

Math CS 122B HW2

Zih-Yu Hsieh

April 8, 2025

1

Question 1 The **Beta function** is defined for $\operatorname{Re}(\alpha) > 0$ and $\Re(\beta) > 0$ by

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt$$

(a) Prove that

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

(b) Show that

$$B(\alpha, \beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$$

Pf:

(a) First, we'll consider $\Gamma(\alpha)\Gamma(\beta)$:

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty t^{\alpha-1} e^{-t} dt \int_0^\infty s^{\beta-1} e^{-s} ds = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-s-t} ds dt$$

If we consider the change of variable $f : (0, 1) \times (0, \infty) \rightarrow (0, \infty) \times (0, \infty)$ by $f(r, u) = (ur, u(1-r)) = (s, t)$, since this is a differentiable function, and its derivative is given by:

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial r}(ur) & \frac{\partial}{\partial u}(ur) \\ \frac{\partial}{\partial r}(u(1-r)) & \frac{\partial}{\partial u}(u(1-r)) \end{pmatrix} = \begin{pmatrix} u & r \\ -u & (1-r) \end{pmatrix}$$
$$\frac{\partial(s, t)}{\partial(r, u)} = \left| \begin{pmatrix} u & r \\ -u & (1-r) \end{pmatrix} \right| = u(1-r) - (-u)r = u$$

Then, the above integral can be given as:

$$\begin{aligned} \Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-s-t} ds dt = \int_0^\infty \int_0^1 (u(1-r))^{\alpha-1} (ur)^{\beta-1} e^{-(ur+u(1-r))} \left| \frac{\partial(s, t)}{\partial(r, u)} \right| dr du \\ &= \int_0^\infty \int_0^1 u^{\alpha+\beta-2} e^{-u} \cdot (1-r)^{\alpha-1} r^{\beta-1} u dr du = \int_0^\infty u^{\alpha+\beta-1} e^{-u} \cdot (1-r)^{\alpha-1} r^{\beta-1} dr du \\ &= \int_0^\infty u^{\alpha+\beta-1} e^{-u} du \cdot \int_0^1 (1-r)^{\alpha-1} r^{\beta-1} dr = \Gamma(\alpha + \beta) \cdot B(\alpha, \beta) \end{aligned}$$

Hence, we can rewrite the above equality to be as follow:

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha + \beta) \cdot B(\alpha, \beta), \quad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

(Recall: $\Gamma(z) \neq 0$ for all $z \in \mathbb{C} \setminus S$, $S = \{0, -1, -2, \dots\}$).

(b) Consider the following expression:

$$\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$$

First, if we do the substitution $(1+u) = e^t$, $du = e^t dt$, which $u = 0 \implies e^t = 1$, $t = 0$, and $\lim_{t \rightarrow \infty} e^t = \infty$, so $\lim_{t \rightarrow \infty} u = \infty$. Then, the integral can be rewrite as:

$$\begin{aligned} \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du &= \int_0^\infty \frac{(e^t - 1)^{\alpha-1}}{(e^t)^{\alpha+\beta}} e^t dt = \int_0^\infty ((1 - e^{-t})e^t)^{\alpha-1} (e^{-t})^{\alpha+\beta} \cdot e^t dt \\ &= \int_0^\infty (1 - e^{-t})^{\alpha-1} \cdot (e^t)^{\alpha-1} \cdot (e^t)^{-\alpha} \cdot (e^t)^{-\beta} \cdot e^t dt = \int_0^\infty (1 - e^{-t})^{\alpha-1} (e^{-t})^\beta dt \end{aligned}$$

Then, for the above expression, if we do the second substitution $r = e^{-t}$, $dr = -e^{-t} dt$, $dt = -e^t dt = -r^{-1} dr$. Which $t = 0 \implies r = e^0$, $r = 1$, and $\lim_{t \rightarrow \infty} e^{-t} = \lim_{t \rightarrow \infty} r = 0$. So, the integral can be rewrite as:

$$\int_0^\infty (1 - e^{-t})^{\alpha-1} (e^{-t})^\beta dt = \int_1^0 (1 - r)^{\alpha-1} r^\beta \cdot (-r^{-1}) dr = \int_0^1 (1 - r)^{\alpha-1} r^{\beta-1} dr = B(\alpha, \beta)$$

Hence, we can conclude the following:

$$\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du = \int_0^\infty (1 - e^{-t})^{\alpha-1} (e^{-t})^\beta dt = B(\alpha, \beta)$$

Question 2 The hypergeometric series $F(\alpha, \beta, \gamma; z)$ was defined as

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1) \cdot \beta(\beta+1)\dots(\beta+n-1)}{n! \cdot \gamma(\gamma+1)\dots(\gamma+n-1)} z^n$$

Here $\alpha > 0, \beta > 0, \gamma > \beta$, and $|z| < 1$. Show that

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-zt)^{-\alpha} dt$$

Show as a result that the hypergeometric function, initially defined by a power series convergent in the unit disc, can be continued analytically to the complex plane slit along the half-line $[1, \infty)$.

Pf:

Properties of Gamma function:

First, we can use induction to verify that given $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, all $n \in \mathbb{N}$ satisfies $\Gamma(z+n) = (z+n-1)\dots(z+1)z\Gamma(z)$.

For base case $n=1$, by the identity of gamma function, $\Gamma(z+1) = z\Gamma(z)$, so the formula is true.

Then, suppose for given $n \in \mathbb{N}$, we have $\Gamma(z+n) = (z+n-1)\dots(z+1)z\Gamma(z)$, which for $(z+n+1)$, it satisfies:

$$\Gamma(z+n+1) = (z+n)\Gamma(z+n) = (z+n)(z+n-1)\dots(z+1)z\Gamma(z)$$

Hence, this completes the induction.

So, for all $n \in \mathbb{N}$, we also have the following identity:

$$(z+n-1)\dots(z+1)z = \frac{\Gamma(z+n)}{\Gamma(z)}$$

By this property, we can rewrite the hypergeometric series as follow:

$$\begin{aligned} F(\alpha, \beta, \gamma; z) &= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1) \cdot \beta(\beta+1)\dots(\beta+n-1)}{n! \cdot \gamma(\gamma+1)\dots(\gamma+n-1)} z^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(\Gamma(\alpha+n)/\Gamma(\alpha))(\Gamma(\beta+n)/\Gamma(\beta))}{n! \cdot (\Gamma(\gamma+n)/\Gamma(\gamma))} z^n = 1 + \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n! \cdot \Gamma(\gamma+n)} z^n \end{aligned}$$

Power series of $(1-\zeta)^{-\alpha}$:

Given the above function, it is analytic within the disk $|\zeta| < 1$. Then, consider its derivatives at $\zeta = 0$, we get:

$$\frac{d}{d\zeta}(1-\zeta)^{-\alpha} = \alpha(1-\zeta)^{-\alpha-1}$$

$$\forall n \in \mathbb{N}, \frac{d^n}{d\zeta^n}(1-\zeta)^{-\alpha} = (\alpha+n-1)\dots(\alpha+1)\alpha(1-\zeta)^{-\alpha-n} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}(1-\zeta)^{-\alpha-n}$$

So, let $f(\zeta) = (1-\zeta)^{-\alpha}$, $f^{(n)}(0) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$. Which, the power series about $\zeta = 0$ is:

$$f(\zeta) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \zeta^n$$

3

Question 3

Pf:

4

Question 4

Pf: