Math 118C HW4

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Question 1 *Rudin Pg. 242 Problem 27:*
Put $f(0,0) = 0$*, and*

$$f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$$

if $(x,y) \neq (0,0)$*. Prove that*

(a) $f,\ D_1f,\ D_2f$ *are continuous in* $\mathbb{R}^2$*.*

(b) $D_{12}f$ *and* $D_{21}f$ *exist at every point of* $\mathbb{R}^2$*, and are continuous except at* $(0,0)$*.*

(c) $D_{12}f(0,0) = 1$*, and* $D_{21}f(0,0) = -1$*.*

Pf:

For all $(x,y) \in \mathbb{R}^2$ with $(x,y) \neq (0,0)$, using polar coordinates, $(x,y) = (r\cos(\theta), r\sin(\theta))$ for some $r > 0$ and $\theta \in [0, 2\pi)$. Which, $|(x,y)| = r$, when consider limit definition, we'll use polar coordinates instead.

(a) f **is continuous:**

For $(x,y) \neq (0,0)$, since f is a defined rational function, it is continuous, so it suffices to show f is continuous at 0. For all $\epsilon > 0$, choose $\delta = \sqrt{\frac{\epsilon}{2}} > 0$, then for all (x,y) satisfying $0 < |(x,y)| = r < \delta$, we get the following:

$$|f(x,y) - f(0,0)| = \left|\frac{(r\cos(\theta))(r\sin(\theta))((r\cos(\theta))^2 - (r\sin(\theta))^2)}{(r\cos(\theta))^2 + (r\sin(\theta))^2} - 0\right|$$

$$= \left|\frac{r^4\sin(\theta)\cos(\theta)(\cos^2(\theta) - \sin^2(\theta))}{r^2}\right| \leq r^2|sin(\theta)| \cdot |\cos(\theta)| \cdot (|\cos(\theta)|^2 + |\sin(\theta)|^2)$$

$$\leq 2r^2 < 2\left(\sqrt{\frac{\epsilon}{2}}\right)^2 = 2 \cdot \frac{\epsilon}{2} = \epsilon$$

This shows that f is continuous at $(0,0)$, hence f in continuous in $\mathbb{R}^2$.

D_1f **is continuous:**

First, using basic differentiation rule, for $(x,y) \neq (0,0)$, we get the following:

$$D_1f(x,y) = \frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) = \frac{(3x^2y - y^3)(x^2+y^2) - xy(x^2-y^2)2x}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

Which, at $(0,0)$, $D_1 f$ could be obtained through limit:

$$D_1 f(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h \cdot 0(h^2 - 0^2)}{(h^2+0^2)h} = \lim_{h\to 0} 0 = 0$$

Which, $D_1 f(x,y)$ for $(x,y) \neq (0,0)$ is again a rational function, which is continuous, so to verify continuity, it suffices to check $(0,0)$. For all $\epsilon > 0$, choose $\delta = \frac{\epsilon}{6} > 0$, then for all (x,y) satisfying $0 < |(x,y)| = r < \delta$, we get the following:

$$|D_1 f(x,y) - D_1 f(0,0)| = \left| \frac{(r\cos(\theta))^4 (r\sin(\theta)) + 4(r\cos(\theta))^2 (r\sin(\theta))^3 - (r\sin(\theta)^5)}{((r\cos(\theta))^2 + (r\sin(\theta))^2)^2} - 0 \right|$$

$$= \left| \frac{r^5(\cos^4(\theta)\sin(\theta) + 4\cos^2(\theta)\sin^3(\theta)) - \sin^5(\theta)}{r^4} \right| \leq r(|cos^4(\theta)\sin(\theta)| + 4|cos^2(\theta)\sin^3(\theta)| + |\sin^5(\theta)|)$$

$$\leq r(1+4+1) < 6 \cdot \frac{\epsilon}{6} = \epsilon$$

This proves the continuity of $D_1 f$ at $(0,0)$, so $D_1 f$ is continuous in $\mathbb{R}^2$.

$D_2 f$ is continuous:

Using differentiation rule, for $(x,y) \neq (0,0)$, we get the following:

$$D_2 f(x,y) = \frac{\partial}{\partial y} \left(\frac{xy(x^2 - y^2)}{x^2+y^2} \right) = \frac{(x^3 - 3xy^2)(x^2+y^2) - xy(x^2-y^2)2y}{(x^2+y^2)^2} = \frac{x^5 - xy^4 - 4x^3y^2}{(x^2+y^2)^2}$$

Again, at $(0,0)$, $D_2 f$ could be obtained through limit:

$$D_2 f(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0 \cdot h(0^2 - h^2)}{(0^2+h^2)h} = \lim_{h\to 0} 0 = 0$$

Notice that $D_2 f(x,y)$ for $(x,y) \neq (0,0)$ is a rational function, which is continuous, so to verify continuity, it suffices to check $(0,0)$. For all $\epsilon > 0$, choose $\delta = \frac{\epsilon}{6} > 0$, then for all (x,y) satisfying $0 < |(x,y)| = r < \delta$, we get the following:

$$|D_2 f(x,y) - D_2 f(0,0)| = \left| \frac{(r\cos(\theta))^5 - (r\cos(\theta))(r\sin(\theta))^4 - 4(r\cos(\theta))^3 (r\sin(\theta))^2}{((r\cos(\theta))^2 + (r\sin(\theta))^2)^2} - 0 \right|$$

$$= \left| \frac{r^5(\cos^5(\theta) - \cos(\theta)\sin^4(\theta) - 4\cos^3(\theta)\sin^2(\theta))}{r^4} \right| \leq r(|\cos^5(\theta)| + |\cos(\theta)\sin^4(\theta)| + 4|\cos^3(\theta)\sin^2(\theta)|)$$

$$\leq r(1+1+4) < 6 \cdot \frac{\epsilon}{6} = \epsilon$$

This proves the continuity of $D_2 f$ at $(0,0)$, hence $D_2 f$ is continuous in $\mathbb{R}^2$.

(b) **Function $D_{21} f$:**

Given that $D_1 f(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$ and $D_1 f(0,0) = 0$, apply differentiation rule for $(x,y) \neq (0,0)$, we get:

$$D_{21} f(x,y) = \frac{\partial}{\partial y} \left(\frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \right) = \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2+y^2)^2 - (x^4y + 4x^2y^3 - y^5)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

Which, $D_{21}f(x,y)$ is continuous for $(x,y) \neq (0,0)$ (since it's a rational function).

Now, to get $D_{21}f(0,0)$, we'll use limit definition:

$$D_{21}f(0,0) = \lim_{h\to 0} \frac{D_1f(0,h) - D_1f(0,0)}{h} = \lim_{h\to 0} \frac{0^4 \cdot h + 4 \cdot 0^2 \cdot h^3 - h^5}{(0^2+h^2)^2 h} = \lim_{h\to 0} -\frac{h^5}{h^5} = -1$$

Hence, $D_{21}f$ exists on the whole $\mathbb{R}^2$, and is continuous at all $(x,y) \neq (0,0)$. But, it is not continuous at $(0,0)$, since choosing $x \neq 0$ and $y = 0$, $D_{21}f$ becomes:

$$D_{21}f(x,0) = \frac{x^8}{x^8} = 1$$

Hence, $\lim_{x\to 0} D_{21}f(x,0) = 1 \neq -1 = D_{21}f(0,0)$, showing the discontinuity at $(0,0)$.

So, $D_{21}f$ exists on $\mathbb{R}^2$, while being continuous on $\mathbb{R}^2 \setminus \{0\}$.

Function $D_{12}f$:

Given that $D_2f(x,y) = \frac{x^5 - xy^4 - 4x^3y^2}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$ and $D_2f(0,0) = 0$, apply differentiation rule for $(x,y) \neq (0,0)$, we get:

$$D_{12}f(x,y) = \frac{\partial}{\partial x}\left(\frac{x^5 - xy^4 - 4x^3y^2}{(x^2+y^2)^2}\right) = \frac{(5x^4 - y^4 - 12x^2y^2)(x^2+y^2)^2 - (x^5 - xy^4 - 4x^3y^2)2(x^2+y^2)2x}{(x^2+y^2)^4}$$

Hence, $D_{12}f$ is continuous for $(x,y) \neq (0,0)$, since it's also a rational function.

Now, to get $D_{12}f(0,0)$, we'll again use limit definition:

$$D_{12}f(0,0) = \lim_{h\to 0} \frac{D_2f(h,0) - D_2f(0,0)}{h} = \lim_{h\to 0} \frac{h^5 - h \cdot 0^4 - 4h^3 \cdot 0^2}{(h^2+0^2)^2 h} = \lim_{h\to 0} \frac{h^5}{h^5} = 1$$

Hence, $D_{12}f$ exists on the whole $\mathbb{R}^2$, and is continuous at all $(x,y) \neq (0,0)$. But again, it's not continuous at $(0,0)$, since choosing $x = 0$ and $y \neq 0$, $D_{12}f$ becomes:

$$D_{12}f(0,y) = \frac{-y^8}{y^8} = -1$$

Hence, $\lim_{y\to 0} D_{12}f(0,y) = -1 \neq 1 = D_{12}f(0,0)$, showing the discontinuity at $(0,0)$.

So, $D_{12}f$ exists on $\mathbb{R}^2$, while being continuous on $\mathbb{R}^2 \setminus \{0\}$.

(c) From **part (b)**, when verifying that the existence of $D_{12}f(0,0)$ and $D_{21}f(0,0)$, we've shown that $D_{12}f(0,0) = 1$, and $D_{21}f(0,0) = -1$.

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Question 2 *Rudin Pg. 242 Problem 28:*
For $t \geq 0$, put

$$\varphi(x,t) = \begin{cases} x & 0 \leq x \leq \sqrt{t} \\ -x + 2\sqrt{t} & \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0 & otherwise \end{cases}$$

and put $\varphi(x,t) = -\varphi(x,|t|)$ if $t < 0$.
Show that φ is continuous on $\mathbb{R}^2$, and $D_2\varphi(x,0) = 0$ for all x. Define

$$f(t) = \int_{-1}^{1} \varphi(x,t)dx$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^{1} D_2\varphi(x,0)dx$$

Pf:
Continuity of φ:

$D_2\varphi$ when $t = 0$:
For all $x \in \mathbb{R}$, if $x \leq 0$, then we get $\varphi(x,t) = 0$ regardless of $t \in \mathbb{R}$, showing that $D_2\varphi(x,0) = \frac{\partial \varphi}{\partial t}(x,0) = 0$.

Now for $x > 0$, since for all $t \in \mathbb{R}$ satisfying $4|t| < x^2$, we have $2\sqrt{|t|} < x$, then $\varphi(x,t) = 0$ when $t \in (-\frac{x^2}{4}, \frac{x^2}{4})$. So, $D_2\varphi(x,0) = 0$ (since $\lim_{t\to 0} \frac{\varphi(x,t)-\varphi(x,0)}{t} = \lim_{t\to 0} 0 = 0$, because for small enough t, it lies in the range $(-\frac{x^2}{4}, \frac{x^2}{4})$).

So, regardless of $x \in \mathbb{R}$, we have $D_2\varphi(x,0) = 0$.

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Question 3 *Rudin Pg. 243 Problem 30:*
Let $f \in \mathcal{C}^{(m)}(E)$, where E is an open subset of $\mathbb{R}^n$. Fix $a \in E$, and suppose $x \in \mathbb{R}^n$ is so close to 0 that the points $p(t) = a + tx$ lie in E whenever $0 \le t \le 1$. Define $h(t) = f(p(t))$ for all $t \in \mathbb{R}$ for which $p(t) \in E$.

(a) For $1 \le k \le m$, show (by repeated application of the chain rule) that

$$h^{(k)}(t) = \sum (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

The sum extends over all order k-tuples $(l_1, \dots, l_k)$ in which each l_j is one of the integers $1, \dots, n$.

Pf:

Given $a, x \in \mathbb{R}^n$ (where $x = (x_1, \dots, x_n)$ for fixed $x_1, \dots, x_n \in \mathbb{R}^n$) and $p(t) = a + tx$ for $t \in [0, 1]$, then $p'(t) = x$.

Now, we'll use induction to verify the formula (and we'll use matrix representation of the differentials).

First, for $k = 1$, using chain rule, we get the following:

$$h'(t) = Df(p(t))p'(t) = \begin{pmatrix} D_1 f & \dots & D_n f \end{pmatrix}\Big|_{p(t)} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} D_i f(p(t)) x_i$$

Since all the possible 1-tuple is included in the summation, the $h'(t)$ satisfies the given formula.

Now, suppose for given $1 \le k \le (m-1)$, $h^{(k)}(t)$ satisfies the following formula:

$$h^{(k)}(t) = \sum (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

Since for each k-tuple $(l_1, \dots, l_k)$ (where each $l_i \in \{1, \dots, n\}$) has the function $x_{l_1} \dots x_{l_k} D_{l_1 \dots l_k} f(p(t))$ being a differentiable function from $(0, 1)$ to $\mathbb{R}$ (where $D_{l_1 \dots l_k} f(z)$ for $z \in E$ is a differentiable function, since it has only been differentiated $k < m$ times, while $f \in C^{(m)}(E)$). Then, to calculate the $(k+1)^{th}$ derivative, we get:

$$h^{(k+1)}(t) = \sum \frac{d}{dt} (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

$$\begin{aligned} \forall (l_1, \dots, l_k), \quad \frac{d}{dt} (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k} &= x_{l_1} \dots x_{l_k} D \left(D_{l_1 \dots l_k} f \right) (p(t)) p'(t) \\ = x_{l_1} \dots x_{l_k} \sum_{i=1}^{n} D_i \left(D_{l_1 \dots l_k} f \right) (p(t)) x_i &= \sum_{i=1}^{n} D_{i l_1 \dots l_k} f(p(t)) x_i x_{l_1} \dots x_{l_k} \\ \implies h^{(k+1)}(t) &= \sum \left(\sum_{i=1}^{n} D_{i l_1 \dots l_k} f(p(t)) x_i x_{l_1} \dots x_{l_k} \right) \end{aligned}$$

Which, the first summation indicates all possible k-tuple $(l_1, \dots, l_k)$ for $l_j \in \{1, \dots, n\}$.

Now, for all $(k+1)$-tuple $(j_0, j_1, \dots, j_k)$ where each $j_l \in \{1, \dots, n\}$, choose the unique k-tuple $(j_1, \dots, j_k)$, then $D_{j_0 j_1 \dots j_k} f(p(t)) x_{j_0} x_{j_1} \dots x_{j_k}$ appears precisely once in the summation of $h^{(k+1)}(t)$ given above; similarly, since each k-tuple $(l_1, \dots, l_k)$ and $i \in \{1, \dots, n\}$ corresponds to a unique $(k+1)$-tuple $(i, l_1, \dots, l_k)$, so the summation in $h^{(k+1)}(t)$ has a 1-to-1 correspondance to all $(k+1)$-tuple. Then, the summation $h^{(k+1)}(t)$ can also be described as:

$$h^{(k+1)}(t) = \sum D_{l_1 \dots l_k l_{k+1}} f(p(t)) x_{l_1} \dots x_{l_k} x_{l_{k+1}}$$

Where each $(l_1, \dots, l_k, l_{k+1})$ is a $(k+1)$-tuple with entries from $\{1, \dots, n\}$.

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Question 4 *Rudin Pg. 288 Problem 2:*
For $i = 1, 2, 3, ...,$ let $\varphi_i \in \mathcal{C}(\mathbb{R})$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x,y) = \sum_{i=1}^{\infty} (\varphi_i(x) - \varphi_{i+1}(x))\varphi_i(y)$$

Then f has compact support in $\mathbb{R}^2$, f is cotinuous except at $(0,0)$, and

$$\int dy \int f(x,y)dx = 0, \quad \textit{but} \int dx \int f(x,y)dy = 1$$

Observe that f is unbounded in every neighborhood of $(0,0)$.

Pf:
The function f is well-defined: