

# Math 118C HW4

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**Question 1** Rudin Pg. 242 Problem 27:

Put  $f(0, 0) = 0$ , and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if  $(x, y) \neq (0, 0)$ . Prove that

(a)  $f$ ,  $D_1f$ ,  $D_2f$  are continuous in  $\mathbb{R}^2$ .

(b)  $D_{12}f$  and  $D_{21}f$  exist at every point of  $\mathbb{R}^2$ , and are continuous except at  $(0, 0)$ .

(c)  $D_{12}f(0, 0) = 1$ , and  $D_{21}f(0, 0) = -1$ .

**Pf:**

For all  $(x, y) \in \mathbb{R}^2$  with  $(x, y) \neq (0, 0)$ , using polar coordinates,  $(x, y) = (r \cos(\theta), r \sin(\theta))$  for some  $r > 0$  and  $\theta \in [0, 2\pi)$ . Which,  $|(x, y)| = r$ , when consider limit definition, we'll use polar coordinates instead.

(a)  **$f$  is continuous:**

For  $(x, y) \neq (0, 0)$ , since  $f$  is a defined rational function, it is continuous, so it suffices to show  $f$  is continuous at 0. For all  $\epsilon > 0$ , choose  $\delta = \sqrt{\frac{\epsilon}{2}} > 0$ , then for all  $(x, y)$  satisfying  $0 < |(x, y)| = r < \delta$ , we get the following:

$$\begin{aligned} |f(x, y) - f(0, 0)| &= \left| \frac{(r \cos(\theta))(r \sin(\theta))((r \cos(\theta))^2 - (r \sin(\theta))^2)}{(r \cos(\theta))^2 + (r \sin(\theta))^2} - 0 \right| \\ &= \left| \frac{r^4 \sin(\theta) \cos(\theta)(\cos^2(\theta) - \sin^2(\theta))}{r^2} \right| \leq r^2 |\sin(\theta)| \cdot |\cos(\theta)| \cdot (|\cos(\theta)|^2 + |\sin(\theta)|^2) \\ &\leq 2r^2 < 2 \left( \sqrt{\frac{\epsilon}{2}} \right)^2 = 2 \cdot \frac{\epsilon}{2} = \epsilon \end{aligned}$$

This shows that  $f$  is continuous at  $(0, 0)$ , hence  $f$  is continuous in  $\mathbb{R}^2$ .

**$D_1f$  is continuous:**

First, using basic differentiation rule, for  $(x, y) \neq (0, 0)$ , we get the following:

$$D_1f(x, y) = \frac{\partial}{\partial x} \left( \frac{xy(x^2 - y^2)}{x^2 + y^2} \right) = \frac{(3x^2y - y^3)(x^2 + y^2) - xy(x^2 - y^2)2x}{(x^2 + y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$$

Which, at  $(0,0)$ ,  $D_1f$  could be obtained through limit:

$$D_1f(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{h \cdot 0(h^2 - 0^2)}{(h^2 + 0^2)h} = \lim_{h \rightarrow 0} 0 = 0$$

Which,  $D_1f(x,y)$  for  $(x,y) \neq (0,0)$  is again a rational function, which is continuous, so to verify continuity, it suffices to check  $(0,0)$ . For all  $\epsilon > 0$ , choose  $\delta = \frac{\epsilon}{6} > 0$ , then for all  $(x,y)$  satisfying  $0 < |(x,y)| = r < \delta$ , we get the following:

$$\begin{aligned} |D_1f(x,y) - D_1f(0,0)| &= \left| \frac{(r \cos(\theta))^4(r \sin(\theta)) + 4(r \cos(\theta))^2(r \sin(\theta))^3 - (r \sin(\theta))^5}{((r \cos(\theta))^2 + (r \sin(\theta))^2)^2} - 0 \right| \\ &= \left| \frac{r^5(\cos^4(\theta) \sin(\theta) + 4 \cos^2(\theta) \sin^3(\theta)) - \sin^5(\theta)}{r^4} \right| \leq r(|\cos^4(\theta) \sin(\theta)| + 4|\cos^2(\theta) \sin^3(\theta)| + |\sin^5(\theta)|) \\ &\leq r(1 + 4 + 1) < 6 \cdot \frac{\epsilon}{6} = \epsilon \end{aligned}$$

This proves the continuity of  $D_1f$  at  $(0,0)$ , so  $D_1f$  is continuous in  $\mathbb{R}^2$ .

**$D_2f$  is continuous:**

Using differentiation rule, for  $(x,y) \neq (0,0)$ , we get the following:

$$D_2f(x,y) = \frac{\partial}{\partial y} \left( \frac{xy(x^2 - y^2)}{x^2 + y^2} \right) = \frac{(x^3 - 3xy^2)(x^2 + y^2) - xy(x^2 - y^2)2y}{(x^2 + y^2)^2} = \frac{x^5 - xy^4 - 4x^3y^2}{(x^2 + y^2)^2}$$

Again, at  $(0,0)$ ,  $D_2f$  could be obtained through limit:

$$D_2f(0,0) = \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0 \cdot h(0^2 - h^2)}{(0^2 + h^2)h} = \lim_{h \rightarrow 0} 0 = 0$$

Notice that  $D_2f(x,y)$  for  $(x,y) \neq (0,0)$  is a rational function, which is continuous, so to verify continuity, it suffices to check  $(0,0)$ . For all  $\epsilon > 0$ , choose  $\delta = \frac{\epsilon}{6} > 0$ , then for all  $(x,y)$  satisfying  $0 < |(x,y)| = r < \delta$ , we get the following:

$$\begin{aligned} |D_2f(x,y) - D_2f(0,0)| &= \left| \frac{(r \cos(\theta))^5 - (r \cos(\theta))(r \sin(\theta))^4 - 4(r \cos(\theta))^3(r \sin(\theta))^2}{((r \cos(\theta))^2 + (r \sin(\theta))^2)^2} - 0 \right| \\ &= \left| \frac{r^5(\cos^5(\theta) - \cos(\theta) \sin^4(\theta) - 4 \cos^3(\theta) \sin^2(\theta))}{r^4} \right| \leq r(|\cos^5(\theta)| + |\cos(\theta) \sin^4(\theta)| + 4|\cos^3(\theta) \sin^2(\theta)|) \\ &\leq r(1 + 1 + 4) < 6 \cdot \frac{\epsilon}{6} = \epsilon \end{aligned}$$

This proves the continuity of  $D_2f$  at  $(0,0)$ , hence  $D_2f$  is continuous in  $\mathbb{R}^2$ .

(b) **Function  $D_{21}f$ :**

Given that  $D_1f(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$  for  $(x,y) \neq (0,0)$  and  $D_1f(0,0) = 0$ , apply differentiation rule for  $(x,y) \neq (0,0)$ , we get:

$$D_{21}f(x,y) = \frac{\partial}{\partial y} \left( \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} \right) = \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2 + y^2)^2 - (x^4y + 4x^2y^3 - y^5)2(x^2 + y^2)2y}{(x^2 + y^2)^4}$$

Which,  $D_{21}f(x, y)$  is continuous for  $(x, y) \neq (0, 0)$  (since it's a rational function).

Now, to get  $D_{21}f(0, 0)$ , we'll use limit definition:

$$D_{21}f(0, 0) = \lim_{h \rightarrow 0} \frac{D_1f(0, h) - D_1f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0^4 \cdot h + 4 \cdot 0^2 \cdot h^3 - h^5}{(0^2 + h^2)^2 h} = \lim_{h \rightarrow 0} -\frac{h^5}{h^5} = -1$$

Hence,  $D_{21}f$  exists on the whole  $\mathbb{R}^2$ , and is continuous at all  $(x, y) \neq (0, 0)$ . But, it is not continuous at  $(0, 0)$ , since choosing  $x \neq 0$  and  $y = 0$ ,  $D_{21}f$  becomes:

$$D_{21}f(x, 0) = \frac{x^8}{x^8} = 1$$

Hence,  $\lim_{x \rightarrow 0} D_{21}f(x, 0) = 1 \neq -1 = D_{21}f(0, 0)$ , showing the discontinuity at  $(0, 0)$ .

So,  $D_{21}f$  exists on  $\mathbb{R}^2$ , while being continuous on  $\mathbb{R}^2 \setminus \{0\}$ .

**Function  $D_{12}f$ :**

Given that  $D_2f(x, y) = \frac{x^5 - xy^4 - 4x^3y^2}{(x^2 + y^2)^2}$  for  $(x, y) \neq (0, 0)$  and  $D_2f(0, 0) = 0$ , apply differentiation rule for  $(x, y) \neq (0, 0)$ , we get:

$$D_{12}f(x, y) = \frac{\partial}{\partial x} \left( \frac{x^5 - xy^4 - 4x^3y^2}{(x^2 + y^2)^2} \right) = \frac{(5x^4 - y^4 - 12x^2y^2)(x^2 + y^2)^2 - (x^5 - xy^4 - 4x^3y^2)2(x^2 + y^2)2x}{(x^2 + y^2)^4}$$

Hence,  $D_{12}f$  is continuous for  $(x, y) \neq (0, 0)$ , since it's also a rational function.

Now, to get  $D_{12}f(0, 0)$ , we'll again use limit definition:

$$D_{12}f(0, 0) = \lim_{h \rightarrow 0} \frac{D_2f(h, 0) - D_2f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h^5 - h \cdot 0^4 - 4h^3 \cdot 0^2}{(h^2 + 0^2)^2 h} = \lim_{h \rightarrow 0} \frac{h^5}{h^5} = 1$$

Hence,  $D_{12}f$  exists on the whole  $\mathbb{R}^2$ , and is continuous at all  $(x, y) \neq (0, 0)$ . But again, it's not continuous at  $(0, 0)$ , since choosing  $x = 0$  and  $y \neq 0$ ,  $D_{12}f$  becomes:

$$D_{12}f(0, y) = \frac{-y^8}{y^8} = -1$$

Hence,  $\lim_{y \rightarrow 0} D_{12}f(0, y) = -1 \neq 1 = D_{12}f(0, 0)$ , showing the discontinuity at  $(0, 0)$ .

So,  $D_{12}f$  exists on  $\mathbb{R}^2$ , while being continuous on  $\mathbb{R}^2 \setminus \{0\}$ .

- (c) From **part (b)**, when verifying that the existence of  $D_{12}f(0, 0)$  and  $D_{21}f(0, 0)$ , we've shown that  $D_{12}f(0, 0) = 1$ , and  $D_{21}f(0, 0) = -1$ .

**Question 2** Rudin Pg. 242 Problem 28:

For  $t \geq 0$ , put

$$\varphi(x, t) = \begin{cases} x & 0 \leq x \leq \sqrt{t} \\ -x + 2\sqrt{t} & \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0 & \text{otherwise} \end{cases}$$

and put  $\varphi(x, t) = -\varphi(x, |t|)$  if  $t < 0$ .

Show that  $\varphi$  is continuous on  $\mathbb{R}^2$ , and  $D_2\varphi(x, 0) = 0$  for all  $x$ . Define

$$f(t) = \int_{-1}^1 \varphi(x, t) dx$$

Show that  $f(t) = t$  if  $|t| < \frac{1}{4}$ . Hence

$$f'(0) \neq \int_{-1}^1 D_2\varphi(x, 0) dx$$

**Pf:**

**Continuity of  $\varphi$ :**

First, for the open half plane  $x < 0$ , since  $\varphi(x, t) = 0$ , then  $\varphi$  is continuous.

**$D_2\varphi$  when  $t = 0$ :**

For all  $x \in \mathbb{R}$ , if  $x \leq 0$ , then we get  $\varphi(x, t) = 0$  regardless of  $t \in \mathbb{R}$ , showing that  $D_2\varphi(x, 0) = \frac{\partial \varphi}{\partial t}(x, 0) = 0$ .

Now for  $x > 0$ , since for all  $t \in \mathbb{R}$  satisfying  $4|t| < x^2$ , we have  $2\sqrt{|t|} < x$ , then  $\varphi(x, t) = 0$  when  $t \in (-\frac{x^2}{4}, \frac{x^2}{4})$ . So,  $D_2\varphi(x, 0) = 0$  (since  $\lim_{t \rightarrow 0} \frac{\varphi(x, t) - \varphi(x, 0)}{t} = \lim_{t \rightarrow 0} 0 = 0$ , because for small enough  $t$ , it lies in the range  $(-\frac{x^2}{4}, \frac{x^2}{4})$ ).

So, regardless of  $x \in \mathbb{R}$ , we have  $D_2\varphi(x, 0) = 0$ .

**Function  $f(t)$ :**

Given  $f(t) = \int_{-1}^1 \varphi(x, t) dx$ , when  $|t| < \frac{1}{4}$ , there are several cases to consider:

- when  $t \geq 0$ , then  $0 \leq \sqrt{t} < \sqrt{\frac{1}{4}} = \frac{1}{2}$ , while  $0 \leq 2\sqrt{t} < 1$ . Hence, the integral expression can be broken down as the following pieces:

$$\begin{aligned} \int_{-1}^1 \varphi(x, t) dx &= \int_{-1}^0 0 dx + \int_0^{\sqrt{t}} x dx + \int_{\sqrt{t}}^{2\sqrt{t}} (-x + 2\sqrt{t}) dx + \int_{2\sqrt{t}}^1 0 dx \\ &= \frac{1}{2} x^2 \Big|_0^{\sqrt{t}} + \left( -\frac{1}{2} x^2 + 2\sqrt{t}x \right) \Big|_{\sqrt{t}}^{2\sqrt{t}} = \frac{1}{2} t + ((4t - 2t) - (2t - \frac{1}{2}t)) = t \end{aligned}$$

- when  $t < 0$  (where  $t = -|t|$ ), since  $\varphi(x, t) = -\varphi(x, |t|)$  with  $|t| > 0$ , then inheriting from the above expression, we get:

$$\int_{-1}^1 \varphi(x, t) dx = - \int_{-1}^1 \varphi(x, |t|) dx = -|t| = t$$

Hence, for  $|t| < \frac{1}{4}$ , we can deduce that  $f(t) = t$ , which  $f'(t) = 1$ . So, the following inequality is true:

$$f'(0) = 1 \neq 0 = \int_{-1}^1 0 dx = \int_{-1}^1 D_2\varphi(x, 0) dx$$

This shows that differentiation under integral sign fails under certain situation.

### 3

**Question 3** Rudin Pg. 243 Problem 30:

Let  $f \in C^{(m)}(E)$ , where  $E$  is an open subset of  $\mathbb{R}^n$ . Fix  $a \in E$ , and suppose  $x \in \mathbb{R}^n$  is so close to 0 that the points  $p(t) = a + tx$  lie in  $E$  whenever  $0 \leq t \leq 1$ . Define  $h(t) = f(p(t))$  for all  $t \in \mathbb{R}$  for which  $p(t) \in E$ .

(a) For  $1 \leq k \leq m$ , show (by repeated application of the chain rule) that

$$h^{(k)}(t) = \sum (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

The sum extends over all order  $k$ -tuples  $(l_1, \dots, l_k)$  in which each  $l_j$  is one of the integers  $1, \dots, n$ .

**Pf:**

Given  $a, x \in \mathbb{R}^n$  (where  $x = (x_1, \dots, x_n)$  for fixed  $x_1, \dots, x_n \in \mathbb{R}$ ) and  $p(t) = a + tx$  for  $t \in [0, 1]$ , then  $p'(t) = x$ .

Now, we'll use induction to verify the formula (and we'll use matrix representation of the differentials).

First, for  $k = 1$ , using chain rule, we get the following:

$$h'(t) = Df(p(t))p'(t) = \begin{pmatrix} D_1 f & \dots & D_n f \end{pmatrix} \Big|_{p(t)} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^n D_i f(p(t)) x_i$$

Since all the possible 1-tuple is included in the summation, the  $h'(t)$  satisfies the given formula.

Now, suppose for given  $1 \leq k \leq (m-1)$ ,  $h^{(k)}(t)$  satisfies the following formula:

$$h^{(k)}(t) = \sum (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

Since for each  $k$ -tuple  $(l_1, \dots, l_k)$  (where each  $l_i \in \{1, \dots, n\}$ ) has the function  $x_{l_1} \dots x_{l_k} D_{l_1 \dots l_k} f(p(t))$  being a differentiable function from  $(0, 1)$  to  $\mathbb{R}$  (where  $D_{l_1 \dots l_k} f(z)$  for  $z \in E$  is a differentiable function, since it has only been differentiated  $k < m$  times, while  $f \in C^{(m)}(E)$ ). Then, to calculate the  $(k+1)^{th}$  derivative, we get:

$$\begin{aligned} h^{(k+1)}(t) &= \sum \frac{d}{dt} (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k} \\ \forall (l_1, \dots, l_k), \quad \frac{d}{dt} (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k} &= x_{l_1} \dots x_{l_k} D (D_{l_1 \dots l_k} f)(p(t)) p'(t) \\ &= x_{l_1} \dots x_{l_k} \sum_{i=1}^n D_i (D_{l_1 \dots l_k} f)(p(t)) x_i = \sum_{i=1}^n D_{il_1 \dots l_k} f(p(t)) x_i x_{l_1} \dots x_{l_k} \\ \implies h^{(k+1)}(t) &= \sum \left( \sum_{i=1}^n D_{il_1 \dots l_k} f(p(t)) x_i x_{l_1} \dots x_{l_k} \right) \end{aligned}$$

Which, the first summation indicates all possible  $k$ -tuple  $(l_1, \dots, l_k)$  for  $l_j \in \{1, \dots, n\}$ .

Now, for all  $(k+1)$ -tuple  $(j_0, j_1, \dots, j_k)$  where each  $j_l \in \{1, \dots, n\}$ , choose the unique  $k$ -tuple  $(j_1, \dots, j_k)$ , then  $D_{j_0 j_1 \dots j_k} f(p(t)) x_{j_0} x_{j_1} \dots x_{j_k}$  appears precisely once in the summation of  $h^{(k+1)}(t)$  given above; similarly, since each  $k$ -tuple  $(l_1, \dots, l_k)$  and  $i \in \{1, \dots, n\}$  corresponds to a unique  $(k+1)$ -tuple  $(i, l_1, \dots, l_k)$ , so the summation in  $h^{(k+1)}(t)$  has a 1-to-1 correspondance to all  $(k+1)$ -tuple. Then, the summation  $h^{(k+1)}(t)$  can also be described as:

$$h^{(k+1)}(t) = \sum D_{l_1 \dots l_k l_{k+1}} f(p(t)) x_{l_1} \dots x_{l_k} x_{l_{k+1}}$$

Where each  $(l_1, \dots, l_k, l_{k+1})$  is a  $(k+1)$ -tuple with entries from  $\{1, \dots, n\}$ .

**Question 4** *Rudin Pg. 288 Problem 2:*

For  $i = 1, 2, 3, \dots$ , let  $\varphi_i \in \mathcal{C}(\mathbb{R})$  have support in  $(2^{-i}, 2^{1-i})$ , such that  $\int \varphi_i = 1$ . Put

$$f(x, y) = \sum_{i=1}^{\infty} (\varphi_i(x) - \varphi_{i+1}(x)) \varphi_i(y)$$

Then  $f$  has compact support in  $\mathbb{R}^2$ ,  $f$  is continuous except at  $(0, 0)$ , and

$$\int dy \int f(x, y) dx = 0, \quad \text{but} \quad \int dx \int f(x, y) dy = 1$$

Observe that  $f$  is unbounded in every neighborhood of  $(0, 0)$ .

**Pf:**

**The function  $f$  is well-defined:**