Math CS 122B HW3

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Question 1 *Freitag Chap. IV.3 Exercise 3:*
Show:

$$\frac{\pi}{\cos(\pi z)} = 4\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2-4z^2}$$

and derive from this

$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...$$

Pf:

We'll complete this by the following trigonometric identity, and the expression of $\frac{\pi}{\sin(\pi\zeta)}$ under partial fraction series:

$$\cos(\zeta) = \sin\left(\frac{\pi}{2}-\zeta\right)$$

$$\frac{\pi}{\sin(\pi\zeta)} = \frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\zeta-n}+\frac{1}{\zeta+n}\right)$$

Then, $\frac{\pi}{\cos(\pi z)}$ can be expressed as:

$$\frac{\pi}{\cos(\pi z)} = \frac{\pi}{\sin(\pi/2-\pi z)} = \frac{\pi}{\sin(\pi(1/2-z))} = \frac{1}{1/2-z}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{(1/2-z)-n}+\frac{1}{(1/2-z)+n}\right)$$

$$= \frac{2}{1-2z}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{2}{(1-2z)-2n}+\frac{2}{(1-2z)+2n}\right)$$

$$= \frac{2}{1-2z}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{2}{-(2n-1)-2z}+\frac{2}{(2n+1)-2z}\right)$$

Which, for all $z\notin\frac{1}{2}+\mathbb{Z}$, if we view the partial sum of the above series, we get:

$$\forall N\in\mathbb{N},\ N\geq 2,\quad S_N = \frac{2}{1-2z}+\sum_{n=1}^{N}(-1)^n\left(\frac{2}{-(2n-1)-2z}+\frac{2}{(2n+1)-2z}\right)$$

$$= \frac{2}{1-2z}+\frac{(-1)^1\cdot 2}{-(2\cdot 1-1)-2z}+\sum_{n=2}^{N}\frac{(-1)^n\cdot 2}{-(2n-1)-2z}+\sum_{n=1}^{N-1}\frac{(-1)^n\cdot 2}{(2n+1)-2z}+\frac{(-1)^N\cdot 2}{(2N+1)-2z}$$

$$= \frac{2}{1-2z}-\frac{2}{-1-2z}+\sum_{n=1}^{N-1}\frac{(-1)^{n+1}\cdot 2}{-(2(n+1)-1)-2z}+\sum_{n=1}^{N-1}\frac{(-1)^n\cdot 2}{(2n+1)-2z}+\frac{(-1)^N\cdot 2}{(2N+1)-2z}$$

$$
\begin{aligned}
&= \left(\frac{2}{1-2z} - \frac{2}{-1-2z}\right) + \sum_{n=1}^{N-1}(-1)^n\left(\frac{2}{(2n+1)-2z} - \frac{2}{-(2n+1)-2z}\right) + \frac{(-1)^N\cdot 2}{(2N+1)-2z} \\
&= \sum_{n=0}^{N-1}(-1)^n\left(\frac{2}{(2n+1)-2z} - \frac{2}{-(2n+1)-2z}\right) + \frac{(-1)^N\cdot 2}{(2N+1)-2z} \\
&= \sum_{n=0}^{N-1} 2\cdot(-1)^n\cdot\frac{(-(2n+1)-2z)-((2n+1)-2z)}{4z^2-(2n+1)^2} + \frac{(-1)^N\cdot 2}{(2N+1)-2z} \\
&= 2\sum_{n=0}^{N-1}(-1)^n\cdot\frac{-2(2n+1)}{4z^2-(2n+1)^2} + \frac{(-1)^N\cdot 2}{(2N+1)-2z} \\
&= 4\sum_{n=0}^{N-1}(-1)^n\cdot\frac{(2n+1)}{(2n+1)^2-4z^2} + \frac{(-1)^N\cdot 2}{(2N+1)-2z}
\end{aligned}
$$

So, we get:

$$
\begin{aligned}
\lim_{N\to\infty} s_N &= \lim_{N\to\infty} 4\sum_{n=0}^{N-1}(-1)^n\cdot\frac{(2n+1)}{(2n+1)^2-4z^2} + \frac{(-1)^N\cdot 2}{(2N+1)-2z} \\
&= 4\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2-4z^2}
\end{aligned}
$$

(Note: The above series converges, because before modifying the series, the partial sum already converges, and our modification provides the same sum for each $N \in \mathbb{N}$).

Hence, we can conclude the following:

$$
\frac{\pi}{\cos(\pi z)} = 4\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2-4z^2}
$$

Now, based on this formula, plugging in $z = 0$, we get the following:

$$
\pi = \frac{\pi}{\cos(\pi\cdot 0)} = 4\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2-4\cdot 0^2} = 4\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2} = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}
$$

Hence, we get the following expression of $\frac{\pi}{4}$:

$$
\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}
$$

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Question 2 *Freitag Chap. IV.3 Exercise 4:*
Find a meromorphic function f in $\mathbb{C}$ which has simple poles in

$$S = \{\sqrt{n} \mid n \in \mathbb{N}\}$$

with corresponding residues $Res(f; \sqrt{n}) = \sqrt{n}$, and is analytic in $\mathbb{C} \setminus S$.

Pf:

With the given condition, one could guess that for each $n \in \mathbb{N}$, at $z = \sqrt{n}$, the principal part is described using $\frac{\sqrt{n}}{z-\sqrt{n}}$ (which is a simple pole, and has residue $\lim_{z\to\sqrt{n}}(z-\sqrt{n})\frac{\sqrt{n}}{(z-\sqrt{n})} = \sqrt{n}$). However, the series of such function potentially diverges, hence we need to do some modification.

For all $z \in \mathbb{C} \setminus \{\sqrt{n} \mid n \in \mathbb{N}\}$, there exists $N \in \mathbb{N}$, such that $n \geq N$ implies $\frac{|z|}{|\sqrt{n}|} \leq \frac{1}{2}$ (which, we're working within the compact disk $|z| \leq \frac{\sqrt{N}}{2}$). Then, for $n \geq N$, since $\frac{z}{\sqrt{n}}$ is within the radius of convergence of the geometric series (since $\left|\frac{z}{\sqrt{n}}\right| < 1$), we get:

$$\frac{\sqrt{n}}{z-\sqrt{n}} = \frac{-1}{1-z/\sqrt{n}} = -\sum_{k=0}^{\infty}\left(\frac{z}{\sqrt{n}}\right)^k$$

Then, if we subtract out the terms up to degree 3, we get the following:

$$\frac{\sqrt{n}}{z-\sqrt{n}} + \sum_{k=0}^{3}\left(\frac{z}{\sqrt{n}}\right)^k = -\sum_{k=0}^{\infty}\left(\frac{z}{\sqrt{n}}\right)^k + \sum_{k=0}^{3}\left(\frac{z}{\sqrt{n}}\right)^k = -\sum_{k=4}^{\infty}\left(\frac{z}{\sqrt{n}}\right)^k = -\left(\frac{z}{\sqrt{n}}\right)^4\sum_{k=0}^{\infty}\left(\frac{z}{\sqrt{n}}\right)^k$$

Which, compare the modulus, we get the following inequality:

$$\left|\frac{\sqrt{n}}{z-\sqrt{n}} + \sum_{k=0}^{3}\left(\frac{z}{\sqrt{n}}\right)^k\right| = \left|\left(\frac{z}{\sqrt{n}}\right)^4\sum_{k=0}^{\infty}\left(\frac{z}{\sqrt{n}}\right)^k\right| \leq \frac{|z|^4}{n^2}\sum_{k=0}^{\infty}\left|\frac{z}{\sqrt{n}}\right|^k$$

$$\leq \frac{(\sqrt{N}/2)^4}{n^2}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{N^2/(16)}{n^2}\cdot 2 = \frac{N^2}{8n^2}$$

Hence, the following series of functions converges uniformly within the compact disk $|z| \leq \frac{\sqrt{N}}{2}$:

$$\left|\sum_{n=N}^{\infty}\left(\frac{\sqrt{n}}{z-\sqrt{n}} + \sum_{k=0}^{3}\left(\frac{z}{\sqrt{n}}\right)^k\right)\right| \leq \sum_{n=N}^{\infty}\left|\frac{\sqrt{n}}{z-\sqrt{n}} + \sum_{k=0}^{3}\left(\frac{z}{\sqrt{n}}\right)^k\right| \leq \sum_{n=N}^{\infty}\frac{N^2}{8n^2} < \infty$$

So, we can conclude that $\sum_{n=1}^{\infty}\left(\frac{\sqrt{n}}{z-\sqrt{n}} + \sum_{k=0}^{3}\left(\frac{z}{\sqrt{n}}\right)^k\right)$ converges normally on $\mathbb{C} \setminus \{\sqrt{n}\}_{n\in\mathbb{N}}$.

Since each $\sqrt{n}$, $n \in \mathbb{N}$ has the principal part given by $\frac{\sqrt{n}}{z-\sqrt{n}}$, while this principal part satisfies the desired properties, then this partial fraction series (which is a meromorphic function in this case) is a solution of the principal part distribution (simple poles at each $\sqrt{n}$, while having residue $\sqrt{n}$).

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Question 3 *Freitag Chap. IV.3 Exercise 5:*
Prove the following refinement of the Mittag-Leffler Theorem:

Theorem 1 *Mittag-Leffler Let $S \subset \mathbb{C}$ be a discrete subset. Then one can construct an analytic function $f : \mathbb{C} \setminus S \to \mathbb{C}$ which has at any $s \in S$, not only given principal parts but also finitely many Laurent coefficients for nonnegative indices.*

i.e. For each point, finitely many lauent coefficients with nonnegative indices are predetermined.

Pf:

For every $s \in S$, if we want to construct a function with the given principal parts and finitely many laurent coefficients for nonnegative indices being predetermined, then the goal is to create h (a partial fraction series) and g (a Weierstrass product), such that their product $f = hg$ provides a laurent series at s, with the first several determined coefficients being $a_N, a_{N+1}, ..., a_M$, where N is ther order of the pole of f at s (so every coefficient a_n with $n < N$ is 0), and N, M are dependent on s.

The Weierstrass Product g:

For each $s \in S$, the largest index of the predetermined coefficient is M (dependent on s).

If $M < 0$, we simply don't include this point as a zero for g (so the Taylor Series of g about s has nonzero constant term).

Else if $M \geq 0$, include s as a zero of g, with order being $(M + 1)$ (provide higher degrees for the product hg to construct all the predetermined a_n with $n \geq 0$).

Construction of Principal Parts for h:

For fixed $s \in S$, g has a Taylor Series about s being $\sum_{k=m}^{\infty} b_k(z - s)^k$, where b_m (with $m \geq 0$) is the first nonzero coefficient (so from the previous part, if $M < 0$ for given s, then $m = 0$; else if $M \geq 0$, then $m = M + 1$. Which, $m > M$ for each $s \in S$).

Now, we can construct the principal part of h at s, described by $\sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n}$ (where $\{c_n\}_{n=N-m}^{-1}$ are yet to be determined).

Our goal is to let the product $(\sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n})(\sum_{k=m}^{\infty} b_k(z - s)^k)$ (which has all coefficients $n \geq N$) to produce $a_N, a_{N+1}, ..., a_M$ as the first several coefficients. Which, for $N \leq n \leq M$, it suffices to solve the following equation (with all c_u being unknown variables):

$$\sum_{u+v=n} c_u b_v = a_n, \quad m \leq v \leq n - u, \quad N - m \leq u \leq n - m$$

For $n = N$, the only choice is $u = (N - m)$ and $v = m$ (since all other $u > (N - m)$ and $v > m$, hence $u + v > N$), so $c_{N-m} b_m = a_N$, or $c_{N-m} = a_N / b_m$.

Then, for $N < n \leq M$, we can recursively solve the expression for each c_{n-m} (since each equation about a_n only has finitely many b_v involved, while dependent on $c_{N-m}, ..., c_{n-m}$, while the coefficients before c_{n-m} are solved by previous steps).

The remaining argument to make is why this determines $a_N, ..., a_M$ as the first several Laurent coefficients of $f = hg$ when expanding about s.

For each $s \in S$, the principal part for s is $\sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n}$ provided above, hence $h(z) - \sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n}$ can be extended analytically to s, which has Taylor Series (within some radius of convergence) as follow:

$$h(z) - \sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n} = \sum_{v=0}^{\infty} c_v (z-s)^v$$

$$\implies h(z) = \left(\sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n} \right) + \left(\sum_{v=0}^{\infty} c_v (z-s)^v \right)$$

Then, with $g(z) = \sum_{k=m}^{\infty} b_k (z-s)^k$, the Laurent Series of $f = hg$ about s is given as:

$$f(z) = \left[\left(\sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n} \right) + \left(\sum_{v=0}^{\infty} c_v (z-s)^v \right) \right] \left(\sum_{k=m}^{\infty} b_k (z-s)^k \right)$$

$$= \left(\sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n} \right) \left(\sum_{k=m}^{\infty} b_k (z-s)^k \right) + \left(\sum_{v=0}^{\infty} c_v (z-s)^v \right) \left(\sum_{k=m}^{\infty} b_k (z-s)^k \right)$$

Which, the product on the left provides the first several coefficients to be $a_N, ..., a_M$ based on our construction, while the product on the right provides coefficients for degree $v+k$, with $v \geq 0$ and $k \geq m$ (so $v+k \geq m > M$).

So, the product on the right only affects coefficients with index $n > M$, hence the coefficients with index $n \leq M$ are all determined by the product on the left, showing that the first several coefficients are indeed $a_N, ..., a_M$.

Hence, it's possible to construct such function $f : \mathbb{C} \setminus S \to \mathbb{C}$, such that at each $s \in S$, finitely many laurent coefficients are determined.

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Question 4 *Stein and Shakarchi Chap. 8 Problem 7: (Too long I don't want to copy it)*

Pf:

(a) **Expansion satisfies $r_{f(K)} \geq r_K$:**

For all radius $0 < r < r_K$, the circle c_r (with radius r) is fully contained in K (since $c_r \subset \mathbb{D}(0, r_K) \subseteq K$). Since f is an expansion, then for all $z \neq 0$, $f(z) \neq 0$ (since it is injective, and $f(0) = 0$). Hence, by argument principle, the following integral shows he number of zeros enclosed by curve c_r:

$$\frac{1}{2\pi i}\int_{c_r} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}\int_{f(c_r)} \frac{1}{w} dw$$

Since c_r encloses only one zero (enclosing the origin, the only point that gets mapped to 0 by f), then the above integral yields value 1. This also implies that $f(c_r)$ is a closed curve satisfying $n(f(c_r), 0) = 1$ (same argument applies to all points enclosed by c_r, hence $f(c_r)$ is a simple closed curve enclosing region with 0).

On the other hand, $f(c_r)$ is fully contained in the range $f(K)$, while $f(K)$ is also simply connected, hence the curve $f(c_r)$ is homologous to 0, the open region enclosed by $f(c_r)$ (denoted as D) is also fully contained in $f(K)$.

Then, since $f(c_r)$ is compact, there exists $w_0 \in f(c_r)$ that yields a minimum modulus; which, if consider $|w_0|$ as a radius, since $\mathbb{D}(0, |w_0|)$ again contains no points in $f(c_r)$ (since $z \in \mathbb{D}(0, |w_0|)$ satisfies $|z| < |w_0|$, while all $w \in f(c_r)$ satisfies $|w_0| \leq |w|$), then $\mathbb{D}(0, |w_0|) \subseteq D \subseteq f(K)$, hence $|w_0| \leq r_{f(K)}$.

However, if consider the point $z_0 \in c_r$ that satisfies $f(z_0) = w_0$ (which, $|z_0| = r$), then we have the following inequality:

$$r = |z_0| < |f(z_0)| = |w_0| \leq r_{f(K)}$$

Hence, all $0 < r < r_K$ satisfies $r < r_{f(K)}$, which implies that $r_K \leq r_{f(K)}$ (since r_K is the supremum of $(0, r_K)$).

Expansion satisfies $|f'(0)| > 1$:

Since $f : K \to \mathbb{D}$ is an expansion, implies that $f(0) = 0$, then $f(z) = zg(z)$ for some analytic $g : K \to \mathbb{C}$.

Now, if consider the fact that all $z \in K \setminus \{0\}$ satisfies $|f(z)| > |z|$, we get the following inequality:

$$|g(z)| = \frac{|zg(z)|}{|z|} = \frac{|f(z)|}{|z|} > \frac{|z|}{|z|} = 1$$

Hence, if consider $f'(0)$ using limit definition, we get:

$$|f'(0)| = \left|\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}\right| = \left|\lim_{z \to 0} \frac{zg(z)}{z}\right| = |\lim_{z \to 0} g(z)| \geq 1$$

Now, we'll prove that $|f'(0)| = |g(0)| \neq 1$: Suppose the contrary that $|g(0)| = 1$, since the above statement implies that all $z \in K$ (including $z = 0$) satisfies $|g(z)| \geq 1$, then $g(z) \neq 0$ in K, hence $1/g : K \to \mathbb{C}$ is a well-defined analytic function, satisfying $|1/g(z)| \leq 1$.

However, since K is an open set, while g is nonconstant (if g is constant, and $|g(0)| = 1$, then $f(z) = zg(z) = g(0)z$, which $|f(z)| = |g(0)z| = |z|$, contradicting the fact that f is an expansion), then $|1/g(z)|$ shouldn't obtain a maximum on any point $z \in K$. Yet, since we assume $g(0) = 1$, while $|1/g(z)| \le 1$, hence $|1/g(z)| \le |1/g(0)|$ for all $z \in K$, showing that $0 \in K$ is in fact a maximum of $1/g$ on K, which violates the maximum principle.

Hence, our assumption must be false, $|g(0)| \neq 1$, showing that $|g(0)| = |f'(0)| > 1$.

(b) Given Koebe domain K_0, and a sequence of expansion $\{f_0, f_1, ...\}$ satisfying $K_{n+1} = f_n(K_n) \subseteq \mathbb{D}$, define $F_n : K_0 \to \mathbb{D}$ by $F_n = f_n \circ ... \circ f_0$.

F_n is an expansion:

We'll show this by induction (Note: expansion f satisfies $f(0) = 0$).

First, for $n = 1$, $F_1 = f_1 \circ f_0$, which for all $z \in K_0 \setminus \{0\}$, based on the fact that f_0, f_1 are expansions, it satisfies:

$$|F_1(z)| = |f_1(f_0(z))| > |f_0(z)| > |z|$$

Also, $F_1(0) = f_1(f_0(0)) = f_1(0) = 0$. Which, F_1 is an expansion.

Now, for given $n \in \mathbb{N}$, suppose F_n is an expansion. Then, $F_{n+1} = f_{n+1} \circ (f_n \circ ... \circ f_0) = f_{n+1} \circ F_n$. Again, since both f_{n+1}, F_n are expansions, all $z \in K_0 \setminus \{0\}$ satisfies:

$$|F_{n+1}(z)| = |f_{n+1}(F_n(z))| > |F_n(z)| > |z|$$

Also, $F_{n+1}(0) = f_{n+1}(F_n(0)) = f_{n+1}(0) = 0$. Hence, F_{n+1} is an expansion, and this completes the induction. (Note: since each f_n is injective, their finite composition is also injective, which completes the injectivity of all F_n).

Formula for $F_n'(0)$:

Again, we can show by induction, that $F_n'(0) = \prod_{k=0}^{n} f_k'(0)$.

First, for $n = 1$, $F_1 = f_1 \circ f_0$, then by chain rule, $F_1'(0) = f_1'(f_0(0)) \cdot f_0'(0) = f_1'(0) \cdot f_0'(0)$.

Now, suppose for given $n \in \mathbb{N}$, $F_n'(0) = \prod_{k=0}^{n} f_k'(0)$, then for $F_{n+1} = f_{n+1} \circ F_n$ satisfies:

$$F_{n+1}'(0) = f_{n+1}'(F_n(0)) \cdot F_n'(0) = f_{n+1}'(0) \cdot \prod_{k=0}^{n} f_k'(0) = \prod_{k=0}^{n+1} f_k'(0)$$

Which, this proves the case for $(n + 1)$, and it completes the induction.

Limit of $|f_n'(0)|$:

First, based on the above formula of $F_n'(0)$, since for all $n \in \mathbb{N}$, f_{n+1} is an expansion, then $|f_{n+1}'(0)| > 1$ (based on **part (a)**). Hence, the following is true:

$$|F_{n+1}'(0)| = \left|\prod_{k=0}^{n+1} f_k'(0)\right| = |f_{n+1}'(0)| \cdot \prod_{k=0}^{n} |f_k'(0)| > \prod_{k=0}^{n} |f_k'(0)| = \left|\prod_{k=0}^{n} f_k'(0)\right| = |F_n'(0)|$$

This proves that $\{|F_n'(0)|\}_{n\in\mathbb{N}}$ is a strictly increasing sequence.

Also, recall that since $\mathbb{D}(0, r_{K_0}) \subseteq K_0$, for each n, define $\bar{F}_n : \mathbb{D} \to \mathbb{D}$ by $\bar{F}_n(z) = F_n(r_{K_0} z)$ (Note: each $z \in \mathbb{D}$, since $|z| < 1$, then $|r_{K_0} z| < r_{K_0}$, hence $r_{K_0} z \in \mathbb{D}(0, r_{K_0}) \subseteq K_0$). Since $\bar{F}_n$ is an analytic map from $\mathbb{D}$ to $\mathbb{D}$, and it satisfies $\bar{F}_n(0) = F_n(r_{K_0} \cdot 0) = 0$, then by Schwarz Lemma, $|\bar{F}_n'(0)| \le 1$. So, we get the following:

$$\bar{F}_n'(z) = r_{K_0} F_n'(r_{K_0} z), \quad |\bar{F}_n'(0)| = r_{K_0} |F_n'(0)| \le 1, \quad |F_n'(0)| \le \frac{1}{r_{K_0}}$$

This proves that $\{|F_n'(0)|\}_{n\in\mathbb{N}}$ is bounded above by $\frac{1}{r_{K_0}} > 0$ (Note: since K_0 is open, $r_{K_0} > 0$).

Hence, since the sequence is srictly increasing while bounded from abouve, $\lim_{n\to\infty} |F_n'(0)| = L \in \mathbb{R}$.

Then, since the limit exists, while $|F_n'(0)|$ is based on products of $|f_k'(0)|$, then:

$$\lim_{n\to\infty} |F_n'(0)| = \lim_{n\to\infty} \prod_{k=0}^{n} |f_k'(0)| = L \in \mathbb{R} \implies \lim_{n\to\infty} |f_n'(0)| = 1$$

(c) Given the family of expansions $\{F_n : K_0 \to \mathbb{D} \mid n \in \mathbb{N}\}$ with $\lim_{n\to\infty} r_{K_n} = 1$, unfortunately we cannot conclude that the sequence of functions converges (if F_n converges to F, since one can add a constant rotation of radians $\pi/2$ in between f_n and f_{n+1} for each $n \in \mathbb{N}$, which for all $z \in K_0 \setminus \{0\}$, since $F_n(z)$ now becomes a sequence that's constantly rotating, while the modulus $|F_n(z)|$ is still increasing, then the modified sequence no longer converges).

However, based on **Montel's Theorem**, since the family of expansions are uniformly bounded (for all $z \in K_0$, all $n \in \mathbb{N}$ satisfies $|F_n(z)| < 1$, because $F_n(z) \in \mathbb{D}$), then there exists a subsequence $\{F_{n_k}\}_{k\in\mathbb{N}}$ that converges locally uniformly to some analytic function $F : K_0 \to \mathbb{D}$ (i.e. on any compact subsets of K_0, the sequence of functions converges uniformly).

Now, since $\lim_{k\to\infty} r_{K_{n_k}} = 1$ (subsequential limit agrees with the sequential limit if the original sequence converges), then $r_{f(K_0)} \geq 1$: For all $0 < r < 1$, because of the limit, there exists $K \in \mathbb{N}$, such that $k \geq K$ implies $r < r_{K_{n_k}} \leq 1$. Then, based on the result in **part (a)**, we know $r_{K_{n_k}} \leq r_{f_{n_k}(K_{n_k})}$, which $\mathbb{D}(0,r) \subseteq \mathbb{D}(0, r_{f_{n_k}(K_{n_k})}) = \mathbb{D}(0, r_{F_{n_k}(K_0)}) \subseteq F_{n_k}(K_0)$ (Note: recall that f_{n_k} has the image being the same as F_{n_k}). Hence, as F_{n_k} converges to F, this implies that $\mathbb{D}(0,r) \subseteq F(K_0)$, showing that $r \leq r_{F(K_0)}$. Since for all $0 < r < 1$, $r \leq r_{F(K_0)}$, then $1 \leq r_{F(K_0)}$. So, this implies that $\mathbb{D} \subseteq \mathbb{D}(0, r_{F(K_0)}) \subseteq F(K_0)$, showing that F is surjective.

Moreover, since the collection $\{F_{n_k}\}_{k\in\mathbb{N}}$ are a sequence of expansions (analytic injective fnctions) that converges locally uniformly, then by **Hurwitz's Theorem**, the limit is either constant or injective; however, since it is surjective onto $\mathbb{D}$, the map F is not constant, hence it must be injective.

Because F is both injective and surjective while being analytic, it is a conformal map.

(d) Given Koebe domain K, with $\alpha \in \partial K$ such that $|\alpha| = r_K > 0$, and $\beta \in \mathbb{D}$ such that $\beta^2 = \alpha$.

If $|\alpha| = 1$, this implies that $r_K = 1$, or $\mathbb{D} \subseteq K$, which $K = \mathbb{D}$, so there's no need for constructing a conformal map. Hence, can assume $|\alpha| < 1$ (consequently, since $\beta^2 = \alpha$, then $|\beta| < 1$ also).

Then, the following transformation is a conformal map from $\mathbb{D} \to \mathbb{D}$:

$$\psi_\alpha : \mathbb{D} \to \mathbb{D}, \quad \psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$$

If we restrict the domain to be K, since K is open, then $K = K^\circ$. Which $K^\circ \cap \partial K = \emptyset$, showing that $\alpha \notin K$.

Then, if consider $\psi_\alpha(K)$, since for $\psi_\alpha(z) = 0$, we need $\alpha - z = 0$, or $z = \alpha$, then since $\alpha \notin K$, then $0 \notin \psi_\alpha(K)$.

Hence, because ψ_α is a conformal map, K is open and simply connected, implies that $\psi_\alpha(K)$ is also open and simply connected. Therefore, it's possible to define a single-valued branch of square root on $\psi_\alpha(K)$, or $S : \psi_\alpha(K) \to \mathbb{D}$ that satisfies:

$$\forall w \in \psi_\alpha(K), \quad (S(w))^2 = w, \quad S(\alpha) = \beta$$

(Note: Since $\psi_\alpha(0) = \alpha$, then $\alpha \in \psi_\alpha(K)$, hence we can define the branch such that $S(\alpha) = \beta$).

(Note 2: Because all $w \in \psi_\alpha(K) \subseteq \mathbb{D}$, then $|w| < 1$; hence, $|S(w)|^2 = |w| < 1$ implies $|S(w)| < 1$, showing that $S(\psi_\alpha(K)) \subseteq \mathbb{D}$).

Also, since $\alpha \in \partial K$ is a limit point of K (since $|\alpha| = r_K$, all $0 < r < 1$ satisfies $|r\alpha| < |\alpha| = r_K$, which $r\alpha \in K$; hence for all $\epsilon > 0$, choose $r > 0$ such that $1 - \epsilon < r < 1$, then $|\alpha - r\alpha| = |\alpha| \cdot |1 - r| < r_K \epsilon < \epsilon$, showing that $r\alpha \in B_\epsilon(\alpha)$, hence α is a limit point), then we can find a sequence $(c_n)_{n\in\mathbb{N}} \subset K$ converging to α. By continuity of ψ_α on $\mathbb{D}$, we get:

$$\lim_{n\to\infty} \psi_\alpha(c_n) = \psi_\alpha(\alpha) = 0, \quad \lim_{n\to\infty} |S(\psi_\alpha(c_n))^2| = \lim_{n\to\infty} |\psi_\alpha(c_n)| = 0$$

Hence, $\lim_{n\to\infty} \sqrt{|S(\psi_\alpha(c_n))^2|} = \lim_{n\to\infty} |S(\psi_\alpha(c_n))| = 0$, showing that $\lim_{n\to\infty} S(\psi_\alpha(c_n)) = 0$. So, without considering if the extension is analytic or not, we can define $S(0) = 0$. Which, the desired square root is defined.

The expansion based on α and β:

Consider $f = \psi_\beta \circ S \circ \psi_\alpha : K \to \mathbb{D}$.

To prove that it is an expansion, we'll first prove that $f(0) = 0$:

$$f(0) = \psi_\beta \circ S \circ \psi_\alpha(0) = \psi_\beta \circ S(\alpha) = \psi_\beta(\beta) = 0$$

(Note: the mobius transformation ψ_α swaps α and 0, while we define $S(\alpha) = \beta$).

Then, we'll consider its inverse: For ψ_α as an automorphism on $\mathbb{D}$, its inverse is itself (the property of mobius transformation given that $|\alpha| < 1$), and the same logic applies to ψ_β. Now, if we consider $g(z) = z^2$, the composition $h = \psi_\alpha \circ g \circ \psi_\beta : f(K) \to \mathbb{C}$ becomes a left inverse of f:

$$\forall z \in K, \quad h(f(z)) = \psi_\alpha \circ g \circ \psi_\beta \circ \psi_\beta \circ S \circ \psi_\alpha(z) = \psi_\alpha \circ g \circ S(\psi_\alpha(z)) = \psi_\alpha((S(\psi_\alpha(z)))^2)$$
$$= \psi_\alpha(\psi_\alpha(z)) = z$$

Also, since $h = \psi_\alpha \circ g \circ \psi_\beta$ is in fact a map from $\mathbb{D} \to \mathbb{D}$ (since ψ_α, ψ_β are automorphisms of $\mathbb{D}$, while $g(z) = z^2$ has all $z \in \mathbb{D}$ with $|g(z)| = |z|^2 < |z| < 1$, hence $g(z) \in \mathbb{D}$), and it satisfies $h(0) = 0$ (since $\psi_\alpha \circ g \circ \psi_\beta(0) = \psi_\alpha \circ g(\beta) = \psi_\alpha(\beta^2) = \psi_\alpha(\alpha) = 0$), then apply Schwarz Lemma, we know all $w \in f(K) \subseteq \mathbb{D}$ satisfies $|h(w)| \leq |w|$.

To prove that it is a strict inequality above, recall that Schwarz Lemma also states that if any nonzero $w \in \mathbb{D}$ satisfies $|h(w)| = |w|$, then h must be a rotation (i.e. all $w \in \mathbb{D}$ satisfies $|h(w)| = |w|$). However, this is not the case for $\beta \in \mathbb{D}$:

$$h(\beta) = \psi_\alpha \circ g \circ \psi_\beta(\beta) = \psi_\alpha \circ g(0) = \psi_\alpha(0) = \alpha, \quad |h(\beta)| = |\alpha| = |\beta|^2 < |\beta|$$

Since $\beta \in \mathbb{D}$ satisfies a strict inequality $|h(\beta)| < |\beta|$, then h cannot be a rotation, hence all nonzero $w \in \mathbb{D}$ cannot satisfy $|h(w)| = |w|$, which enforces $|h(w)| < |w|$.

Hence, for all nonzero $z \in K$, we have:

$$|z| = |h(f(z))| < |f(z)|$$

Lastly, because both ψ_α, ψ_β are injective (automorphisms of $\mathbb{D}$), and S as a square root on $\psi_\alpha(K)$ is also injective (if $z, w \in \psi_\alpha(K)$ satisfies $S(z) = S(w)$, then $z = S(z)^2 = S(w)^2 = w$), then f as a composition of them is also injective.

Becuse the function f satisfies $f(0) = 0$, $|z| < |f(z)|$ for all $z \in K \setminus \{0\}$, and is injective, then f is an expansion.

Formula of $|f'(0)|$:

First, the derivative of ψ_α is given as follow:

$$\psi'_\alpha(z) = \frac{-(1-\bar{\alpha}z) - (\alpha - z)(-\bar{\alpha})}{(1-\bar{\alpha}z)^2}, \quad \psi'_\alpha(0) = \frac{-1-\alpha(-\bar{\alpha})}{1^2} = |\alpha|^2 - 1$$

Similarly, derivative of ψ_β is given as follow:

$$\psi'_\beta(z) = \frac{-(1-\bar{\beta}z) - (\beta - z)(-\bar{\beta})}{(1-\bar{\beta}z)^2}, \quad \psi'_\beta(\beta) = \frac{-(1-|\beta|^2) - 0}{(1-|\beta|^2)^2} = \frac{-1}{(1-|\beta|^2)} = \frac{-1}{1-|\alpha|}$$

Then, derivative of S is given as follow:

$$\forall w \in \psi_\alpha(K), \quad (S(w))^2 = w, \quad 2S(w)\cdot S'(w) = 1, \quad S'(w) = \frac{1}{2S(w)}, \quad S'(\alpha) = \frac{1}{2S(\alpha)} = \frac{1}{2\beta}$$

Then, the derivative of f at 0 is given as:

$$f'(0) = (\psi_\beta \circ S \circ \psi_\alpha)'(0) = \psi'_\beta(S\circ\psi_\alpha(0))\cdot S'(\psi_\alpha(0))\cdot\psi'_\alpha(0)$$
$$= \psi'_\beta(S(\alpha))\cdot S'(\alpha)\cdot(|\alpha|^2-1) = \psi'_\beta(\beta)\cdot\frac{1}{2\beta}\cdot(|\alpha|^2-1) = \frac{-1}{1-|\alpha|}\cdot\frac{1}{2\beta}\cdot(|\alpha|^2-1)$$
$$= \frac{(1-|\alpha|^2)}{2\beta(1-|\alpha|)} = \frac{1+|\alpha|}{2\beta}$$

Which, it has modulus given by:

$$|f'(0)| = \frac{|1+|\alpha||}{|2\beta|} = \frac{1+|\alpha|}{2\sqrt{|\beta^2|}} = \frac{1+r_K}{2\sqrt{|\alpha|}} = \frac{1+r_K}{2\sqrt{r_K}}$$

(e) To construct the expansion, we'll do this in an iterative manner (and we'll assume initially given Koebe Domain K_0, $r_{K_0} < 1$, since the case $r_{K_0} \geq 1$ implies $K_0 = \mathbb{D}$).

0. First, find an $\alpha_0 \in \partial K_0$ and its corresponding $\beta_0 \in \mathbb{D}$ satisfying $|\alpha_0| = r_{K_0}$ and $\beta_0^2 = \alpha_0$, then $f_0 = \psi_{\beta_0} \circ S_{\alpha_0} \circ \psi_{\alpha_0} : K_0 \to \mathbb{D}$ is an expansion. Let $K_1 = f_0(K_0) \subseteq \mathbb{D}$. (Note: S_{α_0} is the described square root in **part (d)**).

n. With integer $n \geq 1$, in the previous step, we have new Koebe Domain K_n. Repeat the same process, choose $\alpha_n \in \partial K$ with $|\alpha_n| = r_{K_{n-1}}$ and $\beta_n \in \mathbb{D}$ with $\beta_n^2 = \alpha_n$. Then, $f_n = \psi_{\beta_n} \circ S_{\alpha_n} \circ \psi_{\alpha_n}$: $K_n \to \mathbb{D}$ is again an expansion. Let $K_{n+1} = f_n(K_n) \subseteq \mathbb{D}$.

The above constructs a sequence of expansion described in the problem. Then, for all $n \in \mathbb{N}$, $F_n = f_n \circ ... \circ f_0$ is an expansion, and it satisfies $\lim_{n\to\infty} |f'_n(0)| = 1$ (both proven in **part (b)**).

By the statement proven in **part (c)**, the sequence $\{F_n\}_{n\in\mathbb{N}}$ has a subsequence converges locally uniformly onto some injective analytic function $F : K \to \mathbb{D}$; also, since the sequence r_{K_n} is strictly increasing (proven in **part (a)**) while bounded above by 1, then $\lim_{n\to\infty} r_{K_n} = d \leq 1$. Which, based on the formula of $|f'_n(0)|$ give in **part (d)**, we have the following:

$$1 = \lim_{n\to\infty} |f'_n(0)| = \lim_{n\to\infty} \frac{1+r_{K_n}}{2\sqrt{r_{K_n}}} = \frac{1+d}{2\sqrt{d}}$$

Hence, $2\sqrt{d} = 1+d$, $4d = (1+d)^2 = 1+2d+d^2$, $(1-2d+d^2) = (1-d)^2 = 0$, so $(1-d) = 0$, or $d = 1$.

Because $\lim_{n\to\infty} r_{K_n} = 1$, then based on the statement in **part (c)**, the function F that the subsequence converges to, is in fact conformal. Which, this is the result we want to show.