Math CS 122B HW7

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Question 1 *The functional equation of the ζ-function can also be written in the following form:*

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

Deduce from this: In the half-plane $\sigma \leq 0$, the function $\zeta(s)$ has exactly the zeros $s = -2k,\ k \in \mathbb{N}$. All other zeros of the ζ-function are located in the vertical strip $0 < Res < 1$.

Pf:

First, recall that for the half plane $\sigma > 1$, the following inequality is given:

$$\left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4 |\zeta(\sigma+2it)|[\zeta(\sigma)(\sigma-1)]^3 \geq (\sigma-1)^{-1}$$

Since for $\sigma > 1$, the expressiong $(\sigma-1)^{-1} > 0$, this enforces all $s = \sigma + it$ in the half plane to have $\zeta(s) \neq 0$ (or else the left side of the inequality is 0, which violates the inequality). Similarly, this inequality can be extended onto the line $\mathrm{Re}(s) = 1$, where $\zeta(s)$ has no zeros on this line also. So, for $\sigma \geq 1$, $\zeta(s)$ has no zero.

Now, in the half plane $\sigma \leq 0$, for all $s' \neq 0$, since it can be written as $s' = 1-s$, where $s = 1-s'$ has $\mathrm{Re}(s) = 1 - \mathrm{Re}(s') \geq 1$ (and since $s' \neq 0$, then $s \neq 1$). So, $\zeta(s)$ after the continuation past $\mathrm{Re}(s) = 1$, has $\zeta(s)$ being well-defined.

Then, by the functional equation, we get the following:

$$\zeta(s') = \zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

Since $\mathrm{Re}(s) \geq 1$ with $s \neq 1$, then $\zeta(s) \neq 0$ based on what is mentioned during the start; also, $\Gamma(s) \neq 0$ for all $s \in \mathbb{C} \setminus \{0, -1, -2, ...\}$, while $2(2\pi)^{-s} \neq 0$ for all $s \in \mathbb{C}$. Hence, in case for $\zeta(1-s) = 0$, we must have $\cos(\frac{\pi s}{2}) = 0$, which enforces $\frac{\pi s}{2} = k\pi + \frac{\pi}{2}$ for some $k \in \mathbb{Z}$, or $s = 2k+1$ fo some $k \in \mathbb{Z}$. Now, under this assumption, since $\mathrm{Re}(s) \geq 1$ while $s \neq 1$, then $k \geq 1$. So, when transfering back to $s' = 1-s$, we get $s' = 1-(2k+1) = -2k$ for integer $k \geq 1$.

Hence, for $\mathrm{Re}(s') \leq 0$, for $\zeta(s') = 0$, then $s' = -2k$ for some $k \in \mathbb{N}$ (this is an iff since at all these points, $\cos(\frac{\pi s}{2}) = 0$, which $\zeta(s') = \zeta(1-s) = 0$).

Finally, for $s' = 0$ (where if $s' = 1-s$, $s = 1$). Recall that $\zeta(s)$ has a simple pole at $s = 1$, while $\cos(\frac{\pi s}{2})$ has a simple zero at $s = 1$ (where the input is $\frac{\pi}{2}$, where cos is 0). Hence, $\cos(\frac{\pi s}{2}) = (s-1)h(z)$ for some analytic function h where $h(1) \neq 0$. Also, we know $\lim_{s \to 1}(s-1)\zeta(s) = 1$ (has been given in the textbook). Then, we get the following:

$$\lim_{s \to 1} \zeta(1-s) = \lim_{s \to 1} 2(2\pi)^{-s}\Gamma(s)h(s)(s-1)\zeta(s) = 2(2\pi)^{-1}\Gamma(1)h(1) \cdot \lim_{s \to 1}(s-1)\zeta(s) = 2(2\pi)^{-1}\Gamma(1)h(1) \neq 0$$

Hence, we can deduce that at $s = 1$ (where $s' = 1 - s = 0$), $\zeta(s')$ has a removable singularity that has limit not being 0, henc $\zeta(s')$ as an extension has $\zeta(0) \neq 0$.

The above casees proves that when $\sigma \geq 1$ or $\sigma \leq 0$, $\zeta(s) = 0$ iff $s = -2k$ for some $k \in \mathbb{N}$, where for any other input ζ is nonzero.

Hence, if there are any other zeros, it must exist in the vertical strip $0 < \mathrm{Re}(s) < 1$.

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Question 2 *The following special case of the Hecke Theorem was already known to B. Riemann (1859):*

$$\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t} t^{s/2}\frac{dt}{t}$$

$$= \frac{1}{2}\int_1^{\infty}(\theta(it) - 1)(t^{s/2} + t^{(1-s)/2})\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

Deduce directly this special case, and use it to prove the meromorphic continuation and the functional equation.

Pf: