Math CS 122B HW1

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Question 1 *Ahlfors Pg. 178 Problem 2:*
Show that the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

converges for $Re(z) > 1$, and represent its derivative in series form.

Pf:

The following proof would assume the domain of the above series is the half plane $Re(z) > 1$.

The series converges pointwise:

For all $z \in \mathbb{C}$ satisfying $Re(z) > 1$, $z = a + bi$ for $a, b \in \mathbb{R}$, and $a > 1$. Then, for any $n \in \mathbb{N}$, the number $n^{-z} = e^{-z\log(n)} = e^{-(a+bi)\ln(n)} = e^{-a\ln(n)} \cdot e^{i(-b\ln(n))} = n^{-a} \cdot e^{i(-b\ln(a))}$. Hence, if taken the modulus $e^{-a\ln(n)}$, since $a > 1$, by p-series test, the following series converges:

$$\sum_{n=1}^{\infty} n^{-a}$$

Which, since the original series satisfies the following:

$$\sum_{n=1}^{\infty} |n^{-z}| = \sum_{n=1}^{\infty} \left| n^{-a} \cdot e^{i(-b\ln(n))} \right| = \sum_{n=1}^{\infty} n^{-a}$$

Hence, the series absolutely converges, which $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ is defined on $Re(z) > 1$.

Partial Sum converges uniformly on any Compact Subset:

Suppose $K \subset \mathbb{C}$ is a compact subset of the plane $Re(z) > 1$, each component $n^{-z} = e^{-z\ln(n)} = n^{-a} \cdot e^{i(-b\ln(n))}$ is analytic on the half plane (also on K), hence there exists $z_0 \in K$, such that $|n^{-z_0}|$ yields the maximum.

Since for $z_0 = a+bi$, $|n^{-z_0}| = \left|n^{-a} \cdot e^{i(-b\ln(n))}\right| = n^{-a}$, and since z_0 is in the half plane, so $Re(z_0) = a > 1$. Then, the series $\sum_{n=1}^{\infty} n^{-a}$ converges.

Now, notice that for each $n \in \mathbb{N}$, $M_n = \sup_{z\in K} |n^{-z}| = \max_{z\in K} |n^{-z}| = n^{-a}$ satisfies $\sum_{n=1}^{\infty} M_n$ converges, then by Weierstrass M-Test, the series $\sum_{n=1}^{\infty} n^{-z}$ in fact converges uniformly on K.

Then, because the series $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ converges absolutely on $Re(z) > 1$, converging uniformly on all compact subset of the half plane, and each component is analytic on the half plane, then by the theorem in Ahlfors pg. 176, $\zeta(z)$ is analytic, and the partial sum $\sum_{n=1}^{N} n^{-z}$ (for $N \in \mathbb{N}$) has derivative converges to $\zeta'(z)$ uniformly on all compact subsets of the half plane. Hence, based on the same theorem again, we can claim the folloing on the chosen half plane:

$$\zeta'(z) = \lim_{N\to\infty} \frac{d}{dz} \sum_{n=1}^{N} n^{-z} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{d}{dz}(e^{-z\ln(n)}) = \lim_{N\to\infty} \sum_{n=1}^{N} -\ln(n)e^{-z\ln(n)} = -\sum_{n=1}^{\infty} ln(n)n^{-z}$$

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Question 2 *Ahlfors Pg. 184 Problem 5:*
The Fibonacci numbers are defined by $c_0 = 0$, $c_1 = 1$, and $c_n = c_{n-1} + c_{n-2}$ for all $n \geq 2$.
Show that the c_n are Taylor Coefficients of a rational function, and determine a closed expression for c_n.

Pf:

Consider the generating function, a formal power series defined as $F(z) = \sum_{n=0}^{\infty} c_n z^n$.

Radius of Convergence of the Power Series:

Recall that radius of convergence of power series $R = \limsup(|c_n|^{\frac{1}{n}})^{-1} \in [0, \infty]$, where c_n is the coefficients of each degree.

First, we can verify that for all $n \in \mathbb{N}$, $c_n < 2^n$:

For base case $n = 1$, $c_1 = 1 < 2^1$.

Now, suppose for given $n \geq 1$, $c_n < 2^n$, then for case $(n+1) \geq 2$, since $c_{n+1} = c_n + c_{n-1} < 2 \cdot c_n$ (since $c_n > c_{n-1}$), then $c_{n+1} < 2 \cdot c_n < 2 \cdot 2^n = 2^{n+1}$ by induction hypothesis, which this completes the induction.

Since all $n \in \mathbb{N}$ has $0 < c_n < 2^n$, then $|c_n|^{\frac{1}{n}} = c_n^{\frac{1}{n}} < (2^n)^{\frac{1}{n}} < 2$, so $\limsup(|c_n|^{\frac{1}{n}}) \leq 2$, hence $R = \limsup(|c_n|^{\frac{1}{n}})^{-1} \geq \frac{1}{2}$. Thus, we can claim that the power series $F(z) = \sum_{n=0}^{\infty} c_n z^n$ in fact converges absolutely for disk $|z| < \frac{1}{2}$ (since $|z| < \frac{1}{2}$ is contained in the radius of convergence).

Closed Expression of c_n:

Now, consider power series $F(z)$ on $|z| < \frac{1}{2}$: Since $F(z)$ can be rewritten as $c_0 + c_1 z + \sum_{n=2}^{\infty} c_n z^n = 1 + z + \sum_{n=2}^{\infty} c_n z^n$. Then, based on the definition of Fibonnaci numbers, it can be rewritten as:

$$F(z) = 1 + z + \sum_{n=2}^{\infty}(c_{n-1} + c_{n-2})z^n = 1 + z + \sum_{n=2}^{\infty} c_{n-1}z^n + \sum_{n=2}^{\infty} c_{n-2}z^n$$

$$= 1 + z + \sum_{n=1}^{\infty} c_n z^{n+1} + \sum_{n=0}^{\infty} c_n z^{n+2} = 1 + z\left(1 + \sum_{n=1}^{\infty} c_n z^n\right) + z^2 \sum_{n=0}^{\infty} c_n z^n$$

$$= 1 + z\sum_{n=0}^{\infty} c_n z^n + z^2 F(z) = 1 + zF(z) + z^2 F(z)$$

Then, we can yield the following:

$$F(z) = 1 + zF(z) + z^2F(z), \quad F(z)(1 - z - z^2) = 1, \quad F(z) = \frac{1}{1 - z - z^2}$$

Now, if $1-z-z^2=0$ (or $z^2+z-1=0$), we have $z=\frac{-1\pm\sqrt{5}}{2}$. Hence, $1-z-z^2=-\left(\frac{-1+\sqrt{5}}{2}-z\right)\left(\frac{-1-\sqrt{5}}{2}-z\right)$. Then, $F(z)$ can be decomposed using partial fraction:

$$F(z)=\frac{A}{\frac{-1+\sqrt{5}}{2}-z}+\frac{B}{\frac{-1-\sqrt{5}}{2}-z}=\frac{1}{1-z-z^2},\quad B\left(\frac{-1+\sqrt{5}}{2}-z\right)+A\left(\frac{-1-\sqrt{5}}{2}-z\right)=-1$$

So, from the above expression, we get:

$$B\cdot\frac{-1+\sqrt{5}}{2}+A\cdot\frac{-1-\sqrt{5}}{2}=-1,\quad -B-A=0$$

$$\implies A=-B,\quad B\cdot\frac{-1+\sqrt{5}}{2}-B\cdot\frac{-1-\sqrt{5}}{2}=-1$$

$$\implies B\left(\frac{-1+\sqrt{5}}{2}-\frac{-1-\sqrt{5}}{2}\right)=B\cdot\sqrt{5}=-1,\quad B=-\frac{1}{\sqrt{5}},\quad A=\frac{1}{\sqrt{5}}$$

So, $F(z)$ can be expressed as:

$$F(z)=\frac{1}{\sqrt{5}}\left(\frac{1}{\frac{-1+\sqrt{5}}{2}-z}-\frac{1}{\frac{-1-\sqrt{5}}{2}-z}\right)$$

Now, notice that for any $k\neq 0$, on $|z|<|k|$, since $|z/k|<1$, then $\sum_{n=0}^{\infty}(z/k)^n$ converges absolutely to $\frac{1}{1-z/k}=\frac{k}{k-z}$, which $\frac{1}{k-z}=\frac{1}{k}\sum_{n=0}^{\infty}(z/k)^n$.

Because both $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$ has abslute values greater than $\frac{1}{2}$ (first one is approximately 0.618, the second one is approximately -1.618), hence, on the disk $|z|<\frac{1}{2}$, both equations below are true based on the above formula:

$$\frac{1}{\frac{-1+\sqrt{5}}{2}-z}=\frac{1}{\frac{-1+\sqrt{5}}{2}}\sum_{n=0}^{\infty}\left(\frac{z}{\frac{-1+\sqrt{5}}{2}}\right)^n,\quad \frac{1}{\frac{-1-\sqrt{5}}{2}-z}=\frac{1}{\frac{-1-\sqrt{5}}{2}}\sum_{n=0}^{\infty}\left(\frac{z}{\frac{-1-\sqrt{5}}{2}}\right)^n$$

Hence, $F(z)$ can be expressed as:

$$F(z)=\frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}\left(\frac{1}{\frac{-1+\sqrt{5}}{2}}\right)^{n+1}z^n-\sum_{n=0}^{\infty}\left(\frac{1}{\frac{-1-\sqrt{5}}{2}}\right)^{n+1}z^n\right)$$

$$=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}-\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}}z^n=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}-\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{(-1)^2-(\sqrt{5})^2}{4}\right)^{n+1}}z^n$$

$$=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}-\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}{(-1)^{n+1}}z^n=\sum_{n=0}^{\infty}\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}z^n$$

Then, by the uniqueness of Taylor Series, the following is the closed expression of c_n:

$$\forall n\in\mathbb{N},\quad c_n=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$$

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Question 3 *Ahlfors Pg. 186 Problem 4:*

Pf:

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Question 4 *Stein and Shakarchi Pg. 86 Problem 2:*

Pf: