisfies $A(V_i) = V_i$ for all $i = 1,...,n$, hence $A(e_i) \in A(V_i) = V_i = \mathrm{span}\{e_1,...,e_i\}$ for each index i, showing A is in fact uppertriangular with respect to basis $\{e_1,...,e_n\}$, hence $A \in B(n,\mathbb{C})$.

Else, if $A \in B(n,\mathbb{C})$, each e_i satisfies $A(e_i) \in V_i = \mathrm{span}\{e_1,...,e_i\}$, hence $A(V_1) = A(\mathrm{span}\{e_1\}) \subseteq \mathrm{span}\{e_1\} = V_1$ (while $A(V_1)$ and V_1 have the same dimension due to the fact that $A \in \mathrm{GL}(n,\mathbb{C})$), showing that $A(V_1) = V_1$. Inductively one can show that all $A(V_i) = V_i$, hence $\mu(A,\mathcal{F}_0) = A(\mathcal{F}_0) = \mathcal{F}_0$.

So, this concludes that $\mathcal{F}_0$ is stable under (and only under) $B(n,\mathbb{C})$, which $B(n,\mathbb{C})$ is a stabilizer of $\mathcal{F}_0$.

2. μ is a Transitive Action:

To show such, consider a flag $\mathcal{F}$ formed by $\{0\} = U_0 \subsetneq U_1 \subsetneq ... \subsetneq U_n = \mathbb{C}^n$. Choose u_1 so that $U_1 = \mathrm{span}\{u_1\}$, and inductively choose $u_i \in U_i \setminus U_{i-1}$ so that $U_i = \mathrm{span}\{u_1,...,u_i\}$, then the list $\{u_1,...,u_n\}$ eventually forms a basis of $\mathbb{C}^n$ (**Rmk:** in particular using Gram Schmidt Formula one can restrict $\{u_1,...,u_n\}$ to be an orthonormal basis of $\mathbb{C}^n$ with Euclidean Inner Product).

Then, take a linear operator $T \in \mathrm{GL}(n,\mathbb{C})$ that satisfies $T(e_i) = u_i$ for all index i. It satisfies $T(V_1) = T(\mathrm{span}\{e_1\}) \subseteq \mathrm{span}\{u_1\} = U_1$, while the two have the same dimension, hence $T(V_1) = U_1$. Then, inductively one can derive $T(V_i) = \mathrm{span}\{u_1,...,u_i\} = U_i$ using similar logic. Hence, $\mu(T,\mathcal{F}_0) = T(\mathcal{F}_0) = \mathcal{F}$, which shows that μ is a transitive action. (**Rmk 2:** Because u_is can be chosen as orthonormal basis, T in fact can be chosen as a unitary operator, or $T \in U(n)$. Hence, resrict the action μ to an sub-action $U(n) \times \mathcal{F}_n(\mathbb{C}) \to \mathcal{F}_n(\mathbb{C})$ is still a transitive action).

Then, since the orbit $\mathrm{Orb}(\mathcal{F}_0) = \mathcal{F}_n(\mathbb{C})$ while $G_{\mathcal{F}_0} = B(n, \mathbb{C})$ (the stabilizer), there is a one-to-one correspondance between $\mathcal{F}_n$ and cosets of $B(n, \mathbb{C})$, hence set wise $\mathcal{F}_n(\mathbb{C}) \cong \mathrm{GL}(n, \mathbb{C})/B(n, \mathbb{C})$.

On the other hand, one has $B(n, \mathbb{C}) \cap U(n) = T(n)$: Recall that $U(n)$ collects all unitary operators, so $A \in B(n, \mathbb{C})$ satisfies $A \in U(n)$, iff as an uppertriangular matrix, it's also unitary (which equivalently requires A to have orthonormal column vectors). Which, let u_j be the j^{th} column vector of A, $u_j = \sum_{i=1}^{j} a_{i,j} e_i$. Since $u_1 \cdot u_2 = 0$, it satisfies $\left(a_{1,1} e_1\right) \cdot \left(a_{1,2} e_1 + a_{2,2} e_2\right) = a_{1,2} a_{1,2} = 0$, with $u_1 \neq \mathbf{0}$ (or $a_{1,1} \neq 0$), it requires $a_{1,2} = 0$, showing $u_2 = a_{2,2} e_2$ Inductively one can verify $u_i = a_{i,i} e_i$ by the fact that $\{u_1, ..., u_n\}$ is an orthonormal list. Hence, A is in fact a diagonal matrix, showing $A \in T(n)$, which concludes that $B(n, \mathbb{C}) \cap U(n) \subseteq T(n)$ (while $T(n) \subseteq B(n, \mathbb{C}) \cap U(n)$ by definition). Hence, $B(n, \mathbb{C}) \cap U(n) = T(n)$.

As a consequence, the sub-action $\mu : U(n) \times \mathcal{F}_n(\mathbb{C}) \to \mathcal{F}_n(\mathbb{C})$ has stabilizer of $\mathcal{F}_0$, $G_{\mathcal{F}_0} = B(n, \mathbb{C}) \cap U(n) = T(n)$, hence set wise $\mathcal{F}_n(\mathbb{C}) \cong U(n)/T(n)$ also.

Since $\mathrm{GL}(n, \mathbb{C}), U(n)$ are Lie groups, while $B(n, \mathbb{C}), T(n)$ are their closed Lie subgroups (since they're stabilizers of the given action / sub-action μ, and all stabilizers are closed Lie subgroups b **Proposition 4.12** in Etingof's lecture notes), then can classify $\mathcal{F}_n(\mathbb{C}) \coloneqq \mathrm{GL}(n, \mathbb{C})/B(n, \mathbb{C})$ or $\mathcal{F}_n(\mathbb{C}) \coloneqq U(n)/T(n)$ as homogeneous space (and these two structures are compatible, since the cosets in $U(n)/T(n)$ is simply a restriction of cosets in $\mathrm{GL}(n, \mathbb{C})/B(n, \mathbb{C})$ onto $U(n)$). Hence, with $U(n)$ being a compact Lie group, its quotient $\mathcal{F}_n(\mathbb{C}) = U(n)/T(n)$ is also compact.

(**Note:** The reason why $U(n)$ is compact, since chosen the smooth endomorphism on $M_n(\mathbb{C})$ by $A \mapsto AA^\dagger$ (here it represents conjugate transpose), we have the preimage of $\{\mathrm{id}\}$ being $U(n)$, hence $\{\mathrm{id}\}$ is closed implies $U(n)$ is closed; similarly, $U(n)$ is compact, since for all $A \in U(n)$, its operator norm $\|A\| \coloneqq \sup_{\|v\|=1} \|Av\| = 1$, and this operator norm is compactible with Euclidean norm, showing that $U(n)$ is also bounded under Euclidean Norm).

Finally, to collect the dimension of $\mathcal{F}_n(\mathbb{C})$, we claim that $B(n, \mathbb{C})$ has real dimension $n(n+1)$: Given any $A \in B(n, \mathbb{C}) \subseteq \mathrm{GL}(n, \mathbb{C})$, since $\mathrm{GL}(n, \mathbb{C})$ is open in $M_n(\mathbb{C})$, there exists $r > 0$ such that with respect to the Euclidean Norm $B_r(A) \subseteq \mathrm{GL}(n, \mathbb{C})$. Take the smooth inclusion $\iota : \mathbb{C}^{\frac{n(n+1)}{2}} \hookrightarrow M_n(\mathbb{C})$, by sending $\frac{n(n+1)}{2}$ entries to an upper triangular matrix in some particular order, then take $\iota^{-1}(B_r(A))$ as the desired open set, it has a natural 1-to-1 correspondance with $B_r(A) \cap B(n, \mathbb{C})$, and ι (when restricting onto $B(n, \mathbb{C})$) has inverse naturally given by canonical projection $\pi : B(n, \mathbb{C}) \to \mathbb{C}^{\frac{n(n+1)}{2}}$ that projects the $\frac{n(n+1)}{2}$ upper triangular entries back. Hence, this ι characterizes a homeomorphism from an open subset of $\mathbb{C}^{\frac{n(n+1)}{2}}$ to an open neighborhood of A, showing that $B(n, \mathbb{C})$ has real dimension of $n(n+1)$ (since $\mathbb{C}^{\frac{n(n+1)}{2}}$ has real dimension $n(n+1)$).

Then, with $\mathrm{GL}(n, \mathbb{C})$ having real dimension of $2n^2$, $\mathcal{F}_n(\mathbb{C}) = \mathrm{GL}(n, \mathbb{C})/B(n, \mathbb{C})$ has real dimension $2n^2 - n(n+1) = n^2 - n$.

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Problem 3

Etingof Problem Sets 2.5:

Let $G_{n,k}$ be the set of all dimension k subspaces in $\mathbb{R}^n$ (usually called the Grassmanian). Show that $G_{n,k}$ is a homogeneous space of the group $O(n, \mathbb{R})$ and thus can be identified with coset space $O(n, \mathbb{R})/H$ for appropriate H. Use it to prove that $G_{n,k}$ is a manifold and find its dimension.

Solution: First, let $\{e_1, ..., e_n\}$ be the ordered elementary basis of $\mathbb{R}^n$, and we'll set $V_s :=$ span$\{e_1, ..., e_k\}$ as the standard subspace with dimension k. We aim to show that $O(n, \mathbb{R})$ has a transitive action on $G_{n,k}$.

1. **Action of $O(n, \mathbb{R})$ on $G_{n,k}$:**

First, given $V \in G_{n,k}$ (a k-dimensional subspace), it's clear that all $A \in O(n, \mathbb{R})$ satisfies $\dim(A(V)) = k$ (since the linear operator is invertible). So, define the map $\mu : O(n, \mathbb{R}) \times G_{n,k} \to O(n, \mathbb{R})$ by $\mu(A, V) = A(V)$, it forms a left action since $\mu(B, \mu(A, V)) = B(A(V)) = (BA)(V) = \mu(BA, V)$ for all $A, B \in O(n, \mathbb{R})$ and $V \in G_{n,k}$, while $\mu(\mathrm{id}, V) = \mathrm{id}(V) = V$.

2. **Action μ is Transitive:**

To show it is transitive, it suffices to show that $O(n, \mathbb{R})$ can permute the standard subspace V_s to any k-dimensional subspace $V \in G_{n,k}$.

For any $V \in G_{n,k}$, let $\{v_1, ..., v_k\}$ be a a basis of V, and extend it to a basiss $\{v_1, ..., v_n\}$ of the whole space $\mathbb{R}^n$. Using Gram Schmidt formula, $\{v_1, ..., v_n\}$ can specifically be modified to an orthonormal basis $\{f_1, ..., f_n\}$, such that $V = \mathrm{span}\{f_1, ..., f_k\}$ still, so WLOG, we'll say $\{v_1, ..., v_n\}$ is orthonormal.

Define a linear operator $A \in M_n(\mathbb{R})$ satisfying $A(e_i) = v_i$ for all index $i = 1, ..., n$, since it sends orthonormal basis $\{e_1, ..., e_n\}$ to orthonormal basis $\{v_1, ..., v_n\}$, it's a real unitary operator, hence $A \in O(n, \mathbb{R})$. And, since $A(e_i) = v_i$, then $A(V_s) = A(\mathrm{span}\{e_1, ..., e_k\}) = \mathrm{span}\{A(e_1), ..., A(e_k)\} = \mathrm{span}\{v_1, ..., v_k\} = V$, showing that $\mu(A, V_s) = A(V_s) = V$. This shows the transitivity of the action μ.

Hence, let $H = G_{V_s}$, the stabilizer of V_s under the action μ, $H \leq O(n, \mathbb{R})$ is a closed Lie subgroup (by **Proposition 4.12** in Etingof's lecture notes), and since $G_{n,k} = \mathrm{Orb}(V_s)$ has a natural set isomorphism to the let cosets of G_{V_s}, we have $G_{n,k} \cong O(n, \mathbb{R})/H$ as sets, showing that $G_{n,k}$ can be identified as a homogeneous space of $O(n, \mathbb{R})$ by $O(n, \mathbb{R})/H$, with dimension $\dim(O(n, \mathbb{R})) - \dim(H)$.

Now, to calculate the dimension of $G_{n,k}$, it requires both the dimension of $O(n, \mathbb{R})$ and H. For this, we'll explicitly calculate H. Recall that dimension of $O(l, \mathbb{R})$ for all $l \in \mathbb{N} \setminus \{0\}$ is given by $\frac{l(l-1)}{2}$. We'll eventually show that $H \cong O(k, \mathbb{R}) \times O(n-k, \mathbb{R})$.

For all $A \in H = G_{V_s}$, let $u_i = A(e_i)$ for all index i, we have $\{u_1, ..., u_n\}$ being an orthonormal basis, such that $\mathrm{span}\{u_1, ..., u_k\} = \mathrm{span}\{A(e_1), ..., A(e_k)\} = A(\mathrm{span}\{e_1, ..., e_k\}) = A(V_s) = V_s$, hence if restrict A as a linear operator to V_s, since $\{u_1, ..., u_k\}$ is an orthonormal basis of $V_s = \mathrm{span}\{e_1, ..., e_k\}$ (so each component can be expressed as unique linear combination of $e_1, ..., e_k$), hence $A|_{V_s} \in O(V_s)$ (or it's a unitary operator on V_s).

Similarly, since $u_{k+1}, ..., u_n$ are all orthogonal to $u_1, ..., u_k$, then $\mathrm{span}\{u_{k+1}, ..., u_n\} \subseteq \mathrm{span}\{u_1, ..., u_k\}^\perp = V_s^\perp$; and since $\dim(V_s^\perp) = n - \dim(V_s) = n - k$, then $\{u_{k+1}, ..., u_n\} \subset V_s^\perp$ (an orthonormal list of $n-k$ vectors) is a basis of $V_s^\perp$. Notice that $e_{k+1}, ..., e_n \in V_s^\perp$ (since $V_s = \mathrm{span}\{e_1, ..., e_k\}$, and the standard basis forms an orthonormal basis), hence we again get $A(V_s^\perp) = V_s^\perp$ (since it sends $\{e_{k+1}, ..., e_n\}$ an orthonormal basis of $V_s^\perp$ to $\{u_{k+1}, ..., u_n\}$ another orthonormal basis of $V_s^\perp$). So, when restricting to $V_s^\perp$, $A|_{V_s^\perp} \in O(V_s^\perp)$.

Then, given $O(V_s) = O(k, \mathbb{R})$ (if using $e_1, ..., e_k$ as its basis) and $O(V_s^\perp) \cong O(n-k, \mathbb{R})$ (if using $e_{k+1}, ..., e_n$ as its basis) by similar reason, then, A in fact can be decomposed as follow:

$$A = \begin{pmatrix} A|_{V_s} & 0 \\ 0 & A|_{V_s^\perp} \end{pmatrix} \tag{3.1}$$

This is due to the fac that each $u_1, ..., u_k \in V_s$ can be written as linear combination of $e_1, ..., e_k$ (and such linear combination is unique), and similar reason for $u_{k+1}, ..., u_n \in V_s^\perp$ being written as unique linear combination of $e_{k+1}, ..., e_n$.

Hence, there is a natural group homomorphism $\rho : H \to O(k, \mathbb{R}) \times O(n-k, \mathbb{R})$ given by $\rho(A) = \left(A|_{V_s}, A|_{V_s^\perp}\right)$, since it satisfies the following for all $A, B \in H$:

$$AB = \begin{pmatrix} A|_{V_s} & 0 \\ 0 & A|_{V_s^\perp} \end{pmatrix} \begin{pmatrix} B|_{V_s} & 0 \\ 0 & B|_{V_s^\perp} \end{pmatrix} = \begin{pmatrix} A|_{V_s} B|_{V_s} & 0 \\ 0 & A|_{V_s^\perp} \ B|_{V_s^\perp} \end{pmatrix} = \begin{pmatrix} (AB)|_{V_s} & 0 \\ 0 & (AB)|_{V_s^\perp} \end{pmatrix} \tag{3.2}$$

Which, $\rho(AB) = \left(A|_{V_s} \ B|_{V_s}, A|_{V_s^\perp} \ B|_{V_s^\perp}\right) = \left(A|_{V_s}, A|_{V_s^\perp}\right) \cdot \left(B|_{V_s}, B|^\perp_{V_s}\right) = \rho(A) \cdot \rho(B)$.

This morphism is injective, simply because if $\rho(A) = \left(\mathrm{id}|_{V_s}, \mathrm{id}|^\perp_{V_s}\right)$, with $\mathbb{R}^n = V_s \oplus V_s^\perp$ and restriction of A fixing both subspaces, we must have $A = \mathrm{id}$; and, this morphism is surjective, simply because any unitary operators $A_k \in O(k, \mathbb{R})$ and $A_{n-k} \in O(n-k, \mathbb{R})$, the following is a unitary operator fixing V_s:

$$A := \begin{pmatrix} A_k & 0 \\ 0 & A_{n-k} \end{pmatrix} \tag{3.3}$$

Since first k and last $(n-k)$ column vectors form two orthonormal lists, while any column vector u_i $(i \leq k)$ and u_j $(j > k)$ are orthonormal because their nonzero entries are mutually disjoint. Hence, the column vectors of A form an orthonormal list of n vectors, which is an orthonormal basis, showing $A \in O(n, \mathbb{R})$; also, since every e_i for $1 \leq i \leq k$ satisfies $A(e_i) \in \mathrm{span}\{e_1, ..., e_k\} = V_s$, it stabilizes V_s, hence $A \in H$.

As a result, one has $H \cong O(k, \mathbb{R}) \times O(n-k, \mathbb{R})$, and the map is in fact smooth (since it's essentially projecting onto the top left $k \times k$ minor, and the bottom right $(n-k) \times (n-k)$ minor), which demonstrates that $\dim(H) = \dim(O(k, \mathbb{R})) + \dim(O(n-k, \mathbb{R})) = \frac{k(k-1)}{2} + (n-k)\frac{n-k-1}{2}$. Finally, we conclude that the dimension of $O(n, \mathbb{R})/H$ is given as follow:

$$\dim(O(n, \mathbb{R})/H) = \dim(O(n, \mathbb{R})) - \dim(H) \tag{3.4}$$

$$= \frac{n(n-1)}{2} - \left(\frac{k(k-1)}{2} + (n-k)\frac{n-k-1}{2}\right) = k(n-k) \tag{3.5}$$

4 D

Problem 4

Etingof Problem Sets 2.6:

Show that if $G = \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{End}(\mathbb{R}^n)$ so that each tangent space is canonically identified with $\mathrm{End}(\mathbb{R}^n)$, then $\left(L_g\right)_* v = gv$ (or $\left(dL_g\right)_{\mathrm{id}} v = gv$) where the product in the right-hand side is the usual product of matrices, and similarly for the right action. Also, the adjoint action is given by $\mathrm{Ad}\ g(v) = gvg^{-1}$.

Solution: Given $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathrm{End}(\mathbb{R}^n) = M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ (the $n \times n$ matrix space), then for any $g \in \mathrm{GL}(n, \mathbb{R})$, the left multiplication $L_g : \mathrm{GL}(n, \mathbb{R}) \rightsquigarrow \mathrm{GL}(n, \mathbb{R})$ can be viewed as a restriction of a left multiplication $L_g : M(n, \mathbb{R}) \rightsquigarrow M(n, \mathbb{R})$, and since $\mathrm{GL}(n, \mathbb{R})$ is viewed as an open subset, so the tangent space $T_g\ \mathrm{GL}(n, \mathbb{R})$ can be identified as $T_g M(n, \mathbb{R}) \cong M(n, \mathbb{R})$, hence it suffices to calculate $\left(dL_g\right)_{\mathrm{id}} : T_{\mathrm{id}} M(n, \mathbb{R}) \to T_g M(n, \mathbb{R})$.

For this question specifically, calculating using directional derivative would be a lot easier: Recall that $L_g : M(n, \mathbb{R}) \to M(n, \mathbb{R})$ is a linear map, since for all $a, b \in \mathbb{R}$ and $S, T \in M(n, \mathbb{R})$, it satisfies $L_{g(aS+bT)} = g \cdot (aS + bT) = a(g \cdot S) + b(g \cdot T) = aL_{g(S)} + bL_{g(T)}$. Then, for each entry x_{ij}, denote $e_{ij} \in M(n, \mathbb{R})$ as the elementary matrix that has 1 at the ij entry, while 0 everywhere else (so that e_{ij} for $i \leq i, j \leq n$ forms a basis for for $M_{n,\mathbb{R}}$).

Then, for each basis direction e_{ij} (which corresponds to a tangent vector $e_{ij} \in T_{\mathrm{id}} M(n, \mathbb{R})$), its corresponding differential in $T_g M(n, \mathbb{R})$ can be calculated as follow:

$$\left(dL_g\right)_{\mathrm{id}(e_{ij})} = \lim_{t \to 0} \frac{L_g\left(\mathrm{id} + t \cdot e_{ij}\right) - L_g(\mathrm{id})}{t} = \lim_{t \to 0} \frac{t \cdot L_g\left(e_{ij}\right)}{t} = g e_{ij} \tag{4.1}$$

Hence, since L_g is smooth, arbitrary differential (the directional derivative) can be expressed as linear combinations of the differentials of the basis elements. Hence, given $v = \sum_{1 \leq i,j \leq n} a_{ij} e_{ij} \in M(n, \mathbb{R})$, we have the following:

$$\left(dL_g\right)_{\mathrm{id}}(v) = \sum_{1 \leq i,j \leq n} a_{ij} \left(dL_g\right)_{\mathrm{id}(e_{ij})} = \sum_{1 \leq i,j \leq n} a_{ij} g \cdot e_{ij} = g \cdot \left(\sum_{1 \leq i,j \leq n} a_{ij} e_{ij} \right) = g \cdot v \tag{4.2}$$

Showing that the differential of L_g acts canonically on the tangent space, the same way as doing a left multipliation.

Apply similar proof for right multiplication R_g one would get similar results.

(Note: In general given any linear map on $\mathbb{R}^n$ to $\mathbb{R}^m$, its differential with respect to standard basis is automatically given by its own matrix with respect to the standard basis).

5 D

Problem 5

Let $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ be the morphism defined in the previous problem. Compute explicitly the map of tangent spaces $\varphi_* : \mathfrak{su}(2) \to \mathfrak{so}(3,\mathbb{R})$ and show that φ_* is an isomorphism. Deduce from this that $\ker\varphi$ is a discrete normal subgroup in $\mathrm{SU}(2)$ and that $\mathrm{im}\ \varphi$ is an open subgroup in $\mathrm{SO}(3,\mathbb{R})$.

Solution: We'll compute the differential of φ, by doing an explicit calculation of its map with respect to $\mathrm{SU}(2)$ (when characterizing it as S^3), and later on when computing the differential we'll utilize stereographic projection for calculation.

1. Explicit Map of φ:
Given that all matrix $A \in \mathrm{SU}(2)$ are in the form $A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, where $a = x + iy$ and $b = z + iw$ satisfies $(x, y, z, w) \in S^3$.

Given also the basis $i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ for Lie algebra $\mathfrak{su}(2)$. $\varphi(A) = \mathcal{M}(\mathrm{Ad}\ A)$ under the basis $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ is given as follow:

$$\varphi(A) = \begin{pmatrix} \mathrm{Re}(a^2 - b^2) & \mathrm{Im}(a^2 + b^2) & -2\ \mathrm{Re}(ab) \\ -\mathrm{Im}(a^2 - b^2) & \mathrm{Re}(a^2 + b^2) & 2\ \mathrm{Im}(ab) \\ 2\ \mathrm{Re}\left(a\bar{b}\right) & 2\ \mathrm{Im}\ \left(a\bar{b}\right) & |a|^2 - |b|^2 \end{pmatrix} \tag{5.1}$$

$$= \begin{pmatrix} (x^2 - y^2 - z^2 + w^2) & 2(xy + zw) & -2(xz - yw) \\ -2(xy - zw) & (x^2 - y^2 + z^2 - w^2) & 2(yz + xw) \\ 2(xz + yw) & 2(yz - xw) & (x^2 + y^2 - z^2 - w^2) \end{pmatrix} \tag{5.2}$$

2. Stereographic Projection onto S^3:
If we specifically consider $\mathbb{R}^3$ as $\mathbb{R}^3 \times \{0\}$ (the affine plane that's tangent to the south pole of $S^3 \subset \mathbb{R}^4$), then the stereographic projection is given as follow: Let $v = (x, y, z) \in \mathbb{R}^3$, $t = \frac{2}{1+\|v\|^2}$, we have the following:

$$(x, y, z) \mapsto (tx, ty, tz, 1 - t) = t\left(x, y, z, \frac{1}{t} - 1\right) \tag{5.3}$$

Hence, if mapping to the point $(1, 0, 0, 0) \in S^3$, one needs $y, z = 0$ (since $t > 0$ in general) and $t = 1$ (since $1 - t = 0$), so $tx = 1 \implies x = 1$. Hence, $(1, 0, 0) \mapsto (1, 0, 0, 0) \in S^3$.

Now, to do the explicit calculation, since $S^3 \cong \mathrm{SU}(2)$ by $(x, y, z, w) \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ (where $a = x + iy, b = z + iw$), and $(1, 0, 0, 0) \mapsto \mathrm{id}$ (since then $a = 1$ and $b = 0$), then if consider the map $\mathbb{R}^3 \to \mathrm{SO}(3,\mathbb{R})$ by compose φ with the map of $S^3 \to \mathrm{SU}(2)$, and the stereographic projection $\mathbb{R}^3 \to S^3$, we get the following map:

$$(x, y, z) \mapsto t^2 \begin{pmatrix} (x^2 - y^2 - z^2 + w^2) & 2(xy + zw) & -2(xz - yw) \\ -2(xy - zw) & (x^2 - y^2 + z^2 - w^2) & 2(yz + xw) \\ 2(xz + yw) & 2(yz - xw) & (x^2 + y^2 - z^2 - w^2) \end{pmatrix}, \quad w = \frac{1}{t} - 1 \tag{5.4}$$

If taking the partial derivative with respect to x, y, z and evaluated at $(x, y, z) = (1, 0, 0)$ (which corresponds to the point that maps to $(1, 0, 0, 0) \in S^3$, or $\mathrm{id} \in \mathrm{SU}(2)$), we get the following (where $t = 1$ and $w = 0$, $\frac{\partial}{\partial k} w^2 = 0$, $\frac{\partial}{\partial k} w = k$, $\frac{\partial}{\partial k} t = k$ for all $k = x, y, z$, when evaluated at the point):

$$\frac{\partial}{\partial x} = -2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} \tag{5.5}$$

$$\frac{\partial}{\partial y} = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.6}$$

$$\frac{\partial}{\partial z} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \tag{5.7}$$

Notice that the three tangent vectors that're spanning the image of the differential in $\mathfrak{so}(3, \mathbb{R})$ are linearly independent, hence they span a 3-dimensional subspace of $\mathfrak{su}(3, \mathbb{R})$; on the other hand, because $\mathrm{SO}(3, \mathbb{R})$ has dimension 3, then its tangent space $\mathfrak{so}(3, \mathbb{R})$ is also 3-dimensional. Hence, this shows that φ_* maps 3 basis of tangent vectors (in the original space $\mathfrak{su}(2)$) to the above 3 linearly independent tangent vectors in $\mathfrak{so}(3, \mathbb{R})$ (which forms a basis since $\dim(\mathfrak{so}(3, \mathbb{R})) = 3$), hence φ_* is an isomorphism.

Finally, since φ_* is an isomorphism, in particular it's a submersion, so if we take $\varphi^{-1}(\mathrm{id}) = \ker(\varphi)$, it is naturally a smooth manifold with dimension $\dim(\ker(\varphi)) = \dim(\mathrm{SU}(2)) - \dim(\mathrm{SO}(3, \mathbb{R})) = 3 - 3 = 0$, hence $\ker(\varphi)$ must be a discrete normal subgroup in $\mathrm{SU}(2)$.

Also, for all $A \in \mathrm{SU}(2)$, hence (θ, U) be a local chart in $\mathrm{SU}(2)$ with $A \in U$, and let (ψ, V) be a local chart of $\mathrm{SO}(3, \mathbb{R})$ with $\varphi(A) \in V$ (WLOG, say $\varphi(U) \subseteq V$ for simplicity). Then, given $\theta : U \xrightarrow{\sim} \tilde{U} \subseteq \mathbb{R}^3$ and $\psi : V \xrightarrow{\sim} \tilde{V} \subseteq \mathbb{R}^3$ (based on the dimension of the manifold), the map $\psi \circ \varphi \circ \theta^{-1} : \tilde{U} \to V$ is in fact a real smooth function with its differential being an isomorphism (since ψ, θ^{-1} are both diffeomorphisms, which have differentials being isomorphism; and, φ_* by our verification has differential also being isomorphism). Then, apply Inverse Function Theorem, there exists open subsets $W \subseteq \tilde{V}$ and $X \subseteq \tilde{U}$ open, such that restricting to X and W has $\psi \circ \varphi \circ \theta^{-1} : X \xrightarrow{\sim} W$ be a diffeomorphism. Hence, $\psi^{-1}(W)$ is an open neighborhood of $\varphi(A)$ that is also contained in the image of φ, showing that $\mathrm{im}(\varphi)$ is actually an open subgroup of $\mathrm{SO}(3, \mathbb{R})$.