

Math CS 122B HW7

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Question 1 *The functional equation of the ζ -function can also be written in the following form:*

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s)$$

Deduce from this: In the half-plane $\sigma \leq 0$, the function $\zeta(s)$ has exactly the zeros $s = -2k$, $k \in \mathbb{N}$. All other zeros of the ζ -function are located in the vertical strip $0 < \text{Re } s < 1$.

Pf:

First, recall that for the half plane $\sigma > 1$, the following inequality is given:

$$\left| \frac{\zeta(\sigma + it)}{\sigma - 1} \right|^4 |\zeta(\sigma + 2it)| [\zeta(\sigma)(\sigma - 1)]^3 \geq (\sigma - 1)^{-1}$$

Since for $\sigma > 1$, the expression $(\sigma - 1)^{-1} > 0$, this enforces all $s = \sigma + it$ in the half plane to have $\zeta(s) \neq 0$ (or else the left side of the inequality is 0, which violates the inequality). Similarly, this inequality can be extended onto the line $\text{Re}(s) = 1$, where $\zeta(s)$ has no zeros on this line also. So, for $\sigma \geq 1$, $\zeta(s)$ has no zero.

Now, in the half plane $\sigma \leq 0$, for all $s' \neq 0$, since it can be written as $s' = 1 - s$, where $s = 1 - s'$ has $\text{Re}(s) = 1 - \text{Re}(s') \geq 1$ (and since $s' \neq 0$, then $s \neq 1$). Then, by the functional equation, we get the following:

$$\zeta(s') = \zeta(1-s) = 2(2\pi)^{-s}\Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s)$$

Since $\text{Re}(s) \geq 1$ with $s \neq 1$, then $\zeta(s) \neq 0$ based on what is mentioned during the start; also, $\Gamma(s) \neq 0$ for all $s \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, while $2(2\pi)^{-s} \neq 0$ for all $s \in \mathbb{C}$. Hence, in case for $\zeta(1-s) = 0$, we must have $\cos(\frac{\pi s}{2}) = 0$, which enforces $\frac{\pi s}{2} = k\pi + \frac{\pi}{2}$ for some $k \in \mathbb{Z}$, or $s = 2k + 1$ for some $k \in \mathbb{Z}$. Now, under this assumption, since $\text{Re}(s) \geq 1$ while $s \neq 1$, then $k \geq 1$. So, when transferring back to $s' = 1 - s$, we get $s' = 1 - (2k + 1) = -2k$ for integer $k \geq 1$.

Hence, for $\text{Re}(s') \leq 0$, if $\zeta(s') = 0$, then $s' = -2k$ for some $k \in \mathbb{N}$ (this is an iff since at all these points, $s = 2 - s'$ satisfies $\cos(\frac{\pi s}{2}) = 0$, which $\zeta(s') = \zeta(1-s) = 0$).

Finally, for $s' = 0$ (where if $s' = 1 - s$, $s = 1$). Recall that $\zeta(s)$ has a simple pole at $s = 1$, while $\cos(\frac{\pi s}{2})$ has a simple zero at $s = 1$ (where the input is $\frac{\pi}{2}$, where \cos is 0). Hence, $\cos(\frac{\pi s}{2}) = (s - 1)h(z)$ for some analytic function h where $h(1) \neq 0$. Also, we know $\lim_{s \rightarrow 1} (s - 1)\zeta(s) = 1$ (has been given in the textbook). Then, we get the following:

$$\lim_{s \rightarrow 1} \zeta(1-s) = \lim_{s \rightarrow 1} 2(2\pi)^{-s}\Gamma(s)h(s)(s-1)\zeta(s) = 2(2\pi)^{-1}\Gamma(1)h(1) \cdot \lim_{s \rightarrow 1} (s-1)\zeta(s) = 2(2\pi)^{-1}\Gamma(1)h(1) \neq 0$$

Hence, we can deduce that at $s = 1$ (where $s' = 1 - s = 0$), $\zeta(s')$ has a removable singularity that has limit not being 0, so $\zeta(s')$ as an extension has $\zeta(0) \neq 0$.

The above casees proves that when $\sigma \geq 1$ or $\sigma \leq 0$, $\zeta(s) = 0$ iff $s = -2k$ for some $k \in \mathbb{N}$, where for any other input ζ is nonzero.

Hence, if there are any other zeros, it must exist in the vertical strip $0 < \text{Re}(s) < 1$.

Question 2 *The following special case of the Hecke Theorem was already known to B. Riemann (1859):*

$$\begin{aligned}\xi(s) &:= \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} \\ &= \frac{1}{2} \int_1^{\infty} (\vartheta(it) - 1)(t^{s/2} + t^{(1-s)/2}) \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}\end{aligned}$$

Deduce directly this special case, and use it to prove the meromorphic continuation and the functional equation.

Pf:

For this problem, first assume $\operatorname{Re}(s) > 1$ (where $\zeta(s)$ is defined with the original series form). We'll break down into two different equations:

Equation 1:

We'll first prove the following:

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{dt}{t}$$

For each $n \in \mathbb{N}$, the integral within the right hand side summation, after doing the substitution $u = \pi n^2 t$ (where $du = \pi n^2 dt$), we get the following:

$$\begin{aligned}\int_0^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} &= \int_0^{\infty} e^{-u} \left(\frac{u}{\pi n^2}\right)^{s/2} \frac{du/(\pi n^2)}{u/(\pi n^2)} = (\pi n^2)^{-s/2} \int_0^{\infty} e^{-u} u^{s/2} \frac{du}{u} \\ &= \pi^{-s/2} n^{-s} \int_0^{\infty} e^{-u} u^{s/2-1} du = \pi^{-s/2} n^{-s} \Gamma\left(\frac{s}{2}\right)\end{aligned}$$

Hence, for the series, since $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ converges normally within $\operatorname{Re}(s) > 1$, we get the following:

$$\sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \sum_{n=1}^{\infty} \pi^{-s/2} n^{-s} \Gamma\left(\frac{s}{2}\right) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} n^{-s} = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

So, the first equatity holds.

Equation 2:

Our second goal is to prove the following:

$$\sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \int_1^{\infty} (\vartheta(it) - 1)(t^{s/2} + t^{(1-s)/2}) \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

Since the summation is absolutely convergent for $\operatorname{Re}(s) > 1$ (based on how $\xi(s)$ is defined), then swapping the order of summation causes no issue. Hence, for all $n \in \mathbb{N}$, since $n^2 = (-n)^2$, the series can also be decomposed as:

$$\sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} + \frac{1}{2} \sum_{n=-\infty}^{-1} \int_0^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{dt}{t}$$

$$= \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_1^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} + \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t}$$

For the first summation, recall that the Theta Series is defined as $\vartheta : \mathbb{H} \rightarrow \mathbb{C}$, $\vartheta(\tau) = \sum_{n=-\infty}^\infty e^{i\pi n^2 \tau}$. Since for $n = 0$, $e^{i\pi n^2 \tau} = e^0 = 1$, then we get the following:

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{i\pi n^2 \tau} = \vartheta(\tau) - 1$$

If consider $\tau = it$ for $t \in (0, \infty)$, we get the following:

$$\vartheta(it) - 1 = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{i\pi n^2 \cdot it} = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{-\pi n^2 t}$$

So, for the first summation in the equation, since ϑ is a normally convergent series of function, the summation and integral can change the order of operation. Hence:

$$\frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_1^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \int_1^\infty \left(\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{-\pi n^2 t} \right) t^{s/2} \frac{dt}{t} = \frac{1}{2} \int_1^\infty (\vartheta(it) - 1) t^{s/2} \frac{dt}{t}$$

Now, to work on the second summation, for all $n \in \mathbb{Z}$ with $n \neq 0$, given the substitution $u = \frac{1}{t}$, $du = -\frac{1}{t^2} dt$ (or $\frac{dt}{t} = -t du = -\frac{du}{u}$), we get the following integral representation:

$$\int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = - \int_\infty^1 e^{-\pi n^2 \cdot 1/u} u^{-s/2} \frac{du}{u} = \int_1^\infty e^{-\pi n^2 \cdot 1/u} u^{-s/2} \frac{du}{u}$$

Which, based on the above change of variable and the property of Theta Series, the second summation in the equation can be rewrite as:

$$\begin{aligned} \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} &= \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_1^\infty e^{-\pi n^2 \cdot 1/u} u^{-s/2} \frac{du}{u} = \frac{1}{2} \int_1^\infty \left(\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{-\pi n^2 \cdot 1/u} \right) u^{-s/2} \frac{du}{u} \\ &= \frac{1}{2} \int_1^\infty \left(\vartheta\left(\frac{i}{u}\right) - 1 \right) u^{-s/2} \frac{du}{u} \end{aligned}$$

Which, there is a property of Theta Series given below:

$$\vartheta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}} \vartheta(z)$$

Hence, for $u \in (0, \infty)$, let $z = iu$, we get the following:

$$\vartheta\left(\frac{i}{u}\right) = \vartheta\left(-\frac{1}{iu}\right) = \sqrt{\frac{iu}{i}} \vartheta(iu) = u^{1/2} \vartheta(iu)$$

So, the summation can be modified as follow:

$$\begin{aligned} \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} &= \frac{1}{2} \int_1^\infty \left(u^{1/2} \vartheta(iu) - 1 \right) u^{-s/2} \frac{du}{u} \\ &= \frac{1}{2} \int_1^\infty (\vartheta(iu) - 1) u^{1/2} u^{-s/2} \frac{du}{u} + \frac{1}{2} \int_1^\infty u^{1/2} u^{-s/2} \frac{du}{u} - \frac{1}{2} \int_1^\infty u^{-s/2} \frac{du}{u} \end{aligned}$$

For the integrals at the middle and the right, since the power of u is given by $\frac{1-s}{2} - 1$ and $\frac{-s}{2} - 1$, then because $\operatorname{Re}(s) > 1$, the two powers of u both have the real parts being less than -1 . Hence, the integral absolutely converges, and using power rule, we yield:

$$\begin{aligned}\frac{1}{2} \int_1^\infty u^{1/2} u^{-s/2} \frac{du}{u} &= \frac{1}{2} \int_1^\infty u^{\frac{1-s}{2}-1} du = \frac{1}{2} \cdot \frac{2}{1-s} u^{\frac{1-s}{2}} \Big|_1^\infty = -\frac{1}{1-s} \\ \frac{1}{2} \int_1^\infty u^{-s/2} \frac{du}{u} &= \frac{1}{2} \int_1^\infty u^{-s/2-1} du = -\frac{1}{2} \cdot \frac{2}{s} u^{-s/2} \Big|_1^\infty = \frac{1}{s} u^{-s/2} \Big|_\infty = \frac{1}{s}\end{aligned}$$

(Note: since $\operatorname{Re}(s) > 1$, then $\operatorname{Re}(\frac{1-s}{2}) < 0$ and $\operatorname{Re}(-\frac{s}{2}) < 0$, so $\lim_{u \rightarrow \infty} u^{\frac{1-s}{2}} = 0$ and $\lim_{u \rightarrow \infty} u^{-s/2} = 0$). So, combining the pieces for the second summation, we get:

$$\begin{aligned}\frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} &= \frac{1}{2} \int_1^\infty (\vartheta(iu) - 1) u^{1/2} u^{-s/2} \frac{du}{u} + \frac{1}{2} \int_1^\infty u^{1/2} u^{-s/2} \frac{du}{u} - \frac{1}{2} \int_1^\infty u^{-s/2} \frac{du}{u} \\ &= \frac{1}{2} \int_1^\infty (\vartheta(it) - 1) t^{(1-s)/2} \frac{dt}{t} + \left(-\frac{1}{1-s} \right) - \frac{1}{s} = \frac{1}{2} \int_1^\infty (\vartheta(it) - 1) t^{(1-s)/2} \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}\end{aligned}$$

Finally, the original equation can be obtained as follow:

$$\begin{aligned}\sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} &= \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_1^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} + \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} \\ &= \left(\frac{1}{2} \int_1^\infty (\vartheta(it) - 1) t^{s/2} \frac{dt}{t} \right) + \left(\frac{1}{2} \int_1^\infty (\vartheta(it) - 1) t^{(1-s)/2} \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s} \right) \\ &= \frac{1}{2} \int_1^\infty (\vartheta(it) - 1) (t^{s/2} + t^{(1-s)/2}) \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}\end{aligned}$$

This verifies the second equation. Together with the first and the second equation proven, the function $\xi(s)$ defined in the question for $\operatorname{Re}(s) > 1$ satisfies the following equation:

$$\begin{aligned}\xi(s) &:= \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} \\ &= \frac{1}{2} \int_1^\infty (\vartheta(it) - 1) (t^{s/2} + t^{(1-s)/2}) \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}\end{aligned}$$

Meromorphic Continuation & Functional equation of ζ :

The textbook had introduced the meromorphic continuation of ζ onto the half plane $\operatorname{Re}(s) > 0$. Now, consider any s satisfying $\frac{1}{2} < \operatorname{Re}(s) < 1$, then $(1-s)$ satisfies $\operatorname{Re}(1-s) = 1 - \operatorname{Re}(s)$, $0 < \operatorname{Re}(1-s) < \frac{1}{2}$. Since both $s, (1-s)$ are within the domain of the extended ζ , then if plug in the ξ -function defined above, using the integral expression, we get:

$$\begin{aligned}\xi(s) &= \int_1^\infty (\vartheta(it) - 1) (t^{s/2} + t^{(1-s)/2}) \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s} \\ \xi(1-s) &= \int_1^\infty (\vartheta(it) - 1) (t^{(1-s)/2} + t^{(1-(1-s))/2}) \frac{dt}{t} - \frac{1}{1-s} - \frac{1}{1-(1-s)} \\ &= \int_1^\infty (\vartheta(it) - 1) (t^{s/2} + t^{(1-s)/2}) \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s} = \xi(s)\end{aligned}$$

Hence, for s satisfying $\frac{1}{2} < \operatorname{Re}(s) < 1$, $\xi(1-s) = \xi(s)$, which leads to the following functional equation:

$$\begin{aligned} \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) &= \xi(1-s) = \xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \\ \implies \zeta(1-s) &= \pi^{1/2-s} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \zeta(s) \end{aligned}$$

Now, recall the Duplication Formula of Γ -function:

$$\forall z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}, \quad \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{1+z}{2}\right) = \frac{\sqrt{\pi}}{2^{z-1}} \Gamma(z)$$

Then, take $z = s$ (since $\frac{1}{2} < \operatorname{Re}(s) < 1$ by assumption now, it is valid), we get:

$$\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1+s}{2}\right) = \frac{\sqrt{\pi}}{2^{s-1}} \Gamma(s) \implies \Gamma\left(\frac{s}{2}\right) = \frac{\sqrt{\pi} \cdot \Gamma(s)}{2^{s-1} \Gamma\left(\frac{1+s}{2}\right)}$$

Hence, the expression $\zeta(1-s)$ becomes:

$$\begin{aligned} \zeta(1-s) &= \pi^{1/2-s} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \zeta(s) = \frac{\sqrt{\pi}}{\pi^s \Gamma\left(\frac{1-s}{2}\right)} \cdot \frac{\sqrt{\pi} \cdot \Gamma(s)}{2^{s-1} \Gamma\left(\frac{1+s}{2}\right)} \zeta(s) \\ &= \frac{2\pi \cdot \Gamma(s)}{(2\pi)^s} \cdot \frac{1}{\Gamma\left(\frac{1+s}{2}\right) \Gamma\left(1 - \left(\frac{1+s}{2}\right)\right)} \zeta(s) \end{aligned}$$

Which, this expression allows the use of another property of Γ -function:

$$\forall z \in \mathbb{C}, \quad \frac{1}{\Gamma(z) \Gamma(1-z)} = \frac{\sin(\pi z)}{\pi}$$

Take $z = \frac{1+s}{2}$, we get:

$$\frac{1}{\Gamma\left(\frac{1+s}{2}\right) \Gamma\left(1 - \left(\frac{1+s}{2}\right)\right)} = \frac{\sin\left(\pi \left(\frac{1+s}{2}\right)\right)}{\pi} = \frac{\sin\left(\frac{\pi s}{2} + \frac{\pi}{2}\right)}{\pi} = \frac{\cos\left(\frac{\pi s}{2}\right)}{\pi}$$

So, the following functional equation of ζ (on $\frac{1}{2} < \operatorname{Re}(s) < 1$) can be deduced:

$$\begin{aligned} \zeta(1-s) &= \frac{2\pi \cdot \Gamma(s)}{(2\pi)^s} \cdot \frac{1}{\Gamma\left(\frac{1+s}{2}\right) \Gamma\left(1 - \left(\frac{1+s}{2}\right)\right)} \zeta(s) \\ &= \frac{2\pi \cdot \Gamma(s)}{(2\pi)^s} \cdot \frac{\cos\left(\frac{\pi s}{2}\right)}{\pi} \zeta(s) = 2(2\pi)^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s) \end{aligned}$$

This finishes the proof for the functional equation. Moreover, for all s' with $\operatorname{Re}(s') < \frac{1}{2}$, since given $s' = 1-s$, or $s = 1-s'$, we have $\operatorname{Re}(s) = \operatorname{Re}(1-s') = 1 - \operatorname{Re}(s') > \frac{1}{2}$, then the following expression is well-defined:

$$2(2\pi)^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s)$$

Hence, the meromorphic continuation of ζ onto the whole plane can be followed by the functional equation:

$$\forall s' \in \mathbb{C}, \operatorname{Re}(s') < \frac{1}{2}, \text{ let } s' = 1-s, \operatorname{Re}(s) > \frac{1}{2}, \quad \text{Define } \zeta(1-s) := 2(2\pi)^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s)$$