Lie Algebra of a Lie Group

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Tangent Vectors as Derivations

When embedding smooth manifolds into $\mathbb{R}^n$, tangent vectors are associated with directional derivatives. To generalize tangent vectors into abstract smooth manifold, we need an analogy:

Definition

Any point $u \in M$, a **Derivation at** u, is a linear map $v_u : C^\infty(M) \to \mathbb{R}$, that satisfies the product rule:

$$\forall f, g \in C^\infty(M), \quad v_u(fg) = f(u)(v_u g) + g(u)(v_u f)$$

Which, the set of all derivations at u, denoted as $T_u(M)$, is the **Tangent Space** of M at u, and each derivation $v_u \in T_u(M)$ is a **Tangent Vector** of u.

Vector Fields & Smooth Conditions

Given smooth manifold M, a vector field assigns each point with a tangent vector. More precisely:

Definition

a vector field is a map $X : M \to TM$ (TM denotes the **Tangent Bundle**), with $X(u) = X_u \in T_u(M)$.
Which, X is a **Smooth Vector Field**, if $X : M \to TM$ is a smooth map.
A collection of smooth vector fields on M is denoted as $\mathfrak{X}(M)$, which is an $\mathbb{R}$-vector space.

insert image

An equivalent condition of saying a vector field X is smooth, is through smooth functions $f \in C^\infty(M)$: For all $u \in M$, $X(u) = X_u \in T_u(M)$ is a derivation at u, define $Xf : M \to \mathbb{R}$ by $Xf(u) = X_u(f)$, then X is a smooth vector field iff $Xf \in C^\infty(M)$. Which, a smooth vector field can be viewed as a **Derivation:**

Property

For all $f, g \in C^\infty(M)$, given $X \in \mathfrak{X}(M)$, any $u \in M$ satisfies product rule:

$$X(fg)(u) = X_u(fg) = f(u)(X_u g) + g(u)(X_u f) = f(u)Xg(u) + g(u)Xf(u)$$
$$\implies X(fg) = f(Xg) + g(Xf)$$

Vector Fields of Different Manifolds

Given M, N two smooth manifolds, and smooth map $F : M \to N$. Let $X \in \mathfrak{X}(M)$, an ideal situation is mapping X to a smooth vector field of N through F. Yet, this requires F to be bijective:
Insert an example for bijective
So, we'll consider a weaker notion:

Definition

Given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, the two are ***F*-related**, if for all $u \in M$, the following is true:

$$dF_u(X_u) = Y_{F(u)}$$

Simply speaking, F maps the tangent vectors collected by X, to be compatible with tangent vectors collected by Y.

Insert another example

Lie Brackets of Vector Fields

The initial motivation is to combine two vector fields $X, Y \in \mathfrak{X}(M)$ to be another vector field. For all $f \in C^\infty(M)$, since $Yf \in C^\infty(M)$, then $XYf := X(Yf) \in C^\infty(M)$. But, in general XY is not a derivation, hence not a vector field:

Example

Define vector fields $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Take smooth functions $f(x, y) = x$ and $g(x, y) = y$, then we get the following:

$$XY(fg) = X\left(x\frac{\partial}{\partial y}(xy)\right) = \frac{\partial}{\partial x}(x^2) = 2x$$

But, product rule doesn't hold for this example:

$$f(XYg) + g(XYf) = x\left(X\left(x\frac{\partial}{\partial y}(y)\right)\right) + y\left(X\left(x\frac{\partial}{\partial y}(x)\right)\right) = x$$

So, we need to define a new operation on vector fields:

Definition

The **Lie Bracket** $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, is defined as:

$$\forall X, Y \in \mathfrak{X}(M), \quad [X, Y] = XY - YX$$

Which, the output $[X, Y] \in \mathfrak{X}(M)$, since it satisfies product rule:

$$[X, Y](fg) = X(Y(fg)) - Y(X(fg)) = X(f(Yg) + g(Yf)) - Y(f(Xg) + g(Xf))$$
$$= f(XYg)+(Yg)(Xf)+g(XYf)+(Yf)(Xg)-f(YXg)-(Xg)(Yf)-g(YXf)-(Xf)(Yg)$$
$$= f(XYg - YXg) + g(XYf - YXf) = f[X, Y](g) + g[X, Y](f)$$

Lie Bracket also satisfies these properties:

- **Bilinearity:** $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- **Antisymmetry:** $[X, Y] = -[Y, X]$
- **Jacobi's Identity:** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Moreover, Lie Bracket inherits relation of smooth maps:

Property

Given smooth map $F : M \to N$, if $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ are F-related respectively, then $[X_1, X_2] \in \mathfrak{X}(M)$ and $[Y_1, Y_2] \in \mathfrak{X}(N)$ are also F-related.

Lie Group & Left-Invariant Vector Fields

The initial motivation is to study group structures in some smooth manifolds.

Definition

A **Lie Group** G, is a smooth manifold along with group structure, such that the group operation $P : G \times G \to G$ by $P(g, h) = gh$, and the inversion map $i : G \to G$ by $i(g) = g^{-1}$ are both smooth maps between manifolds.

For all $g \in G$, denote the left multiplication $L_g : G \to G$ by $L_g(h) = gh$, since $L_g = P\big|_{\{g\}\times G}$, it is a smooth map. Hence, there's a notion of X being L_g-related to itself:

Definition

Given any $X \in \mathfrak{X}(G)$ and all $g \in G$, X is a **Left-Invariant Vector Field**, if for all $g \in G$, X is L_g-related to itself.
The collection of Left-Invariant vector fields $\mathfrak{g} \subseteq \mathfrak{X}(G)$, is a linear subspace.

Recall that Lie Bracket of vector field preserves F-relation between manifolds, so:

Property

For all $X, Y \in \mathfrak{X}(G)$ that are left-invariant, since for all $g \in G$, X and Y are L_g related to themselves, then the Lie Bracket $[X, Y]$ is also L_g related to $[X, Y]$. Hence, $[X, Y]$ is also left-invariant, so Left-Invariant vector fields $\mathfrak{g}$ is closed under Lie Bracket's operation.

Lie Algebra on a Lie Group

Definition

Given a vector space $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$, with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that the following holds:

- **Bilinearity:** $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- **Antisymmetry:** $[X, Y] = -[Y, X]$
- **Jacobi's Identity:** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Then, the pair $(\mathfrak{g}, [\cdot, \cdot])$ is a **Lie Algebra**.

In general, Lie Algebra is non-associative, which Jacobi's Identity is an alternative condition for Lie Algebra.
Finally, we can define **Lie Algebra of a Lie Group:**

Definition

Given a lie group G, since the subset of left-invariant vector fields $\mathfrak{g} \subseteq \mathfrak{X}(G)$ forms a linear subspace, while closed under Lie Bracket's operation, then the pair $(\mathfrak{g}, [\cdot, \cdot])$ forms a **Lie Algebra** of G, denoted as Lie(G).

Here's an example of Lie Algebra on a Lie Group:

Example

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Reference: John M. Lee, Introduction to Smooth Manifolds, 2nd Edition