Math 111C HW8

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Question 1 *Show that every finite separable extension K/F has only finitely many sub-extensions.*

Pf:

Since K/F is a finite separable extension, there exists $\alpha_1, ..., \alpha_n \in K$, such that $F(\alpha_1, ..., \alpha_n) = K$ (and all α_i are separable over F by definition).

Which, fix $\overline{F}$ such that $K \subseteq \overline{F}$, and take $A = \{m_{\alpha_1,F}(x), ..., m_{\alpha_n,F}(x)\} \subset F[x]$. Consider $L \subseteq \overline{F}$ to be the splitting field of A, then since each polynomial in A must split completely over L, it must necessarily contain all the roots of all polynomials in A; on the other hand, since each $\alpha_i \in K \subseteq \overline{F}$ is a root of $m_{\alpha_i,F}(x) \in A$, then $\alpha_i \in L$, hence $K = F(\alpha_1, ..., \alpha_n) \subseteq L$.

Now, notice that L is a splitting field of $A \subseteq F[x]$, hence L/F is a normal extension;

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Question 2 *Let $L \subseteq \mathbb{C}$ be the splitting field of $f(x) = x^3 - 3x + 1$ over $\mathbb{Q}$. Let $\alpha, \beta, \gamma \in L$ be roots of $f(x)$.*

(a) Calculate Gal$(L/\mathbb{Q})$ as a group of permutations of $\{\alpha, \beta, \gamma\}$.

(b) Is there an automorphism of L that acts on $\{\alpha, \beta, \gamma\}$ as the transposition (α, β)?

*(**Hint:-** For a polynomial $f(x) = x^3 + ax^2 + bx + c \in \mathbb{Q}[x]$ wth roots $\alpha, \beta, \gamma \in \mathbb{C}$, the discriminant of $f(x)$, D is defined as*

$$D = (\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2$$

It is known that $D = 18abc + a^2b^2 - 4b^3 - 4a^3c - 27c^2$).

Pf:

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Question 3 *Repeat the above question with* $f(x) = x^3 - 4x + 1$.

Pf:

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Question 4 *Let $L \subseteq \mathbb{C}$ be the splitting field of $f(x) = (x^3 - 2)(x^2 - 3)$ over $\mathbb{Q}$.*

(a) Show that $L = \mathbb{Q}(\sqrt{2} + \sqrt{3})$ and $[L : \mathbb{Q}] = 4$.

(b) Find $Gal(L/\mathbb{Q})$ as a group of permutations of the roots of f.

(c) Which elements of your ansewr to (b) belong to the subgroup $Gal(L/\mathbb{Q}(\sqrt{6}))$?

Pf:

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Question 5 *The Galois group of a polynomial $f(x)$ over a perfect field F is defined as $Gal(K/F)$ where K is a splitting field of $f(x)$. Find the Galois groups of $x^6 - 1$ over $\mathbb{F}_5$, $\mathbb{F}_{5^2}$, and $\mathbb{F}_{5^3}$.*

Pf: