Math 118C HW4

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Question 1 *Rudin Pg. 242 Problem 27:*
Put $f(0,0)=0$, *and*

$$f(x,y)=\frac{xy(x^2-y^2)}{x^2+y^2}$$

if $(x,y)\neq(0,0)$. *Prove that*

(a) $f,\ D_1f,\ D_2f$ *are continuous in* $\mathbb{R}^2$.

(b) $D_{12}f$ *and* $D_{21}f$ *exist at every point of* $\mathbb{R}^2$, *and are continuous except at* $(0,0)$.

(c) $D_{12}f(0,0)=1$, *and* $D_{21}f(0,0)=-1$.

Pf:

For all $(x,y)\in\mathbb{R}^2$ with $(x,y)\neq(0,0)$, using polar coordinates, $(x,y)=(r\cos(\theta),r\sin(\theta))$ for some $r>0$ and $\theta\in[0,2\pi)$. Which, $|(x,y)|=r$, when consider limit definition, we'll use polar coordinates instead.

(a) f **is continuous:**

For $(x,y)\neq(0,0)$, since f is a defined rational function, it is continuous, so it suffices to show f is continuous at 0. For all $\epsilon>0$, choose $\delta=\sqrt{\frac{\epsilon}{2}}>0$, then for all (x,y) satisfying $0<|(x,y)|=r<\delta$, we get the following:

$$|f(x,y)-f(0,0)|=\left|\frac{(r\cos(\theta))(r\sin(\theta))((r\cos(\theta))^2-(r\sin(\theta))^2)}{(r\cos(\theta))^2+(r\sin(\theta))^2}-0\right|$$

$$=\left|\frac{r^4\sin(\theta)\cos(\theta)(\cos^2(\theta)-\sin^2(\theta))}{r^2}\right|\leq r^2|sin(\theta)|\cdot|\cos(\theta)|\cdot(|\cos(\theta)|^2+|\sin(\theta)|^2)$$

$$\leq 2r^2<2\left(\sqrt{\frac{\epsilon}{2}}\right)^2=2\cdot\frac{\epsilon}{2}=\epsilon$$

This shows that f is continuous at $(0,0)$, hence f in continuous in $\mathbb{R}^2$.

D_1f **is continuous:**

First, using basic differentiation rule, for $(x,y)\neq(0,0)$, we get the following:

$$D_1f(x,y)=\frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right)=\frac{(3x^2y-y^3)(x^2+y^2)-xy(x^2-y^2)2x}{(x^2+y^2)^2}=\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$$

Which, at $(0,0)$, $D_1 f$ could be obtained through limit:

$$D_1 f(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h\cdot 0(h^2-0^2)}{(h^2+0^2)h} = \lim_{h\to 0} 0 = 0$$

Which, $D_1 f(x,y)$ for $(x,y)\neq(0,0)$ is again a rational function, which is continuous, so to verify continuity, it suffices to check $(0,0)$. For all $\epsilon>0$, choose $\delta=\frac{\epsilon}{6}>0$, then for all (x,y) satisfying $0<|(x,y)|=r<\delta$, we get the following:

$$|D_1 f(x,y)-D_1 f(0,0)| = \left|\frac{(r\cos(\theta))^4(r\sin(\theta))+4(r\cos(\theta))^2(r\sin(\theta))^3-(r\sin(\theta)^5)}{((r\cos(\theta))^2+(r\sin(\theta))^2)^2}-0\right|$$

$$= \left|\frac{r^5(\cos^4(\theta)\sin(\theta)+4\cos^2(\theta)\sin^3(\theta))-\sin^5(\theta)}{r^4}\right| \leq r(|cos^4(\theta)\sin(\theta)|+4|cos^2(\theta)\sin^3(\theta)|+|\sin^5(\theta)|)$$

$$\leq r(1+4+1) < 6\cdot\frac{\epsilon}{6}=\epsilon$$

This proves the continuity of $D_1 f$ at $(0,0)$, so $D_1 f$ is continuous in $\mathbb{R}^2$.

$D_2 f$ is continuous:

Using differentiation rule, for $(x,y)\neq(0,0)$, we get the following:

$$D_2 f(x,y) = \frac{\partial}{\partial y}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) = \frac{(x^3-3xy^2)(x^2+y^2)-xy(x^2-y^2)2y}{(x^2+y^2)^2} = \frac{x^5-xy^4-4x^3y^2}{(x^2+y^2)^2}$$

Again, at $(0,0)$, $D_2 f$ could be obtained through limit:

$$D_2 f(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0\cdot h(0^2-h^2)}{(0^2+h^2)h} = \lim_{h\to 0} 0 = 0$$

Notice that $D_2 f(x,y)$ for $(x,y)\neq(0,0)$ is a rational function, which is continuous, so to verify continuity, it suffices to check $(0,0)$. For all $\epsilon>0$, choose $\delta=\frac{\epsilon}{6}>0$, then for all (x,y) satisfying $0<|(x,y)|=r<\delta$, we get the following:

$$|D_2 f(x,y)-D_2 f(0,0)| = \left|\frac{(r\cos(\theta))^5-(r\cos(\theta))(r\sin(\theta))^4-4(r\cos(\theta))^3(r\sin(\theta))^2}{((r\cos(\theta))^2+(r\sin(\theta))^2)^2}-0\right|$$

$$= \left|\frac{r^5(\cos^5(\theta)-\cos(\theta)\sin^4(\theta)-4\cos^3(\theta)\sin^2(\theta))}{r^4}\right| \leq r(|\cos^5(\theta)|+|\cos(\theta)\sin^4(\theta)|+4|\cos^3(\theta)\sin^2(\theta)|)$$

$$\leq r(1+1+4) < 6\cdot\frac{\epsilon}{6}=\epsilon$$

This proves the continuity of $D_2 f$ at $(0,0)$, hence $D_2 f$ is continuous in $\mathbb{R}^2$.

(b) **Function $D_{21} f$:**

Given that $D_1 f(x,y)=\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$ for $(x,y)\neq(0,0)$ and $D_1 f(0,0)=0$, apply differentiation rule for $(x,y)\neq(0,0)$, we get:

$$D_{21} f(x,y) = \frac{\partial}{\partial y}\left(\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}\right) = \frac{(x^4+12x^2y^2-5y^4)(x^2+y^2)^2-(x^4y+4x^2y^3-y^5)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

Which, $D_{21}f(x,y)$ is continuous for $(x,y) \neq (0,0)$ (since it's a rational function).

Now, to get $D_{21}f(0,0)$, we'll use limit definition:

$$D_{21}f(0,0) = \lim_{h\to 0} \frac{D_1 f(0,h) - D_1 f(0,0)}{h} = \lim_{h\to 0} \frac{0^4 \cdot h + 4 \cdot 0^2 \cdot h^3 - h^5}{(0^2+h^2)^2 h} = \lim_{h\to 0} -\frac{h^5}{h^5} = -1$$

Hence, $D_{21}f$ exists on the whole $\mathbb{R}^2$, and is continuous at all $(x,y) \neq (0,0)$. But, it is not continuous at $(0,0)$, since choosing $x \neq 0$ and $y = 0$, $D_{21}f$ becomes:

$$D_{21}f(x,0) = \frac{x^8}{x^8} = 1$$

Hence, $\lim_{x\to 0} D_{21}f(x,0) = 1 \neq -1 = D_{21}f(0,0)$, showing the discontinuity at $(0,0)$.

So, $D_{21}f$ exists on $\mathbb{R}^2$, while being continuous on $\mathbb{R}^2 \setminus \{0\}$.

Function $D_{12}f$:

Given that $D_2 f(x,y) = \frac{x^5 - xy^4 - 4x^3y^2}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$ and $D_2 f(0,0) = 0$, apply differentiation rule for $(x,y) \neq (0,0)$, we get:

$$D_{12}f(x,y) = \frac{\partial}{\partial x}\left(\frac{x^5 - xy^4 - 4x^3y^2}{(x^2+y^2)^2}\right) = \frac{(5x^4 - y^4 - 12x^2y^2)(x^2+y^2)^2 - (x^5 - xy^4 - 4x^3y^2)2(x^2+y^2)2x}{(x^2+y^2)^4}$$

Hence, $D_{12}f$ is continuous for $(x,y) \neq (0,0)$, since it's also a rational function.

Now, to get $D_{12}f(0,0)$, we'll again use limit definition:

$$D_{12}f(0,0) = \lim_{h\to 0} \frac{D_2 f(h,0) - D_2 f(0,0)}{h} = \lim_{h\to 0} \frac{h^5 - h \cdot 0^4 - 4h^3 \cdot 0^2}{(h^2+0^2)^2 h} = \lim_{h\to 0} \frac{h^5}{h^5} = 1$$

Hence, $D_{12}f$ exists on the whole $\mathbb{R}^2$, and is continuous at all $(x,y) \neq (0,0)$. But again, it's not continuous at $(0,0)$, since choosing $x = 0$ and $y \neq 0$, $D_{12}f$ becomes:

$$D_{12}f(0,y) = \frac{-y^8}{y^8} = -1$$

Hence, $\lim_{y\to 0} D_{12}f(0,y) = -1 \neq 1 = D_{12}f(0,0)$, showing the discontinuity at $(0,0)$.

So, $D_{12}f$ exists on $\mathbb{R}^2$, while being continuous on $\mathbb{R}^2 \setminus \{0\}$.

(c) From **part (b)**, when verifying that the existence of $D_{12}f(0,0)$ and $D_{21}f(0,0)$, we've shown that $D_{12}f(0,0) = 1$, and $D_{21}f(0,0) = -1$.

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Question 2 *Rudin Pg. 242 Problem 28:*
For $t \geq 0$, put

$$\varphi(x,t) = \begin{cases} x & 0 \leq x \leq \sqrt{t} \\ -x + 2\sqrt{t} & \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0 & \text{otherwise} \end{cases}$$

and put $\varphi(x,t) = -\varphi(x,|t|)$ if $t < 0$.
Show that φ is continuous on $\mathbb{R}^2$, and $D_2\varphi(x,0) = 0$ for all x. Define

$$f(t) = \int_{-1}^{1} \varphi(x,t)dx$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^{1} D_2\varphi(x,0)dx$$

Pf:

Continuity of φ:

First, in the open half plane $x < 0$, since $\varphi(x,t) = 0$, then φ is continuous.

Similarly, in the open region where all (x,t) satisfies $x > 2\sqrt{|t|}$, since again $\varphi(x,t) = 0$ by the restriction, then φ is again continuous.

Then, for the open region where all (x,t) satisfies $0 < x < \sqrt{|t|}$, since the function φ is described by x for $t > 0$, and $-x$ for $t < 0$, then the addition φ is also continuous within this region.

Also, for the open region where all (x,t) satisfies $\sqrt{|t|} < x < 2\sqrt{|t|}$, since the function φ is described by $-x + 2\sqrt{|t|}$ for $t > 0$, while described by $-(-x + 2\sqrt{|t|})$ when $t < 0$, so since both $x, \sqrt{|t|}$ are continuous functions, φ as their linear combination is again continuous within this region.

Hence, the only regions left to check, is the lines where (x,t) satisfies $x = 0$, $x = \sqrt{|t|}$, or $x = 2\sqrt{|t|}$. (Note: Since both x and $2\sqrt{|t|}$ are continuous functions, then for any given (x_0, t_0), for all $\epsilon > 0$, there exists $\delta > 0$, such that $|x - x_0| < \delta \implies |x - x_0| < \frac{\epsilon}{2}$, and $|t - t_0| < \delta \implies |2\sqrt{|t|} - 2\sqrt{|t_0|}| < \frac{\epsilon}{2}$). (Note 2: below when $\pm$ appears, it considers the case where t could be positive or negative). (Note 3: below we'll directly assume the choice of δ relates to $\epsilon > 0$).

- For the line $x = 0$, we have $\varphi(0,t) = 0$. Which, for any $(0, t_0)$:

  If $t_0 = 0$, for all (x,t) with $|(x,t) - (0,0)| < \delta$, since $|x - 0|, |t - 0| < \delta$, we get the following three cases:

  $$\varphi(x,t) = \pm x, \quad |\varphi(x,t) - \varphi(0,0)| = |x - 0| < \frac{\epsilon}{2} < \epsilon$$

  $$\varphi(x,t) = \pm(-x + 2\sqrt{|t|}), \quad |\varphi(x,t) - \varphi(0,0)| = |-x + 2\sqrt{|t|}| \leq |x| + |2\sqrt{|t|}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

  $$\varphi(x,t) = 0, \quad |\varphi(x,t) - \varphi(0,0)| = |0 - 0| < \epsilon$$

  For $t_0 \neq 0$ instead, we can add an extra condition, not only $|(x,t) - (0,t_0)| < \delta$, but shrink δ so that $|x| < \sqrt{|t|}$ for all point in the region. Hence, we no longer need to consider the second case of the function, which left with the following two cases:

  $$\varphi(x,t) = \pm x, \quad |\varphi(x,t) - \varphi(0,t_0)| = |x - 0| < \frac{\epsilon}{2} < \epsilon$$

$$\varphi(x,t)=0, \quad |\varphi(x,t)-\varphi(0,t_0)|=|0|<\epsilon$$

This shows that φ is continuous at all $(0,t_0)$.

- For the line $x=\sqrt{|t|}$ (assume $(x,t)\neq(0,0)$, which has checked before). Then, for all (x_0,t_0) on this line, since $x_0=\sqrt{|t_0|}$, then $\varphi(x_0,t_0)=\pm x_0$. Then, choose $\delta>0$, such that for all (x,t) satisfying $|(x,t)-(x_0,t_0)|<\delta$, $0<x<2\sqrt{|t|}$, and t has the same sign with t_0. Then, we don't need to consider the case where $\varphi(x,t)=0$, and $\varphi(x,t)$ and $\varphi(x_0,t_0)$ are following the same sign (since assuming t,t_0 have the same sign). So, we get the following two cases:

$$\varphi(x,t)=\pm x, \quad |\varphi(x,t)-\varphi(x_0,t_0)|=|\pm x-\pm x_0|=|x-x_0|<\frac{\epsilon}{2}<\epsilon$$

$$\varphi(x,t)=\pm(-x+2\sqrt{t}), \quad \text{since } x=x_0+\delta',\ |\delta'-0|<\delta,\ |\delta'-0|<\frac{\epsilon}{2}$$

$$\implies |\varphi(x,t)-\varphi(x_0,t_0)|=|\pm(-(x_0+\delta')+2\sqrt{|t|})-\pm x_0|=|2\sqrt{|t|}-2x_0-\delta'|\leq|2\sqrt{|t|}-2\sqrt{|t_0|}|+|\delta'|$$
$$<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

(Note: the second case has$x_0=\sqrt{|t_0|}$, and $|2\sqrt{|t|}-2\sqrt{|t_0|}|<\frac{\epsilon}{2}$ since assuming $|t-t_0|<\delta$).

This proves continuity on the line $x=\sqrt{|t|}$.

- For the line $x=2\sqrt{|t|}$, for all (x_0,t_0) on the line (again, assume $(x_0,t_0)\neq(0,0)$), since $x_0=2\sqrt{|t_0|}$, then $\varphi(x_0,t_0)=\pm(-x_0+2\sqrt{|t_0|})=\pm(-2\sqrt{|t_0|}+2\sqrt{|t_0|})=0$. Which, choose $\delta>0$, such that not only satisfy the relationship with ϵ, but also for any (x,t) with $|(x,t)-(x_0,t_0)|$, we have $x>\sqrt{|t|}$. This avoids the case where $\varphi(x,t)=x$. Then, we get the following two cases:

$$\varphi(x,t)=\pm(-x+2\sqrt{|t|}), \quad \text{since } x=x_0+\delta', \quad |\delta'-0|<\delta,\ |\delta'-0|<\frac{\epsilon}{2}$$

$$\implies |\varphi(x,t)-\varphi(x_0,t_0)|=|-x+2\sqrt{|t|}|=|-(x_0+\delta')+2\sqrt{|t|}|\leq|2\sqrt{|t|}-2\sqrt{|t_0|}|+|\delta'|$$
$$<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

$$\varphi(x,t)=0, \quad |\varphi(x,t)-\varphi(x_0,t_0)|=0<\epsilon$$

(Note: the first case has $x_0=2\sqrt{|t_0|}$, while $|2\sqrt{|t|}-2\sqrt{|t_0|}|<\frac{\epsilon}{2}$ since $|t-t_0|<\delta$).

This proves continuity on the line $x=2\sqrt{|t|}$.

The above situation covers the all points in $\mathbb{R}^2$, hence φ is continuous on $\mathbb{R}^2$.

$D_2\varphi$ when $t=0$:

For all $x\in\mathbb{R}$, if $x\leq 0$, then we get $\varphi(x,t)=0$ regardless of $t\in\mathbb{R}$, showing that $D_2\varphi(x,0)=\frac{\partial\varphi}{\partial t}(x,0)=0$.

Now for $x>0$, since for all $t\in\mathbb{R}$ satisfying $4|t|<x^2$, we have $2\sqrt{|t|}<x$, then $\varphi(x,t)=0$ when $t\in(-\frac{x^2}{4},\frac{x^2}{4})$. So, $D_2\varphi(x,0)=0$ (since $\lim_{t\to 0}\frac{\varphi(x,t)-\varphi(x,0)}{t}=\lim_{t\to 0}0=0$, because for small enough t, it lies in the range $(-\frac{x^2}{4},\frac{x^2}{4})$).

So, regardless of $x\in\mathbb{R}$, we have $D_2\varphi(x,0)=0$.

Function $f(t)$:

Given $f(t)=\int_{-1}^{1}\varphi(x,t)dt$, when $|t|<\frac{1}{4}$, there are several cases to consider:

- when $t \geq 0$, then $0 \leq \sqrt{t} < \sqrt{\frac{1}{4}} = \frac{1}{2}$, while $0 \leq 2\sqrt{t} < 1$. Hence, the integral expression can be broken down as the following pieces:

$$\int_{-1}^{1} \varphi(x,t)dx = \int_{-1}^{0} 0dx + \int_{0}^{\sqrt{t}} xdx + \int_{\sqrt{t}}^{2\sqrt{t}} (-x + 2\sqrt{t})dx + \int_{2\sqrt{t}}^{1} 0dx$$

$$= \frac{1}{2}x^2 \Big|_{0}^{\sqrt{t}} + \left(-\frac{1}{2}x^2 + 2\sqrt{t}x\right) \Big|_{\sqrt{t}}^{2\sqrt{t}} = \frac{1}{2}t + ((4t - 2t) - (2t - \frac{1}{2}t)) = t$$

- when $t < 0$ (where $t = -|t|$), since $\varphi(x,t) = -\varphi(x,|t|)$ with $|t| > 0$, then inheriting from the above expression, we get:

$$\int_{-1}^{1} \varphi(x,t)dx = -\int_{-1}^{1} \varphi(x,|t|)dx = -|t| = t$$

Hence, for $|t| < \frac{1}{4}$, we can deduce that $f(t) = t$, which $f'(t) = 1$. So, the following inequality is true:

$$f'(0) = 1 \neq 0 = \int_{-1}^{1} 0dx = \int_{-1}^{1} D_2\varphi(x,0)dx$$

This shows that differentiation under integral sign fails under certain situation.

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Question 3 *Rudin Pg. 243 Problem 30:*
Let $f \in \mathcal{C}^{(m)}(E)$, where E is an open subset of $\mathbb{R}^n$. Fix $a \in E$, and suppose $x \in \mathbb{R}^n$ is so close to 0 that the points $p(t) = a + tx$ lie in E whenever $0 \le t \le 1$. Define $h(t) = f(p(t))$ for all $t \in \mathbb{R}$ for which $p(t) \in E$.

(a) For $1 \le k \le m$, show (by repeated application of the chain rule) that

$$h^{(k)}(t) = \sum (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

The sum extends over all order k-tuples $(l_1, \dots, l_k)$ in which each l_j is one of the integers $1, \dots, n$.

Pf:

Given $a, x \in \mathbb{R}^n$ (where $x = (x_1, \dots, x_n)$ for fixed $x_1, \dots, x_n \in \mathbb{R}^n$) and $p(t) = a + tx$ for $t \in [0, 1]$, then $p'(t) = x$.

Now, we'll use induction to verify the formula (and we'll use matrix representation of the differentials).

First, for $k = 1$, using chain rule, we get the following:

$$h'(t) = Df(p(t))p'(t) = \begin{pmatrix} D_1 f & \dots & D_n f \end{pmatrix}\Big|_{p(t)} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^n D_i f(p(t)) x_i$$

Since all the possible 1-tuple is included in the summation, the $h'(t)$ satisfies the given formula.

Now, suppose for given $1 \le k \le (m-1)$, $h^{(k)}(t)$ satisfies the following formula:

$$h^{(k)}(t) = \sum (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

Since for each k-tuple $(l_1, \dots, l_k)$ (where each $l_i \in \{1, \dots, n\}$) has the function $x_{l_1} \dots x_{l_k} D_{l_1 \dots l_k} f(p(t))$ being a differentiable function from $(0, 1)$ to $\mathbb{R}$ (where $D_{l_1 \dots l_k} f(z)$ for $z \in E$ is a differentiable function, since it has only been differentiated $k < m$ times, while $f \in C^{(m)}(E)$). Then, to calculate the $(k+1)^{th}$ derivative, we get:

$$h^{(k+1)}(t) = \sum \frac{d}{dt} (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

$$\begin{aligned}
\forall (l_1, \dots, l_k), \quad \frac{d}{dt} (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k} &= x_{l_1} \dots x_{l_k} D \left(D_{l_1 \dots l_k} f \right)(p(t)) p'(t) \\
= x_{l_1} \dots x_{l_k} \sum_{i=1}^n D_i \left(D_{l_1 \dots l_k} f \right)(p(t)) x_i &= \sum_{i=1}^n D_{i l_1 \dots l_k} f(p(t)) x_i x_{l_1} \dots x_{l_k} \\
\implies h^{(k+1)}(t) &= \sum \left(\sum_{i=1}^n D_{i l_1 \dots l_k} f(p(t)) x_i x_{l_1} \dots x_{l_k} \right)
\end{aligned}$$

Which, the first summation indicates all possible k-tuple $(l_1, \dots, l_k)$ for $l_j \in \{1, \dots, n\}$.

Now, for all $(k+1)$-tuple $(j_0, j_1, \dots, j_k)$ where each $j_l \in \{1, \dots, n\}$, choose the unique k-tuple $(j_1, \dots, j_k)$, then $D_{j_0 j_1 \dots j_k} f(p(t)) x_{j_0} x_{j_1} \dots x_{j_k}$ appears precisely once in the summation of $h^{(k+1)}(t)$ given above; similarly, since each k-tuple $(l_1, \dots, l_k)$ and $i \in \{1, \dots, n\}$ corresponds to a unique $(k+1)$-tuple $(i, l_1, \dots, l_k)$, so the summation in $h^{(k+1)}(t)$ has a 1-to-1 correspondance to all $(k+1)$-tuple. Then, the summation $h^{(k+1)}(t)$ can also be described as:

$$h^{(k+1)}(t) = \sum D_{l_1 \dots l_k l_{k+1}} f(p(t)) x_{l_1} \dots x_{l_k} x_{l_{k+1}}$$

Where each $(l_1, \dots, l_k, l_{k+1})$ is a $(k+1)$-tuple with entries from $\{1, \dots, n\}$.

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Question 4 *Rudin Pg. 288 Problem 2:*
For $i = 1, 2, 3, ...,$ let $\varphi_i \in \mathcal{C}(\mathbb{R})$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x,y) = \sum_{i=1}^{\infty} (\varphi_i(x) - \varphi_{i+1}(x))\varphi_i(y)$$

Then f has compact support in $\mathbb{R}^2$, f is cotinuous except at $(0,0)$, and

$$\int dy \int f(x,y)dx = 0, \quad \text{but} \int dx \int f(x,y)dy = 1$$

Observe that f is unbounded in every neighborhood of $(0,0)$.

Pf:

The function f is well-defined, with compact support:

First, notice that for $x \leq 0$ or $x \geq 1$, since for all $i \in \mathbb{N}$, we have $(2^{-i}, 2^{1-i}) \subseteq (0,1)$, then x is not in the support of φ_i, hence $\varphi_i(x) = 0$. So, for $(x,y) \notin (0,1) \times (0,1)$, since $\varphi_i(x), \varphi_i(y) = 0$, then $f(x,y) = 0$.

Now, for all $x \in (0,1)$, since $\lim_{i\to\infty} 2^{-i} = 0$, then take the smallest $i \in \mathbb{N}$ such that $2^{-i} < x$, then $x \leq 2^{1-i}$. So, if $x \neq 2^{1-i}$. Which, for other $j \neq i$, since $(2^{-j}, 2^{1-j}) \cap (2^{-i}, 2^{1-i}) = \emptyset$, this indicates $x \notin (2^{-j}, 2^{1-j})$ (x is not in the support of φ_j), hence $\varphi_j(x) = 0$.

So, for all $(x,y) \in (0,1) \times (0,1)$, since there exists $i, j \in \mathbb{N}$, such that $x \in (2^{-i}, 2^{1-i})$ and $y \in (2^{-j}, 2^{1-j})$, then if $k \neq i$, $\varphi_k(x) = 0$; and if $k \neq j$, $\varphi_k(y) = 0$. So, consider the infinite summation, we get:

$$f(x,y) = \sum_{k=1}^{\infty} (\varphi_k(x) - \varphi_{k+1}(x))\varphi_k(y) = (\varphi_j(x) - \varphi_{j+1}(x))\varphi_j(y)$$

(Note: if $k \neq j$, then $\varphi_k(y) = 0$, so the other terms are trivial).

Hence, regardles of $(x,y) \in \mathbb{R}^2$, $f(x,y)$ is well-defined. And, since for $(x,y) \notin (0,1) \times (0,1)$, $f(x,y) = 0$, this shows that the support of f is contained in $[0,1] \times [0,1]$, which is bounded. Then, because support is chosen to be closed, the support of f is in fact compact.

Continuity of f except at $(0,0)$:

For all $(x,y) \neq (0,0)$, there are several cases to consider:

- If $y < 0$ or $y > 1$, then since for any $i \in \mathbb{N}$, y is not in the support of φ_i (given by $(2^{-i}, 2^{1-i})$), then $f(x,y) = 0$ (since every term in the series include $\varphi_i(y)$ for some i), hence f is continuous. (similarly, if $x < 0$ or $x > 1$, then $\varphi_i(x) = 0$ for all $i \in \mathbb{N}$ also, then $f(x,y) = 0$, since every term in the summation includes $(\varphi_i(x) - \varphi_{i+1}(x))$ for some i). So, for the region where $(x,y) \notin [0,1] \times [0,1]$, f is continuous (and the below cases we'll assume the points are in $[0,1] \times [0,1]$).

- For $0 < y < 1$, then choose $i, j \in \mathbb{N} \cup \{0\}$, with $i < j$, such that $2^{-j} < y < 2^{-i}$. Then, for any (x_0, y_0) within this region, choose an open neighborhood U that's also contained in the region. Since for this neighborhood $2^{-j} < y < 2^{-i}$ for all points, there's only finitely many index $k \in \mathbb{N}$ satisfying $\varphi_k(y) \neq 0$ for some $(x,y) \in U$ (based on the formula initially derived), hence, f can be expressed as finite product and summation of φ_k with input x or y, which f is continuous in this region.

- For any point with $y = 1$ (and $x \neq 0$), choose open neighborhood U such that all $(x, y) \in U$ satisfies $y > 2^{-1}$. Then, since for all index $i > 1$, any $(x, y) \in U$ has y outside the support of φ_i, then $\varphi_i(y) = 0$, forcing $f(x, y) = (\varphi_1(x) - \varphi_2(x))\varphi_1(y)$, hence f is continuous in this region.

- Then, for any point with $y = 0$, since $x > 0$, choose $j \in \mathbb{N}$ such that $2^{-j} < x$. Then, choose an open neighborhood U such that all $(x', y') \in U$ has $2^{-j} < x'$, then again only finitely many index $i \in \mathbb{N}$ (specifically, with $i \leq j$) has $\varphi_i(x) \neq 0$ for some $(x, y) \in U$. Then, within U, f can again be expressed as finite sum and product of $\varphi_k(x)$ and $\varphi_j(y)$, hence f is still continuous in this region.

The above verifies the continuity of f under various cases. Now, to prove that f is discontinuous at $(0, 0)$, it suffices to show that f is unbounded in any neighborhood of $(0, 0)$.

For all $i \in \mathbb{N}$, since φ_i has support contained in $(2^{-i}, 2^{1-i})$, then along with the fact that φ_i has integral being 1, we get the following:

$$\int_{2^{-i}}^{2^{1-i}} \varphi_i(x)dx = 1$$

Which, if we define the function $\overline{\varphi}_i : [2^{-i}, 2^{1-i}] \to \mathbb{R}$ as follow:

$$\overline{\varphi}_i(t) = \int_{2^{-i}}^{t} \varphi_i(x)dx$$

Then, since φ_i is continuous, $\overline{\varphi}_i$ is differentiable, with $\varphi_i(t)$ being its derivative. Then, with Mean Value Theorem, there exists $t_i \in (2^{-i}, 2^{1-i})$, such that the following is true:

$$1 - 0 = 1 = \int_{2^{-i}}^{2^{1-i}} \varphi_i(x)dx = \overline{\varphi}_i(2^{1-i}) - \overline{\varphi}_i(2^{-i}) = \varphi(t_i)(2^{1-i} - 2^{-i}) = \varphi_i(t_i) \cdot 2^{-i}$$

$$\implies \varphi_i(t_i) = 2^i$$

Which, if consider $f(t_i, t_i)$, since only index $i \in \mathbb{N}$ has t_i being in the support of φ_i, then we get:

$$f(t_i, t_i) = \sum_{k=1}^{\infty} (\varphi_k(t_i) - \varphi_{k+1}(t_i))\varphi_k(t_i) = (\varphi_i(t_i) - \varphi_{i+1}(t_i))\varphi_i(t_i) = (2^i - 0)2^i = 2^{2i}$$

Which, for all $M > 0$ and $r > 0$, for the open neighborhood $B_r(0, 0)$, choose $i \in \mathbb{N}$ such that $2^{1-i} < \frac{r}{\sqrt{2}}$ and $2^i > M$. Then, the point (t_i, t_i) satisfies:

$$|(t_i, t_i)| = \sqrt{2t_i^2} < \sqrt{2 \cdot (2^{1-i})^2} < \sqrt{2 \cdot \left(\frac{r}{\sqrt{2}}\right)^2} = \sqrt{2 \cdot \frac{r^2}{2}} = \sqrt{r^2} = r$$

Hence, $(t_i, t_i) \in B_r(0, 0)$. Also, $f(t_i, t_i) = 2^{2i} > 2^i > M$. This shows that f is unbounded within any neighborhood of $(0, 0)$, which f is not continuous at $(0, 0)$.

Integral of f:

Since f has a support in $[0, 1] \times [0, 1]$, it suffices to consider the integral over this region. (Also, for all $i \in \mathbb{N}$, since φ_i has support $(2^{-i}, 2^{1-i}) \subseteq [0, 1]$, then integration along one variable can be taken from 0 to 1, and $\int_0^1 \varphi_i(x)dx = 1$ based on assumption).

First, fix $y \in (0, 1)$, since in the first section we've proven that $f(x, y) = (\varphi_j(x) - \varphi_{j+1}(x))\varphi_j(y)$ for some $j \in \mathbb{N}$, then:

$$\int_0^1 f(x, y)dx = \int_0^1 (\varphi_j(x) - \varphi_{j+1}(x))\varphi_j(y)dx = \varphi_j(y) \left(\int_0^1 \varphi_j(x)dx - \int_0^1 \varphi_{j+1}(x)dx\right) = \varphi_j(y)(1 - 1) = 0$$

This indicates that $\int f(x,y)dx = 0$. Hence, we get the following:

$$\int dy \left(\int f(x,y)dx \right) = \int 0dy = 0$$

Else, if fix $x \in (0,1)$, there are two cases:

- For $x \in (2^{-1}, 2^{1-1})$, since the only index $i \in \mathbb{N}$ such that x is in φ_i's support is $i = 1$, then for index $i > 2$, $\varphi_i(x) = 0$. So, we get:

$$f(x,y) = \sum_{i=1}^{\infty} (\varphi_i(x) - \varphi_{i+1}(x))\varphi_i(y) = (\varphi_1(x) - \varphi_2(x))\varphi_1(y) = \varphi_1(x)\varphi_1(y)$$

  (Note: $\varphi_2(x) = 0$ for the fixed $x \in (2^{-1}, 2^{1-1})$). Which, its integral with respect to y becomes:

$$\int_0^1 f(x,y)dy = \int_0^1 \varphi_1(x)\varphi_1(y)dy = \varphi_1(x)$$

- If $x \notin (2^{-1}, 2^{1-1})$, either $x \notin (2^{-i}, 2^{1-i})$ for all $i \in \mathbb{N}$ (which x is not in the support of any φ_i, showing that $f(x,y) = 0$, so $\int_0^1 f(x,y)dy = 0$), or $x \in (2^{-i}, 2^{1-i})$ for some integer $i > 1$. Which, for the second case, since $i - 1 \geq 1$, we get:

$$f(x,y) = \sum_{i=1}^{\infty} (\varphi_i(x) - \varphi_{i+1}(x))\varphi_i(y) = (\varphi_{i-1}(x) - \varphi_i(x))\varphi_{i-1}(y) + (\varphi_i(x) - \varphi_{i+1}(x))\varphi_i(y)$$
$$= \varphi_i(x)(\varphi_i(y) - \varphi_{i-1}(y))$$

  So, its integral with respect to y becomes:

$$\int_0^1 f(x,y)dy = \int_0^1 \varphi_i(x)(\varphi_i(y) - \varphi_{i-1}(y))dy = \varphi_i(x) \left(\int_0^1 \varphi_i(y)dy - \int_0^1 \varphi_{i+1}(y)dy \right) = \varphi_i(x)(1-1) = 0$$

So, if consider the integral, we get:

$$\int dx \left(\int f(x,y)dy \right) = \int_0^1 dx \left(\int f(x,y)dy \right) = \int_0^{2^{-1}} dx \left(\int f(x,y)dy \right) + \int_{2^{-1}}^{2^{1-1}} dx \left(\int f(x,y)dy \right)$$
$$= \int_0^{2^{-1}} 0dx + \int_{2^{-1}}^{2^{1-1}} \varphi_1(x)dx = \int \varphi_1(x)dx = 1$$

(Note: since φ_1 has support $(2^{-1}, 2^{1-1})$, the above integral is valid).

So, this shows that integrating x or y in different order actually causes a difference for this function.