Math CS 122B HW2

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Question 1 *The **Beta function** is defined for $Re(\alpha) > 0$ and $\Re(\beta) > 0$ by*

$$B(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt$$

(a) Prove that

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(b) Show that

$$B(\alpha,\beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$$

Pf:

(a) First, we'll consider $\Gamma(\alpha)\Gamma(\beta)$:

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty t^{\alpha-1}e^{-t}dt \int_0^\infty s^{\beta-1}e^{-s}ds = \int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-s-t}dsdt$$

If we consider the change of variable $f : (0,1)\times(0,\infty) \to (0,\infty)\times(0,\infty)$ by $f(r,u) = (ur, u(1-r)) = (s,t)$, since this is a differentiable function, and its derivative is given by:

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial r}(ur) & \frac{\partial}{\partial u}(ur) \\ \frac{\partial}{\partial r}(u(1-r)) & \frac{\partial}{\partial u}(u(1-r)) \end{pmatrix} = \begin{pmatrix} u & r \\ -u & (1-r) \end{pmatrix}$$

$$\frac{\partial(s,t)}{\partial(r,u)} = |\begin{vmatrix} u & r \\ -u & (1-r) \end{vmatrix}| = u(1-r) - (-u)r = u$$

Then, the above integral can be given as:

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-s-t}dsdt = \int_0^\infty \int_0^1 (u(1-r))^{\alpha-1}(ur)^{\beta-1}e^{-(ur+u(1-r))}\left|\frac{\partial(s,t)}{\partial(r,u)}\right| drdu$$

$$= \int_0^\infty \int_0^1 u^{\alpha+\beta-2}e^{-u}\cdot(1-r)^{\alpha-1}r^{\beta-1}udrdu = \int_0^\infty u^{\alpha+\beta-1}e^{-u}\cdot(1-r)^{\alpha-1}r^{\beta-1}drdu$$

$$= \int_0^\infty u^{\alpha+\beta-1}e^{-u}du \cdot \int_0^1 (1-r)^{\alpha-1}r^{\beta-1}dr = \Gamma(\alpha+\beta)\cdot B(\alpha,\beta)$$

Hence, we can rewrite the above equality to be as follow:

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)\cdot B(\alpha,\beta), \quad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(Recall: $\Gamma(z) \neq 0$ for all $z \in \mathbb{C} \setminus S$, $S = \{0, -1, -2, ...\}$).

(b) Consider the following expression:

$$\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$$

First, if we do the substitution $(1+u) = e^t$, $du = e^t dt$, which $u = 0 \implies e^t = 1$, $t = 0$, and $\lim_{t\to\infty} e^t = \infty$, so $\lim_{t\to\infty} u = \infty$. Then, the integral can be rewrite as:

$$\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du = \int_0^\infty \frac{(e^t-1)^{\alpha-1}}{(e^t)^{\alpha+\beta}} e^t dt = \int_0^\infty ((1-e^{-t})e^t)^{\alpha-1}(e^{-t})^{\alpha+\beta} \cdot e^t dt$$

$$= \int_0^\infty (1-e^{-t})^{\alpha-1} \cdot (e^t)^{\alpha-1} \cdot (e^t)^{-\alpha} \cdot (e^t)^{-\beta} \cdot e^t dt = \int_0^\infty (1-e^{-t})^{\alpha-1}(e^{-t})^\beta dt$$

Then, for the above expression, if we do the second substitution $r = e^{-t}$, $dr = -e^{-t}dt$, $dt = -e^t dt = -r^{-1}dr$. Which $t = 0 \implies r = e^0$, $r = 1$, and $\lim_{t\to\infty} e^{-t} = \lim_{t\to\infty} r = 0$. So, the integral can be rewrite as:

$$\int_0^\infty (1-e^{-t})^{\alpha-1}(e^{-t})^\beta dt = \int_1^0 (1-r)^{\alpha-1} r^\beta \cdot (-r^{-1}) dr = \int_0^1 (1-r)^{\alpha-1} r^{\beta-1} dr = B(\alpha,\beta)$$

Hence, we can conclude the following:

$$\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du = \int_0^\infty (1-e^{-t})^{\alpha-1}(e^{-t})^\beta dt = B(\alpha,\beta)$$

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Question 2 *The hypergeometric series $F(\alpha,\beta,\gamma;z)$ was defined as*

$$F(\alpha,\beta,\gamma;z) = 1 + \sum_{n=1}^\infty \frac{\alpha(\alpha+1)...(\alpha+n-1)\cdot\beta(\beta+1)...(\beta+n-1)}{n!\cdot\gamma(\gamma+1)...(\gamma+n-1)} z^n$$

Here $\alpha > 0, \beta > 0, \gamma > \beta$, and $|z| < 1$. Show that

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt$$

Show as a result that the hypergeometric function, initially defined by a power series convergent in the unit disc, can be continued analytically to the complex plane slit along the half-line $[1,\infty)$.

Pf:

Properties of Gamma function:

First, we can use induction to verity that given $z \in \mathbb{C} \setminus \{0, -1, -2, ...\}$, all $n \in \mathbb{N}$ satisfies $\Gamma(z+n) = (z+n-1)...(z+1)z\Gamma(z)$.

For base case $n = 1$, by the identity of gamma function, $\Gamma(z+1) = z\Gamma(z)$, so the formula is true.

Then, suppose for given $n \in \mathbb{N}$, we have $\Gamma(z+n) = (z+n-1)...(z+1)z\Gamma(z)$, which for $(z+n+1)$, it satisfies:

$$\Gamma(z+n+1) = (z+n)\Gamma(z+n) = (z+n)(z+n-1)...(z+1)z\Gamma(z)$$

Hence, this completes the induction.

So, for all $n \in \mathbb{N}$, we also have the following identity:

$$(z+n-1)...(z+1)z = \frac{\Gamma(z+n)}{\Gamma(z)}$$

By this property, we can rewrite the hypergeometric series as follow:

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+n-1) \cdot \beta(\beta+1)...(\beta+n-1)}{n! \cdot \gamma(\gamma+1)...(\gamma+n-1)} z^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(\Gamma(\alpha+n)/\Gamma(\alpha))(\Gamma(\beta+n)/\Gamma(\beta))}{n! \cdot (\Gamma(\gamma+n)/\Gamma(\gamma))} z^n = 1 + \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n! \cdot \Gamma(\gamma+n)} z^n$$

Power series of $(1-\zeta)^{-\alpha}$:

Given the above function, it is analytic within the disk $|\zeta| < 1$. Then, consider its derivatives at $\zeta = 0$, we get:

$$\frac{d}{d\zeta}(1-\zeta)^{-\alpha} = \alpha(1-\zeta)^{-\alpha-1}$$

$$\forall n \in \mathbb{N}, \ \frac{d^n}{d\zeta^n}(1-\zeta)^{-\alpha} = (\alpha+n-1)...(\alpha+1)\alpha(1-\zeta)^{-\alpha-n} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}(1-\zeta)^{-\alpha-n}$$

So, let $f(\zeta) = (1-\zeta)^{-\alpha}$, $f^{(n)}(0) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$. Which, the power series about $\zeta = 0$ is:

$$f(\zeta) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \zeta^n = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n! \cdot \Gamma(\alpha)} \zeta^n$$

The Integral:

Then, since the power series converges uniformly for any compact region within the unit disk $|\zeta| < 1$, while the integral of the function with $(1-zt)^{-\alpha}$ being defined with $|z| < 1$, $t \in (0,1)$, hence the function is integrated over a region that can be covered by a compact set (choose closed disk with radius $|\zeta| \leq R < 1$, where $|z| < R$). As the power series converges uniformly on this region, integral and summation in fact commutes, hence the integral mentioned in the question can be represented as:

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\gamma-\beta-1}(1-t)^{\beta-1} \left(\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n! \cdot \Gamma(\alpha)} (zt)^n \right) dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!} z^n \int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1} dt$$

With the identity verified in **Question 1**, we know the following:

$$\forall \alpha, \beta \in \mathbb{C}, \ Re(\alpha), Re(\beta) > 0, \quad B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Hence, the above form of integral becomes:

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{n!}z^n\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}dt$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{n!}z^n\cdot B(\gamma-\beta,\beta+n)$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{n!}z^n\cdot\frac{\Gamma(\gamma-\beta)\Gamma(\beta+n)}{\Gamma(\gamma+n)}$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)}z^n=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha+0)\Gamma(\beta+0)}{0!\Gamma(\gamma+0)}+\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)}\sum_{n=1}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)}z^n$$

$$=1+\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)}\sum_{n=1}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)}z^n$$

Hence, we can conclude the following:

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{n!}z^n\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}dt$$

$$=1+\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)}\sum_{n=1}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)}z^n=F(\alpha,\beta,\gamma;z)$$

The identity proposed in the question is showned above.

Analytic Continuation:

For all $z\in\mathbb{C}\setminus[1,\infty)$ and all $t\in(0,1)$, then since $z\notin[1,\infty)$, then $tz\notin[1,\infty)$ (since if $tz\in[1,\infty)$, $z\in[1/t,\infty)\subseteq[1,\infty)$, which is a contradiction), hence $(1-tz)\notin(-\infty,0]$. So, if define a $\log(z)$ to have a branch cut on $(-\infty,0]$, then $\log(1-tz)$ is analytic.

Which, on this new domain, the following function is defined, and analytic:

$$\bar{F}(\alpha,\beta,\gamma;z)=\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}e^{-\alpha\log(1-zt)}dt$$

Which, on the unit disk $|z|<1$, the above function agrees with the hypergeometric functions. Hence, it is an analytic continuation of the hypergeometric function on the domain $\mathbb{C}\setminus[1,\infty)$.

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Question 3 *Prove that*

$$\frac{d^2}{ds^2}(\log(\Gamma(s))) = \sum_{n=0}^{\infty} \frac{1}{(s+n)^2}$$

whenever s is a positive number. Show that if the left-hand side is interpreted as $(\Gamma'/\Gamma)'$, then the above formula also holds for all complex numbers s with $s \neq 0, -1, -2, \ldots$

Pf:

We'll directly prove the case for viewing it as Γ'/Γ (which applies to the case for positive real inputs). First, recall the following characterization of Γ:

$$\forall z \in \mathbb{C}, \quad \frac{1}{\Gamma(z)} = G(z) = ze^{\gamma z}H(z), \quad H(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

(Note: γ is the Euler-Mascheroni Constant).

Which, the derivative of $1/\Gamma(z)$ given as $-\frac{\Gamma'(z)}{(\Gamma(z))^2}$, while the derivative of $G(z)$ is given as follow:

$$G'(z) = \left(e^{\gamma z} + \gamma z e^{\gamma z}\right)H(z) + ze^{\gamma z}H'(z) = \frac{1}{z}\cdot ze^{\gamma z}H(z) + \gamma \cdot ze^{\gamma z}H(z) + ze^{\gamma z}H'(z)$$

$$= \left(\frac{1}{z}+\gamma\right)G(z) + ze^{\gamma z}H'(z)$$

Since the derivatives match up, the only thing left is finding a precise formula for $H'(z)$.

Expression of $H'(z)$:

For all $z \in \mathbb{C}$, choose $N \in \mathbb{N}$, such that for all $n \in \mathbb{N}$, $n \geq N$ implies $\left|\frac{z}{n}\right| \leq \frac{1}{2}$ (in other words, we're working in the disk $|z| \leq \frac{N}{2}$, which is compact). Then, we can define a single-valued branch for $\log(1+\zeta)$ for $|\zeta| < 1$. Then, by grouping the components of the product in $H(z)$, we get the following:

$$H(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} = \left(\prod_{n=1}^{N}\left(1+\frac{z}{n}\right)e^{-z/n}\right)\cdot\left(\prod_{n=N+1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right)$$

$$= \left(\prod_{n=1}^{N}\left(1+\frac{z}{n}\right)\right)\cdot\exp\left(\sum_{n=1}^{N}-\frac{z}{n}\right)\cdot\left(\prod_{n=N+1}^{\infty}\exp\left(\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right)\right)$$

$$= \left(\prod_{n=1}^{N}\left(1+\frac{z}{n}\right)\right)\cdot\exp\left(\sum_{n=1}^{N}-\frac{z}{n}\right)\cdot\exp\left(\sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right)\right)$$

$$= \left(\prod_{n=1}^{N}\left(1+\frac{z}{n}\right)\right)\cdot\exp\left(-\sum_{n=1}^{N}\frac{z}{n}+\sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right)\right)$$

Before continuing, we need to argue why the infinite series of function in the above exponent converges normally in the disk: Since $\left|\frac{z}{n}\right| \leq \frac{1}{2}$ for all $n \geq N$, then the power series of $\log(1+\frac{z}{n})$ is $-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{z}{n}\right)^k$. Then, each index $n \geq N$ satisfies the following:

$$\left|\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right| = \left|-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{z}{n}\right)^k-\frac{z}{n}\right| = \left|\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\left(\frac{z}{n}\right)^k\right| \leq \sum_{k=2}^{\infty}\frac{1}{k}\left|\frac{z}{n}\right|^k$$

$$\leq \sum_{k=2}^{\infty}\left|\frac{z}{n}\right|^k \leq \left|\frac{z}{n}\right|^2 \sum_{k=2}^{\infty}\left|\frac{z}{n}\right|^{k-2} = \left|\frac{z}{n}\right|^2 \sum_{k=0}^{\infty}\left|\frac{z}{n}\right|^k \leq \left|\frac{z}{n}\right|^2 \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 2\left|\frac{z}{n}\right|^2$$

With the assumption that we're working over the disk $|z| \leq \frac{N}{2}$, the above bound can be simplified as:

$$\left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right| \leq 2\left|\frac{z}{n}\right|^2 \leq 2\left(\frac{N}{2}\right)^2 \cdot \frac{1}{n^2} = \frac{N^2}{2} \cdot \frac{1}{n^2}$$

Hence, the series of function converges normally in the disk because of the following inequality:

$$\sum_{n=N+1}^{\infty}\left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right| \leq \sum_{n=N+1}^{\infty} \frac{N^2}{2} \cdot \frac{1}{n^2} < \infty$$

So, it's valid to talk about the way we organize the infinite product in $H(z)$ (and more conveniently, the above infinite series can be differentiated term by term).

Now, define the two functions $A(z), B(z)$ on the disk $|z| \leq \frac{N}{2}$ as follow:

$$A(z) = \prod_{n=1}^{N}\left(1+\frac{z}{n}\right), \quad B(z) = \exp\left(-\sum_{n=1}^{N}\frac{z}{n} + \sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right)\right)$$

Then, the function $H = AB$, hence the derivative is given by $H' = A'B + B'A$.

For $A'(z)$, it is expressed as follow:

$$A'(z) = \sum_{n=1}^{N}\left(\frac{d}{dz}\left(1+\frac{z}{n}\right)\right) \cdot \left(\prod_{k=1,\ k\neq n}^{N}\left(1+\frac{z}{k}\right)\right) = \sum_{n=1}^{N}\frac{1}{n} \cdot \left(\prod_{k=1,\ k\neq n}^{N}\left(1+\frac{z}{k}\right)\right)$$

$$= \sum_{n=1}^{N}\frac{1}{n} \cdot \frac{1}{1+\frac{z}{n}} \cdot \left(\prod_{k=1}^{N}\left(1+\frac{z}{k}\right)\right) = \sum_{n=1}^{N}\frac{1}{z+n} \cdot A(z)$$

For $B'(z)$, it is expressed as follow:

$$B'(z) = \frac{d}{dz}\exp\left(-\sum_{n=1}^{N}\frac{z}{n} + \sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right)\right)$$

$$= \exp\left(-\sum_{n=1}^{N}\frac{z}{n} + \sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right)\right) \cdot \frac{d}{dz}\left(-\sum_{n=1}^{N}\frac{z}{n} + \sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right)\right)$$

$$= B(z)\left(-\sum_{n=1}^{N}\frac{1}{n} + \sum_{n=N+1}^{\infty}\left(\frac{1/n}{1+z/n} - \frac{1}{n}\right)\right) = B(z)\left(-\sum_{n=1}^{N}\frac{1}{n} + \sum_{n=N+1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)\right)$$

Then, $H'(z)$ is then given by:

$$H'(z) = A'B + B'A = \left(\sum_{n=1}^{N}\frac{1}{z+n}\right) \cdot A(z) \cdot B(z) + B(z)\left(-\sum_{n=1}^{N}\frac{1}{n} + \sum_{n=N+1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)\right) \cdot A(z)$$

$$= A(z)B(z) \cdot \left(\sum_{n=1}^{N}\left(\frac{1}{z+n} - \frac{1}{n}\right) + \sum_{n=N+1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)\right) = H(z)\left(\sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)\right)$$

Expression of Γ'/Γ:

Now, plug back into the original expression,

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Question 4

Pf:

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Question 5

Pf: