Math 118C HW3

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1

Question 1 *Rudin Pg. 241 Problem 19:*
Show that the system of equations

$$3x + y - z + u^2 = 0$$

$$x - y + 2z + u = 0$$

$$2x + 2y - 3z + 2u = 0$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

Pf:

2

Question 2 *Rudin Pg. 242 Problem 23:*
Define f *in* $\mathbb{R}^3$ *by*

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2$$

Show that $f(0, 1, -1) = 0$,$(D_1 f)(0, 1, -1) \neq 0$, *and that there exists therefore a differentiable function* g *in some neighborhood of* $(1, -1)$ *in* $\mathbb{R}^2$, *such that* $g(1, -1) = 0$ *and*

$$f(g(y_1, y_2), y_1, y_2) = 0$$

Find $(D_1 g)(1, -1)$ *and* $(D_2 g)(1, -1)$.

Pf:

3

Question 3 *Rudin Pg. 242 Problem 24:*
For $(x, y) \neq (0, 0)$, define $f = (f_1, f_2)$ by

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad f_2(x, y) = \frac{xy}{x^2 + y^2}$$

Compute the rank of $f'(x, y)$, and find the range of f.

Pf:

4

Question 4 *Rudin Pg. 242 Problem 25:*
Suppose $A \in \mathcal{L}(\mathbb{R}^n, \mathcal{R}^m)$, let r be the rank of A.

(a) Define S as in the proof of Theorem 9.32. Show that SA is a projection in $\mathbb{R}^n$ whose null space is null(A) and whose range is range(S).

(b) Use (a) to show that

$$\dim(\mathit{null}(A)) + \dim(\mathit{range}(A)) = n$$

Pf:

(a) Given that A has rank r, then its range range$(A) \subseteq \mathbb{R}^m$ is an r-dimensional linear subspace, hence there exists $y_1, ..., y_r \in$ range(A) that forms a basis of it.

Then, by the text in Rudin, choose $z_1, ..., z_r \in \mathbb{R}^n$, so for each index $i \in \{1, ..., r\}$, $Az_i = y_i$. Which, the collection $z_1, ..., z_r \in \mathbb{R}^n$ is linearly independent, since if $a_1, ..., a_r \in \mathbb{R}$ satisfies $\sum_{i=1}^r a_i z_i = \bar{0}$, then the following is true:

$$A\left(\sum_{i=1}^r a_i z_i\right) = \sum_{i=1}^r a_i(Az_i) = \sum_{i=1}^r a_i y_i$$

By the linear independence of $y_1, ..., y_r \in$ range(A), each $a_i = 0$, which proves the linear independence of $z_1, ..., z_r \in \mathbb{R}^n$.

Now, define $S \in \mathcal{L}(\text{range}(A), \mathbb{R}^n)$ the same as in the text, which has the following formula:

$$\forall c_1, ..., c_r \in \mathbb{R}, \quad S\left(\sum_{i=1}^r c_i y_i\right) = \sum_{i=1}^r c_i z_i$$

Then, for all $x \in \mathbb{R}^n$, since $Ax \in$ range(A), it is spanned by $y_1, ..., y_r$, hence there exists unique $a_1, ..., a_r \in \mathbb{R}$, such that the following is true:

$$Ax = \sum_{i=1}^r a_i y_i$$

Hence, we get the following:

$$SAx = S\left(\sum_{i=1}^r a_i y_i\right) = \sum_{i=1}^r a_i z_i$$

Hence, applying SA twice, we get:

$$SA(SAx) = SA\left(\sum_{i=1}^r a_i z_i\right) = S\left(\sum_{i=1}^r a_i y_i\right) = \sum_{i=1}^r a_i z_i$$

This shows that $SA(SAx) = SAx$ for all $x \in \mathbb{R}^n$, hence SA is a projection on $\mathbb{R}^n$.

Now, to find the null space and range, consider the following:

- For all $x \in$ null(A), since $Ax = 0$, then $SAx = S(0) = 0$, so $x \in$ null(SA), or null$(A) \subseteq$ null(SA).

On the other hand, for all $\in$ null(SA), since $S(Ax) = 0$, $Ax \in$ null(S). But, since $Ax \in$ range(A), there exists unique $a_1, ..., a_r \in \mathbb{R}$, with $Ax = \sum_{i=1}^{r} a_i y_i$. Hence, we have the following:

$$0 = S(Ax) = S\left(\sum_{i=1}^{r} a_i y_i\right) = \sum_{i=1}^{r} a_i z_i$$

Hence, by linear independence of $z_1, ..., z_r \in \mathbb{R}^n$, we must have $a_i = 0$ for all index $i \in \{1, ..., r\}$. This proves that $Ax = \sum_{i=1}^{r} a_i y_i = 0$, so $x \in$ null(A). Hence, null(SA) $\subseteq$ null(A), showing that null(SA) = null(A).

- For all $z \in$ range(SA), there exists $x \in \mathbb{R}^n$ with $SAx = z$. Since $z = S(Ax) \in$ range(S), then range(SA) $\subseteq$ range(S).

  Similarly, for all $z \in$ range(S), there exists $y \in$ range(A) (the domain of S), with $Sy = z$; then because there exists $x \in \mathbb{R}^n$, with $Ax = y$ by the definition of range, we have $SAx = S(Ax) = Sy = z$, hence $z \in$ range(SA), proving that range(S) $\subseteq$ range(SA), or range(S) = range(SA).

Hence, the above to cases proves that null(SA) = null(A), while range(S) = range(SA). So, SA is a projection in $\mathbb{R}^n$ with null space being null(A), and range being range(S).

(b)

5

Question 5 *Rudin Pg. 242 Problem 26:*
Show that the existence (and even the continuity) of $D_{12}f$ does not imlpy the existence of $D_1 f$. For example, let $f(x, y) = g(x)$, where g is nowhere differentiable.

Pf:

Consider the Weierstrass Functon $g : \mathbb{R} \to \mathbb{R}$, which is uniformly continuous, while being differentiable nowhere.

Then, given the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = g(x)$, since g is not differentiable with respect to its variable x, then $D_1 f$ does not exist; yet, since $D_2 f \equiv 0$ (due to the fact that g is a constant when x is fixed), then $D_{12}f = D_1(D_2 f) = 0$.

Hence, even though $D_{12}f$ is continuous, $D_1 f$ doesn't exist in this case.