Math CS 122B HW8 Part 2

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Question 1 *Stein and Shakarchi Pg. 200-201 Exercise 4:*
Suppose $\{a_n\}_{n\in\mathbb{N}}$ *is a sequence of complex numbers such that* $a_n = a_m$ *iff* $n \equiv m \mod q$ *for some positive integer* q. *Define the* ***Dirichlet*** L***-series*** *associated to* $\{a_n\}$ *by*

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \textit{for } Re(s) > 1$$

Also, with $a_0 = a_q$, *let*

$$Q(x) = \sum_{m=0}^{q-1} a_{q-m} e^{mx}$$

Show, as in Exercises 15 *and* 16 *of the previous chapter, that*

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{Q(x)x^{s-1}}{e^{qx}-1} dx \quad \textit{for } Re(s) > 1$$

Prove as a result that $L(s)$ *is continuable into the complex plane, with the only possible singularity a pole at* $s = 1$. *In fact,* $L(s)$ *is regular at* $s = 1$ *if and only if* $\sum_{m=0}^{q-1} a_m = 0$. *Note the conection with the Direchlet* $L(s,\chi)$ *series, taken up to BOok I Chapter 8, and that as a consequence,* $L(s,\chi)$ *is regular at* $s = 1$ *if and only if* χ *is a non-trivial character.*

Pf:

1.1 Integral Representation of $L(s)$:

Given $\mathrm{Re}(s) > 1$, and $x \in (0,\infty)$, notice that $\frac{1}{e^{qx}-1} = \frac{e^{-qx}}{1-e^{-qx}}$, with the fact that $-qx < 0$, then $e^{-qx} < 1$. Hence, the following expression is absolutely convergent, and converging normally for any compact subset of $(0,\infty)$:

$$\frac{1}{e^{qx}-1} = \frac{e^{-qx}}{1-e^{-qx}} = \sum_{n=1}^{\infty} (e^{-qx})^n \tag{1}$$

Since it converges normally within any compact subset of $(0, \infty)$ (the domain of integration), then the integral expression in the question can be rewritten as:

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx &= \frac{1}{\Gamma(s)} \int_0^\infty Q(x)x^{s-1} \left(\sum_{n=1}^\infty e^{-qx} \right) dx \\ &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \int_0^\infty \left(\sum_{m=0}^{q-1} a_{q-m} e^{mx} \right) x^{s-1} \cdot e^{-nqx} dx \\ &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \sum_{m=0}^{q-1} a_{q-m} \int_0^\infty x^{s-1} e^{-(nq-m)x} dx \end{aligned} \tag{2}$$

Which, by swapping $r = q - m$ (where r ranges from 1 to q), extending from (2), we get the following:

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \sum_{r=1}^{q} a_r \int_0^\infty x^{s-1} e^{-(nq-(q-r))x} dx \\ &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \sum_{r=1}^{q} a_r \int_0^\infty x^{s-1} e^{-((n-1)q+r)x} dx \\ &= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^{q} a_r \int_0^\infty x^{s-1} e^{-(nq+r)x} dx \end{aligned} \tag{3}$$

Then, performing substitution $u = (nq + r)x$ for each index n and r, $du = (nq + r)dx$, which (3) becomes:

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx &= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^{q} a_r \int_0^\infty \left(\frac{u}{nq+r} \right)^{s-1} \cdot e^{-u} \frac{du}{nq+r} \\ &= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^{q} a_r \cdot \frac{1}{(nq+r)^s} \int_0^\infty u^{s-1} e^{-u} du \\ &= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^{q} \frac{a_r}{(nq+r)^s} \cdot \Gamma(s) = \sum_{n=0}^\infty \sum_{r=1}^{q} \frac{a_r}{(nq+r)^s} \end{aligned} \tag{4}$$

Now, in terms of the original $L(s)$, recall that $a_n = a_m$ iff $n \equiv m \mod q$, so the original series expression can be rearranged as:

$$\begin{aligned} L(s) &= \sum_{k=1}^\infty \frac{a_k}{k^s} = \sum_{n=1}^\infty \frac{a_{nq}}{(nq)^s} + \sum_{n=0}^\infty \sum_{r=1}^{q-1} \frac{a_{nq+r}}{(nq+r)^s} \\ &= \sum_{n=0}^\infty \frac{a_q}{(nq+q)^s} + \sum_{n=0}^\infty \sum_{r=1}^{q-1} \frac{a_r}{(nq+r)^s} = \sum_{n=0}^\infty \sum_{r=1}^{q} \frac{a_r}{(nq+r)^s} \end{aligned} \tag{5}$$

Then, combining the results in (4) and (5), we get $L(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx$ (for $\mathrm{Re}(s) > 1$).

1.2 Continuation to $\mathbb{C} \setminus \{1\}$:

With the above integral expression for $\mathrm{Re}(s) > 1$, one can separate the integration as follow:

$$\begin{aligned} L_1(s) :=& \frac{1}{\Gamma(s)} \int_0^1 \frac{q(x)x^{s-1}}{e^{qx}-1}dx, \quad L_2(s) := \frac{1}{\Gamma(s)} \int_1^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx \\ L(s) =& \frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx = L_1(s) + L_2(s) \end{aligned} \tag{6}$$

Since $Q(x) = \sum_{m=0}^{q-1} a_{q-m} e^{mx}$, it is with the order of $e^{(q-1)x}$. Then, for $x > 1$ and $\mathrm{Re}(s) > 1$, since $qx > 1$, then $e^{qx} > e > 2$, so $\frac{1}{2}e^{qx} > 1$. Then, $L_2(s)$ satisfies the following inequality:

$$|L_2(s)| \leq \frac{1}{|\Gamma(s)|} \int_1^{\infty} \frac{|Q(x)| \cdot |x^{s-1}|}{|e^{qx} - 1|} dx \leq \frac{1}{|\Gamma(s)|} \int_1^{\infty} \frac{K e^{(q-1)x} \cdot x^{\mathrm{Re}(s)-1}}{e^{qx} - 1} dx \tag{7}$$

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Question 2 *Stein and Shakarchi Pg. 204 Problem 4:*
One can combine ideas from the prime number theorem with the proof of Dirichlet's Theorem about primes in arithmetic progression (given in Book I) to prove the following: Let q *and* l *be relatively prime integers. We consider the primes belonging to the arithmetic progression* $\{qk + ll\}_{k\in\mathbb{N}}$*, and let* $\pi_{q,l}(x)$ *denote the number of such primes* $\leq x$*. Then one has*

$$\pi_{q,l}(x) \sim \frac{x}{\varphi(q)\log(x)} \quad \text{as } x \to \infty$$

where $\varphi(q)$ *denotes the number of positive integers less than* q *and relatively prime to* q *(i.e. the Euler Totient Function).*

Pf: