Math CS 122b HW8 Part 1

Zih-Yu Hsieh

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1 (need slight modification)

Question 1 *Stein and Shakarchi Pg. 201-202 Exercise 8:*
The function ζ has infinitely many zeros in the critical strip. This can be seen as follows.

(a) Let $F(s) = \xi(1/2 + s)$, where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Show that $F(s)$ is an even function of s, and as a result, there exists G so that $G(s^2) = F(s)$.

(b) Show that the function $(s-1)\zeta(s)$ is an entire function of growth order 1, that is

$$|(s-1)\zeta(s)| \leq A_\epsilon e^{a_\epsilon |s|^{1+\epsilon}}$$

As a consequence $G(s)$ is of growth order $1/2$.

(c) Deduce from the above that ζ has infinitely many zeros in the critical strip.

[Hint: To prove (a) and (b) use the functional equation for $\zeta(s)$. For (c), use a result of Hadamard, which states that an entire function with fractional order has infinitely many zeros (Exercise 14 in Chapter 5)].

Pf:

(a) Recall that in **HW 7 Question 1** (**Freitag Chap. VII.5 Problem 5**), to deduce the functional equation of ζ, we've proven the functional equation $\xi(s') = \xi(1-s')$. As a result, for any $s \in \mathbb{C}$, if treating F as a meromorphic function, we get:

$$F(s) = \xi\left(\frac{1}{2} + s\right) = \xi\left(1 - \left(\frac{1}{2} + s\right)\right) = \xi\left(\frac{1}{2} - s\right) = F(-s)$$

Hence, this proves that $F(s)$ is an even function.

(b) Recall that $\zeta(s)$ is analytic on $\mathbb{C} \setminus \{1\}$, with a simple pole at $s = 1$ with residue 1, then $(s-1)\zeta(s)$ is in fact having a removable singularity at $s = 1$, hence can be extended to an entire function.

1. $(s-1)\zeta(s)$ **Has growth order** 1 **for** $\mathrm{Re}(s) \geq \frac{1}{2}$**:**

In **Freitag Lemma VII.5.2**, the following functions are well defined:

$$\forall t \in \mathbb{R}, \quad \beta(t) = t - [t] - \frac{1}{2}, \quad [t] := \max n \in \mathbb{Z},\ n \leq t$$

$$\forall s \in \mathbb{C},\ \mathrm{Re}(s) > 0, \quad F(s) := \int_1^\infty t^{-s-1}\beta(t)dt$$

Then as a result, the following equation is true for $\mathrm{Re}(s) > 1$, hence defines an analytic continuation for $\zeta(s)$ on $\mathrm{Re}(s) > 0$:

$$\forall s \in \mathbb{C},\ \mathrm{Re}(s) > 1, \quad \zeta(s) = \frac{1}{2} + \frac{1}{s-1} - sF(s)$$

So, if multiply with $(s-1)$, for $\mathrm{Re}(s) \geq \frac{1}{2}$, $(s-1)\zeta(s)$ is well-defined, and can be given as the following formula:

$$(s-1)\zeta(s) = \frac{(s-1)}{2} + 1 - (s-1)sF(s)$$

Which, let $s = x + iy$ for $x, y \in \mathbb{R}$, on $\mathrm{Re}(s) = x \geq \frac{1}{2}$ (which $\frac{1}{x} \leq 2$), $F(s)$ can be bounded as follow:

$$|F(s)| = \left|\int_1^\infty t^{-s-1}\beta(t)dt\right| \leq \int_1^\infty |t^{-(x+iy)-1}\beta(t)|dt \leq \int_1^\infty |t^{-x-1} \cdot t^{iy}|dt = \int_1^\infty t^{-x-1}dt$$

$$= \frac{-1}{x}t^{-x}\Big|_1^\infty = \frac{1}{x} \leq 2$$

(Note: for any $t \in \mathbb{R}$, $|\beta(t)| \leq \frac{1}{2} < 1$, and since $x \geq \frac{1}{2}$, then the integral of t^{-x-1} has power < -1, which is absolutely convergent).

So, if considering the modulus of $(s-1)\zeta(s)$ on $\mathrm{Re}(s) \geq \frac{1}{2}$, we get the following:

$$|(s-1)\zeta(s)| = \left|\frac{(s-1)}{2} + 1 - (s-1)sF(s)\right| \leq \frac{|s-1|}{2} + 1 + |(s-1)s| \cdot |F(s)| \leq \frac{|s|+1}{2} + 1 + 2(|s|^2 + |s|)$$

$$\leq 2|s|^2 + \frac{3}{2}|s| + \frac{3}{2}$$

Which, take $4e^{|s|} = 4 + 4|s| + 2|s|^2 + \sum_{n=3}^\infty \frac{4}{n!}|s|^n$, since for any $s \in \mathbb{C}$ each term is nonnegative, then we can deduce:

$$|(s-1)\zeta(s)| \leq 2|s|^2 + \frac{3}{2}|s| + \frac{3}{2} \leq 4 + 4|s| + 2|s|^2 \leq 4 + 4|s| + 2|s|^2 + \sum_{n=3}^\infty \frac{4}{n!}|s|^n = 4e^{|s|}$$

This shows that $(s-1)\zeta(s)$ has growth order 1 on the half plane $\mathrm{Re}(s) \geq \frac{1}{2}$.

2. $(s-1)\zeta(s)$ **Has growth order** 1 **for the whole plane:**

In the previous part the growth order is verified for $\mathrm{Re}(s) \geq \frac{1}{2}$. so the rest suffices to show it for the half plane $\mathrm{Re}(s') < \frac{1}{2}$. (And, we'll utilize the fact that for all $s \in \mathbb{C}$, $|e^s| \leq e^{|s|}$, which can be seen using Taylor Series).

Recall that in **HW 7**, we've proven the following functional equation of ζ:

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

Hence, for any s' with $\mathrm{Re}(s') < \frac{1}{2}$, let $s' = 1-s$ for some $s \in \mathbb{C}$, then $s = 1-s'$, so $\mathrm{Re}(s) = \mathrm{Re}(1-s') > \frac{1}{2}$. Then, the equation $(s'-1)\zeta(s')$ becomes:

$$(s'-1)\zeta(s') = ((1-s)-1)\zeta(1-s) = -s \cdot 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

And, since $\cos(\frac{\pi}{2}) = 0$, $\cos(\frac{\pi s}{2}$ has a zero at $s = 1$, then $\cos\left(\frac{\pi s}{2}\right) = (s-1)h(s)$ for some analytic function h. So, the above formula can be further written as:

$$((1-s)-1)\zeta(1-s) = -s \cdot 2(2\pi)^{-s}\Gamma(s)h(s) \cdot (s-1)\zeta(s)$$

Which, $|s| = |1-s'| \leq |s'|+1$, so the growth order in terms of $|s|$ can be replaced using $|s'|$ instead. From the above equality, we do need to talk about the growth order of different components:

- For $(2\pi)^{-s} = e^{-\log(2\pi)s} = e^{-\log(2\pi)(x+iy)} = e^{-\log(2\pi)x} \cdot e^{-\log(2\pi)iy}$, it satisfies the following:

$$|(2\pi)^{-s}| = |e^{-\log(2\pi)s}| \leq e^{\log(2\pi)|s|}$$

  This proves that $(2\pi)^{-s}$ has growth order 1.

- For $\Gamma(s)$, since we're working with the half plane $\mathrm{Re}(s) > \frac{1}{2}$, then it's valid to apply **Stir ing's Formula** (given in **Freitag Proposition IV.1.14**):
  Let $H(s) = \sum_{n=0}^{\infty}\left(\left(s+n+\frac{1}{2}\right)\log\left(1+\frac{1}{s+n}\right)-1\right)$, then $\Gamma(s)$ can be expressed as:

$$\Gamma(s) = \sqrt{2\pi}s^{s-\frac{1}{2}}e^{-s}e^{H(z)} = \sqrt{2\pi}e^{(s-1/2)\log(s)-s+H(s)}$$

  and $s\to\infty$ implies $H(s)\to 0$ (within the given half plane $\mathrm{Re}(s) > \frac{1}{2}$).
  Which, notice that for s in the half plane, since $s\to\infty$ implies $H(s)\to 0$, then there exists $M>0$, such that $|s|>M$ implies $|H(s)|<1$. And, since for all $\epsilon>0$ (specifically, can limit to $\epsilon<1$), there exists $M'>0$, such that $|\log(s)| \leq |s|^{\epsilon}$, then for all s in the half plant satisfies $|s|>M, M'$, we get:

$$\left|\left(s-\frac{1}{2}\right)\log(s)-s+H(s)\right| \leq \left(|s|+\frac{1}{2}\right)|\log(s)|+|s|+|H(s)| \leq \left(|s|+\frac{1}{2}\right)|s|^{\epsilon}+|s|+1$$

$$\leq |s|^{1+\epsilon}+\frac{1}{2}|s|^{\epsilon}+|s|^{1+\epsilon}+1 \leq \frac{5}{2}|s|^{1+\epsilon}+1$$

  Hence, $\Gamma(s)$ satisfies:

$$|\Gamma(s)| = \left|\sqrt{2\pi}e^{(s-1/2)\log(s)-s+H(s)}\right| \leq \sqrt{2\pi}\exp\left(\left|\left(s-\frac{1}{2}\right)\log(s)-s+H(s)\right|\right)$$

$$\leq \sqrt{2\pi}\exp\left(\frac{5}{2}|s|^{1+\epsilon}+1\right) = e\sqrt{2\pi}e^{\frac{5}{2}|s|^{1+\epsilon}}$$

  Hence, for any $\epsilon>0$, with suitable constant $A_\epsilon, a_\epsilon > 0$, on the half plane $\mathrm{Re}(s) > \frac{1}{2}$, $|\Gamma(s)| \leq A_\epsilon e^{a_\epsilon |s|^{1+\epsilon}}$, showing that $\Gamma(s)$ has growth order 1.

- For $h(s)$ mentioned above, since $(s-1)h(s) = \cos\left(\frac{\pi s}{2}\right)$, and $\cos\left(\frac{\pi s}{2}\right)$ can be written as:

$$\cos\left(\frac{\pi s}{2}\right) = \frac{e^{i\frac{\pi s}{2}}+e^{-i\frac{\pi s}{2}}}{2}$$

  Hence, the following inequality is true:

$$\left|\cos\left(\frac{\pi s}{2}\right)\right| \leq \frac{1}{2}\left(|e^{i\frac{\pi s}{2}}|+|e^{-i\frac{\pi s}{2}}|\right) \leq \frac{1}{2}\left(e^{\frac{\pi}{2}|s|}+e^{\frac{\pi}{2}|s|}\right) = e^{\frac{\pi}{2}|s|}$$

  Hence, $\cos\left(\frac{\pi s}{2}\right)$ is with growth order 1, which also implies that $h(s)$ is with growth order 1.

Finally, back to the original equation, since for any s' with $\mathrm{Re}(s') < \frac{1}{2}$, writing $s' = 1-s$ for $\mathrm{Re}(s) > \frac{1}{2}$ yields the following expresion:

$$(s'-1)\zeta(s') = ((1-s)-1)\zeta(1-s) = -s\cdot 2(2\pi)^{-s}\Gamma(s)h(s)\cdot(s-1)\zeta(s)$$

Then, since s, $(2\pi)^{-s}$, $\Gamma(s)$, $h(s)$ are all with growth order 1, and $(s-1)\zeta(s)$ has been proven to have growth order 1 also in the previous part, then the whole product $(s'-1)\zeta(s')$ is with growth order 1 (with input s). However, since $|s| = |1 - s'| \le |s'| + 1$ as mentioned before, then it is also with growth order 1 with respect to s'.

Regardless of the choise of s (either $\mathrm{Re}(s) \ge \frac{1}{2}$ or $\mathrm{Re}(s) < \frac{1}{2}$), we eventually get that $(s-1)\zeta(s)$ is with growth order 1.

(c) In **Part (b)**, it was proven that F has growth order 1, while G (after being modified into an entire function) has growth order 1/2. So based on Hadamard's result, it has infinitely many zeros, which also implies that $F(s) = G(s^2)$ has infinitely many zeros. However, since $F(s)$ is given as:

$$F(s) = \xi(1/2 + s) = \pi^{-(1/2+s)/2}\Gamma\left(\frac{(1/2+s)}{2}\right)\zeta\left(\frac{1}{2} + s\right)$$

Which, because $F(s)$ is even, it is enough to consider the half plane $\mathrm{Re}(s) \ge 0$: Because π^z, $\Gamma(z)$ are both nonzero functions, then these zeros of F must be contributed by $\zeta(1/2 + s)$; On the other hand, it is well-known that $\zeta(z)$ has no zeros for $\mathrm{Re}(z) \ge 1$, hence for $\mathrm{Re}(1/2 + s) \ge 1$, or $\mathrm{Re}(s) \ge \frac{1}{2}$, since $\zeta(1/2+s)$ has no zeros, then $F(s)$ has no zeros. Hence, the zeros of $F(s)$ (on the half plane $\mathrm{Re}(s) \ge 0$) must appear in the range $0 \le \mathrm{Re}(s) < \frac{1}{2}$, which eventually implies that there are infinitely many s in this strip (which satisfies $\frac{1}{2} \le \mathrm{Re}(1/2 + s) < 1$, with $(1/2 + s)$ being in the critical strip) satisfying $\zeta(1/2 + s) = 0$.

So, we can conclude that $\zeta(s)$ has infinitely many zeros in the critical strip.

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Question 2 *Stein and Shakarchi Pg. 202-203 Exercise 10:*
In the theory of primes, a better approximation fo $\pi(x)$ (instead of $x/\log(x)$) turns out to be $Li(x)$ defined by

$$Li(x) = \int_2^x \frac{dt}{\log(t)}$$

(a) Prove that

$$Li(x) = \frac{x}{\log(x)} + O\left(\frac{x}{(\log(x))^2}\right) \quad \text{as } x \to \infty$$

and that as a consequence

$$\pi(x) \sim Li(x) \quad \text{as } x \to \infty$$

(b) Refine the previous analysis by showing that for every integer $N > 0$ one has the following asymptotic expansion

$$Li(x) = \frac{x}{\log(x)} + \frac{x}{(\log(x))^2} + 2\frac{x}{(\log(x))^3} + \cdots (N-1)!\frac{x}{(\log(x))^N} + O\left(\frac{x}{(\log(x))^{N+1}}\right)$$

as $x \to \infty$.

Pf:

(a) First, using integration by parts, for all $x \geq 4$ (where $x \geq \sqrt{x} \geq 2$), $\mathrm{Li}(x)$ can be expressed as follow:

$$\begin{aligned}
\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)} &= \frac{t}{\log(t)}\bigg|_2^x - \int_2^x t \cdot \frac{d}{dt}\left(\frac{1}{\log(t)}\right) dt \\
&= \frac{x}{\log(x)} - \frac{2}{\log(2)} - \int_2^x t \cdot \frac{-1}{(\log(t))^2} \cdot \frac{1}{t} dt \\
&= \frac{x}{\log(x)} - \frac{2}{\log(2)} + \int_2^x \frac{1}{(\log(t))^2} dt
\end{aligned}$$

Which, for the last integral expression, it can be reformulate as follow:

$$\int_2^x \frac{dt}{(\log(t))^2} = \int_2^{\sqrt{x}} \frac{dt}{(\log(t))^2} + \int_{\sqrt{x}}^x \frac{dt}{(\log(t))^2}$$

Since $\log(t)$ is a strictly increasing function on $(1, \infty)$ and is strictly positive, then $\frac{1}{(\log(t))^2}$ is a strictly decreasing function on this interval instead. Hence, for all $t \in [2, \sqrt{x}]$, $\frac{1}{(\log(t))^2} \leq \frac{1}{(\log(2))^2}$, while any $t \in [\sqrt{x}, x]$ satisfies $\frac{1}{(\log(t))^2} \leq \frac{1}{(\log(\sqrt{x}))^2} = \frac{4}{(\log(x))^2}$. Hence, the above expression satisfies:

$$\begin{aligned}
\int_2^x \frac{dt}{(\log(t))^2} &= \int_2^{\sqrt{x}} \frac{dt}{(\log(t))^2} + \int_{\sqrt{x}}^x \frac{dt}{(\log(t))^2} \leq \int_2^{\sqrt{x}} \frac{dt}{(\log(2))^2} + \int_{\sqrt{x}}^x \frac{4dt}{(\log(x))^2} \\
&= \frac{\sqrt{x}-2}{(\log(2))^2} + \frac{4(x-\sqrt{x})}{(\log(x))^2} \leq \frac{4x}{(\log(x))^2} + \frac{\sqrt{x}}{(\log(2))^2}
\end{aligned}$$

Which, if evaluate the following limit, we get:

$$\lim_{x\to\infty} \frac{\sqrt{x}}{x/(\log(x))^2} = \lim_{x\to\infty} \frac{(\log(x))^2}{\sqrt{x}} = \lim_{x\to\infty} \frac{2\log(x)/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{4\log(x)}{\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{4/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{8}{\sqrt{x}} = 0$$

Hence, for some $x_1 > 4$ and $A_1 > 0$, we have $x > x_1$ implies $\sqrt{x} \le A_1 \frac{x}{(\log(x))^2}$. So, the integral follows the inequality below for $x > x_0$:

$$\int_2^x \frac{dt}{(\log(t))^2} \le \frac{\sqrt{x}}{(\log(2))^2} + \frac{4x}{(\log(x))^2} \le \frac{1}{(\log(2))^2} \cdot \frac{A_1 x}{(\log(x))^2} + \frac{4x}{(\log(x))^2}$$

$$\le \left(\frac{A_1}{(\log(2))^2} + 4 \right) \frac{x}{(\log(x))^2}$$

So, this shows that $\int_2^x \frac{dt}{(\log(t))^2} = O\left(\frac{x}{(\log(x))^2}\right)$. Hence:

$$\mathrm{Li}(x) = \frac{x}{\log(x)} - \frac{2}{\log(2)} + \int_2^x \frac{dt}{(\log(t))^2} \le \frac{x}{\log(x)} + \int_2^x \frac{dt}{(\log(t))^2} = \frac{x}{\log(x)} + O\left(\frac{x}{(\log(x))^2}\right)$$

This shows that $\mathrm{Li}(x) = \frac{x}{\log(x)} + O\left(\frac{x}{(\log(x))^2}\right)$.

(b) First, we'll consider the following formula about the integral of $\frac{1}{(\log(t))^n}$ using integration by parts:

$$\forall n \in \mathbb{N}, \quad \int_2^x \frac{dt}{(\log(t))^n} = \left.\frac{t}{(\log(t))^n}\right|_2^x - \int_2^x t \cdot \frac{d}{dt}\left(\frac{1}{(\log(t))^n}\right) dt$$

$$= \frac{x}{(\log(x))^n} - \frac{2}{(\log(2))^n} - \int_2^x t \cdot \frac{-n}{(\log(t))^{n+1}} \cdot \frac{1}{t} dt = \frac{x}{(\log(x))^n} - \frac{2}{(\log(2))^n} + n \int_2^x \frac{dt}{(\log(t))^{n+1}}$$

Which, using the same argument used in **part (a)** about $\frac{1}{(\log(t))^n}$ is a decreasing function for all $n \in \mathbb{N}$, for all $x \ge 4$ (where $x > \sqrt{x} \ge 2$), we get:

$$\int_2^x \frac{dt}{(\log(t))^{n+1}} = \int_2^{\sqrt{x}} \frac{dt}{(\log(t))^{n+1}} + \int_{\sqrt{x}}^x \frac{dt}{(\log(t))^{n+1}} \le \int_2^{\sqrt{x}} \frac{dt}{(\log(2))^{n+1}} + \int_{\sqrt{x}}^x \frac{dt}{(\log(\sqrt{x}))^{n+1}}$$

$$= \frac{(\sqrt{x} - 2)}{(\log(2))^{n+1}} + \frac{2^{n+1}(x - \sqrt{x})}{(\log(x))^{n+1}} \le \frac{2^{n+1} x}{(\log(x))^{n+1}} + \frac{\sqrt{x}}{(\log(2))^{n+1}}$$

Now, since the base case $\lim_{x\to\infty} \frac{\sqrt{x}}{x/(\log(x))^2} = 0$ is proven in **part (a)**, using induction, we can get the following relationship:

$$\forall n \in \mathbb{N}, \quad \lim_{x\to\infty} \frac{\sqrt{x}}{x/(\log(x))^{n+1}} = \lim_{x\to\infty} \frac{(\log(x))^{n+1}}{\sqrt{x}} = \lim_{x\to\infty} \frac{(n+1)(\log(x))^n/x}{1/(2\sqrt{x})}$$

$$= \lim_{x\to\infty} 2(n+1) \frac{\sqrt{x}}{x/(\log(x))^n} = 0$$

Hence, there exists $x_n > 4$ and $A_n > 0$, such that $x > x_n$ implies $\sqrt{x} \le A_n \frac{x}{(\log(x))^{n+1}}$. Hence, we get:

$$\int_2^x \frac{dt}{(\log(t))^{n+1}} \le \frac{2^{n+1} x}{(\log(x))^{n+1}} + \frac{\sqrt{x}}{(\log(2))^{n+1}}$$

$$\le \frac{2^{n+1} x}{(\log(x))^{n+1}} + \frac{A_n}{(\log(2))^{n+1}} \frac{x}{(\log(x))^{n+1}} = \left(2^{n+1} + \frac{A_n}{(\log(2))^{n+1}}\right) \frac{x}{(\log(x))^{n+1}}$$

This shows that $\int_2^x \frac{dt}{(\log(t))^{n+1}} = O\left(\frac{x}{(\log(x))^{n+1}}\right)$.

Finally, using the case proven in **part (a)**, we know $\mathrm{Li}(x) = \frac{x}{\log(x)} - \frac{2}{\log(2)} + \int_2^x \frac{dt}{(\log(t))^2}$. Which utilizing the above equation, by induction, one can show that for any integer $n \geq 2$, the following formula holds:

$$\mathrm{Li}(x) = \sum_{k=1}^{n}(k-1)!\frac{x}{(\log(x))^k} - \sum_{k=1}^{n}(k-1)!\frac{2}{(\log(2))^k} + n!\int_2^x \frac{dt}{(\log(t))^{n+1}} \leq \sum_{k=1}^{n}(k-1)!\frac{x}{(\log(x))^k} + n!\int_2^x \frac{dt}{(\log(t))^{n+1}}$$

Then, with the statement that $\int_2^x \frac{dt}{(\log(t))^{n+1}} = O\left(\frac{x}{(\log(x))^{n+1}}\right)$ deduced previously, for any $n \in \mathbb{N}$, we get the following:

$$\mathrm{Li}(x) = \sum_{k=1}^{n}(k-1)!\frac{x}{(\log(x))^k} + O\left(\frac{x}{(\log(x))^{n+1}}\right)$$

$$= \frac{x}{\log(x)} + \frac{x}{(\log(x))^2} + 2!\frac{x}{(\log(x))^3} + \cdots + (n-1)!\frac{x}{(\log(x))^n} + O\left(\frac{x}{(\log(x))^{n+1}}\right)$$

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Question 3 *Stein and Shakarchi Pg. 204 Problem 2:*
One of the "explicit formulas" in the theory of primes is as follows: if ψ_1 is the integrated Tchebychev function considered in Section 2, then

$$\psi_1(x) = \frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} - E(x)$$

where the sum is taken over all zeros ρ of the ζ-function in the critical strip. The error term is given by $E(x) = c_1 x + c_0 + \sum_{k=1}^{\infty} x^{1-2k}/(2k(2k-1))$, where $c_1 = \zeta'(0)/\zeta(0)$ and $c_0 = -\zeta'(-1)/\zeta(-1)$. Note that $\sum_{\rho} 1/|\rho|^{1+\epsilon} < \infty$ for every $\epsilon > 0$, because $(1-s)\zeta(s)$ has order of growth 1. Also, obviously $E(x) = O(x)$ as $x \to \infty$.

Pf:

First, recall that the following formula of $\psi_1(x)$ holds for any $x > 1$ and $c > 1$:

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds$$

Which, to get a closed expression, we'll utilize Hadamard's product formula for ζ and Residue Theorem.

1. Product Formula for ζ and $-\frac{\zeta'}{\zeta}$:

Based on **Question 1 part (b)** in this assignment, we've proven that $(s-1)\zeta(s)$ is an entire function with growth order 1, and it is zero precisely at all the zeros of $\zeta(s)$ since at $s = 1$, $\zeta(s)$ has residue 1. Which, $(s-1)\zeta(s)$ has zeros at $-2k$ for $k \in \mathbb{N}$, and all zeros of ζ, denoted as ρ in the critical strip.

Then, based on **Hadamard's Factorization Theorem** (can be seen in **Stein and Shakarchi Chapter 5.5**), since $(s-1)\zeta(s)$ has growth order 1 with the zeros mentioned above (which the zeros are all nonzero), then there exists polynomial $P(s) = ls + d$ with degree 1 (at most the growth order), such that the following holds:

$$\begin{aligned}(s-1)\zeta(s) &= e^{ls+d} \left(\prod_{k=1}^{\infty} E_1\left(-\frac{s}{2k}\right) \right) \left(\prod_{\rho} E_1\left(\frac{s}{\rho}\right) \right) \\ &= e^{ls+d} \left(\prod_{k=1}^{\infty} \left(1 + \frac{s}{2k}\right) e^{-s/(2k)} \right) \left(\prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \right)\end{aligned}$$

Where the second product contains all nontrivial zeros of ζ in the critical strip. Hence, the following is a formula for $\zeta(s)$ in terms of products of zeros and poles:

$$\zeta(s) = (s-1)^{-1} e^{ls+d} \left(\prod_{k=1}^{\infty} \left(1 + \frac{s}{2k}\right) e^{-s/(2k)} \right) \left(\prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \right)$$

Then, utilizing logarithmic derivative, we get the following:

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + l + \sum_{k=1}^{\infty} \left(\frac{1}{s+2k} - \frac{1}{2k} \right) + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - l + \sum_{k=1}^{\infty} \frac{s}{(s+2k)2k} - \sum_{\rho} \frac{s}{(s-\rho)\rho}$$

And, this formula is normally convergent within any compact subset of the domain (not containing the zeros and the poles of ζ), so integration can be exchanged with summation.

2. Contour Integration:

Now, choose any $c_0 > 1$, and restrict the domain to the open half plane $\mathrm{Re}(s) < c_0$. CHoose any $c \in \mathbb{R}$ such that $1 < c < c_0$, and define the contour γ_r as the following semicircle for any $r \in \mathbb{R}_{>0}$:

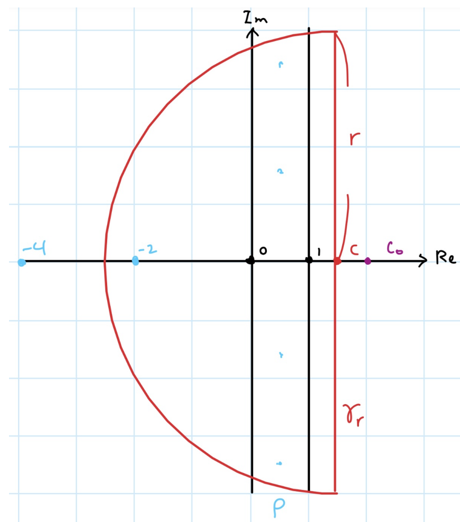

Figure 1: The contour used for integration

Which, γ_r is involved with semicircle c_r with radius r centered at $s = c$, and the straight line ℓ_r parametrized by $c + it$, for $t \in [-r, r]$.

Temporarily, assume r is chosen so that γ_r contains no zeros or poles of ζ, and let D_r be the region enclosed by γ_r. Then, if perform the contour integration, we get the following:

$$\frac{1}{2\pi i}\int_{\gamma_r} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds = \frac{1}{2\pi i}\int_{\gamma_r} \frac{x^{s+1}}{s(s+1)}\left(\frac{1}{s-1} - l + \sum_{k=1}^{\infty}\frac{s}{(s+2k)2k} - \sum_{\rho}\frac{s}{(s-\rho)\rho}\right) ds$$

$$= \frac{1}{2\pi i}\int_{\gamma_r} \frac{x^{s+1}}{s(s+1)(s-1)} ds - \frac{1}{2\pi i}\int_{\gamma_r} \frac{l x^{s+1}}{s(s+1)} ds + \sum_{k=1}^{\infty}\frac{1}{2\pi i}\int_{\gamma_r}\frac{x^{s+1}}{(s+1)(s+2k)2k} ds - \sum_{\rho}\frac{1}{2\pi i}\int_{\gamma_r}\frac{x^{s+1}}{(s+1)(s-\rho)\rho} ds$$

Now, since we've restricted the domain to $\mathrm{Re}(s) < c_0$, then for any $s = u+iv$ in this region (which $u < c_0$), for any fixed $x > 1$, we get $x^{s+1} = x^{(u+iv)+1} = x^{u+1} \cdot x^{iv} = x^{u+1} \cdot e^{iv\log(x)}$, which $|x^{s+1}| = x^{u+1} < x^{c_0+1}$.

Then, for the first integral above, there involves some fixed $\alpha \in \mathbb{C}$, where the denominator involve the terms $(s - \alpha)$. Then, one can choose radius $R > 0$, such that for all radius $r > R$, the involved term $(s - \alpha)$ satisfies $|s - \alpha| > \frac{r}{2}$ (which $\frac{1}{|s-\alpha|} < \frac{2}{r}$). So, for the integration over the semicircle c_r, we get the following:

$$\left|\frac{1}{2\pi i}\int_{c_r}\frac{x^{s+1}}{s(s+1)(s-1)} ds\right| \le \frac{1}{2\pi}\int_{c_r}\frac{|x^{s+1}|}{|s(s+1)(s-1)|}|ds| < \frac{1}{2\pi}\int_{c_r}\frac{2^3 \cdot x^{c_0+1}}{r^3}|ds|$$

$$= \frac{1}{2\pi}\cdot\frac{2^3 \cdot x^{c_0+1}}{r^3}\cdot \pi r = \frac{2^2 \cdot x^{c_0+1}}{r^2}$$

(Note: We're integrating over a semicircle, so eventually the integration of a constnat multiplies by πr).

Then, take $r \to \infty$, since $\frac{1}{r^2} \to 0$, the above inequality proves that $\left|\frac{1}{2\pi i}\int_{C_r} \frac{x^{s+1}}{s(s+1)(s-1)}ds\right| \to 0$, or $\frac{1}{2\pi i}\int_{C_r} \frac{x^{s+1}}{s(s+1)(s-1)}ds \to 0$. Hence, we get the following:

$$\lim_{r\to\infty} \frac{1}{2\pi i}\int_{\gamma_r} \frac{x^{s+1}}{s(s+1)(s-1)}ds = \lim_{r\to\infty}\left(\frac{1}{2\pi i}\int_{C_r} \frac{x^{s+1}}{s(s+1)(s-1)}ds + \frac{1}{2\pi i}\int_{\ell_r} \frac{x^{s+1}}{s(s+1)(s-1)}ds\right)$$

$$= \lim_{r\to\infty} \frac{1}{2\pi i}\int_{c-ir}^{c+ir} \frac{x^{s+1}}{s(s+1)(s-1)}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)(s-1)}ds$$

Now, if we apply similar methods to the other integrals (integrating functions with polynomial of degree 2 on the denominator), then we get the following instead (we'll use the second one as an example):

$$\left|\frac{1}{2\pi i}\int_{C_r} \frac{lx^{s+1}}{s(s+1)}ds\right| \le \frac{1}{2\pi}\int_{C_r} \frac{l|x^{s+1}|}{|s(s+1)|}|ds| < \frac{1}{2\pi}\int_{C_r} \frac{2^2 \cdot l \cdot x^{c_0+1}}{r^2}|ds|$$

$$= \frac{1}{2\pi} \cdot \frac{2^2 \cdot l \cdot x^{c_0+1}}{r^2} \cdot \pi r = \frac{2l \cdot x^{c_0+1}}{r}$$

Take $r \to \infty$, since $\frac{1}{r} \to 0$, the above inequality proves that $\left|\frac{1}{2\pi i}\int_{C_r} \frac{lx^{s+1}}{s(s+1)}ds\right| \to 0$, or $\frac{1}{2\pi i}\int_{C_r} \frac{lx^{s+1}}{s(s+1)}ds \to 0$. Hence, we again get the following:

$$\lim_{r\to\infty} \frac{1}{2\pi i}\int_{\gamma_r} \frac{lx^{s+1}}{s(s+1)}ds = \lim_{r\to\infty}\left(\frac{1}{2\pi i}\int_{C_r} \frac{lx^{s+1}}{s(s+1)}ds + \frac{1}{2\pi i}\int_{\ell_r} \frac{lx^{s+1}}{s(s+1)}ds\right)$$

$$= \lim_{r\to\infty} \frac{1}{2\pi i}\int_{c-ir}^{c+ir} \frac{lx^{s+1}}{s(s+1)}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{lx^{s+1}}{s(s+1)}ds$$

Apply the similar formulas to the other two sums, then we get the following:

$$\lim_{r\to\infty} \frac{1}{2\pi i}\int_{\gamma_r} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds$$

$$= \lim_{r\to\infty} \frac{1}{2\pi i}\int_{\gamma_r} \frac{x^{s+1}}{s(s+1)(s-1)}ds - \lim_{r\to\infty} \frac{1}{2\pi i}\int_{\gamma_r} \frac{lx^{s+1}}{s(s+1)}ds$$
$$+ \lim_{r\to\infty}\sum_{k=1}^{\infty} \frac{1}{2\pi i}\int_{\gamma_r} \frac{x^{s+1}}{(s+1)(s+2k)2k}ds - \lim_{r\to\infty}\sum_{\rho} \frac{1}{2\pi i}\int_{\gamma_r} \frac{x^{s+1}}{(s+1)(s-\rho)\rho}ds$$

$$= \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \cdot \frac{1}{s-1}ds - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \cdot l ds$$
$$+ \sum_{k=1}^{\infty} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \cdot \frac{s}{(s+2k)2k}ds - \sum_{\rho} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \cdot \frac{s}{(s-\rho)\rho}ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(\frac{1}{s-1} - l + \sum_{k=1}^{\infty} \frac{s}{(s+2k)2k} - \sum_{\rho} \frac{s}{(s-\rho)\rho}\right)ds$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds = \psi_1(x)$$

So, it suffices to show that the above limit provides the explicit formula mentioned in the question.

3. Value of each integration:
As a quick recap, we get the following formula:

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds$$

$$= \lim_{r\to\infty}\frac{1}{2\pi i}\int_{\gamma_r}\frac{x^{s+1}}{s(s+1)(s-1)}ds - \lim_{r\to\infty}\frac{1}{2\pi i}\int_{\gamma_r}\frac{lx^{s+1}}{s(s+1)}ds$$
$$+\lim_{r\to\infty}\sum_{k=1}^{\infty}\frac{1}{2\pi i}\int_{\gamma_r}\frac{x^{s+1}}{(s+1)(s+2k)2k}ds - \lim_{r\to\infty}\sum_{\rho}\frac{1}{2\pi i}\int_{\gamma_r}\frac{x^{s+1}}{(s+1)(s-\rho)\rho}ds$$

Which, for each integration, there exists finitely many points with nontrivial residues. Which for each integration, can assume for some $R > 0$, every $r > R$ satisfies D_r containing all points with nontrivial residues. So, as the limit $r \to \infty$, can assume all the points of residues are contained in the region being integrated.

- Let $f(s) = \frac{x^{s+1}}{s(s+1)(s-1)}$, which it has simple poles at $s = 1,\ 0,\ -1$. Then, for the integration we get:

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{x^{s+1}}{s(s+1)(s-1)}ds = \mathrm{Res}(f, s=1) + \mathrm{Res}(f, s=0) + \mathrm{Res}(f, s=-1)$$
$$= \frac{x^{1+1}}{1(1+1)} + \frac{x^{0+1}}{(0+1)(0-1)} + \frac{x^{-1+1}}{-1(-1-1)} = \frac{x^2}{2} - x + \frac{1}{2}$$

- Let $g(s) = \frac{lx^{s+1}}{s(s+1)}$, which has simple poles at $s = 0,\ -1$. Then, we get:

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{lx^{s+1}}{s(s+1)}ds = \mathrm{Res}(g, s=0) + \mathrm{Res}(g, s=-1)$$
$$= \frac{lx^{0+1}}{(0+1)} + \frac{lx^{-1+1}}{-1} = lx - l$$

- For any $k \in \mathbb{N}$, let $h_k(s) = \frac{x^{s+1}}{(s+1)(s+2k)2k}$, which has simple poles at $s = -1,\ -2k$. Then, we get:

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{x^{s+1}}{(s+1)(s+2k)2k}ds = \mathrm{Res}(h_k, s=-1) + \mathrm{Res}(h_k, s=-2k)$$
$$\frac{x^{-1+1}}{(-1+2k)2k} + \frac{x^{-2k+1}}{(-2k+1)2k} = \frac{1}{2k(2k-1)} - \frac{x^{1-2k}}{2k(2k-1)}$$

- For any zero ρ of ζ in the critical strip, let $p_\rho(s) = \frac{x^{s+1}}{(s+1)(s-\rho)\rho}$, which has simple poles at $s = -1,\ \rho$. Then, we get:

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{x^{s+1}}{(s+1)(s-\rho)\rho}ds = \mathrm{Res}(p_\rho, s=-1) + \mathrm{Res}(p_\rho, s=\rho)$$
$$= \frac{x^{-1+1}}{(-1-\rho)\rho} + \frac{x^{\rho+1}}{(\rho+1)\rho} = -\frac{1}{\rho(\rho+1)} + \frac{x^{\rho+1}}{\rho(\rho+1)}$$

So, $\psi_1(x)$ can be expressed as:

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$

$$= \lim_{r\to\infty}\frac{1}{2\pi i}\int_{\gamma_r}\frac{x^{s+1}}{s(s+1)(s-1)}ds - \lim_{r\to\infty}\frac{1}{2\pi i}\int_{\gamma_r}\frac{lx^{s+1}}{s(s+1)}ds$$
$$+\lim_{r\to\infty}\sum_{k=1}^{\infty}\frac{1}{2\pi i}\int_{\gamma_r}\frac{x^{s+1}}{(s+1)(s+2k)2k}ds - \lim_{r\to\infty}\sum_{\rho}\frac{1}{2\pi i}\int_{\gamma_r}\frac{x^{s+1}}{(s+1)(s-\rho)\rho}ds$$

$$= \left(\frac{x^2}{2} - x + \frac{1}{2}\right) - (lx - l) + \left(\sum_{k=1}^{\infty}\frac{1}{2k(2k-1)} - \frac{x^{1-2k}}{2k(2k-1)}\right) - \left(\sum_{\rho} -\frac{1}{\rho(\rho+1)} + \frac{x^{\rho+1}}{\rho(\rho+1)}\right)$$

$$\frac{x^2}{2} - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} + x\left(-1-l\right) + \left(\frac{1}{2} + l + \sum_{k=1}^{\infty}\frac{1}{2k(2k-1)} + \sum_{\rho}\frac{1}{\rho(\rho+1)}\right) - \sum_{k=1}^{\infty}\frac{x^{1-2k}}{2k(2k-1)}$$

Which, consider the previous product formula of $\frac{\zeta'(s)}{\zeta(s)}$, we get the following two expressions:

$$\frac{\zeta'(0)}{\zeta(0)} = -\frac{1}{0-1} + l + \sum_{k=1}^{\infty}\left(\frac{1}{0+2k} - \frac{1}{2k}\right) + \sum_{\rho}\left(\frac{1}{0-\rho} + \frac{1}{\rho}\right) = 1 + l$$

$$\frac{\zeta'(-1)}{\zeta(-1)} = -\frac{1}{-1-1} + l + \sum_{k=1}^{\infty}\left(\frac{1}{-1+2k} - \frac{1}{2k}\right) + \sum_{\rho}\left(\frac{1}{-1-\rho} + \frac{1}{\rho}\right)$$
$$= \frac{1}{2} + l + \sum_{k=1}^{\infty}\frac{1}{2k(2k-1)} + \sum_{\rho}\frac{1}{\rho(\rho+1)}$$

Hence, the previous formula can be rewritten as:

$$\psi_1(x) = \frac{x^2}{2} - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} + x\left(-1-l\right) + \left(\frac{1}{2} + l + \sum_{k=1}^{\infty}\frac{1}{2k(2k-1)} + \sum_{\rho}\frac{1}{\rho(\rho+1)}\right) - \sum_{k=1}^{\infty}\frac{x^{1-2k}}{2k(2k-1)}$$
$$= \frac{x^2}{2} - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} - \frac{\zeta'(0)}{\zeta(0)}x + \frac{\zeta'(-1)}{\zeta(-1)} - \sum_{k=1}^{\infty}\frac{x^{1-2k}}{2k(2k-1)}$$

Which, if define the error term $E(x)$ as follow:

$$c_1 = \frac{\zeta'(0)}{\zeta(0)},\ c_0 = -\frac{\zeta'(-1)}{\zeta(-1)}, \quad E(x) = c_1 x + c_0 + \sum_{k=1}^{\infty}\frac{x^{1-2k}}{2k(2k-1)}$$

Then, $\psi_1(x)$ has the following explicit formula:

$$\psi_1(x) = \frac{x^2}{2} - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} - E(x)$$

Which is the desired formula.

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Question 4 *Stein and Shakarchi Pg. 204 Problem 3:*
Using the previous problem one can show that

$$\pi(x) - Li(x) = O(x^{\alpha+\epsilon}) \quad \text{as } x \to \infty$$

for every $\epsilon > 0$, where α is fixed and $1/2 \leq \alpha < 1$ if and only if $\zeta(s)$ has no zeros in the strip $\alpha < Re(s) < 1$. The case $\alpha = 1/2$ corresponds to the Riemann Hypothesis.

Pf: