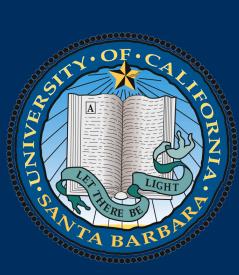
LIE ALGEBRA OF A LIE GROUP

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Tangent Space, Tangent Vectors and Derivations

In simplest case, if embedd manifold M^n into \mathbb{R}^m , for any chart (U,ϕ) of M, since $\phi:U\to\phi(U)\subseteq\mathbb{R}^n$ has its inverse ϕ^{-1} being smooth, for any $u\in U\subseteq M$, a tangent vector v_u associates with vector $v\in\mathbb{R}^n$, is characterized by differential of ϕ^{-1} :

$$v_u := D\phi^{-1}(\phi(u))(v) = \lim_{t \to 0} \frac{\phi^{-1}(\phi(u) + tv) - \phi^{-1}(\phi(u))}{t}$$

A collection of all such vector is the **Geometric Tangent Space** of u, denoted as $T_u(M)$.

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Notice that for any smooth function $f \in C^{\infty}(M)$, it has a notion of directional derivative at u depending on the tangent vector $v_u \in T_u(M)$, and such derivative satisfies genral differentiation rules (for instance, product rule).

To generalize such notion into abstract manifold (space with no definition of vectors), we need a notion of **Derivation**: For any point $u \in M$, a **Derivation at** u, is a linear map $v_u : C^{\infty}(M) \to \mathbb{R}$, that satisfies the product rule:

$$\forall f, g \in C^{\infty}(M), \quad v_u(fg) = f(u)(v_ug) + g(u)(v_uf)$$

Which, the set of all derivations at u, denoted as $T_u(M)$, is the **Tangent Space** of M at u, and each derivation $v_u \in T_u(M)$ is called the **Tangent Vector** of u.

Vector Fields & Smooth Conditions

Given smooth manifold M, a vector field X is a function associating each point $u \in M$ with a tangent vector of u, so $X(u) \in T_u(M)$. More precisely, a vector field is a map $X: M \to TM$ (where TM denotes the **Tangent Bundle** of M), such that with the canonical projection map $\pi: TM \to M$, $\pi \circ X: M \to M$ is an identity.

Which, X is a **Smooth Vector Field**, if $X:M\to TM$ is a smooth map. And, a collection of smooth vector fields on M is denoted as $\mathfrak{X}(M)$.

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An equivalent condition of saying a vector field X is smooth, is through smooth functions $f \in C^{\infty}(M)$: Since for all $u \in M$, $X(u) = X_u \in T_u(M)$ is a derivation at u, define $Xf : M \to \mathbb{R}$ by $Xf(u) = X_u(f)$. Then, X is a smooth vector field iff $Xf \in C^{\infty}(M)$.

Vector Fields of Different Manifolds

Given M,N two smooth manifolds, and smooth map $F:M\to N$. Let $X\in\mathfrak{X}(M)$, it would be ideal if we can send vector field X to be a vector field of N. Yet, this requires both injectivity and surjectivity, which is too much to assume.

Insert an example

So, we'll consider a weaker notion, called an F-Relation: Given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, the two are F-related, if for all $u \in M$, the following is true:

$$dF_u(X_u) = Y_{F(u)}$$

Simply speaking, F maps the tangent vectors collected by X, to be compatible with tangent vectors collected by Y.

Insert another example

Lie Brackets on Vector Fields

The initial motivation is to combine two vector fields $X,Y\in\mathfrak{X}(M)$ to be another vector field. Which, for all $f\in C^\infty(M)$, since $Yf\in C^\infty(M)$ from previous characterization, then $XYf=X(Yf)\in C^\infty(M)$.

Surfaces

Acknowledgements

References