

Math 118C HW4

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Question 1 Rudin Pg. 242 Problem 27:

Put $f(0, 0) = 0$, and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$. Prove that

(a) f , D_1f , D_2f are continuous in \mathbb{R}^2 .

(b) $D_{12}f$ and $D_{21}f$ exist at every point of \mathbb{R}^2 , and are continuous except at $(0, 0)$.

(c) $D_{12}f(0, 0) = 1$, and $D_{21}f(0, 0) = -1$.

Pf:

For all $(x, y) \in \mathbb{R}^2$ with $(x, y) \neq (0, 0)$, using polar coordinates, $(x, y) = (r \cos(\theta), r \sin(\theta))$ for some $r > 0$ and $\theta \in [0, 2\pi)$. Which, $|(x, y)| = r$, when consider limit definition, we'll use polar coordinates instead.

(a) **f is continuous:**

For $(x, y) \neq (0, 0)$, since f is a defined rational function, it is continuous, so it suffices to show f is continuous at 0. For all $\epsilon > 0$, choose $\delta = \sqrt{\frac{\epsilon}{2}} > 0$, then for all (x, y) satisfying $0 < |(x, y)| = r < \delta$, we get the following:

$$\begin{aligned} |f(x, y) - f(0, 0)| &= \left| \frac{(r \cos(\theta))(r \sin(\theta))((r \cos(\theta))^2 - (r \sin(\theta))^2)}{(r \cos(\theta))^2 + (r \sin(\theta))^2} - 0 \right| \\ &= \left| \frac{r^4 \sin(\theta) \cos(\theta) (\cos^2(\theta) - \sin^2(\theta))}{r^2} \right| \leq r^2 |\sin(\theta)| \cdot |\cos(\theta)| \cdot (|\cos(\theta)|^2 + |\sin(\theta)|^2) \\ &\leq 2r^2 < 2 \left(\sqrt{\frac{\epsilon}{2}} \right)^2 = 2 \cdot \frac{\epsilon}{2} = \epsilon \end{aligned}$$

This shows that f is continuous at $(0, 0)$, hence f is continuous in \mathbb{R}^2 .

D_1f is continuous:

First, using basic differentiation rule, for $(x, y) \neq (0, 0)$, we get the following:

$$D_1f(x, y) = \frac{\partial}{\partial x} \left(\frac{xy(x^2 - y^2)}{x^2 + y^2} \right) = \frac{(3x^2y - y^3)(x^2 + y^2) - xy(x^2 - y^2)2x}{(x^2 + y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$$

Which, at $(0,0)$, D_1f could be obtained through limit:

$$D_1f(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{h \cdot 0(h^2 - 0^2)}{(h^2 + 0^2)h} = \lim_{h \rightarrow 0} 0 = 0$$

Which, $D_1f(x,y)$ for $(x,y) \neq (0,0)$ is again a rational function, which is continuous, so to verify continuity, it suffices to check $(0,0)$. For all $\epsilon > 0$, choose $\delta = \frac{\epsilon}{6} > 0$, then for all (x,y) satisfying $0 < |(x,y)| = r < \delta$, we get the following:

$$\begin{aligned} |D_1f(x,y) - D_1f(0,0)| &= \left| \frac{(r \cos(\theta))^4(r \sin(\theta)) + 4(r \cos(\theta))^2(r \sin(\theta))^3 - (r \sin(\theta))^5}{((r \cos(\theta))^2 + (r \sin(\theta))^2)^2} - 0 \right| \\ &= \left| \frac{r^5(\cos^4(\theta) \sin(\theta) + 4 \cos^2(\theta) \sin^3(\theta)) - \sin^5(\theta)}{r^4} \right| \leq r(|\cos^4(\theta) \sin(\theta)| + 4|\cos^2(\theta) \sin^3(\theta)| + |\sin^5(\theta)|) \\ &\leq r(1 + 4 + 1) < 6 \cdot \frac{\epsilon}{6} = \epsilon \end{aligned}$$

This proves the continuity of D_1f at $(0,0)$, so D_1f is continuous in \mathbb{R}^2 .

D_2f is continuous:

Using differentiation rule, for $(x,y) \neq (0,0)$, we get the following:

$$D_2f(x,y) = \frac{\partial}{\partial y} \left(\frac{xy(x^2 - y^2)}{x^2 + y^2} \right) = \frac{(x^3 - 3xy^2)(x^2 + y^2) - xy(x^2 - y^2)2y}{(x^2 + y^2)^2} = \frac{x^5 - xy^4 - 4x^3y^2}{(x^2 + y^2)^2}$$

Again, at $(0,0)$, D_2f could be obtained through limit:

$$D_2f(0,0) = \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0 \cdot h(0^2 - h^2)}{(0^2 + h^2)h} = \lim_{h \rightarrow 0} 0 = 0$$

Notice that $D_2f(x,y)$ for $(x,y) \neq (0,0)$ is a rational function, which is continuous, so to verify continuity, it suffices to check $(0,0)$. For all $\epsilon > 0$, choose $\delta = \frac{\epsilon}{6} > 0$, then for all (x,y) satisfying $0 < |(x,y)| = r < \delta$, we get the following:

$$\begin{aligned} |D_2f(x,y) - D_2f(0,0)| &= \left| \frac{(r \cos(\theta))^5 - (r \cos(\theta))(r \sin(\theta))^4 - 4(r \cos(\theta))^3(r \sin(\theta))^2}{((r \cos(\theta))^2 + (r \sin(\theta))^2)^2} - 0 \right| \\ &= \left| \frac{r^5(\cos^5(\theta) - \cos(\theta) \sin^4(\theta) - 4 \cos^3(\theta) \sin^2(\theta))}{r^4} \right| \leq r(|\cos^5(\theta)| + |\cos(\theta) \sin^4(\theta)| + 4|\cos^3(\theta) \sin^2(\theta)|) \\ &\leq r(1 + 1 + 4) < 6 \cdot \frac{\epsilon}{6} = \epsilon \end{aligned}$$

This proves the continuity of D_2f at $(0,0)$, hence D_2f is continuous in \mathbb{R}^2 .

(b) **Function $D_{21}f$:**

Given that $D_1f(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$ for $(x,y) \neq (0,0)$ and $D_1f(0,0) = 0$, apply differentiation rule for $(x,y) \neq (0,0)$, we get:

$$D_{21}f(x,y) = \frac{\partial}{\partial y} \left(\frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} \right) = \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2 + y^2)^2 - (x^4y + 4x^2y^3 - y^5)2(x^2 + y^2)2y}{(x^2 + y^2)^4}$$

Which, $D_{21}f(x, y)$ is continuous for $(x, y) \neq (0, 0)$ (since it's a rational function).

Now, to get $D_{21}f(0, 0)$, we'll use limit definition:

$$D_{21}f(0, 0) = \lim_{h \rightarrow 0} \frac{D_1f(0, h) - D_1f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0^4 \cdot h + 4 \cdot 0^2 \cdot h^3 - h^5}{(0^2 + h^2)^2 h} = \lim_{h \rightarrow 0} -\frac{h^5}{h^5} = -1$$

Hence, $D_{21}f$ exists on the whole \mathbb{R}^2 , and is continuous at all $(x, y) \neq (0, 0)$. But, it is not continuous at $(0, 0)$, since choosing $x \neq 0$ and $y = 0$, $D_{21}f$ becomes:

$$D_{21}f(x, 0) = \frac{x^8}{x^8} = 1$$

Hence, $\lim_{x \rightarrow 0} D_{21}f(x, 0) = 1 \neq -1 = D_{21}f(0, 0)$, showing the discontinuity at $(0, 0)$.

So, $D_{21}f$ exists on \mathbb{R}^2 , while being continuous on $\mathbb{R}^2 \setminus \{0\}$.

Function $D_{12}f$:

Given that $D_2f(x, y) = \frac{x^5 - xy^4 - 4x^3y^2}{(x^2 + y^2)^2}$ for $(x, y) \neq (0, 0)$ and $D_2f(0, 0) = 0$, apply differentiation rule for $(x, y) \neq (0, 0)$, we get:

$$D_{12}f(x, y) = \frac{\partial}{\partial x} \left(\frac{x^5 - xy^4 - 4x^3y^2}{(x^2 + y^2)^2} \right) = \frac{(5x^4 - y^4 - 12x^2y^2)(x^2 + y^2)^2 - (x^5 - xy^4 - 4x^3y^2)2(x^2 + y^2)2x}{(x^2 + y^2)^4}$$

Hence, $D_{12}f$ is continuous for $(x, y) \neq (0, 0)$, since it's also a rational function.

Now, to get $D_{12}f(0, 0)$, we'll again use limit definition:

$$D_{12}f(0, 0) = \lim_{h \rightarrow 0} \frac{D_2f(h, 0) - D_2f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h^5 - h \cdot 0^4 - 4h^3 \cdot 0^2}{(h^2 + 0^2)^2 h} = \lim_{h \rightarrow 0} \frac{h^5}{h^5} = 1$$

Hence, $D_{12}f$ exists on the whole \mathbb{R}^2 , and is continuous at all $(x, y) \neq (0, 0)$. But again, it's not continuous at $(0, 0)$, since choosing $x = 0$ and $y \neq 0$, $D_{12}f$ becomes:

$$D_{12}f(0, y) = \frac{-y^8}{y^8} = -1$$

Hence, $\lim_{y \rightarrow 0} D_{12}f(0, y) = -1 \neq 1 = D_{12}f(0, 0)$, showing the discontinuity at $(0, 0)$.

So, $D_{12}f$ exists on \mathbb{R}^2 , while being continuous on $\mathbb{R}^2 \setminus \{0\}$.

- (c) From **part (b)**, when verifying that the existence of $D_{12}f(0, 0)$ and $D_{21}f(0, 0)$, we've shown that $D_{12}f(0, 0) = 1$, and $D_{21}f(0, 0) = -1$.

Question 2 Rudin Pg. 242 Problem 28:

For $t \geq 0$, put

$$\varphi(x, t) = \begin{cases} x & 0 \leq x \leq \sqrt{t} \\ -x + 2\sqrt{t} & \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0 & \text{otherwise} \end{cases}$$

and put $\varphi(x, t) = -\varphi(x, |t|)$ if $t < 0$.

Show that φ is continuous on \mathbb{R}^2 , and $D_2\varphi(x, 0) = 0$ for all x . Define

$$f(t) = \int_{-1}^1 \varphi(x, t) dx$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^1 D_2\varphi(x, 0) dx$$

Pf:

Continuity of φ :

First, in the open half plane $x < 0$, since $\varphi(x, t) = 0$, then φ is continuous.

Similarly, in the open region where all (x, t) satisfies $x > 2\sqrt{|t|}$, since again $\varphi(x, t) = 0$ by the restriction, then φ is again continuous.

Then, for the open region where all (x, t) satisfies $0 < x < \sqrt{|t|}$, since the function φ is described by x for $t > 0$, and $-x$ for $t < 0$, then the addition φ is also continuous within this region.

Also, for the open region where all (x, t) satisfies $\sqrt{|t|} < x < 2\sqrt{|t|}$, since the function φ is described by $-x + 2\sqrt{|t|}$ for $t > 0$, while described by $-(-x + 2\sqrt{|t|})$ when $t < 0$, so since both $x, \sqrt{|t|}$ are continuous functions, φ as their linear combination is again continuous within this region.

Hence, the only regions left to check, is the lines where (x, t) satisfies $x = 0$, $x = \sqrt{|t|}$, or $x = 2\sqrt{|t|}$. (Note: Since both x and $\sqrt{|t|}$ are continuous functions, then for any given (x_0, t_0) , for all $\epsilon > 0$, there exists $\delta > 0$, such that $|x - x_0| < \delta \implies |x - x_0| < \frac{\epsilon}{2}$, and $|t - t_0| < \delta \implies |\sqrt{|t|} - \sqrt{|t_0|}| < \frac{\epsilon}{2}$).

- For the line $x = 0$, we have $\varphi(0, t) = 0$.

$D_2\varphi$ when $t = 0$:

For all $x \in \mathbb{R}$, if $x \leq 0$, then we get $\varphi(x, t) = 0$ regardless of $t \in \mathbb{R}$, showing that $D_2\varphi(x, 0) = \frac{\partial \varphi}{\partial t}(x, 0) = 0$.

Now for $x > 0$, since for all $t \in \mathbb{R}$ satisfying $4|t| < x^2$, we have $2\sqrt{|t|} < x$, then $\varphi(x, t) = 0$ when $t \in (-\frac{x^2}{4}, \frac{x^2}{4})$. So, $D_2\varphi(x, 0) = 0$ (since $\lim_{t \rightarrow 0} \frac{\varphi(x, t) - \varphi(x, 0)}{t} = \lim_{t \rightarrow 0} 0 = 0$, because for small enough t , it lies in the range $(-\frac{x^2}{4}, \frac{x^2}{4})$).

So, regardless of $x \in \mathbb{R}$, we have $D_2\varphi(x, 0) = 0$.

Function $f(t)$:

Given $f(t) = \int_{-1}^1 \varphi(x, t) dx$, when $|t| < \frac{1}{4}$, there are several cases to consider:

- when $t \geq 0$, then $0 \leq \sqrt{t} < \sqrt{\frac{1}{4}} = \frac{1}{2}$, while $0 \leq 2\sqrt{t} < 1$. Hence, the integral expression can be broken

down as the following pieces:

$$\begin{aligned}\int_{-1}^1 \varphi(x, t) dx &= \int_{-1}^0 0 dx + \int_0^{\sqrt{t}} x dx + \int_{\sqrt{t}}^{2\sqrt{t}} (-x + 2\sqrt{t}) dx + \int_{2\sqrt{t}}^1 0 dx \\ &= \frac{1}{2} x^2 \Big|_0^{\sqrt{t}} + \left(-\frac{1}{2} x^2 + 2\sqrt{t} x \right) \Big|_{\sqrt{t}}^{2\sqrt{t}} = \frac{1}{2} t + ((4t - 2t) - (2t - \frac{1}{2} t)) = t\end{aligned}$$

- when $t < 0$ (where $t = -|t|$), since $\varphi(x, t) = -\varphi(x, |t|)$ with $|t| > 0$, then inheriting from the above expression, we get:

$$\int_{-1}^1 \varphi(x, t) dx = - \int_{-1}^1 \varphi(x, |t|) dx = -|t| = t$$

Hence, for $|t| < \frac{1}{4}$, we can deduce that $f(t) = t$, which $f'(t) = 1$. So, the following inequality is true:

$$f'(0) = 1 \neq 0 = \int_{-1}^1 0 dx = \int_{-1}^1 D_2 \varphi(x, 0) dx$$

This shows that differentiation under integral sign fails under certain situation.

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Question 3 Rudin Pg. 243 Problem 30:

Let $f \in C^{(m)}(E)$, where E is an open subset of \mathbb{R}^n . Fix $a \in E$, and suppose $x \in \mathbb{R}^n$ is so close to 0 that the points $p(t) = a + tx$ lie in E whenever $0 \leq t \leq 1$. Define $h(t) = f(p(t))$ for all $t \in \mathbb{R}$ for which $p(t) \in E$.

(a) For $1 \leq k \leq m$, show (by repeated application of the chain rule) that

$$h^{(k)}(t) = \sum (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

The sum extends over all order k -tuples (l_1, \dots, l_k) in which each l_j is one of the integers $1, \dots, n$.

Pf:

Given $a, x \in \mathbb{R}^n$ (where $x = (x_1, \dots, x_n)$ for fixed $x_1, \dots, x_n \in \mathbb{R}$) and $p(t) = a + tx$ for $t \in [0, 1]$, then $p'(t) = x$.

Now, we'll use induction to verify the formula (and we'll use matrix representation of the differentials).

First, for $k = 1$, using chain rule, we get the following:

$$h'(t) = Df(p(t))p'(t) = \begin{pmatrix} D_1 f & \dots & D_n f \end{pmatrix} \Big|_{p(t)} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^n D_i f(p(t)) x_i$$

Since all the possible 1-tuple is included in the summation, the $h'(t)$ satisfies the given formula.

Now, suppose for given $1 \leq k \leq (m-1)$, $h^{(k)}(t)$ satisfies the following formula:

$$h^{(k)}(t) = \sum (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

Since for each k -tuple (l_1, \dots, l_k) (where each $l_i \in \{1, \dots, n\}$) has the function $x_{l_1} \dots x_{l_k} D_{l_1 \dots l_k} f(p(t))$ being a differentiable function from $(0, 1)$ to \mathbb{R} (where $D_{l_1 \dots l_k} f(z)$ for $z \in E$ is a differentiable function, since it has only been differentiated $k < m$ times, while $f \in C^{(m)}(E)$). Then, to calculate the $(k+1)^{th}$ derivative, we get:

$$\begin{aligned} h^{(k+1)}(t) &= \sum \frac{d}{dt} (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k} \\ \forall (l_1, \dots, l_k), \quad \frac{d}{dt} (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k} &= x_{l_1} \dots x_{l_k} D (D_{l_1 \dots l_k} f)(p(t)) p'(t) \\ &= x_{l_1} \dots x_{l_k} \sum_{i=1}^n D_i (D_{l_1 \dots l_k} f)(p(t)) x_i = \sum_{i=1}^n D_{il_1 \dots l_k} f(p(t)) x_i x_{l_1} \dots x_{l_k} \\ \implies h^{(k+1)}(t) &= \sum \left(\sum_{i=1}^n D_{il_1 \dots l_k} f(p(t)) x_i x_{l_1} \dots x_{l_k} \right) \end{aligned}$$

Which, the first summation indicates all possible k -tuple (l_1, \dots, l_k) for $l_j \in \{1, \dots, n\}$.

Now, for all $(k+1)$ -tuple (j_0, j_1, \dots, j_k) where each $j_l \in \{1, \dots, n\}$, choose the unique k -tuple (j_1, \dots, j_k) , then $D_{j_0 j_1 \dots j_k} f(p(t)) x_{j_0} x_{j_1} \dots x_{j_k}$ appears precisely once in the summation of $h^{(k+1)}(t)$ given above; similarly, since each k -tuple (l_1, \dots, l_k) and $i \in \{1, \dots, n\}$ corresponds to a unique $(k+1)$ -tuple (i, l_1, \dots, l_k) , so the summation in $h^{(k+1)}(t)$ has a 1-to-1 correspondance to all $(k+1)$ -tuple. Then, the summation $h^{(k+1)}(t)$ can also be described as:

$$h^{(k+1)}(t) = \sum D_{l_1 \dots l_k l_{k+1}} f(p(t)) x_{l_1} \dots x_{l_k} x_{l_{k+1}}$$

Where each $(l_1, \dots, l_k, l_{k+1})$ is a $(k+1)$ -tuple with entries from $\{1, \dots, n\}$.

Question 4 *Rudin Pg. 288 Problem 2:*

For $i = 1, 2, 3, \dots$, let $\varphi_i \in \mathcal{C}(\mathbb{R})$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x, y) = \sum_{i=1}^{\infty} (\varphi_i(x) - \varphi_{i+1}(x)) \varphi_i(y)$$

Then f has compact support in \mathbb{R}^2 , f is continuous except at $(0, 0)$, and

$$\int dy \int f(x, y) dx = 0, \quad \text{but} \quad \int dx \int f(x, y) dy = 1$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

Pf:

The function f is well-defined: