

Math CS 122B HW3

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1 (not done)

Question 1 Freitag Chap. IV.3 Exercise 3:

Show:

$$\frac{\pi}{\cos(\pi z)} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 - 4z^2}$$

and derive from this

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Pf:

We'll complete this by the following trigonometric identity, and the expression of $\frac{\pi}{\sin(\pi\zeta)}$ under partial fraction series:

$$\begin{aligned} \cos(\zeta) &= \sin\left(\frac{\pi}{2} - \zeta\right) \\ \frac{\pi}{\sin(\pi\zeta)} &= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\zeta - n} + \frac{1}{\zeta + n} \right) \end{aligned}$$

Then, $\frac{\pi}{\cos(\pi z)}$ can be expressed as:

$$\begin{aligned} \frac{\pi}{\cos(\pi z)} &= \frac{\pi}{\sin(\pi/2 - \pi z)} = \frac{\pi}{\sin(\pi(1/2 - z))} = \frac{1}{1/2 - z} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{(1/2 - z) - n} + \frac{1}{(1/2 - z) + n} \right) \\ &= \frac{2}{1 - 2z} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{(1 - 2z) - 2n} + \frac{2}{(1 - 2z) + 2n} \right) \\ &= \frac{2}{1 - 2z} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{-(2n - 1) - 2z} + \frac{2}{(2n + 1) - 2z} \right) \end{aligned}$$

Which, for all $z \notin \frac{1}{2} + \mathbb{Z}$, if we view the partial sum of the above series, we get:

$$\begin{aligned} \forall N \in \mathbb{N}, N \geq 3, \quad S_N &= \frac{2}{1 - 2z} + \sum_{n=1}^N (-1)^n \left(\frac{2}{-(2n - 1) - 2z} + \frac{2}{(2n + 1) - 2z} \right) \\ &= \frac{2}{1 - 2z} + \frac{(-1)^1 \cdot 2}{-(2 \cdot 1 - 1) - 2z} + \sum_{n=2}^N \frac{(-1)^n \cdot 2}{-(2n - 1) - 2z} + \sum_{n=1}^{N-1} \frac{(-1)^n \cdot 2}{(2n + 1) - 2z} + \frac{(-1)^N \cdot 2}{(2N + 1) - 2z} \\ &= \frac{2}{1 - 2z} - \frac{2}{-1 - 2z} + \sum_{n=1}^{N-1} \frac{(-1)^{n+1} \cdot 2}{-(2(n+1) - 1) - 2z} + \sum_{n=1}^{N-1} \frac{(-1)^n \cdot 2}{(2n + 1) - 2z} + \frac{(-1)^N \cdot 2}{(2N + 1) - 2z} \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{2}{1-2z} - \frac{2}{-1-2z} \right) + \sum_{n=1}^{N-1} (-1)^n \left(\frac{2}{(2n+1)-2z} - \frac{2}{-(2n+1)-2z} \right) + \frac{(-1)^N \cdot 2}{(2N+1)-2z} \\
&= \sum_{n=0}^{N-1} (-1)^n \left(\frac{2}{(2n+1)-2z} - \frac{2}{-(2n+1)-2z} \right) + \frac{(-1)^N \cdot 2}{(2N+1)-2z} \\
&= \sum_{n=0}^{N-1} 2 \cdot (-1)^n \cdot \frac{(-(2n+1)-2z) - ((2n+1)-2z)}{4z^2 - (2n+1)^2} + \frac{(-1)^N \cdot 2}{(2N+1)-2z} \\
&= 2 \sum_{n=0}^{N-1} (-1)^n \cdot \frac{-2(2n+1)}{4z^2 - (2n+1)^2} + \frac{(-1)^N \cdot 2}{(2N+1)-2z} \\
&= 4 \sum_{n=0}^{N-1} (-1)^n \cdot \frac{(2n+1)}{(2n+1)^2 - 4z^2} + \frac{(-1)^N \cdot 2}{(2N+1)-2z}
\end{aligned}$$

So, if the above partial sum converges, we get:

$$\begin{aligned}
\lim_{N \rightarrow \infty} s_N &= \lim_{N \rightarrow \infty} 4 \sum_{n=0}^{N-1} (-1)^n \cdot \frac{(2n+1)}{(2n+1)^2 - 4z^2} + \frac{(-1)^N \cdot 2}{(2N+1)-2z} \\
&= 4 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 - 4z^2}
\end{aligned}$$

Which is the desired result. Now, the only thing remain is to argue the convergence of the series.

Question 2 Freitag Chap. IV.3 Exercise 4:

Find a meromorphic function f in \mathbb{C} which has simple poles in

$$S = \{\sqrt{n} \mid n \in \mathbb{N}\}$$

with corresponding residues $\text{Res}(f; \sqrt{n}) = \sqrt{n}$, and is analytic in $\mathbb{C} \setminus S$.

Pf:

With the given condition, one could guess that for each $n \in \mathbb{N}$, at $z = \sqrt{n}$, the principal part is described using $\frac{\sqrt{n}}{z - \sqrt{n}}$ (which is a simple pole, and has residue $\lim_{z \rightarrow \sqrt{n}} (z - \sqrt{n}) \frac{\sqrt{n}}{z - \sqrt{n}} = \sqrt{n}$). However, the series of such function diverges, hence we need to do some modification.

For all $z \in \mathbb{C}$, there exists $N \in \mathbb{N}$, such that $n \geq N$ implies $\frac{|z|}{\sqrt{n}} \leq \frac{1}{2}$ (which, we're working within the compact disk $|z| \leq \frac{\sqrt{N}}{2}$). Then, for $n \geq N$, since $\frac{z}{\sqrt{n}}$ is within the radius of convergence of the geometric series, we get:

$$\frac{\sqrt{n}}{z - \sqrt{n}} = \frac{-1}{1 - z/\sqrt{n}} = -\sum_{k=0}^{\infty} \left(\frac{z}{\sqrt{n}}\right)^k$$

Then, if we subtract out the terms up to degree 3, we get the following:

$$\frac{\sqrt{n}}{z - \sqrt{n}} + \sum_{k=0}^3 \left(\frac{z}{\sqrt{n}}\right)^k = -\sum_{k=0}^{\infty} \left(\frac{z}{\sqrt{n}}\right)^k + \sum_{k=0}^3 \left(\frac{z}{\sqrt{n}}\right)^k = \sum_{k=4}^{\infty} \left(\frac{z}{\sqrt{n}}\right)^k = \left(\frac{z}{\sqrt{n}}\right)^4 \sum_{k=0}^{\infty} \left(\frac{z}{\sqrt{n}}\right)^k$$

Which, compare the modulus, we get the following inequality:

$$\begin{aligned} \left| \frac{\sqrt{n}}{z - \sqrt{n}} + \sum_{k=0}^3 \left(\frac{z}{\sqrt{n}}\right)^k \right| &= \left| \left(\frac{z}{\sqrt{n}}\right)^4 \sum_{k=0}^{\infty} \left(\frac{z}{\sqrt{n}}\right)^k \right| \leq \frac{|z|^4}{n^2} \sum_{k=0}^{\infty} \left| \frac{z}{\sqrt{n}} \right|^k \\ &\leq \frac{(\sqrt{N}/2)^4}{n^2} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{N^2/(16)}{n^2} \cdot 2 = \frac{N^2}{8n^2} \end{aligned}$$

Hence, the following series of functions converges uniformly within the compact disk $|z| \leq \frac{\sqrt{N}}{2}$:

$$\left| \sum_{n=N}^{\infty} \left(\frac{\sqrt{n}}{z - \sqrt{n}} + \sum_{k=0}^3 \left(\frac{z}{\sqrt{n}}\right)^k \right) \right| \leq \sum_{n=N}^{\infty} \left| \frac{\sqrt{n}}{z - \sqrt{n}} + \sum_{k=0}^3 \left(\frac{z}{\sqrt{n}}\right)^k \right| \leq \sum_{n=N}^{\infty} \frac{N^2}{8n^2} < \infty$$

So, we can conclude that $\sum_{n=1}^{\infty} \left(\frac{\sqrt{n}}{z - \sqrt{n}} + \sum_{k=0}^3 \left(\frac{z}{\sqrt{n}}\right)^k \right)$ converges normally on $\mathbb{C} \setminus \{\sqrt{n}\}_{n \in \mathbb{N}}$.

Since each \sqrt{n} , $n \in \mathbb{N}$ has the principal part given by $\frac{\sqrt{n}}{z - \sqrt{n}}$, while this principal part satisfies the desired properties, then this partial fraction series (which is a meromorphic function in this case) is a solution of the principal part distribution.

3 (Not done)

Question 3 Freitag Chap. IV.3 Exercise 5:

Prove the following refinement of the Mittag-Leffler Theorem:

Theorem 1 Mittag-Leffler *Let $S \subset \mathbb{C}$ be a discrete subset. Then one can construct an analytic function $f : \mathbb{C} \setminus S \rightarrow \mathbb{C}$ which has at any $s \in S$, not only given principal parts but also finitely many Laurent coefficients for nonnegative indices.*

Pf:

Question 4 *Stein and Shakarchi Chap. 8 Problem 7: (Too long I don't want to copy it)*

Pf:

(a) **Expansion satisfies $r_{f(K)} \geq r_K$:**

For all radius $0 < r < r_K$, the circle c_r (with radius r) is fully contained in K (since $\mathbb{D}(0, r_K) \subseteq K$). Since f is an expansion, then for all $z \neq 0$, $f(z) \neq 0$ (since it is injective, and $f(0) = 0$). Hence, by argument principle, the following integral shows the number of zeros enclosed by curve c_r :

$$\frac{1}{2\pi i} \int_{c_r} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{f(c_r)} \frac{1}{w} dw$$

Since c_r encloses only one zero (enclosing the origin, the only point that gets mapped to 0 by f), then the above integral yields value 1. This also implies that $f(c_r)$ is a closed curve satisfying $n(f(c_r), 0) = 1$ (a simple closed curve enclosing region with 0).

On the other hand, $f(c_r)$ is fully contained in the range $f(K)$, while $f(K)$ is also simply connected, hence the curve $f(c_r)$ is homologous to 0, the open region enclosed by $f(c_r)$ (denoted as D) is also fully contained in $f(K)$.

Then, since $f(c_r)$ is closed, there exists $w_0 \in f(c_r)$ that yields a minimum modulus; which, if consider $|w_0|$ as a radius, since $\mathbb{D}(0, |w_0|)$ again contains no points in $f(c_r)$ (since $z \in \mathbb{D}(0, |w_0|)$ satisfies $|z| < |w_0|$, while all $w \in f(c_r)$ satisfies $|w_0| \leq |w|$), then $\mathbb{D}(0, |z_0|) \subseteq D \subseteq f(K)$, hence $|w_0| \leq r_{f(K)}$.

However, if consider the point $z_0 \in K$ that satisfies $f(z_0) = w_0$ (which, $z_0 \in c_r$, so $|z_0| = r$), then we have the following inequality:

$$r = |z_0| < |f(z_0)| = |w_0| \leq r_{f(K)}$$

Hence, all $0 < r < r_K$ satisfies $r < r_{f(K)}$, which implies that $r_K \leq r_{f(K)}$.

Expansion satisfies $|f'(0)| > 1$:

Since $f : K \rightarrow \mathbb{D}$ is an expansion, implies that $f(0) = 0$, then $f(z) = zg(z)$ for some analytic $g : K \rightarrow \mathbb{C}$.

Now, if consider the fact that all $z \in K \setminus \{0\}$ satisfies $|f(z)| > |z|$, we get the following inequality:

$$|g(z)| = \frac{|zg(z)|}{|z|} = \frac{|f(z)|}{|z|} > \frac{|z|}{|z|} = 1$$

Hence, if consider $f'(0)$ using limit definition, we get:

$$|f'(0)| = \left| \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z - 0} \right| = \left| \lim_{z \rightarrow 0} \frac{zg(z)}{z} \right| = \left| \lim_{z \rightarrow 0} g(z) \right| \geq 1$$

Now, we'll prove that $|f'(0)| = |g(0)| \neq 1$: Suppose the contrary that $|g(0)| = 1$, since the above statement implies that all $z \in K$ (including $z = 0$) satisfies $|g(z)| \geq 1$, then $g(z) \neq 0$ in K , hence $1/g : K \rightarrow \mathbb{C}$ is a well-defined analytic function, satisfying $|1/g(z)| \leq 1$.

However, since K is an open set, while g is nonconstant (if g is constant, and $|g(0)| = 1$, then $f(z) = zg(z) = g(0)z$, which $|f(z)| = |g(0)z| = |z|$, contradicting the fact that f is an expansion), then $|1/g(z)|$ shouldn't obtain a maximum on any point $z \in K$. Yet, since we assume $g(0) = 1$, while

$|1/g(z)| \leq 1$, hence $|1/g(z)| \leq |1/g(0)|$ for all $z \in K$, showing that $0 \in K$ is in fact a maximum, which violates the maximum principle.

Hence, our assumption must be false, $|g(0)| \neq 1$, showing that $|g(0)| = |f'(0)| > 1$.

- (b) Given Koebe domain K_0 , and a sequence of expansion $\{f_0, f_1, \dots\}$ satisfying $K_{n+1} = f_n(K_n) \subseteq \mathbb{D}$, define $F_n : K_0 \rightarrow \mathbb{D}$ by $F_n = f_n \circ \dots \circ f_0$.

F_n is an expansion:

We'll show this by induction (Note: expansion f satisfies $f(0) = 0$).

First, for $n = 1$, $F_1 = f_1 \circ f_0$, which for all $z \in K_0 \setminus \{0\}$, based on the fact that f_0, f_1 are expansions, it satisfies:

$$|F_1(z)| = |f_1(f_0(z))| > |f_0(z)| > |z|$$

Also, $F_1(0) = f_1(f_0(0)) = f_1(0) = 0$. Which, F_1 is an expansion.

Now, for given $n \in \mathbb{N}$, suppose F_n is an expansion. Then, $F_{n+1} = f_{n+1} \circ (f_n \circ \dots \circ f_0) = f_{n+1} \circ F_n$. Again, since both f_{n+1}, F_n are expansions, all $z \in K_0 \setminus \{0\}$ satisfies:

$$|F_{n+1}(z)| = |f_{n+1}(F_n(z))| > |F_n(z)| > |z|$$

Also, $F_{n+1}(0) = f_{n+1}(F_n(0)) = f_{n+1}(0) = 0$. Hence, F_{n+1} is an expansion, and this completes the induction.

Formula for $F'_n(0)$:

Again, we can show by induction, that $F'_n(0) = \prod_{k=0}^n f'_k(0)$.

First, for $n = 1$, $F_1 = f_1 \circ f_0$, then by chain rule, $F'_1(0) = f'_1(f_0(0)) \cdot f'_0(0) = f'_1(0) \cdot f'_0(0)$.

Now, suppose for given $n \in \mathbb{N}$, $F'_n(0) = \prod_{k=0}^n f'_k(0)$, then for $F_{n+1} = f_{n+1} \circ F_n$ satisfies:

$$F'_{n+1}(0) = f'_{n+1}(F_n(0)) \cdot F'_n(0) = f'_{n+1}(0) \cdot \prod_{k=0}^n f'_k(0) = \prod_{k=0}^{n+1} f'_k(0)$$

Which, this proves the case for $(n + 1)$, which completes the induction.

Limit of $|f'_n(0)|$:

First, based on the above formula of $F'_n(0)$, since for all $n \in \mathbb{N}$, f_{n+1} is an expansion, then $|f'_{n+1}(0)| > 0$ (based on **part (a)**), hence, the following is true:

$$|F'_{n+1}(0)| = \left| \prod_{k=0}^{n+1} f'_k(0) \right| = |f'_{n+1}(0)| \cdot \prod_{k=0}^n |f'_k(0)| > \prod_{k=0}^n |f'_k(0)| = \left| \prod_{k=0}^n f'_k(0) \right| = |F'_n(0)|$$

This proves that $\{|F'_n(0)|\}_{n \in \mathbb{N}}$ is a strictly increasing sequence.

Also, recall that since $\mathbb{D}(0, r_{K_0}) \subseteq K_0$, for each n , define $\bar{F}_n : \mathbb{D} \rightarrow \mathbb{D}$ by $\bar{F}_n(z) = F_n(r_{K_0}z)$ (Note: each $z \in \mathbb{D}$, since $|z| < 1$, then $|r_{K_0}z| < r_{K_0}$, hence $r_{K_0}z \in \mathbb{D}(0, r_{K_0}) \subseteq K_0$). Since \bar{F}_n is an analytic map

from \mathbb{D} to \mathbb{D} , and it satisfies $\bar{F}_n(0) = F_n(r_{K_0} \cdot 0) = 0$, then by Schwarz Lemma, $|\bar{F}'_n(0)| \leq 1$. So, we get the following:

$$\bar{F}'_n(z) = r_{K_0} F'_n(r_{K_0} z), \quad |\bar{F}'_n(0)| = r_{K_0} |F'_n(0)| \leq 1, \quad |F'_n(0)| \leq \frac{1}{r_{K_0}}$$

This proves that $\{|F'_n(0)|\}_{n \in \mathbb{N}}$ is bounded above by $\frac{1}{r_{K_0}} > 0$ (Note: since K_0 is open, $r_{K_0} > 0$).

Hence, since the sequence is strictly increasing while bounded from above, $\lim_{n \rightarrow \infty} |F'_n(0)| = L \in \mathbb{R}$.

Then, since the limit exists, while $|F'_n(0)|$ is based on products of $|f'_k(0)|$, then:

$$\lim_{n \rightarrow \infty} |F'_n(0)| = \lim_{n \rightarrow \infty} \prod_{k=0}^n |f'_k(0)| = L \in \mathbb{R} \implies \lim_{n \rightarrow \infty} |f'_n(0)| = 1$$

- (c) Given the family of expansions $\{F_n : K_0 \rightarrow \mathbb{D} \mid n \in \mathbb{N}\}$ with $\lim_{n \rightarrow \infty} r_{K_n} = 1$, unfortunately we cannot conclude that the sequence of functions converges (if F_n converges to F , since one can add a constant rotation of radians $\pi/2$ in between f_n and f_{n+1} for each $n \in \mathbb{N}$, which for all $z \in K_0 \setminus \{0\}$, since $F_n(z)$ now becomes a sequence that's constantly rotating, while the modulus $|F_n(z)|$ is still increasing, then the modified sequence no longer converges).

However, based on **Montel's Theorem**, since the family of expansions are uniformly bounded (for all $z \in K_0$, all $n \in \mathbb{N}$ satisfies $|F_n(z)| < 1$, because $F_n(z) \in \mathbb{D}$), then there exists a subsequence $\{F_{n_k}\}_{k \in \mathbb{N}}$ that converges locally uniformly to some function $F : K_0 \rightarrow \mathbb{D}$ (i.e. on any compact subsets of K_0 , the sequence of functions converges uniformly).

Now, since $\lim_{k \rightarrow \infty} r_{K_{n_k}} = 1$ (subsequential limit agrees with the sequential limit if the original sequence converges), then $r_{f(K_0)} \geq 1$: For all $0 < r < 1$, because of the limit, there exists $K \in \mathbb{N}$, such that $k \geq K$ implies $r < r_{K_{n_k}} \leq 1$. Then, based on the result in **part (a)**, we know $r_{K_{n_k}} \leq r_{f_{n_k}(K_{n_k})}$, which $\mathbb{D}(0, r) \subseteq \mathbb{D}(0, r_{f_{n_k}(K_{n_k})}) = \mathbb{D}(0, r_{F_{n_k}(K_0)}) \subseteq F_{n_k}(K_0)$ (Note: recall that f_{n_k} has the image being the same as F_{n_k}). Hence, as F_{n_k} converges to F , this implies that $\mathbb{D}(0, r) \subseteq F(K_0)$, showing that $r \leq r_{F(K_0)}$. Since for all $0 < r < 1$, $r \leq r_{F(K_0)}$, then $1 \leq r_{F(K_0)}$. So, this implies that $\mathbb{D} \subseteq \mathbb{D}(0, r_{F(K_0)}) \subseteq F(K_0)$, showing that F is surjective.

Moreover, since the collection $\{F_{n_k}\}_{k \in \mathbb{N}}$ are a sequence of expansions (analytic injective functions) that converges locally uniformly, then by **Hurwitz's Theorem**, the limit is either constant or injective; however, since it is surjective onto \mathbb{D} , the map F is not constant, hence it must be injective.

Because F is both injective and surjective while being analytic, it is a conformal map.

- (d) Given Koebe domain K , with $\alpha \in \partial K$ such that $|\alpha| = r_K > 0$, and $\beta \in \mathbb{D}$ such that $\beta^2 = \alpha$.

If $|\alpha| = 1$, this implies that $r_K = 1$, or $\mathbb{D} \subseteq K$, which $K = \mathbb{D}$, so there's no need for constructing a conformal map. Hence, can assume $|\alpha| < 1$ (consequently, since $\beta^2 = \alpha$, then $|\beta| < 1$ also).

Then, the following transformation is a bijection from $\mathbb{D} \rightarrow \mathbb{D}$ (in fact, a conformal map):

$$\psi_\alpha : \mathbb{D} \rightarrow \mathbb{D}, \quad \psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$

If we restrict the domain to be K , since K is open, then $K = K^\circ$. Which $K^\circ \cap \partial K = \emptyset$, showing that $\alpha \notin K$.

Then, if consider $\psi_\alpha(K)$, since for $\psi_\alpha(z) = 0$, we need $\alpha - z = 0$, or $z = \alpha$, then since $\alpha \notin K$, then $0 \notin \psi_\alpha(K)$.

Hence, because ψ_α is a conformal map, K is open and simply connected, implies that $\psi_\alpha(K)$ is also open and simply connected. Therefore, it's possible to define a single-valued branch of square root on $\psi_\alpha(K)$, or $S : \psi_\alpha(K) \rightarrow \mathbb{D}$ that satisfies:

$$\forall w \in \psi_\alpha(K), \quad (S(w))^2 = w, \quad S(\alpha) = \beta$$

(Note: Since $\psi_\alpha(0) = \alpha$, then $\alpha \in \psi_\alpha(K)$, hence we can define the branch such that $S(\alpha) = \beta$).

(Note 2: Because all $w \in \psi_\alpha(K) \subseteq \mathbb{D}$, then $|w| < 1$; hence, $|S(w)|^2 = |w| < 1$ implies $|S(w)| < 1$, showing that $S(\psi_\alpha(K)) \subseteq \mathbb{D}$).

Also, since $\alpha \in \partial K$ is a limit point of K (since $|\alpha| = r_K$, all $0 < r < 1$ satisfies $|r\alpha| < |\alpha| = r_K$, which $r\alpha \in K$; hence for all $\epsilon > 0$, choose $r > 0$ such that $1 - \epsilon < r < 1$, then $|\alpha - r\alpha| = |\alpha| \cdot |1 - r| < r_K \epsilon < \epsilon$, showing that $r\alpha \in B_\epsilon(\alpha)$, hence α is a limit point), then we can find a sequence $(c_n)_{n \in \mathbb{N}} \subset K$ converging to α . By continuity of ψ_α , we get:

$$\lim_{n \rightarrow \infty} \psi_\alpha(c_n) = \psi_\alpha(\alpha) = 0, \quad \lim_{n \rightarrow \infty} |S(\psi_\alpha(c_n))^2| = \lim_{n \rightarrow \infty} |\psi_\alpha(c_n)| = 0$$

Hence, $\lim_{n \rightarrow \infty} \sqrt{|S(\psi_\alpha(c_n))^2|} = \lim_{n \rightarrow \infty} |S(\psi_\alpha(c_n))| = 0$, showing that $\lim_{n \rightarrow \infty} S(\psi_\alpha(c_n)) = 0$. So, without considering if the extension is analytic or not, we can define $S(0) = 0$. Which, the desired square root is defined.

The expansion based on α and β :

Consider $f = \psi_\beta \circ S \circ \psi_\alpha : K \rightarrow \mathbb{D}$.

To prove that it is an expansion, we'll first prove that $f(0) = 0$:

$$f(0) = \psi_\beta \circ S \circ \psi_\alpha(0) = \psi_\beta \circ S(\alpha) = \psi_\beta(\beta) = 0$$

(Note: the mobius transformation ψ_α swaps α and 0, while we define $S(\alpha) = \beta$).

Then, we'll consider its inverse: For ψ_α as an automorphism on \mathbb{D} , its inverse is itself (the property of mobius transformation given that $|\alpha| < 1$), and the same logic applies to ψ_β . Now, if we restrict the codomain of f to $f(K)$, then it becomes a bijective function (since $\psi_\alpha, S, \psi_\beta$ are all injective), then if we consider $g(z) = z^2$, the composition $h = \psi_\alpha \circ g \circ \psi_\beta : f(K) \rightarrow \mathbb{C}$ becomes a left inverse of f :

$$\begin{aligned} \forall z \in K, \quad h(f(z)) &= \psi_\alpha \circ g \circ \psi_\beta \circ S \circ \psi_\alpha(z) = \psi_\alpha \circ g \circ S(\psi_\alpha(z)) = \psi_\alpha((S(\psi_\alpha(z)))^2) \\ &= \psi_\alpha(\psi_\alpha(z)) = z \end{aligned}$$

Also, since $h = \psi_\alpha \circ g \circ \psi_\beta$ is in fact a map from $\mathbb{D} \rightarrow \mathbb{D}$ (since ψ_α, ψ_β are automorphisms of \mathbb{D} , while $g(z) = z^2$ has all $z \in \mathbb{D}$ with $|g(z)| = |z|^2 < |z| < 1$, hence $g(z) \in \mathbb{D}$), and it satisfies $h(0) = 0$ (since $\psi_\alpha \circ g \circ \psi_\beta(0) = \psi_\alpha \circ g(\beta) = \psi_\alpha(\alpha) = 0$), then apply Schwarz Lemma, we know all $w \in f(K) \subseteq \mathbb{D}$ satisfies $|h(w)| \leq |w|$.

To prove that it is a strict inequality above, recall that Schwarz Lemma also states that if any nonzero $w \in \mathbb{D}$ satisfies $|h(w)| = |w|$, then h must be a rotation (i.e. all $w \in \mathbb{D}$ satisfies $|h(w)| = |w|$). However, this is not the case for $\beta \in \mathbb{D}$:

$$h(\beta) = \psi_\alpha \circ g \circ \psi_\beta(\beta) = \psi_\alpha \circ g(0) = \psi_\alpha(0) = \alpha, \quad |h(\beta)| = |\alpha| = |\beta|^2 < |\beta|$$

Since $\beta \in \mathbb{D}$ satisfies a strict inequality $|h(\beta)| < |\beta|$, then h cannot be a rotation, hence all nonzero $w \in \mathbb{D}$ cannot satisfy $|h(w)| = |w|$, which enforces $|h(w)| < |w|$.

Hence, for all nonzero $z \in K$, we have:

$$|z| = |h(f(z))| < |f(z)|$$

Because the function satisfies $f(0) = 0$ and $|z| < |f(z)|$ for all $z \in K \setminus \{0\}$, then f is an expansion.

Formula of $|f'(0)|$:

First, the derivative of ψ_α is given as follow:

$$\psi'_\alpha(z) = \frac{-(1 - \bar{\alpha}z) - (\alpha - z)(-\bar{\alpha})}{(1 - \bar{\alpha}z)^2}, \quad \psi'_\alpha(0) = \frac{-1 - \alpha(-\bar{\alpha})}{1^2} = |\alpha|^2 - 1$$

Similarly, derivative of ψ_β is given as follow:

$$\psi'_\beta(z) = \frac{-(1 - \bar{\beta}z) - (\beta - z)(-\bar{\beta})}{(1 - \bar{\beta}z)^2}, \quad \psi'_\beta(\beta) = \frac{-(1 - |\beta|^2) - 0}{(1 - |\beta|^2)^2} = \frac{-1}{(1 - |\beta|^2)} = \frac{-1}{1 - |\alpha|}$$

Then, derivative of S is given as follow:

$$\forall w \in \psi_\alpha(K), \quad (S(w))^2 = w, \quad 2S(w) \cdot S'(w) = 1, \quad S'(w) = \frac{1}{2S(w)}, \quad S'(\alpha) = \frac{1}{2S(\alpha)} = \frac{1}{2\beta}$$

Then, the derivative of f at 0 is given as:

$$\begin{aligned} f'(0) &= (\psi_\beta \circ S \circ \psi_\alpha)'(0) = \psi'_\beta(S \circ \psi_\alpha(0)) \cdot S'(\psi_\alpha(0)) \cdot \psi'_\alpha(0) \\ &= \psi'_\beta(S(\alpha)) \cdot S'(\alpha) \cdot (|\alpha|^2 - 1) = \psi'_\beta(\beta) \cdot \frac{1}{2\beta} \cdot (|\alpha|^2 - 1) = \frac{-1}{1 - |\alpha|} \cdot \frac{1}{2\beta} \cdot (|\alpha|^2 - 1) \\ &= \frac{(1 - |\alpha|^2)}{2\beta(1 - |\alpha|)} = \frac{1 + |\alpha|}{2\beta} \end{aligned}$$

Which, it has modulus given by:

$$|f'(0)| = \frac{|1 + |\alpha||}{|2\beta|} = \frac{1 + |\alpha|}{2\sqrt{|\beta|^2}} = \frac{1 + r_K}{2\sqrt{|\alpha|}} = \frac{1 + r_K}{2\sqrt{r_K}}$$

(e) To construct the expansion, we'll do this in an iterative manner (and we'll assume initially given Koebe Domain K_0 , $r_{K_0} < 1$, since the case $r_{K_0} \geq 1$ implies $K_0 = \mathbb{D}$).

0. First, find an $\alpha_0 \in \partial K$ and its corresponding $\beta_0 \in \mathbb{D}$ satisfying $|\alpha_0| = r_{K_0}$ and $\beta_0^2 = \alpha_0$, then $f_0 = \psi_{\beta_0} \circ S_{\alpha_0} \circ \psi_{\alpha_0} : K_0 \rightarrow \mathbb{D}$ is an expansion. Let $K_1 = f_0(K_0) \subseteq \mathbb{D}$. (Note: S_{α_0} is the described square root in **part (d)**).

n. With integer $n \geq 1$, in the previous step, we have new Koebe Domain K_n . Repeat the same process, choose $\alpha_n \in \partial K$ with $|\alpha_n| = r_{K_{n-1}}$ and $\beta_n \in \mathbb{D}$ with $\beta_n^2 = \alpha_n$. Then, $f_n = \psi_{\beta_n} \circ S_{\alpha_n} \circ \psi_{\alpha_n} : K_n \rightarrow \mathbb{D}$ is again an expansion. Let $K_{n+1} = f_n(K_n) \subseteq \mathbb{D}$.

The above constructs a sequence of expansion described in the problem. Then, for all $n \in \mathbb{N}$, let $F_n = f_n \circ \dots \circ f_0$ be an expansion, and it satisfies $\lim_{n \rightarrow \infty} |f'_n(0)| = 1$ (both proven in **part (b)**).

By the statement proven in **part (c)**, the sequence $\{F_n\}_{n \in \mathbb{N}}$ has a subsequence converges locally uniformly onto some injective analytic function F ; also, since the sequence r_{K_n} is strictly increasing (proven in **part (a)**) while bounded above by 1, then $\lim_{n \rightarrow \infty} r_{K_n} = d \leq 1$. Which, based on the formula of $|f'_n(0)|$ give in **part (d)**, we have the following:

$$1 = \lim_{n \rightarrow \infty} |f'_n(0)| = \lim_{n \rightarrow \infty} \frac{1 + r_{K_n}}{2\sqrt{r_{K_n}}} = \frac{1 + d}{2\sqrt{d}}$$

Hence, $2\sqrt{d} = 1 + d$, $4d = (1 + d)^2 = 1 + 2d + d^2$, $(1 - 2d + d^2) = (1 - d)^2 = 0$, so $(1 - d) = 0$, or $d = 1$.

Because $\lim_{n \rightarrow \infty} r_{K_n} = 1$, then based on the statement in **part (c)**, the function F that the subsequence converges to, is in fact conformal. Which, this is the result we want to show.