Math CS 122B HW4

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Question 1 *Freitag Chap. V.1 Exercise 10:*
Let f be an entire function, and let L be a lattice in $\mathbb{C}$. For any lattice point $w \in \mathbb{L}$ let there exists a number $C_w \in \mathbb{C}$ with the property

$$f(z+w) = C_w f(z)$$

Then

$$f(z) = Ce^{az}$$

for suitable constantx C and a.

Pf:

We'll consider the meromorphic function $f'/f : \mathbb{C} \to \overline{\mathbb{C}}$: For all $z \in \mathbb{C}$ and $w \in L$, since $f(z+w) = C_w f(z)$, then $f'(z+w) = C_w f'(z)$. Then, f'/f satisfies:

$$\frac{f'(z+w)}{f(z+w)} = \frac{C_w f'(z)}{C_w f(z)} = \frac{f'(z)}{f(z)}$$

This shows that f'/f is in fact an elliptic function with respect to the given lattice L.

Now, we'll consider the singularities of f'/f: Since f is entire, then f' is also entire, hence the only singularities possible for f'/f, are the zeros of f.

Since the singularities of f'/f must be discrete, then we can choose a fundamental region P of lattice L, such that its boundary ∂P contains no singularities of f'/f. Then, by argument principle, we get:

$$\frac{1}{2\pi i}\int_{\partial P} \frac{f'(z)}{f(z)} dz = (\text{Number of zeros of } f \text{ in } P) - (\text{Number of poles of } f \text{ in } P)$$

Also, since f'/f is an elliptic function, then we also get the following:

$$\int_{\partial P} \frac{f'(z)}{f(z)} dz = 0$$

So, this implies that the number of zeros of f in P, is precisely the same as the number of poles of f in P. Because f is entire, there are no poles in $\mathbb{C}$, hence number of poles in P is 0; this implies that the number of zeros of f in P is also 0, showing that f'/f is in fact entire in P, which further extends to be entire in $\mathbb{C}$ (since f'/f is an elliptic function).

Hence, by the **First Liouville's Theorem**, f'/f is in fact a constant.

Lastly, because $f'/f = a \in \mathbb{C}$, then $f'(z) = af(z)$, showing that $f(z) = Ce^{az}$.

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Question 2 *Freitag Chap. V.2 Exercise 1:*
If $L \subset \mathbb{C}$ is a lattice, then the formula

$$\sum_{w \in L} \frac{1}{(z-w)^n}$$

defines for any $n \geq 3$ an elliptic function of order n. What is the connection with the Weierstrass $\wp$-function?

Pf:

Recall that the Weierstrass $\wp$-function iwth lattice L is given as:

$$\wp : \mathbb{C} \setminus L \to \mathbb{C}, \quad \wp(z) = \frac{1}{z^2} + \sum_{\substack{w \in L \\ w \neq 0}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

Which, the above series converges normally in $\mathbb{C} \setminus L$, hence the derivative to any order in fact can be performed termwise.

For any integer $n \geq 3$, $n-2 \geq 1$, then the $(n-2)^{th}$ derivative is given as:

$$\frac{d^{(n-2)}}{dz^{(n-2)}} \wp(z) = \frac{d^{(n-2)}}{dz^{(n-2)}} \frac{1}{z^2} + \sum_{\substack{w \in L \\ w \neq 0}} \frac{d^{(n-2)}}{dz^{(n-2)}} \frac{d^{(n-2)}}{dz^{(n-2)}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

$$= \frac{(-1)^n \cdot (n-1)!}{z^n} + \sum_{\substack{w \in L \\ w \neq 0}} \frac{(-1)^n \cdot (n-1)!}{(z-w)^n} = (-1)^n \cdot (n-1)! \sum_{w \in L} \frac{1}{(z-w)^n}$$

Hence, for $n \geq 3$, the series $\sum_{w \in L} \frac{1}{(z-w)^n}$ is in fact some multiple of the $(n-2)^{th}$ derivative of Weierstrass $\wp$-function.

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Question 3 *Freitag Chap. V.2 Exercise 5:*
Let $L \subset \mathbb{C}$ be a latice. we denote by $\widehat{L}$ the set of all conformal maps $\mathbb{C} \to \mathbb{C}$ of the form

$$z \mapsto \pm z + w, \quad w \in L$$

We identify (similar to the construction of the torus $\mathbb{C}/L$) two points in $\mathbb{C}$, iff they can be mapped into each other by suitable substitutions of $\widehat{L}$. After identiciation, we obtain $\mathbb{C}/\widehat{L}$, first as a set. Show that the $\wp$-function gives a bijection

$$\mathbb{C}/\widehat{L} \to \overline{\mathbb{C}}$$

The field of all $\widehat{L}$-invariant meromorphic functions is generated by $\wp$.

Pf:

First, consider the surjectivity: Since $\wp : \mathbb{C} \to \overline{\mathbb{C}}$ is a nonconstant elliptic function with L being the lattice, then it is in fact surjective:

For all $b \in \overline{\mathbb{C}}$, if $b = \infty$, we know $\wp$ satisfies $\wp(z) = \infty$ for all $z \in L$.

On the other hand, if $b \in \mathbb{C}$, consider the function $\wp(z) - b$, which is again an elliptic function with poles at all points of L. Then, since its derivative is again given by $\wp'(z)$, consider the elliptic function $\frac{\wp'(z)}{\wp(z)-b}$, with a suitable fundamental region P such that ∂P contains no singularities. Integrate along the boundary ∂P, we get:

$$\frac{1}{2\pi i} \int_{\partial P} \frac{\wp'(z)}{\wp(z) - b} dz = 0$$

And, by argument principle, the above integral provides (Number of zeros of $(\wp - b)$ in P)$-$(Number of poles of $(\wp - b)$ in P). Which, the integral is 0 implies that the number of zeros and the number of poles in P for $\wp - b$ must be the same.

Since in given fundamental region P, there exists precisely one double pole (the point $w \in L$ that's also contained in P), hence this forces $\wp - b$ to have two zeros (including multiplicity) within the region P. So, there exists $z \in P$, with $\wp(z) - b = 0$, or $\wp(z) = b$. This proves surjectivity of $\wp(z)$.

Now, to prove injectivity of $\wp : \mathbb{C}/\widehat{L} \to \overline{\mathbb{C}}$, recall that $z_1, z_2 \in \mathbb{C}$ satisfies $\wp(z_1) = \wp(z_2)$ iff $z_1 \equiv z_2 \mod L$ or $z_1 \equiv -z_2 \mod L$. Hence, $z_1 = z_2 + w$ or $z_1 = -z_2 + w$ for some $w \in L$, which, z_1 and z_2 have the same representation under $\mathbb{C}/\widehat{L}$. This finishes the injectivity of $\wp$ when domain is given by $\mathbb{C} \setminus \widehat{L}$.

As conclusion, $\wp : \mathbb{C}/\widehat{L} \to \overline{\mathbb{C}}$ is in fact a bijection.