Math 237A HW 3

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Problem 1

Lazarsfeld Problem Set 2 (4):
Let $X \subseteq \mathbb{A}^{n^2}$ be the locus

$$X = \{A \in M_{n\times n} | \det(A) = 0\} \tag{1.1}$$

(so $X = M_{n\times n}^{\leq n-1}$). Prove that X is birationally isomorphic to $\mathbb{A}^{n^2-1}$.

Solution:
Let $\mathbb{A}^{n^2} = M_{n\times n}$, where the indeterminates are $\left(x_{ij}\right)_{1\leq i,j\leq n}$. Similarly, characterize $\mathbb{A}^{n^2-1}$ as the space with indeterminates $\left(x_{ij}\right)_{1\leq i,j\leq n}$ while x_{11} is not allowed (i.e. every coordinate besides the top left one for $M_{n\times n}$).

Define projection $\pi : \mathbb{A}^{n^2} \to \mathbb{A}^{n^2-1}$ by projection every coordinate x_{ij} (where $(i,j) \neq (1,1)$) onto $\mathbb{A}^{n^2-1}$ (this is guaranteed to be surjective). We'll first prove that this projection is bijective when restricting to some open subset of $M_{n\times n}^{\leq n-1}$ (under subspace topology) and some open subset in $\mathbb{A}^{n^2-1}$, and derive its rational inverse to show it's a birational equivalence.

First, let X_{ij} denote the $(n-1)\times(n-1)$minor of $X = \left(x_{ij}\right)_{1\leq i,j\leq n}$ by deleting the i^{th} row and j^{th} column. Then, the determinant polynomial can be rewrite as follow:

$$\det(X) = \sum_{j=1}^{n} (-1)^{j+1} x_{1j} \det\left(X_{1j}\right) \tag{1.2}$$

Let $Y = \left\{A \in \mathbb{A}^{n^2-1} \mid \det(A_{11}) = 0\right\}$ (i.e. the set of matrices excluding x_{11} entry, whose $(n-1)\times(n-1)$ minor after deleting row and column 1 has determinant 0), and consider $U = \mathbb{A}^{n^2-1} \setminus Y$ as the open dense subset (which are every matrix excluding x_{11} entry, with the bottom right$(n-1)\times(n-1)$ minor being invertible).

Which, for all $P \in U$, since it gathers n^2-1 elements from k, except for the x_{11} entry, then let X_P denotes the $n\times n$ matrix with x_{11} as indeterminate (while the other entries corresponds to P), then $\det(X_P) \in k[x_{11}]$, and it's a degree 1 polynomial (since x_{11} has coefficient $(-1)^{j+1}\det(P_{11})$ based on the above formula, where P_{11} is the bottom right $(n-1)\times(n-1)$ minor of $P \in \mathbb{A}^{n^2-1}$, hence $\det(P_{11}) \neq 0$). Then, since k is assume to be algebraically closed, $\det(X_P)$ as a degree 1 polynomial in x_{11} has a unique solution, say p_{11}. Then, the matrix P' that has $(1,1)$ entry being p_{11} and other entries corresponding to P, satisfies $\det(P') = 0$, hence $P' \in M_{n\times n}^{\leq n-1}$, while $\det(P'_{11}) = \det(P_{11}) \neq 0$. This shows that $P \in \text{im}(\pi)$, or $\mathbb{A}^{n^2-1} = \text{im}(\pi)$, which π is surjective.

In particular, if we let $U' = \left\{A \in \mathbb{A}^{n^2} \,|\, \det(A_{11}) \neq 0\right\}$ be the corresponding open subset, then $M_{n\times n}^{\leq n-1} \cap U'$ as an open subset under subspace topology of $M_{n\times n}^{\leq n}$, the restriction of the projection $\pi : U' \cap M_{n\times n}^{\leq n-1} \to U$ forms a one-to-one correspondance (since if $\det(X_{11}) \neq 0$, then x_{11} is uniquely determined by other entries, while π is surjective).

Based on the determinant function in (1.2), if $\det(X_{11}) \neq 0$ for a matrix X (satisfying $\det(X) = 0$, or $X \in M_{n\times n}^{\leq n-1}$), then x_{11} is determined as follow:

$$0 = x_{11}\det(X_{11}) + \sum_{j=2}^{n} (-1)^{j+1} x_{1j} \det\big(X_{1j}\big) \Longrightarrow x_{11} = \frac{1}{\det(X_{11})} \sum_{j=2}^{n} (-1)^{j} x_{1j} \det\big(X_{1j}\big) \quad (1.3)$$

Now, let $\varphi : \mathbb{A}^{n^2-1} \dashrightarrow X$ be defined on U, as $\varphi(P) = \overline{P}$, where $\overline{P}$ has entries p_{ij} (where $(i,j) \neq (1,1)$) provided by P, and $p_{11} = \frac{1}{\det(P_{11})} \sum_{j=2}^{n} (-1)^j p_{1j} \det\big(P_{1j}\big)$. Then, for all $A \in M_{n\times n}^{\leq n-1} \cap U'$, one has $\varphi \circ \pi(P') = P'$ for all P' in the intersection (since p_{11} is purely determined the same way), while $\pi \circ \varphi(P) = P$ for all $P \in U$. Hence, this forms a birational equivalence (on the restricted open dense subsets of $M_{n\times n}^{\leq n-1}$ and $U \subseteq \mathbb{A}^{n^2-1}$).

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Problem 2

Lazarsfeld Problem Set 3 (2):
Consider the curve

$$\{Y^2Z - X^3 - X^2Z = 0\} \subseteq \mathbb{P}^2 \quad (2.1)$$

Draw the (restriction of) this curve in each of the affine planes $U_X = \{X \neq 0\}$, $U_Y = \{Y \neq 0\}$ and $U_Z = \{Z \neq 0\}$. Indicate how the pictures fit together, i.e. how asymptotes in one view are reflected in another.

Solution: Let $f(X,Y,Z) = Y^2Z - X^3 - X^2Z \in k[X,Y,Z]$.

Fit U_X, with $X \neq 0$, then define variable $y := \frac{Y}{X}$ and $z := \frac{Z}{X}$, take $f_{x(y,z)} := f(1,y,z)$ in $k[y,z]$, $f_{x(y,z)} = y^2z - 1 - z$. Which, given its slice in $\mathbb{R}^2 \subseteq \mathbb{C}^2$, with horizontal and vertical axis being y and z respectively, we have the following graph:

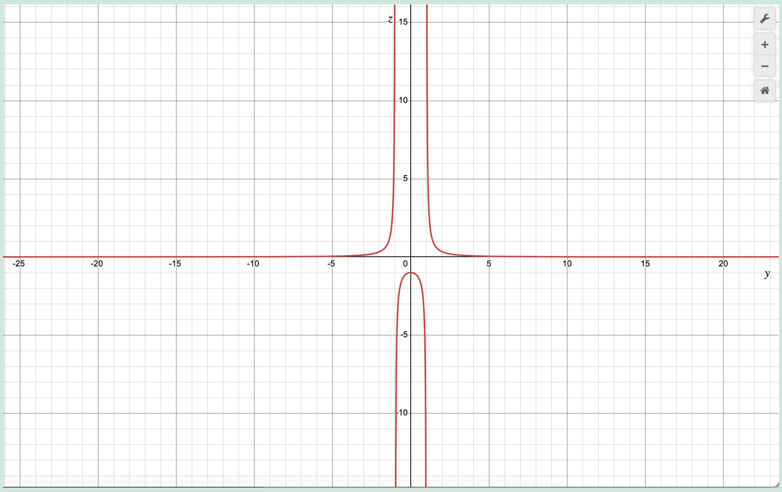

Abb. 1: Section of the curve in $\mathbb{R}^2$ with variable y, z.

Similarly, fit U_Y, with $Y \neq 0$, define variable $x := \frac{X}{Y}$ and $z := \frac{Z}{Y}$, take $f_{y(x,z)} = f(x,1,z)$ in $k[x,z]$, $f_{y(x,z)} = z - x^3 - x^2 z$. Then, the slice in $\mathbb{R}^2$ with horizontal and vertical axis being x and z respectively, we have the following graph:

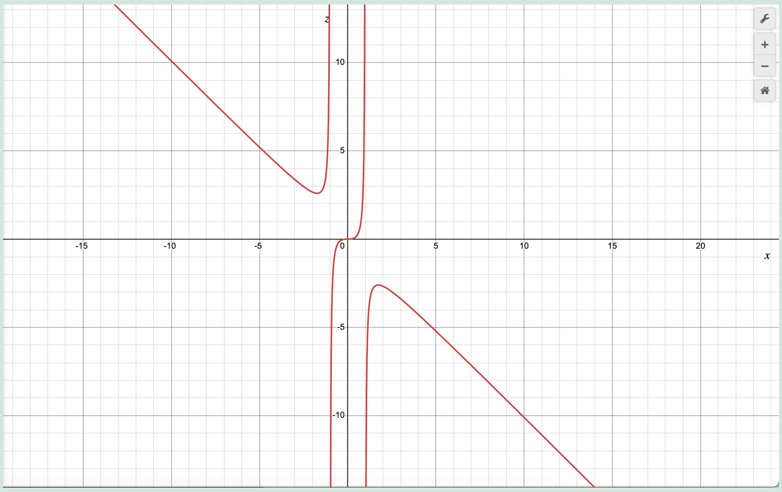

Abb. 2: Section of the curve in $\mathbb{R}^2$ with variable x, z.

Then, fit U_Z, with $Z \neq 0$, define variable $x := \frac{X}{Z}$ and $y := \frac{Y}{Z}$, take $f_{z(x,y)} := f(x,y,1)$ in $k[x,y]$, $f_{z(x,y)} = y^2 - x^3 - x^2$. Then, the slice in $\mathbb{R}^2$ with horizontal and vertical axis being x and y respectively, we have the following graph:

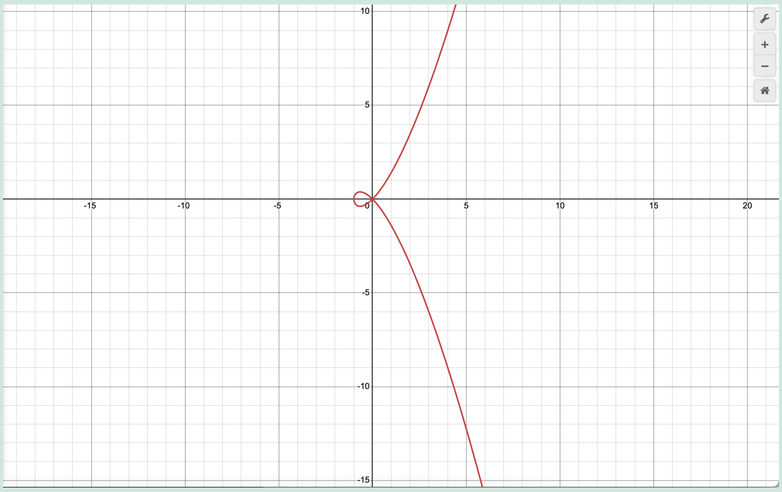

Abb. 3: Section of the curve in $\mathbb{R}^2$ with variable x, y.

To interpret the asymptotic behavior, first as Z diverges, the first and second graph both observe that the other variable ($y = \frac{Y}{X}$ and $x = \frac{X}{Y}$ respectively) are both aproaching 1 or -1 (giving some extra condition), showing that when Z goes unbounded, X and Y must obtain the same growth order (when at least one of them is nonzero).

Then, as Y diverges, for the first graph one can observe that $z = \frac{Z}{X}$ converges toward 0, while the third graph $x = \frac{X}{Z}$ diverges. So, when Y goes unbounded, X in fact has growth order larger than Z.

Finally, as X diverges, in the second graph one observes that $z = \frac{Z}{Y}$ diverges, while in the third graph $y = \frac{Y}{Z}$ also diverges, showing that the two still obtains similar growth order (when one of the other is fixed as 1 or some nonzero constant, since that's the condition of forming the section in $\mathbb{R}^2$ given above).

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Problem 3

Lazarsfeld Problem Set 3 (4):

Given an algebraic set $X \subseteq \mathbb{P}^n$, show that X can be cut out by homogeneous polynomials all having the same degree, say d. (Note that we do not assert that these polynomials actually generate the full homogeneous ideal of X).

Solution: WLOG, say $X \neq \emptyset$ (since if X is empty, then $(x_0, ..., x_n)$ is an ideal corresponding to X, showing that X can be trivially cut out by $x_0, ..., x_n$, which all have degree 1).

Let $I(X) = \mathfrak{a} = (f_1, ..., f_k)$ be the homogeneous ideal corresponding to X (where each generator f_i can be chosen specifically as homogeneous polynomials, and with $\deg(f_i) > 0$ due to the fact that $X \neq \emptyset$, hence $I(X) \neq S$, showing $\mathfrak{a}$ can't contain any nonzero constan polynomial, since they're units). Now, we claim that one can increase the „degree“ of each f_i in a specific way without increasing the zeros.

Notice that given any positive integers $q_1, ..., q_k$, $[p] \in X$ iff each $f_i^{q_i}([p]) = 0$: It is clear that powers of homogeneous polynomial is still homogeneous, hence each $f_i^{q_i}$ is still a homogeneous polynomial.

If $[p] \in X$, since each $f_{i([p])} = 0$, then $f_i^{q_i}([p]) = 0$; conversely, if $f_i^{q_i}([p]) = 0$ for all index i, then each $f_{i([p])} = 0$ (since $f_{i(p)} \neq 0$ implies $f_i^{q_i}(p) \neq 0$), showing that $[p] \in X$ (since $[p]$ is a zero of generators of $I(X)$).

With this in mind, let $d_1, ..., d_k$ be the degrees of $f_1, ..., f_k$ respectively. Let $l = \operatorname{lcm}(d_1, ..., d_k)$, and define $q_i := \frac{l}{d_i} \in \mathbb{N}$. Then, since $\deg(f_i) = d_i$, one has $\deg\left(f_i^{q_i}\right) = \deg(f_i) \cdot q_i = d_i \cdot \frac{l}{d_i} = l$. Hence, if taken the zero set of $\left\{f_1^{q_1}, ..., f_k^{q_k}\right\}$, since the previous statement guarantees that $[p] \in X$ iff $f_i^{q_i}([p]) = 0$, then $Z_{\mathbb{P}^n}\left(\left\{f_1^{q_1}, ..., f_k^{q_k}\right\}\right) = X$, showing that X can be cut out by homogeneous polynomials all with same degree.

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Problem 4

Hartshorne 2.1:

Prove the „homogeneous Nullstellensatz“, which says if $\mathfrak{a} \subsetneq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg(f) > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbb{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$.

(Hint: Interpret the problem in terms of the affine $(n+1)$-space whose affine coordinate ring is S, and use the usual Nullstellensatz).

Solution: Let $Z_{\mathbb{P}^n}(\mathfrak{a})$ denotes all the zeros in $\mathbb{P}^n$, of homogeneous polynomials in $\mathfrak{a}$. Since there is a projection $\pi : \mathbb{A}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{P}^n$, let $C(X) := \pi^{-1}(X) \cup \{0\}$ be the *Cone* of X. We'll verify that $C(X) = Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$ in $\mathbb{A}^{n+1}$.

To address this problem, we'll assume that $X \neq \emptyset$, then it implies $C(X)$ contains some nonzero element (since $\pi^{-1}(X) \subseteq \mathbb{A}^{n+1} \setminus \{0\}$ is nonempty). Since $\mathfrak{a}$ is a homogeneous ideal, it is generated by some nonzero homogeneous polynomials, say $f_1, ..., f_k \in S$. Which, since there exists nonzero $p \in C(X)$ such that $\pi(p) = [p] \in X$ serves as solution for all f_i, then each f_i has degree $d_i > 0$ (since if it's a constant polynomial, the only way it has solution is $f_i = 0$, yet we already eliminated such possibility by assuming $f_i \neq 0$). Hence, with each f_i being homogeneous, 0 is a solution for all f_i, or $0 \in Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$.

For other nonzero points p, suppose $p \in C(X)$, then since $\pi(p) = [p] \in X$ serves as a solution to all f_i, p in particular is a solution to all f_i, hence all polynomials in $\mathfrak{a}$. This shows that $p \in Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$. Else, if $p \in Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$, it's clear that p is a solution for all homogeneous polynomials in $\mathfrak{a}$, so is all λp for any $\lambda \neq 0$, hence $\pi(p) = [p]$ serves as a solution for all homogeneous polynomials in $\mathfrak{a}$, showing that $[p] \in Z_{\mathbb{P}^n}(\mathfrak{a}) = X$. Hence, $p \in \pi^{-1}(X) \subseteq C(X)$.

So, since 0 is in both $C(X)$ and $Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$, while all nonzero point p satisfies $p \in C(X) \iff p \in Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$, then $C(X) = Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$.

Under this statement, given any homogeneous polynomial f with $\deg(f) > 0$ (indicating that 0 is a solution of f), if for all $P \in X$ one has $f(P) = 0$, then every nonzero $p \in C(X)$ satisfies $f(\pi(p)) = f([p]) = 0$ (since $[p] \in X$), showing that $f(p) = 0$. Hence, every point in $C(X) = Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$ is a solution for f, using Hilbert's Nullstellensatz, $f \in I(C(X)) = I(Z_{\mathbb{A}^{n+1}}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, hence there exists positive $q \in \mathbb{N}$, such that $f^q \in \mathfrak{a}$.

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Problem 5

Hartshorne 2.2:

For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(i) $Z(\mathfrak{a}) = \emptyset$;

(ii) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;

(iii) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

Solution:

(i) $\implies$ (iii): Suppose $Z_{\mathbb{P}^n}(\mathfrak{a}) = \emptyset$, using the tools mentioned in class (also in the previous problem), the cone $C(\emptyset) = \{0\} = Z_{\{\mathbb{P}^n\}}(\mathfrak{a})$, hence we get $\sqrt{\mathfrak{a}} = I(\{0\}) = (x_0, x_1, ..., x_n)$. As a consequence, all index $0 \leq i \leq n$ has a $q_i \in \mathbb{N}$, such that $x_i^{q_i} \in \mathfrak{a}$.

Now, let $d = \sum_{i=0}^n q_i$, we claim that $S_d \subseteq \mathfrak{a}$: Recall that S_d are all homogeneous polynomials of degree d, in particular as a k-vector space, it is generated by all monomials $\prod_{i=0}^n x_i^{d_i}$, where $\sum_{i=0}^n d_i = d$. Hence, to prove $S_d \subseteq \mathfrak{a}$, it suffices to show each of the mentioned monomial belongs to $\mathfrak{a}$. And, notice that given such monomial, if for some index i it satisfies $d_i \geq q_i$, then $\prod_{i=0}^n x_i^{d_i}$ is an S-multiple of $x_i^{q_i} \in \mathfrak{a}$, showing that monomial $\prod_{i=0}^n x_i^{d_i} \in \mathfrak{a}$. So, it suffices to show that each monomial with desired property has at least one index i satisfying $d_i \geq q_i$.

Suppose the contrary that for some monomial $\prod_{i=0}^n x_i^{d_i}$ with $\sum_{i=0}^n d_i = d$, has $d_i < q_i$ for all index i, then as a consequence $\sum_{i=0}^n d_i < \sum_{i=0}^n q_i = d$, which directly contradicts the assumption. Hence, given any monomial $\prod_{i=0}^n x_i^{d_i}$ with $\sum_{i=0}^n d_i = d$ (i.e. a generator of S_d), there must exist one index i with $d_i \geq q_i$, showing that $\prod_{i=0}^n x_i^{d_i} \in \mathfrak{a}$. Hence, $S_d \subseteq \mathfrak{a}$.

(iii) $\implies$ (ii): Suppose $\mathfrak{a} \supseteq S_d$ for some $d > 0$, to show (ii), assume that $\sqrt{\mathfrak{a}} \neq S$, our goal is to show $\sqrt{\mathfrak{a}} = S_+$.

Since S_1 is generated by linear combinations of $x_0, x_1, ..., x_n \in S$, since each $x_i^d \in S_d \subseteq \mathfrak{a}$, then $x_i \in \sqrt{\mathfrak{a}}$. Hence, $S_1 \subseteq \sqrt{\mathfrak{a}}$, showing that all monomial with deg > 0 is contained in $\sqrt{\mathfrak{a}}$ (since all deg > 0 monomials are S-multiples of some x_i). As a consequence, all polynomials with constant term being 0 (i.e. k-linear combinations of monomials with deg > 0) must also be contained in $\sqrt{\mathfrak{a}}$, showing that $S_+ = \oplus_{d>0} S_d \subseteq \sqrt{a}$.

Finally, since $\sqrt{\mathfrak{a}} \neq S$, then $\sqrt{\mathfrak{a}}$ cannot contain any unit in S, which implies $\sqrt{\mathfrak{a}}$ contains no nonzero constant polynomials, hence $\sqrt{\mathfrak{a}} \cap S_0 = \{0\}$. With $\mathfrak{a}$ and $\sqrt{\mathfrak{a}}$ both being homogeneous ideal (radical of a homogeneous ideal is still homogeneous), we have $\sqrt{\mathfrak{a}} = \oplus_{d\geq 0} \sqrt{\mathfrak{a}} \cap S_d = \{0\} \oplus_{d>0} \sqrt{\mathfrak{a}} \cap S_d = \oplus_{d>0} \sqrt{\mathfrak{a}} \cap S_d \subseteq S_+$. This shows that $\sqrt{\mathfrak{a}} = S_+$.

Hence, if $\mathfrak{a} \supseteq S_d$ for some $d > 0$, then either $\sqrt{\mathfrak{a}} = S$ (equivalent to $\mathfrak{a} = S$), or $\sqrt{\mathfrak{a}} = S_+$.

(ii) $\implies$ (i): Suppose $\sqrt{\mathfrak{a}} = S$ or $\sqrt{\mathfrak{a}} = S_+$, notice that the degree 1 monomials $x_0, ..., x_n \in \sqrt{\mathfrak{a}}$, hence for some $q_0, ..., q_n \in \mathbb{N}$, one has $x_i^{q_i} \in \mathfrak{a}$.

Which, for all $[p] \in \mathbb{P}^n$, since $p = (p_0, ..., p_n) \in \mathbb{A}^{n+1}$ has at least one entry being nonzero, say p_j for some $j \in \{0, ..., n\}$, then since $p_j^{q_j} \neq 0$, we have $p \in \mathbb{A}^{n+1}$ (or $[p] \in \mathbb{P}^n$) not being a solution to $x_j^{q_j} \in \mathfrak{a}$, showing that $[p]$ is not a solution for some homogeneous polynomial (in particular, monomial) in $\mathfrak{a}$, hence $[p] \notin Z_{\mathbb{P}^n}(\mathfrak{a})$. As a consequence, $Z_{\mathbb{P}^n}(\mathfrak{a}) = \emptyset$.