

LIE ALGEBRA OF A LIE GROUP

Zih-Yu Hsieh, mentored by Arthur Jiang

University of California Santa Barbara



Tangent Vectors as Derivations

When embedding smooth manifolds into \mathbb{R}^n , tangent vectors are associated with directional derivatives. To generalize tangent vectors into abstract smooth manifold, we need an analogy:

Definition

Any point $u \in M$, a **Derivation at u** , is a linear map $v_u : C^\infty(M) \rightarrow \mathbb{R}$, that satisfies the product rule:

$$\forall f, g \in C^\infty(M), \quad v_u(fg) = f(u)(v_u g) + g(u)(v_u f)$$

Which, the set of all derivations at u , denoted as $T_u(M)$, is the **Tangent Space** of M at u , and each derivation $v_u \in T_u(M)$ is a **Tangent Vector** of u .

Vector Fields & Smooth Conditions

Given smooth manifold M , a vector field assigns each point with a tangent vector. More precisely:

Definition

a vector field is a map $X : M \rightarrow TM$ (TM denotes the **Tangent Bundle**), with $X(u) = X_u \in T_u(M)$.
Which, X is a **Smooth Vector Field**, if $X : M \rightarrow TM$ is a smooth map.
A collection of smooth vector fields on M is denoted as $\mathfrak{X}(M)$, which is an \mathbb{R} -vector space.

An equivalent condition of saying a vector field X is smooth, is through smooth functions $f \in C^\infty(M)$: For all $u \in M$, $X(u) = X_u \in T_u(M)$ is a derivation at u , define $Xf : M \rightarrow \mathbb{R}$ by $Xf(u) = X_u(f)$, then X is a smooth vector field iff $Xf \in C^\infty(M)$. Which, a smooth vector field can be viewed as a **Derivation**:

Property

For all $f, g \in C^\infty(M)$, given $X \in \mathfrak{X}(M)$, any $u \in M$ satisfies product rule:

$$X(fg)(u) = X_u(fg) = f(u)(X_u g) + g(u)(X_u f) = f(u)Xg(u) + g(u)Xf(u) \\ \implies X(fg) = f(Xg) + g(Xf)$$

Vector Fields of Different Manifolds

Given M, N two smooth manifolds, and smooth map $F : M \rightarrow N$. Let $X \in \mathfrak{X}(M)$, an ideal situation is mapping X to a smooth vector field of N through F . Yet, this requires F to be bijective:

Insert an example for bijective

So, we'll consider a weaker notion:

Definition

Given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, the two are **F -related**, if for all $u \in M$, the following is true:

$$dF_u(X_u) = Y_{F(u)}$$

Simply speaking, F maps the tangent vectors collected by X , to be compatible with tangent vectors collected by Y .

Insert another example

Lie Brackets of Vector Fields

The initial motivation is to combine two vector fields $X, Y \in \mathfrak{X}(M)$ to be another vector field. For all $f \in C^\infty(M)$, since $Yf \in C^\infty(M)$, then $XYf := X(Yf) \in C^\infty(M)$. But, in general XY is not a derivation, hence not a vector field:

Example

Define vector fields $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial y}$ on \mathbb{R}^2 . Take smooth functions $f(x, y) = x$ and $g(x, y) = y$, then we get the following:

$$XY(fg) = X\left(x\frac{\partial}{\partial y}(xy)\right) = \frac{\partial}{\partial x}(x^2) = 2x$$

But, product rule doesn't hold for this example:

$$f(XYg) + g(XYf) = x\left(X\left(x\frac{\partial}{\partial y}(y)\right)\right) + y\left(X\left(x\frac{\partial}{\partial y}(x)\right)\right) = x$$

So, we need to define a new operation on vector fields:

Definition

The **Lie Bracket** $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$, is defined as:

$$\forall X, Y \in \mathfrak{X}(M), \quad [X, Y] = XY - YX$$

Which, the output $[X, Y] \in \mathfrak{X}(M)$, since it satisfies product rule:

$$[X, Y](fg) = X(Y(fg)) - Y(X(fg)) = X(f(Yg) + g(Yf)) - Y(f(Xg) + g(Xf)) \\ = f(XYg) + (Yg)(Xf) + g(XYf) + (Yf)(Xg) - f(YXg) - (Xg)(Yf) - g(YXf) - (Xf)(Yg) \\ = f(XYg - YXg) + g(XYf - YXf) = f[X, Y](g) + g[X, Y](f)$$

Lie Bracket also satisfies these properties:

- **Bilinearity**: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- **Antisymmetry**: $[X, Y] = -[Y, X]$
- **Jacobi's Identity**: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Moreover, Lie Bracket inherits relation of smooth maps:

Property

Given smooth map $F : M \rightarrow N$, if $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ are F -related respectively, then $[X_1, X_2] \in \mathfrak{X}(M)$ and $[Y_1, Y_2] \in \mathfrak{X}(N)$ are also F -related.

Lie Group & Left-Invariant Vector Fields

The initial motivation is to study group structures in some smooth manifolds.

Definition

A **Lie Group** G , is a smooth manifold along with group structure, such that the group operation $P : G \times G \rightarrow G$ by $P(g, h) = gh$, and the inversion map $i : G \rightarrow G$ by $i(g) = g^{-1}$ are both smooth maps between manifolds.

For all $g \in G$, denote the left multiplication $L_g : G \rightarrow G$ by $L_g(h) = gh$, since $L_g = P|_{\{g\} \times G}$, it is a smooth map. Hence, there's a notion of X being L_g -related to itself:

Definition

Given any $X \in \mathfrak{X}(G)$ and all $g \in G$, X is a **Left-Invariant Vector Field**, if for all $g \in G$, X is L_g -related to itself.
The collection of Left-Invariant vector fields $\mathfrak{g} \subseteq \mathfrak{X}(G)$, is a linear subspace.

Recall that Lie Bracket of vector field preserves F -relation between manifolds, so:

Property

For all $X, Y \in \mathfrak{X}(G)$ that are left-invariant, since for all $g \in G$, X and Y are L_g related to themselves, then the Lie Bracket $[X, Y]$ is also L_g related to $[X, Y]$. Hence, $[X, Y]$ is also left-invariant, so Left-Invariant vector fields \mathfrak{g} is closed under Lie Bracket's operation.

Lie Algebra on a Lie Group

Definition

Given a vector space \mathfrak{g} over \mathbb{R} or \mathbb{C} , with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, such that the following holds:

- **Bilinearity**: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- **Antisymmetry**: $[X, Y] = -[Y, X]$
- **Jacobi's Identity**: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Then, the pair $(\mathfrak{g}, [\cdot, \cdot])$ is a **Lie Algebra**.

In general, Lie Algebra is non-associative, which Jacobi's Identity is an alternative condition for Lie Algebra.
Finally, we can define **Lie Algebra of a Lie Group**:

Definition

Given a lie group G , since the subset of left-invariant vector fields $\mathfrak{g} \subseteq \mathfrak{X}(G)$ forms a linear subspace, while closed under Lie Bracket's operation, then the pair $(\mathfrak{g}, [\cdot, \cdot])$ forms a **Lie Algebra** of G , denoted as $\text{Lie}(G)$.

Here's an example of Lie Algebra on a Lie Group:

Example

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Reference: John M. Lee, Introduction to Smooth Manifolds, 2nd Edition