# Math 237A HW 2

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October 06, 2025

## 1 D

**Problem 1**

Lazarsfeld Problem Set 2 (3):
Let

$$M_{n\times m}^{\le r} \subseteq \mathbb{A}^{nm} \tag{1.1}$$

be the set of all $n \times m$ matrices for rank $\le r$. Prove that $M_{n\times m}^{\le r}$ is irreducible.

**Solution:**

Recall that $\mathrm{GL}_n(k) \subseteq \mathbb{A}^{n^2}$ is irreducible: Given that $\mathbb{A}^{n^2}$ is irreducible (since it corresponds to prime ideal $(0) \subseteq k[x_{11}, ..., x_{nn}]$), then $\mathrm{GL}_n(k) = \det^{-1}(\mathbb{A}^1 \setminus \{0\})$, which is open in $\mathbb{A}^{n^2}$ under Zariski Topology (since $\mathbb{A}^1 \setminus \{0\}$ is open in $\mathbb{A}^1$). Then, since all open subsets of an irreducible space is dense and irreducible, $\mathrm{GL}_n(k)$ is irreducible.

Passing it to product, we have $\mathrm{GL}_n(k) \times \mathrm{GL}_m(k) \subseteq \mathbb{A}^{n^2+m^2}$ to also be irreducible.

Now, recall from linear algebra, that every $K \in M_{n\times m}^r$ (all the $n \times m$ rank $r$ matrix) can be written as $K = A \cdot M \cdot B$, where $A \in \mathrm{GL}_n(k)$, $B \in \mathrm{GL}_m(k)$, and $M \in M_{n\times m}$ is in the following form:

$$M = \begin{pmatrix} \mathrm{id}_r & 0 \\ 0 & 0 \end{pmatrix} \tag{1.2}$$

Where $\mathrm{id}_r \in \mathrm{GL}_r(k)$ is the identity matrix. On the other hand, given any $N \in M_{n\times m}^r$ and $A \in \mathrm{GL}_n(k)$ and $B \in \mathrm{GL}_m(k)$, one has $A \cdot N \cdot B \in M_{n\times m}^r$ (since matrix multiplication with any invertible matrices wouldn't change the rank). Hence, we can define a map $\mu : \mathrm{GL}_n(k) \times \mathrm{GL}_m(k) \to M_{n\times m}^r$ by $\mu(A, B) \to A \cdot M \cdot B$ without ambiguity. And, $\mu$ is surjective, since every $K \in M_{n\times m}$, there exists $A \in \mathrm{GL}_n(k)$ and $B \in \mathrm{GL}_m(k)$ such that $\mu(A, B) = A \cdot M \cdot B = K$ based on the Linear Algebra fact provided above.

Notice that $\mu$ itself is actually a morphism, since given matrices $X_n$ ($n \times n$ indeterminate matrix of $M_{n\times n}$) and $X_m$ ($m \times m$ indeterminate matrix of $M_{m\times m}$), $X_n \cdot M \cdot X_m$ has all the entries being polynomials. Hence, $\mu$ defined above (in general) can be viewed as restrictions of a morphism from $\mathbb{A}^{n^2+m^2} \to \mathbb{A}^{nm}$, and we can work with its subspace topology.

In particular, we can prove $M_{n\times m}^r$ is in fact an irreducible subset under subspace topology of $M_{n\times m}^{\le r}$: Suppose $V_1, V_2 \subseteq M_{n\times m}^r$ are two closed sets (under subspace topology) such that $V_1 \cup V_2 = M_{n\times m}^r$, then since $\mu$ is a morphism, in particular it's also a continuous maps between varieties. Hence:

$$\mathrm{GL}_n(k) \times \mathrm{GL}_m(k) = \mu^{-1}(M^r_{n\times m}) = \mu^{-1}(V_1 \cup V_2) = \mu^{-1}(V_1) \cup \mu^{-1}(V_2) \tag{1.3}$$

Where because $\mu$ is continuous, $\mu^{-1}(V_1), \mu^{-1}(V_2)$ are also closed. Hence, by irreducibility of $\mathrm{GL}_n(k) \times \mathrm{GL}_m(k)$, WLOG $\mu^{-1}(V_1) = \mathrm{GL}_n(k) \times \mathrm{GL}_m(k)$, hence $\mu(\mathrm{GL}_n(k) \times \mathrm{GL}_m(k)) \subseteq \mu(\mu^{-1}(V_1)) \subseteq V_1$.

However, recall that $\mu$ is actually surjective, so $\mu(\mathrm{GL}_n(k) \times \mathrm{GL}_m(k)) = M^r_{n\times m}$, hence $M^r_{n\times m} \subseteq V_1$, showing $V_1 = M^r_{n\times m}$. This shows that under subspace topology of $M^{\leq r}_{n\times m}$, $M^r_{n\times m}$ is irreducible.

Finally, if we first look at $r = 0$, we have $M^{\leq (r-1)}_{n\times m} \subseteq M^{\leq r}_{n\times m}$ being closed in $M_{n\times m}$, hence also closed in $M^{\leq r}_{n\times m}$ under its subspace topology. So, its complement $M^{\leq r}_{n\times m} \setminus M^{\leq (r-1)}_{n\times m} = M^r_{n\times m}$ must be open in $M^{\leq r}_{n\times m}$. Notice that all open subsets under Zariski Topology of $\mathbb{A}^l$ (where $k$ is algebraically closed) has all open subsets being dense, hence we get that, the closure of any open subset is the whole space.

Hence, the closure of $M^r_{n\times m}$ under subspace topology of $M^{\leq r}_{n\times m}$, is the same as taking the closure of $M^r_{n\times m}$ in $\mathbb{A}^{nm}$, then intersect with $M^{\leq r}_{n\times m}$, which implies $\overline{M^r_{n\times m}} = M^{\leq r}_{n\times m}$. Then, since $M^r_{n\times m}$ is proven to be irreducible above, its closure $M^{\leq r}_{n\times m}$ is also irreducible.

(Note: Recall that given $B \subseteq A$ in a topological space $X$, then $\overline{B}_A = \overline{B} \cap A$, where the left side is the closure under subspace topology of $A$, and the right side is the closure in the original space $X$).

## 2 D

**Problem 2**

Hartshorne 1.7:

(a) Show that the following conditions are equivalent for a topological space $X$:
   (i) $X$ is Noetherian.
   (ii) Every nonempty family of closed subsets has a minimal element.
   (iii) $X$ satisfies the ascending chain condition for open subsets.
   (iv) Every nonempty family of open subsets has a maximal element.

(b) A Noetherian topological space is *Quasi-compact*, i.e. every open cover has a finite subcover.

(c) Any subset of a Noetherian topological space is Noetherian in its induced topology.

(d) A Noetherian space which is also Hausdorff must be a finite set with the discrete topology.

**Solution:**

(a) (i) $\implies$ (ii): Suppose $X$ is Noetherian, then every descending chain of closed subsets stabilizes (i.e. collection of closed subsets satisfying D.C.C.). Now, let $\Sigma$ be a nonempty family of closed subsets together with $\subseteq$ being its partial order.

For any chain $C \subseteq \Sigma$, we claim that there exists closed set $V_C \in C$ that serves as an lower bound of $C$: Suppose the contrary, that for some chain $C$, every closed subset $V \in C$ is not a lower bound of $C$. Then, first pick random $V_0 \in C$, there exists $V_1 \in C$ such that $V_0 \supsetneq V_1$ (since $V_0$ is not a lower bound of $C$). Then, recursively every $k \in \mathbb{N}$ one can find $V_k \in C$, such that $V_{k-1} \supsetneq V_k$. So, we eventually form a strict descending chain $V_0 \supsetneq V_1 \supsetneq ... \supsetneq V_k \supsetneq ...$, yet this contradicts the Noetherian Condition of $X$. So, given any chain $C \subseteq \Sigma$, one must find some $V_C \in C$, that serves as a lower bound of $C$.

Then, since all chain $C \subseteq \Sigma$ has a lower bound, by Zorn's Lemma $\Sigma$ has a Minimal Element.

(ii) $\implies$ (iii): Suppose all nonempty family of closed subsets in $X$ has a minimal element. Let $U_1 \subseteq ... \subseteq U_n \subseteq ...$ be an arbitrary ascending chain of open sets in $X$.

Then, let $V_n = X \setminus U_n$ be the closed sets for all $n \in \mathbb{N}$, one generates $V_1 \supseteq ... \supseteq V_n \supseteq ...$, a descending chain of closed sets. Hence by assumption of (ii), there is a minimal element, say $V_m$ for some $m \in \mathbb{N}$. Then, for all index $n \geq m$, we have $V_m \supseteq V_n$ by descending chain's property, then by minimality of $V_m$ in the chain, it enforces $V_m = V_n$. Hence, it implies $U_1 \subseteq ... \subseteq U_n \subseteq ...$ also stabilizes for $n \geq m$ (since for $n \geq m$, one has $X \setminus U_n = V_n = V_m = X \setminus U_m$, so $U_n = U_m$).

Hence, $X$ satisfies Ascending Chain Condition for open subsets.

(iii) $\implies$ (iv): Suppose $X$ satisfies the ascending chain condition for open subsets. Let $\Theta$ be a nonempty collection of open subsets of $X$, and use $\subseteq$ as its partial order.

For any chain $C \subseteq \Theta$, we claim that there exists open set $U_C \in C$, that serves as an upper bound of $C$: Suppose the contrary, for some chain $C$ every open subset $U \in C$ is not an upper bound of $C$. Then, first choose random $U_0 \in C$, there exists $U_1 \in C$ such that $U_0 \subsetneq U_1$ (since $U_0$ is not an upper bound of $C$). Again, recursively every $k \in \mathbb{N}$ one can choose $U_k \in C$ satisfying $U_{k-1} \subsetneq U_k$. So, we form a strict ascending chain of open subsets $U_0 \subsetneq U_1 \subsetneq ... \subsetneq U_i \subsetneq ...$, yet this contradicts the ascending chain condition for open subsets. So, given any chain $C \subseteq \Theta$, one must find some $U_C \in C$ tat serves as an upper bound of $C$.

Since all chain $C \subseteq \Theta$ has an upper bound, by Zorn's Lemma $\Theta$ has a Maximal Element.

(iv) $\implies$ (i): Suppose every nonempty family of open subsets has a maximal element. To prove that $X$ is Noetherian (space with D.C.C for closed subsets), let $V_1 \supseteq ... \supseteq V_n \supseteq ...$ be an arbitrary descending chain of closed subsets. Let $U_n = X \setminus V_n$ be the corresponding open subsets, it forms an ascending chain of open subsets $U_1 \subseteq ... \subseteq U_n \subseteq ...$, hence with the asssumption of (iv), there exists $M \in \mathbb{N}$ with $U_M$ serving as a maximal element of the chain. Which, for all index $n \geq M$, since $U_M \subseteq U_n$ by property of the ascending chain, using the maximality of $U_M$ it enforces $U_M = U_n$. Therefore, it implies $V_1 \supseteq ... \supseteq V_n \supseteq ...$ also stabilizes for $n \geq M$ (Since $X \setminus V_n = U_n = U_M = X \setminus V_M$ implies $V_n = V_M$). So, closed subsets in $X$ satisfies Descending Chain Condition, showing that $X$ is a Noetherian Topological Space.

(b) Let $X$ be a Noetherian Topological Space (i.e. its open subsets satisfy A.C.C). Let $\{U_i\}_{i\in I}$ be any open cover of $X$, one has $X = \bigcup_{i\in I} U_i$. We'll prove by contradiction that $\{U_i\}_{i\in I}$ has a finite subcover.

Suppose $\{U_i\}_{i\in I}$ doesn't induce a finite subcover of $X$, choose arbitrary $U_{i_0}$ for some $i_0 \in I$. Since $U_{i_0}$ doesn't form a subcover of $X$, there exists point $x_1 \in X \setminus U_{i_0}$, hence one can find corresponding $i_1 \in I$ such that $x_1 \in U_{i_1}$ by the open cover condition. Inductively, for each $k \in \mathbb{N}$, there exists $x_k \in X \setminus \left(\bigcup_{j=0}^{k-1} U_{i_j}\right)$, hence there exists corresponding $i_k \in I$, such that $x_k \in U_{i_k}$.

Now, let $W_k = \bigcup_{j=0}^{k} U_{i_j}$ be the open subset for each $k \in \mathbb{N}$, it satisfies $W_k \subsetneq W_{k+1}$ (since $x_{k+1} \in X \setminus \left(\bigcup_{j=0}^{k} U_{i_j}\right) = X \setminus W_k$, while $x_{k+1} \in U_{k+1} \subseteq W_{k+1}$ by construction), hence $W_1 \subsetneq ... \subsetneq W_k \subsetneq ...$ forms a strict ascending chain of open subsets in $X$. Yet, this contradicts the A.C.C. for open subsets in $X$. So, the assumption is false, $\{U_i\}_{i\in I}$ must induce a finite subcover of $X$.

With $\{U_i\}_{i\in I}$ being arbitrary, this concludes that $X$ is compact (or Quasi-compact).

(c) Let $X$ be a Noetherian Space, and $A \subseteq X$ be any nonempty subspace equipped with subspace topology from $X$. To check $A$ is Noetherian, let $V_1 \supseteq ... \supseteq V_n \supseteq ...$ be any descending chain of closed subsets in $A$. For each $n \in \mathbb{N}$, there exists closed subset $C_n \subseteq X$, such that $V_n = A \cap C_n$.

Notice that one can choose $C_n$ specifically to form a descending chain in $X$: Let $C'_n := \bigcap_{i=1}^{n} C_i$ for all $n \in \mathbb{N}$ (where $C'_n$ as an intersection of closed sets, is closed), the base case $n = 1$ satisfies $A \cap C'_1 = A \cap C_1 = V_1$. Now, suppose given $n \in \mathbb{N}$, it satisfies $A \cap C'_n = V_n$, then for the case $(n+1)$, we have the following:

$$A \cap C'_{n+1} = A \cap \left(\bigcap_{i=1}^{n+1} C_i\right) = \left(A \cap \left(\bigcap_{i=1}^{n} C_i\right)\right) \cap (A \cap C_{n+1}) \tag{2.1}$$

$$= (A \cap C'_n) \cap V_{n+1} = V_n \cap V_{n+1} = V_{n+1} \tag{2.2}$$

Hence by induction, all $n \in \mathbb{N}$ satisfies $A \cap C'_n = V_n$. Noice that by definition, each $C'_n = \bigcap_{i=1}^{n} C_i \supseteq \bigcap_{i=1}^{n+1} C_i = C'_{n+1}$, hence $C'_1 \supseteq ... \supseteq C'_n \supseteq ...$ forms a descending chain of closed subsets in $X$, which stabilizes for some $k \in \mathbb{N}$. Then, for all $n \geq k$, since $C'_n = C'_k$, it satisfies $V_n = A \cap C'_n = A \cap C'_k = V_k$, showing the descending chain of closed subsets $V_1 \supseteq ... \supseteq V_n \supseteq ...$ stabilizes past $k$.

This concludes that all descending chain of closed subsets in $A$ (under subspace topology) stabilizes, hence $A$ under subspace topology satisfies D.C.C. for its closed subsets, showing $A$ is a Noetherian subspace.

(d) Let $X$ be a Noetherian and Hausdorff space. Recall the following lemma from Point Set Topology:

***Lemma***

A finite topological space is Hausdorff $\iff$ it's equipped with discrete topology.

**Proof:**

$\Longrightarrow$: Suppose $X$ a finite topological space is Hausdorff, then all its singletons are closed: For any $x \in X$, since for any $y \neq x$ in $X$, there exists open neighborhood $U_x \ni x$ and $U_y \ni y$ satisfying $U_x \cap U_y = \emptyset$ (by Hausdorff Property), then $x \notin U_y$, showing that $y \in U_y \subseteq X \setminus \{x\}$. This shows that $X \setminus \{x\}$ is open (since all point $y \in X \setminus \{x\}$, or $y \neq x$ has an open neighborhood fully contained in $X \setminus \{x\}$). Then, since singletons are closed, any finite union of singletons are also closed. However, since $X$ is finite, any subset of $X$ is finite union of singletons, hence closed. With all subsets of $X$ being closed, $X$ is endowed with discrete topology.

$\Longleftarrow$: Any set equipped with discrete topology is automatically Hausdorff, since for any $x \neq y$, $\{x\} \cap \{y\} = \emptyset$, so $\{x\}, \{y\}$ are open neighborhoods of $x, y$ respectively that're disjoint, showing the space is Hausdorff. $\square$

Now, using the above lemma, if $X$ is finite it is automatically with discrete topology. So, it suffices to show $X$ is finite.

For our purpose, we'll consider another lemma:

***Lemma***

For a Hausdorff Space, singletons are closed. As a consequence, any nonempty irreducible closed subset of $X$ must be singletons.

**Proof:** For all $x \in X$, given any $y \in X \setminus \{x\}$, there exists open neighborhoods $U_x \ni x$ and $U_y \ni y$, such that $U_x \cap U_y = \emptyset$ by Hausdorff property. Hence, $x \notin U_y$, showing $y \in U_y \subseteq X \setminus \{x\}$. This proves that $X \setminus \{x\}$ is open (since all element in $X \setminus \{x\}$ has an open neighborhood fully contained in $X \setminus \{x\}$), or $\{x\}$ is closed.

To show the consequence, given any closed sets $V \subseteq X$ with distinct elements $x, y \in V$, by Hausdorff Property there exists open neighborhood $U_x \ni x$ and $U_y \ni y$ such that $U_x \cap U_y = \emptyset$, then it implies the inclusion $y \in U_y \subseteq X \setminus U_x$, showing that $y \in (X \setminus U_x)^\circ$ (the interior). Hence, $y \notin \overline{U_x}$ (since $\overline{U_x} = X \setminus (X \setminus U_x)^\circ$).

So, if we take $\overline{U_x}$ and $X \setminus U_x$ as two closed sets, we have:

$$\left(V \cap \overline{U_x}\right) \cup (V \cap (X \setminus U_x)) = V \cap \left(\overline{U_x} \cup (X \setminus U_x)\right) = V \cap X = V \tag{2.3}$$

while $\left(V \cap \overline{U_x}\right), (V \cap (X \setminus U_x)) \subsetneq V$ (since $y \notin \overline{U_x}$, showing $y \in V \setminus \left(\overline{U_x}\right)$, while $x \notin X \setminus U_x$, showing $x \in V \setminus (X \setminus U_x)$). So, $V$ can be expressed as intersections of two proper closed sets, showing $V$ is a reducible closed subset of $X$. So, if $V$ is a closed irreducible subset, it cannot contain more than 1 element, hence if $V$ is nonempty, it's automatically a singleton. $\square$

Then, Given $X$ as a Noetherian Hausdorff Space, since any closed subset $V$ can be decomposed into finite nonempty irreducible closed subsets (**Proposition 1.5** in Hartshorne). In particular $X = Y_1 \cup \ldots \cup Y_n$ where each $Y_n$ is an irreducible closed subset. Then with $X$ being Hausdorff, the above lemma guarantees each $Y_i$ to be singleton. Hence, $X$ as finite union of singletons must be finite, and this finishes the proof.

## 3 D

**Problem 3**

Hartshorne 3.2:

A morphism whose undelying map on the topological spaces is a homeomorphism need not be an isomorphism.

(a) For example, let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that $\varphi$ defines a bijective bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that $\varphi$ is not an isomorphism.

(b) For another example, let the characteristic of the base field $k$ be $p > 0$, and define a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ by $t \to t^p$. Show that $\varphi$ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism.*

**Solution:**

(a) First to verify $\varphi$ is surjective, for all $(x, y)$ on the curve $y^2 = x^3$, if $(x, y) = (0, 0)$ the $\varphi(0) = (0^2, 0^3)$ does the job. Else if $x \neq 0$ or $y \neq 0$ (which since $x^2 = y^3$ and $k$ is a field, the two must happen together), let $t = \frac{y}{x}$, it satisfies:

$$\varphi(t) = (t^2, t^3) = \left(\frac{y^2}{x^2}, \frac{y^3}{x^3}\right) = \left(\frac{x^3}{x^2}, \frac{y^3}{y^2}\right) = (x, y) \tag{3.1}$$

This proves that $\varphi$ is surjective.

To show injectivity, if $t, t'$ both gets map to $(0, 0)$, then it's clear that $t^2, t'^2 = 0$, hence $t = t' = 0$. Else, if $t, t'$ gets mapped to $(x, y) \neq (0, 0)$, as said before $x, y \neq 0$, hence they satisfy $t^2 = t'^2 = x$ and $t^3 = t'^3 = y$. Therefore, $t = \frac{t^3}{t^2} = \frac{y}{x} = \frac{t'^3}{t'^2} = t'$, showing $\varphi$ is injective everywhere.

To show bicontinuity, it suffices to show that $\varphi$ is both a closed map (or the inverse is continuous) and itself is continuous. Let $V \subseteq \mathbb{A}^1$ be closed, which if $V = \emptyset$ or $V = \mathbb{A}^1$ the image under $\varphi$ is $\emptyset$ or the whole curve $y^2 = x^3$, hence closed; else if $V \neq \emptyset, \mathbb{A}^1$, then $V$ must be a finite set, hence $\varphi(V)$ is also a finite set, which is again closed. So, $\varphi$ is a closed map.

$\varphi$ is continuous, since for all $V' \subseteq \mathbb{A}^2$ that's closed, so is $V' \cap Y$ (where $Y$ represents the curve $y^2 = x^3$). Then, if consider $\varphi^{-1}(V)$, $t \in \varphi^{-1}(V) \iff \varphi(t) = (t^2, t^3) \in V'$, which for all coresponding polynomial $f_1, \ldots, f_n \in k[x, y]$ for $V$, $t \in \varphi^{-1}(V)$ iff it satisfies $f_1(t^2, t^3) = 0$, for all index $i$. Hence, this shows that $\varphi^{-1}(V)$ is also algebraic, which is closed. Hence, $\varphi$ is continuous since preimage of closed set is closed.

However, even if $\varphi$ is bijective and bicontinuous, it's not an isomorphism: Suppose the contrary that it's indeed an isomorphism, then there exists $g : Y \to \mathbb{A}^1$ (where $Y$ is the algebraic curve $x^3 = y^2$), that serves as an inverse of $f$.

Then, $g$ can be represented as some polynomial in $k[x, y]$ (denoted as $g(x, y)$), such that $g \circ f(t) = g(t^2, t^3) = t$ for all $t \in \mathbb{A}^1 = k$. However, notice that $g(t^2, t^3)$ is a polynomial with indeterminates represented by $t^2$ and $t^3$, in particular it can never be a polynomial of degree 1, hence it's not possible that $g(t^2, t^3) = t$, which is a contradiction.

Therefore, the assumption is false, $f$ cannot be an isomorphism here.

(b) To show that Frobenious morphism is bijective, given any $l \in k$, since $k$ is algebraically closed, then the equation $t^p = l$, or $t^p - l = 0$ has a solution. Let $q \in k$ be a solution for it, then it satisfies $q^p = l$, hence $t^p - l = t^p - q^p = (t - q)^p$ is the unique factorization of $t^p - l \in k[t]$, hence showing that $q$ is the unique solution to $t^p - l = 0$. So, since all $l \in k = \mathbb{A}^1$ has a unique $q \in k = \mathbb{A}^1$ such that $\varphi(q) = q^p = l$, hence $\varphi$ is bijective.

To show it's bicontinuous, it suffices to check it's both closed and continuous. However, since in $\mathbb{A}^1$ besices $\emptyset$ and $\mathbb{A}^1$ as special closed sets, other nonempty proper closed sets are finite. Hence, if closed set $V \neq \emptyset, \mathbb{A}^1$, it has $\varphi(V)$ and $\varphi^{-1}(V)$ both being finite (which is closed in $\mathbb{A}^1$), while $\emptyset = \varphi(\emptyset) = \varphi^{-1}(\emptyset)$ and $\mathbb{A}^1 = \varphi(\mathbb{A}^1) = \varphi^{-1}(\mathbb{A}^1)$, this shows that $\varphi$ is both closed and continuous, hence bicontinuous.

Finally, to show it's not an isomorphism, it suffices to check it doesn't induce an isomorphism on the coordinate ring (which for $\mathbb{A}^1$ since its corresponding ideal is $(0)$, then the coordinate ring $k[\mathbb{A}^1] = k[t]$ the polynomial ring).

Since $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ is given by $\varphi(t) = t^p$, then its induced morphism on coordinate ring $\varphi^* : k[t] \to k[t]$ is given by $\varphi^*(f) = f \circ \varphi(t) = f(t^p)$. However, notice that $\varphi^*$ is not surjective, since for all $g \in \operatorname{im}(\varphi^*)$, $g = f(t^p)$ for some $f \in k[t]$, then $\deg(g)$ is divisible by $p$. Which, choose $t \in k[t]$, it has degree $1 \neq p$, hence not divisible by $p$, showing that $t \notin \operatorname{im}(\varphi^*)$, or $\varphi^*$ is not surjective. Hence, it doesn't induce an isomorphism on coordinate ring, which implies itself is not an isomorphism.

## 4 D

**Problem 4**

Hartshorne 3.15 (a)(b):
*Products of Affine Varieties.* Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.
(a) Show that $X \times Y \subseteq \mathbb{A}^{n+m}$ with its induced topology is irreducible.
(b) Show that $k[X \times Y] \cong k[X] \otimes_k k[Y]$.

**Solution:**

(a) Let $Z_1, Z_2$ be closed sets that satisfy $Z_1 \cup Z_2 = X \times Y$. Notice that for all $x \in X$, the set $\{x\} \times Y$ is closed (since it's $(\{x\} \times \mathbb{A}^m) \cap (X \times Y)$), and as an algebraic set $\{x\} \times Y \cong Y$, which is an affine variety. Hence, $\{x\} \times Y$ is also an affine variety, which is irreducible. Since $(\{x\} \times Y \cap Z_1), (\{x\} \times Y \cap Z_2)$ are two closed sets that union to be $\{x\} \times Y$, then one of them must be $\{x\} \times Y$ by the set's irreducibility. Hence, $\{x\} \times Y \subseteq Z_i$ for one of the index $i$.

Now, let $X_i = \{x \in X \mid \{x\} \times Y \subseteq Z_i\}$. Notice that $X_1 \cup X_2 = X$. Since $X$ is irreducible, our goal is to show either $X_1 = X$ or $X_2 = X$, which based on the irreducibility of $X$, it suffices to show each $X_i$ are closed (so $X_1 \cup X_2 = X$ by irreducibility implies $X = X_i$ for one of the index $i$).

For definiteness, we'll prove the closeness for $X_1$ (since $X_2$ follows the same proof): Given each $y \in Y$, let $X_y := \{x \in X \mid (x, y) \in Z_1\}$, which $X_y$ is closed, since it is isomorphic to $X_y \times \{y\} = (X \times \{y\}) \cap Z_1$, a closed set (since it's an intersection of closed sets). Now, notice that $X_1 = \bigcap_{y \in Y} X_y$ (since for all $x \in X_1$, it satisfies $(x, y) \in Z_1$ for all $y \in Y$, hence $x \in X_y$ for all $y$; conversely, if $x \in \bigcap_{y \in Y} X_y$, then $(x, y) \in Z_1$ for all $y \in Y$, hence $\{x\} \times Y \subseteq Z_1$, showing $x \in X_1$). Hence, since each $X_y$ is closed, $X_1$ as an arbitrary intersection of them is also closed.

Hence, as a consequence we deduced that $X = X_1$ or $X = X_2$ (based on the previous claim), WLOG assume $X = X_1$, then we get $X \times Y = X_1 \times Y \subseteq Z_1$ (while $Z_1 \subseteq X \times Y$ by definition), hence $X \times Y = Z_1$. This concludes that $X \times Y$ is also irreducible, hence an affine variety.

(b) To prove this, we'll recall some categorical statements:

1. The functor $X \mapsto k[X]$ induces an equivalence of categorory of Affine Varieties over $k$, and the opposite category of finitely generated integral domains over $k$ (all finitely generated $k$-algebra that's also an integral domain). This statement is **Corollary 3.8** in Hartshorne.
2. In the category of commutative $k$-algebras, the tensor product over the field $k$ serves as a coproduct.
3. An equivalence of categories preserves limit, and a limit of an opposite category is isomorphic to a colimit in the category.
4. Over an algebraically closed field $k$, tensor products of two integral domains over $k$, is still an integral domain.

With these statements, it suffices to show that product of affine varieties indeed serve as a product in the category of affine varieties:

***Lemma***

In the category of Affine Varieties, given $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, $X \times Y \in \mathbb{A}^{n+m}$ together with natural projections $\pi_x, \pi_y$ onto $X$ and $Y$ respectively is a product of $X$ and $Y$.

**Proof:** For every affine variety $Z$, if $f = (f_1, ..., f_n) : Z \to X$ and $g = (g_1, ..., g_m) : Z \to Y$ are two morphisms, the product map as sets $f \times g = (f_1, ..., f_n, g_1, ..., g_m) : Z \to X \times Y$ is a well-defined morphism that satisfies $\pi_x \circ (f \times g) = f$, and $\pi_y \circ (f \times g) = g$ (simply just because these projection maps are also the projection maps corresponding to the product maps *as sets*, while the product map is also the product of $f, g$ as set functions).

The reason why $f \times g$ must be unique, is because given $h : Z \to X \times Y$ a morphism that satisfies $\pi_x \circ h = f$ and $\pi_y \circ h = g$, then for all $z \in Z$, it satisfies $\pi_x(h(z)) = f(z) \in X$, and $\pi_y(h(z)) = g(z) \in Y$, hence $h(z) = (f(z), g(z)) = (f_1(z), ..., f_n(z), g_1(z), ..., g_n(z)) = f \times g(z)$. This shows that as morphisms, $h$ and $f \times g$ agrees on $Z$, hence $h = f \times g$, showing the uniqueness.

So, $\left(X \times Y, \pi_x, \pi_y\right)$ is indeed a product inside the category of affine variety. $\square$

Now, since $X \times Y$ is a product of $X$ and $Y$ inside the category of affine variety, then its coordinate ring $k[X \times Y]$ serves as a product of of $k[X]$ and $k[Y]$ in the opposite category of *Finitely Generated Integral Domain over $k$* (because equivalence of categories preserve limit), which $k[X \times Y]$ is a coproduct of $k[X]$ and $k[Y]$ in the category of Finitely Generated Integral Domain over $k$.

Finally, since over commutative $k$-algebra, $k[X] \otimes_k k[Y]$ serves as a coproduct of $k[X]$ and $k[Y]$ (two finitely generated integral domains over $k$), while $k$ is algebraically closed, hence the tensor product $k[X] \otimes_k k[Y]$ is also an integral domain that's finitely generated over $k$. Which, $k[X] \otimes_k k[Y]$ and $k[X \times Y]$ are in the same category, that both serve as coproducts of $k[X]$ and $k[Y]$, so the two are isomorphic as $k$-algebra.

As a conclusion, $k[X \times Y] \otimes k[X] \otimes_k k[Y]$.

# 5 D

**Problem 5**

Hartshorne 3.19 (a):

*Automorphisms of* $\mathbb{A}^n$. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ be a morphsim of $\mathbb{A}^n$ to $\mathbb{A}^n$ given by $n$ polynomials $f_1, ..., f_n$ of $n$ variables $x_1, ..., x_n$. Let $J = \det|\frac{\partial f_i}{\partial x_j}|$ be the *Jacobian polynomial* of $\varphi$.

(a) If $\varphi$ is an isomorphism (in which case we call $\varphi$ an automorphism of $\mathbb{A}^n$), show that $J$ is a onzero constant polynomial.

**Solution:** First, we'll observe the case for identity: If $\varphi = \mathrm{id}_{\mathbb{A}^n}$, then each $f_i = x_i$ as polynomial (since $\varphi(t_1, ..., t_n) = (f_1, ..., f_n) = (t_1, ..., t_n)$ for all $(t_1, ..., t_n) \in \mathbb{A}^n$). Then, $\frac{\partial f_i}{\partial x_j} = \delta_{ij}$. Hence, express in matrix form we get $\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i,j \le n} = \mathrm{id} \in M_n(k)$. So, $\det|\frac{\partial f_i}{\partial x_j}| = 1$, which is a nonzero constant polynomial.

Then, recall that *Chain Rule* also applies in differential calculus for polynomial ring over a field, hence given $g, f_1, ..., f_n \in k[x_1, ..., x_n]$, it satisfies:

$$\forall j \in \{1, ..., n\}, \quad \frac{\partial}{\partial x_j} g(f_1, ..., f_n) = \sum_{i=1}^{n} \frac{\partial g(f_1, ..., f_n)}{\partial f_i} \frac{\partial f_i}{\partial x_j} \tag{5.1}$$

Hence, given $\varphi$ as an isomorphism, it equips with $\varphi^{-1}$ also as an isomorphism that's represented by $g_1, ..., g_n$. Which, $g \circ f = \mathrm{id}_{\mathbb{A}^n}$ is represented by $g_i(f_1, ..., f_n) = x_i$. Hence, we get the following:

$$\delta_{ik} = \frac{\partial x_i}{\partial x_k} = \frac{\partial}{\partial x_k} g_i(f_1, ..., f_n) = \sum_{j=1}^{n} \frac{\partial g_i(f_1, ..., f_n)}{\partial f_j} \frac{\partial f_j}{\partial x_k} \tag{5.2}$$

Then, express in matrix form, we get:

$$\mathrm{id} = (\delta_{ik})_{1 \le i,k \le n} = \left(\frac{\partial g_i}{\partial x_j}\right)_{\substack{1 \le i,j \le n \\ x_j = f_j}} \left(\frac{\partial f_j}{\partial x_k}\right)_{1 \le j,k \le n} \tag{5.3}$$

So, we get that $1 = \det|\mathrm{id}| = \det|\frac{\partial g_i(f_1, ..., f_n)}{\partial f_j}| \cdot \det|\frac{\partial f_j}{\partial x_k}|$, hence showing that $\det|\frac{\partial f_j}{\partial f_k}| \in k[x_1, ..., x_n]$ is invertible. Yet, recall that $(k[x_1, ..., x_n])^\times = k^\times$, hence $\det|\frac{\partial f_j}{\partial f_k}|$ (the Jacobian Polynomial of $f$) is a nonzero constant polynomial.

# 6 D

**Problem 6**

Hartshorne 3.21 (a)(b):

*Group Varieties.* A group variety consists of a variety $Y$ together with a morphism $\mu : Y \times Y \to Y$, such that the set of points of $Y$ with the operation given by $\mu$ is a group, and such that the inverse map $y \to y^{-1}$ is also a morphism of $Y \to Y$.

(a) The *additive group* $G_a$ is given by the variety $\mathbb{A}^1$ and the morphism $\mu : \mathbb{A}^2 \to \mathbb{A}^1$ defined by $\mu(a, b) = a + b$. Show it is a group variety.

(b) The *multiplicative group* $G_m$ is given by the variety $\mathbb{A}^1 - \{(0)\}$ and the morphism $\mu(a, b) = ab$. Show it is a group variety.

**Solution:**

(a) To show that the inverse map of the morphism $\mu : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ by $\mu(a, b) = a + b$, since for al $y \in k = \mathbb{A}^1$, the inverse under this morphism is $-y \in \mathbb{A}^1$. Hence, the inverse map $\iota : \mathbb{A}^1 \to \mathbb{A}^1$ is $\iota(y) = -y$, which is a morphism (since it's given by a polynomial). Hence, $G_a$ the additive group is a group variety.

(b) To show that the inverse map of $\mu : \mathbb{A}^1 \setminus \{(0)\} \times \mathbb{A}^1 \setminus \{(0)\} \to \mathbb{A}^1 \setminus \{(0)\}$ by $\mu(a, b) = ab$ defines a group variety (or verify its inverse is continuous), one needs to do extra work, converting $\mathbb{A}^1 \setminus \{(0)\}$ into some variety in higher dimension affine space.

Consider the affine space $\mathbb{A}^2$ together with the algebraic set $Y := Z(xy - 1)$ (where it's over the polynomial ring $k[x, y]$). Notice that $xy - 1$ has degree of $y$ being 1, hence it's automatically irreducible inside $(k[x])[y]$, showing $(xy - 1)$ is prime, or $Y$ is an affine variety.

To show that $Y$ has a 1-to-1 correspondance with $\mathbb{A}^1 \setminus \{(0)\}$, notice that every $(x, y) \in Y$ satisfies $xy - 1 = 0$, or $xy = 1$, hence $x, y \neq 0$, and $y = x^{-1}$. So, define a set function $Y \to \mathbb{A}^{-1} \setminus \{(0)\}$ by $(x, x^{-1}) \mapsto x$ (which is also a morphism onto closure of $\mathbb{A}^1 \setminus \{(0)\}$), it is bijective, since all $x \in \mathbb{A}^1 \setminus \{(0)\}$ has $(x, x^{-1}) \mapsto x$ (which is surjective), while if $(x, y), (x', y') \mapsto x'' \in \mathbb{A}^1 \setminus \{(0)\}$, we must have $x = x' = x''$, hence $y = x^{-1} = (x')^{-1} = y'$, showing the map is also injective.

Hence, there's no ambiguity identifying $\mathbb{A}^1 \setminus \{(0)\}$ as $Y$, where the morphism $\mu$ corresponds to another morphism $\mu' : Y \times Y \to Y$ by $\mu'((x, x^{-1}), (y, y^{-1})) = (xy, x^{-1}y^{-1})$ (since $(x, x^{-1})$ projects to $x$, $(y, y^{-1})$ projects to $y$, while $(xy, x^{-1}y^{-1})$ projects to $xy$, so each input corresponds to the $\mu$'s action on the projection $\mu(x, y) = xy$).

Which, in this variety the inverse map of $\mu'$ is given by $\iota(x, x^{-1}) = (x^{-1}, x)$, which for all $(x, y) \in Y$, $\iota(x, y) = (y, x)$ is a morphism, hence showing that the corresponding structure on $Y$ forms a group variety, hence it defines a group variety on $\mathbb{A}^1 \setminus \{(0)\}$.