Lie Algebra of a Lie Group

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Tangent Space, Tangent Vectors and Derivations

In simplest case, if embedd manifold M^n into $\mathbb{R}^m$, for any chart (U, ϕ) of M, since $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ has its inverse ϕ^{-1} being smooth, for any $u \in U \subseteq M$, a tangent vector v_u associates with vector $v \in \mathbb{R}^n$, is characterized by differential of ϕ^{-1}:

$$v_u := D\phi^{-1}(\phi(u))(v) = \lim_{t\to 0} \frac{\phi^{-1}(\phi(u)+tv) - \phi^{-1}(\phi(u))}{t}$$

A collection of all such vector is the **Geometric Tangent Space** of u, denoted as $T_u(M)$.
insert image
Notice that for any smooth function $f \in C^\infty(M)$, it has a notion of directional derivative at u depending on the tangent vector $v_u \in T_u(M)$, and such derivative satisfies genral differentiation rules (for instance, product rule).

To generalize such notion into abstract manifold (space with no definition of vectors), we need a notion of **Derivation**: For any point $u \in M$, a **Derivation at** u, is a linear map $v_u : C^\infty(M) \to \mathbb{R}$, that satisfies the product rule:

$$\forall f, g \in C^\infty(M), \quad v_u(fg) = f(u)(v_u g) + g(u)(v_u f)$$

Which, the set of all derivations at u, denoted as $T_u(M)$, is the **Tangent Space** of M at u, and each derivation $v_u \in T_u(M)$ is called the **Tangent Vector** of u.

Vector Fields & Smooth Conditions

Given smooth manifold M, a vector field X is a function associating each point $u \in M$ with a tangent vector of u, so $X(u) \in T_u(M)$. More precisely, a vector field is a map $X : M \to TM$ (where TM denotes the **Tangent Bundle** of M), such that with the canonical projection map $\pi : TM \to M$, $\pi \circ X : M \to M$ is an identity.
Which, X is a **Smooth Vector Field**, if $X : M \to TM$ is a smooth map. And, a collection of smooth vector fields on M is denoted as $\mathfrak{X}(M)$.
insert image
An equivalent condition of saying a vector field X is smooth, is through smooth functions $f \in C^\infty(M)$: Since for all $u \in M$, $X(u) = X_u \in T_u(M)$ is a derivation at u, define $Xf : M \to \mathbb{R}$ by $Xf(u) = X_u(f)$. Then, X is a smooth vector field iff $Xf \in C^\infty(M)$.

Vector Fields of Different Manifolds

Given M, N two smooth manifolds, and smooth map $F : M \to N$. Let $X \in \mathfrak{X}(M)$, it would be ideal if we can send vector field X to be a vector field of N. Yet, this requires both injectivity and surjectivity, which is too much to assume.
Insert an example
So, we'll consider a weaker notion, called an F**-Relation**: Given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, the two are F-related, if for all $u \in M$, the following is true:

$$dF_u(X_u) = Y_{F(u)}$$

Simply speaking, F maps the tangent vectors collected by X, to be compatible with tangent vectors collected by Y.
Insert another example
Thm: If F is a diffeomorphism, then for every $X \in \mathfrak{X}(M)$, there exists a unique $Y \in \mathfrak{X}(N)$, such that X and Y are F-related.

Lie Brackets on Vector Fields

The initial motivation is to combine two vector fields $X, Y \in \mathfrak{X}(M)$ to be another vector field. Which, for all $f \in C^\infty(M)$, since $Yf \in C^\infty(M)$ from previous characterization, then $XYf = X(Yf) \in C^\infty(M)$. But, if consider function XY, in general it's not a vector field.
EX: Define vector fields $X = \frac{\partial}{\partial x}$, $y = x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Take smooth functions $f(x,y) = x$ and $g(x,y) = y$, then we get the following:

$$XY(fg) = X(x\frac{\partial}{\partial y}(xy)) = \frac{\partial}{\partial x}(x^2) = 2x$$

But, recall that vector field maps each point on M to a derivation, so product rule should hold:

$$XY(fg) = f(XYg) + g(XYf)$$

With some computation, this equation doesn't hold for the example:

$$f(XYg) + g(XYf) = x(X(x\frac{\partial}{\partial y}(y))) + y(X(x\frac{\partial}{\partial y}(x)))$$

$$= x(\frac{\partial}{\partial x}x) + y(X(x \cdot 0)) = x$$

So, we need to define a new operation, called **Lie Bracket**, which is defined as:

$$\forall X, Y \in \mathfrak{X}(M), \quad [X, Y] = XY - YX$$

$$\forall f \in C^\infty(M), \quad [X, Y]f = X(Yf) - Y(Xf)$$

Which, the output $[X, Y] \in \mathfrak{X}(M)$, and also satisfies the following:

- **Bilinearity:** $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- **Antisymmetry:** $[X, Y] = -[Y, X]$
- **Jacobi's Identity:** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Moreover, given smooth map $F : M \to N$, if $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ are F-related respectively, then $[X_1, X_2] \in \mathfrak{X}(M)$ and $[Y_1, Y_2] \in \mathfrak{X}(N)$ are also F-related. This is the essential tool for defining Lie Algebra on a Lie Group.

Lie Group & Left-Invariant Vector Fields

A **Lie Group** G, is a smooth manifold along with group structure, such that the group operation $P : G \times G \to G$ by $P(g, h) = gh$, and the inversion map $i : G \to G$ by $i(g) = g^{-1}$ are both smooth maps between manifolds.
EX: eucildean space
For all $g \in G$, denote the left multiplication $L_g : G \to G$ by $L_g(h) = gh$, since $L_g = P\big|_{\{g\}\times G}$, all left multiplication is a smooth map; also, since $L_{g^{-1}} \circ L_g(h) = L_{g^{-1}}(gh) = g^{-1}gh = h$, every left multiplication has a smooth inverse, hence it's a **Diffeomorphism**.

Left-Invariant vector fields: Given any $X \in \mathfrak{X}(G)$ and all $g \in G$, since L_g is a diffeomorphism, there's a notion of X being L_g-related to itself. Which, we call X a **Left-Invariant Vector Field**, if for all $g \in G$, X is L_g-related to itself.
EX: euclidean addition, circle or torus

Recall that Lie Bracket of vector field preserves an F-relation. Then, for all $X, Y \in \mathfrak{G}$ that are left-invariant, since for all $g \in G$, X and Y are L_g related to themselves, then the Lie Bracket $[X, Y]$ is also L_g related to $[X, Y]$. Hence, $[X, Y]$ is also left-invariant, Lie Bracket on a Lie Group preserves left invariance of vector fields.

Lie Algebra on a Lie Group

Application

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References

John M. Lee, Introduction to Smooth Manifolds, 2nd Edition