Math CS 122B HW6

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Question 1 *Freitag Chap. V.6 Exercise 5:*
Lef f be an elliptic function for the lattice L. We choose $b_1, ..., b_n$ to be a system of representatives modulo L for the poles of f, and we consider for each j the principal part of f in the pole b_j:

$$\sum_{v=1}^{l_j} \frac{a_{v,j}}{(z-b_j)^v}$$

The Second Liiouville Theorem ensures the relation

$$\sum_{j=1}^{n} a_{1,j} = 0$$

Show:

(a) Let $c_1, ..., c_n \in \mathbb{C}$ b given numbers, and let $b_1, ..., b_n$ modulo L be a set of different points in $\mathbb{C}/L$. The function

$$h(z) := \sum_{j=1}^{n} c_j \zeta(z-b_j)$$

constructed by means of the Weierstrass ζ-function, is then elliptic, iff

$$\sum_{j=1}^{n} c_j = 0$$

(b) Let $b_1, ..., b_n$ be pairwise different modulo L, and let $l_1, ..., l_n$ be prescribed natural numbers. Let $a_{v,j}$ ($1 \le j \le n$, $1 \le v \le l_j$) be complex numbers such that $\sum_{j=1}^{n} a_{1,j} = 0$ and $a_{l_j,j} \neq 0$ for all j.

Then, there exists an elliptic function for the lattice L, having poles modulo L exactly in the points $b_1, ..., b_n$, and having the corresponding principal parts respectively equal to

$$\sum_{v=1}^{l_j} \frac{a_{v,j}}{(z-b_j)^v}$$

Pf:

(a) Given the Weierstrass σ-function below ($\sigma : \mathbb{C} \to \mathbb{C}$), the Weierstrass ζ-function ($\zeta : \mathbb{C} \setminus L \to \mathbb{C}$) is defined as:

$$\sigma(z) = z\prod_{\substack{w\in L\\ w\neq 0}}\left(1-\frac{z}{w}\right)\exp\left(\frac{z}{w}+\frac{1}{2}\left(\frac{z}{w}\right)^2\right), \quad \zeta(z)=\frac{\sigma'(z)}{\sigma(z)}$$

Based on the formula of σ, it has simple zeros at all $w \in L$; and, it implies that ζ is not defined only on L. Now, to prove the statement, consider the following:

$\Longrightarrow$: Suppose the defined $h(z)$ is elliptic. Then, since for each index $j \in \{1, ..., n\}$, $\sigma(z-b_j)$ has a simple zero at $(w+b_j)$ for each $w \in L$ (which the set $b_j + L$ contains all the simple zeros of $\sigma(z-b_j)$, which is discrete). Then, since $\bigcup_{j=1}^n (b_j+L)$ is also discrete, choose the fundamental region P of lattice L such that ∂P contains no points from $\bigcup_{j=1}^n (b_j+L)$ (the set containing all the zeros of each $\sigma(z-b_j)$, also the set of all undefined points of all $\zeta(z-b_j)$), by the Second Liouville's Theorem, we get the following:

$$0=\frac{1}{2\pi i}\int_{\partial P} h(z)dz = \frac{1}{2\pi i}\int_{\partial P}\sum_{j=1}^n c_j\zeta(z-b_j)dz = \sum_{j=1}^n c_j\cdot\frac{1}{2\pi i}\int_{\partial P}\frac{\sigma'(z-b_j)}{\sigma(z-b_j)}dz$$

For each $j \in \{1, ..., n\}$, since P only contains one representative of $b_j \in \mathbb{C}/L$, then it only contains one zero of $\sigma(z-b_j)$. Hence, by argument principle, we get the following:

$$\frac{1}{2\pi i}\int_{\partial P}\frac{\sigma'(z-b_j)}{\sigma(z-b_j)}dz = 1 = \text{Number of zeros of } \sigma(z-b_j) \text{ in } P$$

Hence, the original integral becomes:

$$0=\frac{1}{2\pi i}\int_{\partial P} h(z)dz = \sum_{j=1}^n c_j\cdot\frac{1}{2\pi i}\int_{\partial P}\frac{\sigma'(z-b_j)}{\sigma(z-b_j)}dz = \sum_{j=1}^n c_j$$

So, $\sum_{j=1}^n c_j = 0$.

$\Longleftarrow$: Now, suppose $\sum_{j=1}^n c_j = 0$. For all $w \in L$, since $\sigma(z+w)$ and $\sigma(z)$ both have simple zeros at any $w' \in L$, then $\frac{\sigma(z+w)}{\sigma(z)}$ is an entire function with no zeros in $\mathbb{C}$ (since the zeros cancel out at each $w' \in L$). Hence, there exists an analytic function $h : \mathbb{C} \to \mathbb{C}$, with $\frac{\sigma(z+w)}{\sigma(z)} = e^{h(z)}$. Then, apply derivatives, we get:

$$\frac{\sigma'(z+w)\sigma(z)-\sigma'(z)\sigma(z+w)}{(\sigma(z))^2} = h'(z)e^{h(z)} = h'(z)\cdot\frac{\sigma(z+w)}{\sigma(z)}$$

$$\frac{\sigma'(z+w)\sigma(z+w)}{\sigma(z+w)\sigma(z)} - \frac{\sigma'(z)\sigma(z+w)}{(\sigma(z))^2} = h'(z)\cdot\frac{\sigma(z+w)}{\sigma(z)}$$

$$\frac{\sigma'(z+w)}{\sigma(z+w)} - \frac{\sigma'(z)}{\sigma(z)} = h'(z)$$

On the other hand, since $\left(\frac{\sigma'(z)}{\sigma(z)}\right)' = -\wp(z)$, then:

$$h''(z) = \left(\frac{\sigma'}{\sigma}\right)'(z+w) - \left(\frac{\sigma'}{\sigma}\right)'(z) = (-\wp(z+w)) - (-\wp(z)) = 0$$

Hence, $h(z)$ is in fact a degree 1 polynomial. So, there exists $a_w, b_w \in \mathbb{C}$, such that:

$$\frac{\sigma(z+w)}{\sigma(z)} = e^{h(z)} = e^{a_w z + b_w}, \quad \sigma(z+w) = e^{a_w z+b_w}\sigma(z)$$

Then, apply the derivative, and take its quotient with $\sigma(z+w)$, we get:

$$\sigma'(z+w) = a_w e^{a_w z+b_w}\sigma(z) + e^{a_w z+b_w}\sigma'(z)$$

$$\zeta(z+w) = \frac{\sigma'(z+w)}{\sigma(z+w)} = \frac{a_w e^{a_w z+b_w}\sigma(z) + e^{a_w z+b_w}\sigma'(z)}{e^{a_w z+b_w}\sigma(z)} = a_w + \frac{\sigma'(z)}{\sigma(z)} = a_w + \zeta(z)$$

Which, apply it to the definition of $h(z)$, we get:

$$h(z+w) = \sum_{j=1}^{n} c_j\zeta(z-b_j+w) = \sum_{j=1}^{n} c_j(a_w+\zeta(z-b_j)) = a_j\sum_{j=1}^{n} c_j + \sum_{j=1}^{n} c_j\zeta(z-b_j) = \sum_{j=1}^{n} c_j\zeta(z-b_j) = h(z)$$

(Note: recall that $\sum_{j=1}^{n} c_j$ is assumed to be 0).

Hence, $h(z)$ is an elliptic function.

he above two implication shows that $h(z)$ is an elliptic function iff $\sum_{j=1}^{n} c_j = 0$.

(b) To construct the desired principal part for each point $b_1, ..., b_n$ modulo L, we need to consider the order 1 case separately from the other poles:

For order 1, we have the condition that $\sum_{j=1}^{n} a_{1,j} = 0$, so we can utilize the statement proven in **part (a)**. Notice that $\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$ is the logarithmic derivative of $\sigma(z)$, with the formula given in **part (a)**, we get the following:

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\substack{w\in L\\ w\neq 0}} \left(\frac{-1/w}{1-z/w} + \frac{d}{dz}\left(\frac{z}{w} + \frac{1}{2}\cdot\frac{z^2}{w^2}\right)\right) = \frac{1}{z} + \sum_{\substack{w\in L\\ w\neq 0}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right)$$

This demonstrats that $\zeta(z)$ has its principal part given as $\frac{1}{z-w}$ at all $w \in L$. Hence, $\zeta(z-b_j)$ would have its principal part given as $\frac{1}{z-b_j}$ for all point equivalent to $b_j \mod L$. Which, using the statement in **part (a)**, we know since $\sum_{j=1}^{n} a_{1,j} = 0$, it implies that $h_1(z) = \sum_{j=1}^{n} a_{1,j}\zeta(z-b_j)$ is an elliptic function; moreover, since each b_j is distinct, its principal part is governed by only $a_{1,j}\zeta(z-b_j)$ for each index j, hence this is an elliptic function describing the principal part up to the simple poles at each point.

For order ≥ 2, we could utilize the fact that $\wp(z)$ has a double pole at all $w \in L$. Recall the formula of $\wp(z)$ in series form:

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{w\in L\\ w\neq 0}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right)$$

Which, its principal part is given by $\frac{1}{(z-w)^2}$ at all $w \in L$. So, for any index j with $l_j \geq 2$, to describe the principal part with $\frac{a_{2,j}}{(z-b_j)^2}$ at each point equivalent to $b_j \mod L$, we can use $a_{2,j}\wp(z-b_j)$ (shift the double poles to each point in $b_j + L$).

Besides that, for any $n > 0$, since $\wp(z)$ converges normally within $\mathbb{C} \setminus L$, then its n^{th} order derivative can be performed term by term:

$$\wp^{(n)}(z) = \frac{d^n}{dz^n}\left(\frac{1}{z^2}\right) + \sum_{\substack{w\in L\\ w\neq 0}} \frac{d^n}{dz^n}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right) = \frac{(-1)^n\cdot(n+1)!}{z^{n+2}} + \sum_{\substack{w\in L\\ w\neq 0}} \frac{(-1)^n\cdot(n+1)!}{(z-w)^{n+2}}$$

$$\frac{(-1)^n}{(n+1)!}\wp^{(n)}(z) = \frac{1}{z^{n+2}} + \sum_{\substack{w\in L\\ w\neq 0}} \frac{1}{(z-w)^{(n+2)}}$$

This shows that the function $\frac{(-1)^n}{(n+1)!}\wp^{(n)}(z)$ has principal part $\frac{1}{(z-w)^{n+2}}$ at all $w \in L$. So, for all index j with $l_j > 2$, any $2 < v < l_j$ with its principal part given by $\frac{a_{v,j}}{(z-b_j)^v}$ at each point equivalent to b_j mod L, could be given by $a_{v,j} \cdot \frac{(-1)^{(v-2)}}{(v-1)!}\wp^{(v-2)}(z-b_j)$, based on similar logic as above.

In general, to create an elliptic function with the prescribed principal parts, one explicit formula can be given as:

$$\sum_{j=1}^{n} a_{1,j}\zeta(z-b_j) + \sum_{j=1}^{n}\sum_{v=2}^{l_j} a_{v,j} \cdot \frac{(-1)^{v-2}}{(v-1)!}\wp^{(v-2)}(z-b_j)$$

(Note: if $l_j < 2$, simply ignore the term).

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Question 2 *Freitag Chap. V.6 Exercise 7:*
We are interested in alternating $\mathbb{R}$-bilinear maps (forms)

$$A : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$$

Show:

(a) *Any such map A is of the form*

$$A(z, w) = hIm(z\overline{w})$$

with a uniquely determined real number h. We have explicitly $h = A(1, i)$.

(b) *Let $L \subset \mathbb{C}$ be a lattice. Then A is called a* Riemannian form *with respect to L iff h is positive, and A only takes integral values on $L \times L$. If*

$$L = \mathbb{Z}w_1 + \mathbb{Z}w_2, \quad Im\left(\frac{w_2}{w_1}\right) > 0$$

then the formula

$$A(t_1w_1 + t_2w_2, s_1w_1 + s_2w_2) := \det\begin{pmatrix} t_1 & s_1 \\ t_2 & s_2 \end{pmatrix}$$

defines a Riemannian form A on L.

(c) *A non-constant analytic function $\Theta : \mathbb{C} \to \mathbb{C}$ is called a* theta function *for the lattice $L \subset \mathbb{C}$, iff it satisfies an equation of the type*

$$\Theta(z + w) = e^{a_w z + b_2} \cdot \Theta(z)$$

for all $z \in \mathbb{C}$, and all $w \in L$. Here, a_w and b_w are onstants that may depend on w, but not on z.

Show the existence of a Riemannian form A with respect to L, such that

$$A(w, \lambda) = \frac{1}{2\pi i}(a_w\lambda - wa_\lambda)$$

for all $w, \lambda \in L$.

Pf:

(a) For any $z, w \in \mathbb{C}$, there exists $a, b, c, d \in \mathbb{R}$, with $z = a + bi$ and $w = c + di$. Then, by the property of a bilinear form, we get:

$$A(z, w) = A(a + bi, c + di) = A(a, c + di) + A(bi, c + di) = A(a, c) + A(a, di) + A(bi, c) + A(bi, di)$$

$$= acA(1, 1) + adA(1, i) + bcA(i, 1) + bdA(i, i)$$

Then, because of the property of alternating form, $A(z, w) = -A(w, z)$, which any $u \in \mathbb{C}$ satisfies $A(u, u) = -A(u, u)$, so $A(u, u) = 0$. Hence, we can further reduce the equation to the following:

$$A(z, w) = acA(1, 1) + adA(1, i) + bcA(i, 1) + bdA(i, i) = adA(1, i) - bcA(1, i) = (ad - bc)A(1, i)$$

Now, notice that if we take $z\overline{w}$, we get:

$$z\overline{w} = (a+bi)\overline{(c+di)} = (a+bi)(c-di) = (ac+bd)+(bc-ad)i$$

Which, $\mathrm{Im}(z\overline{w}) = bc - ad$. So in fact, we get the following formula:

$$A(z,w) = (ad-bc)A(1,i) = -A(1,i)\cdot \mathrm{Im}(z\overline{w})$$

So, let $h = -A(1,i) = A(i,1)$ (which is uniquely determined by the alternating form), we get:

$$A(z,w) = A(i,1)\cdot \mathrm{Im}(z\overline{w}) = h\cdot \mathrm{Im}(z\overline{w})$$

(b) If view $\mathbb{C}$ as an $\mathbb{R}$-vector space, it is a two-dimensional vector space. Which, the basis w_1, w_2 of the lattice L is also a basis for $\mathbb{C}$. Then, for all $z, w \in \mathbb{C}$. Then, for all $z, w \in \mathbb{C}$, there exists $t_1, t_2, s_1, s_2 \in \mathbb{R}$, such that $z = t_1w_1 + t_2w_2$, and $w = s_1w_2 + s_2w_2$.

First, we'll check that the given form is an alternating bilinear form:

If consider $A(z,w)$ and $A(w,z)$, we get:

$$A(z,w) = A(t_1w_1+t_2w_2, s_1w_1+s_2w_2) = \det\begin{pmatrix} t_1 & s_1 \\ t_2 & s_2 \end{pmatrix}$$

$$= -\det\begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix} = -A(s_1w_2+s_2w_2, t_1w_2+t_2w_2) = -A(w,z)$$

So, the alternating property is checked. Now, if given $u \in \mathbb{C}$, with $k_1, k_2 \in \mathbb{R}$ satisfying $u = k_1w_1 + k_2w_2$, then we get the following:

$$A(z+u,w) = A((t_1w_1+t_2w_2)+(k_1w_1+k_2w_2), s_1w_1+s_2w_2)$$

$$A((t_1+k_1)w_1+(t_2+k_2)w_2, s_1w_1+s_2w_2) = \det\begin{pmatrix} (t_1+k_1) & s_1 \\ (t_2+k_2) & s_2 \end{pmatrix}$$

$$= (t_1+k_1)s_2 - (t_2+k_2)s_1 = (t_1s_2 - t_2s_1) + (k_1s_2 - k_2s_1)$$

$$= \det\begin{pmatrix} t_1 & s_1 \\ t_2 & s_2 \end{pmatrix} + \det\begin{pmatrix} k_1 & s_1 \\ k_2 & s_2 \end{pmatrix}$$

$$= A(t_1w_1+t_2w_2, s_1w_1+s_2w_2) + A(k_1w_1+k_2w_2, s_1w_1+s_2w_2)$$

$$= A(z,w) + A(u,w)$$

This proves the bilinearity (including the alternating property, this also proves the linearity of the second column).

So, A defined in the question is an alternating bilinear form.

Now, for all $z, w \in L\times L$, since there exists $t_1, t_2, s_1, s_2 \in \mathbb{Z}$, with $z = t_1w_1 + t_2w_2$ and $w = s_1w_1 + s_2w_2$, we get:

$$A(z,w) = A(t_1w_1+t_2w_2, s_1w_1+s_2w_2) = \det\begin{pmatrix} t_1 & s_1 \\ t_2 & s_2 \end{pmatrix} = t_1s_2 - t_2s_1 \in \mathbb{Z}$$

So, A yields integer value for all elements in $L \times L$.

Lastly, consider $h = A(1, i)$ given in **part (a)**. Given that $w_1 = a + bi$, $w_2 = c_d i$ with $a, b, c, d \in \mathbb{R}$, and $\mathrm{Im}(w_2/w_1) > 0$, we get:

$$\frac{w_2}{w_1} = \frac{c+di}{a+bi} = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)} = \frac{(ac+bd)+(ad-bc)i}{a^2+b^2}, \quad \mathrm{Im}\left(\frac{w_2}{w_1}\right) = \frac{ad-bc}{a^2+b^2} > 0$$
$$\implies ad - bc > 0$$

Then, given the definition of A, we know the following:

$$A(w_1, w_2) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$A(w_1, w_2) = A(a+bi, c+di) = acA(1,1) + adA(1,i) + bcA(i,1) + bdA(i,i)$$
$$= adA(1,i) - bcA(1,i) = (ad-bc)h$$

Hence, we derived the following:

$$(ad-bc)h = 1 < 0, \quad ad-bc > 0 \implies h = \frac{1}{ad-bc} > 0$$

Then, since A is an alternating bilinear form, takes integer values on $L \times L$, and has $h > 0$, A is a Riemannian Form.

(c) Given the definition of Θ function, we know for any $z \in \mathbb{C}$, if $\Theta(z) = 0$, then for all $w \in L$, $\Theta(z+w) = e^{a_w z + b_w}\Theta(z) = 0$. Hence, let $b_1, ..., b_n$ represent the zeros of Θ in a fundamental region P, then for all $z \in \mathbb{C}$, we get $\Theta(z) = 0$ iff $z \equiv b_j \mod L$ for some $j \in \{1, ..., n\}$ (since if $z \in P$ satisfies $z \neq b_j$ for all index j, then for all $w \in L$, $\Theta(z+w) = e^{a_w z + b_w}\Theta(z) \neq 0$).

On the other hand, for all $w \in L$, if consider the derivative $\Theta'(z+w)$, we get:

$$\Theta'(z+w) = a_w e^{a_w z + b_w}\Theta(z) + e^{a_w z + b_w}\Theta'(z)$$

Which, the following is true:

$$\frac{\Theta'(z+w)}{\Theta(z+w)} = \frac{a_w e^{a_w z + b_w}\Theta(z) + e^{a_w z + b_w}\Theta'(z)}{e^{a_w z + b_w}\Theta(z)} = a_w + \frac{\Theta'(z)}{\Theta(z)}$$

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Question 3 *Freitag Chap. V.7 Exercise 5:*
Show:

(a) *For the lattice* $L_i = \mathbb{Z} + \mathbb{Z}i$ *we have* $g_3(i) = 0$ *and* $g_2(i) \in \mathbb{R}^\times$, *in particular* $\Delta(i) = g_2^3(i) > 0$.

(b) *For the lattice* $L_w = \mathbb{Z} + \mathbb{Z}w$, $w := e^{2\pi i/3}$, *we have* $g_2(w) = 0$ *and* $g_3(w) \in \mathbb{R}^\times$, *in particular* $\Delta(w) = -27g_3^2(w)$.

Pf:

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Question 4 *Freitag Chap. V.8 Exercise 3:*
The Eisenstein series are "real" functions, i.e. $\overline{G_k(\mathcal{T})} = G_k(-\mathcal{T})$. *This implies*

$$G_k\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\mathcal{T})+\delta}\right) = (\gamma(-\overline{\mathcal{T}})+\delta)\overline{G_k(\mathcal{T})} \qquad \textit{and}$$

$$j\left(\frac{\alpha(-\overline{\mathcal{T}})+\beta}{\gamma(-\overline{\mathcal{T}})+\delta}\right) = \overline{j(\mathcal{T})} \qquad \textit{for } \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in \Gamma$$

On the vertical half-lines $Re(\mathcal{T}) = \pm\frac{1}{2}$ *in* $\mathbb{H}$ *in* $\mathbb{H}$ *the Eisenstein series and the* j*-function are real. if* $\mathcal{T} \in \mathbb{H}$ *lies on the circle line* $|\mathcal{T}| = 1$*, then* $j(\mathcal{T}) = \overline{j(\mathcal{T})}$. *In particular, the* j*-function is real on the boundary of the modular figure, and on the imaginary axis.*

Pf: