Math CS 122b HW8 Part 1

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1 (need slight modification)

Question 1 *Stein and Shakarchi Pg. 201-202 Exercise 8:*
The function ζ has infinitely many zeros in the critical strip. This can be seen as follows.

(a) Let $F(s) = \xi(1/2+s)$, where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Show that $F(s)$ is an even function of s, and as a result, there exists G so that $G(s^2) = F(s)$.

(b) Show that the function $(s-1)\zeta(s)$ is an entire function of growth order 1, that is

$$|(s-1)\zeta(s)| \le A_\epsilon e^{a_\epsilon |s|^{1+\epsilon}}$$

As a consequence $G(s)$ is of growth order $1/2$.

(c) Deduce from the above that ζ has infinitely many zeros in the critical strip.

[Hint: To prove (a) and (b) use the functional equation for $\zeta(s)$. For (c), use a result of Hadamard, which states that an entire function with fractional order has infinitely many zeros (Exercise 14 in Chapter 5)].

Pf:

(a) Recall that in **HW 7 Question 1** (**Freitag Chap. VII.5 Problem 5**), to deduce the functional equation of ζ, we've proven the functional equation $\xi(s') = \xi(1-s')$. As a result, for any $s \in \mathbb{C}$, if treating F as a meromorphic function, we get:

$$F(s) = \xi\left(\frac{1}{2}+s\right) = \xi\left(1-\left(\frac{1}{2}+s\right)\right) = \xi\left(\frac{1}{2}-s\right) = F(-s)$$

Hence, this proves that $F(s)$ is an even function.

(b) Recall that $\zeta(s)$ is analytic on $\mathbb{C}\setminus\{1\}$, with a simple pole at $s=1$ with residue 1, then $(s-1)\zeta(s)$ is in fact having a removable singularity at $s=1$, hence can be extended to an entire function.

1. $(s-1)\zeta(s)$ **Has growth order** 1 **for** $\mathrm{Re}(s) \ge \frac{1}{2}$**:**

In **Freitag Lemma VII.5.2**, the following functions are well defined:

$$\forall t \in \mathbb{R}, \quad \beta(t) = t - [t] - \frac{1}{2}, \quad [t] := \max n \in \mathbb{Z},\ n \le t$$

$$\forall s \in \mathbb{C},\ \mathrm{Re}(s) > 0, \quad F(s) := \int_1^\infty t^{-s-1}\beta(t)dt$$

Then as a result, the following equation is true for $\mathrm{Re}(s) > 1$, hence defines an analytic continuation for $\zeta(s)$ on $\mathrm{Re}(s) > 0$:

$$\forall s \in \mathbb{C},\ \mathrm{Re}(s) > 1, \quad \zeta(s) = \frac{1}{2} + \frac{1}{s-1} - sF(s)$$

So, if multiply with $(s-1)$, for $\mathrm{Re}(s) \geq \frac{1}{2}$, $(s-1)\zeta(s)$ is well-defined, and can be given as the following formula:

$$(s-1)\zeta(s) = \frac{(s-1)}{2} + 1 - (s-1)sF(s)$$

Which, let $s = x+iy$ for $x, y \in \mathbb{R}$, on $\mathrm{Re}(s) = x \geq \frac{1}{2}$ (which $\frac{1}{x} \leq 2$), $F(s)$ can be bounded as follow:

$$|F(s)| = \left|\int_1^\infty t^{-s-1}\beta(t)dt\right| \leq \int_1^\infty |t^{-(x+iy)-1}\beta(t)|dt \leq \int_1^\infty |t^{-x-1}\cdot t^{iy}|dt = \int_1^\infty t^{-x-1}dt$$

$$= \frac{-1}{x}t^{-x}\bigg|_1^\infty = \frac{1}{x} \leq 2$$

(Note: for any $t \in \mathbb{R}$, $|\beta(t)| \leq \frac{1}{2} < 1$, and since $x \geq \frac{1}{2}$, then the integral of t^{-x-1} has power < -1, which is absolutely convergent).

So, if considering the modulus of $(s-1)\zeta(s)$ on $\mathrm{Re}(s) \geq \frac{1}{2}$, we get the following:

$$|(s-1)\zeta(s)| = \left|\frac{(s-1)}{2} + 1 - (s-1)sF(s)\right| \leq \frac{|s-1|}{2} + 1 + |(s-1)s|\cdot|F(s)| \leq \frac{|s|+1}{2} + 1 + 2(|s|^2+|s|)$$

$$\leq 2|s|^2 + \frac{3}{2}|s| + \frac{3}{2}$$

Which, take $4e^{|s|} = 4 + 4|s| + 2|s|^2 + \sum_{n=3}^\infty \frac{4}{n!}|s|^n$, since for any $s \in \mathbb{C}$ each term is nonnegative, then we can deduce:

$$|(s-1)\zeta(s)| \leq 2|s|^2 + \frac{3}{2}|s| + \frac{3}{2} \leq 4 + 4|s| + 2|s|^2 \leq 4 + 4|s| + 2|s|^2 + \sum_{n=3}^\infty \frac{4}{n!}|s|^n = 4e^{|s|}$$

This shows that $(s-1)\zeta(s)$ has growth order 1 on the half plane $\mathrm{Re}(s) \geq \frac{1}{2}$.

2. $(s-1)\zeta(s)$ **Has growth order** 1 **for the whole plane:**

In the previous part the growth order is verified for $\mathrm{Re}(s) \geq \frac{1}{2}$. so the rest suffices to show it for the half plane $\mathrm{Re}(s') < \frac{1}{2}$. (And, we'll utilize the fact that for all $s \in \mathbb{C}$, $|e^s| \leq e^{|s|}$, which can be seen using Taylor Series).

Recall that in **HW 7**, we've proven the following functional equation of ζ:

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

Hence, for any s' with $\mathrm{Re}(s') < \frac{1}{2}$, let $s' = 1-s$ for some $s \in \mathbb{C}$, then $s = 1-s'$, so $\mathrm{Re}(s) = \mathrm{Re}(1-s') > \frac{1}{2}$. Then, the equation $(s'-1)\zeta(s')$ becomes:

$$(s'-1)\zeta(s') = ((1-s)-1)\zeta(1-s) = -s\cdot 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

And, since $\cos(\frac{\pi}{2}) = 0$, $\cos(\frac{\pi s}{2}$ has a zero at $s = 1$, then $\cos\left(\frac{\pi s}{2}\right) = (s-1)h(s)$ for some analytic function h. So, the above formula can be further written as:

$$((1-s)-1)\zeta(1-s) = -s\cdot 2(2\pi)^{-s}\Gamma(s)h(s)\cdot(s-1)\zeta(s)$$

Which, $|s| = |1 - s'| \leq |s'| + 1$, so the growth order in terms of $|s|$ can be replaced using $|s'|$ instead. From the above equality, we do need to talk about the growth order of different components:

- For $(2\pi)^{-s} = e^{-\log(2\pi)s} = e^{-\log(2\pi)(x+iy)} = e^{-\log(2\pi)x} \cdot e^{-\log(2\pi)iy}$, it satisfies the following:

  $$|(2\pi)^{-s}| = |e^{-\log(2\pi)s}| \leq e^{\log(2\pi)|s|}$$

  This proves that $(2\pi)^{-s}$ has growth order 1.

- For $\Gamma(s)$, since we're working with the half plane $\mathrm{Re}(s) > \frac{1}{2}$, then it's valid to apply **Stirling's Formula** (given in **Freitag Proposition IV.1.14**):
  Let $H(s) = \sum_{n=0}^{\infty}\left(\left(s+n+\frac{1}{2}\right)\log\left(1+\frac{1}{s+n}\right)-1\right)$, then $\Gamma(s)$ can be expressed as:

  $$\Gamma(s) = \sqrt{2\pi}s^{s-\frac{1}{2}}e^{-s}e^{H(z)} = \sqrt{2\pi}e^{(s-1/2)\log(s)-s+H(s)}$$

  and $s \to \infty$ implies $H(s) \to 0$ (within the given half plane $\mathrm{Re}(s) > \frac{1}{2}$).
  Which, notice that for s in the half plane, since $s \to \infty$ implies $H(s) \to 0$, then there exists $M > 0$, such that $|s| > M$ implies $|H(s)| < 1$. And, since for all $\epsilon > 0$ (specifically, can limit to $\epsilon < 1$), there exists $M' > 0$, such that $|\log(s)| \leq |s|^{\epsilon}$, then for all s in the half plant satisfies $|s| > M, M'$, we get:

  $$\left|\left(s-\frac{1}{2}\right)\log(s)-s+H(s)\right| \leq \left(|s|+\frac{1}{2}\right)|\log(s)|+|s|+|H(s)| \leq \left(|s|+\frac{1}{2}\right)|s|^{\epsilon}+|s|+1$$

  $$\leq |s|^{1+\epsilon}+\frac{1}{2}|s|^{\epsilon}+|s|^{1+\epsilon}+1 \leq \frac{5}{2}|s|^{1+\epsilon}+1$$

  Hence, $\Gamma(s)$ satisfies:

  $$|\Gamma(s)| = \left|\sqrt{2\pi}e^{(s-1/2)\log(s)-s+H(s)}\right| \leq \sqrt{2\pi}\exp\left(\left|\left(s-\frac{1}{2}\right)\log(s)-s+H(s)\right|\right)$$

  $$\leq \sqrt{2\pi}\exp\left(\frac{5}{2}|s|^{1+\epsilon}+1\right) = e\sqrt{2\pi}e^{\frac{5}{2}|s|^{1+\epsilon}}$$

  Hence, for any $\epsilon > 0$, with suitable constant $A_{\epsilon}, a_{\epsilon} > 0$, on the half plane $\mathrm{Re}(s) > \frac{1}{2}$, $|\Gamma(s)| \leq A_{\epsilon}e^{a_{\epsilon}|s|^{1+\epsilon}}$, showing that $\Gamma(s)$ has growth order 1.

- For $h(s)$ mentioned above, since $(s-1)h(s) = \cos\left(\frac{\pi s}{2}\right)$, and $\cos\left(\frac{\pi s}{2}\right)$ can be written as:

  $$\cos\left(\frac{\pi s}{2}\right) = \frac{e^{i\frac{\pi s}{2}}+e^{-i\frac{\pi s}{2}}}{2}$$

  Hence, the following inequality is true:

  $$\left|\cos\left(\frac{\pi s}{2}\right)\right| \leq \frac{1}{2}\left(|e^{i\frac{\pi s}{2}}|+|e^{-i\frac{\pi s}{2}}|\right) \leq \frac{1}{2}\left(e^{\frac{\pi}{2}|s|}+e^{\frac{\pi}{2}|s|}\right) = e^{\frac{\pi}{2}|s|}$$

  Hence, $\cos\left(\frac{\pi s}{2}\right)$ is with growth order 1, which also implies that $h(s)$ is with growth order 1.

Finally, back to the original equation, since for any s' with $\mathrm{Re}(s') < \frac{1}{2}$, writing $s' = 1-s$ for $\mathrm{Re}(s) > \frac{1}{2}$ yields the following expresion:

$$(s'-1)\zeta(s') = ((1-s)-1)\zeta(1-s) = -s \cdot 2(2\pi)^{-s}\Gamma(s)h(s) \cdot (s-1)\zeta(s)$$

Then, since s, $(2\pi)^{-s}$, $\Gamma(s)$, $h(s)$ are all with growth order 1, and $(s-1)\zeta(s)$ has been proven to have growth order 1 also in the previous part, then the whole product $(s'-1)\zeta(s')$ is with growth order 1 (with input s). However, since $|s| = |1 - s'| \leq |s'| + 1$ as mentioned before, then it is also with growth order 1 with respect to s'.

Regardless of the choise of s (either $\text{Re}(s) \geq \frac{1}{2}$ or $\text{Re}(s) < \frac{1}{2}$), we eventually get that $(s-1)\zeta(s)$ is with growth order 1.

(c) In **Part (b)**, it was proven that F has growth order 1, while G (after being modified into an entire function) has growth order 1/2. So based on Hadamard's result, it has infinitely many zeros, which also implies that $F(s) = G(s^2)$ has infinitely many zeros. However, since $F(s)$ is given as:

$$F(s) = \xi(1/2 + s) = \pi^{-(1/2+s)/2}\Gamma\left(\frac{(1/2+s)}{2}\right)\zeta\left(\frac{1}{2} + s\right)$$

Which, because $F(s)$ is even, it is enough to consider the half plane $\text{Re}(s) \geq 0$: Because π^z, $\Gamma(z)$ are both nonzero functions, then these zeros of F must be contributed by $\zeta(1/2 + s)$; On the other hand, it is well-known that $\zeta(z)$ has no zeros for $\text{Re}(z) \geq 1$, hence for $\text{Re}(1/2 + s) \geq 1$, or $\text{Re}(s) \geq \frac{1}{2}$, since $\zeta(1/2 + s)$ has no zeros, then $F(s)$ has no zeros. Hence, the zeros of $F(s)$ (on the half plane $\text{Re}(s) \geq 0$) must appear in the range $0 \leq \text{Re}(s) < \frac{1}{2}$, which eventually implies that there are infinitely many s in this strip (which satisfies $\frac{1}{2} \leq \text{Re}(1/2 + s) < 1$, with $(1/2 + s)$ being in the critical strip) satisfying $\zeta(1/2 + s) = 0$.

So, we can conclude that $\zeta(s)$ has infinitely many zeros in the critical strip.

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Question 2 *Stein and Shakarchi Pg. 202-203 Exercise 10:*
In the theory of primes, a better approximation fo $\pi(x)$ (instead of $x/\log(x)$) turns out to be $Li(x)$ defined by

$$Li(x) = \int_2^x \frac{dt}{\log(t)}$$

(a) Prove that

$$Li(x) = \frac{x}{\log(x)} + O\left(\frac{x}{(\log(x))^2}\right) \quad \text{as } x \to \infty$$

and that as a consequence

$$\pi(x) \sim Li(x) \quad \text{as } x \to \infty$$

(b) Refine the previous analysis by showing that for every integer $N > 0$ one has the following asymptotic expansion

$$Li(x) = \frac{x}{\log(x)} + \frac{x}{(\log(x))^2} + 2\frac{x}{(\log(x))^3} + \cdots (N-1)!\frac{x}{(\log(x))^N} + O\left(\frac{x}{(\log(x))^{N+1}}\right)$$

as $x \to \infty$.

Pf:

(a) First, using integration by parts, for all $x \geq 4$ (where $x \geq \sqrt{x} \geq 2$), $\mathrm{Li}(x)$ can be expressed as follow:

$$\begin{aligned}\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)} &= \frac{t}{\log(t)}\bigg|_2^x - \int_2^x t \cdot \frac{d}{dt}\left(\frac{1}{\log(t)}\right) dt \\ &= \frac{x}{\log(x)} - \frac{2}{\log(2)} - \int_2^x t \cdot \frac{-1}{(\log(t))^2} \cdot \frac{1}{t} dt \\ &= \frac{x}{\log(x)} - \frac{2}{\log(2)} + \int_2^x \frac{1}{(\log(t))^2} dt\end{aligned}$$

Which, for the last integral expression, it can be reformulate as follow:

$$\int_2^x \frac{dt}{(\log(t))^2} = \int_2^{\sqrt{x}} \frac{dt}{(\log(t))^2} + \int_{\sqrt{x}}^x \frac{dt}{(\log(t))^2}$$

Since $\log(t)$ is a strictly increasing function on $(1, \infty)$ and is strictly positive, then $\frac{1}{(\log(t))^2}$ is a strictly decreasing function on this interval instead. Hence, for all $t \in [2, \sqrt{x}]$, $\frac{1}{(\log(t))^2} \leq \frac{1}{(\log(2))^2}$, while any $t \in [\sqrt{x}, x]$ satisfies $\frac{1}{(\log(t))^2} \leq \frac{1}{(\log(\sqrt{x}))^2} = \frac{4}{(\log(x))^2}$. Hence, the above expression satisfies:

$$\begin{aligned}\int_2^x \frac{dt}{(\log(t))^2} = \int_2^{\sqrt{x}} \frac{dt}{(\log(t))^2} + \int_{\sqrt{x}}^x \frac{dt}{(\log(t))^2} &\leq \int_2^{\sqrt{x}} \frac{dt}{(\log(2))^2} + \int_{\sqrt{x}}^x \frac{4dt}{(\log(x))^2} \\ &= \frac{\sqrt{x}-2}{(\log(2))^2} + \frac{4(x-\sqrt{x})}{(\log(x))^2} \leq \frac{4x}{(\log(x))^2} + \frac{\sqrt{x}}{(\log(2))^2}\end{aligned}$$

Which, if evaluate the following limit, we get:

$$\lim_{x\to\infty} \frac{\sqrt{x}}{x/(\log(x))^2} = \lim_{x\to\infty} \frac{(\log(x))^2}{\sqrt{x}} = \lim_{x\to\infty} \frac{2\log(x)/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{4\log(x)}{\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{4/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{8}{\sqrt{x}} = 0$$

Hence, for some $x_1 > 4$ and $A_1 > 0$, we have $x > x_1$ implies $\sqrt{x} \le A_1 \frac{x}{(\log(x))^2}$. So, the integral follows the inequality below for $x > x_0$:

$$\int_2^x \frac{dt}{(\log(t))^2} \le \frac{\sqrt{x}}{(\log(2))^2} + \frac{4x}{(\log(x))^2} \le \frac{1}{(\log(2))^2} \cdot \frac{A_1 x}{(\log(x))^2} + \frac{4x}{(\log(x))^2}$$

$$\le \left(\frac{A_1}{(\log(2))^2} + 4 \right) \frac{x}{(\log(x))^2}$$

So, this shows that $\int_2^x \frac{dt}{(\log(t))^2} = O\left(\frac{x}{(\log(x))^2}\right)$. Hence:

$$\mathrm{Li}(x) = \frac{x}{\log(x)} - \frac{2}{\log(2)} + \int_2^x \frac{dt}{(\log(t))^2} \le \frac{x}{\log(x)} + \int_2^x \frac{dt}{(\log(t))^2} = \frac{x}{\log(x)} + O\left(\frac{x}{(\log(x))^2}\right)$$

This shows that $\mathrm{Li}(x) = \frac{x}{\log(x)} + O\left(\frac{x}{(\log(x))^2}\right)$.

(b) First, we'll consider the following formula about the integral of $\frac{1}{(\log(t))^n}$ using integration by parts:

$$\forall n \in \mathbb{N}, \quad \int_2^x \frac{dt}{(\log(t))^n} = \frac{t}{(\log(t))^n}\Bigg|_2^x - \int_2^x t \cdot \frac{d}{dt}\left(\frac{1}{(\log(t))^n}\right) dt$$

$$= \frac{x}{(\log(x))^n} - \frac{2}{(\log(2))^n} - \int_2^x t \cdot \frac{-n}{(\log(t))^{n+1}} \cdot \frac{1}{t} dt = \frac{x}{(\log(x))^n} - \frac{2}{(\log(2))^n} + n \int_2^x \frac{dt}{(\log(t))^{n+1}}$$

Which, using the same argument used in **part (a)** about $\frac{1}{(\log(t))^n}$ is a decreasing function for all $n \in \mathbb{N}$, for all $x \ge 4$ (where $x > \sqrt{x} \ge 2$), we get:

$$\int_2^x \frac{dt}{(\log(t))^{n+1}} = \int_2^{\sqrt{x}} \frac{dt}{(\log(t))^{n+1}} + \int_{\sqrt{x}}^x \frac{dt}{(\log(t))^{n+1}} \le \int_2^{\sqrt{x}} \frac{dt}{(\log(2))^{n+1}} + \int_{\sqrt{x}}^x \frac{dt}{(\log(\sqrt{x}))^{n+1}}$$

$$= \frac{(\sqrt{x} - 2)}{(\log(2))^{n+1}} + \frac{2^{n+1}(x - \sqrt{x})}{(\log(x))^{n+1}} \le \frac{2^{n+1} x}{(\log(x))^{n+1}} + \frac{\sqrt{x}}{(\log(2))^{n+1}}$$

Now, since the base case $\lim_{x\to\infty} \frac{\sqrt{x}}{x/(\log(x))^2} = 0$ is proven in **part (a)**, using induction, we can get the following relationship:

$$\forall n \in \mathbb{N}, \quad \lim_{x\to\infty} \frac{\sqrt{x}}{x/(\log(x))^{n+1}} = \lim_{x\to\infty} \frac{(\log(x))^{n+1}}{\sqrt{x}} = \lim_{x\to\infty} \frac{(n+1)(\log(x))^n/x}{1/(2\sqrt{x})}$$

$$= \lim_{x\to\infty} 2(n+1) \frac{\sqrt{x}}{x/(\log(x))^n} = 0$$

Hence, there exists $x_n > 4$ and $A_n > 0$, such that $x > x_n$ implies $\sqrt{x} \le A_n \frac{x}{(\log(x))^{n+1}}$. Hence, we get:

$$\int_2^x \frac{dt}{(\log(t))^{n+1}} \le \frac{2^{n+1} x}{(\log(x))^{n+1}} + \frac{\sqrt{x}}{(\log(2))^{n+1}}$$

$$\le \frac{2^{n+1} x}{(\log(x))^{n+1}} + \frac{A_n}{(\log(2))^{n+1}} \frac{x}{(\log(x))^{n+1}} = \left(2^{n+1} + \frac{A_n}{(\log(2))^{n+1}}\right) \frac{x}{(\log(x))^{n+1}}$$

This shows that $\int_2^x \frac{dt}{(\log(t))^{n+1}} = O\left(\frac{x}{(\log(x))^{n+1}}\right)$.

Finally, using the case proven in **part (a)**, we know $\text{Li}(x) = \frac{x}{\log(x)} - \frac{2}{\log(2)} + \int_2^x \frac{dt}{(\log(t))^2}$. Which utilizing the above equation, by induction, one can show that for any integer $n \geq 2$, the following formula holds:

$$\text{Li}(x) = \sum_{k=1}^{n}(k-1)!\frac{x}{(\log(x))^k} - \sum_{k=1}^{n}(k-1)!\frac{2}{(\log(2))^k} + n!\int_2^x \frac{dt}{(\log(t))^{n+1}} \leq \sum_{k=1}^{n}(k-1)!\frac{x}{(\log(x))^k} + n!\int_2^x \frac{dt}{(\log(t))^{n+1}}$$

Then, with the statement that $\int_2^x \frac{dt}{(\log(t))^{n+1}} = O\left(\frac{x}{(\log(x))^{n+1}}\right)$ deduced previously, for any $n \in \mathbb{N}$, we get the following:

$$\text{Li}(x) = \sum_{k=1}^{n}(k-1)!\frac{x}{(\log(x))^k} + O\left(\frac{x}{(\log(x))^{n+1}}\right)$$

$$= \frac{x}{\log(x)} + \frac{x}{(\log(x))^2} + 2!\frac{x}{(\log(x))^3} + \cdots + (n-1)!\frac{x}{(\log(x))^n} + O\left(\frac{x}{(\log(x))^{n+1}}\right)$$

3

Question 3 *Stein and Shakarchi Pg. 204 Problem 2:*
One of the "explicit formulas" in the theory of primes is as follows: if ψ_1 is the integrated Tchebychev function considered in Section 2, then

$$\psi_1(x) = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - E(x)$$

where the sum is taken over all zeros ρ of the ζ-function in the critical strip. The error term is given by $E(x) = c_1 x + c_0 + \sum_{k=1}^\infty x^{1-2k}/(2k(2k-1))$, where $c_1 = \zeta'(0)/\zeta(0)$ and $c_0 = \zeta'(-1)/\zeta(-1)$. Note that $\sum_\rho 1/|\rho|^{1+\epsilon} < \infty$ for every $\epsilon > 0$, because $(1-s)\zeta(s)$ has order of growth 1. Also, obviously $E(x) = O(x)$ as $x \to \infty$.

Pf:

First, recall that the following formula of $\psi_1(x)$ holds for any $x > 1$ and $c > 1$:

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$

Which, to get a closed expression, we'll utilize Hadamard's product formula for ζ and Residue Theorem.

1. Product Formula for ζ and $-\frac{\zeta'}{\zeta}$:

Based on **Question 1 part (b)** in this assignment, we've proven that $(s-1)\zeta(s)$ is an entire function with growth order 1, and it is zero precisely at all the zeros of $\zeta(s)$ since at $s = 1$, $\zeta(s)$ has residue 1. Which, $(s-1)\zeta(s)$ has zeros at $-2k$ for $k \in \mathbb{N}$, and all zeros of ζ, denoted as ρ in the critical strip.

Then, based on **Hadamard's Factorization Theorem** (can be seen in **Stein and Shakarchi Chapter 5.5**), since $(s-1)\zeta(s)$ has growth order 1 with the zeros mentioned above (which the zeros are all nonzero), then there exists polynomial $P(s) = cs + d$ with degree 1 (at most the growth order), such that the following holds:

$$\begin{aligned}(s-1)\zeta(s) &= e^{cs+d}\left(\prod_{k=1}^\infty E_1\left(\frac{s}{2k}\right)\right)\left(\prod_\rho E_1\left(\frac{s}{\rho}\right)\right) \\ &= e^{cs+d}\left(\prod_{k=1}^\infty \left(1-\frac{s}{2k}\right)e^{s/(2k)}\right)\left(\prod_\rho \left(1-\frac{s}{\rho}\right)e^{s/\rho}\right)\end{aligned}$$

Where the second product contains all nontrivial zeros of ζ in the critical strip. Hence, the following is a formula for $\zeta(s)$ in terms of products of zeros and poles:

$$\zeta(s) = (s-1)^{-1}e^{cs+d}\left(\prod_{k=1}^\infty \left(1-\frac{s}{2k}\right)e^{s/(2k)}\right)\left(\prod_\rho \left(1-\frac{s}{\rho}\right)e^{s/\rho}\right)$$

Then, utilizing logarithmic derivative, we get the following:

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + c + \sum_{k=1}^\infty\left(\frac{1}{s-2k} + \frac{1}{2k}\right) + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

And, this formula is normally convergent within any compact subset of the domain (not containing the zeros and the poles of ζ), so integration can be exchanged with summation.

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Question 4 *Stein and Shakarchi Pg. 204 Problem 3:*
Using the previous problem one can show that

$$\pi(x) - Li(x) = O(x^{\alpha+\epsilon}) \quad \textit{as } x \to \infty$$

for every $\epsilon > 0$*, where* α *is fixed and* $1/2 \leq \alpha < 1$ *if and only if* $\zeta(s)$ *has no zeros in the strip* $\alpha < Re(s) < 1$*. The case* $\alpha = 1/2$ *corresponds to the Riemann Hypothesis.*

Pf: