Math 111C HW4

Zih-Yu Hsieh

May 5, 2025

1

Question 1 *Let F be a field and $f \in F[x]$ be an irreducible polynomial. Prove that all roots of $f(x)$ in $\overline{F}$ have the same multiplicity.*

Pf:

2

Question 2 *(a) Let $\zeta_6 \in \mathbb{C}$ be a primitive 6^{th} root of unity. Find $m_{\zeta_6,\mathbb{Q}}(x)$.*

(b) Let $m, n \in \mathbb{N}$ such that $m \equiv 2 \pmod 6$ and $n \equiv 4 \pmod 6$. Prove that $f(x) = x^m + x^n + 1$ is not irreducible over $\mathbb{Q}$.

Pf:

3

Question 3 *Prove that if F is an infinite field, then its multiplicative group $F^\times$ is never cyclic.*

Pf:

Suppose the contrary, that F is infinite while $F^\times$ is cyclic, then there exists $a \in F^\times$, such that $F^\times = \langle a \rangle$ (under multiplication).

First, notice that $a \neq 0$ (since $a \in F^\times$) and $a \neq 1$ (since $\langle 1 \rangle = \{1\}$, if $a = 1$, then $F^\times$ is finite, contradicting the assumption that $F = F^\times \cup \{0\}$ is infinite). Also, since $F^\times$ must be infinite based on similar reason, then $a \neq -1$ (since $(-1)^2 = 1$, then $|\langle a \rangle| = |a| = 2$ as the order of a, showing that $F^\times$ is again finite, which is a contradiction).

4

Question 4 *Let K/F be a field extension and $m, n \in \mathbb{N}$. Let $\alpha, \beta \in K$ with $[F(\alpha) : F] = m$ and $[F(\beta) : F] = n$.*

(a) Show that $[F(\alpha, \beta) : F] \leq mn$.

(b) If $\gcd(m, n) = 1$, show that $[F(\alpha, \beta) : F] = mn$.

Pf:

5

Question 5 *Let K be a finite field. Show that K is not algebraically closed.*

Pf: