Math CS 122B HW7

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Question 1 *The functional equation of the ζ-function can also be written in the following form:*

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

Deduce from this: In the half-plane $\sigma \le 0$, the function $\zeta(s)$ has exactly the zeros $s = -2k,\ k \in \mathbb{N}$. All other zeros of the ζ-function are located in the vertical strip $0 < Res < 1$.

Pf:

First, recall that for the half plane $\sigma > 1$, the following inequality is given:

$$\left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4 |\zeta(\sigma+2it)|[\zeta(\sigma)(\sigma-1)]^3 \ge (\sigma-1)^{-1}$$

Since for $\sigma > 1$, the expressiong $(\sigma-1)^{-1} > 0$, this enforces all $s = \sigma + it$ in the half plane to have $\zeta(s) \ne 0$ (or else the left side of the inequality is 0, which violates the inequality). Similarly, this inequality can be extended onto the line $\text{Re}(s) = 1$, where $\zeta(s)$ has no zeros on this line also. So, for $\sigma \ge 1$, $\zeta(s)$ has no zero.

Now, in the half plane $\sigma \le 0$, for all $s' \ne 0$, since it can be written as $s' = 1 - s$, where $s = 1 - s'$ has $\text{Re}(s) = 1 - \text{Re}(s') \ge 1$ (and since $s' \ne 0$, then $s \ne 1$). So, $\zeta(s)$ after the continuation past $\text{Re}(s) = 1$, has $\zeta(s)$ being well-defined.

Then, by the functional equation, we get the following:

$$\zeta(s') = \zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

Since $\text{Re}(s) \ge 1$ with $s \ne 1$, then $\zeta(s) \ne 0$ based on what is mentioned during the start; also, $\Gamma(s) \ne 0$ for all $s \in \mathbb{C} \setminus \{0, -1, -2, ...\}$, while $2(2\pi)^{-s} \ne 0$ for all $s \in \mathbb{C}$. Hence, in case for $\zeta(1-s) = 0$, we must have $\cos(\frac{\pi s}{2}) = 0$, which enforces $\frac{\pi s}{2} = k\pi + \frac{\pi}{2}$ for some $k \in \mathbb{Z}$, or $s = 2k+1$ fo some $k \in \mathbb{Z}$. Now, under this assumption, since $\text{Re}(s) \ge 1$ while $s \ne 1$, then $k \ge 1$. So, when transfering back to $s' = 1 - s$, we get $s' = 1 - (2k+1) = -2k$ for integer $k \ge 1$.

Hence, for $\text{Re}(s') \le 0$, for $\zeta(s') = 0$, then $s' = -2k$ for some $k \in \mathbb{N}$ (this is an iff since at all these points, $\cos(\frac{\pi s}{2}) = 0$, which $\zeta(s') = \zeta(1-s) = 0$).

Finally, for $s' = 0$ (where if $s' = 1 - s$, $s = 1$). Recall that $\zeta(s)$ has a simple pole at $s = 1$, while $\cos(\frac{\pi s}{2})$ has a simple zero at $s = 1$ (where the input is $\frac{\pi}{2}$, where cos is 0). Hence, $\cos(\frac{\pi s}{2}) = (s-1)h(z)$ for some analytic function h where $h(1) \neq 0$. Also, we know $\lim_{s\to 1}(s-1)\zeta(s) = 1$ (has been given in the textbook). Then, we get the following:

$$\lim_{s\to 1}\zeta(1-s) = \lim_{s\to 1} 2(2\pi)^{-s}\Gamma(s)h(s)(s-1)\zeta(s) = 2(2\pi)^{-1}\Gamma(1)h(1)\cdot\lim_{s\to 1}(s-1)\zeta(s) = 2(2\pi)^{-1}\Gamma(1)h(1) \neq 0$$

Hence, we can deduce that at $s = 1$ (where $s' = 1 - s = 0$), $\zeta(s')$ has a removable singularity that has limit not being 0, henc $\zeta(s')$ as an extension has $\zeta(0) \neq 0$.

The above casees proves that when $\sigma \geq 1$ or $\sigma \leq 0$, $\zeta(s) = 0$ iff $s = -2k$ for some $k \in \mathbb{N}$, where for any other input ζ is nonzero.

Hence, if there are any other zeros, it must exist in the vertical strip $0 < \mathrm{Re}(s) < 1$.

2 (continuation & functional not done)

Question 2 *The following special case of the Hecke Theorem was already known to B. Riemann (1859):*

$$\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t} t^{s/2}\frac{dt}{t}$$
$$= \frac{1}{2}\int_1^{\infty}(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

Deduce directly this special case, and use it to prove the meromorphic continuation and the functional equation.

Pf:

For this problem, first assume $\mathrm{Re}(s) > 1$ (where $\zeta(s)$ is defined with the original series form). We'll break down into two different equations:

Equation 1:

We'll first prove the following:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t} t^{s/2}\frac{dt}{t}$$

For each $n \in \mathbb{N}$, the integral within the right hand side summation, after doing the substitution $u = \pi n^2 t$ (where $du = \pi n^2 dt$), we get the following:

$$\int_0^{\infty} e^{-\pi n^2 t} t^{s/2}\frac{dt}{t} = \int_0^{\infty} e^{-u}\left(\frac{u}{\pi n^2}\right)^{s/2}\frac{du/(\pi n^2)}{u/(\pi n^2)} = (\pi n^2)^{-s/2}\int_0^{\infty} e^{-u}u^{s/2}\frac{du}{u}$$
$$= \pi^{-s/2}n^{-s}\int_0^{\infty} e^{-u}u^{s/2-1}du = \pi^{-s/2}n^{-s}\Gamma\left(\frac{s}{2}\right)$$

Hence, for the series, since $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ converges normally within $\mathrm{Re}(s) > 1$, we get the following:

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t} t^{s/2}\frac{dt}{t} = \sum_{n=1}^{\infty}\pi^{-s/2}n^{-s}\Gamma\left(\frac{s}{2}\right) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}n^{-s} = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

So, the first equatity holds.

Equation 2:

Our second goal is to prove the following:

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t} t^{s/2}\frac{dt}{t} = \frac{1}{2}\int_1^{\infty}(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

Since the summation is absolutely convergent for $\mathrm{Re}(s) > 1$ (based on how $\xi(s)$ is defined), then swapping the order of summation causes no issue. Hence, since for all $n \in \mathbb{N}$, since $n^2 = (-n)^2$, the infinite summation can also be decomposed as:

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t} t^{s/2}\frac{dt}{t} = \frac{1}{2}\sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t} t^{s/2}\frac{dt}{t} + \frac{1}{2}\sum_{n=-\infty}^{-1}\int_0^{\infty} e^{-\pi n^2 t} t^{s/2}\frac{dt}{t}$$

$$= \frac{1}{2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \int_1^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} + \frac{1}{2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t}$$

For the first summation, recall that the Theta Series is defined as $\vartheta : \mathbb{H} \to \mathbb{C},\ \vartheta(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau}$. Since for $n = 0$, $e^{i\pi n^2 \tau} = e^0 = 1$, then we get the following:

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{i\pi n^2 \tau} = \vartheta(\tau) - 1$$

If consider $\tau = it$ for $t \in (0, \infty)$, we get the following:

$$\vartheta(it) - 1 = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{i\pi n^2 \cdot it} = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-\pi n^2 t}$$

So, for the first summation in the equation, since ϑ is a normally convergent series of function, the summation and integral can change the order of operation. Hence:

$$\frac{1}{2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \int_1^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \int_1^\infty \left(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-\pi n^2 t} \right) t^{s/2} \frac{dt}{t} = \frac{1}{2} \int_1^\infty (\vartheta(it) - 1) t^{s/2} \frac{dt}{t}$$

Now, to work on the second summation, for all $n \in \mathbb{Z}$ with $n \neq 0$, with the substitution $u = \frac{1}{t}$, $du = -\frac{1}{t^2} dt$ (or $\frac{dt}{t} = -t du = -\frac{du}{u}$), we get the following integral representation:

$$\int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = -\int_\infty^1 e^{-\pi n^2 \cdot 1/u} u^{-s/2} \frac{du}{u} = \int_1^\infty e^{-\pi n^2 \cdot 1/u} u^{-s/2} \frac{du}{u}$$

Which, based on the above change of variable and the property of Theta Series, the second summation in the equation can be rewrite as:

$$\frac{1}{2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \int_1^\infty e^{-\pi n^2 \cdot 1/u} u^{-s/2} \frac{du}{u} = \frac{1}{2} \int_1^\infty \left(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-\pi n^2 \cdot 1/u} \right) u^{-s/2} \frac{du}{u}$$

$$= \frac{1}{2} \int_1^\infty \left(\vartheta\left(\frac{i}{u}\right) - 1 \right) u^{-s/2} \frac{du}{u}$$

Which, there is a property of Theta Series given below:

$$\vartheta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}}\vartheta(z)$$

Hence, for $u \in (0, \infty)$, let $z = iu$, we get the following:

$$\vartheta\left(\frac{i}{u}\right) = \vartheta\left(-\frac{1}{iu}\right) = \sqrt{\frac{iu}{i}}\vartheta(iu) = u^{1/2}\vartheta(iu)$$

So, the summation can be modified as follow:

$$\frac{1}{2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \int_0^1 e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} = \frac{1}{2} \int_1^\infty \left(u^{1/2}\vartheta(iu) - 1 \right) u^{-s/2} \frac{du}{u}$$

$$= \frac{1}{2} \int_1^\infty (\vartheta(iu) - 1) u^{1/2} u^{-s/2} \frac{du}{u} + \frac{1}{2} \int_1^\infty u^{1/2} u^{-s/2} \frac{du}{u} - \frac{1}{2} \int_1^\infty u^{-s/2} \frac{du}{u}$$

For the integrals at the middle and the right, since the power of u is given by $\frac{1-s}{2} - 1$ and $\frac{-s}{2} - 1$, then because $\mathrm{Re}(s) > 1$, the two powers of u both have the real parts being less than -1. Hence, the integral absolutely converges, and using power rule, we yield:

$$\frac{1}{2}\int_1^\infty u^{1/2}u^{-s/2}\frac{du}{u} = \frac{1}{2}\int_1^\infty u^{\frac{1-s}{2}-1}du = \frac{1}{2}\cdot\frac{2}{1-s}u^{\frac{1-s}{2}}\Big|_1^\infty = -\frac{1}{1-s}$$

$$\frac{1}{2}\int_1^\infty u^{-s/2}\frac{du}{u} = \frac{1}{2}\int_1^\infty u^{-s/2-1}du = -\frac{1}{2}\cdot\frac{2}{s}u^{-s/2}\Big|_1^\infty = \frac{1}{s}u^{-s/2}\Big|_1^\infty = \frac{1}{s}$$

(Note: since $\mathrm{Re}(s) > 1$, then $\mathrm{Re}(\frac{1-s}{2}) < 0$ and $\mathrm{Re}(-\frac{s}{2}) < 0$, so $\lim_{u\to\infty} u^{\frac{1-s}{2}} = 0$ and $\lim_{u\to\infty} u^{-s/2} = 0$). So, combining the pieces for the second summation, we get:

$$\frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\int_0^1 e^{-\pi n^2 t}t^{s/2}\frac{dt}{t} = \frac{1}{2}\int_1^\infty(\vartheta(iu)-1)u^{1/2}u^{-s/2}\frac{du}{u} + \frac{1}{2}\int_1^\infty u^{1/2}u^{-s/2}\frac{du}{u} - \frac{1}{2}\int_1^\infty u^{-s/2}\frac{du}{u}$$

$$= \frac{1}{2}\int_1^\infty(\vartheta(it)-1)t^{(1-s)/2}\frac{dt}{t} + \left(-\frac{1}{1-s}\right) - \frac{1}{s} = \frac{1}{2}\int_1^\infty(\vartheta(it)-1)t^{(1-s)/2}\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

Finally, the original equation can be obtained as follow:

$$\sum_{n=1}^\infty\int_0^\infty e^{-\pi n^2 t}t^{s/2}\frac{dt}{t} = \frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\int_1^\infty e^{-\pi n^2 t}t^{s/2}\frac{dt}{t} + \frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\int_0^1 e^{-\pi n^2 t}t^{s/2}\frac{dt}{t}$$

$$= \left(\frac{1}{2}\int_1^\infty(\vartheta(it)-1)t^{s/2}\frac{dt}{t}\right) + \left(\frac{1}{2}\int_1^\infty(\vartheta(it)-1)t^{(1-s)/2}\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}\right)$$

$$= \frac{1}{2}\int_1^\infty(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

This verifies the second equation. Together with the first and the second equation proven, the function $\xi(s)$ defined in the question for $\mathrm{Re}(s) > 1$ satisfies the following equation:

$$\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n=1}^\infty\int_0^\infty e^{-\pi n^2 t}t^{s/2}\frac{dt}{t}$$

$$= \frac{1}{2}\int_1^\infty(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

Meromorphic Continuation & Functional equation of ζ:

The textbook had introduced the meromorphic continuation of ζ onto the half plane $\mathrm{Re}(s) > 0$. Now, consider any s satisfying $\frac{1}{2} < \mathrm{Re}(s) < 1$, then $(1-s)$ satisfies $\mathrm{Re}(1-s) = 1 - \mathrm{Re}(s)$, $0 < \mathrm{Re}(1-s) < \frac{1}{2}$. Since both $s, (1-s)$ are within the doamin of the extended ζ, then if plug in the ξ-function defined above, using the integral expression, we get:

$$\xi(s) = \int_1^\infty(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

$$\xi(1-s)\int_1^\infty(\vartheta(it)-1)(t^{(1-s)/2}+t^{(1-(1-s))/2})\frac{dt}{t} - \frac{1}{1-s} - \frac{1}{1-(1-s)}$$

$$= \int_1^\infty(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s} = \xi(s)$$

Hence, for s satisfying $\frac{1}{2} < \mathrm{Re}(s) < 1$, $\xi(1-s) = \xi(s)$, which leads to the following functional equation:

$$\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \xi(1-s) = \xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$\implies \zeta(1-s) = \pi^{1/2-s}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s)$$

Now, recall the Duplication Formula of Γ-function:

$$\forall z \in \mathbb{C} \setminus \{0, -1, -2, ...\}, \quad \Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{1+z}{2}\right) = \frac{\sqrt{\pi}}{2^{z-1}}\Gamma(z)$$

Then, take $z = s$ (since $\frac{1}{2} < \mathrm{Re}(s) < 1$ by assumption now, it is valid), we get:

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right) = \frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s) \implies \Gamma\left(\frac{s}{2}\right) = \frac{\sqrt{\pi}\cdot\Gamma(s)}{2^{s-1}\Gamma\left(\frac{1+s}{2}\right)}$$

Hence, the expression go $\zeta(1-s)$ becomes:

$$\zeta(1-s) = \pi^{1/2-s}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s) = \frac{\sqrt{\pi}}{\pi^s\Gamma\left(\frac{1-s}{2}\right)}\cdot\frac{\sqrt{\pi}\cdot\Gamma(s)}{2^{s-1}\Gamma\left(\frac{1+s}{2}\right)}\zeta(s)$$

$$= \frac{2\pi\cdot\Gamma(s)}{(2\pi)^s}\cdot\frac{1}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(1-\left(\frac{1+s}{2}\right)\right)}\zeta(s)$$

Which, this expression allows the use of another property of Γ-function:

$$\forall z \in \mathbb{C}, \quad \frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{\sin(\pi z)}{\pi}$$

Take $z = \frac{1+s}{2}$, we get:

$$\frac{1}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(1-\left(\frac{1+s}{2}\right)\right)} = \frac{\sin\left(\pi\left(\frac{1+s}{2}\right)\right)}{\pi} = \frac{\sin\left(\frac{\pi s}{2}+\frac{\pi}{2}\right)}{\pi} = \frac{\cos\left(\frac{\pi s}{2}\right)}{\pi}$$

So, the following functional equation of ζ (on $\frac{1}{2} < \mathrm{Re}(s) < 1$) can be deduced:

$$\zeta(1-s) = \frac{2\pi\cdot\Gamma(s)}{(2\pi)^s}\cdot\frac{1}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(1-\left(\frac{1+s}{2}\right)\right)}\zeta(s)$$

$$= \frac{2\pi\cdot\Gamma(s)}{(2\pi)^s}\cdot\frac{\cos\left(\frac{\pi s}{2}\right)}{\pi}\zeta(s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

This finishes the proof for the functional equation. Moreover, for all s' with $\mathrm{Re}(s') < \frac{1}{2}$, since given $s' = 1-s$, or $s = 1-s'$, we have $\mathrm{Re}(s) = \mathrm{Re}(1-s') = 1-\mathrm{Re}(s') > \frac{1}{2}$, then the following expression is well-defined:

$$2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

Hence, the meromorphic continuation of ζ onto the whole plane can be followed by the functional equation:

$$\forall s' \in \mathbb{C},\ \mathrm{Re}(s') < \frac{1}{2},\ \text{let } s' = 1-s,\ \mathrm{Re}(s) > \frac{1}{2}, \quad \text{Define } \zeta(1-s) := 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$