# Math CS 122B HW1

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## 1

**Question 1** *Ahlfors Pg. 178 Problem 2:*
*Show that the series*

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

*converges for $Re(z) > 1$, and represent its derivative in series form.*

**Pf:**
The following proof would assume the domain of the above series is the half plane $Re(z) > 1$.

**The series converges pointwise:**
For all $z \in \mathbb{C}$ satisfying $Re(z) > 1$, $z = a + bi$ for $a, b \in \mathbb{R}$, and $a > 1$. Then, for any $n \in \mathbb{N}$, the number $n^{-z} = e^{-z\log(n)} = e^{-(a+bi)\ln(n)} = e^{-a\ln(n)} \cdot e^{i(-b\ln(n))} = n^{-a} \cdot e^{i(-b\ln(a))}$. Hence, if taken the modulus $e^{-a\ln(n)}$, since $a > 1$, by p-series test, the following series converges:

$$\sum_{n=1}^{\infty} n^{-a}$$

Which, since the original series satisfies the following:

$$\sum_{n=1}^{\infty} |n^{-z}| = \sum_{n=1}^{\infty} \left| n^{-a} \cdot e^{i(-b\ln(n))} \right| = \sum_{n=1}^{\infty} n^{-a}$$

Hence, the series absolutely converges, which $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ is defined on $Re(z) > 1$.

**Partial Sum converges uniformly on any Compact Subset:**
Suppose $K \subset \mathbb{C}$ is a compact subset of the plane $Re(z) > 1$, each component $n^{-z} = e^{-z\ln(n)} = n^{-a} \cdot e^{i(-b\ln(n))}$ is analytic on the half plane (also on $K$), hence there exists $z_0 \in K$, such that $|n^{-z_0}|$ yields the maximum.

Since for $z_0 = a+bi$, $|n^{-z_0}| = \left| n^{-a} \cdot e^{i(-b\ln(n))} \right| = n^{-a}$, and since $z_0$ is in the half plane, so $Re(z_0) = a > 1$. Then, the series $\sum_{n=1}^{\infty} n^{-a}$ converges.

Now, notice that for each $n \in \mathbb{N}$, $M_n = \sup_{z \in K} |n^{-z}| = \max_{z \in K} |n^{-z}| = n^{-a}$ satisfies $\sum_{n=1}^{\infty} M_n$ converges, then by Weierstrass M-Test, the series $\sum_{n=1}^{\infty} n^{-z}$ in fact converges uniformly on $K$.

Then, because the series $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ converges absolutely on $Re(z) > 1$, converging uniformly on all compact subset of the half plane, and each component is analytic on the half plane, then by the theorem in Ahlfors pg. 176, $\zeta(z)$ is analytic, and the partial sum $\sum_{n=1}^{N} n^{-z}$ (for $N \in \mathbb{N}$) has derivative converges to

$\zeta'(z)$ uniformly on all compact subsets of the half plane. Hence, based on the same theorem again, we can claim the folloing on the chosen half plane:

$$\zeta'(z) = \lim_{N\to\infty} \frac{d}{dz}\sum_{n=1}^{N} n^{-z} = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{d}{dz}(e^{-z\ln(n)}) = \lim_{N\to\infty}\sum_{n=1}^{N} -\ln(n)e^{-z\ln(n)} = -\sum_{n=1}^{\infty} ln(n)n^{-z}$$

# 2

**Question 2** *Ahlfors Pg. 184 Problem 5:*
*The Fibonacci numbers are defined by* $c_0 = 0$, $c_1 = 1$, *and* $c_n = c_{n-1} + c_{n-2}$ *for all* $n \geq 2$.
*Show that the* $c_n$ *are Taylor Coefficients of a rational function, and determine a closed expression for* $c_n$.

**Pf:**
Consider the generating function, a formal power series defined as $F(z) = \sum_{n=0}^{\infty} c_n z^n$.

**Radius of Convergence of the Power Series:**
Recall that radius of convergence of power series $R = \limsup(|c_n|^{\frac{1}{n}})^{-1} \in [0,\infty]$, where $c_n$ is the coefficients of each degree.

First, we can verify that for all $n \in \mathbb{N}$, $c_n < 2^n$:
For base case $n = 1$, $c_1 = 1 < 2^1$.
Now, suppose for given $n \geq 1$, $c_n < 2^n$, then for case $(n+1) \geq 2$, since $c_{n+1} = c_n + c_{n-1} < 2 \cdot c_n$ (since $c_n > c_{n-1}$), then $c_{n+1} < 2\cdot c_n < 2\cdot 2^n = 2^{n+1}$ by induction hypothesis, which this completes the induction.
Since all $n \in \mathbb{N}$ has $0 < c_n < 2^n$, then $|c_n|^{\frac{1}{n}} = c_n^{\frac{1}{n}} < (2^n)^{\frac{1}{n}} < 2$, so $\limsup(|c_n|^{\frac{1}{n}}) \leq 2$, hence $R = \limsup(|c_n|^{\frac{1}{n}})^{-1} \geq \frac{1}{2}$. Thus, we can claim that the power series $F(z) = \sum_{n=0}^{\infty} c_n z^n$ in fact converges absolutely for disk $|z| < \frac{1}{2}$ (since $|z| < \frac{1}{2}$ is contained in the radius of convergence).

**Closed Expression of $c_n$:**
Now, consider power series $F(z)$ on $|z| < \frac{1}{2}$: Since $F(z)$ can be rewritten as $c_0 + c_1 z + \sum_{n=2}^{\infty} c_n z^n = 1 + z + \sum_{n=2}^{\infty} c_n z^n$. Then, based on the definition of Fibonnaci numbers, it can be rewritten as:

$$F(z) = 1 + z + \sum_{n=2}^{\infty}(c_{n-1}+c_{n-2})z^n = 1 + z + \sum_{n=2}^{\infty} c_{n-1}z^n + \sum_{n=2}^{\infty} c_{n-2}z^n$$

$$= 1 + z + \sum_{n=1}^{\infty} c_n z^{n+1} + \sum_{n=0}^{\infty} c_n z^{n+2} = 1 + z\left(1 + \sum_{n=1}^{\infty} c_n z^n\right) + z^2\sum_{n=0}^{\infty} c_n z^n$$

$$= 1 + z\sum_{n=0}^{\infty} c_n z^n + z^2 F(z) = 1 + zF(z) + z^2F(z)$$

Then, we can yield the following:

$$F(z) = 1 + zF(z) + z^2F(z), \quad F(z)(1 - z - z^2) = 1, \quad F(z) = \frac{1}{1 - z - z^2}$$

Now, if $1-z-z^2=0$ (or $z^2+z-1=0$), we have $z=\frac{-1\pm\sqrt{5}}{2}$. Hence, $1-z-z^2=-\left(\frac{-1+\sqrt{5}}{2}-z\right)\left(\frac{-1-\sqrt{5}}{2}-z\right)$. Then, $F(z)$ can be decomposed using partial fraction:

$$F(z)=\frac{A}{\frac{-1+\sqrt{5}}{2}-z}+\frac{B}{\frac{-1-\sqrt{5}}{2}-z}=\frac{1}{1-z-z^2},\quad B\left(\frac{-1+\sqrt{5}}{2}-z\right)+A\left(\frac{-1-\sqrt{5}}{2}-z\right)=-1$$

So, from the above expression, we get:

$$B\cdot\frac{-1+\sqrt{5}}{2}+A\cdot\frac{-1-\sqrt{5}}{2}=-1,\quad -B-A=0$$

$$\implies A=-B,\quad B\cdot\frac{-1+\sqrt{5}}{2}-B\cdot\frac{-1-\sqrt{5}}{2}=-1$$

$$\implies B\left(\frac{-1+\sqrt{5}}{2}-\frac{-1-\sqrt{5}}{2}\right)=B\cdot\sqrt{5}=-1,\quad B=-\frac{1}{\sqrt{5}},\quad A=\frac{1}{\sqrt{5}}$$

So, $F(z)$ can be expressed as:

$$F(z)=\frac{1}{\sqrt{5}}\left(\frac{1}{\frac{-1+\sqrt{5}}{2}-z}-\frac{1}{\frac{-1-\sqrt{5}}{2}-z}\right)$$

Now, notice that for any $k\neq 0$, on $|z|<|k|$, since $|z/k|<1$, then $\sum_{n=0}^{\infty}(z/k)^n$ converges absolutely to $\frac{1}{1-z/k}=\frac{k}{k-z}$, which $\frac{1}{k-z}=\frac{1}{k}\sum_{n=0}^{\infty}(z/k)^n$.

Because both $\frac{-1+\sqrt{5}}{2},\frac{-1-\sqrt{5}}{2}$ has abslute values greater than $\frac{1}{2}$ (first one is approximately 0.618, the second one is approximately $-1.618$), hence, on the disk $|z|<\frac{1}{2}$, both equations below are true based on the above formula:

$$\frac{1}{\frac{-1+\sqrt{5}}{2}-z}=\frac{1}{\frac{-1+\sqrt{5}}{2}}\sum_{n=0}^{\infty}\left(\frac{z}{\frac{-1+\sqrt{5}}{2}}\right)^n,\quad \frac{1}{\frac{-1-\sqrt{5}}{2}-z}=\frac{1}{\frac{-1-\sqrt{5}}{2}}\sum_{n=0}^{\infty}\left(\frac{z}{\frac{-1-\sqrt{5}}{2}}\right)^n$$

Hence, $F(z)$ can be expressed as:

$$F(z)=\frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}\left(\frac{1}{\frac{-1+\sqrt{5}}{2}}\right)^{n+1}z^n-\sum_{n=0}^{\infty}\left(\frac{1}{\frac{-1-\sqrt{5}}{2}}\right)^{n+1}z^n\right)$$

$$=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}-\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}}z^n=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}-\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{(-1)^2-(\sqrt{5})^2}{4}\right)^{n+1}}z^n$$

$$=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}-\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}{(-1)^{n+1}}z^n=\sum_{n=0}^{\infty}\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}z^n$$

Then, by the uniqueness of Taylor Series, the following is the closed expression of $c_n$:

$$\forall n\in\mathbb{N},\quad c_n=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$$

## 3

**Question 3** *Ahlfors Pg. 186 Problem 4:*
*Show that the Laurent development of* $(e^z - 1)^{-1}$ *at the origin is of the form*

$$\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$$

*where the numbers* $B_k$ *are known as the Bernoulli numbers. Calculate* $B_1, B_2, B_3$*.*

**Pf:**

Given the function $f(z) = (e^z - 1)^{-1}$, it is analytic on $\mathbb{C} \setminus \{0\}$ (with $0 < |z| < \infty$), hence there exists a laurent development that agrees on the whole $\mathbb{C} \setminus \{0\}$:

$$f(z) = \sum_{n=-\infty}^{\infty} A_n z^n$$

And, for all index $n$, the formula of $A_n$ is given as follow:

$$n \geq 1, \quad A_{-n} = \frac{1}{2\pi i} \int_C f(\zeta)\zeta^{n-1} d\zeta \qquad\qquad n \geq 0, \quad A_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{n+1}} d\zeta$$

Where $C$ is a circle with radius $r > 0$, centered at $z = 0$.

**Coefficients of Negative Degree:**

First, for $n = 1$, the coefficient $A_{-1}$ is given as:

$$A_{-1} = \frac{1}{2\pi i} \int_C f(\zeta)\zeta^{1-1} d\zeta = \frac{1}{2\pi i} \int_C \frac{1}{e^\zeta - 1} d\zeta = \frac{1}{2\pi i} \int_C \frac{1}{\zeta} \cdot \frac{\zeta}{e^\zeta - 1} d\zeta$$

Now, notice that for $\frac{\zeta}{e^\zeta - 1}$ with an isolated singularity at $\zeta = 0$, since $\lim_{\zeta \to 0} \frac{\zeta}{e^\zeta - 1} = 1$ (since the limit of its reciprical is $\lim_{\zeta \to 0} \frac{e^\zeta - e^0}{\zeta} = 1$, which is the derivative of $e^z$ at 0), then, $\lim_{\zeta \to 0} \zeta \cdot \frac{\zeta}{e^\zeta - 1} = 0$, which is a sufficient and necessary condition for $\zeta = 0$ to be a removable singularity of $\frac{\zeta}{e^\zeta - 1}$.

Hence, $\frac{\zeta}{e^\zeta - 1}$ has an analytic extension onto the whole $\mathbb{C}$, with the function being 1 at $\zeta = 0$. Which, by Cauchy's Integral Formula, $A_{-1}$ of the above form, is the evaluation of $\frac{\zeta}{e^\zeta - 1}$ at 0 (more precisely, evaluation of its extension at 0), which provides $A_{-1} = 1$.

Then, for $n > 1$, the coefficient $A_{-n}$ is given by:

$$A_{-n} = \frac{1}{2\pi i} \int_C f(\zeta)\zeta^{n-1} d\zeta = \frac{1}{2\pi i} \int_C \frac{1}{\zeta} \cdot \frac{\zeta^n}{e^\zeta - 1} d\zeta$$

Notice that for $n > 1$, the function $\frac{\zeta^n}{e^\zeta - 1}$ has isolated singularity at 0, since $\lim_{\zeta \to 0} \zeta \cdot \frac{\zeta^n}{e^\zeta - 1} = 0$, then the singularity at 0 is in fact removable. Hence, it has an analytic extension onto $\mathbb{C}$, where the evaluation at $\zeta = 0$ is $\lim_{\zeta \to 0} \frac{\zeta^n}{e^\zeta - 1} = 0$ (since it can be broken down into $\zeta^{n-1}$ and $\frac{\zeta}{e^\zeta - 1}$; the first one has limit 0 since $n - 1 \geq 1$, while the second one has limit 1, hence the product has limit 0).

Again, the integral form of $A_{-n}$ is in fact the evaluation of the extension of $\frac{\zeta^n}{e^\zeta - 1}$ at $\zeta = 0$ by Cauchy's Integral Formula, hence $A_{-n} = 0$.

**Coefficients of Nonnegative Degree:**

For $n \geq 0$, the coefficient $A_n$ is given as:

$$A_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{n+1}}d\zeta = \frac{1}{2\pi i}\int_C \frac{1}{\zeta^{n+1}(e^\zeta - 1)}d\zeta = \frac{1}{2\pi i}\int_C \frac{1}{\zeta^{n+2}}\cdot\frac{\zeta}{e^\zeta - 1}d\zeta$$

Since we've verified above, that $\frac{z}{e^z-1}$ has an analytic extension onto $\mathbb{C}$, it has a power series expansion about 0, $\frac{z}{e^z-1} = \sum_{n=0}^\infty C_n z^n$, and it agrees with the function on the whole $\mathbb{C}$. We'll find the coefficient to help calculate $A_n$.

Notice that since $\frac{e^z-1}{z}\cdot\frac{z}{e^z-1} = 1$, while $\frac{e^z-1}{z}$ also can be extended analytically onto $\mathbb{C}$, given the following power series expansion of $\frac{e^z-1}{z}$:

$$e^z = 1 + \sum_{n=1}^\infty \frac{z^n}{n!}, \quad \frac{e^z-1}{z} = \frac{1}{z}\sum_{n=1}^\infty \frac{z^n}{n!} = \sum_{n=1}^\infty \frac{z^{n-1}}{n!} = \sum_{n=0}^\infty \frac{z^n}{(n+1)!}$$

Which we can conclude the following:

$$\frac{e^z-1}{z}\cdot\frac{z}{e^z-1} = \left(\sum_{n=0}^\infty C_n z^n\right)\left(\sum_{n=0}^\infty \frac{z^n}{(n+1)!}\right) = 1 = 1 + \sum_{n=1}^\infty 0\cdot z^n$$

Since regardless of $z \in \mathbb{C}$, the above equation is true, $\sum_{n=0}^\infty C_n z^n$ is in fact the inverse of the formal power series $\sum_{n=0}^\infty \frac{z^n}{(n+1)!} \in \mathbb{C}[[z]]$. Which, they satisfy the following relationship:

- For coefficient of degree 0, we have $C_0 \cdot \frac{1}{(0+1)!} = 1$, hence $C_0 = 1$.
- For coefficient of degree 1, we have $C_1 \cdot \frac{1}{(0+1)!} + C_0 \cdot \frac{1}{(1+1)!} = 0$, hence $C_1 + \frac{1}{2!} = 0, \quad C_1 = -\frac{1}{2}$.
- For coefficient of degree $n \geq 2$, we have $\sum_{k=0}^n C_k \cdot \frac{1}{((n-k)+1)!} = 0$, hence:

$$C_n = C_n \cdot \frac{1}{(0+1)!} = -\sum_{k=0}^{n-1} C_k \cdot \frac{1}{((n-k)+1)!}$$

Since $g(z) = \frac{z}{e^z-1}$ has its power series $\sum_{n=0}^\infty C_n z^n$ converge to itself on the whole $\mathbb{C}$,then its $n^{th}$ derivative at 0 is given as $g^{(n)}(0) = n!C_n$. Which, $A_n$ can be rewritten as the following using Cauchy's Integral Formula:

$$A_n = \frac{1}{2\pi i}\int_C \frac{1}{\zeta^{n+2}}\cdot\frac{\zeta}{e^\zeta - 1}d\zeta = \frac{1}{2\pi i}\int_C \frac{g(\zeta)}{\zeta^{n+2}}d\zeta = \frac{g^{(n+1)}(0)}{(n+1)!} = \frac{(n+1)!C_{n+1}}{(n+1)!} = C_{n+1}$$

**Forms of Laurent Series of $(e^z - 1)^{-1}$:**

With the information from previous two sections, we can express the laurent series as the following:

$$\sum_{n=-\infty}^\infty A_n z^n = \sum_{n=1}^\infty A_{-n}z^{-n} + \sum_{n=0}^\infty A_n z^n = \frac{1}{z} + \sum_{n=0}^\infty C_{n+1}z^n = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^\infty C_{n+1}z^n$$

Now, recall that $\frac{z}{e^z-1} = \sum_{n=0}^\infty C_n z^n$, then for all $z \in \mathbb{C}$, consider the expression with $-z$, we get:

$$\frac{-z}{e^{-z}-1} = \sum_{n=0}^\infty C_n(-z)^n = \sum_{n=0}^\infty (-1)^n C_n z^n$$

Which, consider the difference of the two terms, we get:

$$\frac{-z}{e^{-z}-1} - \frac{z}{e^z-1} = \frac{-ze^z}{1-e^z} - \frac{z}{e^z-1} = \frac{ze^z}{e^z-1} - \frac{z}{e^z-1} = \frac{z(e^z-1)}{e^z-1} = z$$

(Note: for $z = 0$, consider its extension, where evaluation at $z = 0$ is the limit $\lim_{z\to 0} \frac{z}{e^z-1} = 1$, then the above difference is 0, which agrees with the formula).

Hence, in power series form, we get:

$$\sum_{n=0}^{\infty}(-1)^n C_n z^n - \sum_{n=0}^{\infty} C_n z^n = \sum_{n=0}^{\infty}((-1)^n - 1)C_n z^n = z$$

Then, all the even terms have $(-1)^n - 1 = 0$, we're left with the odd terms. Hence:

$$\sum_{k=1}^{\infty}((-1)^{2k-1} - 1)C_{2k-1}z^{2k-1} = \sum_{k=1}^{\infty} -2C_{2k-1}z^{2k-1} = -2C_1 z + \sum_{k=2}^{\infty} -2C_{2k-1}z^{2k-1} = z$$

By the uniqueness of taylor series, we need $-2C_1 = 1,\ C_1 = \frac{-1}{2}$ (which agrees with our previous calculation), and $-2C_{2k-1} = 0,\ C_{2k-1} = 0$ for all $k \geq 2$. Therefore, all the odd terms of $C_n$ is 0.

Hence, the laurent series can be expressed as follow:

$$\frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} C_{n+1}z^n = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} C_{(2k-1)+1}z^{2k-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} C_{2k}z^{2k-1}$$

(Note: Since now the odd terms of $C_n$ appears as the even degrees' coefficients).

Now, if we do some modification, let $B'_n = n!C_n$ for all $n \in \mathbb{N}$ (or $C_n = \frac{B'_n}{n!}$), then the laurent series can be expressed as:

$$(e^z - 1)^{-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B'_{2k}}{(2k)!}z^{2k-1}$$

Then, let $B_k = (-1)^k B'_{2k}$, we get the desired form:

$$(e^z - 1)^{-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty}(-1)^k \frac{B_k}{(2k)!}z^{2k-1}$$

**Calculation of Bernoulli Numbers:**

For $k = 1, 2, 3$, we'll convert it back into $C_n$ for simplicity. Which, $B_k = (-1)^k B_{2k} = (-1)^k(2k)!C_{2k}$.

For $k = 1, 2k = 2$, we have $B_1$ given as:

$$C_2 = -\sum_{k=0}^{1} C_k \frac{1}{(2-k+1)!} = -\left(\frac{C_0}{3!} + \frac{C_1}{2!}\right) = -\left(1 \cdot \frac{1}{6} - \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{1}{12}$$

$$B_1 = (-1)^1 \cdot 2! \cdot C_2 = -\frac{1}{6}$$

For $k = 2, 2k = 4$, we have $B_2$ given as:

$$C_4 = -\sum_{k=0}^{3} C_k \frac{1}{(4-k+1)!} = -\left(\frac{C_0}{5!} + \frac{C_1}{4!} + \frac{C_2}{3!} + \frac{C_3}{2!}\right) = -\left(\frac{1}{120} - \frac{1}{2 \cdot 24} + \frac{1}{12 \cdot 6}\right)$$

$$= -\frac{1}{12}\left(\frac{1}{10} - \frac{1}{4} + \frac{1}{6}\right) = -\frac{1}{12} \cdot \frac{1}{60}$$

$$B_2 = (-1)^2 \cdot 4! \cdot C_4 = -\frac{24}{12 \cdot 60} = -\frac{1}{30}$$

(Note: recall that for $k \geq 2$, all $C_{2k-1} = 0$, hence all odd index $n \geq 3$ has $C_3 = 0$).

For $k = 3, 2k = 6$, we have $B_3$ given as:

$$C_6 = -\sum_{k=0}^{5} C_k \frac{1}{(6-k+1)!} = -\left(\frac{C_0}{7!} + \frac{C_1}{6!} + \frac{C_2}{5!} + \frac{C_3}{4!} + \frac{C_4}{3!} + \frac{C_5}{2!}\right)$$

$$= -\left(\frac{1}{7!} - \frac{1}{2 \cdot 6!} + \frac{1}{12 \cdot 5!} - \frac{1}{12 \cdot 60 \cdot 3!}\right)$$

$$B_3 = (-1)^3 \cdot 6! \cdot C_6 = -6! \cdot \left(-\left(\frac{1}{7!} - \frac{1}{2 \cdot 6!} + \frac{1}{12 \cdot 5!} - \frac{1}{12 \cdot 60 \cdot 3!}\right)\right)$$

$$= \frac{1}{7} - \frac{1}{2} + \frac{6}{12} - \frac{6!}{12 \cdot 60 \cdot 3!} = \frac{1}{7} - \frac{1}{6} = \frac{1}{42}$$

So, we have the following:

$$B_1 = -\frac{1}{6}, \quad B_2 = -\frac{1}{30}, \quad B_3 = \frac{1}{42}$$

# 4

**Question 4** *Stein and Shakarchi Pg. 86 Problem 2:*
*Let*

$$F(z) = \sum_{n=1}^{\infty} d(n) z^n$$

*for* $|z| < 1$*, where* $d(n)$ *denotes the number of divisors of* $n$*. Observe that the radius of convergence of this series is* $1$*. Verify the identity*

$$\sum_{n=1}^{\infty} d(n) z^n = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n}$$

*Using this identity, show that if* $z = r$ *with* $0 < r < 1$*, then*

$$|F(r)| \geq c \frac{1}{1-r} \log(1/(1-r))$$

*as* $r \to 1$*. Similarly, if* $\theta = 2\pi p/q$ *where* $p$ *and* $q$ *are positive integers and* $z = re^{i\theta}$*, then*

$$|F(re^{i\theta})| \geq c_{p/q} \frac{1}{1-r} \log(1/(1-r))$$

*as* $r \to 1$*. Conclude that* $F$ *cannot be continued analytically past the unit disk.*

**Pf:**