Math CS 122B HW5

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Question 1 *Freitag Chap. V.3 Exercise 5:*
The algebraic differential equation of the $\wp$-function can berewritten as:

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

Here, $e_j,\ 1 \leq j \leq 3$, are the three half lattice values of the $\wp$-function.

Pf:

Given the algebraic differential equation of the $\wp$-function as follow:

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$$

Within the fundamental region P, there are 3 points with the value of $\wp'$ to be zero, which is given by $\frac{w_1}{2},\ \frac{w_2}{2},\ \frac{w_1+w_2}{2}$ (and points congruent to these points $\mod L$) when the lattice $L = w_1\mathbb{Z} + w_2\mathbb{Z}$.

Then, by definition, the given points have the evaluation to be the following:

$$e_1 = \wp\left(\frac{w_1}{2}\right), \quad e_2 = \wp\left(\frac{w_2}{2}\right), \quad e_3 = \wp\left(\frac{w_1 + w_2}{2}\right)$$

Which, let $w = \wp(z)$, then the polynomial $4w^3 - g_2w - g_3 = 0$ iff $\wp'(z) = 0$, which within the fundamental region, only the three distinct points mentioned above are the solution, so the values of $\wp$ of these points are the zeros of the polynomial $4w^3 - g_2w - g_3$.

Then, since e_1, e_2, e_3 are all distinct, while $4w^3 - g_2w - g_3$ has at most 3 distinct zeroes, then they must be all the zeros of the polynomial. Hence, $4w^3 - g_2w - g_3 = 4(w - e_1)(w - e_2)(w - e_3)$, which we get the following:

$$(\wp'(z))^3 = 4(\wp(z))^3 - g_2\wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

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Question 2 *Freitag Chap. V.3 Exercise 6:*
Show the following recursion formulas for the Eisenstein series G_{2m} for $m \geq 4$:

$$(2m+1)(m-3)(2m-1)G_{2m} = 3\sum_{j=2}^{m-2}(2j-1)(2m-2j-1)G_{2j}G_{2m-2j}$$

for instance $G_{10} = \frac{5}{11}G_4G_6$. Any Eisenstein series G_{2m}, $m \geq 4$, is thus representable as a polynomial in G_4 and G_6 with nonnegative coefficients.

Pf:

First, the $\wp$-function is given as follow:

$$\wp(z) = \frac{1}{z^2} + \sum_{m=1}^{\infty}(2m+1)G_{2(m+1)}z^{2m}$$

With the formula of $\wp$-function as series of functions, since it converges normally within $\mathbb{C} \setminus L$ (with L being the lattice), then differentiation can be performed termwise. Hence, its second derivative is given by:

$$\wp''(z) = \frac{d^2}{dz^2}\left(\frac{1}{z^2}\right) + \sum_{m=1}^{\infty}\frac{d^2}{dz^2}\left((2m+1)G_{2(m+1)}z^{2m}\right) = \frac{6}{z^4} + \sum_{m=1}^{\infty}(2m+1)(2m)(2m-1)G_{2(m+1)}z^{2m-2}$$

$$= \frac{6}{z^4} + \sum_{m=2}^{\infty}(2m-1)(2m-2)(2m-3)G_{2m}z^{2m-4}$$

Recall the following second order differential equation of $\wp$-function:

$$2\wp''(z) = 12(\wp(z))^2 - g_2, \quad \wp''(z) = 6(\wp(z))^2 - \frac{g_2}{2}$$

The goal is to get a recursive relation of the coefficient of each power of $\wp''(z)$.

With the expression of $\wp''$ in power series from above, to get an expression of G_{2m} for $m \geq 4$, it suffices to find the coefficient of z^{2m-4} within $6(\wp(z))^2 - \frac{g_2}{2}$. There are two casees to consider:

1. z^{2m-4} can be expressed as $\frac{1}{z^2} \cdot z^{2m-2}$, within $\wp(z)$, the coefficient of $\frac{1}{z^2}$ is 1, while the coefficient of $z^{2m-2} = z^{2(m-1)}$ is $(2(m-1)+1)G_{2((m-1)+1)} = (2m-1)G_{2m}$. Hence, since $(\wp(z))^2$ has two copies of the above expression, then the coefficient of $\frac{1}{z^2} \cdot z^{2m-2}$ is:

$$2 \cdot 1 \cdot (2m-1)G_{2m} = 2(2m-1)G_{2m}$$

2. Since $\wp(z)$ also has all power z^{2m} for $m \geq 1$, then $z^{2m-4} = z^{2(m-2)}$ can also be expressed as $z^{2k} \cdot z^{2(m-k-2)}$, for integers $k \geq 1$ and $(m-k-2) \geq 1$ (or $k \leq (m-3)$). Hence, for the convolution of power series of $(\wp(z))^2$ (excluding the negative powers mentioned above), z^{2m-4} term has the following coefficient:

$$\sum_{k=1}^{m-3}(2k+1)G_{2(k+1)}\cdot(2(m-k-2)+1)G_{2((m-k-2)+1)} = \sum_{k=1}^{m-3}(2(k+1)-1)(2m-2(k+1)-1)G_{2(k+1)}G_{2(m-(k+1))}$$

$$= \sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2(m-k)}$$

(Note: recall that z^{2k} term has coefficient $(2k+1)G_{2(k+1)}$, while $z^{2(m-k-2)}$ term has coefficient given as $(2(m-k-2)+1)G_{2((m-k-2)+1)}$).

So, the coefficient of z^{2m-4} in $(\wp(z))^2$ is recorded as:

$$2(2m-1)G_{2m} + \sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

Hence, based on the equation $\wp''(z) = 6(\wp(z))^2 - \frac{g_2}{2}$, for all $m \geq 4$, the coefficient of z^{2m-4} is given as the following two forms:

$$\text{Coefficient of } z^{2m-4} \text{ in } \wp''(z): \quad (2m-1)(2m-2)(2m-3)G_{2m}$$

$$\text{Coefficient of } z^{2m-4} \text{ in } 6(\wp(z))^2 - \frac{g_2}{2}: \quad 6\left(2(2m-1)G_{2m} + \sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}\right)$$

Which, for the two to be equal, we get the following equality:

$$(2m-1)(2m-2)(2m-3)G_{2m} = 12(2m-1)G_{2m} + 6\sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$(2m-1)(4m^2-10m+6)G_{2m} - 12(2m-1)G_{2m} = 6\sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$(2m-1)(4m^2-10m-6)G_{2m} = 6\sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$(2m-1)(2m-6)(2m+1)G_{2m} = 6\sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$\implies (2m+1)(m-3)(2m-1)G_{2m} = 3\sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

Which, this equation is the desired recursive form.

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Question 3 *Freitag Chap. V.4 Exercise 3:*
Let $L \subset \mathbb{C}$ be a lattice with the property $g_2(L) = 8$ and $g_3(L) = 0$. The point $(2,4)$ lies on the affine elliptic curve $y^2 = 4x^3 - 8x$. Let $+$ be the addition (for points on the corresponding projective curve). SHow that $2 \cdot (2,4) := (2,4) + (2,4)$ is the point $(\frac{9}{4}, \frac{21}{4})$.

Pf:

Consider the tangent of $(2,4)$ on the given elliptic curve $y^2 = 4x^3 - 8x$: By implicit differentiation, we get the following relationship:

$$2y\frac{dy}{dx} = 12x^2 - 8$$

which, for $(x,y) = (2,4)$, $\frac{dy}{dx}\Big|_{(2,4)} = \frac{12x^2-8}{2y}\Big|_{(2,4)} = \frac{12\cdot 2^2 - 8}{2\cdot 4} = 5$. Hence, the tangent is expressed as the following equation:

$$(y-4) = 5(x-2), \quad y = 5x - 6$$

Now, to solve for the third point, it must satisfy the following equations:

$$\begin{cases} y = 5x - 6 \\ y^2 = 4x^3 - 8x \end{cases}$$

Hence, $(5x-6)^2 = 4x^3 - 8x$, which $25x^2 - 60x + 36 = 4x^3 - 8x$, so $4x^3 - 25x^2 + 52x - 36 = 0$. Which, consider the fact that $(x,y) = (2,4)$ appears on the tangent twice (with multiplicity 2), then $(x-2)^2$ is presumably a factor of the above equation. The above polynomial in fact has the following factorization:

$$4x^3 - 25x^2 + 52x - 36 = (x-2)^2(4x-9)$$

This indicates that the third zero happesn when $x = \frac{9}{4}$. Which, the only point lying on the defined tangent above is given as:

$$y = 5 \cdot \frac{9}{4} - 6 = \frac{21}{4}$$

So, the third point lying on the tangent is $(\frac{9}{4}, \frac{21}{4})$.

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Question 4 *Stein and Shakarchi Pg. 281 Problem 3:*
Suppose Ω is a simply connected domain that excludes the three roots of the polynomial $4z^3 - g_2 z - g_3$. For $w_0 \in \Omega$ fixed, define the function I on Ω by

$$I(w) = \int_{w_0}^{w} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}, \quad w \in \Omega$$

Then the function I has an inverse given by $\wp(z+\alpha)$ for some constant α; that is:

$$I(\wp(z+\alpha)) = z$$

for appropriate α.

Pf:

Given that Ω is a simply connected domain that excludes the roots e_1, e_2, e_3 of $4z^3 - g_2 z - g_3$, then since this simply connected open region doesn't include the zeros for the polynomial, hence there exists a well-defined square root for the function (can be denoted by $\sqrt{4z^3 - g_2 z - g_3}$).

Then, given the definition of $I(w)$ above (as an antiderivative of $\frac{1}{\sqrt{4z^3-g_2z-g_3}}$), its derivative $I'(w) = \frac{1}{\sqrt{4z^3-g_2z-g_3}}$.

Now, since $\wp : \mathbb{C} \setminus L \to \mathbb{C}$ is a surjective function, then threre exists $\alpha_1, \alpha_2 \in \mathbb{C} \setminus L$, such that $\wp(\alpha_1) = \wp(\alpha_2) = w_0$, while $\wp'(\alpha_1) = -\wp'(\alpha_2)$ (which they satisfy the relation of $\alpha_1 \equiv \alpha_2 \mod L$).

Then, given the algebraic differential equation $(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$, then for the defined square root, we have $(\wp'(\alpha_1))^2 = (\wp'(\alpha_2))^2 = 4w_0^3 - g_2 w_0 - g_3$. Which, for the defined square root, there are two cases: either $\sqrt{4w_0^3 - g_2 w_0 - g_3} = \wp'(\alpha_1)$, or $\sqrt{4w_0^3 - g_2 w_0 - g_3} = -\wp'(\alpha_1) = \wp'(\alpha_2)$. In either case, we can choose $\alpha \in \{\alpha_1, \alpha_2\}$, such that $\sqrt{4w_0^3 - g_2 w_0 - g_3} = \sqrt{(\wp'(\alpha))^2} = \wp'(\alpha)$ (and it still satisfies $\wp(\alpha) = w_0$).

Hence, given the function $I(\wp(z+\alpha))$ with the domain being the preimage of Ω (which is containing 0, since $\wp(0+\alpha) = \wp(\alpha) = w_0 \in \Omega$), we have the following:

$$I(\wp(0+\alpha)) = I(w_0) = \int_{w_0}^{w_0} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} = 0$$

Also, if differentiate this composition of function, we get:

$$(I(\wp(z+\alpha)))' = I'(\wp(z+\alpha))\wp'(z+\alpha) = \frac{\wp'(z+\alpha)}{\sqrt{4(\wp(z+\alpha))^3 - g_2(\wp(z+\alpha)) - g_3}} = \frac{\wp'(z+\alpha)}{\sqrt{(\wp'(z+\alpha))^2}} = \pm 1$$

Notice that since both I and $\wp$ are analytic function within the given domain, hence the composition and its derivative are both analytic; on the other hand, since $(I(\wp(z+\alpha)))'$ has the value at $z=0$ being the following:

$$(I(\wp(z+\alpha)))' \big|_{z=0} = \frac{\wp'(0+\alpha)}{\sqrt{(\wp'(0+\alpha))^2}} = \frac{\wp'(\alpha)}{\sqrt{(\wp'(\alpha))^2}} = \frac{\wp'(\alpha)}{\wp'(\alpha)} = 1$$

then in case for $(I(\wp(z+\alpha)))'$ to be continuous (in particular, continuous), we need $(I(\wp(z+\alpha)))' = 1$, which implies that $I(\wp(z+\alpha)) = z$. So, α is the desired constant, such that $\wp(z+\alpha)$ is the inverse of I.

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Question 5 *Stein and Shakarchi Pg. 282 Problem 4:*
Suppose $\mathcal{T}$ is purely imaginary, say $\mathcal{T} = it$ with $t > 0$. Consider the division of the complex plane into congruent rectangles obtained by considering the lines $x = n/2$, $y = tm/2$ as n and m range over the integers.

(a) Show that $\wp$ is real-valued on all these lines, adn hence on the boundaries of all these rectangles.

(b) Prove that $\wp$ maps the interior of each rectangle conformally to the uppoer (or lower) half-plane.

Pf: