Singularity of Magic Squares

Investigation with Eigenvalues of Magic Squares

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Even Order Regular Magic Squares are Singular - Bruce Mattingly, AMM, 107:9, 777-782

Singularity Condition of Odd-Order Regular Magic Square

Centrosymmetric and Skew-centrosymmetric Matrices and Regular Magic Squares

Conclusion

References (Need Detailed information)

1. Even Order Regular Magic Squares are Singular - Bruce Mattingly, AMM, 107:9, 777-782
2. On nonsingular regular magic squares of odd order, Lee, Love, Narayan
3. Magic square spectra, Loly, Cameron, Trump, Schindel
4. An investigation of even order magic squares (4, 6, 8): characteristic polynomials, eigenvalues, and encryption, Ashhab, Al-qdah
5. To construct a magic square of order 2n from a given square of order n, Candy
6. Self-complementary magic squares of singly even orders, Chia, Kok

Introduction

What is a Magic Square?

- An $n \times n$ matrix with entries $1, \ldots, n^2$
- Rows, columns, main diagonals add up to the same thing: $\mu = \frac{n^3+n}{2}$

Example 1.
A 4×4 magic square with $\mu = \frac{4^3+4}{2} = 34$.

$$\begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

Regular Magic Square

Definition 2.
Entries a, b are *complements* if

$$a + b = \frac{2\mu}{n} = n^2 + 1$$

.

Definition 3.
If all antipodal entries are complements, then the magic square is *regular*.

Example 4.
A 4×4 magic square, $\mu = \frac{4^3+4}{2} = 34$, $\frac{2\mu}{4} = 17$.

$$\left(\begin{array}{cc|cc} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ \hline 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{array}\right)$$

Observation

Example 5.
MATLAB generated Magic Squares:

$$A = \begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}, \quad \det A = -360$$

$$B = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}, \quad \det B = 0$$

Conjecture

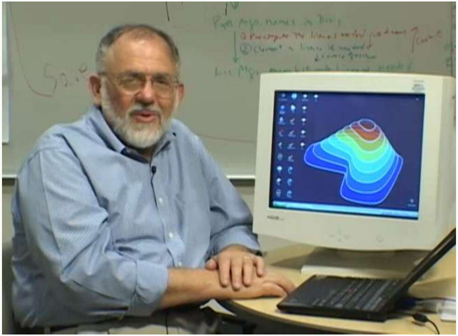

Figure 1: Cleve Moler, Creater of MATLAB

Cleve Moler's Conjecture:

- Even Order magic squares are Singular
- Odd Order magic squares are Nonsingular

Statement of Purpose

Collect and Compile known information about **Singularity of Magic Squares**.

Notation

Define the following:

$$\mu = \text{Sum of each row, column, main diagonal}$$

$$\mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Even Order Regular Magic Squares are Singular - Bruce Mattingly, AMM, 107:9, 777-782

Methodology

Given A, Even Order Regular Magic Square.

Objective: Prove that 0 is an Eigenvalue of A.

Method: Study matrices with similar spectra.

Given A an Even Order Regular Magic Square:

Theorem 6.

- $A\mathbf{e} = \mu\mathbf{e}$, *and* $\mathbf{e}^T A = \mu\mathbf{e}^T$.
- μ *as an eigenvalue of* A, *has multiplicity* 1.

Theorem 7.
Let $Z = A - \frac{\mu}{n}E$, *then:*

- *If* $A\mathbf{x} = \lambda\mathbf{x}$ *and* $\lambda \neq \mu$, *then* $Z\mathbf{x} = \lambda\mathbf{x}$.
- $A\mathbf{e} = \mu\mathbf{e}$, *and* $Z\mathbf{e} = \overline{0}$.

Results

Theorem 8.
Z is **skew-centrosymmetric**, *i.e.* $Z = -JZJ$. *Which:*

- *If* $Z\mathbf{x} = \lambda\mathbf{x}$, *then* $Z(J\mathbf{x}) = -\lambda(J\mathbf{x})$.
- *Disregard signs, each eigenvalue λ of Z has even multiplicity.*
- 0 *an eigenvalue of Z, has even multiplicity.*

Theorem 9.
Given A an Even Order Regular Magic Square, then 0 is an eigenvalue of A, A is singular.

Singularity Condition of Odd-Order Regular Magic Square

Previous Results

Recall:

- If A is a magic square, μ has multiplicity 1
- If A is regular, then $A + JAJ = \frac{2\mu}{n}E$.
- Define $Z := A - \frac{\mu}{n}E$, then $Z = -JZJ$ (**Skew-centrosymmetric**).
- Z inherits the spectra of A, except μ is swapped by 0

EX:

$$M_5 = \begin{pmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 17 & 5 & 13 & 21 & 9 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{pmatrix}, \quad Z_5 = \begin{pmatrix} -2 & 11 & -6 & 7 & -10 \\ -9 & -1 & 12 & -5 & 3 \\ 4 & -8 & 0 & 8 & -4 \\ -3 & 5 & -12 & 1 & 9 \\ 10 & -7 & 6 & -11 & 2 \end{pmatrix}$$

Zero Magic Square

Given A an $n \times n$ matrix, with n^2 distinct entries.

Definition 10.
A is a **Zero Magic Square**, if sum of each row, column, and main diagonal is 0.

EX: Any Magic Square A, $Z = A - \frac{\mu}{n}E$ is a zero magic square.

$$A = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 7.5 & -5.5 & -6.5 & 4.5 \\ -3.5 & 1.5 & 2.5 & -0.5 \\ 0.5 & -2.5 & -1.5 & 3.5 \\ -4.5 & 6.5 & 5.5 & -7.5 \end{pmatrix}$$

Latin Squares

$A, \; B$ are $n \times n$ matrices.

Definition 11.
A is a **Latin Square**, if there are n distinct entries. And, each entry appears once in each row and column.

Definition 12.
$A, \; B$ are Latin Squares. The two are **Orthogonal**, if each pair of matched entries, (a_{ij}, b_{ij}) is unique.

EX:

$$A = \begin{pmatrix} \mathbf{0} & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \mathbf{0} & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

Observation: $(0, 0)$ appears only once.

Circulant Matrices

Definition 13.
A an $n \times n$ matrix is a **Circulant Matrix**, if for all $i < n$:

$$\text{Row } i = \begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \end{pmatrix}$$

$$\implies \text{Row } (i+1) = \begin{pmatrix} a_n & a_1 & \cdots & a_{n-1} \end{pmatrix}$$

EX:

$$B = \begin{pmatrix} \color{red}{0} & 1 & 2 \\ 2 & \color{red}{0} & 1 \\ 1 & 2 & \color{red}{0} \end{pmatrix}$$

Let $S = \left\{ \frac{-(n-1)}{2}, ..., -1, 0, 1, ..., \frac{(n-1)}{2} \right\}$, and $\overline{a} = (a_1, ..., a_n)$ contains all elements of S, with $a_1 = 0$.

Definition 14.
A an $n \times n$ circulant matrix, is S-**Circulant**, if row 1 of A is some $\overline{a}$.

EX: For $n = 5$:

$$A_5 = \begin{pmatrix} 0 & 2 & -1 & 1 & -2 \\ -2 & 0 & 2 & -1 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ -1 & 1 & -2 & 0 & 2 \\ 2 & -1 & 1 & -2 & 0 \end{pmatrix}$$

Singularity Condition of Single-Order Regular Magic Square

Let A be a $n \times n$ regular magic square, $n = 2k + 1$. Partition its Z as:

$$Z = \begin{pmatrix} Z_{11} & a & Z_{13} \\ b^T & 0 & -b^T J \\ -JZ_{13}J & -Ja & -JZ_{11}J \end{pmatrix}$$

Where $a, b \in \mathbb{R}^k$, and $Z_{11}, Z_{13} \in \mathbb{R}^{k \times k}$.

Theorem 15.
A is nonsingular, iff $(Z_{11} + Z_{13}J)$ and $(Z_{11} - Z_{13}J)$ are nonsingular.

Proof Sketch

Goal:

Prove: 0 not an eigenvalue of $A \iff$ 0 has multiplicity 1 for Z

$\iff (Z_{11} + Z_{13}J), (Z_{11} - Z_{13}J)$ are nonsingular.

Proof Sketch

Let $Z' = K^{-1}ZK$ for specific K. Then:

$$\det(Z' - \lambda I) = (-1)^k \lambda \det(C_{21}C_{12} - \lambda^2 I)$$

$$C_{21} = \begin{pmatrix} Z_{11} - Z_{13}J \\ 2b^T \end{pmatrix}, \quad C_{12} = \begin{pmatrix} (Z_{11} + Z_{13}J) & a \end{pmatrix}$$

So, (0 has multiplicity 1) $\iff$ (λ^2 not a factor)

$\iff$ ($\det(C_{21}C_{12}) \neq 0$)

Proof Sketch

$$C_{21}C_{12} = (Z_{11} + Z_{13}J)(I + 2E)(Z_{11} - Z_{13}J)$$

Note: $(I + 2E)$ is invertible.

So, $(\det(C_{21}C_{12}) \neq 0) \iff (C_{21}C_{12}$ invertible$)$

$\iff (Z_{11} + Z_{13}J),\ (Z_{11} - Z_{13}J)$ are invertible.

Construction 1

Given A an $n \times n$ matrix, n odd.

Proposition 16.
If A is skew-centrosymmetric and S-circulant, let $Z = nA + AJ$.

Then, Z is a skew-centrosymmetric zero magic square, with entries

$$\left\{ \frac{-(n^2-1)}{2}, ..., -1, 0, 1, ..., \frac{(n^2-1)}{2} \right\}$$

Construction 1

EX: $n = 5$, $S = \{-2, -1, 0, 1, 2\}$, and $\overline{a} = (0, 2, -1, 1, -2)$.

$$A_5 = \begin{pmatrix} 0 & 2 & -1 & 1 & -2 \\ -2 & 0 & 2 & -1 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ -1 & 1 & -2 & 0 & 2 \\ 2 & -1 & 1 & -2 & 0 \end{pmatrix}, \quad A_5 J = \begin{pmatrix} -2 & 1 & -1 & 2 & 0 \\ 1 & -1 & 2 & 0 & -2 \\ -1 & 2 & 0 & -2 & 1 \\ 2 & 0 & -2 & 1 & -1 \\ 0 & -2 & 1 & -1 & 2 \end{pmatrix}$$

Construction 1

$$Z_5 = nA_5 + A_5 J = 5 \cdot A_5 + A_5 J$$

$$= \begin{pmatrix} 0 & 10 & -5 & 5 & -10 \\ -10 & 0 & 10 & -5 & 5 \\ 5 & -10 & 0 & 10 & -1 \\ -5 & 5 & -10 & 0 & 10 \\ 10 & -5 & 5 & -10 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 1 & -1 & 2 & 0 \\ 1 & -1 & 2 & 0 & -2 \\ -1 & 2 & 0 & -2 & 1 \\ 2 & 0 & -2 & 1 & -1 \\ 0 & -2 & 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 11 & -6 & 7 & -10 \\ -9 & -1 & 12 & -5 & 3 \\ -4 & -8 & 0 & 8 & 4 \\ -3 & 5 & -12 & 1 & 9 \\ 10 & -7 & 6 & -11 & 2 \end{pmatrix}, \quad rank(Z_5) = 4$$

Construction 2

Suppose A an $n \times n$ matrix, is skew-centrosymmetric, and S-circulant.

Proposition 17.

- *If n is an odd prime, then $\text{rank}(Z) = n - 1$.*
- *If n is an odd prime power, and the first row of A, $(a_1, ..., a_n)$ has $a_i = i - 1$ for $1 \leq i \leq \frac{n-1}{2}$, then $\text{rank}(Z) = n - 1$.*

Construction 3

EX: $n = 5$

$$A_5 = \begin{pmatrix} 0 & 2 & -1 & 1 & -2 \\ -2 & 0 & 2 & -1 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ -1 & 1 & -2 & 0 & 2 \\ 2 & -1 & 1 & -2 & 0 \end{pmatrix}, \quad Z_5 = \begin{pmatrix} -2 & 11 & -6 & 7 & -10 \\ -9 & -1 & 12 & -5 & 3 \\ -4 & -8 & 0 & 8 & 4 \\ -3 & 5 & -12 & 1 & 9 \\ 10 & -7 & 6 & -11 & 2 \end{pmatrix}$$

$$M = Z_5 + \frac{\mu}{n}E = Z_5 + 13E$$

$$= \begin{pmatrix} -2 & 11 & -6 & 7 & -10 \\ -9 & -1 & 12 & -5 & 3 \\ -4 & -8 & 0 & 8 & 4 \\ -3 & 5 & -12 & 1 & 9 \\ 10 & -7 & 6 & -11 & 2 \end{pmatrix} + (13) = \begin{pmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 9 & 5 & 13 & 21 & 17 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{pmatrix}$$

$$\det M = 4,680,000$$

Construction 3

Given n odd prime power, A an $n \times n$ skew-centrosymmetric S-circulant matrix.

Theorem 18.
With $Z = nA + AJ$, then define:

$$M = Z + \frac{\mu}{n}E = Z + \frac{n^2+1}{2}E$$

M is a Regular Magic Square, and M is nonsingular.

Centrosymmetric and Skew-centrosymmetric Matrices and Regular Magic Squares

(Skew) Centrosymmetric Matrices

centro = center

Spectra of Certain Matrices

- $H =$ real centrosymmetric matrix
- $S =$ real skew-centrosymmetric
- $H + iS$ real eigenvalues

eigenvalues of $H \pm iS, H \pm JS$
corresponding eigenvectors too

Equivalence Between Symmetric Skew-Centrosymmetric and Doubly Skew Matrices

- Eigenvalues
- Determinant
- Inverse

Takeaways from this Paper

- Symmetries
- Equivalences
- Spectra

Conclusion