Math 118C HW2

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1

Question 1 *Rudin Pg. 239 Problem 9:*
If f is a differentiable mapping of a connected *open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and if $f'(x) = 0$ for every $x \in E$, prove that f is constant in E.*

Pf:

To prove that $f(x) = (f_1(x), ..., f_m(x))$ for $f_1, ..., f_m : E \to \mathbb{R}$ is constant, is suffices to show that each individual f_i is constant in E.

Since f is differentiable, each f_i is also differentiable, hence Df_i exists for all $x \in E$; also, since $f'(x) = 0$, this implies that each $Df_i(x) = 0$ for all $x \in E$.

Now, for all index $i \in \{1, ..., m\}$, the function $f_i : E \to \mathbb{R}$ satisfies:

1. f_i is constant within a Neighborhood:

Consider any $x \in E$, since E is open, then there exists $r > 0$, such that the open ball $B_r(x) \subseteq E$. Since all $x \in E$ satisfies $Df_i(x) = 0$, then $\|Df_i(x)\| \leq 0$ for all $x \in E$, in particular, it also applies to $B_r(x)$.

Now, since $B_r(x)$ is convex, while the differential Df_i is uniformly bounded by 0 in $B_r(x)$, then for all $y \in B_r(x)$, the following inequality is true:

$$0 \leq |f_i(y) - f_i(x)| \leq 0 \cdot |y - x| = 0$$

THis enforces $f_i(y) = f_i(x)$, hence $f_i(x)$ is a constant function when restricting to $B_r(x)$.

2. f_i is constant in E:

Now, fix any $x \in E$, and define $U_x, V_x \subseteq E$ as follow:

$$U_x = \{y \in E \mid f_i(y) = f_i(x)\}, \quad V_x = \{z \in E \mid f_i(z) \neq f_i(x)\}$$

Since for all point $y \in E$, either $f_i(y) = f_i(x)$ or $f_i(y) \neq f_i(x)$, then $y \in U_x \cup V_x$, so $E \subseteq U_x \cup V_x$, or $E = U_x \cup V_x$. Also, by definition, $U_x \cap V_x = \emptyset$, the two sets are disjoint.

Also, both U_x and U_y must be open:

For any $y \in U_x \subseteq E$, since there exists $r' > 0$, such that the open ball $B_{r'}(y) \subseteq E$. Since in the first part, we've proven that f_i is a constant when restricting to any open ball in E, then all $y' \in B_{r'}(y)$ satisfies $f_i(y') = f_i(y)$, while by definition, $y \in U_x$ implies $f_i(y) = f_i(x)$, so $f_i(y') = f_i(x)$, or $y' \in U_x$. Hence, $B_{r'(y)} \subseteq U_x$, proven that U_x is open.

Similarly, for any $z \in V_x \subseteq E$, since there exists $r'' > 0$, such that the open ball $B_{r''}(z) \subseteq E$, then because f_i restricting to any open ball in E is a constant, then all $z' \in B_{r''}(z)$ satisfies $f_i(z') = f_i(z)$, while by definition, $z \in V_x$ implies $f_i(z) \neq f_i(x)$, so $f_i(z') \neq f_i(x)$, showing that $z' \in V_x$. Hence, $B_{r''}(z) \subseteq V_x$, proven that V_x is also open.

Then, since U_x, V_x are two open sets satisfying $U_x \cap V_x = \emptyset$, while $U_x \cap V_x = E$, then they form a separation of E. If both U_x and V_x are not empty, then it contradicts the assumption that E is connected, hence we must have one of the sets being empty.

Which, because there exists $r > 0$, with $B_r(x) \subseteq E$, then from the above proof, we know $B_r(x) \subseteq U_x$, proving that U_x is not empty. Hence, $V_x = \emptyset$, showing that $U_x = E$. So, all $y \in E$ must have $f_i(y) = f_i(x)$, showing that f_i is constant in E.

Since all f_i (with $i \in \{1, ..., m\}$) must be constant, then the original function $f : E \to \mathbb{R}^m$ must also be constant.

2 (not done)

Question 2 *Rudin Pg. 239-240 Problem 12:*
Fix two real numbers a and b, $0 < a < b$. Define a mapping $f = (f_1, f_2, f_3)$ of $\mathbb{R}^2$ into $\mathbb{R}^3$ by

$$f_1(s,t) = (b + a\cos(s))\cos(t), \quad f_2(s,t) = (b + a\cos(s))\sin(t), \quad f_3(s,t) = a\sin(s)$$

Describe the range K of f. (It is a certain compact subset of $\mathbb{R}^3$).

(a) *SHow that there are exactly 4 points $p \in K$ such that*

$$(\nabla f_1)(f^{-1}(p)) = 0$$

Find these points.

(b) *Determine the set of all $q \in K$ such that*

$$(\nabla f_3)(f^{-1}(q)) = 0$$

(c) *Show that one of the points p found in part (a) corresponds to a local maximum of f_1, one corresponds to a local minimum, and that the other two are neither (they are so-called "saddle points").*

Whic of the points q found in part (b) correspond to maxima or minima?

(d) *Let λ be an irrational real number, and define $g(t) = f(t, \lambda t)$. Prove that g is a $1-1$ mapping of $\mathbb{R}$ onto a dense subset of K. Prove that*

$$|g'(t)|^2 = a^2 + \lambda^2(b + a\cos(t))^2$$

Pf:

3

Question 3 *Rudin Pg. 240 Problem 13:*
Suppose f is differentiable mapping of $\mathbb{R}$ into $\mathbb{R}^3$ such that $|f(t)| = 1$ for every 5. Prove that $f'(t) \cdot f(t) = 0$. Interpret this result geometrically.

Pf:

Since $|f(t)| = 1$, then the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(t) = f(t) \cdot f(t) = |f(t)|^2 = 1$, hence $g'(t) = 0$.

On the other hand, since $g'(t) = \frac{d}{dt}(f(t) \cdot f(t))$, while the derivative given by product rule for real dot product is given by:

$$\frac{d}{dt}(f(t) \cdot f(t)) = f'(t) \cdot f(t) + f(t) \cdot f'(t) = 2f'(t) \cdot f(t)$$

Hence, $2f'(t) \cdot f(t) = 0$, or $f'(t) \cdot f(t) = 0$.

Geometrically, since $|f(t)| = 1$, then f is in fact a curve on the 2-dimensional sphere S^2; since $f'(t)$ is the tangent vector (the traveling direction) of the curve at any given point, then $f'(t) \cdot f(t) = 0$ implies the tangent vector and the position of the curve is always orthogonal to each other, showing that to travel on a sphere, the tangent vector is necessarily orthogonal to the surface.

4

Question 4 *Show that the continuity of f' at the point a is needed in the inverse function theorem, even in the case $n = 1$: if*

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

for $t \neq 0$, and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1, 1)$, but f is not $1-1$ in any neighborhood of 0.

Pf:

Derivative at 0:

The derivative of the function at 0 is given as follow:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h + 2h^2 \sin(1/h)}{h} = \lim_{h \to 0}(1 + 2h \sin(1/h)) = 1$$

(Note: since $|h \sin(1/h)| \leq |h|$ for all $0 < h$, then $0 \leq \lim_{h \to 0} |h \sin(1/h)| \leq \lim_{h \to 0} |h| = 0$, so the limit is 0).

Derivative is bounded in $(-1, 1)$, but not continuous at 0:

For any nonzero $t \in (-1, 1)$, based on differentiation rules, we get the following:

$$f'(t) = 1 + 4t \sin\left(\frac{1}{t}\right) + 2t^2 \cos\left(\frac{1}{t}\right) \cdot \frac{-1}{t^2} = 1 + 4t \sin\left(\frac{1}{t}\right) - 2\cos\left(\frac{1}{t}\right)$$

Which, its bound is given as follow:

$$|f'(t)| \leq 1 + \left|4t \sin\left(\frac{1}{t}\right)\right| + \left|2\cos\left(\frac{1}{t}\right)\right| \leq 1 + 4 + 2 = 7$$

(Note: $\sin, \cos$ are both bounded by 1, while $t \in (-1, 1)$ implies $|t| < 1$).

So, conbine with the previous part that $f'(0) = 1$, all $t \in (-1, 1)$ satisfies $|f'(t)| \leq 7$, hence f' is bounded in $(-1, 1)$.

Yet, since $\lim_{t \to 0} f'(t)$ does not exist, then $f'(t)$ is not continuous at 0.

Inverse Function Theorem doesn't apply:

For any open neighborhood $U \subseteq \mathbb{R}$ of 0, there exists $\epsilon > 0$, such that $(-\epsilon, \epsilon) \subseteq U$. Now, by Archimedean's Property, choose $n \in \mathbb{N}$ such that $0 < \frac{1}{2n\pi} < \epsilon$ (which, since $2n\pi < 2n\pi + \pi/2 < 2n\pi + \pi$, then $0 < \frac{1}{2n\pi+\pi} < \frac{1}{2n\pi+\pi/2} < \frac{1}{2n\pi} < \epsilon$, so all of these points are within $(-\epsilon, \epsilon)$).

Let $t_1 = \frac{1}{2n\pi+\pi}$, $t_2 =, \frac{1}{2n\pi+\pi/2}$ $t_3 = \frac{1}{2n\pi}$. If we evaluate f at these points, we get:

$$f_1 = f\left(\frac{1}{2n\pi}\right) = \frac{1}{2n\pi} + 2\left(\frac{1}{2n\pi}\right)^2 \sin\left(\frac{1}{1/(2n\pi)}\right) = \frac{1}{2n\pi} + 2\left(\frac{1}{2n\pi}\right)^2 \sin(2n\pi) = \frac{1}{2n\pi}$$

$$f_2 = f\left(\frac{1}{2n\pi + \pi/2}\right) = \frac{1}{2n\pi + \pi/2} + 2\left(\frac{1}{2n\pi + \pi/2}\right)^2 \sin\left(\frac{1}{1/(2n\pi + \pi/2)}\right)$$
$$= \frac{1}{2n\pi + \pi/2} + 2\left(\frac{1}{2n\pi + \pi/2}\right)^2 \sin\left(2n\pi + \frac{\pi}{2}\right) = \frac{1}{2n\pi + \pi/2} + 2\left(\frac{1}{2n\pi + \pi/2}\right)^2$$

$$f_3 = f\left(\frac{1}{2n\pi + \pi}\right) = \frac{1}{2n\pi + \pi} + 2\left(\frac{1}{2n\pi + \pi}\right)^2 \sin\left(\frac{1}{1/(2n\pi + \pi)}\right)$$
$$= \frac{1}{2n\pi + \pi} + 2\left(\frac{1}{2n\pi + \pi}\right)^2 \sin(2n\pi + \pi) = \frac{1}{2n\pi + \pi}$$

If we compare f_2 and f_3, we get:

$$f_2 = \frac{1}{2n\pi + \pi/2} + 2\left(\frac{1}{2n\pi + \pi/2}\right)^2 > \frac{1}{2n\pi + \pi/2} > \frac{1}{2n\pi + \pi} = f_3$$

On the other hand, if we choose $n > \frac{\pi}{16-4\pi} > 0$, we get the following inequality:

$$16n - 4n\pi > \pi \implies 16n - 4n\pi - \pi > 0 \implies \frac{16n - (4n\pi + \pi)}{4n(4n\pi + \pi)} > 0 \implies \frac{4}{4n\pi + \pi} - \frac{1}{4n} > 0$$

$$\implies \frac{2}{2n\pi + \pi/2} - \frac{\pi/2}{2n\pi} > 0 \implies 2\left(\frac{1}{2n\pi + \pi/2}\right)^2 - \frac{\pi/2}{2n\pi(2n\pi + \pi/2)} > 0$$
$$\implies 2\left(\frac{1}{2n\pi + \pi/2}\right)^2 + \left(\frac{1}{2n\pi + \pi/2} - \frac{1}{2n\pi}\right) > 0$$
$$\implies \frac{1}{2n\pi + \pi/2} + 2\left(\frac{1}{2n\pi + \pi/2}\right)^2 > \frac{1}{2n\pi}$$

Which, this inequality implies the following:

$$f_2 = \frac{1}{2n\pi + \pi/2} + 2\left(\frac{1}{2n\pi + \pi/2}\right)^2 > \frac{1}{2n\pi} = f_1$$

So, $f_2 > f_3$ and $f_2 > f_1$

Now, choose any y satisfying $f_3 < y < f_2$ and $f_1 < y < f_2$. Since by the notation, we have $t_1 < t_2 < t_3$, and $f_1 = f(t_1)$, $f_2 = f(t_2)$, and $f_3 = f(t_3)$, then because f is a continuous function, by Intermediate Value Theorem, there exists $c \in (t_1, t_2)$ and $c' \in (t_2, t_3)$, such that $f(c) = f(c') = y$.

Which, because $c \neq c'$, then f is not injective; also, since $0 < t_1 < c < t_2 < c' < t_3 < \epsilon$, then $c, c' \in (-\epsilon, \epsilon) \subseteq U$. This shows that f restricting to U is not injective. Then, because $f|U$ for any open neighborhood U of 0 is not injective, then Inverse Function Theorem fails.

Beca

5

Question 5 *Lef $f = (f_1, f_2)$ be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by*

$$f_1(x, y) = e^x \cos(y), \quad f_2(x, y) = e^x \sin(y)$$

(a) What is the range of f?

(b) Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$. Thus every point of $\mathbb{R}^2$ has a neighborhood in which f is $1 - 1$. Nevertheless, f is not $1 - 1$ on $\mathbb{R}^2$.

(c) Put $a = (0, \pi/3)$, $b = f(a)$, let g be the continuous inverse of f, defined in a neighborhood of b, such that $g(b) = a$. Find an explicit formula of g, compute $f'(a)$ and $g'(b)$, and verify the formula (52).

(Note: Formula (52) states if g is an inverse of f, then for any y in the given domain of f, $g'(y) = (f'(g(y)))^{-1}$).

(d) What are the images under f of lines parallel to the coordinate axes?

Pf: