Math 118C HW5

Zih-Yu Hsieh

May 31, 2025

1

Question 1 *Let $R_1 \subset R_2 \subset \mathbb{R}^n$ be finite rectangles. Using the definition of volume, prove that the set $R_2 \setminus R_1$ is Riemann-Measurable and $Vol(R_2 \setminus R_1) = Vol(R_2) - Vol(R_1)$.*

Pf:

First, since $R_1, R_2, (R_2 \setminus R_1) \subseteq R_2$, let $\chi_{R_1}, \chi_{R_2}, \chi : R_2 \to \mathbb{R}$ be the characteristic functions of $R_1, R_2, R_1 \setminus R_2$ respectively. Then, they take the following form:

$$\chi_{R_1}(x) = \begin{cases} 1 & x \in R_1 \\ 0 & x \notin R_1 \end{cases}, \quad \chi_{R_2}(x) = 1, \quad \chi(x) = \begin{cases} 1 & x \in R_2 \setminus R_1 \\ 0 & x \in R_2 \setminus (R_2 \setminus R_1) = R_1 \end{cases} \tag{1}$$

Which, if consider the function $\chi_{R_2} - \chi_{R_1}$, we get:

$$\begin{aligned} x \in R_2 \setminus R_1 &\implies \chi_{R_2}(x) - \chi_{R_1}(x) = 1 - 0 = 1 = \chi(x) \\ x \in R_1 \subseteq R_2 &\implies \chi_{R_2}(x) - \chi_{R_1}(x) = 1 - 1 = 0 = \chi(x) \end{aligned} \tag{2}$$

Hence, $\chi_{R_2} - \chi_{R_1} = \chi$ (the characteristic function of $R_2 \setminus R_1$), so since R_2, R_1 are rectangles, then χ_{R_2}, χ_{R_1} are both Riemann-Integrable over R_2, then $\chi = \chi_{R_2} - \chi_{R_1}$ is also Riemann-Integrable over R_2.

Because χ is the characteristic function of $R_2 \setminus R_1$, then being Riemann-Integrable implies $R_2 \setminus R_1$ is Riemann-Measurable. Furthermore, its volume is given as follow:

$$\begin{aligned} \text{Vol}(R_2 \setminus R_1) &= \int_{R_2} \chi(x)dx = \int_{R_2} (\chi_{R_2}(x) - \chi_{R_1}(x))dx \\ &= \int_{R_2} \chi_{R_2}(x)dx - \int_{R_2} \chi_{R_1}(x)dx = \text{Vol}(R_2) - \text{Vol}(R_1) \end{aligned} \tag{3}$$

2

Question 2

1. *Let $R, R_1, R_2, ..., R_m \subset \mathbb{R}^n$ be finite rectangles such that $R \subset \bigcup_{i=1}^m R_i$. Without using characteristic functions and integration, prove that $Vol(R) \leq \sum_{i=1}^m Vol(R_i)$.*

2. *Let $\tilde{R}_1, \tilde{R}_2, ..., \tilde{R}_k \subset \mathbb{R}^n$ be finite rectangles and let $R_1, R_2, ..., R_m \subset \mathbb{R}^n$ be nonoverlapping finite rectangles such that $\bigcup_{i=1}^m R_i \subset \bigcup_{j=1}^k \tilde{R}_j$. Without using characteristic functions and integration, prove that $\sum_{i=1}^m Vol(R_i) \leq \sum_{j=1}^k Vol(\tilde{R}_j)$.*

Pf:

1.

Since $R, R_1, ..., R_m \subset \mathbb{R}^n$ are all bounded, then there exists a finite rectangle $R^* \subset \mathbb{R}^n$, such that $R, R_1, ..., R_m \subset R^*$. Then, we can define a partition P of R^* that is formed by all the sides of R_j and R. (i.e. given the i^{th} side of R^*, say $[a_i^*, b_i^*]$, if R has the i^{th} side $[a_i, b_i]$ and R_j has the i^{th} side $[a_i^{(j)}, b_i^{(j)}]$, the create the i^{th} partition by including a_i, b_i and all $a_i^{(j)}, b_i^{(j)}$, and use these partitions of each side to form a partition of R^*).

Which, since each endpoints of all R_i is contained in P, then some subrectangles form a partition of R_i. Similarly, since the endpoints of each side of R is also within the partition P, then some of the subrectangles also form a partition of R. Let $P' \subseteq P$ collect all the subrectangles of P that is contained in R, which it is precisely the partion of R under P. Notice they satisfy the following equation:

$$\sum_{R' \in P'} \text{Vol}(R') = \text{Vol}(R) \tag{4}$$

Then, because $R \subseteq \bigcup_{i=1}^m R_i$, then each of the subrectangles $R' \in P'$ is contained within some R_i (where $i \in \{1, ..., m\}$). Then, consider the following process:

(1) Let $C_1 = \{R_1^{(1)}, R_2^{(1)}, ..., R_{l_1}^{(i)}\} \subseteq P'$ denote all the rectangles contained in R_1. Then, since these rectangles are not overlapping while contained in R_1, they are part of the partitions of R_1 formed by P mentioned before. Hence, because the subrectangles of a partition of a larger rectangle has the sum of volume at most the larger rectangle's volume, we get the following:

$$\sum_{j=1}^{l_1} \text{Vol}(R_j^{(1)}) \leq \text{Vol}(R_1) \tag{5}$$

Which, take the remaining subrectangles $P' \setminus C_1$ and proceed with the process.

(2) Let $C_2 = \{R_1^{(2)}, R_2^{(2)}, ..., R_{l_2}^{(2)}\} \subseteq (P' \setminus C_1)$ denote all the rectangles contained in R_2 (which C_2 is chosen such that C_1 and C_2 are disjoint). Again, because they're nonoverlapping, while contained in R_2, they're part of the partitions of R_1 formed by P. Hence, we again get the following:

$$\sum_{j=1}^{l_2} \text{Vol}(R_j^{(2)}) \leq \text{Vol}(R_2) \tag{6}$$

Now, take $P' \setminus (C_1 \sqcup C_2)$ and proceed.

(k) For each step $k \leq m$, let $C_k = \{R_1^{(k)}, R_2^{(k)}, ..., R_{l_k}^{(k)}\} \subseteq P' \setminus (C_1 \sqcup C_2 \sqcup ... \sqcup C_{k-1})$ denote all the remaining rectangles contained in R_k. Which, based on similar logics before, due to the properties of partition, we get:

$$\sum_{j=1}^{l_k} \text{Vol}(R_j^{(k)}) \leq \text{Vol}(R_k) \tag{7}$$

Which, the above process groups the partition P' into disjoint subsets of C_i (which $P' = \sqcup_{i=1}^m C_i$), such that each subset C_i is corresponding to a specific rectangle R_i containing the subrectangles in this subset, and each C_i has no intersection, hence no subrectangles in the partition is overcounted. Then, based on the formula from above, we get:

$$\text{Vol}(R) = \sum_{R' \in P'} \text{Vol}(R') = \sum_{i=1}^{m} \left(\sum_{j=1}^{l_i} \text{Vol}(R_j^{(i)}) \right) \leq \sum_{i=1}^{m} \text{Vol}(R_i) \tag{8}$$

Which, this is the desired inequality.

2.

Since each of the finite rectangles $R_1, ..., R_m, \tilde{R}_1, ..., \tilde{R}_k \subset \mathbb{R}^n$ are bounded, there exists a finite rectangles $R \subset \mathbb{R}^n$ that contains all of R_i and $\tilde{R}_j$. Then, take a partition P of R that is formed with all the endpoints of each sides of every R_i and $\tilde{R}_j$ (the same logic mentioned in the previous section), the partition P contains subrectangles that form partitions for each R_i and $\tilde{R}_j$.

Which, let $P' \subseteq P$ be the collection of all the subrectangles in P, that is contained in some R_i, then since P contains all the partitions of R_i, the subcollection P' contains all the subrectangles that covers $R_1, ..., R_m$ with their union. Together with the fact that the R_is are not overlapping, and the collection P' are also formed by nonoverlapping subrectangles, then we get the following:

$$\sum_{R' \in P'} \text{Vol}(R') = \sum_{i=1}^{m} \text{Vol}(R_i) \tag{9}$$

Which, notice that the previous section's method can be used:

(1) Let $C_1\{R_1^{(1)}, ..., R_{l_1}^{(1)}\} \subseteq P'$ be all the subrectangles contained in $\tilde{R}_1$. Then, since none of the subrectangles in C_1 is overlapping, while contained in $\tilde{R}_1$, then they're part of the partitions of $\tilde{R}_1$ formed by P. Hence, we get the following inequality:

$$\sum_{j=1}^{l_1} \text{Vol}(R_j^{(1)}) \leq \text{Vol}(\tilde{R}_1) \tag{10}$$

Which, take $P' \setminus C_1$ and proceed.

(2) Let $C_2 = \{R_1^{(2)}, ..., R_{l_2}^{(2)}\} \subseteq (P' \setminus C_1)$ be all the subrectangles contained in $\tilde{R}_2$ (which C_1, C_2 are disjoint). Then again, following similar logic from step (1), we get:

$$\sum_{j=1}^{l_2} \text{Vol}(R_j^{(2)}) \leq \text{Vol}(\tilde{R}_2) \tag{11}$$

Which, take $P' \setminus (C_1 \sqcup C_2)$ and proceed.

(p) For any step $p \leq k$, let $C_p = \{R_1^{(p)}, ..., R_{l_p}^{(p)}\} \subseteq (P' \setminus (C_1 \sqcup ... \sqcup C_{p-1}))$ be all the subrectangles contained in $\tilde{R}_p$. Then again, we get the following inequality:

$$\sum_{j=1}^{l_p} \text{Vol}(R_j^{(p)}) \leq \text{Vol}(\tilde{R}_p) \tag{12}$$

Hence, P' is separated into disjoint subsets C_p (where $P' = \sqcup_{p=1}^{k} C_p$), such that each C_p corresponds to $\tilde{R}_p$ that contains all the subrectangles in C_p, and no subrectangle in the collection P' is overcounted based on this disjoint way of grouping P'. Then, based on all the inequalities from above, we get:

$$\sum_{i=1}^{m} \text{Vol}(R_i) = \sum_{R' \in P'} \text{Vol}(R') = \sum_{p=1}^{k} \left(\sum_{j=1}^{l_p} \text{Vol}(R_j^{(p)}) \right) \leq \sum_{p=1}^{k} \text{Vol}(\tilde{R}_p) \tag{13}$$

Hence, this proves the desired inequality.

3

Question 3 *Prove that a bounded set $A \subset \mathbb{R}^n$ has zero volume, if and only if for any $\epsilon > 0$, there exist cubes $Q_1, Q_2, ..., Q_m \subset \mathbb{R}^n$ such that $A \subset \bigcup_{i=1}^m Q_i$ and $\sum_{i=1}^m$ Vol$(Q_i) < \epsilon$.*

Pf:

Recall that a bounded set $A \subset \mathbb{R}^n$ satisfies $\text{Vol}(A) = 0$, iff for all $\epsilon > 0$ there exists finitely many rectangles $R_1, ..., R_m \subset \mathbb{R}^n$ such that $A \subseteq \bigcup_{i=1}^m R_i$ and $\sum_{i=1}^m \text{Vol}(R_i) < \epsilon$.

Also, notice that every cube $Q \subseteq \mathbb{R}^n$ is a rectangle.

$\Longrightarrow$: Suppose $A \subset \mathbb{R}^n$ has zero volume, by the equivalent condition, for any $\epsilon > 0$, there eixsts finitely many rectangles $R_1, ..., R_m \subset \mathbb{R}^n$, such that $A \subseteq \bigcup_{i=1}^m R_i$ and $\sum_{i=1}^m \text{Vol}(R_i) < \epsilon$. Which, consider the following lemmas:

Lemma 1 *Given finite rectangle $R \subset \mathbb{R}^n$ with side lengths $l_1, ..., l_k \in \mathbb{R}_{>0}$, if all side length is rational, then R can be partitioned into finitely many cubes.*

Proof of Lemma 1: Suppose $l_1, ..., l_k \in \mathbb{Q}_{>0}$, then for each index $i \in \{1, ..., k\}$, there exists $m_i, n_i \in \mathbb{N}$, such that $l_i = \frac{m_i}{n_i}$. Then, take length $l = \frac{1}{m_1 ... m_k n_1 ... n_k}$, for each index $i \in \{1, ..., k\}$, the division shows the following:

$$\frac{l_i}{l} = \frac{m_i}{n_i}(m_1 ... m_k n_1 ... n_k) = m_i(m_1 ... m_k) \cdot \prod_{\substack{j=1 \\ j \neq i}}^k n_j \in \mathbb{N} \tag{14}$$

So, since each side length l_i can be partitioned into integer copies of l, then each side has a partition P_i, where all interval has length l. Unify all partitions for each side, we get a partition P on R, where each subrectangle has side length provided by an interval of each P_i, which each side length is l. Hence, all subrectangle is in fact a cube, so P partitions R into finitely many cubes.

Lemma 2 *Given finite rectangle $R \subset \mathbb{R}^n$, for all $\epsilon > 0$, there exists a larger rectangle $R' \subset \mathbb{R}^n$ with rational side lengths containing R, and Vol(R') − Vol$(R) < \epsilon$.*

Proof of Lemma 2: If R has all side lengths being rational, then $R' = R$ provides the solution. So, can assume R has at least one side length being irrational.

Let $R = [a_1, b_1] \times ... \times [a_n, b_n]$, and for each index $i \in \{1, ..., n\}$, let $l_i = b_i - a_i$ (the i^{th} side length of R). For $x \in \mathbb{R}_{\geq 0}$, define $R_x = [a_1, b_1 + x] \times ... \times [a_n, b_n + x]$. Then, $\text{Vol}(R_x)$ is given by:

$$\text{Vol}(R_x) = \prod_{i=1}^n ((b_i + x) - a_i) = \prod_{i=1}^n (l_i + x) \in \mathbb{R}[x] \tag{15}$$

(Note: by the definition of R_x with $x \geq 0$, it's clear that $R \subseteq R_x$; and since each side length of R_x has $l_i + x \geq l_i$, this also implies $\text{Vol}(R_x) \geq \text{Vol}(R)$).

Since $\text{Vol}(R_x)$ for $x \geq 0$ can be expressed as a real polynomial, then as a function of x, it is continuous. Notice that given $x = 0$, we have $R_0 = R$, hence $\text{Vol}(R_0) = \text{Vol}(R)$. Which, by continuity, for all $\epsilon > 0$, there exists $\delta > 0$, such that $0 \leq x < \delta$ implies $|\text{Vol}(R_x) - \text{Vol}(R_0)| = \text{Vol}(R_x) - \text{Vol}(R) < \epsilon$.

Now, for each index $i \in \{1, ..., n\}$, consider the interval $[l_i, l_i + \delta)$: By the deneseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists rational number within this interval. Hence, there exists $x_i \in [0, \delta)$, such that $l_i + x_i \in [l_i, l_i + \delta)$

and $l_i + x_i \in \mathbb{Q}$. So, if we consider the rectangle $R' = [a_1, a_1 + l_1 + x_1] \times ... \times [a_n, a_n + l_n + x_n] = [a_1, b_1 + x_1] \times [a_n, b_n + x_n]$, and the number $x = \max\{x_1, ..., x_n\}$ (note: since each $0 \leq x_i < \delta$, then $0 \leq x < \delta$), then one can see $R \subseteq R' \subseteq R_x$ (since $R_x = [a_1, b_1 + x] \times ... \times [a_n, b_n + x]$, and each side of R' is given by $[a_i, b_i + x_i]$ with $0 \leq x_i \leq x$).

Hence, such inclusion implies $\text{Vol}(R) \leq \text{Vol}(R') \leq \text{Vol}(R_x)$; with $0 \leq x < \delta$, this also shows that $0 \leq \text{Vol}(R') - \text{Vol}(R) \leq \text{Vol}(R_x) - \text{Vol}(R) < \epsilon$. On the other hand, since R' has side length $(b_i + x_i) - a_i = l_i + x_i \in \mathbb{Q}$, it has all side lengths being rational, and this finishes the proof.

Now, with the above two lemmas in mind, we can prove the statement.

Given $\text{Vol}(A) = 0$, using the equivalent condition of zero volume, for all $\epsilon > 0$, since $\frac{\epsilon}{2} > 0$, then there exists finite rectangles $R_1, ..., R_m \subset \mathbb{R}^n$, such that $A \subseteq \bigcup_{i=1}^m R_i$, and $\sum_{i=1}^m \text{Vol}(R_i) < \frac{\epsilon}{2}$.

Notice that because $\frac{\epsilon}{2m} > 0$, then using **Lemma 2** from above, each index $i \in \{1, ..., m\}$ has a corresponding R_i' with rational side lengths, such that $R_i \subseteq R_i'$, and $\text{Vol}(R_i') - \text{Vol}(R_i) < \frac{\epsilon}{2m}$. Then, the collection $\{R_1', ..., R_m'\}$ satisfies $A \subseteq \bigcup_{i=1}^m R_i \subseteq \bigcup_{i=1}^m R_i'$, and the sum of volume satisfies the following:

$$\begin{aligned} &\sum_{i=1}^m \text{Vol}(R_i') - \sum_{i=1}^m \text{Vol}(R_i) = \sum_{i=1}^m (\text{Vol}(R_i') - \text{Vol}(R_i)) < \sum_{i=1}^m \frac{\epsilon}{2m} = \frac{\epsilon}{2} \\ &\implies \sum_{i=1}^m \text{Vol}(R_i') < \sum_{i=1}^m \text{Vol}(R_i) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned} \tag{16}$$

On the other hand, each R_i' has rational side lengths, hence by **Lemma 1**, it can be partitioned into finitely many cubes $\{Q_1^{(i)}, ..., Q_{l_i}^{(i)}\}$, such that $R_i' = \bigcup_{j=1}^{l_i} Q_j^{(i)}$, and $\sum_{j=1}^{l_i} \text{Vol}(Q_j^{(i)}) = \text{Vol}(R_i')$.

Hence, take the finite collection of cubes $\mathcal{F} = \bigcup_{i=1}^m \{Q_1^{(i)}, ..., Q_{l_i}^{(i)}\}$, we get the following two relationships:

$$A \subseteq \bigcup_{i=1}^m R_i' = \bigcup_{i=1}^m \left(\bigcup_{j=1}^{l_i} Q_j^{(i)} \right) = \bigcup_{Q \in \mathcal{F}} Q \tag{17}$$

$$\sum_{Q \in \mathcal{F}} \text{Vol}(Q) = \sum_{i=1}^m \left(\sum_{j=1}^{l_i} \text{Vol}(Q_j^{(i)}) \right) = \sum_{i=1}^m \text{Vol}(R_i') < \epsilon \tag{18}$$

Hence, $\mathcal{F}$ is a desired collection of cubes with all satisfying properties.

$\Longleftarrow$: Suppose for any $\epsilon > 0$, there exist cubes $Q_1, ..., Q_m \subset \mathbb{R}^n$ with $A \subseteq \bigcup_{i=1}^m Q_i$ and $\sum_{i=1}^m \text{Vol}(Q_i) < \epsilon$, then since each cube is also a rectangle, this satisfies the equivalent condition for $\text{Vol}(A) = 0$. Hence, A has zero volume.

4

Question 4 *Let $a \in \mathbb{R}^2$ and $r > 0$. Prove that the disc $D_r(a) = \{x \in \mathbb{R}^2 \mid |x - a| < r\} \subset \mathbb{R}^2$ is Riemann-Measurable and Vol$(D_r(a)) = \pi r^2$.*

Pf:
Let $a = (a_x, a_y) \in \mathbb{R}^2$. We'll break the proofs into following sections:

1. $D_r(a)$ is Riemann-Measurable:

Given $D_r(a) = \{x \in \mathbb{R}^2 \mid |x-a| < r\}$, then its boundary $\partial D_r(a) = \{x \in \mathbb{R}^2 \mid |x-a| = r\}$. Which, if consider the map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follow:

$$\varphi(\rho, \theta) = (a_x + \rho\cos(\theta), a_y + \rho\sin(\theta)) \tag{19}$$

Then, φ is continuously differentiable. If we take a straight line segment $I = \{(x, y) \in \mathbb{R}^2 \mid x = r,\ 0 \leq y \leq 2\pi\}$, then since this line segment is bounded, $D\varphi$ as a continuous function is bounded on any compact subset containing I, hence φ is in fact Lipschitz on some compact subset containing I.

Also, since for all $\epsilon > 0$, choose the rectangle $R = [r, r + \frac{\epsilon}{2\pi}] \times [0, 2\pi]$, then it's clear that $I \subseteq R$, and since $\text{Vol}(R) = \frac{\epsilon}{2\pi} \cdot 2\pi = \epsilon$, then I can be covered by rectangle with arbitrary volume, showing that $\text{Vol}(I) = 0$.

Which, if consider the image $\varphi(I)$, since any $(x, y) \in I$ has $(x, y) = (r, \theta)$ for some $\theta \in [0, 2\pi]$, hence $\varphi(x, y) = (a_x + r\cos(\theta), a_y + r\sin(\theta))$, which $|\varphi(x, y) - a| = |(r\cos(\theta), r\sin(\theta))| = r$, showing that $\varphi(x, y) \in \partial D_r(a)$, or $\varphi(I) \subseteq \partial D_r(a)$;

on the other hand, for any $x \in \partial D_r(a)$, since $|x - a| = r$, then there exists $\theta \in [0, 2\pi]$, such that $x - a = (r\cos(\theta), r\sin(\theta))$, hence $(r, \theta) \in I$ satisfies $\varphi(r, \theta) = (a_x + r\cos(\theta), a_y + r\sin(\theta)) = a + (x - a) = x$, showing that $\partial D_r(a) \subseteq \varphi(I)$, or $\partial D_r(a) = \varphi(I)$.

As a conclusion, since φ is Lipschitz on some compact subset containing I, $\text{Vol}(I) = 0$, and $\partial D_r(a) = \varphi(I)$, this shows that $\text{Vol}(\partial D_r(a)) = 0$, which is equivalent to $D_r(a)$ is Riemann-Measurable.

2. Volume of $D_r(a)$:

For this part, we'll utilize Change of Variable.

Let $I = \{(x, y) \in \mathbb{R}^2 \mid y = a_y,\ a_x \leq x < (a_x + r)\}$ (a horizontal line segment cutting through the disk $D_r(a)$ containing the center), and open set $D' = D_r(a) \setminus I$ (which D' defines a disk cutting out a line). Which, using similar proof from section 1, we know that $\text{Vol}(I) = 0$ (volume of a bounded straight line always has volume 0); also, since $\partial D' = \partial D_r(a) \cup I$, with both having volume 0, then $\text{Vol}(D') = 0$, showing that D' is Riemann-Measurable. Then, because $D_r(a) = D' \sqcup I$, then we get the following equation:

$$\text{Vol}(D_r(a)) = \text{Vol}(D') + \text{Vol}(I) = \text{Vol}(D') \tag{20}$$

Now, to find $\text{Vol}(D')$, we'll utilize the function φ defined in Section 1: Define rectangle $R = [0, r] \times [0, 2\pi] \subset \mathbb{R}^2$, and consider the interior $R^\circ = (0, r) \times (0, 2\pi)$: For any $(\rho, \theta) \in R^\circ$, $\varphi(\rho, \theta) = (a_x + \rho\cos(\theta), a_y + \rho\sin(\theta))$, hence $|\varphi(\rho, \theta) - a| = |(\rho\cos(\theta), \rho\sin(\theta))| = \rho < r$, so $\varphi(\rho, \theta) \in D_r(a)$.

On the other hand, if $a_y + \rho\sin(\theta) = a_y$, then $\rho\sin(\theta) = 0$, which enforces $\theta = \pi$; but, this implies $a_x + \rho\cos(\theta) = a_x + \rho\cos(\pi) = a_x - \rho < a_x$, this shows that $\varphi(\rho, \theta) \notin I$ (since it doesn't satisfy the set axiom of I), hence $\varphi(\rho, \varphi) \in D_r(a) \setminus I = D'$, showing that $\varphi : R^\circ \to D'$ is a well-defined C^1 continuous map.

Then, notice that φ is bijective after the restriction:

Suppose (ρ_1, θ_1) and (ρ_2, θ_2) have the same output, then we know since $|\varphi(\rho, \theta) - a| = \rho$ based on some calculations above, then $\varphi(\rho_1, \theta_1) = \varphi(\rho_2, \theta_2)$ enforces $\rho_1 = \rho_2 = \rho$. Then, this implies that $\varphi(\rho, \theta_1) = (a_x + \rho\cos(\theta_1), a_y + \rho\sin(\theta_1)) = \varphi(\rho, \theta_2) = (a_x + \rho\cos(\theta_2), a_y + \rho\sin(\theta_2))$, or $\rho\cos(\theta_1) = \rho\cos(\theta_2)$ and $\rho\sin(\theta_1) = \rho\sin(\theta_2)$. Which, since $\theta_1, \theta_2 \in (0, 2\pi)$, we must have $\theta_1 = \theta_2$, which proves the injectivity.

Now, for each $(x, y) \in D'$, since $(x, y) \notin I$, which implies that $(x, y) - a$ is not on the positive x-axis (I is a horizontal straight line going to the right of the disk from the center); together with the fact that $|(x, y) - a| < r$, then $(x, y) - a$ can be represented with some $(\rho, \theta) \in R^\circ = (0, r) \times (0, 2\pi)$ under polar coordinates, or $(x, y) - a = (\rho\cos(\theta), \rho\sin(\theta))$. This shows that $(x, y) = (a_x + \rho\cos(\theta), a_y + \rho\sin(\theta)) = \varphi(\rho, \theta)$, which proves the surjectivity.

Moreover, φ is in fact a diffeomorphism on R°: For all $(\rho, \theta) \in R^\circ$, we have $\rho > 0$. Which, φ has its differential and determinant given as follow:

$$\varphi(\rho, \theta) = (\varphi_1, \varphi_2) = (a_x + \rho\cos(\theta), a_y + \rho\sin(\theta)) \tag{21}$$

$$\begin{aligned} D\varphi(\rho, \theta) &= \begin{pmatrix} \frac{\partial \varphi_1}{\partial \rho} & \frac{\partial \varphi_1}{\partial \theta} \\ \frac{\partial \varphi_2}{\partial \rho} & \frac{\partial \varphi_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\rho\sin(\theta) \\ \sin(\theta) & \rho\cos(\theta) \end{pmatrix} \\ \implies |\det(D\varphi(\rho, \theta))| &= |\rho\cos^2(\theta) + \rho\sin^2(\theta)| = |\rho| = \rho > 0 \end{aligned} \tag{22}$$

Since at each (ρ, θ), the differential has nonzero determinant, then the differential is invertible. Since $\varphi : R^\circ \to D'$ is bijective, with the differential being invertible at all point of R°, then φ forms a diffeomorphism.

With all the tools established above, using **Change of Variable**, we get the following:

$$\mathrm{Vol}(D') = \int_{\varphi(R^\circ) = D'} 1 dy = \int_{R^\circ} 1 \circ \varphi(x) \cdot |\det(D\varphi(x))| dx = \int_{R^\circ} \rho dx \tag{23}$$

Our final goal is to do this estimation.

First, we'll prove that $\mathrm{Vol}(D') \leq \pi r^2$: Let χ_{R° be the characteristic function of R°, then for any $x = (\rho, \theta) \in R$, $\chi_{R^\circ}(x) = 1$ if $x \in R^\circ$, and 0 elsewhere. So, notice that the function $\rho \cdot \chi_{R^\circ}(x) \leq \rho$ for all $x \in R$. Hence, we get the following based on the definition of characteristic function:

$$\mathrm{Vol}(D') = \int_{R^\circ} \rho dx = \int_R \rho \cdot \chi_{R^\circ}(x) dx \leq \int_R \rho dx \tag{24}$$

Which, the last integral above can be explicitly written as follow using **Fubini's Theorem**:

$$\int_R \rho dx = \int_{\rho=0}^{r} \int_{\theta=0}^{2\pi} \rho d\theta d\rho = 2\pi \int_{\rho=0}^{r} \rho d\rho = 2\pi \cdot \frac{\rho^2}{2}\bigg|_0^r = \pi r^2 \tag{25}$$

Hence, combining the above two expressions, we get:

$$\mathrm{Vol}(D') = \int_{R^\circ} \rho dx \leq \int_R \rho dx = \pi r^2 \tag{26}$$

Now, we'll prove that $\mathrm{Vol}(D') \geq \pi r^2$: Assume for suitable $K \in \mathbb{N}$, any integer $k \geq K$ satisfies $\frac{1}{2k}, (r - \frac{1}{2k}) \in (0, r)$, and $\frac{1}{2k}, (2\pi - \frac{1}{2k}) \in (0, 2\pi)$, and the second value is greater than the first one for both pairs. Then, the following rectangle $R_k = [\frac{1}{2k}, r - \frac{1}{2k}] \times [\frac{1}{2k}, 2\pi - \frac{1}{2k}] \subset R^\circ$. Which, because R_k is Riemann-Measurable and φ is a diffeomorphism, then $\varphi(R_k) \subseteq \varphi(R^\circ) = D'$ is also Riemann-Measurable, with $\mathrm{Vol}(\varphi(R_k)) \leq \mathrm{Vol}(D')$.

Again, apply **Change of Variable** and **Fubini's Theorem**, we get the following:

$$\begin{aligned}\mathrm{Vol}(\varphi(R_k)) &= \int_{\varphi(R_k)} 1dy = \int_{R_k} 1 \circ \varphi(x)|\det(D\varphi)(x)|dx = \int_{R_k} \rho dx \\ &= \int_{\rho=\frac{1}{2k}}^{r-\frac{1}{2k}} \int_{\theta=\frac{1}{2k}}^{2\pi-\frac{1}{2k}} \rho d\theta d\rho = \left(2\pi - \frac{1}{k}\right) \int_{\rho=\frac{1}{2k}}^{r-\frac{1}{2k}} \rho d\rho = \left(2\pi - \frac{1}{k}\right) \frac{\rho^2}{2}\Bigg|_{\frac{1}{2k}}^{r-\frac{1}{2k}} \\ &= \left(\pi - \frac{1}{2k}\right) r \left(r - \frac{1}{k}\right) = \pi r^2 - \frac{\pi r}{k} - \frac{r^2}{2k} + \frac{r}{2k^2}\end{aligned} \tag{27}$$

With the previous inequality, we get the following:

$$\mathrm{Vol}(\varphi(R_k)) = \pi r^2 - \frac{\pi r}{k} - \frac{r^2}{2k} + \frac{r}{2k^2} \leq \mathrm{Vol}(D') \tag{28}$$

Then, if take the limit, we get that $\lim_{k\to\infty} \mathrm{Vol}(\varphi(R_k)) = \pi r^2$, which with the above inequality, we know that the limit $\pi r^2 \leq \mathrm{Vol}(D')$.

So, combining both inequalities about πr^2 and $\mathrm{Vol}(D')$, we get that $\mathrm{Vol}(D') = \pi r^2$.

5

Question 5 *Let $R \subset \mathbb{R}^n$ be a finite rectangle and let $f : R \to [0, \infty)$ be Riemann-integrable. Prove that if $\int_R f(x)dx = 0$, then for any $\epsilon > 0$, the set $\{x \in R \mid f(x) \neq 0\}$ can be covered by an infinite sequence of rectangles $\{R_k\}_{k=1}^{\infty} \subset R$ such that $\sum_{k=1}^{\infty} Vol(R_k) < \epsilon$.*

Pf:

Given all the stated conditions, let $S = \{x \in R \mid f(x) \neq 0\}$. WLOG, assume $S \neq \emptyset$. We'll break the proof down into multiple steps with the following order:

1. Separate S as infinite subsets, and generate associated functions:

First, let $S_1 = \{x \in S \mid f(x) \in (\frac{1}{2}, \infty)\}$, and for each integer $n \geq 2$, let $S_n = \{x \in S \mid f(x) \in (\frac{1}{2^n}, \frac{1}{2^{n-1}}]\}$. Then, it is clear that $\bigcup_{n=1}^{\infty} S_n \subseteq S$; also, for each $x \in S$, since $f(x) \neq 0$ and the codomain of f is $[0, \infty)$, hence $f(x) > 0$, which there exists smallest $k \in \mathbb{N}$, such that $\frac{1}{2^k} < f(x)$, so $\frac{1}{2^k} < f(x) \leq \frac{1}{2^{k-1}}$, or $s \in S_k \subseteq \bigcup_{n=1}^{\infty} S_n$. Hence, we can claim that $S = \bigcup_{n=1}^{\infty} S_n$.

Now, for each S_n, define $f_n : R \to [0, \infty)$ by the following definition:

$$f_n(x) = \begin{cases} f(x) & x \in S_n \\ 0 & x \notin S_n \end{cases} \tag{29}$$

Notice that for all $x \in R$, if $x \in S_n$, then $f_n(x) = f(x)$, otherwise $f_n(x) \leq f(x)$ (since if $x \notin S$, then $f_n(x) = f(x) = 0$; else if $s \in S \setminus S_n$, then $f_n(x) = 0$, while $f(x) > 0$). So, on the domain R, $f_n(x) \leq f(x)$. Also, notice that $f_n(x)$ is Riemann-Integrable: Since f is Riemann-Integrable with $\int_R f(x)dx = 0$, then for any $\epsilon > 0$, there exists partition P of R, such that the following holds:

$$U(f, P) - \int_R f(x)dx = U(f, P) = \sum_{R_i \in P} \sup_{x \in R_i} (f(x)) \cdot \text{Vol}(R_i) < \epsilon \tag{30}$$

Then, since $0 \leq f_n(x) \leq f(x)$ on R, for any $R_i \in P$, we get $\sup_{x \in R_i}(f_n(x)) \leq \sup_{x \in R_i}(f(x))$, hence:

$$\overline{\int_R} f_n(x)dx \leq U(f_n, P) = \sum_{R_i \in P} \sup_{x \in R_i} (f_n(x)) \cdot \text{Vol}(R_i) \leq \sum_{R_i \in P} \sup_{x \in R_i} (f(x)) \cdot \text{Vol}(R_i) < \epsilon \tag{31}$$

On the other hand, since $f_n(x)$ is nonnegative, then for every $R_i \in P$, $\inf_{x \in R_i}(f_n(x)) \geq 0$, hence:

$$0 \leq \sum_{R_i \in P} \inf_{x \in R_i} (f_n(x)) \cdot \text{Vol}(R_i) = L(f_n, P) \leq \underline{\int_R} f_n(x)dx \leq \overline{\int_R} f_n(x)dx < \epsilon \tag{32}$$

Since ϵ is arbitrary, the above inequality shows that the lower and upper integral of $f_n(x)$ are both equal to 0, hence $f_n(x)$ is Riemann-Integrable, and $\int_R f_n(x)dx = 0$.

2. Finite covers for each S_n:

Fix an arbitrary $\epsilon > 0$ for universal purpose.

Now, for each $n \in \mathbb{N}$, since $\frac{\epsilon}{2^{2n}} > 0$, then based on the Rimann Integral of $f_n(x)$, there exists partition $P^{(n)}$ of R, such that the following holds true:

$$0 \leq U(f_n, P^{(n)}) - \int_R f_n(x)dx = U(f_n, P^{(n)}) = \sum_{R_i^{(n)} \in P^{(n)}} \sup_{x \in R_i^{(n)}} (f_n(x)) \cdot \text{Vol}(R_i^{(n)}) < \frac{\epsilon}{2^{2n}} \tag{33}$$

Since $f_n(x) > 0$ iff $x \in S_n$, the above can be rewrite as:

$$0 \leq \sum_{R_i^{(n)} \in P^{(n)}} \sup_{x \in R_i} (f_n(x)) \cdot \mathrm{Vol}(R_i^{(n)}) = \sum_{\substack{R_i^{(n)} \in P^{(n)} \\ R_i^{(n)} \cap S_n \neq \emptyset}} \sup_{x \in R_i^{(n)}} (f_n(x)) \cdot \mathrm{Vol}(R_i^{(n)}) < \frac{\epsilon}{2^{2n}}$$

Moreover, since for all $x \in S_n$, we have $f(x) > \frac{1}{2^n}$ based on the set axiom, then for each $R_i^{(n)}$ with $R_i^{(n)} \cap S_n \neq \emptyset$, $\sup_{x \in R_i^{(n)}} (f_n(x)) > \frac{1}{2^n}$. Hence:

$$0 \leq \sum_{\substack{R_i^{(n)} \in P^{(n)} \\ R_i^{(n)} \cap S_n \neq \emptyset}} \frac{1}{2^n} \cdot \mathrm{Vol}(R_i^{(n)}) \leq \sum_{\substack{R_i^{(n)} \in P^{(n)} \\ R_i^{(n)} \cap S_n \neq \emptyset}} \sup_{x \in R_i^{(n)}} (f_n(x)) \cdot \mathrm{Vol}(R_i^{(n)}) < \frac{\epsilon}{2^{2n}} \tag{34}$$

$$\implies \sum_{\substack{R_i^{(n)} \in P^{(n)} \\ R_i^{(n)} \cap S_n \neq \emptyset}} \mathrm{Vol}(R_i^{(n)}) < 2^n \cdot \frac{\epsilon}{2^{2n}} = \frac{\epsilon}{2^n} \tag{35}$$

WLOG, can assume that the partition $P^{(n)}$ has indexed such that the first j_n subrectangles $R_1^{(n)}, ..., R_{j_n}^{(n)}$ are all the rectangles with nontrivial intersection with S_n. Then, based on the above equation, we get:

$$S_n \subseteq \bigcup_{i=1}^{j_n} R_i^{(n)}, \quad \sum_{i=1}^{j_n} \mathrm{Vol}(R_i^{(n)}) < \frac{\epsilon}{2^n} \tag{36}$$

3. Infinite covers for S:

In the previous section, for each $n \in \mathbb{N}$, we get a cover $C_n = \{R_1^{(n)}, ..., R_{j_n}^{(n)}\}$ for S_n, such that the sum of the volume is less than $\frac{\epsilon}{2^n}$. Then, let $\mathcal{F} = \bigcup_{n \in \mathbb{N}} C_n$, since each C_n is finite, then $\mathcal{F}$ as a collection of rectangles is at most countable. Which, for all $x \in S$, since there exists $k \in \mathbb{N}$ with $x \in S_k$, then $x \in S_k \subseteq \bigcup_{i=1}^{j_n} R_i^{(n)} \subseteq \bigcup_{R_l \in \mathcal{F}} R_l$, hence $S \subseteq \bigcup_{R_l \in \mathcal{F}} R_l$, showing that $\mathcal{F}$ is a collection of at most countable rectangles that covers S.

Now, because $\mathcal{F}$ is at most countable, it can be indexed as $\{R_l\}_{l \in \mathbb{N}}$. To calculate the sum of volume, we'll consider the partial sum first: For each $N \in \mathbb{N}$, since for each index $l \in \{1, ..., N\}$, there exists some $k_l \in \mathbb{N}$, such that $R_l \in C_{k_l}$. Then, let $k_N \in \mathbb{N}$ be the largest integer containing some rectangles from $\{R_1, ..., R_N\}$, then $\{R_1, ..., R_N\} \subseteq \bigcup_{i=1}^{k_N} C_i$, hence when considering the volume, we get:

$$\begin{aligned} 0 \leq \sum_{l=1}^{N} \mathrm{Vol}(R_l) &\leq \sum_{n=1}^{k_N} \sum_{R_i^{(n)} \in C_n} \mathrm{Vol}(R_i^{(n)}) = \sum_{n=1}^{k_N} \left(\sum_{i=1}^{j_n} \mathrm{Vol}(R_i^{(n)}) \right) \\ &\leq \sum_{n=1}^{k_N} \frac{\epsilon}{2^n} < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} < \epsilon \end{aligned} \tag{37}$$

Which, for all $N \in \mathbb{N}$, $\sum_{l=1}^{N} \mathrm{Vol}(R_i)$ is bounded by ϵ; also, since $\mathrm{Vol}(R_l) \geq 0$ for all $l \in \mathbb{N}$, then the partial sum of volume is a nondecreasing sequence. Hence, its limit exist, and:

$$\lim_{N \to \infty} \sum_{l=1}^{N} \mathrm{Vol}(R_i) = \sum_{l=1}^{\infty} \mathrm{Vol}(R_i) \leq \epsilon \tag{38}$$

Which, we constructed a sequence $\{R_l\}_{l \in \mathbb{N}}$, such that $S \subseteq \bigcup_{l \in \mathbb{N}} R_l$, and $\sum_{l=1}^{\infty} \mathrm{Vol}(R_l) \leq \epsilon$, which finishes the claim.