Lie Algebra of a Lie Group

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Tangent Vectors as Derivations

When embedding smooth manifolds into $\mathbb{R}^n$, tangent vectors are associated with directional derivatives. To generalize tangent vectors into abstract smooth manifold, we need an analogy:

Definition

Any point $u \in M$, a **Derivation at** u, is a linear map $v_u : C^\infty(M) \to \mathbb{R}$, that satisfies the product rule:

$$\forall f, g \in C^\infty(M), \quad v_u(fg) = f(u)(v_u g) + g(u)(v_u f)$$

The vector space of all derivations at u, or $T_u(M)$, is the **Tangent Space** of M at u, and each derivation $v_u \in T_u(M)$ is a **Tangent Vector** of u.

Vector Fields & Smooth Condition

Definition

a vector field is a map $X : M \to TM$ (TM denotes the **Tangent Bundle**), with $X(u) = X_u \in T_u(M)$.
Which, X is a **Smooth Vector Field**, if $X : M \to TM$ is a smooth map.
A collection of smooth vector fields on M is $\mathfrak{X}(M)$, which is an $\mathbb{R}$-vector space.

An equivalent condition of saying a vector field X is smooth, is through smooth functions $f \in C^\infty(M)$: For all $u \in M$, $X(u) = X_u \in T_u(M)$ is a derivation at u, define $Xf : M \to \mathbb{R}$ by $Xf(u) = X_u(f)$, then X is a smooth vector field iff $Xf \in C^\infty(M)$. Which, the **Derivation** is an equivalent condition for smooth vector field:

Theorem

For all $f, g \in C^\infty(M)$, given vector field X, $X \in \mathfrak{X}(M)$ iff it satisfies product rule for all $u \in M$:

$$X(fg)(u) = X_u(fg) = f(u)(X_u g) + g(u)(X_u f) = f(u)Xg(u) + g(u)Xf(u)$$
$$\implies X(fg) = f(Xg) + g(Xf)$$

Vector Fields of Different Manifolds

Given M, N two smooth manifolds, and smooth map $F : M \to N$. Let $X \in \mathfrak{X}(M)$, an ideal situation is mapping X to a smooth vector field of N through F. Yet, this requires F to be bijective:

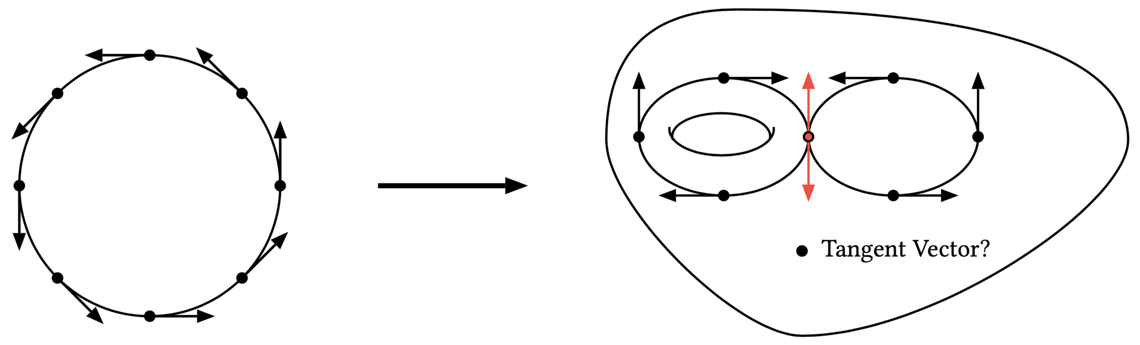

Figure 1: Example of Conflicting Tangent Vectors

So, we'll consider a weaker notion:

Definition

Given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, the two are ***F*-related**, if for all $u \in M$, the following is true:

$$dF_u(X_u) = Y_{F(u)}$$

Simply speaking, F maps the tangent vectors collected by X, to be compatible with tangent vectors collected by Y.

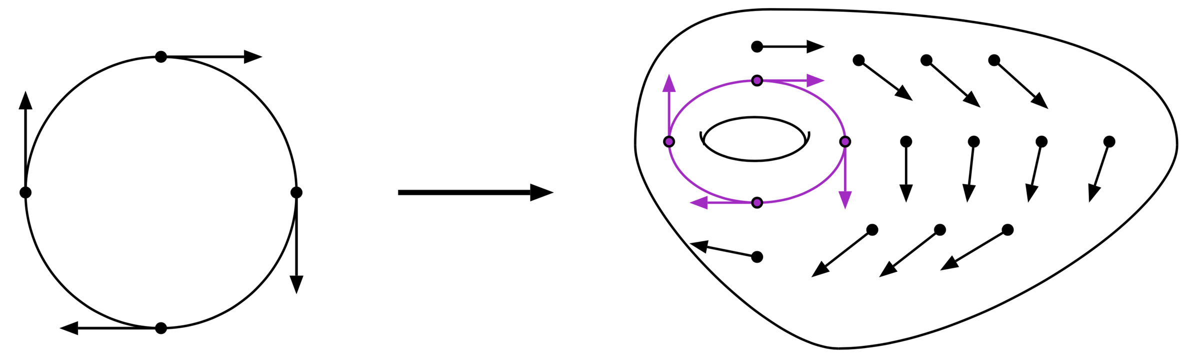

Figure 2: A demonstration of F-Relation

Lie Bracket of Vector Fields

The initial motivation is to combine two vector fields $X, Y \in \mathfrak{X}(M)$ to be another vector field. For all $f \in C^\infty(M)$, since $Yf \in C^\infty(M)$, then $XYf := X(Yf) \in C^\infty(M)$. But, in general XY is not a derivation, hence not a vector field:

Example

Define vector fields $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Take smooth functions $f(x, y) = x$ and $g(x, y) = y$, then we get the following:

$$XY(fg) = X\left(x\frac{\partial}{\partial y}(xy)\right) = \frac{\partial}{\partial x}(x^2) = 2x$$

But, product rule doesn't hold for this example:

$$f(XYg) + g(XYf) = x\left(X\left(x\frac{\partial}{\partial y}(y)\right)\right) + y\left(X\left(x\frac{\partial}{\partial y}(x)\right)\right) = x$$

So, we need to define a new operation on vector fields:

Definition

The **Lie Bracket** $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, is defined as:

$$\forall X, Y \in \mathfrak{X}(M), \quad [X, Y] = XY - YX$$

Which, the output $[X, Y] \in \mathfrak{X}(M)$, since it satisfies product rule:

$$[X, Y](fg) = X(Y(fg)) - Y(X(fg)) = X(f(Yg) + g(Yf)) - Y(f(Xg) + g(Xf))$$
$$= f(XYg) + (Yg)(Xf) + g(XYf) + (Yf)(Xg) - f(YXg) - (Xg)(Yf) - g(YXf) - (Xf)(Yg)$$
$$= f(XYg - YXg) + g(XYf - YXf) = f[X, Y](g) + g[X, Y](f)$$

Lie Bracket also satisfies these properties:

- **Bilinearity:** $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- **Antisymmetry:** $[X, Y] = -[Y, X]$
- **Jacobi's Identity:** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Moreover, Lie Bracket inherits relation of smooth maps:

Theorem

Given smooth map $F : M \to N$, if $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ are F-related respectively, then $[X_1, X_2] \in \mathfrak{X}(M)$ and $[Y_1, Y_2] \in \mathfrak{X}(N)$ are also F-related.

Lie Groups & Left-Invariant Vector Fields

The initial motivation is to study group structures in some smooth manifolds.

Definition

A **Lie Group** G, is a smooth manifold along with group structure, such that the group operation $P : G \times G \to G$ by $P(g, h) = gh$, and the inversion map $i : G \to G$ by $i(g) = g^{-1}$ are both smooth maps between manifolds.

For all $g \in G$, denote the left multiplication $L_g : G \to G$ by $L_g(h) = gh$, since $L_g = P\big|_{\{g\}\times G}$, it is a smooth map. Hence, there's a notion of X being L_g-related to itself:

Definition

Given any $X \in \mathfrak{X}(G)$ and all $g \in G$, X is a **Left-Invariant Vector Field**, if for all $g \in G$, X is L_g-related to itself. Which, for all $g \in G$:

$$d(L_g)_e(X_e) = X_{L_g(e)} = X_g$$

So, X is uniquely determined by its tangent vector at identity, $X_e \in T_e(G)$. In fact, each $v_e \in T_e(G)$ also corresponds to a unique Left-Invariant vector field.
The collection of Left-Invariant vector fields $\mathfrak{g} \subseteq \mathfrak{X}(G)$, is itself a linear subspace, and $\mathfrak{g} \cong T_e(G)$ as vector spaces, based on the above relation.

Recall that Lie Bracket of vector field preserves F-relation between manifolds, so:

Theorem

For all $X, Y \in \mathfrak{X}(G)$ that are left-invariant, since for all $g \in G$, X and Y are L_g related to themselves, then the Lie Bracket $[X, Y]$ is also L_g-related to $[X, Y]$. Hence, $[X, Y]$ is also left-invariant, so $\mathfrak{g}$ is closed under Lie Bracket's operation.

Lie Algebra on a Lie Group

Definition

Given a vector space $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$, with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that the following holds:

- **Bilinearity:** $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- **Antisymmetry:** $[X, Y] = -[Y, X]$
- **Jacobi's Identity:** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Then, the pair $(\mathfrak{g}, [\cdot, \cdot])$ is a **Lie Algebra**.

In general, Lie Algebra is non-associative, so Jacobi's Identity is an alternative condition. Finally, we can define **Lie Algebra of a Lie Group:**

Definition

Given a lie group G, since the subset of left-invariant vector fields $\mathfrak{g} \subseteq \mathfrak{X}(G)$ forms a linear subspace, while closed under Lie Bracket's operation, then the pair $(\mathfrak{g}, [\cdot, \cdot])$ forms a **Lie Algebra** of G, denoted as $Lie(G)$.

Here's an example of Lie Algebra on a Lie Group:

Example

General Linear Group & its Lie Algebra:
Given $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$, since $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and $GL_n(\mathbb{R})$ is an open subset, it's a natural smooth manifold with dimension n^2. The product of matrices and inversion are smooth maps, so $GL_n(\mathbb{R})$ is a Lie Group.
Now, consider $\mathfrak{g} = Lie(GL_n(\mathbb{R}))$: Each $X \in \mathfrak{g}$ is uniquely characterized by $X_{I_n} \in T_{I_n}(GL_n(\mathbb{R}))$. And, as vector spaces, $\mathfrak{g} \cong T_{I_n}(GL_n(\mathbb{R}))$.

Lie Algebra on $M_n(\mathbb{R})$:
Given $M_n(\mathbb{R})$ as $\mathbb{R}$-vector space and the commutator $[A, B] = AB - BA$, the pair $(M_n(\mathbb{R}), [\cdot, \cdot])$ in fact forms a Lie Algebra, denoted as $\mathfrak{gl}_n(\mathbb{R})$.

Lie Algebra Isomorphism between $\mathfrak{g}$ and $\mathfrak{gl}_n(\mathbb{R})$:
$GL_n(\mathbb{R})$ has a global coordinate provided by $M_n(\mathbb{R})$, denote as $(X^i_j)_{1\leq i,j\leq n}$.
For each $A \in \mathfrak{gl}_n(\mathbb{R})$, it corresponds to a tangent vector in $T_{I_n}(GL_n(\mathbb{R}))$:

$$A = (A^i_j) \mapsto A^i_j \frac{\partial}{\partial X^i_j}\bigg|_{I_n}$$

The above tangent vector defines a Left-Invariant vector field $A^L \in \mathfrak{g}$. For all $X \in \mathfrak{g}$, the left multiplication L_X is in fact a linear operator on $M_n(\mathbb{R})$, so its differential is identical to itself. Which, it provides the following relation:

$$A^L_X = d(L_X)_{I_n}\left(A^i_j \frac{\partial}{\partial X^i_j}\bigg|_{I_n}\right) = X^i_j A^j_k \frac{\partial}{\partial X^i_k}\bigg|_X, \quad A^L = X^i_j A^j_k \frac{\partial}{\partial X^i_k}$$

Which, for arbitrary $A, B \in \mathfrak{gl}_n(\mathbb{R})$, Lie Bracket of $A^L,\ B^L \in \mathfrak{g}$ generates:

$$\left[A^L, B^L\right] = X^i_j A^j_k \frac{\partial}{\partial X^i_k}(X^p_q B^q_r)\frac{\partial}{\partial X^p_r} - X^p_q B^q_r \frac{\partial}{\partial X^p_r}(X^i_j A^j_k)\frac{\partial}{\partial X^i_k}$$

Which, each $A^j_k,\ B^q_r$ are constants, while $\frac{\partial}{\partial X^i_k} X^p_r = 1$ iff $(i, k) = (p, r)$ and is 0 otherwise. Then, match $j = q$ for the same intermediate index, we get:

$$\left[A^L, B^L\right] = X^i_j (A^j_k B^k_r - B^j_k A^k_r)\frac{\partial}{\partial X^i_r} = (AB - BA)^L = [A, B]^L$$

Hence, the map $\mathfrak{gl}_n(\mathbb{R}) \to \mathfrak{g}$ by $A \mapsto A^L$ is a Lie Algebra Isomorphism.

Acknowledgements & Reference

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Reference: Lee, J.M. *Introduction to Smooth Manifolds*; 2nd ed.; Springer: New York, 2012; 9781441999825