Math CS 122B HW2

Zih-Yu Hsieh

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Question 1 *The **Beta function** is defined for $Re(\alpha) > 0$ and $\Re(\beta) > 0$ by*

$$B(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt$$

(a) Prove that

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(b) Show that

$$B(\alpha,\beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$$

Pf:

(a) First, we'll consider $\Gamma(\alpha)\Gamma(\beta)$:

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty t^{\alpha-1}e^{-t}dt \int_0^\infty s^{\beta-1}e^{-s}ds = \int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-s-t}dsdt$$

If we consider the change of variable $f : (0,1) \times (0,\infty) \to (0,\infty) \times (0,\infty)$ by $f(r,u) = (ur, u(1-r)) = (s,t)$, since this is a differentiable function, and its derivative is given by:

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial r}(ur) & \frac{\partial}{\partial u}(ur) \\ \frac{\partial}{\partial r}(u(1-r)) & \frac{\partial}{\partial u}(u(1-r)) \end{pmatrix} = \begin{pmatrix} u & r \\ -u & (1-r) \end{pmatrix}$$

$$\frac{\partial(s,t)}{\partial(r,u)} = |\begin{vmatrix} u & r \\ -u & (1-r) \end{vmatrix}| = u(1-r) - (-u)r = u$$

Then, the above integral can be given as:

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-s-t}dsdt = \int_0^\infty \int_0^1 (u(1-r))^{\alpha-1}(ur)^{\beta-1}e^{-(ur+u(1-r))} \left|\frac{\partial(s,t)}{\partial(r,u)}\right| drdu$$

$$= \int_0^\infty \int_0^1 u^{\alpha+\beta-2}e^{-u} \cdot (1-r)^{\alpha-1}r^{\beta-1}udrdu = \int_0^\infty u^{\alpha+\beta-1}e^{-u} \cdot (1-r)^{\alpha-1}r^{\beta-1}drdu$$

$$= \int_0^\infty u^{\alpha+\beta-1}e^{-u}du \cdot \int_0^1 (1-r)^{\alpha-1}r^{\beta-1}dr = \Gamma(\alpha+\beta) \cdot B(\alpha,\beta)$$

Hence, we can rewrite the above equality to be as follow:

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)\cdot B(\alpha,\beta), \quad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(Recall: $\Gamma(z) \neq 0$ for all $z \in \mathbb{C} \setminus S$, $S = \{0, -1, -2, ...\}$).

(b) Consider the following expression:

$$\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$$

First, if we do the substitution $(1+u) = e^t$, $du = e^t dt$, which $u = 0 \implies e^t = 1$, $t = 0$, and $\lim_{t\to\infty} e^t = \infty$, so $\lim_{t\to\infty} u = \infty$. Then, the integral can be rewrite as:

$$\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du = \int_0^\infty \frac{(e^t-1)^{\alpha-1}}{(e^t)^{\alpha+\beta}} e^t dt = \int_0^\infty ((1-e^{-t})e^t)^{\alpha-1}(e^{-t})^{\alpha+\beta}\cdot e^t dt$$

$$= \int_0^\infty (1-e^{-t})^{\alpha-1}\cdot(e^t)^{\alpha-1}\cdot(e^t)^{-\alpha}\cdot(e^t)^{-\beta}\cdot e^t dt = \int_0^\infty (1-e^{-t})^{\alpha-1}(e^{-t})^{\beta} dt$$

Then, for the above expression, if we do the second substitution $r = e^{-t}$, $dr = -e^{-t}dt$, $dt = -e^t dt = -r^{-1}dr$. Which $t = 0 \implies r = e^0$, $r = 1$, and $\lim_{t\to\infty} e^{-t} = \lim_{t\to\infty} r = 0$. So, the integral can be rewrite as:

$$\int_0^\infty (1-e^{-t})^{\alpha-1}(e^{-t})^{\beta} dt = \int_1^0 (1-r)^{\alpha-1} r^{\beta}\cdot(-r^{-1})dr = \int_0^1 (1-r)^{\alpha-1} r^{\beta-1} dr = B(\alpha,\beta)$$

Hence, we can conclude the following:

$$\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du = \int_0^\infty (1-e^{-t})^{\alpha-1}(e^{-t})^{\beta} dt = B(\alpha,\beta)$$

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Question 2 *The hypergeometric series $F(\alpha,\beta,\gamma;z)$ was defined as*

$$F(\alpha,\beta,\gamma;z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+n-1)\cdot\beta(\beta+1)...(\beta+n-1)}{n!\cdot\gamma(\gamma+1)...(\gamma+n-1)} z^n$$

Here $\alpha > 0, \beta > 0, \gamma > \beta$, and $|z| < 1$. Show that

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt$$

Show as a result that the hypergeometric function, initially defined by a power series convergent in the unit disc, can be continued analytically to the complex plane slit along the half-line $[1,\infty)$.

Pf:

Properties of Gamma function:

First, we can use induction to verity that given $z \in \mathbb{C} \setminus \{0,-1,-2,...\}$, all $n \in \mathbb{N}$ satisfies $\Gamma(z+n) = (z+n-1)...(z+1)z\Gamma(z)$.

For base case $n = 1$, by the identity of gamma function, $\Gamma(z+1) = z\Gamma(z)$, so the formula is true.

Then, suppose for given $n \in \mathbb{N}$, we have $\Gamma(z+n) = (z+n-1)...(z+1)z\Gamma(z)$, which for $(z+n+1)$, it satisfies:

$$\Gamma(z+n+1) = (z+n)\Gamma(z+n) = (z+n)(z+n-1)...(z+1)z\Gamma(z)$$

Hence, this completes the induction.

So, for all $n \in \mathbb{N}$, we also have the following identity:

$$(z+n-1)...(z+1)z = \frac{\Gamma(z+n)}{\Gamma(z)}$$

By this property, we can rewrite the hypergeometric series as follow:

$$F(\alpha,\beta,\gamma;z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+n-1)\cdot\beta(\beta+1)...(\beta+n-1)}{n!\cdot\gamma(\gamma+1)...(\gamma+n-1)} z^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(\Gamma(\alpha+n)/\Gamma(\alpha))(\Gamma(\beta+n)/\Gamma(\beta))}{n!\cdot(\Gamma(\gamma+n)/\Gamma(\gamma))} z^n = 1 + \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\cdot\Gamma(\gamma+n)} z^n$$

Power series of $(1-\zeta)^{-\alpha}$:

Given the above function, it is analytic within the disk $|\zeta| < 1$. Then, consider its derivatives at $\zeta = 0$, we get:

$$\frac{d}{d\zeta}(1-\zeta)^{-\alpha} = \alpha(1-\zeta)^{-\alpha-1}$$

$$\forall n \in \mathbb{N},\ \frac{d^n}{d\zeta^n}(1-\zeta)^{-\alpha} = (\alpha+n-1)...(\alpha+1)\alpha(1-\zeta)^{-\alpha-n} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}(1-\zeta)^{-\alpha-n}$$

So, let $f(\zeta) = (1-\zeta)^{-\alpha}$, $f^{(n)}(0) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$. Which, the power series about $\zeta = 0$ is:

$$f(\zeta) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\zeta^n = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!\cdot\Gamma(\alpha)}\zeta^n$$

The Integral:

Then, since the power series converges uniformly for any compact region within the unit disk $|\zeta| < 1$, while the integral of the function with $(1 - zt)^{-\alpha}$ being defined with $|z| < 1$, $t \in (0, 1)$, hence the function is integrated over a region that can be covered by a compact set (choose closed disk with radius $|\zeta| \le R < 1$, where $|z| < R$). As the power series converges uniformly on this region, integral and summation in fact commutes, hence the integral mentioned in the question can be represented as:

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\gamma-\beta-1}(1-t)^{\beta-1}\left(\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!\cdot\Gamma(\alpha)}(zt)^n\right)dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!} z^n \int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}dt$$

With the identity verified in **Question 1**, we know the following:

$$\forall \alpha, \beta \in \mathbb{C},\ Re(\alpha), Re(\beta) > 0, \quad B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1}t^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Hence, the above form of integral becomes:

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!} z^n \int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!} z^n \cdot B(\gamma-\beta, \beta+n)$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!} z^n \cdot \frac{\Gamma(\gamma-\beta)\Gamma(\beta+n)}{\Gamma(\gamma+n)}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)} z^n = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+0)\Gamma(\beta+0)}{0!\Gamma(\gamma+0)} + \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)} z^n$$

$$= 1 + \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)} z^n$$

Hence, we can conclude the following:

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!} z^n \int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}dt$$

$$= 1 + \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)} z^n = F(\alpha, \beta, \gamma; z)$$

The identity proposed in the question is showned above.

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Question 3

Pf:

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Question 4

Pf: