Math CS 122B HW2

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Question 1 *Stein and Shakarchi Chap. 6 Exercise 7:*
*The **Beta function** is defined for $Re(\alpha) > 0$ and $Re(\beta) > 0$ by*

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt$$

(a) Prove that

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(b) Show that

$$B(\alpha, \beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$$

Pf:

(a) First, we'll consider $\Gamma(\alpha)\Gamma(\beta)$:

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty t^{\alpha-1} e^{-t} dt \int_0^\infty s^{\beta-1} e^{-s} ds = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-s-t} ds dt$$

If we consider the change of variable $f : (0,1) \times (0,\infty) \to (0,\infty) \times (0,\infty)$ by $f(r,u) = (ur, u(1-r)) = (s,t)$, since this is a differentiable function, and its derivative is given by:

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial r}(ur) & \frac{\partial}{\partial u}(ur) \\ \frac{\partial}{\partial r}(u(1-r)) & \frac{\partial}{\partial u}(u(1-r)) \end{pmatrix} = \begin{pmatrix} u & r \\ -u & (1-r) \end{pmatrix}$$

$$\frac{\partial(s,t)}{\partial(r,u)} = \begin{vmatrix} u & r \\ -u & (1-r) \end{vmatrix} = u(1-r) - (-u)r = u$$

Then, the above integral can be given as:

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-s-t} ds dt = \int_0^\infty \int_0^1 (u(1-r))^{\alpha-1} (ur)^{\beta-1} e^{-(ur+u(1-r))} \left|\frac{\partial(s,t)}{\partial(r,u)}\right| dr du$$

$$= \int_0^\infty \int_0^1 u^{\alpha+\beta-2} e^{-u} \cdot (1-r)^{\alpha-1} r^{\beta-1} u dr du = \int_0^\infty u^{\alpha+\beta-1} e^{-u} \cdot (1-r)^{\alpha-1} r^{\beta-1} dr du$$

$$= \int_0^\infty u^{\alpha+\beta-1} e^{-u} du \cdot \int_0^1 (1-r)^{\alpha-1} r^{\beta-1} dr = \Gamma(\alpha+\beta) \cdot B(\alpha, \beta)$$

Hence, we can rewrite the above equality to be as follow:

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta) \cdot B(\alpha, \beta), \quad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(b) Consider the following expression:

$$\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$$

First, if we do the substitution $(1+u) = e^t$, $du = e^t dt$, which $u = 0 \implies e^t = 1$, $t = 0$, and $\lim_{t\to\infty} e^t = \infty$, so $\lim_{t\to\infty} u = \infty$. Then, the integral can be rewrite as:

$$\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du = \int_0^\infty \frac{(e^t-1)^{\alpha-1}}{(e^t)^{\alpha+\beta}} e^t dt = \int_0^\infty ((1-e^{-t})e^t)^{\alpha-1}(e^{-t})^{\alpha+\beta} \cdot e^t dt$$

$$= \int_0^\infty (1-e^{-t})^{\alpha-1} \cdot (e^t)^{\alpha-1} \cdot (e^t)^{-\alpha} \cdot (e^t)^{-\beta} \cdot e^t dt = \int_0^\infty (1-e^{-t})^{\alpha-1}(e^{-t})^\beta dt$$

Then, for the above expression, if we do the second substitution $r = e^{-t}$, $dr = -e^{-t}dt$, $dt = -e^t dr = -r^{-1}dr$. Which $t = 0 \implies r = e^0 = 1$, and $\lim_{t\to\infty} e^{-t} = \lim_{t\to\infty} r = 0$. So, the integral can be rewrite as:

$$\int_0^\infty (1-e^{-t})^{\alpha-1}(e^{-t})^\beta dt = \int_1^0 (1-r)^{\alpha-1} r^\beta \cdot (-r^{-1}) dr = \int_0^1 (1-r)^{\alpha-1} r^{\beta-1} dr = B(\alpha,\beta)$$

Hence, we can conclude the following:

$$\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du = \int_0^\infty (1-e^{-t})^{\alpha-1}(e^{-t})^\beta dt = B(\alpha,\beta)$$

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Question 2 *Stein and Shakarchi Chap. 6 Exercise 9:*
The hypergeometric series $F(\alpha,\beta,\gamma;z)$ was defined as

$$F(\alpha,\beta,\gamma;z) = 1 + \sum_{n=1}^\infty \frac{\alpha(\alpha+1)...(\alpha+n-1)\cdot\beta(\beta+1)...(\beta+n-1)}{n!\cdot\gamma(\gamma+1)...(\gamma+n-1)} z^n$$

Here $\alpha > 0, \beta > 0, \gamma > \beta$, and $|z| < 1$. Show that

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt$$

Show as a result that the hypergeometric function, initially defined by a power series convergent in the unit disc, can be continued analytically to the complex plane slit along the half-line $[1,\infty)$.

Pf:
Properties of Gamma function:

First, we can use induction to verity that given $z \in \mathbb{C} \setminus \{0, -1, -2, ...\}$, all $n \in \mathbb{N}$ satisfies $\Gamma(z+n) = (z+n-1)...(z+1)z\Gamma(z)$.

For base case $n = 1$, by the identity of gamma function, $\Gamma(z+1) = z\Gamma(z)$, so the formula is true.

Then, suppose for given $n \in \mathbb{N}$, we have $\Gamma(z+n) = (z+n-1)...(z+1)z\Gamma(z)$, which for $(z+n+1)$, it satisfies:

$$\Gamma(z+n+1) = (z+n)\Gamma(z+n) = (z+n)(z+n-1)...(z+1)z\Gamma(z)$$

Hence, this completes the induction.

So, for all $n \in \mathbb{N}$, we also have the following identity:

$$(z+n-1)...(z+1)z = \frac{\Gamma(z+n)}{\Gamma(z)}$$

By this property, we can rewrite the hypergeometric series as follow:

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+n-1) \cdot \beta(\beta+1)...(\beta+n-1)}{n! \cdot \gamma(\gamma+1)...(\gamma+n-1)} z^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(\Gamma(\alpha+n)/\Gamma(\alpha))(\Gamma(\beta+n)/\Gamma(\beta))}{n! \cdot (\Gamma(\gamma+n)/\Gamma(\gamma))} z^n = 1 + \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n! \cdot \Gamma(\gamma+n)} z^n$$

Power series of $(1-\zeta)^{-\alpha}$:

Given $f(\zeta) = (1-\zeta)^{-\alpha}$, it is analytic within the disk $|\zeta| < 1$. Then, consider its derivatives at $\zeta = 0$, we get:

$$\frac{d}{d\zeta}(1-\zeta)^{-\alpha} = \alpha(1-\zeta)^{-\alpha-1}$$

$$\forall n \in \mathbb{N}, \ \frac{d^n}{d\zeta^n}(1-\zeta)^{-\alpha} = (\alpha+n-1)...(\alpha+1)\alpha(1-\zeta)^{-\alpha-n} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}(1-\zeta)^{-\alpha-n}$$

So, $f^{(n)}(0) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$. Which, the power series about $\zeta = 0$ is:

$$f(\zeta) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\zeta^n = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n! \cdot \Gamma(\alpha)}\zeta^n$$

And, this power series agrees with f in the disk $|\zeta| < 1$.

The Integral:

Since the power series above converges uniformly for any compact region within the unit disk $|\zeta| < 1$, while the integral of the function with $(1-zt)^{-\alpha}$ being defined with $|z| < 1$, $t \in (0,1)$, hence the function is integrated over a region that can be covered by a compact set (choose closed disk with radius $|\zeta| \le R < 1$, where $|z| < R$, then for all $t \in (0,1)$, $|zt| \le |z| < R$, which is in the closed disk $|\zeta| \le R$).

As the power series converges uniformly on this closed disk, integral and summation in fact commutes, hence the integral mentioned in the question can be represented as:

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \left(\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n! \cdot \Gamma(\alpha)}(zt)^n \right) dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!} z^n \int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}dt$$

With the identity verified in **Question 1**, we know the following:

$$\forall \alpha, \beta \in \mathbb{C},\ Re(\alpha), Re(\beta) > 0, \quad B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Hence, the above form of integral becomes:

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!} z^n \int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1} dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!} z^n \cdot B(\gamma-\beta, \beta+n)$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!} z^n \cdot \frac{\Gamma(\gamma-\beta)\Gamma(\beta+n)}{\Gamma(\gamma+n)}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)} z^n = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+0)\Gamma(\beta+0)}{0!\Gamma(\gamma+0)} + \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)} z^n$$

$$= 1 + \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)} z^n$$

Hence, we can conclude the following:

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!} z^n \int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1} dt$$

$$= 1 + \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)} z^n = F(\alpha, \beta, \gamma; z)$$

The identity proposed in the question is showned above.

Analytic Continuation:

For all $z \in \mathbb{C} \setminus [1, \infty)$ and all $t \in (0,1)$, then since $z \notin [1, \infty)$, then $tz \notin [1, \infty)$ (since if $tz \in [1, \infty)$, $z \in [1/t, \infty) \subseteq [1, \infty)$, which is a contradiction), hence $(1 - tz) \notin (-\infty, 0]$. So, if define a $\log(z)$ to have a branch cut on $(-\infty, 0]$, then $\log(1 - tz)$ is well-defined and analytic.

Which, on this new domain, the following function is defined, and analytic:

$$\bar{F}(\alpha, \beta, \gamma; z) = \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} e^{-\alpha \log(1-zt)} dt$$

Which, on the unit disk $|z| < 1$, the above function agrees with the hypergeometric functions (since for $t \in (0,1)$ and $|z| < 1$, $e^{-\alpha \log(1-zt)} = (1-zt)^{-\alpha}$). Hence, $\bar{F}$ is an analytic continuation of the hypergeometric series on the domain $\mathbb{C} \setminus [1, \infty)$.

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Question 3 *Stein and Shakarchi Chap. 6 Exercise 13:*
Prove that

$$\frac{d^2}{ds^2}(\log(\Gamma(s))) = \sum_{n=0}^{\infty} \frac{1}{(s+n)^2}$$

whenever s is a positive number. Show that if the left-hand side is interpreted as $(\Gamma'/\Gamma)'$, then the above formula also holds for all complex numbers s with $s \neq 0, -1, -2, ...$

Pf:

We'll directly prove the case for viewing it as Γ'/Γ (which applies to the case for positive real inputs). First, recall the following characterization of Γ:

$$\forall z \in \mathbb{C}, \quad \frac{1}{\Gamma(z)} = G(z) = ze^{\gamma z}H(z), \quad H(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

(Note: γ is the Euler-Mascheroni Constant).

Which, the derivative of $1/\Gamma(z)$ given as $-\frac{\Gamma'(z)}{(\Gamma(z))^2}$, while the derivative of $G(z)$ is given as follow:

$$G'(z) = \left(e^{\gamma z} + \gamma z e^{\gamma z}\right)H(z) + ze^{\gamma z}H'(z) = \frac{1}{z}\cdot ze^{\gamma z}H(z) + \gamma \cdot ze^{\gamma z}H(z) + ze^{\gamma z}H'(z)$$

$$= \left(\frac{1}{z}+\gamma\right)G(z) + ze^{\gamma z}H'(z)$$

So, to find the derivative of $1/\Gamma$, the only thing left is finding a precise formula for $H'(z)$.

Expression of $H'(z)$:

For all $z \in \mathbb{C}$, choose $N \in \mathbb{N}$, such that for all $n \in \mathbb{N}$, $n \geq N$ implies $\left|\frac{z}{n}\right| \leq \frac{1}{2}$ (in other words, we're working in the disk $|z| \leq \frac{N}{2}$, which is compact). Then, we can define a single-valued branch for $\log(1+\zeta)$ for $|\zeta| < 1$. Which, by grouping the components of the product in $H(z)$, we get the following:

$$H(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} = \left(\prod_{n=1}^{N}\left(1+\frac{z}{n}\right)e^{-z/n}\right)\cdot\left(\prod_{n=N+1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right)$$

$$= \left(\prod_{n=1}^{N}\left(1+\frac{z}{n}\right)\right)\cdot\left(\prod_{n=1}^{N}e^{-z/n}\right)\cdot\left(\prod_{n=N+1}^{\infty}\exp\left(\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right)\right)$$

$$= \left(\prod_{n=1}^{N}\left(1+\frac{z}{n}\right)\right)\cdot\exp\left(\sum_{n=1}^{N}-\frac{z}{n}\right)\cdot\exp\left(\sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right)\right)$$

$$= \left(\prod_{n=1}^{N}\left(1+\frac{z}{n}\right)\right)\cdot\exp\left(-\sum_{n=1}^{N}\frac{z}{n}+\sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right)\right)$$

Before continuing, we need to argue why the infinite series of function in the above exponent converges normally in the disk: Since $\left|\frac{z}{n}\right| \leq \frac{1}{2}$ for all $n \geq N$, then the power series of $\log(1+\frac{z}{n})$ is $-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{z}{n}\right)^k$. Then, each index $n \geq N$ satisfies the following:

$$\left|\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right| = \left|-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{z}{n}\right)^k - \frac{z}{n}\right| = \left|\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\left(\frac{z}{n}\right)^k\right| \leq \sum_{k=2}^{\infty}\frac{1}{k}\left|\frac{z}{n}\right|^k$$

$$\leq \sum_{k=2}^{\infty} \left|\frac{z}{n}\right|^k \leq \left|\frac{z}{n}\right|^2 \sum_{k=2}^{\infty} \left|\frac{z}{n}\right|^{k-2} = \left|\frac{z}{n}\right|^2 \sum_{k=0}^{\infty} \left|\frac{z}{n}\right|^k \leq \left|\frac{z}{n}\right|^2 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2\left|\frac{z}{n}\right|^2$$

With the assumption that we're working over the disk $|z| \leq \frac{N}{2}$, the above bound can be simplified as:

$$\left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right| \leq 2\left|\frac{z}{n}\right|^2 \leq 2\left(\frac{N}{2}\right)^2 \cdot \frac{1}{n^2} = \frac{N^2}{2} \cdot \frac{1}{n^2}$$

Hence, the series of function converges normally in the disk because of the following inequality:

$$\sum_{n=N+1}^{\infty} \left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right| \leq \sum_{n=N+1}^{\infty} \frac{N^2}{2} \cdot \frac{1}{n^2} < \infty$$

So, it's valid to talk about the way we organize the infinite product in $H(z)$ (and more conveniently, the above infinite series can be differentiated term by term).

Now, define the two functions $A(z), B(z)$ on the disk $|z| \leq \frac{N}{2}$ as follow:

$$A(z) = \prod_{n=1}^{N}\left(1+\frac{z}{n}\right), \quad B(z) = \exp\left(-\sum_{n=1}^{N}\frac{z}{n} + \sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right)\right)$$

Then, the function $H = AB$, hence the derivative is given by $H' = A'B + B'A$.

For $A'(z)$, it is expressed as follow:

$$A'(z) = \sum_{n=1}^{N}\left(\frac{d}{dz}\left(1+\frac{z}{n}\right)\right)\cdot\left(\prod_{k=1,\ k\neq n}^{N}\left(1+\frac{z}{k}\right)\right) = \sum_{n=1}^{N}\frac{1}{n}\cdot\left(\prod_{k=1,\ k\neq n}^{N}\left(1+\frac{z}{k}\right)\right)$$

$$= \sum_{n=1}^{N}\frac{1}{n}\cdot\frac{1}{1+\frac{z}{n}}\cdot\left(\prod_{k=1}^{N}\left(1+\frac{z}{k}\right)\right) = \sum_{n=1}^{N}\frac{1}{z+n}\cdot A(z)$$

For $B'(z)$, it is expressed as follow:

$$B'(z) = \frac{d}{dz}\exp\left(-\sum_{n=1}^{N}\frac{z}{n} + \sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right)\right)$$

$$= \exp\left(-\sum_{n=1}^{N}\frac{z}{n} + \sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right)\right)\cdot\frac{d}{dz}\left(-\sum_{n=1}^{N}\frac{z}{n} + \sum_{n=N+1}^{\infty}\left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right)\right)$$

$$= B(z)\left(-\sum_{n=1}^{N}\frac{1}{n} + \sum_{n=N+1}^{\infty}\left(\frac{1/n}{1+z/n} - \frac{1}{n}\right)\right) = B(z)\left(-\sum_{n=1}^{N}\frac{1}{n} + \sum_{n=N+1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)\right)$$

Then, $H'(z)$ is then given by:

$$H'(z) = A'B + B'A = \left(\sum_{n=1}^{N}\frac{1}{z+n}\right)\cdot A(z)\cdot B(z) + B(z)\left(-\sum_{n=1}^{N}\frac{1}{n} + \sum_{n=N+1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)\right)\cdot A(z)$$

$$= A(z)B(z)\cdot\left(\sum_{n=1}^{N}\left(\frac{1}{z+n} - \frac{1}{n}\right) + \sum_{n=N+1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)\right) = H(z)\left(\sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)\right)$$

Expression of Γ'/Γ and its derivative:

Now, for all $z \in \mathbb{C}$, plug $H'(z)$ back into the original expression of derivative, we get the following:

$$\frac{-\Gamma'(z)}{(\Gamma(z))^2} = \left(\frac{1}{\Gamma(z)}\right)' = G'(z) = \left(\frac{1}{z} + \gamma\right) G(z) + ze^{\gamma z} H'(z)$$

$$= \left(\frac{1}{z} + \gamma\right) G(z) + ze^{\gamma z} H(z) \left(\sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right)\right)$$

$$= \left(\gamma + \frac{1}{z}\right) G(z) + G(z) \left(\sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right)\right)$$

$$= G(z) \left(\gamma + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right)\right)$$

Then, since $G(z) = \frac{1}{\Gamma(z)}$, then for all $z \in \mathbb{C} \setminus S$, with $S = \{0, -1, -2, ...\}$, we get:

$$\frac{\Gamma'(z)}{\Gamma(z)} = \frac{-\Gamma'(z)}{(\Gamma(z))^2} \cdot (-\Gamma(z)) = (-\Gamma(z)) \cdot G(z) \left(\gamma + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right)\right)$$

$$= -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right)$$

Finally, the derivative $(\Gamma'/\Gamma)'$ is given as follow:

$$\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)' = \frac{d}{dz}\left(-\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right)\right) = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$$

Then, the above equation is the desired generalization for the problem.

Special Case for real positive inputs:

If restrict the domain to $\mathbb{R}_{>0}$, the function $\log(\Gamma(s))$ is well-defined, and its derivative is given as:

$$\frac{d}{ds}\log(\Gamma(s)) = \frac{\Gamma'(s)}{\Gamma(s)}, \quad \frac{d^2}{ds^2}\log(\Gamma(s)) = \frac{d}{ds}\left(\frac{\Gamma'(s)}{\Gamma(s)}\right) = \sum_{n=0}^{\infty} \frac{1}{(s+n)^2}$$

Which, this finishes the special case for all $s > 0$.

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Question 4 *Stein and Shakarchi Chap. 6 Exercise 14:*
This exercise gives an asymptotic formula for $\log n!$.

(a) Show that

$$\frac{d}{dx}\int_x^{x+1} \log\Gamma(t)dt = \log x, \quad x > 0$$

and as a result

$$\int_x^{x+1} \log\Gamma(t)dt = x\log x - x + c$$

(b) Show as a consequence that $\log\Gamma(n) \approx n\log n$ *as* $n \to \infty$*. In fact, prove that* $\log\Gamma(n) \approx n\log n + O(n)$ *as* $n \to \infty$.

[Hint: Use the fact that $\Gamma(x)$ *is monotonically increasing for all large* x*.]*

Pf:

(a) Given the first derivative in the question part (a), since for $x > 0$, $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt > 0$, then $\log\Gamma(x)$ is defined and continuous on $\mathbb{R} > 0$. Then, by Fundamental Theorem of Calculus, we get:

$$\frac{d}{dx}\int_x^{x+1} \log\Gamma(t)dt = \log\Gamma(x+1) - \log\Gamma(x) = \log(x\Gamma(x)) - \log\Gamma(x)$$

$$= \log(x) + \log\Gamma(x) - \log\Gamma(x) = \log(x)$$

Hence, since the antiderivative of $\log(x)$ is $x\log(x) - x + c$ for arbitrary $c \in \mathbb{R}$, then we get:

$$\int_x^{x+1} \log\Gamma(t)dt = x\log(x) - x + c$$

(b) For all $x > 0$ that's sufficiently large, $\Gamma(x)$ is monotonically increasing, hence for $t \in [x, x+1]$, $\log\Gamma(x+1) \geq \log\Gamma(t) \geq \log\Gamma(x)$. Then, for all $n \in \mathbb{N}$ that's sufficiently large (in particular, $n >> 1$), we have:

$$n\log(n) - n + c = \int_n^{n+1} \log\Gamma(t)dt \geq \int_n^{n+1} \log\Gamma(n)dt = \log\Gamma(n)$$

$$(n-1)\log(n-1) - (n-1) + c = \int_{n-1}^{(n-1)+1} \log\Gamma(t)dt \leq \int_{n-1}^{n} \log\Gamma(n)dt = \log\Gamma(n)$$

(Note: The constant $c \in \mathbb{R}$ can be chosen to satisfy $n\log(n) - n + c = \int_n^{n+1} \log\Gamma(t)dt$).

Then, for the second inequality, after doing some modification to the expression $(n-1)\log(n-1) - (n-1) + c$, we get:

$$n\log(n-1) - \log(n-1) - n + 1 + c = n\log(n) \cdot \frac{\log(n-1)}{\log(n)} - n - \log(n-1) + c$$

As $n \to \infty$, $\frac{\log(n-1)}{\log(n)} \to 1$, then the actual inequality then can be approximated as:

$$n\log(n) - n - \log(n-1) + c \approx (n-1)\log(n-1) - (n-1) + c \leq \log\Gamma(n) \leq n\log(n) - n + c$$

Which, for function $n + \log(n-1) - c$ and $n - c$, both functions are dominated by n as $n \to \infty$ (which can be approximated with $O(n)$), hence, the function is given as:

$$n \log(n) - (n + \log(n-1) - c) \leq \log \Gamma(n) \leq n \log(n) - (n - c)$$

$$\log \Gamma(n) \approx n \log(n) + O(n)$$

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Question 5 *Freitag Chap. IV.2 Exercise 5:*
Let $R = \mathcal{O}(\mathbb{C})$ be the ring of analytic functions in $\mathbb{C}$.

(a) Let $\boldsymbol{a}$ be the set of all entire functions f with the following property. There exists a natural number m, such that f vanishes at all points of $m\mathbb{Z} = \{0, \pm m, \pm 2m, ...\}$. show that $\boldsymbol{a}$ is not finitely generated.

(b) Which are the irreducible elements in $\mathcal{O}(\mathbb{C})$? Which are the prime elements in $\mathcal{O}(\mathbb{C})$?

(c) Which are the inverible elements (i.e. the units) in $\mathcal{O}(\mathbb{C})$?

(d) $\mathcal{O}(\mathbb{C})$ is not a UFD, i.e. there exists elements $\neq 0$ in $\mathcal{O}(\mathbb{C})$ which cannot be written as product of finitely many prime elements.

(e) Any finitely generated ideal in $\mathcal{O}(\mathbb{C})$ with $Af + Bg = 1$.

(For the proof, it can be used that for any discrete subset $S \subset \mathbb{C}$, and for any function $h_0 : S \to \mathbb{C}$ there exists an entire function $h : \mathbb{C} \to \mathbb{C}$ which equals h_0 on S. In fact, more is true, one can even prescribe for any $s \in S$ finitely many Taylor coefficients).

Pf:

(a) Given **a** the ideal described in the problem, we'll prove by contradiction that it's not finitely generated. Suppose the contrary that it is finitely generated, then there exists $f_1, ..., f_n \in \mathbf{a}$, with $\mathbf{a} = (f_1, ..., f_n)$.

For each index $i \in \{1, ..., n\}$, there exists $m_i \in \mathbb{N}$, such that f_i yields 0 for all points in $m_i\mathbb{Z}$. Then, take $m = lcm(m_1, ..., m_n)$, for all $k \in m\mathbb{Z}$, since each index i satisfies $m_i \mid m$, then $m_i \mid k$, showing that $k \in m_i\mathbb{Z}$. Hence, $f_i(k) = 0$.

Since all functions $f \in \mathbf{a} = A_1 f_1 + ... + A_n f_n$ for some $A_1, ..., A_n \in \mathcal{O}(\mathbb{C})$, and every $k \in m\mathbb{Z}$ satisfies $f_i(k) = 0$, regardless of the index i, $f(k) = 0$, hence all f should vanish on the collection $m\mathbb{Z}$.

Yet, here is a counterexample: Consider the function $\sin(\pi z/(2m)) \in \mathcal{O}(\mathbb{C})$: For all $k \in 2m\mathbb{Z}$, since $k = 2ml$ for some $l \in \mathbb{Z}$, then $\sin(\pi k/2m) = \sin(\pi \cdot 2ml/(2m)) = \sin(\pi l) = 0$, so $\sin(\pi z/(2m)) \in \mathbf{a}$. But if we evaluate $m \in m\mathbb{Z}$, we get $\sin(\pi \cdot m/(2m)) = \sin(\pi/2) = 1$, which such function is contained in **a**, while not vanishing on all points of $m\mathbb{Z}$, which contradicts the statement proven before.

Hence, the assumption is false, **a** is not finitely generated.

(b) **Irreducible elements:**

Suppose $f \in \mathcal{O}(\mathbb{C})$ is irreducible, then it's not invertible, which an element is invertible in $\mathcal{O}(\mathbb{C})$ iff it doesn't vanish at all points in $\mathbb{C}$ (will be proven in **Part (c)**). Hence, for f to be non-invertible, $f(a) = 0$ for some $a \in \mathbb{C}$.

Furthermore, if f is irreducible, it cannot have more than one zero, including multiplicity: Suppose f vanishes at $a, b \in \mathbb{C}$ (here, either $b = a$ when a has multiplicity more than 1, or $b \neq a$), then $f(z) = (z-a)(z-b)f_2(z)$ for some $f_2 \in \mathcal{O}(\mathbb{C})$. Hence, since both $(z-a)$ and $(z-b)f_2(z)$ have zeros in $\mathbb{C}$, which are not invertible, f is a product of two non-invertible elements, hence it's not irreducible. So, for f to be irreducible, it must have a unique zero with multiplicity 1.

Lastly, if f has only one zero and with multiplicity 1, it must be irreducible: Suppose it's not irreducible, there exists noninvertible $g, h \in \mathcal{O}(\mathbb{C})$, with $f = gh$. But, since g, h are not invertible, there exists $a, b \in \mathbb{C}$, with $g(a) = 0$, and $h(b) = 0$, hence $g(z) = (z-a)g_1(z)$, $h(z) = (z-b)h_1(z)$ for some $g_1, h_1 \in \mathcal{O}(\mathbb{C})$, or $f(z) = (z-a)g_1(z)(z-b)h_1(z)$. However, this implies f have multiple zeros (counting the case with multiplicity > 1), which is a contradiction. Therefore, f must be irreducible.

With the above statements, we can conclude that f is irreducible iff it has precisely one zero, and with multiplicity 1. So, all irreducible elements are in the form $(z-a)h(z)$, where $h(z) \in \mathcal{O}(\mathbb{C})$ is invertible, which vanishess nowhere on $\mathbb{C}$. (More precisely, all irreducible element is some associates of $(z-a)$ for some $a \in \mathbb{C}$).

Prime elements:

Since all prime elements are irreducible, they must be a subset of the irreducible elements; in this case, we can prove that all irreducible elements are in fact prime.

For all irreducible element in $\mathcal{O}(\mathbb{C})$, it is some associates of $(z-a)$ for some $a \in \mathbb{C}$. Now, suppose $f, g \in \mathcal{O}(\mathbb{C})$ satisfies $(z-a) \mid f(z)g(z)$, then $f(z)g(z) = (z-a)h(z)$ for some $h \in \mathcal{O}(\mathbb{C})$.

Then, since $f(a)g(a) = (a-a)h(a) = 0$, then since $\mathbb{C}$ is a field (in particular, an Integral Domain), either $f(a) = 0$ or $g(a) = 0$. WLOG, suppose $f(a) = 0$, that implies $f(z) = (z-a)f_1(z)$ for some $f_1(z) \in \mathcal{O}(\mathbb{C})$, hence $(z-a) \mid f(z)$. (If $g(a) = 0$ instead, swap $f(z)$ and $g(z)$ then the statement still holds).

Since $(z-a) \mid f(z)g(z)$ implies $(z-a) \mid f(z)$ or $(z-a) \mid g(z)$, then $(z-a)$ is in fact prime, which proves that all irreducible elements are prime elements also in $\mathcal{O}(\mathbb{C})$.

(c) We'll prove that $f \in \mathcal{O}(\mathbb{C})$ is invertible, iff it doesn't vanish at all points in $\mathbb{C}$.

$\Longrightarrow$: Suppose f is invertible, then there exists $h \in \mathcal{O}(\mathbb{C})$, where $f(z)h(z) \equiv 1$. Hence, for all $a \in \mathbb{C}$, since $f(a)g(a) = 1$, then $f(a) \neq 0$, showing that f doesn't vanish for all $a \in \mathbb{C}$.

$\Longleftarrow$: Suppose f doesn't vanish for all $a \in \mathbb{C}$, then $\frac{1}{f(z)}$ is well-defined and analytic on the whole $\mathbb{C}$, and for all $a \in \mathbb{C}$, $f(a) \cdot \frac{1}{f(a)} = 1$, hence $f(z) \cdot \frac{1}{f(z)} \equiv 1$, showing that f is invertible.

(d) Consider the following function $\sin(\pi z) \in \mathcal{O}(\mathbb{C})$, we'll prove by contradiction that the above function can't be factored into finitely many prime numbers.

Suppose it can, there exists prime elements $f_1, ..., f_n \in \mathcal{O}(\mathbb{C})$, such that $\sin(\pi z) = \prod_{i=1}^n f_i(z)$. Since in **Part (b)**, we've proven that each prime element (which is irreducible) has precisely one zero with multiplicity 1, then for each index $i \in \{1, ..., n\}$, there exists unique $a_i \in \mathbb{C}$, with $f_i(a_i) = 0$, which $f_i(z) = (z - a_i)\bar{f}_i(z)$, where $\bar{f}_i \in \mathbb{O}(\mathbb{C})$ has no zeroes, hence it is invertible.

So, $\sin(\pi z)$ can be expressed as:

$$\sin(\pi z) = \prod_{i=1}^n f_i(z) = \left(\prod_{i=1}^n (z - a_i)\right)\left(\prod_{i=1}^n \bar{f}_i(z)\right)$$

Where, the second product is formed by invertible elements, hence it is also invertible (which has no zeros). Then, it implies that $\sin(\pi z)$ has zeros $a_1, ..., a_n$, only n zeros (counting multiplicity), which contradicts the fact that $\sin(\pi z)$ vanishes at all points in $\mathbb{Z}$.

So, the above function is an example that can't be factored into finitely many prime elements, showing that $\mathcal{O}(\mathbb{C})$ is not a UFD.

(e) To show that any finitely generated ideal is principal, we'll show some statements in the following order:

Proposition 1 ***Two nonero functions with no common zeros generate a unit ideal.***

Suppose nonzero $f, g \in \mathcal{O}(\mathbb{C})$ have no common zeros. We can first assume both f, g has zeros (if one of them has no zeros, WLOG, say f has no zeros, then $\frac{1}{f} \cdot f + 0 \cdot g \equiv 1 \in (f, g)$, so (f, g) is a unit ideal). Which, the collection of zeros for f and g must be discrete, since nonconstant analytic function must have isolated zeros.

Let the discrete subsets $S_f = \{f_i \mid i \in I\}$, $S_g = \{g_j \mid j \in J\}$ be the collections of zeros of f and g respectively (which by assumption, $S_f \cap S_g = \emptyset$). Now, for each $i \in I, j \in J$, let $n_i, m_j \in \mathbb{N}$ be the corresponding multiplicity of f_i, g_j respectively. Our goal is to construct two functions $A, B \in \mathcal{O}(\mathbb{C})$, with $Af + Bg \equiv 1$.

Since both S_f, S_g are discrete, $S_f \sqcup S_g$ is also discrete (since for all $z \in \mathbb{C}$, there exists radius $r_1, r_2 > 0$, with $B_{r_1}(z) \setminus \{z\} \cap S_f = B_{r_2}(z) \setminus \{z\} \cap S_g = \emptyset$, then choose $r = \min\{r_1, r_2\} > 0$, $B_r(z) \setminus \{z\} \cap S_f = B_r(z) \setminus \{z\} \cap S_g = \emptyset$, showing that $S_f \sqcup S_g$ has no limit points). Then, by the property given in the question part (e), take a function $h_0 : S_f \sqcup S_g \to \mathbb{C}$ by $h_0(f_i) = 0$ and $h_0(g_j) - 1 = 0$ for all $i \in I, j \in J$ (which h_0 is well-defined, since S_f, S_g are disjoint), we know there exists an entire analytic function $h \in \mathcal{O}(\mathbb{C})$, such that $h|_{S_f \sqcup S_g} = h_0$, and each $f_i \in S_f$ has multiplicity n_i for function $h(z)$, while each $g_j \in S_g$ has multiplicity m_j for function $h(z) - 1$.

Now, consider $B(z) = \frac{1-h(z)}{g(z)}$, and $A(z) = \frac{h(z)}{f(z)}$:

Since B is only not well-defined at the zeros of g, which just needs to resolve the singularity at all points of S_g. However, for all $g_j \in S_g$, it's a zero with multiplicity m_j for g, and by the above construction, it's a zero with multiplicity m_j also for the function $h(z) - 1$. Hence, $h(z) - 1 = (z - g_j)^{m_j}\bar{h}(z)$, and $g(z) = (z - g_j)^{m_j}\bar{g}(z)$, for $\bar{h}, \bar{g} \in \mathcal{O}(\mathbb{C})$ that are not vanishing at g_j. Then, consider the following limit:

$$\lim_{z \to g_j} (z - g_j)B(z) = \lim_{z \to g_j} (z - g_j) \cdot \frac{1 - h(z)}{g(z)} = \lim_{z \to g_j} (z - g_j) \cdot \frac{-(z - g_j)^{m_j}\bar{h}(z)}{(z - g_j)^{m_j}\bar{g}(z)}$$

$$= \lim_{z \to g_j} -(z - g_j) \cdot \frac{\bar{h}(z)}{\bar{g}(z)} = 0$$

Hence, the above limit provides 0, implies that $B(z)$ has a removable singularity at g_j. So, $B(z)$ can be extended analytically onto the whole $\mathbb{C}$.

On the other hand, $A(z)$ is only not well-defined at the zeros of f, which just needs to resolve the singularity at all points of S_f. Again, for all $f_i \in S_f$, it's a zero with multiplicity n_i for f, and again by the construction of h, it's a zero with multiplicity n_i also for function h. Then, $h(z) = (z - f_i)^{n_i}\tilde{h}(z)$, and $f(z) = (z - f_i)^{n_i}\tilde{f}(z)$ for $\tilde{h}, \tilde{f} \in \mathcal{O}(\mathbb{C})$ that are not vanishing at f_i. Hence, the following limit provides:

$$\lim_{z \to f_i}(z - f_i)A(z) = \lim_{z \to f_i}(z - f_i) \cdot \frac{h(z)}{f(z)} = \lim_{z \to f_i}(z - f_i) \cdot \frac{(z - f_i)^{n_i}\tilde{h}(z)}{(z - f_i)^{n_i}\tilde{f}(z)}$$

$$= \lim_{z \to f_i}(z - f_i) \cdot \frac{\tilde{h}(z)}{\tilde{f}(z)} = 0$$

Hence, the above limit is 0 implies that $A(z)$ has a removable singularity at f_i, showing that $A(z)$ can be extended analytically onto the whole $\mathbb{C}$.

Which, if evaluate $Af + Bg$, we get:

$$A(z)f(z) + B(z)g(z) = \frac{h(z)}{f(z)}f(z) + \frac{1 - h(z)}{g(z)}g(z) = h(z) + (1 - h(z)) \equiv 1$$

Hence, $1 \in (f, g)$, showing that f, g generates a unit ideal.

Proposition 2 ***Any two nonzero functions generate a principal ideal.***

Given arbitrary nonzero $f, g \in \mathcal{O}(\mathbb{C})$, which we can assume they share some common zeros (since we've shown above, that two nonzero functions with no common zero generate a unit ideal).

Let discrete subset $S_h = \{h_i \mid i \in I\}$ denotes the common zeros of f, g. For each $i \in I$, let $n_i, m_i \in \mathbb{N}$ be the multiplicity of h_i as a zero of f and g respectively, then $f(z) = (z - h_i)^{n_i} f_i(z),\ g(z) = (z - h_i)^{m_i} g_i(z)$ for $f_i, g_i \in \mathcal{O}(\mathbb{C})$ that are not vanishing at h_i. Which, define $g_i = \min\{n_i, m_i\}$ for each index $i \in I$, and construct a Weierstrass Product function $h(z) \in \mathcal{O}(\mathbb{C})$, such that $h(h_i) = 0$, and h_i has multiplicity g_i for all $i \in I$ (so, h only has zeros on S_h).

Now, consider the function $\frac{f}{h}$, $\frac{g}{h}$: They're only not defined on the zeros of h (namely the set S_h). Which, for all $i \in I$, we know $f(z) = (z - h_i)^{n_i} f_i(z)$, and $g(z) = (z - h_i)^{m_i} g_i(z)$; also, since h_i is a zero with multiplity $g_i = \min\{m_i, n_i\}$ for h, then $h(z) = (z - h_i)^{g_i}\bar{h_i}(z)$ for $\bar{h_i} \in \mathcal{O}(\mathbb{C})$ that's not vanishing at h_i. Then, evaluate the following two limits, we get:

$$\lim_{z \to h_i}(z - h_i)\frac{f(z)}{h(z)} = \lim_{z \to h_i}(z - h_i)\frac{(z - h_i)^{n_i} f_i(z)}{(z - h_i)^{g_i}\bar{h}_i(z)} = \lim_{z \to h_i}(z - h_i)^{n_i - g_i + 1}\frac{f_i(z)}{\bar{h}_i(z)} = 0$$

$$\lim_{z \to h_i}(z - h_i)\frac{g(z)}{h(z)} = \lim_{z \to h_i}(z - h_i)\frac{(z - h_i)^{m_i} g_i(z)}{(z - h_i)^{g_i}\bar{h}_i(z)} = \lim_{z \to h_i}(z - h_i)^{m_i - g_i + 1}\frac{g_i(z)}{\bar{h}_i(z)} = 0$$

(Note: since $n_i, m_i \geq g_i$, then $(n_i - g_i + 1), (m_i - g_i + 1) \geq 1$).

Hence, both $\frac{f}{h}, \frac{g}{h}$ have removable singularity at h_i, showing that both functions can be extended analytically onto $\mathbb{C}$.

On the other hand, since $g_i = \min\{n_i, m_i\}$, then $g_i = n_i$ or $g_i = m_i$ (WLOG, say $g_i = n_i$), then $\frac{f}{h}$ is given as:

$$\frac{f(z)}{h(z)} = \frac{(z-h_i)^{n_i} f_i(z)}{(z-h_i)^{g_i} \bar{h}_i(z)} = \frac{f_i(z)}{\bar{h}_i(z)}$$

Notice that both $f_i, \bar{h_i}$ are not vanishing at $h_i \in S_h$. Hence, $\frac{f}{h}$ doesn't vanish at h_i. (Same statement applies to $\frac{g}{h}$ if $g_i = m_i$).

So, for all $i \in I$, h_i is not vanishing for at least one function in $\frac{f}{h}$ and $\frac{g}{h}$, showing that $\frac{f}{h}, \frac{g}{h}$ sharing no common zeros (since the only possible common zeros are the zeros for both f and g, and we verified that each common zero for f, g is nonvanishing for one of the functions $\frac{f}{h}, \frac{g}{h}$).

Hence, by **Proposition 1**, $\frac{f}{h}, \frac{g}{h}$ generates a unit ideal, there exists $A, B \in \mathcal{O}(\mathbb{C})$ with $A\frac{f}{h} + A\frac{g}{h} = 1$, or $h = Af + Bg$. Therefore, the ideal $(h) \subseteq (f, g)$, while $f = \frac{f}{h} \cdot h$ and $g = \frac{g}{h} \cdot h$, showing that $f, g \in (h)$, or $(f, g) \subseteq (h)$. Hence, $(f, g) = (h)$, showing that f, g generates a principal ideal.

Finally, with the above statements, we can use induction to prove that **All finitely generated ideal in $\mathcal{O}(\mathbb{C})$ is principal**.

For the ideals generated by $n = 2$ functions, we've proven that in **Proposition 2**.

Now, suppose for given $n \in \mathbb{N}$, any $f_1, ..., f_n \in \mathcal{O}(\mathbb{C})$ generates a principal ideal (i.e. there exists $h \in \mathcal{O}(\mathbb{C})$, with $(h) = (f_1, ..., f_n)$), then for the case $(n+1)$, any $f_1, ..., f_n, f_{n+1} \in \mathcal{O}(\mathbb{C})$, since $(f_1, ..., f_n) = (h)$ for some $h \in \mathcal{O}(\mathbb{C})$, then $(f_1, ..., f_n, f_{n+1}) = (h, f_{n+1})$.

Which again by **Proposition 2**, there exists $\bar{h} \in \mathcal{O}(\mathbb{C})$, with $(h, f_{n+1}) = (\bar{h})$, hence this proves that $(f_1, ..., f_n, f_{n+1}) = (\bar{h})$, the ideal is principal.

This completes the induction, shows that all finitely generated ideal is in fact principal.