Math 237A HW 1

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Problem 1

Lazarsfeld Problem Set 1 (1):

Let k be an algebraically closed field, and let $M_{n\times n} = \mathbb{A}^{n^2}(k)$ be the affine space of all $n \times n$ matrices with entries in k. Determine which of the following subsets of $M_{n\times n}$ are algebraic:

(a) $\mathrm{SL}(n) := \{A \in M_{n\times n} \mid \det(A) = 1\}$.

(b) $\mathrm{Diag}(n) := \{A \in M_{n\times n} \mid A \text{ can be diagonalized}\}$.

(c) $\mathrm{Nilp}(n) := \{A \in M_{n\times n} \mid A \text{ is nilpotent}\}$.

Solution:

(a): Given $\det : M_{n\times n} \to k$, it is in fact a polynomial function in $k[x_{11}, ..., x_{nn}]$ (polynomial ring with all entries of $n \times n$ matrix as indeterminates). Which, if consider $\det -1 \in k[x_{11}, ..., x_{nn}]$, for any $A \in M_{n\times n}$, we have $\det(A) - 1 = 0 \Longleftrightarrow A \in \mathrm{SL}(n)$. This shows that $\mathrm{SL}(n) = Z(\det -1)$, the algebraic set corresponding to the polynomial $\det -1$.

(b): For $k = \mathbb{C}$ specifically, we aim to show that $\mathrm{Diag}(n)$ is not an algebraic set. One way of proving this, is showing that $\mathrm{Diag}(n)$ is dense within $M_{n\times n}(\mathbb{C})$ under classical topology.

For all $Y \subseteq \mathbb{A}^{n^2}(\mathbb{C})$ that is an algebraic set, since it can be interpreted as solutions of finitely many polynomials, or solutions to $(f_1, ..., f_k) = I(Y)$ specifically, hence if view each polynomial $f_i : \mathbb{C}^{n^2} \to \mathbb{C}$ as continuous functions, we get $Y = \bigcap_{i=1}^n f_i^{-1}(0)$ is closed under classical topology. So, closed sets in Zariski Topology is closed in classical topology also.

Which, if one can show $\mathrm{Diag}(n)$ is dense under classical topology, the closure of $\mathrm{Diag}(n)$ under Zariski Topology must be a closed set in classical topology containing $\mathrm{Diag}(n)$, which is the whole space. Since $\mathrm{Diag}(n)$ is a proper subset of $M_{n\times n}(\mathbb{C})$, closure being the whole space implies $\mathrm{Diag}(n)$ is not closed under classical topology, consequently it's also not closed under Zariski Topology.

Recall the following norm on $M_{n\times n}(\mathbb{C})$:

$$\|A\|_{\max} := \sup_{v\in\mathbb{C}^n,\ \|v\|=1} \{\|Av\|\}$$

This is a norm equivalent to the Euclidean 2-norm, which they generate the same topology on $M_{n\times n}(\mathbb{C})$ (suppose $c, C > 0$ are two constants such that $c\|A\|_{\max} \le \|A\|_2 \le C\|A\|_{\max}$). For all matrix $A \in M_{n\times n}(\mathbb{C})$, it can be turned into an upper triangular matrix, so there exists invertible $S \in M_{n\times n}(\mathbb{C})$, such that $A = STS^{-1}$ (where T is upper triangular).

Since T is upper triangular, all the eigenvalues are on its diagonal. For all $\varepsilon > 0$, choose T' that is also upper triangular, so that it fixes all nondiagonal entries of T, while perterbate the diagonals so that T' has distinct eigenvalues, and satisfies $\|T - T'\|_{\max} < \frac{\varepsilon}{C\cdot \|S\|_{\max}\cdot \|S^{-1}\|_{\max}}$. Then, the following inequality holds:

$$\|A - ST'S^{-1}\|_2 \le C\|STS^{-1} - ST'S^{-1}\|_{\max} \le C\|S\|_{\max}\cdot\|T - T'\|_{\max}\cdot\|S^{-1}\|_{\max} < \varepsilon$$

(Note: The above property is true if view matrices as linear operators, which for all $\|v\| = 1$, $\|A\cdot Bv\| \le \|A\|_{\max}\cdot\|Bv\| \le \|A\|_{\max}\cdot\|B\|_{\max}$, hence $\|A\cdot B\|_{\max} \le \|A\|_{\max}\cdot\|B\|_{\max}$).

Hence, A and $ST'S^{-1}$ has Euclidean distance $< \varepsilon$, while $ST'S^{-1}$ is diagonalizable (since it has upper triangular form being T', and T' has n distinct eigenvalues based on construction, which has a basis formed by eigenvectors corresponding to distinct eigenvalues of T'). So, this shows that $\mathrm{Diag}(n)$ is dense in $M_{n\times n}(\mathbb{C})$ under classical topology. Which, based on the previous claims we deduce that $\mathrm{Diag}(n)$ can't be cloesd under Zariski Topology, hence not an algebraic set.

(c): First, recall that for any matrix $A \in M_{n\times n}(k)$ (viewed as a linear operator on vector space k^n), its minimal polynomial $m_A(x) \in k[x]$ has $\deg(m_A) \le n = \dim(k^n)$.

On the other hand, if A is nilpotent, that means $A^k = 0$ for some $k \in \mathbb{N}$. Hence, A is a matrix satisfying the polynomial $x^k \in k[x]$, showing that the minimal polynomial $m_A(x)$ divides x^k, or $m_{A(x)} = x^l$ for some $l \in \mathbb{N}$, and $l \le n$ based on the previous conditions. Hence, for all $A \in \mathrm{Nilp}(n)$, we have $A^n = 0$ (since A has minimal polynomial $m_A(x) = x^l$ with $l \le n$, so $A^n = A^{n-l}A^l = A^{n-1}\cdot 0 = 0$); conversely, if $A^n = 0$ by definition we have $A \in \mathrm{Nilp}(n)$. Therefore, we conclude that $A \in \mathrm{Nilp}(n) \Longleftrightarrow A^n = 0$.

Now, let $X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$ be the matrix of indeterminates, and consider the matrix $X^n = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix}$ (where each $f_{ij} \in k[x_{11}, \ldots, x_{nn}]$), we claim that $\mathrm{Nilp}(n) = Z\left(\left(f_{ij}\right)_{1\le i,j\le n}\right)$, the algebraic set generated by all the entries of X^n.

For all $A \in M_{n\times n}$, plug $X = A$ into the polynomials, we get that A^n has each entry $a_{ij} = f_{ij}(A)$ (where the variables are plugged in with entries of A), hence the previous statement states that $A \in \mathrm{Nilp}(n) \Longleftrightarrow A^n = 0 \Longleftrightarrow a_{ij} = f_{ij}(A) = 0$ for all $1 \le i, j \le n$. Therefore, the algebraic set $Z\left(\left(f_{ij}\right)_{1\le i,j\le n}\right) = \mathrm{Nilp}(n)$ (since satisfying these equations is equivalent to the matrix being nilpotent).

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Problem 2

Lazarsfeld Problem Set 1 (4):

Let $n \geq 2$, and let $f \in k[x_1, ..., x_n]$ be a non-constant polynomial over an algebraically closed field k. Show that $X = \{f = 0\} \subseteq \mathbb{A}^n$ is infinite. When $k = \mathbb{C}$, show that X is non-compact in the classical topology.

Solution: For $n \geq 2$, one can view $k[x_1, ..., x_n] = R[x_n]$ (where $R = k[x_1, ..., x_{n-1}]$, and R is not a field, since $n - 1 \geq 1$, so there are indeterminates used in R). Then, for all $(a_1, ..., a_{n-1}) \in \mathbb{A}^{n-1}$, one can consider $f(a_1, ..., a_{n-1}, x_n) \in k[x_n]$ (since plugging in $a_1, ..., a_{n-1}$ for indeterminates $x_1, ..., x_{n-1}$, f is left with only one indeterminate x_n), then because k is algebraically closed, $f(a_1, ..., a_{n-1}, x_n) \in k[x_n]$ has a solution, say $a_n \in k$. Then, $(a_1, ..., a_{n-1}, a_n) \in \mathbb{A}^n$ is a solution of $f(x_1, ..., x_n)$.

Then, since k is algebraically closed (in particular infinite), then $\mathbb{A}^{n-1} = k^{n-1}$ (as set) is infinite. Hence, since for each $(a_1, ..., a_{n-1}) \in \mathbb{A}^{n-1}$, there exists $a_n \in k$ such that $(a_1, ..., a_{n-1}, a_n) \in X$ (being a solution to f), we conclude that X is infinite.

Now, when $k = \mathbb{C}$, to show that X is non-compact in classical topology, it suffices to show that it's not bounded (since in $\mathbb{C}^n$, with Heine-Borel Theorem it guarantees that X is compact iff it is closed and bounded). For all real number $M > 0$, choose $a_1 = ... = a_{n-1} = M \in \mathbb{C}$, since there exists $a_n \in \mathbb{C}$ such that $f(a_1, ..., a_{n-1}, a_n) = 0$, we have $(a_1, ..., a_{n-1}, a_n) \in X$. Which, if consider its norm, we get:

$$\|(a_1, ..., a_{n-1}, a_n)\| = \sqrt{|a_1|^2 + ... + |a_{n-1}|^2 + |a_n|^2} = \sqrt{(n-1) \cdot M^2 + |a_n|^2} \geq M\sqrt{n-1} \geq M$$

(Note: The above requires $n \geq 2$, or $(n - 1) \geq 1$).

Hence, for all $M > 0$, one can choose $(a_1, ..., a_{n-1}, a_n) \in X$, such that $\|(a_1, ..., a_{n-1}, a_n)\| \geq M$, showing that X is in fact not bounded, hence not compact in classical topology of $\mathbb{C}^n$.

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Problem 3

Hartshorne Chapter 1 Exercise 1.1 (a),(b):

(a) Let Y be the plane curve $y = x^2$ (i.e. Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ (or $k[Y]$) is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ (or $k[Z]$) is not isomorphic to a polynomial ring in one variable over k.

Solution:

(a): Let ideal $a = (y - x^2) \subseteq k[x, y]$, then we have $Y = Z(a)$ (the corresponding algebraic set of polynomial $y - x^2$, hence also corresponds to the ideal generated by it). Then, $I(Y) = I(Z(a)) = \sqrt{a}$, so the coordinate ring $k[Y] = k[x, y]/\sqrt{a}$.

However, notice that $y - x^2$ is irreducible in $k[x, y]$: If consider $k[x, y] = (k[x])[y]$ (with base ring $k[x]$), then $y - x^2$ has degree of y being 1, which is irreducible in $(k[x])[y]$. Hence, the ideal $a = (y - x^2)$ is in fact a prime ideal (since the generated element $y - x^2$ is irreducible, and $k[x, y]$ is a UFD), then we get that $\sqrt{a} = a$ (since all prime ideal is its own radical).

Now, to prove that $k[x, y]/\sqrt{a} = k[x, y]/a \cong k[t]$ (where t is an indeterminate), consider a ring homomorphism $\varphi : k[x, y] \to k[t]$ by $\varphi(f(x, y)) = f(t, t^2)$ for all $f(x, y) \in k[x, y]$. Since for all $f(t) \in k[t]$, consider $f(x) \in k[x] \subseteq k[x, y]$, then $\varphi(f(x)) = f(t)$, showing φ is surjective, hence $k[t] \cong k[x, y]/\ker(\varphi)$.

Now, to show that $\ker(\varphi) = a$, first, for all $f(x, y) \in a$, there exists $g(x, y) \in k[x, y]$ such that $f(x, y) = (y - x^2) \cdot g(x, y)$, hence we have $\varphi(f(x, y)) = \varphi((y - x^2) \cdot g(x, y)) = (t^2 - t^2) \cdot g(t, t^2) = 0$, showing $f(x, y) \in \ker(\varphi)$, which proves $a \subseteq \ker(\varphi)$;

On the other hand, if $f(x, y) \in \ker(\varphi)$, then $\varphi(f(x, y)) = f(t, t^2) = 0$. So, for all $x \in k$, with $y = x^2$ we have $f(x, y) = f(x, x^2) = 0$, hence $f(x, y)$ vanishes for all $(x, y) \in Y$. This shows that $f(x, y) \in I(Y) = \sqrt{a} = a$, hence $\ker(\varphi) \subseteq a$.

As a conclusion, we have $\ker(\varphi) = a$, hence $k[t] \cong k[x, y]/\ker(\varphi) = k[x, y]/a$, while $k[x, y]/a = k[Y]$ the coordinate ring (due to the fact that $a = \sqrt{a}$). Hence, $k[Y] \cong k[t]$ (polynomial ring with single indeterminate).

(b): Given that Z is the plane curve $xy = 1$, then Z is the algebraic set corresponding to the polynomial $xy - 1 \in k[x, y]$. Let ideal $b = (xy - 1)$, we have $Z = Z(a)$ (**Note:** the second Z in $Z(a)$ represents the function of mapping ideal to its algebraic set, not the algebraic set Z itself). Which, we get that $I(Z) = I(Z(b)) = \sqrt{b}$, so the corresponding coordinate ring $k[Z] = k[x, y]/\sqrt{b}$.

Now, again if interpreting $k[x, y] = (k[x])[y]$, since $xy - 1$ is a polynomial with degree of y being 1, it is irreducible in $(k[x])[y]$, hence the ideal $b = (xy - 1)$ is in fact a prime ideal, which implies that $\sqrt{b} = b$. So, the coordinate ring $k[Z] = k[x, y]/\sqrt{b} = k[x, y]/b$.

Finally, we'll show that $k[Z] \ncong k[t]$ the polynomial ring in k with one indeterminate. Suppose the contrary that $k[Z] \cong k[t]$, then there exists a ring isomorphism $\psi : k[Z] = k[x, y]/b \to k[t]$. Then, if consider $\psi(\overline{x}), \psi(\overline{y}) \in k[t]$, since $\overline{x} \cdot \overline{y} = \overline{xy} = 1 \in k[Z]$ (due to the fact that $xy - 1 \equiv 0 \bmod b$, so $\overline{xy - 1} = 0 \in k[Z]$), then we get that $\psi(\overline{x}) \cdot \psi(\overline{y}) = \psi(\overline{xy}) = \psi(1) = 1$, hence both $\psi(\overline{x}), \psi(\overline{y}) \in k[t]$ are invertible. Yet, since group of units $(k[t])^\times = k^\times$, this enforces $\psi(\overline{x}), \psi(\overline{y}) \in k^\times$ (nonzero constant polynomials), but this is a contradiction since ψ is supposed to be surjective, while now $\psi\left(\overline{f(x, y)}\right) = f(\psi(\overline{x}), \psi(\overline{y})) \in k$ for all $\overline{f(x, y)} \in k[Z]$, showing that ψ is not surjective. Hence, we conclude that $k[Z] \ncong k[t]$.

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Problem 4

Hartshorne Chapter 1 Exercise 1.2:

The Twisted Cubic Curve. Let $Y \subseteq \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) | t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ (or $k[Y]$) is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t, y = t^2, z = t^3$.

Solution: First, given any $(x, y, z) \in \mathbb{A}^3$, there exists $t \in k$ such that $(x, y, z) = (t, t^2, t^3) \iff y = x^2$ and $z = x^3$:

For $\Longrightarrow$, if there exists $t \in k$ such that $(x, y, z) = (t, t^2, t^3)$, it's clear that $y = t^2 = x^2$ and $z = t^3 = x^3$, so the conditions are satisfied. Conversely (for $\Longleftarrow$), if $y = x^2$ and $z = x^3$, choose $t = x \in k$ we have $(x, y, z) = (x, x^2, x^3) = (t, t^2, t^3)$. Hence, the equivalence is shown. Which, it implies that given the ideal $a = (y - x^2, z - x^3)$, the algebraic set $Z(a) = Y$.

Now, our goal is to prove that $k[x, y, z]/a \cong k[t]$:

Consider the ring homomorphism $\varphi : k[x, y, z] \to k[t]$ by $\varphi(f(x, y, z)) = f(t, t^2, t^3)$. Since for all $f(t) \in k[t]$, one can consider $f(x) \in k[x] \subseteq k[x, y, z]$, which $\varphi(f(x)) = f(t)$, which shows that φ is surjective, and $k[x, y, z]/\ker(\varphi) \cong k[t]$. Which, we want to claim that $\ker(\varphi) = a$.

For one inclusion, we have the equations $\varphi(y - x^2) = t^2 - (t)^2 = 0$ and $\varphi(z - x^3) = t^3 - (t)^3 = 0$, hence $y - x^2, z - x^3 \in \ker(\varphi)$. With all generators of a containing in $\ker(\varphi)$, we have $a \subseteq \ker(\varphi)$.

The other inclusion can be obtained by certain ways of decomposing the polynomials in $k[x, y, z]$. For that, consider the following lemma:

Lemma

For any monomial $f(x, y, z) \in k[x, y, z]$, it can be decomposed into $f_1 \cdot (y - x^2) + f_2 \cdot (z - x^3) + f_3(x)$, where $f_3(x) \in k[x] \subseteq k[x, y, z]$.

Proof: Since all polynomials in $k[x, y, z]$ are finite k-linear combinations of monomials, it suffices to prove the case for each monomial $x^m y^n z^l \in k[x, y, z]$ (where $m, n, l \in \mathbb{N}$ are arbitrary).

Notice that it can be represents as the following form:

$$x^m y^n z^l = x^m \left(x^2 + (y - x^2)\right)^n \left(x^3 + (z - x^3)\right)^l$$

Which, by performing binomial expansion, we can rewrite $\left(x^2 + (y - x^2)\right)^n$ as follow:

$$\left(x^2 + (y - x^2)\right)^n = \sum_{i=0}^{n} \binom{n}{i} \left(x^2\right)^i \cdot \left(y - x^2\right)^{n-i} = x^{2n} + (y - x^2)\left(\sum_{i=0}^{n-1} \binom{n}{i} \left(x^2\right)^{i(y-x^2)^{n-i}}\right)$$

Hence, $\left(x^2 + (y - x^2)\right)^n = x^{2n} + (y - x^2) \cdot h_1$ for some $h_1 \in k[x, y, z]$. Apply similar logic to the second term we also get $\left(x^3 + (z - x^3)\right)^l = x^{3l} + (z - x^3) \cdot h_2$ for some $h_2 \in k[x, y, z]$.

Then, expand out the product, we get:

$$x^m y^n z^l = x^m \left(x^{2n} + (y - x^2) \cdot h_1\right)\left(x^{3l} + (z - x^3) \cdot h_2\right)$$

$$= (y - x^2) \cdot x^m \cdot h_1 \left(x^{3l} + (z - x^3) \cdot h_2\right) + x^{2n} \cdot x^m \left(x^{3l} + (z - x^3) \cdot h_2\right)$$

$$= (y - x^2) \cdot g_1 + (z - x^3) \cdot x^{2n+m} \cdot h_2 + x^{m+2n+3l}$$

$$= (y - x^2) \cdot g_1 + (z - x^3) \cdot g_2 + g_3(x)$$

Where $g_1, g_2 \in k[x, y, z]$ and $g_3(x) \in k[x]$ are chosen so the above equation is true.

Since each monomial can be expressed as some form of $(y - x^2) \cdot g_1 + (z - x^3) \cdot g_2 + g_3(x)$, then for any $f \in k[x, y, z]$, where $f = \sum_{i=1}^{l} a_i x^{m_i} y^{n_i} z^{l_i}$ for some fixed $a_i \in k$ and $m_i, n_i, l_i \in \mathbb{N}$, since each $x^{m_i} y^{n_i} z^{l_i} = (y - x^2) \cdot g_{1,i} + (z - x^3) \cdot g_{2,i} + g_{3,i}(x)$ based on the above derivation, f can ge brepresented as:

$$f = \sum_{i=1}^{n} a_i \left((y - x^2) \cdot g_{1,i} + (z - x^3) \cdot g_{2,i} + g_{3,i}(x)\right)$$

$$= (y - x^2) \sum_{i=1}^{n} a_i \cdot g_{1,i} + (z - x^3) \sum_{i=1}^{n} a_i \cdot g_{2,i} + \sum_{i=1}^{n} a_i \cdot g_{3,i}(x)$$

Hence, $f = (y - x^2) \cdot f_1 + (z - x^3) \cdot f_2 + f_3(x)$ for some $f_1, f_2 \in k[x, y, z]$, and $f_3(x) \in k[x]$. $\square$

Now, based on the above lemma, all $f \in \ker(\varphi)$ can be decomposed into $(y - x^2) \cdot f_1 + (z - x^3) \cdot f_2 + f_3(x)$ for some $f_1, f_2 \in k[x, y, z]$ and $f_3(x) \in k[x]$. Then, plugin to φ we get:

$$0 = \varphi(f) = \varphi((y - x^2) \cdot f_1 + (z - x^3) \cdot f_2 + f_3(x))$$

$$= \varphi(y - x^2) \cdot \varphi(f_1) + \varphi(z - x^3) \cdot \varphi(f_2) + \varphi(f_3(x))$$

$$= f_3(t)$$

Hence, with $f_3(t) = 0 \in k[t]$, $f_3(x) = 0$, so $f = (y - x^2) \cdot f_1 + (z - x^3) \cdot f_2$, showing $f \in a$. Therefore, we conclude that $\ker(\varphi) \subseteq a$.

With the two inclusions deduced, we get $a = \ker(\varphi)$, hence $k[t] \cong k[x, y, z]/\ker(\varphi) = k[x, y, z]/a$. Which, this proves that a is in fact a prime ideal (since $k[t]$ is an integral domain), hence a is a radical. So, as a consequence, $I(Y) = I(Z(a)) = \sqrt{a} = a$, which shows that Y is an affine variety (since the corresponding ideal $I(Y) = a$ is prime, Y is an irreducible closed subset under Zariski Topology, by **Corollary 1.4** in Hartshorne).

The above conclusions show that Y is an affine variety, $I(Y) = a = (y - x^2, z - x^3)$ (so $\{y - x^2, z - x^3\}$ is a set of generators of $I(Y)$), and demonstrated that the coordinate ring $k[Y] = k[x, y, z]/a \cong k[t]$ (polynomial ring in one variable over k). Now, it's left to demonstrate the dimension of Y.

Based on **Proposition 1.7** in Hartshorne, given Y as an affine algebraic set, its dimension is the same as the Krull Dimension of its coordinate ring $k[Y]$. Since here $k[Y] \cong k[t]$, with $k[t]$ being a PID, then (0) is a prime ideal, while any other nonzero prime ideal $P \subseteq k[t]$ is maximal. Hence, the dimension of $k[t]$ is 1 (since the largest strictly increasing chain of prime ideal is $(0) \subsetneq P$ for nonzero prime ideal $P \subseteq k[t]$, due to the maximality of P). Hence, Y is in fact an algebraic variety of dimension 1, and this finishes all the proof.

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Problem 5

Hartshorne Chapter 1 Exercise 1.4:

If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Solution: For this we'll prove by contradiction. First, recall the following lemma from point set topology:

Lemma

Given a topological space X, and consider $X \times X$ under the product topology. Then, the diagonal $\Delta = \{(x, x) \in X \times X \mid x \in X\}$ is closed under product topology $\Longleftrightarrow X$ is Hausdorff.

Proof:

$\Longrightarrow$: First, suppose $\Delta \subseteq X \times X$ is closed, which means $(X \times X) \setminus \Delta$ is open in $X \times X$ under product topology. Hence, for all $(x, y) \in (X \times X) \setminus \Delta$ (with $x \neq y$), there exists open neighborhood $U_x, U_y \subseteq X$ of x, y respectively, such that $(x, y) \in U_x \times U_y \subseteq (X \times X) \setminus \Delta$. Then, for all $z \in U_x$ and $w \in U_y$, since $(z, w) \in U_x \times U_y \subseteq (X \times X) \setminus \Delta$, we have $z \neq w$, hence $U_x \cap U_y = \emptyset$. Since $x, y \in X$ are arbitrary, $x \neq y$, $U_x \ni x$ and $U_y \ni y$ are open neighborhoods that're disjoint, hence X is Hausdorff.

$\Longleftarrow$: Suppose X is Hausdorff, then for all $(x, y) \in (X \times X) \setminus \Delta$ (where $x \neq y$), there exists open neighborhoods $U_x, U_y \subseteq X$ of x, y respectively, such that $U_x \cap U_y = \emptyset$. Hence, for all $(z, w) \in U_x \times U_y$, with $z \in U_x$ and $w \in U_y$, the two sets being disjoint implies $z \neq w$, hence $(z, w) \in (X \times X) \setminus \Delta$. So, $(x, y) \in U_x \times U_y \subseteq (X \times X) \setminus \Delta$, showing that $(X \times X) \setminus \Delta$ is open in $X \times X$ under product topology, hence $\Delta \subseteq X \times X$ is closed under product topology. $\square$

With this lemma in mind, suppose the contrary that the Zariski Topology on $\mathbb{A}^2$ is the same as the product topology of $\mathbb{A}^1 \times \mathbb{A}^1$ (with $\mathbb{A}^1$ equipped with its own Zariski Topology). Then, by the lemma above, the diagonal $\Delta = \{(x,x) \in \mathbb{A}^2 \mid x \in k\} \subseteq \mathbb{A}^2$ is closed in $\mathbb{A}^2 \iff \mathbb{A}^1$ is Hausdorff under Zariski Topology. Now, we can derive the following statements:

1. $\mathbb{A}^1$ is Hausdorff under Zariski Topology:

Notice that with the polynomial $y - x \in k[x,y]$, the corresponding algebraic set $Z(y-x) = \Delta$ (since $(x,y) \in \mathbb{A}^2$ satisfies $y - x = 0$ iff $y = x$ iff $(x,y) \in \Delta$). Hence, Δ itself is closed in $\mathbb{A}^2$ under Zariski Topology, so based on our assumption above, $\mathbb{A}^1$ is Hausdorff.

2. $\mathbb{A}^1$ has Zariski Topology = Finite Complement Topology:

Since $k[x]$ is a PID (given that k is a field), then for any nonempty and proper algebraic set $Y \subsetneq \mathbb{A}^1 = k$, its corresponding ideal $I(Y) = (f(x))$ for some $f(x) \in k[x]$, hence $t \in Y$ iff $f(t) = 0$, or t is a zero of $f(x)$. Since $f(x)$ only has finitely many roots, it follows that Y is finite. Conversely, given any nonempty finite subset $X \subsetneq \mathbb{A}^1$, let $f(x) := \prod_{a \in X}(x-a)$, we have X being the algebraic set corresponding to $f(x)$ (since $a \in X$ iff $f(a) = 0$). Hence, the closed set in $\mathbb{A}^1$ under Zariski Topology (beside $\mathbb{A}^1$ and $\emptyset$) are all finite subsets of $\mathbb{A}^1$, showing that all open sets in $\mathbb{A}^1$ (besides $\emptyset$ and $\mathbb{A}^1$ itself) are precisely the subsets with their complements being finite, hence the Zariski Topology on $\mathbb{A}^1$ is equivalent to the Finite Complement Topology.

3. $\mathbb{A}^1$ with Finite Complement Topology is Not Hausdorff:

Then, given $\mathbb{A}^1 = k$ is infinite (due to the assumption that k is algebraically closed), the Finite Complement Topology on $\mathbb{A}^1$ is not Hausdorff: Suppose the contrary that it is Hausdorff, then for any $x, y \in \mathbb{A}^1$ with $x \neq y$, there exists open neighborhoods $U_x, U_y \subseteq \mathbb{A}^1$ containing x, y respectively, such that $U_x \cap U_y = \emptyset$. However, it implies that $U_y \subseteq \mathbb{A}^1 \setminus U_x$, while $\mathbb{A}^1 \setminus U_x$ is finite, hence U_y is finite. Yet, this implies that $\mathbb{A}^1 \setminus U_y$ is infinite (since $\mathbb{A}^1$ is infinite, while U_y is finite), which reaches a contradiction (since U_y is open, which suppose to have finite complement). So, $\mathbb{A}^1$ cannot be Hausdorff.

However, this contradicts one of the previous conclusions that $\mathbb{A}^1$ is Hausdorff. Hence, the initial assumption must be false, showing that Zariski Topology on $\mathbb{A}^2$ is not the same as product topology of $\mathbb{A}^1 \times \mathbb{A}^1$ (given $\mathbb{A}^1$ is equipped with its own Zariski Topology).

(I think Here we can conclude that $\mathbb{A}^2$ has Zariski Topology being the same as product topology of $\mathbb{A}^1 \times \mathbb{A}^1$ iff the base field k is finite, since this is the only case where the Finite Complement Topology, i.e. the Zariski Topology on $\mathbb{A}^1$, is Hausdorff).

D

Problem 6

Hartshorne Chapter 1 Exercise 1.5:

Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$ for some n if and only if B is a finitely generated k-algebra with no nilpotent elements.

Solution:

$\Longrightarrow$: Suppose B is a k-algebra (here B can be assumed as a commutative algebra) that is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$ for some n.

Then, there exists an algebraic set $Y \subseteq \mathbb{A}^n$, such that $B \cong k[Y]$, where let $J = I(Y) \subseteq k[x_1, ..., x_n]$ the corresponding ideal (which J is a radical), we have $k[Y] = k[x_1, ..., x_n]/J$. This shows that B is a finitely generated k-algebra (since it's isomorphic to a quotient of the polynomial ring $k[x_1, ..., x_n]$), and also B has no nilpotent elements (since J is a radical ideal, so for all $f \in k[x_1, ..., x_n]$, if the quotient $\overline{f} \in k[Y]$ satisfies $\overline{f}^k = 0$ for some $k \in \mathbb{N}$, then $f^k \in J$, hence $f \in J$ since J is a radical, or $\overline{f} = 0$). This proves the forward implication.

$\Longleftarrow$: Now, suppose B is a finitely generated k-algebra with no nilpotent elements, then there exists $a_1, ..., a_n \in B$, such that for every element $b \in B$, there exists a polynomial $f \in k[x_1, ..., x_n]$ such that $b = f(a_1, ..., a_n)$. Hence, if consider the ring homomorphism $\varphi : k[x_1, ..., x_n] \to B$ by $\varphi(f) = f(a_1, ..., a_n)$, the ring homomorphism φ is surjective, showing that $B \cong k[x_1, ..., x_n]/\ker(\varphi)$.

Also, the assumption that B has no nilpotent elements implies that $\ker(\varphi) \subseteq k[x_1, ..., x_n]$ is a radical (since for all $f \in k[x_1, ..., x_n]$, if $f^k \in \ker(\varphi)$ for some $k \in \mathbb{N}$, we have $\varphi(f)^k = \varphi(f^k) = 0$, showing that $\varphi(f) \in B$ is nilpotent, or $\varphi(f) = 0$. Hence $f \in \ker(\varphi)$, therefore $\sqrt{\ker(\varphi)} = \ker(\varphi)$).

Then, if we take $Y = Z(\ker(\varphi)) \subseteq \mathbb{A}^n$ as the algebraic set, since $\ker(\varphi) = I(Y) = I(Z(\ker(\varphi)))$ (due to $\ker(\varphi)$ being a radical), we have the coordinate ring $k[Y] = k[x_1, ..., x_n]/\ker(\varphi)$, hence $B \cong k[x_1, ..., x_n]/\ker(\varphi) = k[Y]$, so B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$ (for some $n \in \mathbb{N}$). This proves the converse.