# Math CS 122B HW7

Zih-Yu Hsieh

May 17, 2025

## 1

**Question 1** *The functional equation of the $\zeta$-function can also be written in the following form:*

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

*Deduce from this: In the half-plane $\sigma \leq 0$, the function $\zeta(s)$ has exactly the zeros $s = -2k,\ k \in \mathbb{N}$. All other zeros of the $\zeta$-function are located in the vertical strip $0 < Res < 1$.*

**Pf:**

First, recall that for the half plane $\sigma > 1$, the following inequality is given:

$$\left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4 |\zeta(\sigma+2it)|[\zeta(\sigma)(\sigma-1)]^3 \geq (\sigma-1)^{-1}$$

Since for $\sigma > 1$, the expressiong $(\sigma-1)^{-1} > 0$, this enforces all $s = \sigma + it$ in the half plane to have $\zeta(s) \neq 0$ (or else the left side of the inequality is 0, which violates the inequality). Similarly, this inequality can be extended onto the line $\text{Re}(s) = 1$, where $\zeta(s)$ has no zeros on this line also. So, for $\sigma \geq 1$, $\zeta(s)$ has no zero.

Now, in the half plane $\sigma \leq 0$, for all $s' \neq 0$, since it can be written as $s' = 1 - s$, where $s = 1 - s'$ has $\text{Re}(s) = 1 - \text{Re}(s') \geq 1$ (and since $s' \neq 0$, then $s \neq 1$). So, $\zeta(s)$ after the continuation past $\text{Re}(s) = 1$, has $\zeta(s)$ being well-defined.

Then, by the functional equation, we get the following:

$$\zeta(s') = \zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

Since $\text{Re}(s) \geq 1$ with $s \neq 1$, then $\zeta(s) \neq 0$ based on what is mentioned during the start; also, $\Gamma(s) \neq 0$ for all $s \in \mathbb{C} \setminus \{0, -1, -2, ...\}$, while $2(2\pi)^{-s} \neq 0$ for all $s \in \mathbb{C}$. Hence, in case for $\zeta(1-s) = 0$, we must have $\cos(\frac{\pi s}{2}) = 0$, which enforces $\frac{\pi s}{2} = k\pi + \frac{\pi}{2}$ for some $k \in \mathbb{Z}$, or $s = 2k+1$ fo some $k \in \mathbb{Z}$. Now, under this assumption, since $\text{Re}(s) \geq 1$ while $s \neq 1$, then $k \geq 1$. So, when transfering back to $s' = 1 - s$, we get $s' = 1 - (2k+1) = -2k$ for integer $k \geq 1$.

Hence, for $\text{Re}(s') \leq 0$, for $\zeta(s') = 0$, then $s' = -2k$ for some $k \in \mathbb{N}$ (this is an iff since at all these points, $\cos(\frac{\pi s}{2}) = 0$, which $\zeta(s') = \zeta(1-s) = 0$).

Finally, for $s' = 0$ (where if $s' = 1 - s$, $s = 1$). Recall that $\zeta(s)$ has a simple pole at $s = 1$, while $\cos(\frac{\pi s}{2})$ has a simple zero at $s = 1$ (where the input is $\frac{\pi}{2}$, where cos is 0). Hence, $\cos(\frac{\pi s}{2}) = (s-1)h(z)$ for some analytic function $h$ where $h(1) \neq 0$. Also, we know $\lim_{s\to 1}(s-1)\zeta(s) = 1$ (has been given in the textbook). Then, we get the following:

$$\lim_{s\to 1}\zeta(1-s) = \lim_{s\to 1} 2(2\pi)^{-s}\Gamma(s)h(s)(s-1)\zeta(s) = 2(2\pi)^{-1}\Gamma(1)h(1)\cdot\lim_{s\to 1}(s-1)\zeta(s) = 2(2\pi)^{-1}\Gamma(1)h(1) \neq 0$$

Hence, we can deduce that at $s = 1$ (where $s' = 1 - s = 0$), $\zeta(s')$ has a removable singularity that has limit not being 0, henc $\zeta(s')$ as an extension has $\zeta(0) \neq 0$.

The above casees proves that when $\sigma \geq 1$ or $\sigma \leq 0$, $\zeta(s) = 0$ iff $s = -2k$ for some $k \in \mathbb{N}$, where for any other input $\zeta$ is nonzero.

Hence, if there are any other zeros, it must exist in the vertical strip $0 < \mathrm{Re}(s) < 1$.

## 2 (continuation & functional not done)

**Question 2** *The following special case of the Hecke Theorem was already known to B. Riemann (1859):*

$$\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t}t^{s/2}\frac{dt}{t}$$
$$= \frac{1}{2}\int_1^{\infty}(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

*Deduce directly this special case, and use it to prove the meromorphic continuation and the functional equation.*

**Pf:**

For this problem, first assume $\mathrm{Re}(s) > 1$ (where $\zeta(s)$ is defined with the original series form). We'll break down into two different equations:

**Equation 1:**

We'll first prove the following:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t}t^{s/2}\frac{dt}{t}$$

For each $n \in \mathbb{N}$, the integral within the right hand side summation, after doing the substitution $u = \pi n^2 t$ (where $du = \pi n^2 dt$), we get the following:

$$\int_0^{\infty} e^{-\pi n^2 t}t^{s/2}\frac{dt}{t} = \int_0^{\infty} e^{-u}\left(\frac{u}{\pi n^2}\right)^{s/2}\frac{du/(\pi n^2)}{u/(\pi n^2)} = (\pi n^2)^{-s/2}\int_0^{\infty} e^{-u}u^{s/2}\frac{du}{u}$$
$$= \pi^{-s/2}n^{-s}\int_0^{\infty} e^{-u}u^{s/2-1}du = \pi^{-s/2}n^{-s}\Gamma\left(\frac{s}{2}\right)$$

Hence, for the series, since $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ converges normally within $\mathrm{Re}(s) > 1$, we get the following:

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t}t^{s/2}\frac{dt}{t} = \sum_{n=1}^{\infty}\pi^{-s/2}n^{-s}\Gamma\left(\frac{s}{2}\right) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}n^{-s} = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

So, the first equatity holds.

**Equation 2:**

Our second goal is to prove the following:

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t}t^{s/2}\frac{dt}{t} = \frac{1}{2}\int_1^{\infty}(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

Since the summation is absolutely convergent for $\mathrm{Re}(s) > 1$ (based on how $\xi(s)$ is defined), then swapping the order of summation causes no issue. Hence, since for all $n \in \mathbb{N}$, since $n^2 = (-n)^2$, the infinite summation can also be decomposed as:

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t}t^{s/2}\frac{dt}{t} = \frac{1}{2}\sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t}t^{s/2}\frac{dt}{t} + \frac{1}{2}\sum_{n=-\infty}^{-1}\int_0^{\infty} e^{-\pi n^2 t}t^{s/2}\frac{dt}{t}$$

$$= \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_1^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} + \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t}$$

For the first summation, recall that the Theta Series is defined as $\vartheta : \mathbb{H} \to \mathbb{C},\ \vartheta(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau}$. Since for $n = 0$, $e^{i\pi n^2 \tau} = e^0 = 1$, then we get the following:

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{i\pi n^2 \tau} = \vartheta(\tau) - 1$$

If consider $\tau = it$ for $t \in (0, \infty)$, we get the following:

$$\vartheta(it) - 1 = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{i\pi n^2 \cdot it} = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{-\pi n^2 t}$$

So, for the first summation in the equation, since $\vartheta$ is a normally convergent series of function, the summation and integral can change the order of operation. Hence:

$$\frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_1^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \int_1^\infty \left( \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{-\pi n^2 t} \right) t^{s/2} \frac{dt}{t} = \frac{1}{2} \int_1^\infty (\vartheta(it) - 1) t^{s/2} \frac{dt}{t}$$

Now, to work on the second summation, for all $n \in \mathbb{Z}$ with $n \neq 0$, with the substitution $u = \frac{1}{t}$, $du = -\frac{1}{t^2} dt$ (or $\frac{dt}{t} = -t du = -\frac{du}{u}$), we get the following integral representation:

$$\int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = -\int_\infty^1 e^{-\pi n^2 \cdot 1/u} u^{-s/2} \frac{du}{u} = \int_1^\infty e^{-\pi n^2 \cdot 1/u} u^{-s/2} \frac{du}{u}$$

Which, based on the above change of variable and the property of Theta Series, the second summation in the equation can be rewrite as:

$$\frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_1^\infty e^{-\pi n^2 \cdot 1/u} u^{-s/2} \frac{du}{u} = \frac{1}{2} \int_1^\infty \left( \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{-\pi n^2 \cdot 1/u} \right) u^{-s/2} \frac{du}{u}$$

$$= \frac{1}{2} \int_1^\infty \left( \vartheta \left( \frac{i}{u} \right) - 1 \right) u^{-s/2} \frac{du}{u}$$

Which, there is a property of Theta Series given below:

$$\vartheta \left( -\frac{1}{z} \right) = \sqrt{\frac{z}{i}} \vartheta(z)$$

Hence, for $u \in (0, \infty)$, let $z = iu$, we get the following:

$$\vartheta \left( \frac{i}{u} \right) = \vartheta \left( -\frac{1}{iu} \right) = \sqrt{\frac{iu}{i}} \vartheta(iu) = u^{1/2} \vartheta(iu)$$

So, the summation can be modified as follow:

$$\frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \int_1^\infty \left( u^{1/2} \vartheta(iu) - 1 \right) u^{-s/2} \frac{du}{u}$$

$$= \frac{1}{2} \int_1^\infty (\vartheta(iu) - 1) u^{1/2} u^{-s/2} \frac{du}{u} + \frac{1}{2} \int_1^\infty u^{1/2} u^{-s/2} \frac{du}{u} - \frac{1}{2} \int_1^\infty u^{-s/2} \frac{du}{u}$$

For the integrals at the middle and the right, since the power of $u$ is given by $\frac{1-s}{2}-1$ and $\frac{-s}{2}-1$, then because $\mathrm{Re}(s)>1$, the two powers of $u$ both have the real parts being less than $-1$. Hence, the integral absolutely converges, and using power rule, we yield:

$$\frac{1}{2}\int_1^\infty u^{1/2}u^{-s/2}\frac{du}{u}=\frac{1}{2}\int_1^\infty u^{\frac{1-s}{2}-1}du=\frac{1}{2}\cdot\frac{2}{1-s}u^{\frac{1-s}{2}}\bigg|_1^\infty=-\frac{1}{1-s}$$

$$\frac{1}{2}\int_1^\infty u^{-s/2}\frac{du}{u}=\frac{1}{2}\int_1^\infty u^{-s/2-1}du=-\frac{1}{2}\cdot\frac{2}{s}u^{-s/2}\bigg|_1^\infty=\frac{1}{s}u^{-s/2}\bigg|_1^\infty=\frac{1}{s}$$

(Note: since $\mathrm{Re}(s)>1$, then $\mathrm{Re}(\frac{1-s}{2})<0$ and $\mathrm{Re}(-\frac{s}{2})<0$, so $\lim_{u\to\infty}u^{\frac{1-s}{2}}=0$ and $\lim_{u\to\infty}u^{-s/2}=0$). So, combining the pieces for the second summation, we get:

$$\frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\n\neq 0}}\int_0^1 e^{-\pi n^2 t}t^{s/2}\frac{dt}{t}=\frac{1}{2}\int_1^\infty(\vartheta(iu)-1)u^{1/2}u^{-s/2}\frac{du}{u}+\frac{1}{2}\int_1^\infty u^{1/2}u^{-s/2}\frac{du}{u}-\frac{1}{2}\int_1^\infty u^{-s/2}\frac{du}{u}$$

$$=\frac{1}{2}\int_1^\infty(\vartheta(it)-1)t^{(1-s)/2}\frac{dt}{t}+\left(-\frac{1}{1-s}\right)-\frac{1}{s}=\frac{1}{2}\int_1^\infty(\vartheta(it)-1)t^{(1-s)/2}\frac{dt}{t}-\frac{1}{s}-\frac{1}{1-s}$$

Finally, the original equation can be obtained as follow:

$$\sum_{n=1}^\infty\int_0^\infty e^{-\pi n^2 t}t^{s/2}\frac{dt}{t}=\frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\n\neq 0}}\int_1^\infty e^{-\pi n^2 t}t^{s/2}\frac{dt}{t}+\frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\n\neq 0}}\int_0^1 e^{-\pi n^2 t}t^{s/2}\frac{dt}{t}$$

$$=\left(\frac{1}{2}\int_1^\infty(\vartheta(it)-1)t^{s/2}\frac{dt}{t}\right)+\left(\frac{1}{2}\int_1^\infty(\vartheta(it)-1)t^{(1-s)/2}\frac{dt}{t}-\frac{1}{s}-\frac{1}{1-s}\right)$$

$$=\frac{1}{2}\int_1^\infty(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t}-\frac{1}{s}-\frac{1}{1-s}$$

This verifies the second equation. Together with the first and the second equation proven, the function $\xi(s)$ defined in the question for $\mathrm{Re}(s)>1$ satisfies the following equation:

$$\xi(s):=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\sum_{n=1}^\infty\int_0^\infty e^{-\pi n^2 t}t^{s/2}\frac{dt}{t}$$

$$=\frac{1}{2}\int_1^\infty(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t}-\frac{1}{s}-\frac{1}{1-s}$$

**Meromorphic Continuation:**