Math 118C HW4

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May 16, 2025

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Question 1 *Rudin Pg. 242 Problem 27:*
Put $f(0,0) = 0$, *and*

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x,y) \neq (0,0)$. *Prove that*

(a) $f,\ D_1 f,\ D_2 f$ *are continuous in* $\mathbb{R}^2$.

(b) $D_{12} f$ *and* $D_{21} f$ *exist at every point of* $\mathbb{R}^2$, *and are continuous except at* $(0,0)$.

(c) $D_{12} f(0,0) = 1$, *and* $D_{21} f(0,0) = -1$.

Pf:

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Question 2 *Rudin Pg. 242 Problem 28:*
For $t \geq 0$, *put*

$$\varphi(x,t) = \begin{cases} x & 0 \leq x \leq \sqrt{t} \\ -x + 2\sqrt{t} & \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0 & \textit{otherwise} \end{cases}$$

and put $\varphi(x,t) = -\varphi(x,|t|)$ *if* $t < 0$.
Show that φ *is continuous on* $\mathbb{R}^2$, *and* $D_2\varphi(x,0) = 0$ *for all* x. *Define*

$$f(t) = \int_{-1}^{1} \varphi(x,t)dx$$

Show that $f(t) = t$ *if* $|t| < \frac{1}{4}$. *Hence*

$$f'(0) \neq \int_{-1}^{1} D_2\varphi(x,0)dx$$

Pf:

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Question 3 *Rudin Pg. 243 Problem 30:*
Let $f \in \mathcal{C}^{(m)}(E)$, where E is an open subset of $\mathbb{R}^n$. Fix $a \in E$, and suppose $x \in \mathbb{R}^n$ is so close to 0 that the points $p(t) = a + tx$ lie in E whenever $0 \leq t \leq 1$. Define $h(t) = f(p(t))$ for all $t \in \mathbb{R}$ for which $p(t) \in E$.

(a) *For $1 \leq k \leq m$, show (by repeated application of the chain rule) that*

$$h^{(k)}(t) = \sum (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

The sum extends over all order k-tuples $(l_1, \dots, l_k)$ in which each l_j is one of the integers $1, \dots, n$.

(b) *By Taylor's Theorem:*

$$h(1) = \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m)}(t)}{m!}$$

for some $t \in (0,1)$. Use this to prove Taylor's Theorem in n variables by showing that the formula

$$f(a+x) = \sum_{k=0}^{m-1} \frac{1}{k!} \left(\sum (D_{l_1 \dots l_k} f)(a) x_{l_1} \dots x_{l_k} \right) + r(x)$$

represents $f(a+x)$ as the sum of its so-called "Taylor polynomial of degree $m-1$" plus a remainder that satisfies

$$\lim_{x \to 0} \frac{r(x)}{|x|^{m-1}} = 0$$

Each of the inner sums extends over all ordered k-tuples $(l_1, \dots, l_k)$, as in part (a); as usual, the zero-order derivative of f is simply f, so that the constant term of the Taylor polynomial of f at a is $f(a)$.

(c) *Exercise 29 shows that repetition occurs in the Taylor polynomial as written in part (b). For instance, D_{113} occurs three times, as $D_{113}, D_{131}, D_{311}$. The sum of the corresponding three terms can be written in the form*

$$3(D_1^2 D_3 f)(a) x_1^2 x_3$$

Prove (by calculating how often each derivative occurs) that the Taylor polynomial in (b) can be written in the form

$$\sum \frac{(D_1^{s_1} \dots D_n^{s_n} f)(a)}{s_1! \dots s_n!} x_1^{s_1} \dots x_n^{s_n}$$

Here the summation extends over all ordered n-tuples $(s_1, \dots, s_n)$ such that each s_i is a nonnegative integer and $s_1 + \dots + s_n \leq m - 1$.

Pf:

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Question 4 *Rudin Pg. 288 Problem 2:*
For $i = 1, 2, 3, ...,$ let $\varphi_i \in \mathcal{C}(\mathbb{R})$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x, y) = \sum_{i=1}^{\infty} (\varphi_i(x) - \varphi_{i+1}(x))\varphi_i(y)$$

Then f has compact support in $\mathbb{R}^2$, f is cotinuous except at $(0, 0)$, and

$$\int dy \int f(x, y) dx = 0, \quad \textit{but} \int dx \int f(x, y) dy = 1$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

Pf: