

# Math CS 122B HW5

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**Question 1** Freitag Chap. V.3 Exercise 5:

The algebraic differential equation of the  $\wp$ -function can be rewritten as:

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

Here,  $e_j$ ,  $1 \leq j \leq 3$ , are the three half lattice values of the  $\wp$ -function.

**Pf:**

Given the algebraic differential equation of the  $\wp$ -function as follow:

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$$

Within the fundamental region  $P$ , there are 3 points with the value of  $\wp'$  to be zero, which is given by  $\frac{w_1}{2}$ ,  $\frac{w_2}{2}$ ,  $\frac{w_1+w_2}{2}$  (and points congruent to these points mod  $L$ ) when the lattice  $L = w_1\mathbb{Z} + w_2\mathbb{Z}$ .

Then, by definition, the given points have the evaluation to be the following:

$$e_1 = \wp\left(\frac{w_1}{2}\right), \quad e_2 = \wp\left(\frac{w_2}{2}\right), \quad e_3 = \wp\left(\frac{w_1 + w_2}{2}\right)$$

Which, let  $w = \wp(z)$ , then the polynomial  $4w^3 - g_2w - g_3 = 0$  iff  $\wp'(z) = 0$ , which within the fundamental region, only the three distinct points mentioned above are the solution, so the values of  $\wp$  of these points are the zeros of the polynomial  $4w^3 - g_2w - g_3$ .

Then, since  $e_1, e_2, e_3$  are all distinct, while  $4w^3 - g_2w - g_3$  has at most 3 distinct zeroes, then they must be all the zeros of the polynomial. Hence,  $4w^3 - g_2w - g_3 = 4(w - e_1)(w - e_2)(w - e_3)$ , which we get the following:

$$(\wp'(z))^3 = 4(\wp(z))^3 - g_2\wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

**Question 2** Freitag Chap. V.3 Exercise 6:

Show the following recursion formulas for the Eisenstein series  $G_{2m}$  for  $m \geq 4$ :

$$(2m+1)(m-3)(2m-1)G_{2m} = 3 \sum_{j=2}^{m-2} (2j-1)(2m-2j-1)G_{2j}G_{2m-2j}$$

for instance  $G_{10} = \frac{5}{11}G_4G_6$ . Any Eisenstein series  $G_{2m}$ ,  $m \geq 4$ , is thus representable as a polynomial in  $G_4$  and  $G_6$  with nonnegative coefficients.

**Pf:**

First, the  $\wp$ -function is given as follow:

$$\wp(z) = \frac{1}{z^2} + \sum_{m=1}^{\infty} (2m+1)G_{2(m+1)}z^{2m}$$

With the formula of  $\wp$ -function as series of functions, since it converges normally within  $\mathbb{C} \setminus L$  (with  $L$  being the lattice), then differentiation can be performed termwise. Hence, its second derivative is given by:

$$\begin{aligned} \wp''(z) &= \frac{d^2}{dz^2} \left( \frac{1}{z^2} \right) + \sum_{m=1}^{\infty} \frac{d^2}{dz^2} ((2m+1)G_{2(m+1)}z^{2m}) = \frac{6}{z^4} + \sum_{m=1}^{\infty} (2m+1)(2m)(2m-1)G_{2(m+1)}z^{2m-2} \\ &= \frac{6}{z^4} + \sum_{m=2}^{\infty} (2m-1)(2m-2)(2m-3)G_{2m}z^{2m-4} \end{aligned}$$

Recall the following second order differential equation of  $\wp$ -function:

$$2\wp''(z) = 12(\wp(z))^2 - g_2, \quad \wp''(z) = 6(\wp(z))^2 - \frac{g_2}{2}$$

The goal is to get a recursive relation of the coefficient of each power of  $\wp''(z)$ .

With the expression of  $\wp''$  in power series from above, to get an expression of  $G_{2m}$  for  $m \geq 4$ , it suffices to find the coefficient of  $z^{2m-4}$  within  $6(\wp(z))^2 - \frac{g_2}{2}$ . There are two cases to consider:

1.  $z^{2m-4}$  can be expressed as  $\frac{1}{z^2} \cdot z^{2m-2}$ , within  $\wp(z)$ , the coefficient of  $\frac{1}{z^2}$  is 1, while the coefficient of  $z^{2m-2} = z^{2(m-1)}$  is  $(2(m-1)+1)G_{2((m-1)+1)} = (2m-1)G_{2m}$ . Hence, since  $(\wp(z))^2$  has two copies of the above expression, then the coefficient of  $\frac{1}{z^2} \cdot z^{2m-2}$  is:

$$2 \cdot 1 \cdot (2m-1)G_{2m} = 2(2m-1)G_{2m}$$

2. Since  $\wp(z)$  also has all power  $z^{2m}$  for  $m \geq 1$ , then  $z^{2m-4} = z^{2(m-2)}$  can also be expressed as  $z^{2k} \cdot z^{2(m-k-2)}$ , for integers  $k \geq 1$  and  $(m-k-2) \geq 1$  (or  $k \leq (m-3)$ ). Hence, for the convolution of power series of  $(\wp(z))^2$  (excluding the negative powers mentioned above),  $z^{2m-4}$  term has the following coefficient:

$$\begin{aligned} \sum_{k=1}^{m-3} (2k+1)G_{2(k+1)} \cdot (2(m-k-2)+1)G_{2((m-k-2)+1)} &= \sum_{k=1}^{m-3} (2(k+1)-1)(2m-2(k+1)-1)G_{2(k+1)}G_{2(m-(k+1))} \\ &= \sum_{k=2}^{m-2} (2k-1)(2m-2k-1)G_{2k}G_{2(m-k)} \end{aligned}$$

(Note: recall that  $z^{2k}$  term has coefficient  $(2k+1)G_{2(k+1)}$ , while  $z^{2(m-k-2)}$  term has coefficient given as  $(2(m-k-2)+1)G_{2((m-k-2)+1)}$ ).

So, the coefficient of  $z^{2m-4}$  in  $(\wp(z))^2$  is recorded as:

$$2(2m-1)G_{2m} + \sum_{k=2}^{m-2} (2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

Hence, based on the equation  $\wp''(z) = 6(\wp(z))^2 - \frac{g_2}{2}$ , for all  $m \geq 4$ , the coefficient of  $z^{2m-4}$  is given as the following two forms:

$$\text{Coefficient of } z^{2m-4} \text{ in } \wp''(z) : (2m-1)(2m-2)(2m-3)G_{2m}$$

$$\text{Coefficient of } z^{2m-4} \text{ in } 6(\wp(z))^2 - \frac{g_2}{2} : 6 \left( 2(2m-1)G_{2m} + \sum_{k=2}^{m-2} (2k-1)(2m-2k-1)G_{2k}G_{2m-2k} \right)$$

Which, for the two to be equal, we get the following equality:

$$(2m-1)(2m-2)(2m-3)G_{2m} = 12(2m-1)G_{2m} + 6 \sum_{k=2}^{m-2} (2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$(2m-1)(4m^2 - 10m + 6)G_{2m} - 12(2m-1)G_{2m} = 6 \sum_{k=2}^{m-2} (2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$(2m-1)(4m^2 - 10m - 6)G_{2m} = 6 \sum_{k=2}^{m-2} (2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$(2m-1)(2m-6)(2m+1)G_{2m} = 6 \sum_{k=2}^{m-2} (2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$\implies (2m+1)(m-3)(2m-1)G_{2m} = 3 \sum_{k=2}^{m-2} (2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

Which, this equation is the desired recursive form.

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**Question 3** Freitag Chap. V.4 Exercise 3:

Let  $L \subset \mathbb{C}$  be a lattice with the property  $g_2(L) = 8$  and  $g_3(L) = 0$ . The point  $(2, 4)$  lies on the affine elliptic curve  $y^2 = 4x^3 - 8x$ . Let  $+$  be the addition (for points on the corresponding projective curve). Show that  $2 \cdot (2, 4) := (2, 4) + (2, 4)$  is the point  $(\frac{9}{4}, \frac{21}{4})$ .

**Pf:**

Consider the tangent of  $(2, 4)$  on the given elliptic curve  $y^2 = 4x^3 - 8x$ : By implicit differentiation, we get the following relationship:

$$2y \frac{dy}{dx} = 12x^2 - 8$$

which, for  $(x, y) = (2, 4)$ ,  $\frac{dy}{dx} \Big|_{(2,4)} = \frac{12x^2 - 8}{2y} \Big|_{(2,4)} = \frac{12 \cdot 2^2 - 8}{2 \cdot 4} = 5$ . Hence, the tangent is expressed as the following equation:

$$(y - 4) = 5(x - 2), \quad y = 5x - 6$$

Now, to solve for the third point, it must satisfy the following equations:

$$\begin{cases} y = 5x - 6 \\ y^2 = 4x^3 - 8x \end{cases}$$

Hence,  $(5x - 6)^2 = 4x^3 - 8x$ , which  $25x^2 - 60x + 36 = 4x^3 - 8x$ , so  $4x^3 - 25x^2 + 52x - 36 = 0$ . Which, consider the fact that  $(x, y) = (2, 4)$  appears on the tangent twice (with multiplicity 2), then  $(x - 2)^2$  is presumably a factor of the above equation. The above polynomial in fact has the following factorization:

$$4x^3 - 25x^2 + 52x - 36 = (x - 2)^2(4x - 9)$$

This indicates that the third zero happens when  $x = \frac{9}{4}$ . Which, the only point lying on the defined tangent above is given as:

$$y = 5 \cdot \frac{9}{4} - 6 = \frac{21}{4}$$

So, the third point lying on the tangent is  $(\frac{9}{4}, \frac{21}{4})$ .

**Question 4** *Stein and Shakarchi Pg. 281 Problem 3:*

Suppose  $\Omega$  is a simply connected domain that excludes the three roots of the polynomial  $4z^3 - g_2z - g_3$ . For  $w_0 \in \Omega$  fixed, define the function  $I$  on  $\Omega$  by

$$I(w) = \int_{w_0}^w \frac{dz}{\sqrt{4cz^3 - g_2z - g_3}}, \quad w \in \Omega$$

Then the function  $I$  has an inverse given by  $\wp(z + \alpha)$  for some constant  $\alpha$ ; that is:

$$I(\wp(z + \alpha)) = z$$

for appropriate  $\alpha$ .

**Pf:**

**Question 5** *Stein and Shakarchi Pg. 282 Problem 4:*

Suppose  $\mathcal{T}$  is purely imaginary, say  $\mathcal{T} = it$  with  $t > 0$ . Consider the division of the complex plane into congruent rectangles obtained by considering the lines  $x = n/2$ ,  $y = tm/2$  as  $n$  and  $m$  range over the integers.

- (a) Show that  $\wp$  is real-valued on all these lines, and hence on the boundaries of all these rectangles.
- (b) Prove that  $\wp$  maps the interior of each rectangle conformally to the upper (or lower) half-plane.

**Pf:**