Math 118C HW3

Zih-Yu Hsieh

May 4, 2025

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Question 1 *Rudin Pg. 241 Problem 19:*
Show that the system of equations

$$3x + y - z + u^2 = 0$$

$$x - y + 2z + u = 0$$

$$2x + 2y - 3z + 2u = 0$$

can be solved for x, y, u *in terms of* z*; for* x, z, u *in terms of* y*; for* y, z, u *in terms of* x*; but not for* x, y, z *in terms of* u*.*

Pf:

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Question 2 *Rudin Pg. 242 Problem 23:*
Define f in $\mathbb{R}^3$ by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2$$

Show that $f(0,1,-1) = 0$,$(D_1 f)(0,1,-1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1,-1)$ in $\mathbb{R}^2$, such that $g(1,-1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0$$

Find $(D_1 g)(1,-1)$ and $(D_2 g)(1,-1)$.

Pf:

f and $D_1 f$ at $(0,1,-1)$:

Evaluate at $(0,1,-1)$, we get $f(0,1,-1) = 0^2 \cdot 1 + e^0 + (-1) = 0 + 1 - 1 = 0$.

On the other hand, $D_1 f(x, y_1, y_2)$ is given as follow:

$$D_1 f(x, y_1, y_2) = \frac{\partial}{\partial x}(x^2 y_1 + e^x + y_2) = 2xy_1 + e^x$$

Hence, $D_1 f(0,1,-1) = 2 \cdot 0 \cdot 1 + e^{-1} = e^{-1} \neq 0$.

Validity of Implicit Function Theorem:

If we consider the partial derivatives with respect to all variables, we get:

$$\frac{\partial f}{\partial x} = 2xy_1 + e^x, \quad \frac{\partial f}{\partial y_1} = x^2, \quad \frac{\partial f}{\partial x} = 1$$

Hence, since the partial derivative of f with respect to any variable is continuous, then f is continuously differentiable, with $Df(x, y_1, y_2) = (2xy_1 + e^x, x^2, 1) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$.

Then, since in the previous part, we get that $D_1 f(0,1,-1) \neq 0$, which if view $\mathbb{R}^3 = \mathbb{R}^{1+2}$, with $(0,1,-1) = (0, \bar{0}) + (0, (1,-1))$, then $A = D_1 f(0,1,-1) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$, can be broken down as follow:

$$\forall x \in \mathbb{R},\ \bar{y} \in \mathbb{R}^2, \quad A(x, \bar{y}) = A_x(x) + A_y(\bar{y}), \quad A_x = D_1 f(0,1,-1) \in \mathcal{L}(\mathbb{R}), \quad A_y = \left(\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}\right) \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$$

Then, since f is continuously differentiable, while $A_x = D_1 f(0,1,-1)$ for $A = Df(0,1,-1)$ is invertible (since $D_1 f(0,1,-1) \neq 0$), then by Implicit Function Theorem, there exists open neighborhood $U \subseteq \mathbb{R}^3$ of $(0,1,-1)$, open neighborhood $V \subseteq \mathbb{R}^2$ of $(1,-1)$, such that for all $\bar{y} \in V$, there exists a unique $x \in \mathbb{R}$, such that $f(x, \bar{y}) = 0$.

Which, define $g : V \to \mathbb{R}$ by $g(\bar{y}) = x$ got from the previous part, then g is continuously differentiable.

And, by the equation of $Dg(0,1,-1)$, given $A = Df(0,1,-1)$ and A_x and A_y provided above, it is given by the following:

$$\begin{aligned} Dg(1,-1) &= A_x^{-1} A_y = (D_1 f(0,1,-1))^{-1} \begin{pmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \end{pmatrix} \Big|_{(0,1,-1)} \\ &= \frac{1}{e^{-1}} \begin{pmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \end{pmatrix} \Big|_{(0,1,-1)} = e \begin{pmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \end{pmatrix} \Big|_{(0,1,-1)} \end{aligned}$$

So, by the uniqueness of the differential, we know the following:

$$Dg(y_1, y_2) = \begin{pmatrix} \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \end{pmatrix}$$

Then, we get the following relations:

$$D_1g(1,-1) = \frac{\partial g}{\partial y_1} \mid_{(1,-1)} = e\frac{\partial f}{\partial y_1} \mid_{(0,1,-1)} = e \cdot 0^2 = 0$$

$$D_2g(1,-1) = \frac{\partial g}{\partial y_2} \mid_{(1,-1)} = e\frac{\partial f}{\partial y_2} \mid_{(0,1,-1)} = e \cdot 1 = e$$

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Question 3 *Rudin Pg. 242 Problem 24:*
For $(x,y) \neq (0,0)$, define $f = (f_1, f_2)$ by

$$f_1(x,y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad f_2(x,y) = \frac{xy}{x^2 + y^2}$$

Compute the rank of $f'(x,y)$, and find the range of f.

Pf:
Rank of $f'(x,y) = Df(x,y)$:
Given f_1 and f_2, their partial derivatives are given as follow:

$$\frac{\partial f_1}{\partial x} = \frac{2x(x^2+y^2) - 2x(x^2-y^2)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}, \quad \frac{\partial f_1}{\partial y} = \frac{-2y(x^2+y^2) - 2y(x^2-y^2)}{(x^2+y^2)^2} = \frac{-4x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial f_2}{\partial x} = \frac{y(x^2+y^2) - 2x(xy)}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}, \quad \frac{\partial f_2}{\partial y} = \frac{x(x^2+y^2) - 2y(xy)}{(x^2+y^2)^2} = \frac{x^3 - xy^2}{(x^2+y^2)^2}$$

Hence, the differential Df is given as:

$$Df(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \frac{1}{(x^2+y^2)^2} \begin{pmatrix} 4xy^2 & -4x^2y \\ (y^3 - x^2y) & (x^3 - xy^2) \end{pmatrix}$$

- If $x = 0$, the matrix is given by:

$$Df(0,y) = \frac{1}{y^4} \begin{pmatrix} 0 & 0 \\ y^3 & 0 \end{pmatrix}$$

  Which, with $y \neq 0$ (due to the condition $(x,y) \neq (0,0)$), the above matrix has rank 1.

- If $y = 0$, the matrix is given by:

$$Df(x,0) = \frac{1}{x^4} \begin{pmatrix} 0 & 0 \\ 0 & x^3 \end{pmatrix}$$

  Which again, with $x \neq 0$, the above matrix has rank 1.

- If both $x, y \neq 0$, then consider any vector $r(x,-y)$ for $r \in \mathbb{R}$, we get::

$$Df(x,y)r\begin{pmatrix} x \\ -y \end{pmatrix} = \frac{1}{(x^2+y^2)^2} \begin{pmatrix} xy^2 & -x^2y \\ (y^3 - x^2y) & (x^3 - xy^2) \end{pmatrix} r \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$\frac{r}{(x^2+y^2)^2} \begin{pmatrix} x^2y^2 - x^2y^2 \\ (xy^3 - x^3y) - (-x^3y + xy^3) \end{pmatrix} = \bar{0}$$

  So the span$\{(x,-y)\}$ is within the null space of $Df(x,y)$, so the dimension of null space is at least 1. On the other hand, consider $(1,0) \in \mathbb{R}^2$ (that is linearly independent with $(x,-y)$, since $(x,-y)$ has both entries being nonzero), it has the following:

$$Df(x,y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{(x^2+y^2)^2} \begin{pmatrix} xy^2 & -x^2y \\ (y^3 - x^2y) & (x^3 - xy^2) \end{pmatrix} r \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{(x^2+y^2)^2} \begin{pmatrix} xy^2 \\ (y^3 - x^2y) \end{pmatrix} \neq \bar{0}$$

  Which, it shows that the range of $Df(x,y)$ is nontrivial, hence it has dimension at least 1 also.

Because both the null space and the range have dimension ≥ 1, while $\mathbb{R}^2$ has dimension 2, by Rank Nullity Theorem, it enforces both the null space and the range must have dimension precisely 1 (since the sum of the dimension of the null space and the range must be 2). So, the rank of $Df(x,y)$ is again 1.

So, regardless of the case, $Df(x,y)$ has rank 1.

Range of f:

When fixing $(x,y) \neq (0,0)$ in $\mathbb{R}^2$, there exists $r > 0$, and $\theta \in [0, 2\pi)$, such that $x = r\cos(\theta)$ and $y = r\sin(\theta)$ (under polar coordinates). Then, we get the output of f_1, f_2 as:

$$f_1(x,y) = \frac{(r\cos(\theta))^2 - (r\sin(\theta))^2}{(r\cos(\theta))^2 + (r\sin(\theta))^2} = \frac{r^2(\cos^2(\theta) - \sin^2(\theta))}{r^2} = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$$

$$f_2(x,y) = \frac{(r\cos(\theta))(r\sin(\theta))}{(r\cos(\theta))^2 + (r\sin(\theta))^2} = \frac{r^2\cos(\theta)\sin(\theta)}{r^2} = \sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$$

Hence, for all $(x,y) \neq (0,0)$, $f = (f_1, f_2)$ satisfies the following equation:

$$f_1^2 + 4f_2^2 = \cos^2(2\theta) + 4 \cdot \frac{1}{4}\sin^2(2\theta) = \cos^2(2\theta) + \sin^2(2\theta) = 1$$

Hence, $(u,v) = f(x,y)$ is a solution to $u^2 + 4v^2 = 1$, so the range of f is contained in the ellipse characterized by $u^2 + 4v^2 = 1$.

On the other hand, for all point (u,v) satisfying $u^2+4v^2 = 1$, there exists $\theta \in [0, 2\pi)$, such that $u = \cos(\theta)$ and $v = \frac{1}{2}\sin(\theta)$. Then, consider the point $(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})) \in \mathbb{R}^2$, we have:

$$f(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})) = (f_1(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})), f_2(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2}))) = \left(\cos\left(2 \cdot \frac{\theta}{2}\right), \frac{1}{2}\sin\left(2 \cdot \frac{\theta}{2}\right)\right)$$

$$= \left(\cos(\theta), \frac{1}{2}\sin(\theta)\right) = (u,v)$$

Hence, (u,v) is also in the range of f. This proves that f has the range precisely described by the ellipse $u^2 + 4v^2 = 1$.

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Question 4 *Rudin Pg. 242 Problem 25:*
Suppose $A \in \mathcal{L}(\mathbb{R}^n, \mathcal{R}^m)$, let r be the rank of A.

(a) Define S as in the proof of Theorem 9.32. Show that SA is a projection in $\mathbb{R}^n$ whose null space is null(A) and whose range is range(S).

(b) Use (a) to show that

$$\dim(\textit{null}(A)) + \dim(\textit{range}(A)) = n$$

Pf:

(a) Given that A has rank r, then its range range$(A) \subseteq \mathbb{R}^m$ is an r-dimensional linear subspace, hence there exists $y_1, ..., y_r \in$ range(A) that forms a basis of it.

Then, by the text in Rudin, choose $z_1, ..., z_r \in \mathbb{R}^n$, so for each index $i \in \{1, ..., r\}$, $Az_i = y_i$. Which, the collection $z_1, ..., z_r \in \mathbb{R}^n$ is linearly independent, since if $a_1, ..., a_r \in \mathbb{R}$ satisfies $\sum_{i=1}^r a_i z_i = \bar{0}$, then the following is true:

$$A\left(\sum_{i=1}^r a_i z_i\right) = \sum_{i=1}^r a_i(Az_i) = \sum_{i=1}^r a_i y_i$$

By the linear independence of $y_1, ..., y_r \in$ range(A), each $a_i = 0$, which proves the linear independence of $z_1, ..., z_r \in \mathbb{R}^n$.

Now, define $S \in \mathcal{L}(\text{range}(A), \mathbb{R}^n)$ the same as in the text, which has the following formula:

$$\forall c_1, ..., c_r \in \mathbb{R}, \quad S\left(\sum_{i=1}^r c_i y_i\right) = \sum_{i=1}^r c_i z_i$$

Then, for all $x \in \mathbb{R}^n$, since $Ax \in$ range(A), it is spanned by $y_1, ..., y_r$, hence there exists unique $a_1, ..., a_r \in \mathbb{R}$, such that the following is true:

$$Ax = \sum_{i=1}^r a_i y_i$$

Hence, we get the following:

$$SAx = S\left(\sum_{i=1}^r a_i y_i\right) = \sum_{i=1}^r a_i z_i$$

Hence, applying SA twice, we get:

$$SA(SAx) = SA\left(\sum_{i=1}^r a_i z_i\right) = S\left(\sum_{i=1}^r a_i y_i\right) = \sum_{i=1}^r a_i z_i$$

This shows that $SA(SAx) = SAx$ for all $x \in \mathbb{R}^n$, hence SA is a projection on $\mathbb{R}^n$.

Now, to find the null space and range, consider the following:

- For all $x \in$ null(A), since $Ax = 0$, then $SAx = S(0) = 0$, so $x \in$ null(SA), or null$(A) \subseteq$ null(SA).

On the other hand, for all $\in \text{null}(SA)$, since $S(Ax) = 0$, $Ax \in \text{null}(S)$. But, since $Ax \in \text{range}(A)$, there exists unique $a_1, ..., a_r \in \mathbb{R}$, with $Ax = \sum_{i=1}^{r} a_i y_i$. Hence, we have the following:

$$0 = S(Ax) = S\left(\sum_{i=1}^{r} a_i y_i\right) = \sum_{i=1}^{r} a_i z_i$$

Hence, by linear independence of $z_1, ..., z_r \in \mathbb{R}^n$, we must have $a_i = 0$ for all index $i \in \{1, ..., r\}$. This proves that $Ax = \sum_{i=1}^{r} a_i y_i = 0$, so $x \in \text{null}(A)$. Hence, $\text{null}(SA) \subseteq \text{null}(A)$, showing that $\text{null}(SA) = \text{null}(A)$.

- For all $z \in \text{range}(SA)$, there exists $x \in \mathbb{R}^n$ with $SAx = z$. Since $z = S(Ax) \in \text{range}(S)$, then $\text{range}(SA) \subseteq \text{range}(S)$.

  Similarly, for all $z \in \text{range}(S)$, there exists $y \in \text{range}(A)$ (the domain of S), with $Sy = z$; then because there exists $x \in \mathbb{R}^n$, with $Ax = y$ by the definition of range, we have $SAx = S(Ax) = Sy = z$, hence $z \in \text{range}(SA)$, proving that $\text{range}(S) \subseteq \text{range}(SA)$, or $\text{range}(S) = \text{range}(SA)$.

Hence, the above to cases proves that $\text{null}(SA) = \text{null}(A)$, while $\text{range}(S) = \text{range}(SA)$. So, SA is a projection in $\mathbb{R}^n$ with null space being $\text{null}(A)$, and range being $\text{range}(S)$.

(b) With the linearly independent set $z_1, ..., z_r \in \mathbb{R}^n$ given in **part (a)**, we'll consider an extra list $x_1, ..., x_k \in \text{null}(A) \subseteq \mathbb{R}^n$ that forms a basis of $\text{null}(A)$. Our goal is to prove that $x_1, ..., x_k, z_1, ..., z_r$ forms a basis of $\mathbb{R}^n$.

First, consider $a_1, ..., a_k, b_1, ..., b_r \in \mathbb{R}$, suppose the vector $\sum_{i=1}^{k} a_i x_i + \sum_{j=1}^{r} b_j z_j = \bar{0} \in \mathbb{R}^n$, then we have the following:

$$0 = A(\bar{0}) = A\left(\sum_{i=1}^{k} a_i x_i + \sum_{j=1}^{r} b_j z_j\right) = A\left(\sum_{i=1}^{k} a_i x_i\right) + A\left(\sum_{j=1}^{r} b_j z_j\right) = \sum_{j=1}^{r} b_j (Az_j) = \sum_{j=1}^{r} b_j y_j$$

Which, by the linear independence of $y_1, ..., y_r \in \text{range}(A)$ assumed in **part (a)**, we must have $b_j = 0$ for all $j \in \{1, ..., n\}$. So, $\sum_{i=1}^{k} a_i x_i + \sum_{j=1}^{r} b_j z_j = \sum_{i=1}^{k} a_i x_i = \bar{0}$. But again, based on the linear independence ov $x_1, ..., x_k$ by assumption, we get $a_i = 0$ for all $i \in \{1, ..., k\}$. This proves that all $a_i, b_j = 0$, which the collection $x_1, ..., x_k, z_1, ..., z_r$ is linearly independent.

Then, for all $x \in \mathbb{R}^n$, since $Ax \in \text{range}(A)$, then there exists unique $b_1, ..., b_r \in \mathbb{R}$, with $Ax = \sum_{j=1}^{r} b_j y_j$. Hence, we get the following:

$$Ax = \sum_{j=1}^{r} b_j y_j = \sum_{j=1}^{r} b_j (Az_j) = A\left(\sum_{j=1}^{r} b_j z_j\right)$$

So, we can reduce to the following:

$$Ax - A\left(\sum_{j=1}^{r} b_j z_j\right) = A\left(x - \sum_{j=1}^{r} b_j z_j\right) = 0, \quad x - \sum_{j=1}^{r} b_j z_j \in \text{null}(A)$$

Then, since $x_1, ..., x_k$ forms a basis of $\text{null}(A)$, then there exists unique $a_1, ..., a_k \in \mathbb{R}$, with:

$$x - \sum_{j=1}^{r} b_j z_j = \sum_{i=1}^{k} a_i x_i, \quad x = \sum_{i=1}^{k} a_i x_i + \sum_{j=1}^{r} b_j z_j$$

So, $x \in \text{span}\{x_1, ..., x_k, z_1, ..., z_r\}$, proving that $\mathbb{R}^n = \text{span}\{x_1, ..., x_k, z_1, ..., z_r\}$.

Hence, since $x_1, ..., x_k, z_1, ..., z_r$ spans $\mathbb{R}^n$ while being linearly independent, it is a basis of $\mathbb{R}^n$. Hence, the length of the basis, $k + r = \dim(\mathbb{R}^n) = n$.

Which, since $x_1, ..., x_k$ is a basis of $\text{null}(A)$, then $\dim(\text{null}(A)) = k$.

On the other hand, since $S \in \mathcal{L}(\text{range}(A), \mathbb{R}^n)$ is injective (since it maps $y_1, ..., y_r$ a basis of $\text{range}(A)$, to $z_1, ..., z_r \in \mathbb{R}^n$ a linearly independent set), then the domain $\text{range}(A)$ and range are in fact isomorphic as vector spaces, while the range of S is precisely $\text{span}\{z_1, ..., z_r\}$ (since the definition of S is maps y_i to z_i for each $i \in \{1, ..., r\}$, showing that the output value must be a linear combination of all z_i). Hence, $\dim(\text{range}(A)) = \dim(\text{span}\{z_1, ..., z_r\}) = r$ (since $z_1, ..., z_r$ is linearly independent, it forms a basis of the span).

So, compile the information from above, we get:

$$k + r = n, \quad k = \dim(\text{null}(A)), \quad r = \dim(\text{range}(A))$$

$$\implies \dim(\text{null}(A)) + \dim(\text{range}(A)) = n$$

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Question 5 *Rudin Pg. 242 Problem 26:*
Show that the existence (and even the continuity) of $D_{12}f$ does not imlpy the existence of $D_1 f$. For example, let $f(x, y) = g(x)$, where g is nowhere differentiable.

Pf:

Consider the Weierstrass Functon $g : \mathbb{R} \to \mathbb{R}$, which is uniformly continuous, while being differentiable nowhere.

Then, given the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = g(x)$, since g is not differentiable with respect to its variable x, then $D_1 f$ does not exist; yet, since $D_2 f \equiv 0$ (due to the fact that g is a constant when x is fixed), then $D_{12}f = D_1(D_2 f) = 0$.

Hence, even though $D_{12}f$ is continuous, $D_1 f$ doesn't exist in this case.