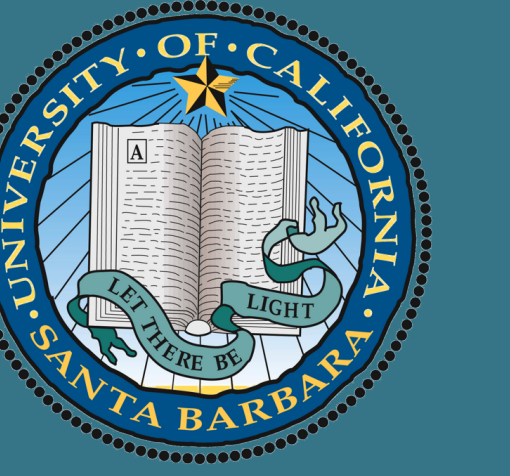


# LIE ALGEBRA OF A LIE GROUP

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## Tangent Vectors as Derivations

When embedding smooth manifolds into  $\mathbb{R}^n$ , tangent vectors are associated with directional derivatives. To generalize tangent vectors into abstract smooth manifold, we need an analogy:

### Definition

Any point  $u \in M$ , a **Derivation at  $u$** , is a linear map  $v_u : C^\infty(M) \rightarrow \mathbb{R}$ , that satisfies the product rule:

$$\forall f, g \in C^\infty(M), \quad v_u(fg) = f(u)(v_u g) + g(u)(v_u f)$$

The vector space of all derivations at  $u$ , or  $T_u(M)$ , is the **Tangent Space** of  $M$  at  $u$ , and each derivation  $v_u \in T_u(M)$  is a **Tangent Vector** of  $u$ .

## Vector Fields & Smooth Condition

### Definition

a vector field is a map  $X : M \rightarrow TM$  ( $TM$  denotes the **Tangent Bundle**), with  $X(u) = X_u \in T_u(M)$ . Which,  $X$  is a **Smooth Vector Field**, if  $X : M \rightarrow TM$  is a smooth map. A collection of smooth vector fields on  $M$  is denoted as  $\mathfrak{X}(M)$ , which is an  $\mathbb{R}$ -vector space.

An equivalent condition of saying a vector field  $X$  is smooth, is through smooth functions  $f \in C^\infty(M)$ : For all  $u \in M$ ,  $X(u) = X_u \in T_u(M)$  is a derivation at  $u$ , define  $Xf : M \rightarrow \mathbb{R}$  by  $Xf(u) = X_u(f)$ , then  $X$  is a smooth vector field iff  $Xf \in C^\infty(M)$ . Which, a smooth vector field can be viewed as a **Derivation**:

### Theorem

For all  $f, g \in C^\infty(M)$ , given  $X \in \mathfrak{X}(M)$ , any  $u \in M$  satisfies product rule:

$$\begin{aligned} X(fg)(u) &= X_u(fg) = f(u)(X_u g) + g(u)(X_u f) = f(u)Xg(u) + g(u)Xf(u) \\ \implies X(fg) &= f(Xg) + g(Xf) \end{aligned}$$

## Vector Fields of Different Manifolds

Given  $M, N$  two smooth manifolds, and smooth map  $F : M \rightarrow N$ . Let  $X \in \mathfrak{X}(M)$ , an ideal situation is mapping  $X$  to a smooth vector field of  $N$  through  $F$ . Yet, this requires  $F$  to be bijective:

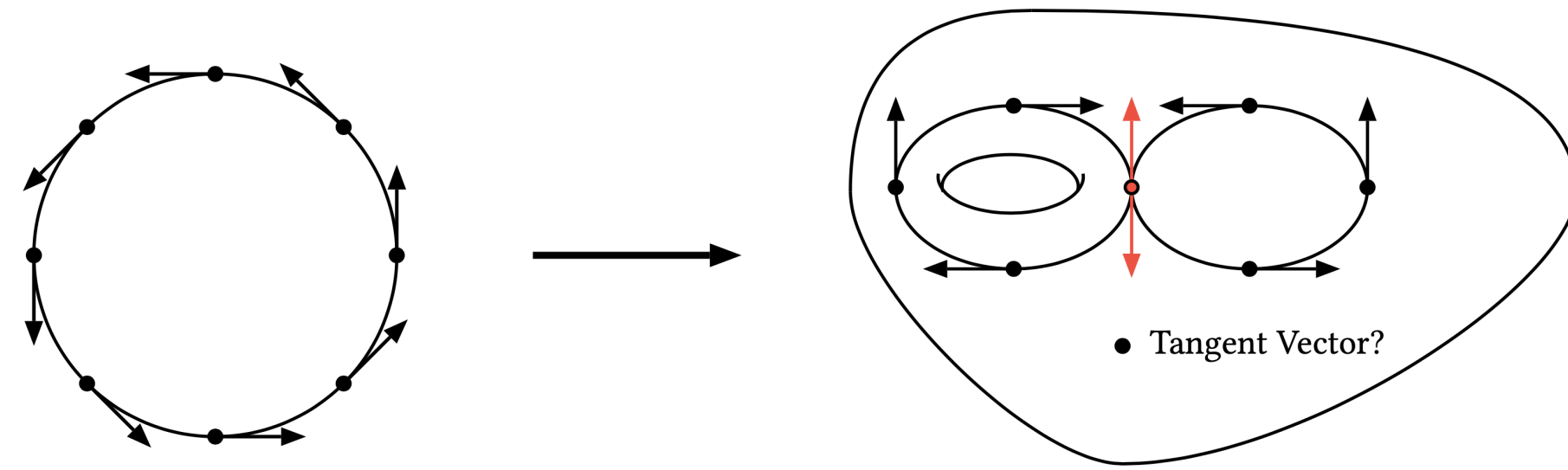


Figure 1: Example of Conflicting Tangent Vectors

So, we'll consider a weaker notion:

### Definition

Given  $X \in \mathfrak{X}(M)$  and  $Y \in \mathfrak{X}(N)$ , the two are **F-related**, if for all  $u \in M$ , the following is true:

$$dF_u(X_u) = Y_{F(u)}$$

Simply speaking,  $F$  maps the tangent vectors collected by  $X$ , to be compatible with tangent vectors collected by  $Y$ .

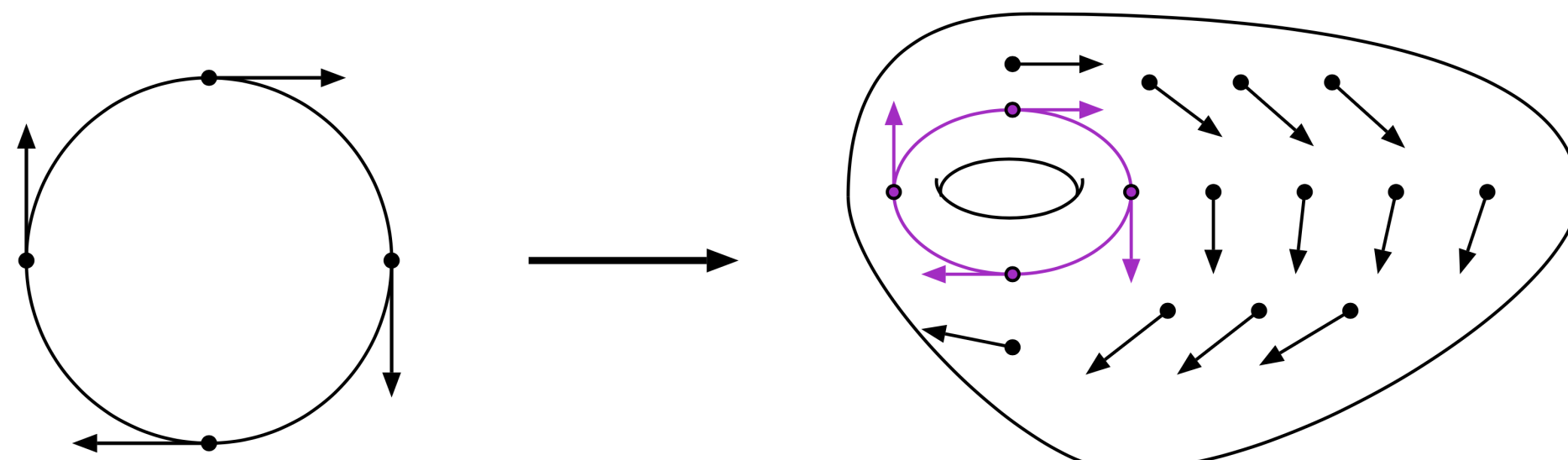


Figure 2: A demonstration of  $F$ -Relation

## Lie Bracket of Vector Fields

The initial motivation is to combine two vector fields  $X, Y \in \mathfrak{X}(M)$  to be another vector field. For all  $f \in C^\infty(M)$ , since  $Yf \in C^\infty(M)$ , then  $XYf := X(Yf) \in C^\infty(M)$ . But, in general  $XY$  is not a derivation, hence not a vector field:

### Example

Define vector fields  $X = \frac{\partial}{\partial x}$ ,  $Y = x \frac{\partial}{\partial y}$  on  $\mathbb{R}^2$ . Take smooth functions  $f(x, y) = x$  and  $g(x, y) = y$ , then we get the following:

$$XY(fg) = X \left( x \frac{\partial}{\partial y}(xy) \right) = \frac{\partial}{\partial x}(x^2) = 2x$$

But, product rule doesn't hold for this example:

$$f(XYg) + g(XYf) = x \left( X \left( x \frac{\partial}{\partial y}(y) \right) \right) + y \left( X \left( x \frac{\partial}{\partial y}(x) \right) \right) = x$$

So, we need to define a new operation on vector fields:

### Definition

The **Lie Bracket**  $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ , is defined as:

$$\forall X, Y \in \mathfrak{X}(M), \quad [X, Y] = XY - YX$$

Which, the output  $[X, Y] \in \mathfrak{X}(M)$ , since it satisfies product rule:

$$\begin{aligned} [X, Y](fg) &= X(Y(fg)) - Y(X(fg)) = X(f(Yg) + g(Yf)) - Y(f(Xg) + g(Xf)) \\ &= f(XYg) + (Yg)(Xf) + g(XYf) + (Yf)(Xg) - f(YXg) - (Xg)(Yf) - g(YXf) - (Xf)(Yg) \\ &= f(XYg - YXg) + g(XYf - YXf) = f[X, Y](g) + g[X, Y](f) \end{aligned}$$

Lie Bracket also satisfies these properties:

- **Bilinearity**:  $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- **Antisymmetry**:  $[X, Y] = -[Y, X]$
- **Jacobi's Identity**:  $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Moreover, Lie Bracket inherits relation of smooth maps:

### Theorem

Given smooth map  $F : M \rightarrow N$ , if  $X_1, X_2 \in \mathfrak{X}(M)$  and  $Y_1, Y_2 \in \mathfrak{X}(N)$  are  $F$ -related respectively, then  $[X_1, X_2] \in \mathfrak{X}(M)$  and  $[Y_1, Y_2] \in \mathfrak{X}(N)$  are also  $F$ -related.

## Lie Groups & Left-Invariant Vector Fields

The initial motivation is to study group structures in some smooth manifolds.

### Definition

A **Lie Group**  $G$ , is a smooth manifold along with group structure, such that the group operation  $P : G \times G \rightarrow G$  by  $P(g, h) = gh$ , and the inversion map  $i : G \rightarrow G$  by  $i(g) = g^{-1}$  are both smooth maps between manifolds.

For all  $g \in G$ , denote the left multiplication  $L_g : G \rightarrow G$  by  $L_g(h) = gh$ , since  $L_g = P|_{\{g\} \times G}$ , it is a smooth map. Hence, there's a notion of  $X$  being  $L_g$ -related to itself:

### Definition

Given any  $X \in \mathfrak{X}(G)$  and all  $g \in G$ ,  $X$  is a **Left-Invariant Vector Field**, if for all  $g \in G$ ,  $X$  is  $L_g$ -related to itself. Which, for all  $g \in G$ :

$$d(L_g)_e(X_e) = X_{L_g(e)} = X_g$$

So,  $X$  is uniquely determined by its tangent vector at identity,  $X_e \in T_e(G)$ . The collection of Left-Invariant vector fields  $\mathfrak{g} \subseteq \mathfrak{X}(G)$ , is a linear subspace.

Recall that Lie Bracket of vector field preserves  $F$ -relation between manifolds, so:

### Theorem

For all  $X, Y \in \mathfrak{X}(G)$  that are left-invariant, since for all  $g \in G$ ,  $X$  and  $Y$  are  $L_g$  related to themselves, then the Lie Bracket  $[X, Y]$  is also  $L_g$ -related to  $[X, Y]$ . Hence,  $[X, Y]$  is also left-invariant, so  $\mathfrak{g}$  is closed under Lie Bracket's operation.

## Lie Algebra on a Lie Group

### Definition

Given a vector space  $\mathfrak{g}$  over  $\mathbb{R}$  or  $\mathbb{C}$ , with a binary operation  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ , such that the following holds:

- **Bilinearity**:  $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- **Antisymmetry**:  $[X, Y] = -[Y, X]$
- **Jacobi's Identity**:  $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Then, the pair  $(\mathfrak{g}, [\cdot, \cdot])$  is a **Lie Algebra**.

In general, Lie Algebra is non-associative, so Jacobi's Identity is an alternative condition. Finally, we can define **Lie Algebra of a Lie Group**:

### Definition

Given a lie group  $G$ , since the subset of left-invariant vector fields  $\mathfrak{g} \subseteq \mathfrak{X}(G)$  forms a linear subspace, while closed under Lie Bracket's operation, then the pair  $(\mathfrak{g}, [\cdot, \cdot])$  forms a **Lie Algebra** of  $G$ , denoted as  $Lie(G)$ .

Here's an example of Lie Algebra on a Lie Group:

### Example

**General Linear Group & its Lie Algebra**:

Given  $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ , since  $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$  and  $GL_n(\mathbb{R})$  is an open subset, it's a smooth manifold with dimension  $n^2$ . Which, the product of matrices and inversion are smooth functions, so  $GL_n(\mathbb{R})$  is a Lie Group.

Now, consider  $\mathfrak{g} = Lie(GL_n(\mathbb{R}))$ : Each Left-Invariant vector field  $X \in \mathfrak{g}$  is uniquely characterized by  $X_{I_n} \in T_{I_n}(GL_n(\mathbb{R}))$ . In fact, such characterization is a 1-to-1 correspondence. So, as vector spaces,  $\mathfrak{g} \cong T_{I_n}(GL_n(\mathbb{R}))$ .

**Lie Algebra on  $M_n(\mathbb{R})$** :

Given  $M_n(\mathbb{R})$  as  $\mathbb{R}$ -vector space and the commutator  $[A, B] = AB - BA$ , the pair  $(M_n(\mathbb{R}), [\cdot, \cdot])$  in fact forms a Lie Algebra, denoted as  $\mathfrak{gl}_n(\mathbb{R})$ .

**Lie Algebra Isomorphism between  $\mathfrak{g}$  and  $\mathfrak{gl}_n(\mathbb{R})$** :

$GL_n(\mathbb{R})$  has a global coordinate provided by  $M_n(\mathbb{R})$ , denote as  $(X_j^i)_{1 \leq i, j \leq n}$ . For each  $A \in \mathfrak{gl}_n(\mathbb{R})$ , it corresponds to a tangent vector in  $T_{I_n}(GL_n(\mathbb{R}))$ :

$$A = (A_j^i) \mapsto A_j^i \frac{\partial}{\partial X_j^i} \Big|_{I_n}$$

Which, the above tangent vector defines a Left-Invariant vector field  $A^L \in \mathfrak{g}$ . For all  $X \in \mathfrak{g}$ , the left multiplication  $L_X$  provides the following relation:

$$A_X^L = d(L_X)_{I_n} \left( A_j^i \frac{\partial}{\partial X_j^i} \Big|_{I_n} \right) = X_j^i A_k^j \frac{\partial}{\partial X_k^i} \Big|_X, \quad A^L = X_j^i A_k^j \frac{\partial}{\partial X_k^i}$$

Which, for arbitrary  $A, B \in \mathfrak{gl}_n(\mathbb{R})$ , Lie Bracket of  $A^L, B^L \in \mathfrak{g}$  generates:

$$[A^L, B^L] = X_j^i A_k^j \frac{\partial}{\partial X_k^i} (X_q^p B_r^q) \frac{\partial}{\partial X_r^p} - X_q^p B_r^q \frac{\partial}{\partial X_r^p} (X_j^i A_k^j) \frac{\partial}{\partial X_k^i}$$

Which, each  $A_k^j, B_r^q$  are constants, while  $\frac{\partial}{\partial X_k^i} X_r^p = 1$  iff  $(i, k) = (p, r)$  and is 0 otherwise. Then, match  $j = q$  for the same intermediate index, we get:

$$[A^L, B^L] = X_j^i (A_k^j B_r^q - B_k^j A_r^q) \frac{\partial}{\partial X_r^p} = (AB - BA)^L = [A, B]^L$$

Hence,  $\mathfrak{gl}_n(\mathbb{R})$  and  $\mathfrak{g}$  are isomorphic as Lie Algebra, since  $A \mapsto A^L$  preserves Lie Bracket.

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**Reference**: Lee, J.M. *Introduction to Smooth Manifolds*; 2nd ed.; Springer: New York, 2012; 9781441999825