Math CS 122B HW8 Part 2

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Question 1 *Stein and Shakarchi Pg. 200-201 Exercise 4:*
Suppose $\{a_n\}_{n\in\mathbb{N}}$ *is a sequence of complex numbers such that* $a_n = a_m$ *iff* $n \equiv m \mod q$ *for some positive integer* q. *Define the* ***Dirichlet*** ***L-series*** *associated to* $\{a_n\}$ *by*

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{for } Re(s) > 1$$

Also, with $a_0 = a_q$, *let*

$$Q(x) = \sum_{m=0}^{q-1} a_{q-m} e^{mx}$$

Show, as in Exercises 15 *and* 16 *of the previous chapter, that*

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{Q(x)x^{s-1}}{e^{qx}-1} dx \quad \text{for } Re(s) > 1$$

Prove as a result that $L(s)$ *is continuable into the complex plane, with the only possible singularity a pole at* $s = 1$. *In fact,* $L(s)$ *is regular at* $s = 1$ *if and only if* $\sum_{m=0}^{q-1} a_m = 0$. *Note the conection with the Direchlet* $L(s,\chi)$ *series, taken up to BOok I Chapter 8, and that as a consequence,* $L(s,\chi)$ *is regular at* $s = 1$ *if and only if* χ *is a non-trivial character.*

Pf:

1.1 Integral Representation of $L(s)$:

Given $\text{Re}(s) > 1$, and $x \in (0,\infty)$, notice that $\frac{1}{e^{qx}-1} = \frac{e^{-qx}}{1-e^{-qx}}$, with the fact that $-qx < 0$, then $e^{-qx} < 1$. Hence, the following expression is absolutely convergent, and converging normally for any compact subset of $(0,\infty)$:

$$\frac{1}{e^{qx}-1} = \frac{e^{-qx}}{1-e^{-qx}} = \sum_{n=1}^{\infty} (e^{-qx})^n \tag{1}$$

Since it converges normally within any compact subset of $(0,\infty)$ (the domain of integration), then the integral expression in the question can be rewritten as:

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx &= \frac{1}{\Gamma(s)}\int_0^\infty Q(x)x^{s-1}\left(\sum_{n=1}^\infty e^{-qx}\right)dx\\ &= \frac{1}{\Gamma(s)}\sum_{n=1}^\infty\int_0^\infty\left(\sum_{m=0}^{q-1}a_{q-m}e^{mx}\right)x^{s-1}\cdot e^{-nqx}dx\\ &= \frac{1}{\Gamma(s)}\sum_{n=1}^\infty\sum_{m=0}^{q-1}a_{q-m}\int_0^\infty x^{s-1}e^{-(nq-m)x}dx\end{aligned}\tag{2}$$

Which, by swapping $r=q-m$ (where r ranges from 1 to q), extending from (2), we get the following:

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx &= \frac{1}{\Gamma(s)}\sum_{n=1}^\infty\sum_{r=1}^{q}a_r\int_0^\infty x^{s-1}e^{-(nq-(q-r))x}dx\\ &= \frac{1}{\Gamma(s)}\sum_{n=1}^\infty\sum_{r=1}^{q}a_r\int_0^\infty x^{s-1}e^{-((n-1)q+r)x}dx\\ &= \frac{1}{\Gamma(s)}\sum_{n=0}^\infty\sum_{r=1}^{q}a_r\int_0^\infty x^{s-1}e^{-(nq+r)x}dx\end{aligned}\tag{3}$$

Then, performing substitution $u=(nq+r)x$ for each index n and r, $du=(nq+r)dx$, which (3) becomes:

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx &= \frac{1}{\Gamma(s)}\sum_{n=0}^\infty\sum_{r=1}^{q}a_r\int_0^\infty\left(\frac{u}{nq+r}\right)^{s-1}\cdot e^{-u}\frac{du}{nq+r}\\ &= \frac{1}{\Gamma(s)}\sum_{n=0}^\infty\sum_{r=1}^{q}a_r\cdot\frac{1}{(nq+r)^s}\int_0^\infty u^{s-1}e^{-u}du\\ &= \frac{1}{\Gamma(s)}\sum_{n=0}^\infty\sum_{r=1}^{q}\frac{a_r}{(nq+r)^s}\cdot\Gamma(s) = \sum_{n=0}^\infty\sum_{r=1}^{q}\frac{a_r}{(nq+r)^s}\end{aligned}\tag{4}$$

Now, in terms of the original $L(s)$, recall that $a_n=a_m$ iff $n\equiv m \mod q$, so the original series expression can be rearranged as:

$$\begin{aligned}L(s) &= \sum_{k=1}^\infty\frac{a_k}{k^s} = \sum_{n=1}^\infty\frac{a_{nq}}{(nq)^s}+\sum_{n=0}^\infty\sum_{r=1}^{q-1}\frac{a_{nq+r}}{(nq+r)^s}\\ &= \sum_{n=0}^\infty\frac{a_q}{(nq+q)^s}+\sum_{n=0}^\infty\sum_{r=1}^{q-1}\frac{a_r}{(nq+r)^s} = \sum_{n=0}^\infty\sum_{r=1}^{q}\frac{a_r}{(nq+r)^s}\end{aligned}\tag{5}$$

Then, combining the results in (4) and (5), we get $L(s)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{Q(x)x^{s-1}}{e^{qx}-1}dx$ (for $\mathrm{Re}(s)>1$).

1.2 Continuation to $\mathbb{C}\setminus\{1\}$:

With the above integral expression for $\mathrm{Re}(s)>1$, one can separate the integration as follow:

$$\begin{aligned}L_1(s) :=&\frac{1}{\Gamma(s)}\int_0^1\frac{q(x)x^{s-1}}{e^{qx}-1}dx,\quad L_2(s):=\frac{1}{\Gamma(s)}\int_1^\infty\frac{Q(x)x^{s-1}}{e^{qx}-1}dx\\ L(s) =& \frac{1}{\Gamma(s)}\int_0^\infty\frac{Q(x)x^{s-1}}{e^{qx}-1}dx = L_1(s)+L_2(s)\end{aligned}\tag{6}$$

Since $Q(x) = \sum_{m=0}^{q-1} a_{q-m}e^{mx}$, it is with the order of $e^{(q-1)x}$. Then, for $x > 1$ and $\text{Re}(s) > 1$, since $qx > 1$, then $e^{qx} > e > 2$, so $\frac{1}{2}e^{qx} > 1$. Then, $L_2(s)$ satisfies the following inequality:

$$\begin{aligned} |L_2(s)| &\leq \frac{1}{|\Gamma(s)|}\int_1^\infty \frac{|Q(x)| \cdot |x^{s-1}|}{|e^{qx}-1|}dx \leq \frac{1}{|\Gamma(s)|}\int_1^\infty \frac{Ke^{(q-1)x} \cdot x^{\text{Re}(s)-1}}{e^{qx}-1}dx \\ &\leq \frac{1}{|\Gamma(s)|}\int_1^\infty \frac{Ke^{(q-1)x} \cdot x^{Re(s)-1}}{} \end{aligned} \tag{7}$$

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Question 2 *Stein and Shakarchi Pg. 204 Problem 4:*
One can combine ideas from the prime number theorem with the proof of Dirichlet's Theorem about primes in arithmetic progression (given in Book I) to prove the following: Let q and l be relatively prime integers. We consider the primes belonging to the arithmetic progression $\{qk + ll\}_{k\in\mathbb{N}}$, and let $\pi_{q,l}(x)$ denote the number of such primes $\leq x$. Then one has

$$\pi_{q,l}(x) \sim \frac{x}{\varphi(q)\log(x)} \quad \text{as } x \to \infty$$

where $\varphi(q)$ denotes the number of positive integers less than q and relatively prime to q (i.e. the Euler Totient Function).

Pf:

Given $q, l \in \mathbb{N}$ with $\gcd(q, l) = 1$. Find an expression of Dirichlet Series, that produces the following formula:

$$L(s) := \prod_p \frac{1}{1 - \delta_l(p)p^{-s}}$$

Where p ranges through all primes, and $\delta_l : \mathbb{N} \to \mathbb{N}$ is defined as follow:

$$\delta_l(n) = \begin{cases} 1 & n \equiv l \mod q \\ 0 & \text{otherwise} \end{cases}$$