Math CS 121 HW 3

Zih-Yu Hsieh

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Question 1. *3.3.18:*
Suppose that 10 good and 3 dead batteries are mixed up. Jack tests them one by one, at random and without replacement. But before testing the fifth battery he realizes that he does not remember whether the first one tested is good or is dead. All he remembers is that the last three that were tested were all good. What is the probability that the first one is also good?

Proof. Let A denotes the event of the first battery being good, while B denotes the event of the 2^{nd} to 4^{th} ones being good. Then, the scenario described has event B being true, which serves as the condition.

Notice that $A \cap B$ denotes the event that the first four tested batteries are all good, which since there are total of 13 batteries with 10 being good (and 3 being bad), $\mathbb{P}(A \cap B) = \dfrac{\binom{10}{4}}{\binom{13}{4}} = \frac{42}{143}$ (since we need to choose 4 good batteries out of the total of 10 good ones, while there are toal of 13 choose 4 ways of doing so).

Similarly, the probability of B happening is given as follow (where we care about the second to the fourth ones):

$$\mathbb{P}(B) = \mathbb{P}(\text{First one being good}) \cdot \mathbb{P}(\text{Next 2 to 4 being good}) + \mathbb{P}(\text{First one being bad}) \cdot \mathbb{P}(\text{Next 2 to 4 being good}) \tag{1}$$

$$= \frac{10}{13} \cdot \frac{\binom{9}{3}}{\binom{12}{3}} + \frac{3}{13} \cdot \frac{\binom{10}{3}}{\binom{12}{3}} = \frac{60}{143} \tag{2}$$

(Note: After taking out the first battery without replacement, there are 12 batteries left in each case. If the first one is good, only 9 goods are left, while if the first one is bad, 10 goods are left instead, that's why it's in the above form).

Hence, the conditional probability $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{42}{60} = \frac{7}{10}$. □

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Question 2. *3.3.19:*
A box contains 18 tennis balls, of which eight are new. Suppose that three balls are selected randomly, played with, and after play are returned to the box. If other three balls are selected for play a second time, what is the probability that they are all new?

Proof. Given three balls are selected randomly initially, then there are four disjoint events: either all three are old (denoted as A_0), one is new and two are old (denoted as A_1), two are new and one is old (denoted as A_2), or all three are new (denoted as A_3). If we calculated their probability, we get (Note: there are initially 8 new balls and 10 old balls, total of 18 balls. And here, we're choosing the balls without order):

$$\mathbb{P}(A_0) = \frac{\binom{10}{3}}{\binom{18}{3}} = \frac{5}{34}, \quad \mathbb{P}(A_1) = \frac{\binom{10}{2}\binom{8}{1}}{\binom{18}{3}} = \frac{15}{34}, \quad \mathbb{P}(A_2) = \frac{\binom{10}{1}\binom{8}{2}}{\binom{18}{3}} = \frac{35}{102}, \quad \mathbb{P}(A_3) = \frac{\binom{8}{3}}{\binom{18}{3}} = \frac{7}{102} \tag{3}$$

Then, if A_0 happens, there are still 8 new balls (since no new balls are selected); if A_1 happens, there are 7 new balls left (since one new ball is selected and became an old one); if A_2 happens, there are 6 new balls left (since two new balls are selected); similarly, if A_3 happens, there are only 5 new balls left.

So, let W denotes the event that during the second selection, all three balls newly selected are new, then it's probability is given as follow:

$$\mathbb{P}(W) = \mathbb{P}(W|A_0)\cdot\mathbb{P}(A_0) + \mathbb{P}(W|A_1)\cdot\mathbb{P}(A_1) + \mathbb{P}(W|A_2)\cdot\mathbb{P}(A_2) + \mathbb{P}(W|A_3)\cdot\mathbb{P}(A_3) \tag{4}$$

$$= \frac{\binom{8}{3}}{\binom{18}{3}}\cdot\frac{5}{34} + \frac{\binom{7}{3}}{\binom{18}{3}}\cdot\frac{15}{34} + \frac{\binom{6}{3}}{\binom{18}{3}}\cdot\frac{35}{102} + \frac{\binom{5}{3}}{\binom{18}{3}}\cdot\frac{7}{102} = \frac{3185}{83232} \approx 0.0383 \tag{5}$$

(Note: The conditional probability is calculated based on the "updated scenario" and the corresponding number of new balls). $\square$

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Question 3. *3.4.8:*

Proof. □

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Question 4.

Proof. ∎

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Question 5.

Proof. □

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Question 6.

Proof. ∎

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Question 7.

Proof. ☐

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Question 8.

Proof. □