Math CS 122B HW5

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Question 1 *Freitag Chap. V.3 Exercise 5:*
The algebraic differential equation of the $\wp$-function can berewritten as:

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

Here, $e_j,\ 1 \leq j \leq 3$, are the three half lattice values of the $\wp$-function.

Pf:

Given the algebraic differential equation of the $\wp$-function as follow:

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$$

Within the fundamental region P, there are 3 points with the value of $\wp'$ to be zero, which is given by $\frac{w_1}{2},\ \frac{w_2}{2},\ \frac{w_1+w_2}{2}$ (and points congruent to these points $\mod L$) when the lattice $L = w_1\mathbb{Z} + w_2\mathbb{Z}$.

Then, by definition, the given points have the evaluation to be the following:

$$e_1 = \wp\left(\frac{w_1}{2}\right), \quad e_2 = \wp\left(\frac{w_2}{2}\right), \quad e_3 = \wp\left(\frac{w_1+w_2}{2}\right)$$

Which, let $w = \wp(z)$, then the polynomial $4w^3 - g_2w - g_3 = 0$ iff $\wp'(z) = 0$, which within the fundamental region, only the three distinct points mentioned above are the solution, so the values of $\wp$ of these points are the zeros of the polynomial $4w^3 - g_2w - g_3$.

Then, since e_1, e_2, e_3 are all distinct, while $4w^3 - g_2w - g_3$ has at most 3 distinct zeroes, then they must be all the zeros of the polynomial. Hence, $4w^3 - g_2w - g_3 = 4(w-e_1)(w-e_2)(w-e_3)$, which we get the following:

$$(\wp'(z))^3 = 4(\wp(z))^3 - g_2\wp(z) - g_3 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$$

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Question 2 *Freitag Chap. V.3 Exercise 6:*
Show the following recursion formulas for the Eisenstein series G_{2m} for $m \geq 4$:

$$(2m+1)(m-3)(2m-1)G_{2m} = 3\sum_{j=2}^{m-2}(2j-1)(2m-2j-1)G_{2j}G_{2m-2j}$$

for instance $G_{10} = \frac{5}{11}G_4G_6$. Any Eisenstein series G_{2m}, $m \geq 4$, is thus representable as a polynomial in G_4 and G_6 with nonnegative coefficients.

Pf:

First, the $\wp$-function is given as follow:

$$\wp(z) = \frac{1}{z^2} + \sum_{m=1}^{\infty}(2m+1)G_{2(m+1)}z^{2m}$$

With the formula of $\wp$-function as series of functions, since it converges normally within $\mathbb{C} \setminus L$ (with L being the lattice), then differentiation can be performed termwise. Hence, its second derivative is given by:

$$\wp''(z) = \frac{d^2}{dz^2}\left(\frac{1}{z^2}\right) + \sum_{m=1}^{\infty}\frac{d^2}{dz^2}\left((2m+1)G_{2(m+1)}z^{2m}\right) = \frac{6}{z^4} + \sum_{m=1}^{\infty}(2m+1)(2m)(2m-1)G_{2(m+1)}z^{2m-2}$$

$$= \frac{6}{z^4} + \sum_{m=2}^{\infty}(2m-1)(2m-2)(2m-3)G_{2m}z^{2m-4}$$

Recall the following second order differential equation of $\wp$-function:

$$2\wp''(z) = 12(\wp(z))^2 - g_2, \quad \wp''(z) = 6(\wp(z))^2 - \frac{g_2}{2}$$

The goal is to get a recursive relation of the coefficient of each power of $\wp''(z)$.

With the expression of $\wp''$ in power series from above, to get an expression of G_{2m} for $m \geq 4$, it suffices to find the coefficient of z^{2m-4} within $6(\wp(z))^2 - \frac{g_2}{2}$. There are two cases to consider:

1. z^{2m-4} can be expressed as $\frac{1}{z^2} \cdot z^{2m-2}$, within $\wp(z)$, the coefficient of $\frac{1}{z^2}$ is 1, while the coefficient of $z^{2m-2} = z^{2(m-1)}$ is $(2(m-1)+1)G_{2((m-1)+1)} = (2m-1)G_{2m}$. Hence, since $(\wp(z))^2$ has two copies of the above expression, then the coefficient of $\frac{1}{z^2} \cdot z^{2m-2}$ is:

$$2 \cdot 1 \cdot (2m-1)G_{2m} = 2(2m-1)G_{2m}$$

2. Since $\wp(z)$ also has all power z^{2m} for $m \geq 1$, then $z^{2m-4} = z^{2(m-2)}$ can also be expressed as $z^{2k} \cdot z^{2(m-k-2)}$, for integers $k \geq 1$ and $(m-k-2) \geq 1$ (or $k \leq (m-3)$). Hence, for the convolution of power series of $(\wp(z))^2$ (excluding the negative powers mentioned above), z^{2m-4} term has the following coefficient:

$$\sum_{k=1}^{m-3}(2k+1)G_{2(k+1)} \cdot (2(m-k-2)+1)G_{2((m-k-2)+1)} = \sum_{k=1}^{m-3}(2(k+1)-1)(2m-2(k+1)-1)G_{2(k+1)}G_{2(m-(k+1))}$$

$$= \sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2(m-k)}$$

(Note: recall that z^{2k} term has coefficient $(2k+1)G_{2(k+1)}$, while $z^{2(m-k-2)}$ term has coefficient given as $(2(m-k-2)+1)G_{2((m-k-2)+1)}$).

So, the coefficient of z^{2m-4} in $(\wp(z))^2$ is recorded as:

$$2(2m-1)G_{2m} + \sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

Hence, based on the equation $\wp''(z) = 6(\wp(z))^2 - \frac{g_2}{2}$, for all $m \geq 4$, the coefficient of z^{2m-4} is given as the following two forms:

$$\text{Coefficient of } z^{2m-4} \text{ in } \wp''(z): \quad (2m-1)(2m-2)(2m-3)G_{2m}$$

$$\text{Coefficient of } z^{2m-4} \text{ in } 6(\wp(z))^2 - \frac{g_2}{2}: \quad 6\left(2(2m-1)G_{2m} + \sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}\right)$$

Which, for the two to be equal, we get the following equality:

$$(2m-1)(2m-2)(2m-3)G_{2m} = 12(2m-1)G_{2m} + 6\sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$(2m-1)(4m^2-10m+6)G_{2m} - 12(2m-1)G_{2m} = 6\sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$(2m-1)(4m^2-10m-6)G_{2m} = 6\sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$(2m-1)(2m-6)(2m+1)G_{2m} = 6\sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

$$\implies (2m+1)(m-3)(2m-1)G_{2m} = 3\sum_{k=2}^{m-2}(2k-1)(2m-2k-1)G_{2k}G_{2m-2k}$$

Which, this equation is the desired recursive form.

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Question 3 *Freitag Chap. V.4 Exercise 3:*
Let $L \subset \mathbb{C}$ be a lattice with the property $g_2(L) = 8$ and $g_3(L) = 0$. The point $(2, 4)$ lies on the affine elliptic curve $y^2 = 4x^3 - 8x$. Let $+$ be the addition (for points on the corresponding projective curve). SHow that $2 \cdot (2, 4) := (2, 4) + (2, 4)$ is the point $(\frac{9}{4}, -\frac{21}{4})$.

Pf:

Consider the tangent of $(2, 4)$ on the given elliptic curve $y^2 = 4x^3 - 8x$: By implicit differentiation, we get the following relationship:

$$2y\frac{dy}{dx} = 12x^2 - 8$$

which, for $(x, y) = (2, 4)$, $\frac{dy}{dx}\Big|_{(2,4)} = \frac{12x^2-8}{2y}\Big|_{(2,4)} = \frac{12\cdot 2^2-8}{2\cdot 4} = 5$. Hence, the tangent is expressed as the following equation:

$$(y - 4) = 5(x - 2), \quad y = 5x - 6$$

Now, to solve for the third point, it must satisfy the following equations:

$$\begin{cases} y = 5x - 6 \\ y^2 = 4x^3 - 8x \end{cases}$$

Hence, $(5x - 6)^2 = 4x^3 - 8x$, which $25x^2 - 60x + 36 = 4x^3 - 8x$, so $4x^3 - 25x^2 + 52x - 36 = 0$. Which, consider the fact that $(x, y) = (2, 4)$ appears on the tangent twice (with multiplicity 2), then $(x - 2)^2$ is presumably a factor of the above equation. The above polynomial in fact has the following factorization:

$$4x^3 - 25x^2 + 52x - 36 = (x - 2)^2(4x - 9)$$

This indicates that the third zero exists at $x = \frac{9}{4}$. Which, the only point lying on the defined tangent above with $x = \frac{9}{4}$ is given as:

$$y = 5 \cdot \frac{9}{4} - 6 = \frac{21}{4}$$

So, the third point lying on the tangent is $(\frac{9}{4}, \frac{21}{4})$.

Finally, by Addition Theorem, since three points are on the same projective line iff the three points adds up to be 0 on the elliptic curve, so since $(2, 4), (2, 4), (\frac{9}{4}, \frac{21}{4})$ are on the same line, under the addition the three points become 0. Hence, $2 \cdot (2, 4) + (\frac{9}{4}, \frac{21}{4}) \equiv 0$, showing that $2 \cdot (2, 4) \equiv (\frac{9}{4}, -\frac{21}{4})$ based on the addition on the elliptic curve.

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Question 4 *Stein and Shakarchi Pg. 281 Problem 3:*
Suppose Ω is a simply connected domain that excludes the three roots of the polynomial $4z^3 - g_2 z - g_3$. For $w_0 \in \Omega$ fixed, define the function I on Ω by

$$I(w) = \int_{w_0}^{w} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}, \quad w \in \Omega$$

Then the function I has an inverse given by $\wp(z+\alpha)$ for some constant α; that is:

$$I(\wp(z+\alpha)) = z$$

for appropriate α.

Pf:

Given that Ω is a simply connected domain that excludes the roots e_1, e_2, e_3 of $4z^3 - g_2 z - g_3$, then since this simply connected open region doesn't include the zeros for the polynomial, hence there exists a well-defined square root for the function (can be denoted by $\sqrt{4z^3 - g_2 z - g_3}$).

Then, given the definition of $I(w)$ above (as an antiderivative of $\frac{1}{\sqrt{4z^3 - g_2 z - g_3}}$), its derivative $I'(w) = \frac{1}{\sqrt{4z^3 - g_2 z - g_3}}$.

Now, since $\wp : \mathbb{C} \setminus L \to \mathbb{C}$ is an order 2 even elliptic function, then threre exists $\alpha_1 \in \mathbb{C} \setminus L$, such that $\wp(\alpha_1) = \wp(-\alpha_1) = w_0$, while $\wp'(\alpha_1) = -\wp'(-\alpha_1)$.

Then, given the algebraic differential equation $(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$, we have $(\wp'(\alpha_1))^2 = (\wp'(-\alpha_1))^2 = 4w_0^3 - g_2 w_0 - g_3$. Which, for the defined square root, there are two cases: either $\sqrt{4w_0^3 - g_2 w_0 - g_3} = \wp'(\alpha_1)$, or $\sqrt{4w_0^3 - g_2 w_0 - g_3} = -\wp'(\alpha_1) = \wp'(-\alpha_1)$. In either case, we can choose $\alpha \in \{\alpha_1, -\alpha_1\}$, such that $\sqrt{4w_0^3 - g_2 w_0 - g_3} = \sqrt{(\wp'(\alpha))^2} = \wp'(\alpha)$ (and it still satisfies $\wp(\alpha) = w_0$).

Hence, given the function $I(\wp(z+\alpha))$ with the domain containing the preimage of Ω (which defines it to contain 0, since $\wp(0+\alpha) = \wp(\alpha) = w_0 \in \Omega$), we have the following:

$$I(\wp(0+\alpha)) = I(w_0) = \int_{w_0}^{w_0} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} = 0$$

Also, if differentiate this composition of function, we get:

$$(I(\wp(z+\alpha)))' = I'(\wp(z+\alpha))\wp'(z+\alpha) = \frac{\wp'(z+\alpha)}{\sqrt{4(\wp(z+\alpha))^3 - g_2(\wp(z+\alpha)) - g_3}} = \frac{\wp'(z+\alpha)}{\sqrt{(\wp'(z+\alpha))^2}} = \pm 1$$

Notice that since both I and $\wp$ are analytic function within the given domain, hence the composition and its derivative are both analytic; on the other hand, since $(I(\wp(z+\alpha)))'$ has the value at $z = 0$ being the following:

$$(I(\wp(z+\alpha)))' \big|_{z=0} = \frac{\wp'(0+\alpha)}{\sqrt{(\wp'(0+\alpha))^2}} = \frac{\wp'(\alpha)}{\sqrt{(\wp'(\alpha))^2}} = \frac{\wp'(\alpha)}{\wp'(\alpha)} = 1$$

then in case for $(I(\wp(z+\alpha)))'$ to be analytic, we need $(I(\wp(z+\alpha)))' = 1$, which implies that $I(\wp(z+\alpha)) = z$. So, α is the desired constant, such that $\wp(z+\alpha)$ is the inverse of I.

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Question 5 *Stein and Shakarchi Pg. 282 Problem 4:*
Suppose $\mathcal{T}$ is purely imaginary, say $\mathcal{T} = it$ with $t > 0$. Consider the division of the complex plane into congruent rectangles obtained by considering the lines $x = n/2$, $y = tm/2$ as n and m range over the integers.

(a) Show that $\wp$ is real-valued on all these lines, and hence on the boundaries of all these rectangles.

(b) Prove that $\wp$ maps the interior of each rectangle conformally to the uppoer (or lower) half-plane.

Pf:

(a) Assume the lattice is given by $L = \mathbb{Z} + \mathbb{Z}it$ for the $\wp$-function. Which, for all $w = n + i \cdot tm \in L$, its conjugate $\overline{w} = n - i \cdot tm \in L$. On the other hand, $-w = -n - i \cdot tm \in L$. Since the $\wp$-function converges normally on $\mathbb{C} \setminus L$, it converges absolutely everywhere in this domain, hence it's fine to exchange the order of the summation. (For the proof below, we'll exchange the order of summation by replacing w with $\overline{w}$ or some other corresponding points in L).

Horizontal Line:

For all point (that's not a lattice point) on the horizontal line (the line $y = \frac{tm}{2}$ for some $m \in \mathbb{Z}$), $z = x + i \cdot \frac{tm}{2}$ for some $x \in \mathbb{R}$. Which, since $itm \in L$, then $\wp(x - i \cdot \frac{tm}{2}) = \wp((x + i \cdot \frac{tm}{2}) - itm) = \wp(x + i \cdot \frac{tm}{2})$. Then, consider the expression $2\wp(x + i \cdot \frac{tm}{2})$, we get:

$$2\wp\left(x + i \cdot \frac{tm}{2}\right) = \wp\left(x + i \cdot \frac{tm}{2}\right) + \wp\left(x - i \cdot \frac{tm}{2}\right)$$

$$= \left[\frac{1}{(x + itm/2)^2} + \sum_{\substack{w \in L \\ w \neq 0}} \left(\frac{1}{((x + itm/2) - w)^2} - \frac{1}{w^2}\right)\right] + \left[\frac{1}{(x - itm/2)^2} + \sum_{\substack{w \in L \\ w \neq 0}} \left(\frac{1}{((x - itm/2) - \overline{w})^2} - \frac{1}{\overline{w}^2}\right)\right]$$

$$= \left(\frac{1}{(x + itm/2)^2} + \frac{1}{(x + \overline{itm/2})^2}\right) + \sum_{\substack{w \in L \\ w \neq 0}} \left[\left(\frac{1}{(x + itm/2 - w)^2} - \frac{1}{w^2}\right) + \left(\frac{1}{(x + \overline{itm/2} - \overline{w})^2} - \frac{1}{\overline{w}^2}\right)\right]$$

$$= \left(\frac{1}{(x + itm/2)^2} + \frac{1}{\overline{(x + itm/2)}^2}\right) + \sum_{\substack{w \in L \\ w \neq 0}} \left[\left(\frac{1}{(x + itm/2 - w)^2} - \frac{1}{w^2}\right) + \left(\frac{1}{\overline{(x + itm/2 - w)}^2} - \frac{1}{\overline{w}^2}\right)\right]$$

$$= 2\mathrm{Re}\left(\frac{1}{(x + itm/2)^2}\right) + \sum_{\substack{w \in L \\ w \neq 0}} 2\mathrm{Re}\left(\frac{1}{(x + itm/2 - w)^2} - \frac{1}{w^2}\right)$$

$$2\mathrm{Re}\left(\frac{1}{(x + itm/2)^2}\right) + 2\sum_{\substack{w \in L \\ w \neq 0}} \mathrm{Re}\left(\frac{1}{(x + itm/2 - w)^2} - \frac{1}{w^2}\right)$$

(Note: the above series converges, because for each component z of the series, $|Re(z)| \leq |z|$, hence if the original series converges absolutely, the above series also converges; and, the original series $\wp(x + i \cdot \frac{tm}{2})$ is absolutely convergent).

Then, since $2\wp(x + i \cdot \frac{tm}{2})$ is real (since each component in the series is real), so does $\wp(x + i \cdot \frac{tm}{2})$. This proves that $\wp$ is purely real on the line $y = \frac{tm}{2},\ m \in \mathbb{Z}$ with the given lattice.

Vertical Line:

For all non-lattice point on the vertical line (the line $x = \frac{n}{2}$ for some $n \in \mathbb{Z}$), $z = \frac{n}{2} + iy$ for some $y \in \mathbb{R}$. Which, since $n \in L$, then $\wp(-\frac{n}{2} + iy) = \wp((\frac{n}{2} + iy) - n) = \wp(\frac{n}{2} + iy)$. Then, if we consider the term $\wp(\frac{n}{2} + iy) - \overline{\wp(\frac{n}{2} + iy)} = 2\text{Im}(\wp(\frac{n}{2} + iy))$, we get:

$$\wp\left(\frac{n}{2} + iy\right) - \overline{\wp\left(\frac{n}{2} + iy\right)} = \wp\left(\frac{n}{2} + iy\right) - \overline{\wp\left(-\frac{n}{2} + iy\right)}$$

$$= \left[\frac{1}{(n/2 + iy)^2} + \sum_{\substack{w \in L \\ w \neq 0}} \left(\frac{1}{(n/2 + iy - w)^2} - \frac{1}{w^2}\right)\right] - \overline{\left[\frac{1}{(-n/2 + iy)^2} + \sum_{\substack{w \in L \\ w \neq 0}} \left(\frac{1}{(-n/2 + iy - (-\overline{w}))^2} - \frac{1}{(-\overline{w})^2}\right)\right]}$$

$$= \left(\frac{1}{(n/2 + iy)^2} - \frac{1}{\overline{(-n/2 + iy)}^2}\right) + \sum_{\substack{w \in L \\ w \neq 0}} \left[\left(\frac{1}{(n/2 + iy - w)^2} - \frac{1}{w^2}\right) - \overline{\left(\frac{1}{(-n/2 + iy + \overline{w})^2} - \frac{1}{\overline{w}^2}\right)}\right]$$

$$= \left(\frac{1}{(n/2 + iy)^2} - \frac{1}{(-n/2 - iy)^2}\right) + \sum_{\substack{w \in L \\ w \neq 0}} \left[\left(\frac{1}{(n/2 + iy - w)^2} - \frac{1}{w^2}\right) - \left(\frac{1}{\overline{(-n/2 + iy + \overline{w})}^2} - \frac{1}{\overline{(\overline{w})}^2}\right)\right]$$

$$= \left(\frac{1}{(n/2 + iy)^2} - \frac{1}{(n/2 + iy)^2}\right) + \sum_{\substack{w \in L \\ w \neq 0}} \left[\left(\frac{1}{(n/2 + iy - w)^2} - \frac{1}{w^2}\right) - \left(\frac{1}{(-n/2 - iy + w)^2} - \frac{1}{w^2}\right)\right]$$

$$= 0 + \sum_{\substack{w \in L \\ w \neq 0}} \left[\left(\frac{1}{(n/2 + iy - w)^2} - \frac{1}{w^2}\right) - \left(\frac{1}{(n/2 + iy - w)^2} - \frac{1}{w^2}\right)\right] = 0$$

This shows that $2 \cdot \text{Im}(\wp(\frac{n}{2} + iy)) = 0$, hence $\text{Im}(\wp(\frac{n}{2} + iy)) = 0$, so $\wp(\frac{n}{2} + iy)$ is purely real.

Then, this proves that $\wp$ is purely real on the line $x = \frac{n}{2},\ n \in \mathbb{Z}$ with the given lattice.

(b) To prove the problem, we'll consider only the fundamental region given in the graph (which is made up of 4 rectangles).

1. The Boundary of the rectangle is injective:

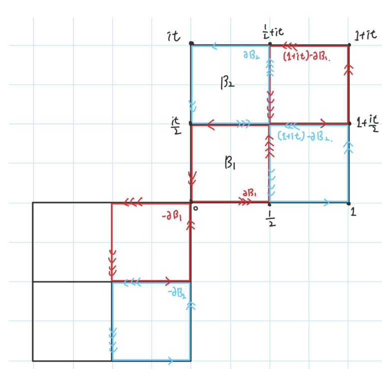
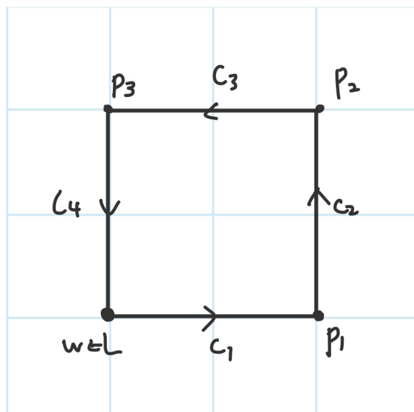


Figure 1: Illustration of the rectangles in the fundamental region

Based on the above graph, within the fundamental region (with vertices $0,\ 1,\ (1+it)$, and it), the region $B_1,\ B_2$ are the two rectangles with distinct characterization (which, based on the relation of $\wp$, the two points z, w in the fundamental region have the same image iff $z \equiv w$ or $z \equiv -w$ under modulo L). Also, because $\wp$ has order 2, at most 2 distinct points in the fundamental region can be evaluated to be the same. Hence, for all points $z \in B_1$, the other point w in the fundamental region with $\wp(z) = \wp(z)$ must occur in $(1+it) - B_1$ (the same case applies for B_2 and $(1+it) - B_2$).

Then, since the boundary of B_1 and $(1+it) - B_1$ only intersects at the midpoint of the fundamental region (which by the property of $\wp$, it has order 2, so no other points evaluated to be the same as the midpoint), then for the other point on the boundary of B_1, since the corresponding point with the same value lies in the boundary of $(1+it) - B_1$ (so they are not in the same boundary), then restricting to ∂B_1, the function $\wp$ is in fact injective (and same logic applies to ∂B_2).

2. Boundary surjects onto $\mathbb{R}$ by $\wp$:



Given a rectangle with boundary, WLOG, up to certain rotation and reflection, can assume under this orientation, the bottom left corner is a point in the lattice (so $\wp(w) = \infty$), $p_1, p_2, p_3 \notin L$ are the midpoints, with $2p_i \in L$ for each index i (which corresponds to the values $\wp(p_1) = e_1,\ \wp(p_2) = e_2$, and $\wp(p_3) = e_3$ respectively, and $\wp'$ evaluated to be 0 at these points), and $e_1 < e_3$.

Which, for each c_i, since it is a closed straight line, can generate continuous path $f_i : [0,1] \to c_i$ that satisfies the given orientation in the graph, and f_i' being a nonzero constant in $(0,1)$ (i.e. can view each c_i as a unit interval). And, since c_i is contained in the boundary of the rectangle, then $\wp(c_i) \subseteq \mathbb{R} \cup \{\infty\}$. Hence, if exclude the point w, when restricting the domain to each c_i, can view $\wp$ as a real valued function from interval $[0,1]$ to $\mathbb{R}$ (so we're treating each c_i as an interval in $\mathbb{R}$). Then, there are some properties we can derive:

- $e_2 \in (e_1, e_3)$: Suppose the contrary that this is false, then either $e_2 < e_1, e_3$ or $e_2 > e_1, e_3$ (for definiteness, consider the first case). Yet, if we choose $y \in \mathbb{R}$ such that $y \in (e_2, e_1)$ and $y \in (e_2, e_3)$, since p_1, p_2, p_3 maps to e_1, e_2, e_3 respectively, while they're the endpoints of c_2 and c_3, then by Intermediate Value Theorem, there exists $z_2 \in c_2$ and $z_3 \in c_3$ (which are not the endpoints p_1, p_2, p_3), such that $\wp(z_2) = \wp(z_3) = y$ (since each c_i can be mapped to by the unit interval $[0,1]$ in a linear manner, can treat c_i as an interval in $\mathbb{R}$). But, since $z_2 \neq z_3$ (because they're not p_2 by assumption, while c_2, c_3 only intersect at p_2), this violates the injectivity of $\wp$ on the boundary of the rectangle. Hence, $e_2 \in (e_1, e_3)$ is enforced.

- $\wp$ is monotonic on each c_i: Since $\wp'$ only evaluates to be 0 at the midpoints (the points with $a \notin L$, but $2a \in L$), then on the boundary, the only part with $\wp' = 0$ is p_1, p_2, p_3. Which, if viewing each c_i as an interval in $\mathbb{R}$, since $\wp' \neq 0$ on these intervals except at the endpoints, then the derivative (in form of $\wp' \cdot f_i'$) is either > 0 or < 0 for all points in the interior of c_i. Hence, the function must be monotonic.

  Also, based on the fact that $\wp(p_1) = e_1 < e_2 = \wp(p_2)$ and $\wp(p_2) = e_2 < e_3 = \wp(p_3)$ derived above, on c_2 and c_3 with the specified orientation, $\wp$ is monotonically increasing.

  Then, this also enforces $\wp$ to be increasing on c_1 and c_4, since if they're decreasing instead, there exists $z_1 \in c_1$, $z_4 \in c_4$, such that $\wp(z_1) > \wp(p_1) = e_1$, while $\wp(z_4) < \wp(p_3) = e_3$, then again by Intermediate Value Theorem, some points on the segmant $\overline{z_1 p_1}$ and some other points on the segmant $\overline{z_4 p_3}$ will be mapped into the interval (e_1, e_3), which contradicts the injectivity of $\wp$ on the boundary.

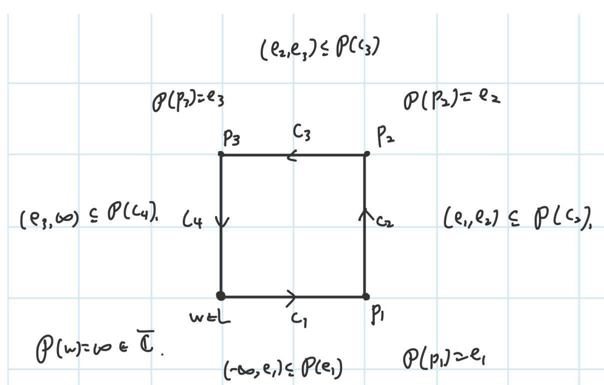
Now, to prove that the boundary rectangle surjects onto $\mathbb{R}$ by $\wp$, consider the following:

First, since $\wp(w) = \infty$, for all $M > 0$, there exists $r > 0$, such that $|z - w| < r$ implies $|\wp(z)| > M$. So, for any $x < e_1$, choose $M > \max\{|e_1|, |x|\} \geq 0$, then any $z_1 \in c_1$ ($z_1 \neq p_1$) that satisfies $0 < |w - z_1| < r$ (where r corresponds to M), has $|\wp(z_1)| > M$ (so, $\wp(z_1) > M$ or $\wp(z_1) < -M$). But, with the monotonicity of $\wp$ on c_1, we must have $\wp(z_1) < \wp(p_1) = e_1$. Then, since $e_1 \leq |e_1| < M$, $\wp(z_1) > M$ would cause a contradiction. So, $\wp(z_1) < -M$, while $-M < -|x| \leq x$. So, given $\wp(z_1) < -M < x < e_1 = \wp(p_1)$, by Intermediate Value Theorem, there exists z in the segmant $\overline{z_1 p_1} \subseteq c_1$, such that $\wp(z) = x$.

Then, for any $x \in (e_1, e_3)$, we can also use Intermediate Value Theorem on c_2 (if $x \in (e_1, e_2)$) or on c_3 (if $x \in (e_2, e_3)$) to get that x is also in the image of $\wp$ on the boundary of the rectangle. (Note: if $x = e_2$, then $\wp(p_2) = e_2$).

Lastly, for $x > e_3$, using the same logic as the first case, choose $M > \max\{|e_3|, |x|\} \geq 0$, for any $z_4 \in c_4$ ($z_4 \neq p_3$) that satisfies $0 < |w - z_4| < r$, satisfies $|\wp(z_4)| > M$. Hence, since $\wp$ is increasing on c_4, with $z_4 \neq p_3$, we have $\wp(z_4) > \wp(p_3) = e_3$. Which, since $M > |e_3| \geq e_3$, we must have $\wp(z_4) > M$ for $|\wp(z_4)| > M$ to be satisfied. So, because $\wp(z_4) > M > |x| \geq x > e_3$, by Intermediate Value Theorem again, there exists $z \in \overline{p_3 z_4} \subseteq c_4$, with $\wp(z) = x$.

The above three cases cover all $x \in \mathbb{R}$, which for all such x, there exists z on the boundary of the rectangle, such that $\wp(z) = x$, showing that the boundary surjects onto $\mathbb{R}$ with $\wp$.



3. The interior of the rectangle is within a half plane:

From the prvious part, we got that for all $x \in \mathbb{R}$, there exists z on the boundary, with $\wp(z) = x$, then since within a fundamental region, the other point w with $\wp(w) = \wp(z) = x$ lies within the other boundary based on the statement in **1**, it implies that all u in the interior of the rectangle (not on the boundary of any rectangle) must have $\wp(u) \notin \mathbb{R}$, so $\wp(u)$ must be in upper or lower half plane.

However, since the interior of the rectangle is connected, while $\wp$ is continuous, this enforces the image of the interior to be continuous, hence the image of the interior must be fully contained in one of the upper or lower half plane, but not both (since the upper half plane $\operatorname{Im}(z) > 0$ and lower half plane $\operatorname{Im}(z) < 0$ are separated, if a connected set has image lying in both half planes, it forms a nontrivial separation of the image, which contradicts the fact that continuous function maps connected set to connected set).

4. The interior has its image cover the whole half plane:

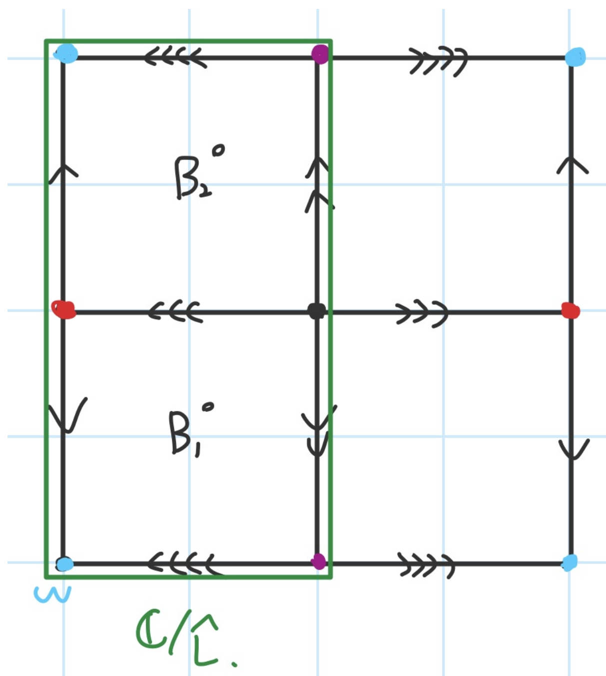


Figure 2: The Fundamental Region with $\mathbb{C}/\widehat{L}$

Recall that in **HW 4 Problem 3**, under the quotient $\mathbb{C}/\widehat{L}$ (where $z \sim z'$ iff $z \equiv \pm z' \mod L$), $\wp : \mathbb{C}/\widehat{L} \to \overline{\mathbb{C}}$ forms a bijection, hence consider just the region shown in the above figure (removing the point $w \in L$), $\wp$ is in fact bijective onto $\mathbb{C}$.

Then, since $\mathbb{R}$ has its preimage being precisely the boundary (since the boundary surjects onto $\mathbb{R}$, and other points in the interior of the rectangles have image not in $\mathbb{R}$), then for all $z \in \mathbb{C} \setminus \mathbb{R}$, its preimage $\wp^{-1}(z)$ in the region above, must be in one of the rectangles' interior.

Which, if the interior B_1° gets mapped into the upper half plane, every z in the upper half plane must have its preimage (in the region drawn above) being in B_1°: Suppose the contrary that this is false, then there exists z in the upper half plane, with its preimage being in B_2° instead. But, since B_1° gets mapped into the upper half plane, all w in the lower half plane must have their preimage being in B_2°, showing that the image of B_2° is not connected (since there are points in the image being in the upper half plane, while some others are in the lower half plane). Yet, this contradicts the fact that B_2° is connected, and $\wp$ is continuous. So, B_1° must have its image being the whole upper half plane, to avoid the contradiction (same logic applies to B_2°, which has its image being the whole lower half plane).

Finally, since each interior of the rectangle must have its image being the whole upper or lower half plane, while $\wp$ is injective on each rectangle's interior and $\wp' \neq 0$ in the interior (since the only 3 points in the fundamental region with $\wp' = 0$, are the three midpoints that lie on the boundary of the rectangles), then $\wp$ maps each interior of the rectangle conformally onto the upper or lower half plane.