Math 118C HW4

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Question 1 *Rudin Pg. 242 Problem 27:*
Put $f(0,0) = 0$, *and*

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x,y) \neq (0,0)$. *Prove that*

(a) $f,\ D_1 f,\ D_2 f$ *are continuous in* $\mathbb{R}^2$.

(b) $D_{12}f$ *and* $D_{21}f$ *exist at every point of* $\mathbb{R}^2$, *and are continuous except at* $(0,0)$.

(c) $D_{12}f(0,0) = 1$, *and* $D_{21}f(0,0) = -1$.

Pf:

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Question 2 *Rudin Pg. 242 Problem 28:*
For $t \geq 0$, *put*

$$\varphi(x,t) = \begin{cases} x & 0 \leq x \leq \sqrt{t} \\ -x + 2\sqrt{t} & \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0 & \text{otherwise} \end{cases}$$

and put $\varphi(x,t) = -\varphi(x,|t|)$ *if* $t < 0$.
Show that φ *is continuous on* $\mathbb{R}^2$, *and* $D_2\varphi(x,0) = 0$ *for all* x. *Define*

$$f(t) = \int_{-1}^{1} \varphi(x,t)dx$$

Show that $f(t) = t$ *if* $|t| < \frac{1}{4}$. *Hence*

$$f'(0) \neq \int_{-1}^{1} D_2\varphi(x,0)dx$$

Pf:

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Question 3 *Rudin Pg. 243 Problem 30:*
Let $f \in \mathcal{C}^{(m)}(E)$, where E is an open subset of $\mathbb{R}^n$. Fix $a \in E$, and suppose $x \in \mathbb{R}^n$ is so close to 0 that the points $p(t) = a + tx$ lie in E whenever $0 \leq t \leq 1$. Define $h(t) = f(p(t))$ for all $t \in \mathbb{R}$ for which $p(t) \in E$.

(a) For $1 \leq k \leq m$, show (by repeated application of the chain rule) that

$$h^{(k)}(t) = \sum (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

The sum extends over all order k-tuples $(l_1, \dots, l_k)$ in which each l_j is one of the integers $1, \dots, n$.

Pf:

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Question 4 *Rudin Pg. 288 Problem 2:*
For $i = 1, 2, 3, ...,$ *let* $\varphi_i \in \mathcal{C}(\mathbb{R})$ *have support in* $(2^{-i}, 2^{1-i})$, *such that* $\int \varphi_i = 1$. *Put*

$$f(x,y) = \sum_{i=1}^{\infty} (\varphi_i(x) - \varphi_{i+1}(x))\varphi_i(y)$$

Then f *has compact support in* $\mathbb{R}^2$, f *is cotinuous except at* $(0,0)$, *and*

$$\int dy \int f(x,y)dx = 0, \quad \textit{but} \int dx \int f(x,y)dy = 1$$

Observe that f *is unbounded in every neighborhood of* $(0,0)$.

Pf: