Math 118C HW5

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Question 1 *Let $R_1 \subset R_2 \subset \mathbb{R}^n$ be finite rectangles. Using the definition of volume, prove that the set $R_2 \setminus R_1$ is Riemann-Measurable and $Vol(R_2 \setminus R_1) = Vol(R_2) - Vol(R_1)$.*

Pf:

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Question 2

1. *Let* $R, R_1, R_2, ..., R_m \subset \mathbb{R}^n$ *be finite rectangles such that* $R \subset \bigcup_{i=1}^m R_i$. *Without using characteristic functions and integration, prove that* $Vol(R) \leq \sum_{i=1}^m Vol(R_i)$.

2. *Let* $\tilde{R}_1, \tilde{R}_2, ..., \tilde{R}_k \subset \mathbb{R}^n$ *be finite rectangles and let* $R_1, R_2, ..., R_m \subset \mathbb{R}^n$ *be nonoverlapping finite rectangles such that* $\bigcup_{i=1}^m R_i \subset \bigcup_{j=1}^k \tilde{R}_j$. *Without using characteristic functions and integration, prove that* $\sum_{i=1}^m Vol(R_i) \leq \sum_{j=1}^k Vol(\tilde{R}_j)$.

Pf:

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Question 3 *Prove that a bounded set $A \subset \mathbb{R}^n$ has zero volume, if and only if for any $\epsilon > 0$, there exist cubes $Q_1, Q_2, ..., Q_m \subset \mathbb{R}^n$ such that $A \subset \bigcup_{i=1}^{m} Q_i$ and $\sum_{i=1}^{m} Vol(Q_i) < \epsilon$.*

Pf:

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Question 4 *Let $a \in \mathbb{R}^2$ and $r > 0$. Prove that the disc $D_r(a) = \{x \in \mathbb{R}^2 \mid |x - a| < r\} \subset \mathbb{R}^2$ is Riemann-measurable and $Vol(D_r(a)) = \pi r^2$.*

Pf:

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Question 5 *Let $R \subset \mathbb{R}^n$ be a finite rectangle and let $F : R \to [0, \infty)$ be Riemann-integrable. Prove that if $\int_R f(x)dx = 0$, then for any $\epsilon > 0$, the set $\{x \in R \mid f(x) \neq 0\}$ can be covered by an infinite sequence of rectangles $\{R_k\}_{k=1}^{\infty} \subset R$ such that $\sum_{k=1}^{\infty} Vol(R_k) < \epsilon$.*

Pf: