Math CS 122B HW8 Part 2

Zih-Yu Hsieh

June 4, 2025

1

Question 1 *Stein and Shakarchi Pg. 200-201 Exercise 4:*
*Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers such that $a_n = a_m$ iff $n \equiv m \bmod q$ for some positive integer q. Define the **Dirichlet L-series** associated to $\{a_n\}$ by*

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{for } Re(s) > 1$$

Also, with $a_0 = a_q$, let

$$Q(x) = \sum_{m=0}^{q-1} a_{q-m} e^{mx}$$

Show, as in Exercises 15 and 16 of the previous chapter, that

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{Q(x)x^{s-1}}{e^{qx}-1} dx \quad \text{for } Re(s) > 1$$

Prove as a result that $L(s)$ is continuable into the complex plane, with the only possible singularity a pole at $s = 1$. In fact, $L(s)$ is regular at $s = 1$ if and only if $\sum_{m=0}^{q-1} a_m = 0$. Note the conection with the Direchlet $L(s,\chi)$ series, taken up to Book I Chapter 8, and that as a consequence, $L(s,\chi)$ is regular at $s = 1$ if and only if χ is a non-trivial character.

Pf:

1.1 Integral Representation of $L(s)$:

Given $\mathrm{Re}(s) > 1$, and $x \in (0,\infty)$, notice that $\frac{1}{e^{qx}-1} = \frac{e^{-qx}}{1-e^{-qx}}$, with the fact that $-qx < 0$, then $e^{-qx} < 1$. Hence, the following expression is absolutely convergent, and converging normally for any compact subset of $(0,\infty)$:

$$\frac{1}{e^{qx}-1} = \frac{e^{-qx}}{1-e^{-qx}} = \sum_{n=1}^{\infty} (e^{-qx})^n \tag{1}$$

Since it converges normally within any compact subset of $(0, \infty)$ (the domain of integration), then the integral expression in the question can be rewritten as:

$$\begin{aligned}
\frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1} dx &= \frac{1}{\Gamma(s)} \int_0^\infty Q(x)x^{s-1} \left(\sum_{n=1}^\infty e^{-qx} \right) dx \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \int_0^\infty \left(\sum_{m=0}^{q-1} a_{q-m} e^{mx} \right) x^{s-1} \cdot e^{-nqx} dx \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \sum_{m=0}^{q-1} a_{q-m} \int_0^\infty x^{s-1} e^{-(nq-m)x} dx
\end{aligned} \tag{2}$$

Which, by swapping $r = q - m$ (where r ranges from 1 to q), extending from (2), we get the following:

$$\begin{aligned}
\frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1} dx &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \sum_{r=1}^{q} a_r \int_0^\infty x^{s-1} e^{-(nq-(q-r))x} dx \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \sum_{r=1}^{q} a_r \int_0^\infty x^{s-1} e^{-((n-1)q+r)x} dx \\
&= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^{q} a_r \int_0^\infty x^{s-1} e^{-(nq+r)x} dx
\end{aligned} \tag{3}$$

Then, performing substitution $u = (nq+r)x$ for each index n and r, $du = (nq+r)dx$, which (3) becomes:

$$\begin{aligned}
\frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1} dx &= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^{q} a_r \int_0^\infty \left(\frac{u}{nq+r} \right)^{s-1} \cdot e^{-u} \frac{du}{nq+r} \\
&= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^{q} a_r \cdot \frac{1}{(nq+r)^s} \int_0^\infty u^{s-1} e^{-u} du \\
&= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^{q} \frac{a_r}{(nq+r)^s} \cdot \Gamma(s) = \sum_{n=0}^\infty \sum_{r=1}^{q} \frac{a_r}{(nq+r)^s}
\end{aligned} \tag{4}$$

Now, in terms of the original $L(s)$, recall that $a_n = a_m$ iff $n \equiv m \mod q$, so the original series expression can be rearranged as:

$$\begin{aligned}
L(s) = \sum_{k=1}^\infty \frac{a_k}{k^s} &= \sum_{n=1}^\infty \frac{a_{nq}}{(nq)^s} + \sum_{n=0}^\infty \sum_{r=1}^{q-1} \frac{a_{nq+r}}{(nq+r)^s} \\
&= \sum_{n=0}^\infty \frac{a_q}{(nq+q)^s} + \sum_{n=0}^\infty \sum_{r=1}^{q-1} \frac{a_r}{(nq+r)^s} = \sum_{n=0}^\infty \sum_{r=1}^{q} \frac{a_r}{(nq+r)^s}
\end{aligned} \tag{5}$$

Then, combining the results in (4) and (5), we get $L(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1} dx$ (for $\mathrm{Re}(s) > 1$).

1.2 Continuation to $\mathbb{C} \setminus \{1\}$:

With the above integral expression, one can separate the integration as follow:

$$\begin{aligned}
L_1(s) :=& \frac{1}{\Gamma(s)} \int_0^{\frac{1}{q}} \frac{Q(x)x^{s-1}}{e^{qx}-1} dx, \quad L_2(s) := \frac{1}{\Gamma(s)} \int_{\frac{1}{q}}^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1} dx \\
L(s) =& \frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1} dx = L_1(s) + L_2(s)
\end{aligned} \tag{6}$$

Since $Q(x)=\sum_{m=0}^{q-1}a_{q-m}e^{mx}$, it is dominated by $e^{(q-1)x}$. Hence, there exists $K>0$, such that $|Q(x)|\leq\sum_{m=0}^{q-1}|a_{q-m}|e^{mx}\leq Ke^{q-1}x$. Then, for $x>\frac{1}{q}$ and any $s\in\mathbb{C}$, since $qx>1$, then $e^{qx}>e>2$, so $\frac{1}{2}e^{qx}>1$, or $\frac{1}{2}e^{qx}=e^{qx}-\frac{1}{2}e^{qx}<e^{qx}-1$. Then, $L_2(s)$ satisfies the following inequality:

$$
\begin{aligned}
|L_2(s)| &\leq \frac{1}{|\Gamma(s)|}\int_{\frac{1}{q}}^{\infty}\frac{|Q(x)|\cdot|x^{s-1}|}{|e^{qx}-1|}dx \leq \frac{1}{|\Gamma(s)|}\int_{\frac{1}{q}}^{\infty}\frac{Ke^{(q-1)x}\cdot x^{\mathrm{Re}(s)-1}}{e^{qx}-1}dx \\
&\leq \frac{1}{|\Gamma(s)|}\int_{\frac{1}{q}}^{\infty}\frac{Ke^{(q-1)x}\cdot x^{Re(s)-1}}{\frac{1}{2}e^{qx}}dx = \frac{2K}{|\Gamma(s)|}\int_{\frac{1}{q}}^{\infty}x^{\mathrm{Re}(s)-1}\cdot e^{-x}dx<\infty
\end{aligned}
\tag{7}
$$

Hence, $L_2(s)$ is an entire function.

Now, if consider $L_1(s)$, we'll utilize the power series of e^{mz} (for all $z\in\mathbb{C}$) and $\frac{z}{e^z-1}$ (for $|z|<2\pi$) given as follow:

$$
e^{mz}=\sum_{k=0}^{\infty}\frac{(mz)^n}{n!},\qquad \frac{z}{e^z-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}z^k \tag{8}
$$

(Note: $\{B_k\}_{k\in\mathbb{N}}$ denotes the sequence of **Bernoulli Numbers**, and $B_0=1$).

Then, under the power series, the integral in $L_1(s)$ can be expressed as follow for $\mathrm{Re}(s)>1$:

$$
\begin{aligned}
\int_0^{\frac{1}{q}}\frac{Q(x)x^{s-1}}{e^{qx}-1}dx &= \sum_{m=0}^{q-1}a_{q-m}\int_0^{\frac{1}{q}}\frac{e^{mx}x^{s-1}}{e^{qx}-1}dx \\
&= a_q\int_0^{\frac{1}{q}}\frac{x^{s-1}}{e^{qx}-1}dx+\sum_{m=1}^{q-1}a_{q-m}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\frac{1}{q}}\frac{(mx)^n x^{s-1}}{e^{qx}-1}dx\right) \\
&= a_q\int_0^{1}\frac{(u/q)^{s-1}}{e^{u}-1}\frac{du}{q}+\sum_{m=1}^{q-1}\sum_{n=0}^{\infty}\frac{a_{q-m}\cdot m^n}{n!}\int_0^{\frac{1}{q}}\frac{x^{s+n-1}}{e^{qx}-1}dx \\
&= \frac{a_q}{q^s}\int_0^{1}\frac{u\cdot u^{s-2}}{e^{u}-1}du+\sum_{m=1}^{q-1}\sum_{n=0}^{\infty}\frac{a_{q-m}\cdot m^n}{n!}\int_0^{1}\frac{(u/q)^{s+n-1}}{e^{u}-1}\frac{du}{q} \\
&= \frac{a_q}{q^s}\sum_{k=0}^{\infty}\frac{B_k}{k!}\int_0^{1}u^{s+k-2}du+\frac{1}{q^s}\sum_{m=1}^{q-1}\sum_{n=0}^{\infty}\frac{a_{q-m}\cdot m^n}{n!\cdot q^n}\int_0^{1}\frac{u\cdot u^{s+n-2}}{e^{u}-1}du \\
&= \frac{a_q}{q^s}\sum_{k=0}^{\infty}\frac{B_k}{k!(s+k-1)}u^{s+k-1}\Big|_0^1+\frac{1}{q^s}\sum_{m=1}^{q-1}\sum_{n=0}^{\infty}\frac{a_{q-m}\cdot m^n}{n!\cdot q^n}\left(\sum_{k=0}^{\infty}\int_0^{1}u^{s+n+k-2}du\right) \\
&= \frac{a_q}{q^s}\sum_{k=0}^{\infty}\frac{B_k}{k!(s+k-1)}+\frac{1}{q^s}\sum_{m=1}^{q-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_{q-m}\cdot m^n}{n!\cdot q^n(s+n+k-1)}u^{s+n+k-1}\Big|_0^1 \\
&= \frac{a_q}{q^s}\sum_{k=0}^{\infty}\frac{B_k}{k!(s+k-1)}+\frac{1}{q^s}\sum_{m=1}^{q-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_{q-m}\cdot(m/q)^n}{n!(s+n+k-1)} \\
&= \frac{a_q}{q^s(s-1)}+\frac{a_q}{q^s}\sum_{k=1}^{\infty}\frac{B_k}{k!(s+k-1)}+\frac{1}{q^s}\sum_{m=1}^{q-1}\left(\frac{a_{q-m}}{s-1}+\sum_{\substack{n,k=0\\(n,k)\neq(0,0)}}^{\infty}\frac{a_{q-m}(m/q)^n}{n!(s+n+k-1)}\right) \\
&= \sum_{m=0}^{q-1}\frac{a_{q-m}}{q^s(s-1)}+\frac{a_q}{q^s}\sum_{k=1}^{\infty}\frac{B_k}{k!(s+k-1)}+\frac{1}{q^s}\sum_{m=1}^{q-1}\sum_{\substack{n,k=0\\(n,k)\neq(0,0)}}^{\infty}\frac{a_{q-m}(m/q)^n}{n!(s+n+k-1)}
\end{aligned}
\tag{9}
$$

Which, using the infinite series expression above as the analytic continuation, we see that the integral as potential simple pole at $s=1$ (the first summation), and has simple pole at all integer $\mathbb{Z}_{\leq 0}$ (the second and third summation with $(s+k-1)$ or $(s+n+k-1)$ as denominator, for $k\geq1$ and $(n,k)\neq(0,0)$).

Then, because $\frac{1}{\Gamma(s)}$ only has simple zeros at all $\mathbb{Z}_{\leq 0}$, the expression $L_1(s) = \frac{1}{\Gamma(s)} \int_1^{\frac{1}{q}} \frac{Q(x)x^{s-1}}{e^{qx}-1} dx$ has all the simple poles of the integral at $\mathbb{Z}_{\leq 0}$ cancelled out by $\frac{1}{\Gamma(s)}$, and left with a potential pole at $s = 1$.

Finally, for the summation expression of the integral in (9), we get that the potential pole at $s = 1$ is described by $\sum_{m=0}^{q-1} \frac{a_{q-m}}{q^s(s-1)} = \frac{1}{q^s(s-1)} \sum_{m=0}^{q-1}$. Since at $s = 1$, $\frac{1}{\Gamma(s)} \neq 0$, then after multiplying with $\frac{1}{\Gamma(s)}$, this expression is regular at $s = 1$ iff the coefficient $\sum_{m=0}^{q-1} a_{q-m} = \sum_{r=1}^{q} a_r = 0$.

Hence, we can conclude that $L(s)$ can be continued onto the whole plane, except possibly at $s = 1$. And, with the last information from above, $L(s)$ is regular at $s = 1$ iff $\sum_{r=1}^{q} a_r = 0$.

2

Question 2 *Stein and Shakarchi Pg. 204 Problem 4:*
One can combine ideas from the prime number theorem with the proof of Dirichlet's Theorem about primes in arithmetic progression (given in Book I) to prove the following: Let q and l be relatively prime integers. We consider the primes belonging to the arithmetic progression $\{qk + ll\}_{k\in\mathbb{N}}$, and let $\pi_{q,l}(x)$ denote the number of such primes $\leq x$. Then one has

$$\pi_{q,l}(x) \sim \frac{x}{\varphi(q)\log(x)} \quad \text{as } x \to \infty$$

where $\varphi(q)$ denotes the number of positive integers less than q and relatively prime to q (i.e. the Euler Totient Function).

Pf:

Given $q, l \in \mathbb{N}$ with $\gcd(q, l) = 1$. Find an expression of Dirichlet Series, that produces the following formula:

$$L(s) := \prod_p \frac{1}{1 - \delta_l(p)p^{-s}}$$

Where p ranges through all primes, and $\delta_l : \mathbb{N} \to \mathbb{N}$ is defined as follow:

$$\delta_l(n) = \begin{cases} 1 & n \equiv l \mod q \\ 0 & \text{otherwise} \end{cases}$$