Math 118C HW3

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1

Question 1 *Rudin Pg. 241 Problem 19:*
Show that the system of equations

$$3x + y - z + u^2 = 0$$

$$x - y + 2z + u = 0$$

$$2x + 2y - 3z + 2u = 0$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

Pf:

If we define the function $f : \mathbb{R}^4 \to \mathbb{R}^3$ by $f = (f_1, f_2, f_3)$, with $f_1, f_2, f_3 : \mathbb{R}^4 \to \mathbb{R}$ defined as:

$$f_1(x, y, z, u) = 3x + y - z + u^2, \quad f_2(x, y, z, u) = x - y + 2z + u, \quad f_3(x, y, z, u) = 2x + 2y - 3z + 2u$$

So, in the order of x, y, z, u from the left to right, the differential of f is given as follow:

$$A = Df(x, y, z, u) = \begin{pmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{pmatrix}$$

Notice that there exists an obvious solution to $f(\bar{x}) = \bar{0} \in \mathbb{R}^3$, which is $\bar{x} = (0, 0, 0, 0) \in \mathbb{R}^4$, so there exists $\bar{x} \in \mathbb{R}^4$, with $f(\bar{x}) = 0$.

Now, since f_1, f_2, f_3 are all multivariable polynomials of variables x, y, z, u, then the partial derivatives all exist and are all continuous, hence f is in fact continuously differentiable. To apply Implicit Function Theorem, we just need to verify when isolating one variable, the singularity of f's differential with respect to the other variables, at all points $\bar{x} \in \mathbb{R}^4$ with $f(\bar{x}) = 0$.

Isolating z:

If isolate z, with respect to x, y, u, A has a corresponding linear operator $A_{xyu} \in \mathcal{L}(\mathbb{R}^3)$ given by:

$$A_{xyu} = \begin{pmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$

Which has its determinant given by:

$$\det A_{xyu} = 2u(2 - (-2)) - 1(2 - 2) + 3((-2) - 2) = 8u - 12$$

So, A_{xyu} is not invertible iff $\det A_{xyu} = 0$ iff $u = \frac{3}{2}$.

Isolating y:
If isolate y, with respect to x, z, u, A has a corresponding linear operator $A_{xzu} \in \mathcal{L}(\mathbb{R}^3)$ given by:

$$A_{xzu} = \begin{pmatrix} 3 & -1 & 2u \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix}$$

Which has its determinant given by:

$$\det A_{xzu} = 2u((-3) - 4) - (-1)(2 - 2) + 3(4 - (-3)) = -14u + 21$$

So, A_{xzu} is not invertible iff $\det A_{xzu} = 0$ iff $u = \frac{3}{2}$.

Isolating x:
If isolate x, with respect to y, z, u, A has a corresponding linear operator $A_{yzu} \in \mathcal{L}(\mathbb{R}^3)$ given by:

$$A_{yzu} = \begin{pmatrix} 1 & -1 & 2u \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix}$$

Which has its determinant given by:

$$\det A_{yzu} = 2u(3 - 4) - (-1)((-2) - 2) + 1(4 - (-3)) = -2u - 4 + 7 = -2u + 3$$

So, A_{yzu} is not invertible iff $\det A_{yzu} = 0$ iff $u = \frac{3}{2}$.

Isolating u:
If isolate u, with respect to x, y, z, A has a corresponding linear operator $A_{xyz} \in \mathcal{L}(\mathbb{R}^3)$ given by:

$$A_{xyz} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix}$$

Which has its determinant given by:

$$\det A_{xyz} = 3(3 - 4) - 1((-3) - 4) - 1(2 - (-2)) = -3 + 7 - 4 = 0$$

So regardless of the point, when isolating u, the differential with respect to x, y, z is always noninvertible.

Notice that in the first three cases, the only possibility for the linear operators to be noninvertible, is when $u = \frac{3}{2}$. Which, to see if there are solutions corresponding to $u = \frac{3}{2}$, plug into the original equation, it's equivalent to the following systems of linear equation:

$$\begin{cases} 3x + y - z = -u^2 = -9/4 \\ x - y + 2z = -u = -3/2 \\ 2x + 2y - 3z = -2u = -3 \end{cases}$$

Which, after row reduction, we get:

$$\begin{cases} 2x+2y-3z=-3/4 \\ x-y+2z=-3/2 \\ 2x+2y-3z=-3 \end{cases} \implies \begin{cases} 2x+2y-3z=-3/4 \\ x-y+2z=-3/2 \\ 0x+0y+0z=-9/4 \end{cases}$$

Since there is an inconsistency in the system, there is no solution to $f(x,y,z,\frac{3}{2})=0$. Hence, for all $\bar{x}\in\mathbb{R}^4$ satisfying $f(\bar{x})=0$ (which implies $u\neq\frac{3}{2}$), the first three cases (when isolating z,y, or x) has the linear operator of the remaining variables (corresponding to the differential $Df(\bar{x})$) being invertible, hence Implicit Function Theorem applys.

So, when solving x,y,u in terms of z, by Implicit Function Theorem, if given $(x,y,z,u)\in\mathbb{R}^4$ satisfies $f(x,y,z,u)$, there exists open neighborhood $U\subseteq\mathbb{R}^4$ of (x,y,z,u) and open neighborhood $V\subseteq\mathbb{R}$ of u, such that all $u'\in V$ corresponds to a unique $(x',y',z')\in\mathbb{R}^3$, such that $(x',y',z',u')\in U$, and $f(x',y',z',u')=0$.

Hence, for each $z\in\mathbb{R}$ that corresponds to some $(x,y,u)\in\mathbb{R}^3$ satisfying $f(x,y,z,u)=0$, such correspondance is unique based on Implicit Function Theorem (since within an open neighborhood V of u, each u' corresponds to a unique $(x',y',z')\in\mathbb{R}^3$ such that $f(x',y',z',u')=0$), therefore it's possible to solve x,y,u in terms of z. And, same logic applies when solving x,z,u in terms of y, and solving y,z,u in terms of x (since in all three cases, Implicit Function Theorem applies).

Finally, the reason why it's not possible to solve x,y,z in terms of u, is because for some $u\in\mathbb{R}$, there exists more than one solution to the corresponding $(x,y,z)\in\mathbb{R}^3$: Fix $u=0$, then the original system of equations become:

$$\begin{cases} 3x+y-z=-u^2=0 \\ x-y+2z=-u=0 \\ 2x+2y-3z=-2u=0 \end{cases}$$

Written in matrix equation, we get:

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Notice that this matrix is exactly A_{xyz} calculated in one of the previous parts, with $\det A_{xyz}=0$, hence, this matrix is not invertible, which also implies that it is not injective, therefore its null space is nontrivial, there exists nonzero $(x,y,z)\in\mathbb{R}^3$, such that the above equation holds.

But, this implies $f(x,y,z,0)=f(0,0,0,0)=\bar{0}\in\mathbb{R}^3$, showing that there are multiple $(x,y,z)\in\mathbb{R}^3$ corresponding to $u=0$, such that the system of equations hold, therefore it's not possible to solve for x,y,z in terms of u, due to the existence of multiple solutions.

2

Question 2 *Rudin Pg. 242 Problem 23:*
Define f in $\mathbb{R}^3$ by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2$$

Show that $f(0,1,-1) = 0$,$(D_1 f)(0,1,-1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1,-1)$ in $\mathbb{R}^2$, such that $g(1,-1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0$$

Find $(D_1 g)(1,-1)$ and $(D_2 g)(1,-1)$.

Pf:

f and $D_1 f$ at $(0,1,-1)$:

Evaluate at $(0,1,-1)$, we get $f(0,1,-1) = 0^2 \cdot 1 + e^0 + (-1) = 0 + 1 - 1 = 0$.

On the other hand, $D_1 f(x, y_1, y_2)$ is given as follow:

$$D_1 f(x, y_1, y_2) = \frac{\partial}{\partial x}(x^2 y_1 + e^x + y_2) = 2xy_1 + e^x$$

Hence, $D_1 f(0,1,-1) = 2 \cdot 0 \cdot 1 + e^0 = 1 \neq 0$.

Validity of Implicit Function Theorem:

If we consider the partial derivatives with respect to all variables, we get:

$$\frac{\partial f}{\partial x} = 2xy_1 + e^x, \quad \frac{\partial f}{\partial y_1} = x^2, \quad \frac{\partial f}{\partial y_2} = 1$$

Hence, since the partial derivative of f with respect to any variable is continuous, then f is continuously differentiable, with $Df(x, y_1, y_2) = (2xy_1 + e^x, x^2, 1) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$.

Then, since in the previous part, we get that $D_1 f(0,1,-1) \neq 0$, which if view $\mathbb{R}^3 = \mathbb{R}^{1+2}$, with $(0,1,-1) = (0, \bar{0}) + (0, (1,-1))$, then $A = D_1 f(0,1,-1) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$, can be broken down as follow:

$$\forall x \in \mathbb{R},\ \bar{y} \in \mathbb{R}^2, \quad A(x, \bar{y}) = A_x(x) + A_y(\bar{y}), \quad A_x = D_1 f(0,1,-1) \in \mathcal{L}(\mathbb{R}), \quad A_y = \left(\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2} \right) \Big|_{(0,1,-1)} \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$$

Then, since f is continuously differentiable, while $A_x = D_1 f(0,1,-1)$ for $A = Df(0,1,-1)$ is invertible (since $D_1 f(0,1,-1) \neq 0$), then by Implicit Function Theorem, there exists open neighborhood $U \subseteq \mathbb{R}^3$ of $(0,1,-1)$, open neighborhood $V \subseteq \mathbb{R}^2$ of $(1,-1)$, such that for all $\bar{y} \in V$, there exists a unique $x \in \mathbb{R}$, such that $f(x, \bar{y}) = 0$.

Which, define $g : V \to \mathbb{R}$ by $g(\bar{y}) = x$ got from the previous part, then g is continuously differentiable, with $g(1,-1) = 0$ (since $g(0,1,-1) = 0$, $(1,-1)$ corresponds to $x = 0$).

And, by the equation of $Dg(0,1,-1)$, given $A = Df(0,1,-1)$ and A_x and A_y provided above, it is given by the following:

$$Dg(1,-1) = -A_x^{-1} A_y = -\left(D_1 f(0,1,-1)\right)^{-1} \begin{pmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \end{pmatrix} \Big|_{(0,1,-1)} = \begin{pmatrix} -\frac{\partial f}{\partial y_1} & -\frac{\partial f}{\partial y_2} \end{pmatrix} \Big|_{(0,1,-1)}$$

Also, by the uniqueness of the differential, we know the following:

$$Dg(y_1, y_2) = \begin{pmatrix} \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \end{pmatrix}$$

Then, we get the following relations:

$$D_1 g(1,-1) = \frac{\partial g}{\partial y_1} \Big|_{(1,-1)} = -\frac{\partial f}{\partial y_1} \Big|_{(0,1,-1)} = -0^2 = 0$$

$$D_2 g(1,-1) = \frac{\partial g}{\partial y_2} \Big|_{(1,-1)} = -\frac{\partial f}{\partial y_2} \Big|_{(0,1,-1)} = -1$$

So, $D_1 g(1,-1) = 0$, and $D_2 g(1,-1) = -1$.

3

Question 3 *Rudin Pg. 242 Problem 24:*
For $(x, y) \neq (0,0)$, define $f = (f_1, f_2)$ by

$$f_1(x,y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad f_2(x,y) = \frac{xy}{x^2 + y^2}$$

Compute the rank of $f'(x,y)$, and find the range of f.

Pf:
Rank of $f'(x,y) = Df(x,y)$:
Given f_1 and f_2, their partial derivatives are given as follow:

$$\frac{\partial f_1}{\partial x} = \frac{2x(x^2+y^2) - 2x(x^2-y^2)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}, \quad \frac{\partial f_1}{\partial y} = \frac{-2y(x^2+y^2) - 2y(x^2-y^2)}{(x^2+y^2)^2} = \frac{-4x^2 y}{(x^2+y^2)^2}$$

$$\frac{\partial f_2}{\partial x} = \frac{y(x^2+y^2) - 2x(xy)}{(x^2+y^2)^2} = \frac{y^3 - x^2 y}{(x^2+y^2)^2}, \quad \frac{\partial f_2}{\partial y} = \frac{x(x^2+y^2) - 2y(xy)}{(x^2+y^2)^2} = \frac{x^3 - xy^2}{(x^2+y^2)^2}$$

Hence, the differential Df is given as:

$$Df(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \frac{1}{(x^2+y^2)^2} \begin{pmatrix} 4xy^2 & -4x^2 y \\ (y^3 - x^2 y) & (x^3 - xy^2) \end{pmatrix}$$

- If $x = 0$, the matrix is given by:

  $$Df(0,y) = \frac{1}{y^4} \begin{pmatrix} 0 & 0 \\ y^3 & 0 \end{pmatrix}$$

  Which, with $y \neq 0$ (due to the condition $(x,y) \neq (0,0)$), the above matrix has rank 1.

- If $y = 0$, the matrix is given by:

  $$Df(x,0) = \frac{1}{x^4} \begin{pmatrix} 0 & 0 \\ 0 & x^3 \end{pmatrix}$$

  Which again, with $x \neq 0$, the above matrix has rank 1.

- If both $x, y \neq 0$, then consider any vector $r(x,y)$ for $r \in \mathbb{R}$, we get::

  $$Df(x,y) r \begin{pmatrix} x \\ -y \end{pmatrix} = \frac{1}{(x^2+y^2)^2} \begin{pmatrix} xy^2 & -x^2 y \\ (y^3 - x^2 y) & (x^3 - xy^2) \end{pmatrix} r \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{r}{(x^2+y^2)^2}\begin{pmatrix} x^2y^2 - x^2y^2 \\ (xy^3 - x^3y) + (x^3y - xy^3) \end{pmatrix} = \bar{0}$$

So the span$\{(x,y)\}$ is within the null space of $Df(x,y)$, so the dimension of null space is at least 1. On the other hand, consider $(1,0) \in \mathbb{R}^2$ (which is linearly independent with (x,y), since (x,y) has both entries being nonzero), it has the following:

$$Df(x,y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{(x^2+y^2)^2}\begin{pmatrix} xy^2 & -x^2y \\ (y^3 - x^2y) & (x^3 - xy^2) \end{pmatrix} r \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{(x^2+y^2)^2}\begin{pmatrix} xy^2 \\ (y^3 - x^2y) \end{pmatrix} \neq \bar{0}$$

Which, it shows that the range of $Df(x,y)$ is nontrivial, hence it has dimension at least 1 also.

Because both the null space and the range have dimension ≥ 1, while $\mathbb{R}^2$ has dimension 2, by Rank Nullity Theorem, it enforces both the null space and the range must have dimension precisely 1 (since the sum of the dimension of the null space and the range must be 2). So, the rank of $Df(x,y)$ is again 1.

So, regardless of the case, $Df(x,y)$ has rank 1.

Range of f:

When fixing $(x,y) \neq (0,0)$ in $\mathbb{R}^2$, there exists $r > 0$, and $\theta \in [0, 2\pi)$, such that $x = r\cos(\theta)$ and $y = r\sin(\theta)$ (under polar coordinates). Then, we get the output of f_1, f_2 as:

$$f_1(x,y) = \frac{(r\cos(\theta))^2 - (r\sin(\theta))^2}{(r\cos(\theta))^2 + (r\sin(\theta))^2} = \frac{r^2(\cos^2(\theta) - \sin^2(\theta))}{r^2} = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$$

$$f_2(x,y) = \frac{(r\cos(\theta))(r\sin(\theta))}{(r\cos(\theta))^2 + (r\sin(\theta))^2} = \frac{r^2\cos(\theta)\sin(\theta)}{r^2} = \sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$$

Hence, for all $(x,y) \neq (0,0)$, $f = (f_1, f_2)$ satisfies the following equation:

$$f_1^2 + 4f_2^2 = \cos^2(2\theta) + 4 \cdot \frac{1}{4}\sin^2(2\theta) = \cos^2(2\theta) + \sin^2(2\theta) = 1$$

Hence, $(u,v) = f(x,y)$ is a solution to $u^2 + 4v^2 = 1$, so the range of f is contained in the ellipse characterized by $u^2 + 4v^2 = 1$.

On the other hand, for all point (u,v) satisfying $u^2 + 4v^2 = 1$, there exists $\theta \in [0, 2\pi)$, such that $u = \cos(\theta)$ and $v = \frac{1}{2}\sin(\theta)$. Then, consider the point $(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})) \in \mathbb{R}^2$, we have:

$$f(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})) = (f_1(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})), f_2(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2}))) = \left(\cos\left(2 \cdot \frac{\theta}{2}\right), \frac{1}{2}\sin\left(2 \cdot \frac{\theta}{2}\right)\right)$$

$$= \left(\cos(\theta), \frac{1}{2}\sin(\theta)\right) = (u,v)$$

Hence, (u,v) is also in the range of f. This proves that f has the range precisely described by the ellipse $u^2 + 4v^2 = 1$.

4

Question 4 *Rudin Pg. 242 Problem 25:*
Suppose $A \in \mathcal{L}(\mathbb{R}^n, \mathcal{R}^m)$, let r be the rank of A.

(a) Define S as in the proof of Theorem 9.32. Show that SA is a projection in $\mathbb{R}^n$ whose null space is null(A) and whose range is range(S).

(b) Use (a) to show that
$$\dim(\textit{null}(A)) + \dim(\textit{range}(A)) = n$$

Pf:

(a) Given that A has rank r, then its range range$(A) \subseteq \mathbb{R}^m$ is an r-dimensional linear subspace, hence there exists $y_1, ..., y_r \in$ range(A) that forms a basis of it.

Then, by the text in Rudin, choose $z_1, ..., z_r \in \mathbb{R}^n$, so for each index $i \in \{1, ..., r\}$, $Az_i = y_i$. Which, the collection $z_1, ..., z_r \in \mathbb{R}^n$ is linearly independent, since if $a_1, ..., a_r \in \mathbb{R}$ satisfies $\sum_{i=1}^r a_i z_i = \bar{0}$, then the following is true:

$$0 = A\bar{0} = A\left(\sum_{i=1}^r a_i z_i\right) = \sum_{i=1}^r a_i (Az_i) = \sum_{i=1}^r a_i y_i$$

By the linear independence of $y_1, ..., y_r \in$ range(A), each $a_i = 0$, which proves the linear independence of $z_1, ..., z_r \in \mathbb{R}^n$.

Now, define $S \in \mathcal{L}(\text{range}(A), \mathbb{R}^n)$ the same as in the text, which has the following formula:

$$\forall c_1, ..., c_r \in \mathbb{R}, \quad S\left(\sum_{i=1}^r c_i y_i\right) = \sum_{i=1}^r c_i z_i$$

Then, for all $x \in \mathbb{R}^n$, since $Ax \in$ range(A), it is spanned by $y_1, ..., y_r$, hence there exists unique $a_1, ..., a_r \in \mathbb{R}$, such that the following is true:

$$Ax = \sum_{i=1}^r a_i y_i$$

Hence, with $SA \in \mathcal{L}(\mathbb{R}^n)$, we get the following:

$$SAx = S\left(\sum_{i=1}^r a_i y_i\right) = \sum_{i=1}^r a_i z_i$$

Hence, applying SA twice, we get:

$$SA(SAx) = SA\left(\sum_{i=1}^r a_i z_i\right) = S\left(\sum_{i=1}^r a_i y_i\right) = \sum_{i=1}^r a_i z_i$$

This shows that $SA(SAx) = SAx$ for all $x \in \mathbb{R}^n$, hence SA is a projection on $\mathbb{R}^n$.

Now, to find the null space and range, consider the following:

- For all $x \in$ null(A), since $Ax = 0$, then $SAx = S(0) = 0$, so $x \in$ null(SA), or null$(A) \subseteq$ null(SA).

On the other hand, for all $\in \text{null}(SA)$, since $S(Ax) = 0$, $Ax \in \text{null(S)}$. But, since $Ax \in \text{range}(A)$, there exists unique $a_1, ..., a_r \in \mathbb{R}$, with $Ax = \sum_{i=1}^r a_i y_i$. Hence, we have the following:

$$0 = S(Ax) = S\left(\sum_{i=1}^r a_i y_i\right) = \sum_{i=1}^r a_i z_i$$

Hence, by linear independence of $z_1, ..., z_r \in \mathbb{R}^n$, we must have $a_i = 0$ for all index $i \in \{1, ..., r\}$. This proves that $Ax = \sum_{i=1}^r a_i y_i = 0$, so $x \in \text{null}(A)$. Hence, $\text{null}(SA) \subseteq \text{null}(A)$, showing that $\text{null}(SA) = \text{null}(A)$.

– For all $z \in \text{range}(SA)$, there exists $x \in \mathbb{R}^n$ with $SAx = z$. Since $z = S(Ax) \in \text{range}(S)$, then $\text{range}(SA) \subseteq \text{range}(S)$.

  Similarly, for all $z \in \text{range}(S)$, there exists $y \in \text{range}(A)$ (the domain of S), with $Sy = z$; then because there exists $x \in \mathbb{R}^n$, with $Ax = y$ by the definition of range, we have $SAx = S(Ax) = Sy = z$, hence $z \in \text{range}(SA)$, proving that $\text{range}(S) \subseteq \text{range}(SA)$, or $\text{range}(S) = \text{range}(SA)$.

Hence, the above two cases proves that $\text{null}(SA) = \text{null}(A)$, while $\text{range}(S) = \text{range}(SA)$. So, SA is a projection in $\mathbb{R}^n$ with null space being $\text{null}(A)$, and range being $\text{range}(S)$.

(b) With the linearly independent set $z_1, ..., z_r \in \mathbb{R}^n$ given in **part (a)**, we'll consider an extra list $x_1, ..., x_k \in \text{null}(A) \subseteq \mathbb{R}^n$ that forms a basis of $\text{null}(A)$. Our goal is to prove that $x_1, ..., x_k, z_1, ..., z_r$ forms a basis of $\mathbb{R}^n$.

First, consider $a_1, ..., a_k, b_1, ..., b_r \in \mathbb{R}$, such that the vector $\sum_{i=1}^k a_i x_i + \sum_{j=1}^r b_j z_j = \bar{0} \in \mathbb{R}^n$, then we have the following:

$$0 = A(\bar{0}) = A\left(\sum_{i=1}^k a_i x_i + \sum_{j=1}^r b_j z_j\right) = A\left(\sum_{i=1}^k a_i x_i\right) + A\left(\sum_{j=1}^r b_j z_j\right) = \sum_{j=1}^r b_j (Az_j) = \sum_{j=1}^r b_j y_j$$

(Note: since $x_1, ..., x_k \in \text{null}(A)$, then $\sum_{i=1}^k a_i x_i \in \text{null}(A)$).

Which, by the linear independence of $y_1, ..., y_r \in \text{range}(A)$ assumed in **part (a)**, we must have $b_j = 0$ for all $j \in \{1, ..., n\}$. So, $\sum_{i=1}^k a_i x_i + \sum_{j=1}^r b_j z_j = \sum_{i=1}^k a_i x_i = \bar{0}$. But again, based on the linear independence ov $x_1, ..., x_k$ by assumption, we get $a_i = 0$ for all $i \in \{1, ..., k\}$. This proves that all $a_i, b_j = 0$, which the collection $x_1, ..., x_k, z_1, ..., z_r$ is linearly independent.

Then, for all $x \in \mathbb{R}^n$, since $Ax \in \text{range}(A)$, then there exists unique $b_1, ..., b_r \in \mathbb{R}$, with $Ax = \sum_{j=1}^r b_j y_j$. Hence, we get the following:

$$Ax = \sum_{j=1}^r b_j y_j = \sum_{j=1}^r b_j (Az_j) = A\left(\sum_{j=1}^r b_j z_j\right)$$

So, we can reduce to the following:

$$Ax - A\left(\sum_{j=1}^r b_j z_j\right) = A\left(x - \sum_{j=1}^r b_j z_j\right) = 0, \quad x - \sum_{j=1}^r b_j z_j \in \text{null}(A)$$

Then, since $x_1, ..., x_k$ forms a basis of $\text{null}(A)$, then there exists unique $a_1, ..., a_k \in \mathbb{R}$, with:

$$x - \sum_{j=1}^r b_j z_j = \sum_{i=1}^k a_i x_i, \quad x = \sum_{i=1}^k a_i x_i + \sum_{j=1}^r b_j z_j$$

So, $x \in \text{span}\{x_1, ..., x_k, z_1, ..., z_r\}$, proving that $\mathbb{R}^n = \text{span}\{x_1, ..., x_k, z_1, ..., z_r\}$.

Hence, since $x_1, ..., x_k, z_1, ..., z_r$ spans $\mathbb{R}^n$ while being linearly independent, it is a basis of $\mathbb{R}^n$. Hence, the length of the basis, $k + r = \dim(\mathbb{R}^n) = n$.

Which, since $x_1, ..., x_k$ is a basis of $\text{null}(A)$, then $\dim(\text{null}(A)) = k$.

On the other hand, since $S \in \mathcal{L}(\text{range}(A), \mathbb{R}^n)$ is injective (since it maps $y_1, ..., y_r$ a basis of $\text{range}(A)$, to $z_1, ..., z_r \in \mathbb{R}^n$ a linearly independent set), then the domain of S, $\text{range}(A)$ and range of S are in fact isomorphic as vector spaces, while the range of S is precisely $\text{span}\{z_1, ..., z_r\}$ (since the definition of S is maps y_i to z_i for each $i \in \{1, ..., r\}$, showing that the output value must be a linear combination of all z_i). Hence, $\dim(\text{range}(A)) = \dim(\text{span}\{z_1, ..., z_r\}) = r$ (since $z_1, ..., z_r$ is linearly independent, it forms a basis of the span).

So, compile the information from above, we get:

$$k + r = n, \quad k = \dim(\text{null}(A)), \quad r = \dim(\text{range}(A))$$

$$\implies \dim(\text{null}(A)) + \dim(\text{range}(A)) = n$$

5

Question 5 *Rudin Pg. 242 Problem 26:*
Show that the existence (and even the continuity) of $D_{12}f$ does not imlpy the existence of $D_1 f$. For example, let $f(x, y) = g(x)$, where g is nowhere differentiable.

Pf:

Consider the Weierstrass Functon $g : \mathbb{R} \to \mathbb{R}$, which is uniformly continuous, while being differentiable nowhere.

Then, given the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = g(x)$, since g is not differentiable with respect to its variable x, then $D_1 f$ does not exist; yet, since $D_2 f \equiv 0$ (due to the fact that g is a constant when x is fixed), then $D_{12}f = D_1(D_2 f) = 0$.

Hence, even though $D_{12}f$ is continuous, $D_1 f$ doesn't exist in this case.