Math CS 122B HW6

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Question 1 *Freitag Chap. V.6 Exercise 5:*
Lef f be an elliptic function for the lattice L. We choose $b_1, ..., b_n$ to be a system of representatives modulo L for the poles of f, and we consider for each j the principal part of f in the pole b_j:

$$\sum_{v=1}^{l_j} \frac{a_{v,j}}{(z-b_j)^v}$$

The Second Liiouville Theorem ensures the relation

$$\sum_{j=1}^{n} a_{1,j} = 0$$

Show:

(a) *Let $c_1, ..., c_n \in \mathbb{C}$ b given numbers, and let $b_1, ..., b_n$ modulo L be a set of different points in $\mathbb{C}/L$. The function*

$$h(z) := \sum_{j=1}^{n} c_j \zeta(z-b_j)$$

constructed by means of the Weierstrass ζ-function, is then elliptic, iff

$$\sum_{j=1}^{n} c_j = 0$$

(b) *Let $b_1, ..., b_n$ be pairwise different modulo L, and let $l_1, ..., l_n$ be prescribed natural numbers. Let $a_{v,j}$ ($1 \leq j \leq n$, $1 \leq v \leq l_j$) be complex numbers such that $\sum_{j=1}^{n} a_{1,j} = 0$ and $a_{l_j,j} \neq 0$ for all j.*

Then, there exists an elliptic function for the lattice L, having poles modulo L exactly in the points $b_1, ..., b_n$, and having the corresponding principal parts respectively equal to

$$\sum_{v=1}^{l_j} \frac{a_{v,j}}{(z-b_j)^v}$$

Pf:

(a) Given the Weierstrass σ-function below ($\sigma : \mathbb{C} \to \mathbb{C}$), the Weierstrass ζ-function ($\zeta : \mathbb{C} \setminus L \to \mathbb{C}$) is defined as:

$$\sigma(z) = z \prod_{\substack{w \in L \\ w \neq 0}} \left(1 - \frac{z}{w}\right) \exp\left(\frac{z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2\right), \quad \zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$$

Based on the formula of σ, it has simple zeros at all $w \in L$; and, it implies that ζ is not defined only on L. Now, to prove the statement, consider the following:

$\Longrightarrow$: Suppose the defined $h(z)$ is elliptic. Then, since for each index $j \in \{1, ..., n\}$, $\sigma(z - b_j)$ has a simple zero at $(w + b_j)$ for each $w \in L$ (which the set $b_j + L$ contains all the simple zeros of $\sigma(z - b_j)$, which is discrete). Then, since $\bigcup_{j=1}^n (b_j + L)$ is also discrete, choose the fundamental region P of lattice L such that ∂P contains no points from $\bigcup_{j=1}^n (b_j + L)$ (the set containing all the zeros of each $\sigma(z - b_j)$, also the set of all undefined points of all $\zeta(z - b_j)$), by the Second Liouville's Theorem, we get the following:

$$0 = \frac{1}{2\pi i} \int_{\partial P} h(z) dz = \frac{1}{2\pi i} \int_{\partial P} \sum_{j=1}^n c_j \zeta(z - b_j) dz = \sum_{j=1}^n c_j \cdot \frac{1}{2\pi i} \int_{\partial P} \frac{\sigma'(z - b_j)}{\sigma(z - b_j)} dz$$

For each $j \in \{1, ..., n\}$, since P only contains one representative of $b_j \in \mathbb{C}/L$, then it only contains one zero of $\sigma(z - b_j)$. Hence, by argument principle, we get the following:

$$\frac{1}{2\pi i} \int_{\partial P} \frac{\sigma'(z - b_j)}{\sigma(z - b_j)} dz = 1 = \text{Number of zeros of } \sigma(z - b_j) \text{ in } P$$

Hence, the original integral becomes:

$$0 = \frac{1}{2\pi i} \int_{\partial P} h(z) dz = \sum_{j=1}^n c_j \cdot \frac{1}{2\pi i} \int_{\partial P} \frac{\sigma'(z - b_j)}{\sigma(z - b_j)} dz = \sum_{j=1}^n c_j$$

So, $\sum_{j=1}^n c_j = 0$.

$\Longleftarrow$: Now, suppose $\sum_{j=1}^n c_j = 0$. For all $w \in L$, since $\sigma(z + w)$ and $\sigma(z)$ both have simple zeros at any $w' \in L$, then $\frac{\sigma(z+w)}{\sigma(z)}$ is an entire function with no zeros in $\mathbb{C}$ (since the zeros cancel out at each $w' \in L$). Hence, there exists an analytic function $h : \mathbb{C} \to \mathbb{C}$, with $\frac{\sigma(z+w)}{\sigma(z)} = e^{h(z)}$. Then, apply derivatives, we get:

$$\frac{\sigma'(z + w)\sigma(z) - \sigma'(z)\sigma(z + w)}{(\sigma(z))^2} = h'(z) e^{h(z)} = h'(z) \cdot \frac{\sigma(z + w)}{\sigma(z)}$$

$$\frac{\sigma'(z + w)\sigma(z + w)}{\sigma(z + w)\sigma(z)} - \frac{\sigma'(z)\sigma(z + w)}{(\sigma(z))^2} = h'(z) \cdot \frac{\sigma(z + w)}{\sigma(z)}$$

$$\frac{\sigma'(z + w)}{\sigma(z + w)} - \frac{\sigma'(z)}{\sigma(z)} = h'(z)$$

On the other hand, since $\left(\frac{\sigma'(z)}{\sigma(z)}\right)' = -\wp(z)$, then:

$$h''(z) = \left(\frac{\sigma'}{\sigma}\right)'(z + w) - \left(\frac{\sigma'}{\sigma}\right)'(z) = (-\wp(z + w)) - (-\wp(z)) = 0$$

Hence, $h(z)$ is in fact a degree 1 polynomial. So, there exists $a_w, b_w \in \mathbb{C}$, such that:

$$\frac{\sigma(z + w)}{\sigma(z)} = e^{h(z)} = e^{a_w z + b_w}, \quad \sigma(z + w) = e^{a_w z + b_w} \sigma(z)$$

Then, apply the derivative, and take its quotient with $\sigma(z+w)$, we get:

$$\sigma'(z+w) = a_w e^{a_w z + b_w}\sigma(z) + e^{a_w z + b_w}\sigma'(z)$$

$$\zeta(z+w) = \frac{\sigma'(z+w)}{\sigma(z+w)} = \frac{a_w e^{a_w z + b_w}\sigma(z) + e^{a_w z + b_w}\sigma'(z)}{e^{a_w z + b_w}\sigma(z)} = a_w + \frac{\sigma'(z)}{\sigma(z)} = a_w + \zeta(z)$$

Which, apply it to the definition of $h(z)$, we get:

$$h(z+w) = \sum_{j=1}^{n} c_j\zeta(z-b_j+w) = \sum_{j=1}^{n} c_j(a_w+\zeta(z-b_j)) = a_j\sum_{j=1}^{n} c_j + \sum_{j=1}^{n} c_j\zeta(z-b_j) = \sum_{j=1}^{n} c_j\zeta(z-b_j) = h(z)$$

(Note: recall that $\sum_{j=1}^{n} c_j$ is assumed to be 0).

Hence, $h(z)$ is an elliptic function.

he above two implication shows that $h(z)$ is an elliptic function iff $\sum_{j=1}^{n} c_j = 0$.

(b) To construct the desired principal part for each point $b_1, ..., b_n$ modulo L, we need to consider the order 1 case separately from the other poles:

For order 1, we have the condition that $\sum_{j=1}^{n} a_{1,j} = 0$, so we can utilize the statement proven in **part (a)**. Notice that $\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$ is the logarithmic derivative of $\sigma(z)$, with the formula given in **part (a)**, we get the following:

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\substack{w\in L\\ w\neq 0}} \left(\frac{-1/w}{1-z/w} + \frac{d}{dz}\left(\frac{z}{w} + \frac{1}{2}\cdot\frac{z^2}{w^2}\right)\right) = \frac{1}{z} + \sum_{\substack{w\in L\\ w\neq 0}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right)$$

This demonstrats that $\zeta(z)$ has its principal part given as $\frac{1}{z-w}$ at all $w \in L$. Hence, $\zeta(z-b_j)$ would have its principal part given as $\frac{1}{z-b_j}$ for all point equivalent to $b_j \mod L$. Which, using the statement in **part (a)**, we know since $\sum_{j=1}^{n} a_{1,j} = 0$, it implies that $h_1(z) = \sum_{j=1}^{n} a_{1,j}\zeta(z-b_j)$ is an elliptic function; moreover, since each b_j is distinct, its principal part is governed by only $a_{1,j}\zeta(z-b_j)$ for each index j, hence this is an elliptic function describing the principal part up to the simple poles at each point.

For order ≥ 2, we could utilize the fact that $\wp(z)$ has a double pole at all $w \in L$. Recall the formula of $\wp(z)$ in series form:

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{w\in L\\ w\neq 0}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right)$$

Which, its principal part is given by $\frac{1}{(z-w)^2}$ at all $w \in L$. So, for any index j with $l_j \geq 2$, to describe the principal part with $\frac{a_{2,j}}{(z-b_j)^2}$ at each point equivalent to $b_j \mod L$, we can use $a_{2,j}\wp(z-b_j)$ (shift the double poles to each point in $b_j + L$).

Besides that, for any $n > 0$, since $\wp(z)$ converges normally within $\mathbb{C} \setminus L$, then its n^{th} order derivative can be performed term by term:

$$\wp^{(n)}(z) = \frac{d^n}{dz^n}\left(\frac{1}{z^2}\right) + \sum_{\substack{w\in L\\ w\neq 0}} \frac{d^n}{dz^n}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right) = \frac{(-1)^n\cdot(n+1)!}{z^{n+2}} + \sum_{\substack{w\in L\\ w\neq 0}} \frac{(-1)^n\cdot(n+1)!}{(z-w)^{n+2}}$$

$$\frac{(-1)^n}{(n+1)!}\wp^{(n)}(z) = \frac{1}{z^{n+2}} + \sum_{\substack{w\in L\\ w\neq 0}} \frac{1}{(z-w)^{(n+2)}}$$

This shows that the function $\frac{(-1)^n}{(n+1)!}\wp^{(n)}(z)$ has principal part $\frac{1}{(z-w)^{n+2}}$ at all $w \in L$. So, for all index j with $l_j > 2$, any $2 < v < l_j$ with its principal part given by $\frac{a_{v,j}}{(z-b_j)^v}$ at each point equivalent to b_j mod L, could be given by $a_{v,j} \cdot \frac{(-1)^{(v-2)}}{(v-1)!}\wp^{(v-2)}(z-b_j)$, based on similar logic as above.

In general, to create an elliptic function with the prescribed principal parts, one explicit formula can be given as:

$$\sum_{j=1}^{n} a_{1,j}\zeta(z-b_j) + \sum_{j=1}^{n}\sum_{v=2}^{l_j} a_{v,j} \cdot \frac{(-1)^{v-2}}{(v-1)!}\wp^{(v-2)}(z-b_j)$$

(Note: if $l_j < 2$, simply ignore the term).

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Question 2 *Freitag Chap. V.6 Exercise 7:*
We are interested in alternating $\mathbb{R}$-bilinear maps (forms)

$$A : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$$

Show:

(a) *Any such map A is of the form*

$$A(z, w) = hIm(z\overline{w})$$

with a uniquely determined real number h. We have explicitly $h = A(1, i)$.

(b) *Let $L \subset \mathbb{C}$ be a lattice. Then A is called a* Riemannian form *with respect to L iff h is positive, and A only takes integral values on $L \times L$. If*

$$L = \mathbb{Z}w_1 + \mathbb{Z}w_2, \quad Im\left(\frac{w_2}{w_1}\right) > 0$$

then the formula

$$A(t_1w_1 + t_2w_2, s_1w_1 + s_2w_2) := \det \begin{pmatrix} t_1 & s_1 \\ t_2 & s_2 \end{pmatrix}$$

defines a Riemannian form A on L.

(c) *A non-constant analytic function $\Theta : \mathbb{C} \to \mathbb{C}$ is called a* theta function *for the lattice $L \subset \mathbb{C}$, iff it satisfies an equation of the type*

$$\Theta(z + w) = e^{a_w z + b_2} \cdot \Theta(z)$$

for all $z \in \mathbb{C}$, and all $w \in L$. Here, a_w and b_w are onstants that may depend on w, but not on z.

Show the existence of a Riemannian form A with respect to L, such that

$$A(w, \lambda) = \frac{1}{2\pi i}(a_w\lambda - wa_\lambda)$$

for all $w, \lambda \in L$.

Pf:

(a) For any $z, w \in \mathbb{C}$, there exists $a, b, c, d \in \mathbb{R}$, with $z = a + bi$ and $w = c + di$. Then, by the property of a bilinear form, we get:

$$A(z, w) = A(a + bi, c + di) = A(a, c + di) + A(bi, c + di) = A(a, c) + A(a, di) + A(bi, c) + A(bi, di)$$
$$= acA(1, 1) + adA(1, i) + bcA(i, 1) + bdA(i, i)$$

Then, because of the property of alternating form, $A(z, w) = -A(w, z)$, which any $u \in \mathbb{C}$ satisfies $A(u, u) = -A(u, u)$, so $A(u, u) = 0$. Hence, we can further reduce the equation to the following:

$$A(z, w) = acA(1, 1) + adA(1, i) + bcA(i, 1) + bdA(i, i) = adA(1, i) - bcA(1, i) = (ad - bc)A(1, i)$$

Now, notice that if we take $z\overline{w}$, we get:

$$z\overline{w} = (a+bi)\overline{(c+di)} = (a+bi)(c-di) = (ac+bd) + (bc-ad)i$$

Which, $\text{Im}(z\overline{w}) = bc - ad$. So in fact, we get the following formula:

$$A(z,w) = (ad-bc)A(1,i) = -A(1,i)\cdot \text{Im}(z\overline{w})$$

So, let $h = -A(1,i) = A(i,1)$ (which is uniquely determined by the alternating form), we get:

$$A(z,w) = A(i,1)\cdot \text{Im}(z\overline{w}) = h\cdot \text{Im}(z\overline{w})$$

(b) If view $\mathbb{C}$ as an $\mathbb{R}$-vector space, it is a two-dimensional vector space. Which, the basis w_1, w_2 of the lattice L is also a basis for $\mathbb{C}$. Then, for all $z, w \in \mathbb{C}$. Then, for all $z, w \in \mathbb{C}$, there exists $t_1, t_2, s_1, s_2 \in \mathbb{R}$, such that $z = t_1w_1 + t_2w_2$, and $w = s_1w_2 + s_2w_2$.

First, we'll check that the given form is an alternating bilinear form:

If consider $A(z,w)$ and $A(w,z)$, we get:

$$A(z,w) = A(t_1w_1 + t_2w_2, s_1w_1 + s_2w_2) = \det\begin{pmatrix} t_1 & s_1 \\ t_2 & s_2 \end{pmatrix}$$

$$= -\det\begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix} = -A(s_1w_2 + s_2w_2, t_1w_2 + t_2w_2) = -A(w,z)$$

So, the alternating property is checked. Now, if given $u \in \mathbb{C}$, with $k_1, k_2 \in \mathbb{R}$ satisfying $u = k_1w_1 + k_2w_2$, then given arbitrary $k, l \in \mathbb{R}$, we get the following:

$$A(kz + lu, w) = A(k(t_1w_1 + t_2w_2) + l(k_1w_1 + k_2w_2), s_1w_1 + s_2w_2)$$

$$A((kt_1 + lk_1)w_1 + (kt_2 + lk_2)w_2, s_1w_1 + s_2w_2) = \det\begin{pmatrix} (kt_1 + lk_1) & s_1 \\ (kt_2 + lk_2) & s_2 \end{pmatrix}$$

$$= (kt_1 + lk_1)s_2 - (kt_2 + lk_2)s_1 = k(t_1s_2 - t_2s_1) + l(k_1s_2 - k_2s_1)$$

$$= k\det\begin{pmatrix} t_1 & s_1 \\ t_2 & s_2 \end{pmatrix} + l\det\begin{pmatrix} k_1 & s_1 \\ k_2 & s_2 \end{pmatrix}$$

$$= kA(t_1w_1 + t_2w_2, s_1w_1 + s_2w_2) + lA(k_1w_1 + k_2w_2, s_1w_1 + s_2w_2)$$

$$= kA(z,w) + lA(u,w)$$

This proves the bilinearity (including the alternating property, this also proves the linearity of the second column).

So, A defined in the question is an alternating bilinear form.

Now, for all $z, w \in L\times L$, since there exists $t_1, t_2, s_1, s_2 \in \mathbb{Z}$, with $z = t_1w_1 + t_2w_2$ and $w = s_1w_1 + s_2w_2$, we get:

$$A(z,w) = A(t_1w_1 + t_2w_2, s_1w_1 + s_2w_2) = \det\begin{pmatrix} t_1 & s_1 \\ t_2 & s_2 \end{pmatrix} = t_1s_2 - t_2s_1 \in \mathbb{Z}$$

So, A yields integer value for all elements in $L \times L$.

Lastly, consider $h = A(1, i)$ given in **part (a)**. Given that $w_1 = a + bi$, $w_2 = c_d i$ with $a, b, c, d \in \mathbb{R}$, and $\text{Im}(w_2/w_1) > 0$, we get:

$$\frac{w_2}{w_1} = \frac{c+di}{a+bi} = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)} = \frac{(ac+bd)+(ad-bc)i}{a^2+b^2}, \quad \text{Im}\left(\frac{w_2}{w_1}\right) = \frac{ad-bc}{a^2+b^2} > 0$$

$$\implies ad - bc > 0$$

Then, given the definition of A, we know the following:

$$A(w_1, w_2) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$A(w_1, w_2) = A(a+bi, c+di) = acA(1,1) + adA(1,i) + bcA(i,1) + bdA(i,i)$$
$$= adA(1,i) - bcA(1,i) = (ad-bc)h$$

Hence, we derived the following:

$$(ad-bc)h = 1 < 0, \quad ad - bc > 0 \implies h = \frac{1}{ad-bc} > 0$$

Then, since A is an alternating bilinear form, takes integer values on $L \times L$, and has $h > 0$, A is a Riemannian Form.

(c) Let $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$, with $\text{Im}(\frac{w_2}{w_1}) > 0$. Given the definition of Θ function, we know for any $z \in \mathbb{C}$, if $\Theta(z) = 0$, then for all $w \in L$, $\Theta(z+w) = e^{a_w z + b_w}\Theta(z) = 0$. Hence, let $b_1, ..., b_n$ represent the zeros of Θ in a fundamental region P, then for all $z \in \mathbb{C}$, we get $\Theta(z) = 0$ iff $z \equiv b_j \mod L$ for some $j \in \{1, ..., n\}$ (since if $z \in P$ satisfies $z \neq b_j$ for all index j, then for all $w \in L$, $\Theta(z+w) = e^{a_w z + b_w}\Theta(z) \neq 0$).

On the other hand, for all $w \in L$, if consider the derivative $\Theta'(z+w)$, we get:

$$\Theta'(z+w) = a_w e^{a_w z + b_w}\Theta(z) + e^{a_w z + b_w}\Theta'(z)$$

Which, the following is true:

$$\frac{\Theta'(z+w)}{\Theta(z+w)} = \frac{a_w e^{a_w z + b_w}\Theta(z) + e^{a_w z + b_w}\Theta'(z)}{e^{a_w z + b_w}\Theta(z)} = a_w + \frac{\Theta'(z)}{\Theta(z)}$$

1. Relations of a_w with basis w_1, w_2:

For any $w \in L$, there exists $k, l \in \mathbb{Z}$, such that $w = kw_1 + lw_2$. Which, notice the following:

$$\frac{\Theta'(z+w_1)}{\Theta(z+w_1)} = a_{w_1} + \frac{\Theta'(z)}{\Theta(z)}$$

Which, by induction, any $k \in \mathbb{Z}$ with $k \geq 0$ satisfies:

$$\frac{\Theta'(z+kw_1)}{\Theta(z+kw_1)} = ka_{w_1} + \frac{\Theta'(z)}{\Theta(z)}$$

Then, for $k < 0$, since $z = (z + kw_1) - kw_1$ with $-k > 0$, we get the following relation:

$$\frac{\Theta'(z)}{\Theta(z)} = \frac{\Theta'((z+kw_1)-kw_1)}{\Theta((z+kw_1)-kw_1)} = (-k)a_{w_1} + \frac{\Theta'(z+kw_1)}{\Theta(z+kw_1)}, \quad \frac{\Theta'(z+kw_1)}{\Theta(z+kw_1)} = ka_{w_1} + \frac{\Theta'(z)}{\Theta(z)}$$

Hence, the above formula can be generalize to any $k \in \mathbb{Z}$. Then, apply similar logic to w_2, we also get the following:

$$\forall l \in \mathbb{Z}, \quad \frac{\Theta'(z+lw_2)}{\Theta(z+lw_2)} = la_{w_2} + \frac{\Theta'(z)}{\Theta(z)}$$

So, for arbitrary $w \in L$, since there exists $k, l \in \mathbb{Z}$, with $w = kw_1 + lw_2$, then the following relation is true:

$$a_w + \frac{\Theta'(z)}{\Theta(z)} = \frac{\Theta'(z+w)}{\Theta(z+w)} = \frac{\Theta'(z+kw_1+lw_2)}{\Theta(z+kw_1+lw_2)}$$

$$= la_{w_2} + \frac{\Theta'(z+kw_1)}{\Theta'(z+kw_1)} = ka_{w_1} + la_{w_2} + \frac{\Theta'(z)}{\Theta(z)}$$

$$\implies a_w = ka_{w_1} + la_{w_2}$$

2. Define the Riemannian Form:

Since L is a lattice, w_1 and w_2 are linearly independent when viewing $\mathbb{C}$ as an $\mathbb{R}$-vector space, hence w_1, w_2 forms a basis of $\mathbb{C}$. Which, for all $u, v \in \mathbb{C}$, there exists $t_1, t_2, s_1, s_2 \in \mathbb{R}$, such that $u = t_1w_1 + t_2w_2$ and $v = s_1w_1 + s_2w_2$. So, define the map $A : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ as follow:

$$A(u, v) = A(t_1w_1 + t_2w_2, s_1w_1 + s_2w_2) = \frac{1}{2\pi i}((t_1a_{w_1} + t_2a_{w_2})v - (s_1a_{w_1} + s_2a_{w_2})u)$$

Notice that the image isn't guaranteed to be in $\mathbb{R}$. Temporarily, we'll postpone the proof of $A(\mathbb{C}\times\mathbb{C}) \subseteq \mathbb{R}$, and verify that A satisfies all the other properties of being a Riemannian Form first (except the part that $h > 0$).

- **Alternating Property:**

  Given the definition of A, we get:

$$A(v, u) = A(s_1w_1 + s_2w_2, t_1w_1 + t_2w_2) = \frac{1}{2\pi i}((s_1a_{w_1} + s_2a_{w_2})u - (t_1a_{w_1} + t_2a_{w_2})v)$$

$$= -\frac{1}{2\pi i}((t_1a_{w_1} + t_2a_{w_2})v - (s_1a_{w_1} + s_2a_{w_2})u) = -A(u, v)$$

- **Bilinearity:**

  Given arbitrary $w \in \mathbb{C}$, there exists $r_1, r_2 \in \mathbb{R}$, with $w = r_1w_1 + r_2w_2$. Which, for arbitrary $k, l \in \mathbb{R}$ we get:

$$A(ku+lw, v) = A(k(t_1w_1+t_2w_2)+l(r_1w_1+r_2w_2), s_1w_1+s_2w_2) = A((kt_1+lr_1)w_1+(kt_2+lr_2)w_2, s_1w_1+s_2w_2)$$

$$= \frac{1}{2\pi i}(((kt_1 + lr_1)a_{w_1} + (kt_2 + lr_2)a_{w_2})v - (s_1a_{w_1} + s_2a_{w_2})(ku + lw))$$

$$= \frac{1}{2\pi i}\left(k(t_1a_{w_1} + t_2a_{w_2})v - (s_1a_{w_1} + s_2a_{w_2})ku\right) + \frac{1}{2\pi i}\left(l(r_1a_{w_1} + r_2a_{w_2})v - (s_1a_{w_1} + s_2a_{w_2})lw\right)$$

$$= kA(u, v) + lA(w, v)$$

  Combining the alternating property, the linearity in the second column is also given.

– A **yields integer values on** $L \times L$**:**

Given any $w, \lambda \in L$, there exists $t_1, t_2, s_1, s_2 \in \mathbb{Z}$, $w = t_1 w_1 + t_2 w_2$, and $\lambda = s_1 w_1 + s_2 w_2$. Which, with $a_w = t_1 a_{w_1} + t_2 a_{w_2}$ and $a_\lambda = s_1 a_{w_1} + a_2 a_{w_2}$ proven in statement **1**, we get:

$$A(w,\lambda) = A(t_1 a_{w_1} + t_2 a_{w_2}, s_1 a_{w_1} + s_2 a_{w_2})$$

$$= \frac{1}{2\pi i}((t_1 a_{w_1} + t_2 a_{w_2})\lambda - (s_1 a_{w_1} + s_2 a_{w_2})w) = \frac{1}{2\pi i}(a_w \lambda - a_\lambda w)$$

This is the desired formula for any $w, \lambda \in L$. To prove that the value is an integer, consider the parallelagram P spanned by w and λ. Which, given ∂P, since it's closed and Θ has discrete zeros, then there existts $a \in \mathbb{C}$, such that $P' = a + P$ with $\partial P'$ containing no zeros of Θ (hence $\frac{\Theta'}{\Theta}$ is well-defined on $\partial P'$).

Insert image

Which, WLOG, assume the orientation of $\partial P'$ is given by $a \to (a+w) \to (a+w+\lambda) \to (a+\lambda) \to a$, along with argument principal, we get the following:

$$\text{Number of zeros of } \Theta \text{ in } P' = \frac{1}{2\pi i}\int_{\partial P'} \frac{\Theta'(z)}{\Theta(z)} dz$$

$$= \frac{1}{2\pi i}\left(\int_a^{a+w} \frac{\Theta'(z)}{\Theta(z)} dz + \int_{a+w}^{a+w+\lambda} \frac{\Theta'(z)}{\Theta(z)} dz + \int_{a+w+\lambda}^{a+\lambda} \frac{\Theta'(z)}{\Theta(z)} dz + \int_{a+\lambda}^{a} \frac{\Theta'(z)}{\Theta(z)} dz\right)$$

$$= \frac{1}{2\pi i}\left(\left(\int_a^{a+w} \frac{\Theta'(z)}{\Theta(z)} dz - \int_a^{a+w} \frac{\Theta'(z+\lambda)}{\Theta(z+\lambda)} dz\right) + \left(\int_a^{a+\lambda} \frac{\Theta'(z+w)}{\Theta(z+w)} dz - \int_a^{a+\lambda} \frac{\Theta'(z)}{\Theta(z)} dz\right)\right)$$

$$= \frac{1}{2\pi i}\left(\int_a^{a+w} \left(\frac{\Theta'(z)}{\Theta(z)} - \left(a_\lambda + \frac{\Theta'(z)}{\Theta(z)}\right)\right) dz + \int_a^{a+\lambda} \left(a_w + \frac{\Theta'(z)}{\Theta(z)} - \frac{\Theta'(z)}{\Theta(z)}\right) dz\right)$$

$$= \frac{1}{2\pi i}\left(\int_a^{a+\lambda} a_w dz - \int_a^{a+w} a_\lambda dz\right) = \frac{1}{2\pi i}(a_w \lambda - a_\lambda w) = A(w,\lambda)$$

This shows that $A(w,\lambda)$ is in fact an integer (the sign depends on the orientation of P' described above).

Second to last, to prove the image is contained in $\mathbb{R}$, we'll utilize the continuity of A (since fixing one entry, A becomes a linear map that is continuous).

First, we'll prove that any $w, \lambda \in (\mathbb{Q}w_1 + \mathbb{Q}w_2)$ satisfies $A(w,\lambda) \in \mathbb{R}$: Since both w, λ have the coefficients of w_1, w_2 being rational, then for large enough $k, l \in \mathbb{N}$, $kw, l\lambda \in L$ (EX: choose k, l to be the multiples of the denominators of the ratioal coefficients of w, λ respectively, the each coefficient of $kw, l\lambda$ is an integer). So, evaluate in A, we get:

$$A(kw, l\lambda) \in \mathbb{Z}, \quad A(w,\lambda) = \frac{1}{kl}A(kw, l\lambda) \in \mathbb{R}$$

Now, let $L' = \mathbb{Q}w_1 + \mathbb{Q}w_2$, since L' is a dense set in $\mathbb{C}$ (due to the denseness of $\mathbb{Q}$ in $\mathbb{R}$), then $L' \times L'$ is a dense set iin $\mathbb{C} \times \mathbb{C}$. So, for any $(u,v) \in \mathbb{C} \times \mathbb{C}$, it is a limit point of $L' \times L'$, hence there exists a sequence $(u_n, v_n)_{n\in\mathbb{N}} \subset L' \times L'$, with $\lim_{n\to\infty}(u_n, v_n) = (u,v)$. Hence, by continuity of A, we get:

$$\lim_{n\to\infty} A(u_n, v_n) = A(u,v)$$

And, since each index n satisfies $A(u_n, v_n) \in \mathbb{R}$ (recall that $(u_n, v_n) \in L' \times L'$), then by completeness of $\mathbb{R}$, $A(u,v)$ as a limit of sequence in $\mathbb{R}$, must also belong to $\mathbb{R}$. Hence, $A(u,v) \in \mathbb{R}$.

This proves that A has an image in $\mathbb{R}$, hence it's in fact an alternating bilinear form $A : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$, which yields integer values on $L \times L$ (based on the above information).

The last task is to verify that h corresponding to A is in fact positive. Recall that $h = A(1, i)$, with the assumption that $w_1 = a + bi$, $w_2 = c + di$, and $\mathrm{Im}(w_2/w_1) > 0$, we know $ad - bc > 0$. Hence, we get the following:

$$A(w_1, w_2) = A(a+bi, c+di) = acA(1, 1)+adA(1, i)+bcA(i, 1)+bdA(i, i) = (ad-bc)A(1, i) = (ad-bc)h$$

Which, given the assumption that $\mathrm{Im}(w_2/w_1) > 0$, the orientation of the parallelagram is given as follow:

insert image

Hence, $A(w_1, w_2)$ when representing as the integral form mentioned before, it is nonnegative (since the integration is along a counterclockwise contour like above). So, given $(ad-bc)h = A(w_1, w_2) \geq 0$, then $h \geq 0$.

In terms for h to be positive, we need $A(w_1, w_2) > 0$, hence an extra condition imposed is that Θ needs to have at least a zero (so there is a zero within the fundamental region of lattice L, causing the integral form of $A(w_1, w_2)$ to be nonzero).

3

Question 3 *Freitag Chap. V.7 Exercise 5:*
Show:

(a) *For the lattice* $L_i = \mathbb{Z} + \mathbb{Z}i$ *we have* $g_3(i) = 0$ *and* $g_2(i) \in \mathbb{R}^{\times}$*, in particular* $\Delta(i) = g_2^3(i) > 0$.

(b) *For the lattice* $L_w = \mathbb{Z} + \mathbb{Z}w$, $w := e^{2\pi i/3}$*, we have* $g_2(w) = 0$ *and* $g_3(w) \in \mathbb{R}^{\times}$*, in particular* $\Delta(w) = -27g_3^2(w)$.

Pf:

Given any lattice $L = \mathbb{Z} + \mathbb{Z}\mathcal{T}$ with $\mathcal{T} \in \mathbb{H}$, recall that $g_2(\mathcal{T}) = 60G_4(\mathcal{T})$ and $g_3(\mathcal{T}) = 140G_6(\mathcal{T})$, and $\Delta(\mathcal{T}) \neq 0$ (since in case to have lattice, the half lattices yield distinct values e_1, e_2, e_3, which are roots of the polynomial $4w^3 - g_2w - g_3$ proven before (when talking about the algebraic differential equation of $\wp$-function). Hence as $\Delta(\mathcal{T}) = g_2^3(\mathcal{T}) - 27g_3^2(\mathcal{T})$ denotes the discriminant of the cubic polynomial above, having distinct roots implies $\Delta(\mathcal{T}) \neq 0$).

To prove that g_2, g_3 yields real values for specific lattices, it suffices to prove the case for G_4 and G_6.

(a) Given lattice $L_i = \mathbb{Z} + \mathbb{Z}i$, there are two properties:

- For all $w = a + bi \in L_i$, its conjugate $\overline{w} = a - bi \in L_i$ (since both $a, b \in \mathbb{Z}$).
- Given same w, $iw = -b + ai \in L_i$ based on the same reason above.

Notice that the above pairing is a one-to-one correspondance. Hence, for $k \geq 3$, we get the following formula for $2G_k(i)$:

$$2G_k(i) = \sum_{\substack{w \in L_i \\ w \neq 0}} \frac{1}{w^k} + \sum_{\substack{w \in L_i \\ w \neq 0}} \frac{1}{\overline{w}^k} = \sum_{\substack{w \in L_i \\ w \neq 0}} 2\text{Re}\left(\frac{1}{w^4}\right) = 2\sum_{\substack{w \in L_i \\ w \neq 0}} \text{Re}\left(\frac{1}{w^4}\right) \in \mathbb{R}$$

(Note: since for $k \geq 3$, $G_k(z)$ converges normally within $\mathbb{H}$).

So, both $G_4(i)$ and $G_6(i)$ are real, implying that $g_2(i) = 60G_4(i)$, $g_3(i) = 140G_6(i)$ are also real.

Then, consider the similar formulation with iw instead, we get:

$$2G_6(i) = \sum_{\substack{w \in L_i \\ w \neq 0}} \frac{1}{w^6} + \sum_{\substack{w \in L_i \\ w \neq 0}} \frac{1}{(iw)^6} = \sum_{\substack{w \in L \\ w \neq 0}} \left(\frac{1}{w^6} - \frac{1}{w^6}\right) = 0$$

This implies that $G_6(i) = 0$, which $g_3(i) = 140G_6(i) = 0$.

Finally, given the discriminant formula, since the following is true:

$$\Delta(i) = g_2^3(i) - 27g_3^2(i) = g_2^3(i)$$

On the other hand, $\Delta(i) \neq 0$ (since L_i is a lattice), which the above equation implies that $g_2(i) \neq 0$. Also, recall that from **HW 5 Problem 5**, we've proven that given the lattice formed by $\mathbb{Z}+\mathbb{Z}ti$ (where $t \in \mathbb{R}$), $\wp(z)$ yields real values on all the half lines; specifically, on all the half points (the intersection of half lines), the value of $\wp$ (denoted as e_1, e_2, e_3) are all real. Since these three values are the roots of the cubic polynomial $4w^3 - g_2(i)w - g_3(i)$, then because the cubic polynomial have three real distinct roots, its discriminant $\Delta(i) > 0$, showing that $g_2^3(i) > 0$, or $g_2(i) > 0$.

Hence, for lattice L_i, we have $g_2(i) > 0$, and $g_3(i) = 0$ (so, $g_2(i) \in \mathbb{R}^{\times}$).

(b) Given $L_w = \mathbb{Z} + \mathbb{Z}w$ with $w = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (which $w^3 = 1$), here are several of its properties:

- Since $e^{2\pi i/6} = \frac{1}{2} + \frac{\sqrt{3}}{2}i = 1 + (-\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 1 + w \in L_w$, while $-\overline{w} = -(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) = \frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{2\pi i/6}$, so $-\overline{w} \in L$. Then, for all $\lambda = a + bw \in L_w$ (for some $a, b \in \mathbb{Z}$), we get:

$$-\overline{\lambda} = -(a + b\overline{w}) = -a + b(-\overline{w}) \in L_w \implies \overline{\lambda} \in L_w$$

- Since $e^{2\pi i/6} \in L_w$, then $-e^{2\pi i/6} = e^{2\pi i \cdot 2/3} = w^2 \in L_w$. Hence, given the same λ above, we also get:

$$w\lambda = w(a + bw) = aw + bw^2 \in L_w$$

$$w^2\lambda = w^2(a + bw) = aw^2 + bw^3 = b + aw^2 \in L_2$$

Notice that all pairings above are one-to-one correspondance. Hence, for all $k \geq 3$, we get the following formula for $2G_k(w)$:

$$2G_k(w) = \sum_{\substack{\lambda \in L_w \\ \lambda \neq 0}} \frac{1}{\lambda^k} + \sum_{\substack{\lambda \in L_w \\ \lambda \neq 0}} \frac{1}{\overline{\lambda}^k} = \sum_{\substack{\lambda \in L_w \\ \lambda \neq 0}} 2\mathrm{Re}\left(\frac{1}{\lambda^k}\right) = 2 \sum_{\substack{\lambda \in L_\lambda \\ w \neq 0}} \mathrm{Re}\left(\frac{1}{\lambda^k}\right) \in \mathbb{R}$$

Hence, $G_k(w) \in \mathbb{R}$ for all $k \geq 3$. In particular, $G_4(w), G_6(w) \in \mathbb{R}$, implying that $g_2(w) = 60G_4(w)$ and $g_3(w) = 140G_6(w)$ are also real.

Now, if we consider $3G_4(w)$ specifically, based on the second property of L_w listed above (where $\lambda \in L_w$ implies $w\lambda$ and $w^2\lambda$ are in L_w), we get:

$$3G_4(w) = \sum_{\substack{\lambda \in L_w \\ w \neq 0}} \frac{1}{\lambda^4} + \sum_{\substack{\lambda \in L_w \\ w \neq 0}} \frac{1}{(w\lambda)^4} + \sum_{\substack{\lambda \in L_w \\ w \neq 0}} \frac{1}{(w^2\lambda)^4} = \sum_{\substack{\lambda \in L_w \\ \lambda \neq 0}} \left(\frac{1}{\lambda^4} + \frac{1}{w^4\lambda^4} + \frac{1}{w^8\lambda^4}\right)$$

$$\sum_{\substack{\lambda \in L_w \\ \lambda \neq 0}} \left(\frac{1}{\lambda^4} + \frac{w^2}{w^6\lambda^4} + \frac{w}{w^9\lambda^4}\right) = \sum_{\substack{\lambda \in L_w \\ \lambda \neq 0}} \left(\frac{1}{\lambda^4} + \frac{w^2}{\lambda^4} + \frac{w}{\lambda^4}\right) = (1 + w + w^2) \sum_{\substack{\lambda \in L_w \\ \lambda \neq 0}} \frac{1}{\lambda^4}$$

Since $w = e^{2\pi i/3}$ is a primitive 3^{rd} root of unity, then $1 + w + w^2 = 0$, showing that $3G_4(w) = 0$ based on the above equation. Hence, $G_4(w) = 0$, so $g_2(w) = 60G_4(w) = 0$ also.

Finally, since $\Delta(w) = g_2^3(w) - 27g_3^2(w) = -27g_3^2(w)$, and $\Delta(w) \neq 0$, then $g_3^2(w) \neq 0$.

So, in conclusion, we have $g_2(w) = 0$, $g_3(w) \neq 0$, and $g_3(w) \in \mathbb{R}$, so $g_3(w) \in \mathbb{R}^\times$.

4

Question 4 *Freitag Chap. V.8 Exercise 3:*
The Eisenstein series are "real" functions, i.e. $\overline{G_k(\mathcal{T})} = G_k(-\overline{\mathcal{T}})$. This implies

$$G_k\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\mathcal{T})+\delta}\right) = (\gamma(-\overline{\mathcal{T}})+\delta)^k\overline{G_k(\mathcal{T})} \qquad \textit{and}$$

$$j\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\overline{\mathcal{T}})+\delta}\right) = \overline{j(\mathcal{T})} \qquad \textit{for } \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma$$

On the vertical half-lines $Re(\mathcal{T}) = \pm\frac{1}{2}$ in $\mathbb{H}$ in $\mathbb{H}$ the Eisenstein series and the j-function are real. if $\mathcal{T}\in\mathbb{H}$ lies on the circle line $|\mathcal{T}|=1$, then $j(\mathcal{T}) = \overline{j(\mathcal{T})}$. In particular, the j-function is real on the boundary of the modular figure, and on the imaginary axis.

Pf:

For all $\mathcal{T}\in\mathbb{H}$ (and $k\geq 3$), since $G_k(\mathcal{T})$ is a series of functions that converges normally within $\mathbb{H}$, then the following is true:

$$\overline{G_k(\mathcal{T})} = \overline{\sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}}\frac{1}{(a+b\mathcal{T})^k}} = \sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}}\frac{1}{\overline{(a-b\mathcal{T})}^k} = \sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}}\frac{1}{(a+b(-\overline{\mathcal{T}}))^k} = G_k(-\overline{\mathcal{T}})$$

This verifies the first property in the problem.

Then, based on the relations given as follow:

$$\forall\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma,\quad G_k\left(\frac{\alpha\mathcal{T}+\beta}{\gamma\mathcal{T}+\delta}\right) = (\gamma\mathcal{T}+\delta)^kG_k(\mathcal{T})$$

$$g_2(\mathcal{T}) = 60G_4(\mathcal{T}),\quad g_3(\mathcal{T}) = 140G_6(\mathcal{T}),\quad j(\mathcal{T}) = \frac{g_2^3(\mathcal{T})}{g_2^3(\mathcal{T})-27g_3^2(\mathcal{T})}$$

We can conclude the following:

$$G_k\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\mathcal{T})+\delta}\right) = (\gamma(-\overline{\mathcal{T}})+\delta)^kG_k(-\overline{\mathcal{T}}) = (\gamma(-\overline{\mathcal{T}})+\delta)^k\overline{G_k(\mathcal{T})}$$

$$\begin{aligned}
j\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\mathcal{T})+\delta}\right) &= \frac{\left(g_2\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\mathcal{T})+\delta}\right)\right)^3}{\left(g_2\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\mathcal{T})+\delta}\right)\right)^3 - 27\left(g_3\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\mathcal{T})+\delta}\right)\right)^2}\\
&= \frac{\left(60G_4\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\mathcal{T})+\delta}\right)\right)^3}{\left(60G_4\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\mathcal{T})+\delta}\right)\right)^3 - 27\left(140G_6\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\mathcal{T})+\delta}\right)\right)^2}\\
&= \frac{(60(\gamma(-\overline{\mathcal{T}})+\delta)^4\overline{G_4(\mathcal{T})})^3}{(60(\gamma(-\overline{\mathcal{T}})+\delta)^4\overline{G_4(\mathcal{T})})^3 - 27(140(\gamma(-\overline{\mathcal{T}})+\delta)^6\overline{G_6(\mathcal{T})})^2}\\
&= \frac{(\gamma(-\overline{\mathcal{T}})+\delta)^{12}\overline{(60G_4(\mathcal{T}))}^3}{(\gamma(-\overline{\mathcal{T}})+\delta)^{12}\overline{(60G_4(\mathcal{T}))}^3 - 27\cdot(\gamma(-\overline{\mathcal{T}})+\delta)^{12}\overline{(140G_6(\mathcal{T}))}^2}\\
&= \frac{\overline{g_2(\mathcal{T})}^3}{\overline{g_2(\mathcal{T})}^3 - 27\overline{g_3(\mathcal{T})}^2} = \overline{\frac{g_2^3(\mathcal{T})}{g_2^3(\mathcal{T})-27g_3^2(\mathcal{T})}} = \overline{j(\mathcal{T})}
\end{aligned}$$

So, the second and third properties are verified.

Now, given $\mathcal{T} \in \mathbb{H}$ with $\text{Re}(\mathcal{T}) = \pm\frac{1}{2}$ (WLOG, assume it is $\frac{1}{2}$, since if $\mathcal{T} = -\frac{1}{2} + yi$ for some $y > 0$, then $\mathcal{T} + 1 = \frac{1}{2} + yi = -\overline{\mathcal{T}}$, which since $-\overline{\mathcal{T}} = \frac{\mathcal{T}+1}{0\cdot\mathcal{T}+1}$, the two values are equivalent. Hence, swap $\mathcal{T}$ and $-\overline{\mathcal{T}}$, we still get the same case).

Then, since $\mathcal{T} = \frac{1}{2} + yi$ for some $y > 0$, $-\overline{\mathcal{T}} = -\frac{1}{2} + yi$, so $-\overline{\mathcal{T}} + 1 = \mathcal{T}$. Using the previous properties, we get:

$$G_k(\mathcal{T}) = G_k\left(\frac{(-\overline{\mathcal{T}}) + 1}{0 \cdot (-\overline{\mathcal{T}}) + 1}\right) = (0 \cdot (-\overline{\mathcal{T}}) + 1)^k \overline{G_k(\mathcal{T})} = \overline{G_k(\mathcal{T})}$$

$$G_k(\mathcal{T}) = \overline{G_k(\mathcal{T})} \implies G_k(\mathcal{T}) \in \mathbb{R}$$

$$j(\mathcal{T}) = j\left(\frac{(-\overline{\mathcal{T}}) + 1}{0 \cdot (-\overline{\mathcal{T}}) + 1}\right) = \overline{j(\mathcal{T})}$$

$$j(\mathcal{T}) = \overline{j(\mathcal{T})} \implies j(\mathcal{T}) \in \mathbb{R}$$

This proves that both Eisenstein series and the j-function are real on $\text{Re}(\mathcal{T}) = \pm\frac{1}{2}$.

Finally, given $\mathcal{T} \in \mathbb{H}$ with $|\mathcal{T}| = 1$, then since $\mathcal{T} = \frac{1}{\overline{\mathcal{T}}} = \frac{-1}{(-\overline{\mathcal{T}})}$, we get the following:

$$j(\mathcal{T}) = j\left(\frac{0 \cdot (-\overline{\mathcal{T}}) - 1}{(-\overline{\mathcal{T}}) + 0}\right) = \overline{j(\mathcal{T})} \implies j(\mathcal{T}) \in \mathbb{R}$$

So, on the half circle $|\mathcal{T}| = 1$ in $\mathbb{H}$, j-function is also real.