Math CS 122B HW1

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April 6, 2025

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Question 1 *Ahlfors Pg. 178 Problem 2:*
Show that the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

converges for $Re(z) > 1$, and represent its derivative in series form.

Pf:

The following proof would assume the domain of the above series is the half plane $Re(z) > 1$.

The series converges pointwise:

For all $z \in \mathbb{C}$ satisfying $Re(z) > 1$, $z = a + bi$ for $a, b \in \mathbb{R}$, and $a > 1$. Then, for any $n \in \mathbb{N}$, the number $n^{-z} = e^{-z\log(n)} = e^{-(a+bi)\ln(n)} = e^{-a\ln(n)} \cdot e^{i(-b\ln(n))} = n^{-a} \cdot e^{i(-b\ln(a))}$. Hence, if taken the modulus $e^{-a\ln(n)}$, since $a > 1$, by p-series test, the following series converges:

$$\sum_{n=1}^{\infty} n^{-a}$$

Which, since the original series satisfies the following:

$$\sum_{n=1}^{\infty} |n^{-z}| = \sum_{n=1}^{\infty} \left| n^{-a} \cdot e^{i(-b\ln(n))} \right| = \sum_{n=1}^{\infty} n^{-a}$$

Hence, the series absolutely converges, which $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ is defined on $Re(z) > 1$.

Partial Sum converges uniformly on any Compact Subset:

Suppose $K \subset \mathbb{C}$ is a compact subset of the plane $Re(z) > 1$, all $z = a + bi \in K$ has the map $z \mapsto a$ being a continuous map, hence there exists $z_0 = a_0 + b_0 i$, such that a_0 is the minimum (or, $1 < a_0 \leq Re(z)$ for all $z \in K$).

Then, each component $n^{-z} = e^{-z\ln(n)} = n^{-a} \cdot e^{i(-b\ln(n))}$ has $|n^{-z}| = n^{-a} \leq n^{-a_0}$ (since $a = Re(z) \geq a_0$, hence $n^{-a} \leq n^{-a_0}$). So, the series $\sum_{n=1}^{\infty} n^{-a_0}$ converges.

Now, notice that for each $n \in \mathbb{N}$, $M_n = \sup_{z\in K} |n^{-z}| = \max_{z\in K} |n^{-z}| = n^{-a_0}$ satisfies $\sum_{n=1}^{\infty} M_n$ converges, then by Weierstrass M-Test, the series $\sum_{n=1}^{\infty} n^{-z}$ in fact converges uniformly on K.

Then, because the series $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ converges absolutely on $Re(z) > 1$, converging uniformly on all compact subset of the half plane, and each component is analytic on the half plane, then by the theorem in Ahlfors pg. 176, $\zeta(z)$ is analytic, and the partial sum $\sum_{n=1}^{N} n^{-z}$ (for $N \in \mathbb{N}$) has derivative converges to

$\zeta'(z)$ uniformly on all compact subsets of the half plane. Hence, based on the same theorem again, we can claim the folloing on the chosen half plane:

$$\zeta'(z) = \lim_{N\to\infty} \frac{d}{dz} \sum_{n=1}^{N} n^{-z} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{d}{dz}(e^{-z\ln(n)}) = \lim_{N\to\infty} \sum_{n=1}^{N} -\ln(n)e^{-z\ln(n)} = -\sum_{n=1}^{\infty} ln(n)n^{-z}$$

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Question 2 *Ahlfors Pg. 184 Problem 5:*
The Fibonacci numbers are defined by $c_0 = 0$, $c_1 = 1$, and $c_n = c_{n-1} + c_{n-2}$ for all $n \geq 2$.
Show that the c_n are Taylor Coefficients of a rational function, and determine a closed expression for c_n.

Pf:
Consider the generating function, a formal power series defined as $F(z) = \sum_{n=0}^{\infty} c_n z^n$.

Radius of Convergence of the Power Series:
Recall that radius of convergence of power series $R = \limsup(|c_n|^{\frac{1}{n}})^{-1} \in [0, \infty]$, where c_n is the coefficients of each degree.

First, we can verify that for all $n \in \mathbb{N}$, $c_n < 2^n$:

For base case $n = 1$, $c_1 = 1 < 2^1$.

Now, suppose for given $n \geq 1$, $c_n < 2^n$, then for case $(n+1) \geq 2$, since $c_{n+1} = c_n + c_{n-1} < 2 \cdot c_n$ (since $c_n > c_{n-1}$), then $c_{n+1} < 2 \cdot c_n < 2 \cdot 2^n = 2^{n+1}$ by induction hypothesis, which this completes the induction.

Since all $n \in \mathbb{N}$ has $0 < c_n < 2^n$, then $|c_n|^{\frac{1}{n}} = c_n^{\frac{1}{n}} < (2^n)^{\frac{1}{n}} < 2$, so $\limsup(|c_n|^{\frac{1}{n}}) \leq 2$, hence $R = \limsup(|c_n|^{\frac{1}{n}})^{-1} \geq \frac{1}{2}$. Thus, we can claim that the power series $F(z) = \sum_{n=0}^{\infty} c_n z^n$ in fact converges absolutely for disk $|z| < \frac{1}{2}$ (since $|z| < \frac{1}{2}$ is contained in the radius of convergence).

Closed Expression of c_n:
Now, consider power series $F(z)$ on $|z| < \frac{1}{2}$: Since $F(z)$ can be rewritten as $c_0 + c_1 z + \sum_{n=2}^{\infty} c_n z^n = 1 + z + \sum_{n=2}^{\infty} c_n z^n$. Then, based on the definition of Fibonnaci numbers, it can be rewritten as:

$$F(z) = 1 + z + \sum_{n=2}^{\infty}(c_{n-1} + c_{n-2})z^n = 1 + z + \sum_{n=2}^{\infty} c_{n-1}z^n + \sum_{n=2}^{\infty} c_{n-2}z^n$$

$$= 1 + z + \sum_{n=1}^{\infty} c_n z^{n+1} + \sum_{n=0}^{\infty} c_n z^{n+2} = 1 + z\left(1 + \sum_{n=1}^{\infty} c_n z^n\right) + z^2 \sum_{n=0}^{\infty} c_n z^n$$

$$= 1 + z\sum_{n=0}^{\infty} c_n z^n + z^2 F(z) = 1 + zF(z) + z^2 F(z)$$

Then, we can yield the following, showing that $F(z)$ is in fact a rational function:

$$F(z) = 1 + zF(z) + z^2F(z), \quad F(z)(1 - z - z^2) = 1, \quad F(z) = \frac{1}{1 - z - z^2}$$

Now, if $1-z-z^2=0$ (or $z^2+z-1=0$), we have $z=\frac{-1\pm\sqrt{5}}{2}$. Hence, $1-z-z^2=-\left(\frac{-1+\sqrt{5}}{2}-z\right)\left(\frac{-1-\sqrt{5}}{2}-z\right)$. Then, $F(z)$ can be decomposed using partial fraction:

$$F(z)=\frac{A}{\frac{-1+\sqrt{5}}{2}-z}+\frac{B}{\frac{-1-\sqrt{5}}{2}-z}=\frac{1}{1-z-z^2},\quad B\left(\frac{-1+\sqrt{5}}{2}-z\right)+A\left(\frac{-1-\sqrt{5}}{2}-z\right)=-1$$

So, from the above expression, we get:

$$B\cdot\frac{-1+\sqrt{5}}{2}+A\cdot\frac{-1-\sqrt{5}}{2}=-1,\quad -B-A=0$$

$$\implies A=-B,\quad B\cdot\frac{-1+\sqrt{5}}{2}-B\cdot\frac{-1-\sqrt{5}}{2}=-1$$

$$\implies B\left(\frac{-1+\sqrt{5}}{2}-\frac{-1-\sqrt{5}}{2}\right)=B\cdot\sqrt{5}=-1,\quad B=-\frac{1}{\sqrt{5}},\quad A=\frac{1}{\sqrt{5}}$$

So, $F(z)$ can be expressed as:

$$F(z)=\frac{1}{\sqrt{5}}\left(\frac{1}{\frac{-1+\sqrt{5}}{2}-z}-\frac{1}{\frac{-1-\sqrt{5}}{2}-z}\right)$$

Now, notice that for any $k\neq 0$, on $|z|<|k|$, since $|z/k|<1$, then $\sum_{n=0}^{\infty}(z/k)^n$ converges absolutely to $\frac{1}{1-z/k}=\frac{k}{k-z}$, which $\frac{1}{k-z}=\frac{1}{k}\sum_{n=0}^{\infty}(z/k)^n$.

Because both $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$ has abslute values greater than $\frac{1}{2}$ (first one is approximately 0.618, the second one is approximately -1.618), hence, on the disk $|z|<\frac{1}{2}$, both equations below are true based on the above formula:

$$\frac{1}{\frac{-1+\sqrt{5}}{2}-z}=\frac{1}{\frac{-1+\sqrt{5}}{2}}\sum_{n=0}^{\infty}\left(\frac{z}{\frac{-1+\sqrt{5}}{2}}\right)^n,\quad \frac{1}{\frac{-1-\sqrt{5}}{2}-z}=\frac{1}{\frac{-1-\sqrt{5}}{2}}\sum_{n=0}^{\infty}\left(\frac{z}{\frac{-1-\sqrt{5}}{2}}\right)^n$$

Hence, $F(z)$ can be expressed as:

$$F(z)=\frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}\left(\frac{1}{\frac{-1+\sqrt{5}}{2}}\right)^{n+1}z^n-\sum_{n=0}^{\infty}\left(\frac{1}{\frac{-1-\sqrt{5}}{2}}\right)^{n+1}z^n\right)$$

$$=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}-\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}}z^n=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}-\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{(-1)^2-(\sqrt{5})^2}{4}\right)^{n+1}}z^n$$

$$=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}-\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}{(-1)^{n+1}}z^n=\sum_{n=0}^{\infty}\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}z^n$$

Then, by the uniqueness of Taylor Series, the following is the closed expression of c_n:

$$\forall n\in\mathbb{N},\quad c_n=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$$

This shows that c_n is in fact taylor coefficients of rational function $\frac{1}{1-z-z^2}$.

(Note: here the index $c_0=1$ instead of $c_0=0$. Which, if $c_0=0$ instead, the function is then given by $\frac{z}{1-z-z^2}$, since we need to shift the index by 1).

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Question 3 *Ahlfors Pg. 186 Problem 4:*
Show that the Laurent development of $(e^z - 1)^{-1}$ at the origin is of the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$$

where the numbers B_k are known as the Bernoulli numbers. Calculate B_1, B_2, B_3.

Pf:

Given the function $f(z) = (e^z - 1)^{-1}$, it is analytic on $\mathbb{C} \setminus \{0\}$ (with $0 < |z| < \infty$), hence there exists a laurent development that agrees on the whole $\mathbb{C} \setminus \{0\}$:

$$f(z) = \sum_{n=-\infty}^{\infty} A_n z^n$$

And, for all index n, the formula of A_n is given as follow:

$$n \geq 1, \quad A_{-n} = \frac{1}{2\pi i} \int_C f(\zeta)\zeta^{n-1} d\zeta \qquad\qquad n \geq 0, \quad A_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{n+1}} d\zeta$$

Where C is a circle with radius $r > 0$, centered at $z = 0$.

Coefficients of Negative Degree:

First, for $n = 1$, the coefficient A_{-1} is given as:

$$A_{-1} = \frac{1}{2\pi i} \int_C f(\zeta)\zeta^{1-1} d\zeta = \frac{1}{2\pi i} \int_C \frac{1}{e^\zeta - 1} d\zeta = \frac{1}{2\pi i} \int_C \frac{1}{\zeta} \cdot \frac{\zeta}{e^\zeta - 1} d\zeta$$

Now, notice that for $\frac{\zeta}{e^\zeta - 1}$ with an isolated singularity at $\zeta = 0$, since $\lim_{\zeta \to 0} \frac{\zeta}{e^\zeta - 1} = 1$ (since the limit of its reciprical is $\lim_{\zeta \to 0} \frac{e^\zeta - e^0}{\zeta} = 1$, which is the derivative of e^z at 0), then, $\lim_{\zeta \to 0} \zeta \cdot \frac{\zeta}{e^\zeta - 1} = 0$, which is a sufficient and necessary condition for $\zeta = 0$ to be a removable singularity of $\frac{\zeta}{e^\zeta - 1}$.

Hence, $\frac{\zeta}{e^\zeta - 1}$ has an analytic extension onto the whole $\mathbb{C}$, with the function being 1 at $\zeta = 0$. Which, by Cauchy's Integral Formula, A_{-1} of the above form, is the evaluation of $\frac{\zeta}{e^\zeta - 1}$ at 0 (more precisely, evaluation of its extension at 0), which provides $A_{-1} = 1$.

Then, for $n > 1$, the coefficient A_{-n} is given by:

$$A_{-n} = \frac{1}{2\pi i} \int_C f(\zeta)\zeta^{n-1} d\zeta = \frac{1}{2\pi i} \int_C \frac{\zeta^{n-1}}{e^\zeta - 1} d\zeta$$

Notice that for $n > 1$, the function $\frac{\zeta^{n-1}}{e^\zeta - 1}$ has isolated singularity at 0, since $\lim_{\zeta \to 0} \zeta \cdot \frac{\zeta^{n-1}}{e^\zeta - 1}$ is defined due to the fact that $(n-1) \geq 1$, then the singularity at 0 is in fact removable. Hence, it has an analytic extension onto $\mathbb{C}$.

Then, the integral form of A_{-n} is in fact a closed contour integral of an analytic function on $\mathbb{C}$, which $A_{-n} = 0$.

Coefficients of Nonnegative Degree:

For $n \geq 0$, the coefficient A_n is given as:

$$A_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{n+1}}d\zeta = \frac{1}{2\pi i}\int_C \frac{1}{\zeta^{n+1}(e^\zeta - 1)}d\zeta = \frac{1}{2\pi i}\int_C \frac{1}{\zeta^{n+2}}\cdot\frac{\zeta}{e^\zeta - 1}d\zeta$$

Since we've verified above, that $\frac{z}{e^z-1}$ has an analytic extension onto $\mathbb{C}$, it has a power series expansion about 0, $\frac{z}{e^z-1} = \sum_{n=0}^\infty C_n z^n$, and it agrees with the function on the whole $\mathbb{C}$. We'll find the coefficient to help calculate A_n.

Notice that since $\frac{e^z-1}{z}\cdot\frac{z}{e^z-1} = 1$, while $\frac{e^z-1}{z}$ also can be extended analytically onto $\mathbb{C}$, given the following power series expansion of $\frac{e^z-1}{z}$:

$$e^z = 1 + \sum_{n=1}^\infty \frac{z^n}{n!}, \quad \frac{e^z-1}{z} = \frac{1}{z}\sum_{n=1}^\infty \frac{z^n}{n!} = \sum_{n=1}^\infty \frac{z^{n-1}}{n!} = \sum_{n=0}^\infty \frac{z^n}{(n+1)!}$$

Which we can conclude the following:

$$\frac{e^z-1}{z}\cdot\frac{z}{e^z-1} = \left(\sum_{n=0}^\infty C_n z^n\right)\left(\sum_{n=0}^\infty \frac{z^n}{(n+1)!}\right) = 1 = 1 + \sum_{n=1}^\infty 0\cdot z^n$$

Since regardless of $z \in \mathbb{C}$, the above equation is true, $\sum_{n=0}^\infty C_n z^n$ is in fact the inverse of the formal power series $\sum_{n=0}^\infty \frac{z^n}{(n+1)!} \in \mathbb{C}[[z]]$. Which, they satisfy the following relationship:

- For coefficient of degree 0, we have $C_0 \cdot \frac{1}{(0+1)!} = 1$, hence $C_0 = 1$.
- For coefficient of degree 1, we have $C_1 \cdot \frac{1}{(0+1)!} + C_0 \cdot \frac{1}{(1+1)!} = 0$, hence $C_1 + \frac{1}{2!} = 0, \quad C_1 = -\frac{1}{2}$.
- For coefficient of degree $n \geq 2$, we have $\sum_{k=0}^n C_k \cdot \frac{1}{((n-k)+1)!} = 0$, hence:

$$C_n = C_n \cdot \frac{1}{(0+1)!} = -\sum_{k=0}^{n-1} C_k \cdot \frac{1}{((n-k)+1)!}$$

Since $g(z) = \frac{z}{e^z-1}$ has its power series $\sum_{n=0}^\infty C_n z^n$ converge to itself on the whole $\mathbb{C}$,then its n^{th} derivative at 0 is given as $g^{(n)}(0) = n!C_n$. Which, A_n can be rewritten as the following using Cauchy's Integral Formula:

$$A_n = \frac{1}{2\pi i}\int_C \frac{1}{\zeta^{n+2}}\cdot\frac{\zeta}{e^\zeta-1}d\zeta = \frac{1}{2\pi i}\int_C \frac{g(\zeta)}{\zeta^{n+2}}d\zeta = \frac{g^{(n+1)}(0)}{(n+1)!} = \frac{(n+1)!C_{n+1}}{(n+1)!} = C_{n+1}$$

Forms of Laurent Series of $(e^z-1)^{-1}$:

With the information from previous two sections, we can express the laurent series as the following:

$$\sum_{n=-\infty}^\infty A_n z^n = \sum_{n=1}^\infty A_{-n}z^{-n} + \sum_{n=0}^\infty A_n z^n = \frac{1}{z} + \sum_{n=0}^\infty C_{n+1}z^n = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^\infty C_{n+1}z^n$$

Now, recall that $\frac{z}{e^z-1} = \sum_{n=0}^\infty C_n z^n$, then for all $z \in \mathbb{C}$, consider the expression with $-z$, we get:

$$\frac{-z}{e^{-z}-1} = \sum_{n=0}^\infty C_n(-z)^n = \sum_{n=0}^\infty (-1)^n C_n z^n$$

Which, consider the difference of the two terms, we get:

$$\frac{-z}{e^{-z}-1} - \frac{z}{e^z-1} = \frac{-ze^z}{1-e^z} - \frac{z}{e^z-1} = \frac{ze^z}{e^z-1} - \frac{z}{e^z-1} = \frac{z(e^z-1)}{e^z-1} = z$$

(Note: for $z = 0$, consider the extension of $\frac{z}{e^z-1}$, where evaluation at $z = 0$ is the limit $\lim_{z\to 0} \frac{z}{e^z-1} = 1$, then the above difference is $1 - 1 = 0$, which agrees with the formula).

Hence, in power series form, we get:

$$\sum_{n=0}^{\infty}(-1)^n C_n z^n - \sum_{n=0}^{\infty} C_n z^n = \sum_{n=0}^{\infty}((-1)^n - 1)C_n z^n = z$$

Then, all the even terms have $(-1)^n - 1 = 0$, we're left with the odd terms. Hence:

$$\sum_{k=1}^{\infty}((-1)^{2k-1} - 1)C_{2k-1}z^{2k-1} = \sum_{k=1}^{\infty} -2C_{2k-1}z^{2k-1} = -2C_1 z + \sum_{k=2}^{\infty} -2C_{2k-1}z^{2k-1} = z$$

By the uniqueness of Taylor series, we need $-2C_1 = 1,\ C_1 = \frac{-1}{2}$ (which agrees with our previous calculation), and $-2C_{2k-1} = 0,\ C_{2k-1} = 0$ for all $k \geq 2$. Therefore, all the odd terms of C_n is 0.

Hence, the laurent series can be expressed as follow:

$$\frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} C_{n+1}z^n = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} C_{(2k-1)+1}z^{2k-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} C_{2k}z^{2k-1}$$

(Note: Since now the odd terms of C_n appears as the even degrees' coefficients).

Now, if we do some modification, let $B'_n = n!C_n$ for all $n \in \mathbb{N}$ (or $C_n = \frac{B'_n}{n!}$), then the laurent series can be expressed as:

$$(e^z - 1)^{-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B'_{2k}}{(2k)!}z^{2k-1}$$

Then, let $B_k = (-1)^k B'_{2k}$, we get the desired form:

$$(e^z - 1)^{-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty}(-1)^k \frac{B_k}{(2k)!}z^{2k-1}$$

Calculation of Bernoulli Numbers:

For $k = 1, 2, 3$, we'll convert it back into C_n for simplicity. Which, $B_k = (-1)^k B'_{2k} = (-1)^k(2k)!C_{2k}$.

For $k = 1, 2k = 2$, we have B_1 given as:

$$C_2 = -\sum_{k=0}^{1} C_k \frac{1}{(2-k+1)!} = -\left(\frac{C_0}{3!} + \frac{C_1}{2!}\right) = -\left(1 \cdot \frac{1}{6} - \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{1}{12}$$

$$B_1 = (-1)^1 \cdot 2! \cdot C_2 = -\frac{1}{6}$$

For $k = 2, 2k = 4$, we have B_2 given as:

$$C_4 = -\sum_{k=0}^{3} C_k \frac{1}{(4-k+1)!} = -\left(\frac{C_0}{5!} + \frac{C_1}{4!} + \frac{C_2}{3!} + \frac{C_3}{2!}\right) = -\left(\frac{1}{120} - \frac{1}{2 \cdot 24} + \frac{1}{12 \cdot 6}\right)$$

$$= -\frac{1}{12}\left(\frac{1}{10} - \frac{1}{4} + \frac{1}{6}\right) = -\frac{1}{12} \cdot \frac{1}{60}$$

$$B_2 = (-1)^2 \cdot 4! \cdot C_4 = -\frac{24}{12 \cdot 60} = -\frac{1}{30}$$

(Note: recall that for $k \geq 2$, all $C_{2k-1} = 0$, hence all odd index $n \geq 3$ has $C_3 = 0$).

For $k = 3, 2k = 6$, we have B_3 given as:

$$C_6 = -\sum_{k=0}^{5} C_k \frac{1}{(6-k+1)!} = -\left(\frac{C_0}{7!} + \frac{C_1}{6!} + \frac{C_2}{5!} + \frac{C_3}{4!} + \frac{C_4}{3!} + \frac{C_5}{2!}\right)$$

$$= -\left(\frac{1}{7!} - \frac{1}{2 \cdot 6!} + \frac{1}{12 \cdot 5!} - \frac{1}{12 \cdot 60 \cdot 3!}\right)$$

$$B_3 = (-1)^3 \cdot 6! \cdot C_6 = -6! \cdot \left(-\left(\frac{1}{7!} - \frac{1}{2 \cdot 6!} + \frac{1}{12 \cdot 5!} - \frac{1}{12 \cdot 60 \cdot 3!}\right)\right)$$

$$= \frac{1}{7} - \frac{1}{2} + \frac{6}{12} - \frac{6!}{12 \cdot 60 \cdot 3!} = \frac{1}{7} - \frac{1}{6} = \frac{1}{42}$$

So, we have the following:

$$B_1 = -\frac{1}{6}, \quad B_2 = -\frac{1}{30}, \quad B_3 = \frac{1}{42}$$

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Question 4 *Stein and Shakarchi Pg. 86 Problem 2:*
Let

$$F(z) = \sum_{n=1}^{\infty} d(n) z^n$$

for $|z| < 1$, where $d(n)$ denotes the number of divisors of n. Observe that the radius of convergence of this series is 1. Verify the identity

$$\sum_{n=1}^{\infty} d(n) z^n = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$$

Using this identity, show that if $z = r$ with $0 < r < 1$, then

$$|F(r)| \geq c \frac{1}{1-r} \log(1/(1-r))$$

as $r \to 1$. Similarly, if $\theta = 2\pi p/q$ where p and q are positive integers and $z = re^{i\theta}$, then

$$|F(re^{i\theta})| \geq c_{p/q} \frac{1}{1-r} \log(1/(1-r))$$

as $r \to 1$. Conclude that F cannot be continued analytically past the unit disk.

Pf:

The other form of the function:

Consider the series $\sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$, since inside the radius of convergence $|\zeta| < 1$, we have $\frac{\zeta}{1-\zeta} = \sum_{n=1}^{\infty} \zeta^n$, then since for all $|z| < 1$, $|z^n| < 1$ for all $n \in \mathbb{N}$, the sum can also be expressed as:

$$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} (z^n)^k\right)$$

Which, for all $q \in \mathbb{N}$, any $n \in \mathbb{N}$ satisfies $n \mid q$ iff $z^q = (z^n)^k$ for some unique $k \in \mathbb{N}$, hence z^q appears precisely once in the series $\sum_{k=1}^{\infty}(z^n)^k$ for all $n \mid q$, and appear 0 times if n doesn't divide q. Therefore, z^q appears total of $d(q)$ times in the above double series (appear once for each $n \mid q$, and there are total of $d(q)$ natural numbers n with $n \mid q$). Hence:

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(z^n)^k\right) = \sum_{q=1}^{\infty} d(q) z^q$$

The first inequality for $0 < r < 1$:

Before starting, we'll consider the power series of $\frac{1}{1-z}$, namely $\sum_{n=0}^{\infty} z^n$: This power series have radius of convergence $R = 1$, which inside $|z| < 1$, not only $\frac{1}{1-r}$ is analytic, we can also define a single-valued branch of $-\log(1-z) = \log(1/(1-z))$, which has the derivative being $\frac{1}{1-z}$. Then, express $\log(1/(1-z))$ in power series, we get:

$$\log(1/(1-z)) = -\log(1-z) = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=1}^{\infty} c_n z^n$$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} = \frac{d}{dz}\log(1/(1-z)) = \sum_{n=1}^{\infty}\frac{d}{dz}(c_n z^n) = \sum_{n=1}^{\infty} n c_n z^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1} z^n$$

$$\forall n \in \mathbb{N}\cup\{0\}, \quad (n+1)c_{n+1} = 1, \quad c_{n+1} = \frac{1}{n+1}$$

Hence, we can conclude that for all $n \in \mathbb{N}$, $c_n = \frac{1}{n}$.

Then, if consider the power series of $\frac{1}{1-z}\log(1/(1-z))$, it has radius of convergence $R = 1$ (since the two series in the product has radius of convergence $R = 1$), which we get:

$$\frac{1}{1-z}\log(1/(1-z)) = \left(\sum_{n=0}^{\infty} z^n\right)\left(0 + \sum_{n=1}^{\infty}\frac{1}{n} z^n\right) = \sum_{n=0}^{\infty} a_n z^n$$

Which, we have the coefficient $a_0 = 0 \cdot 1 = 0$, and for all $n \in \mathbb{N}$, $a_n = 0 \cdot 1 + \sum_{k=1}^{n}\frac{1}{k}\cdot 1 = \sum_{k=1}^{n}\frac{1}{k} = H_n$, where H_n is the n^{th} harmonic sum. Hence:

$$\frac{1}{1-z}\log(1/(1-z)) = \sum_{n=1}^{\infty} H_n z^n$$

Now, to prove the inequality, we'll show that for all $N \in \mathbb{N}$, there exists $0 < \delta < 1$, with $1-\delta < r < 1$ implies $\frac{1}{1-r^n} \geq H_n$ for all $1 \leq n \leq N$.

Given $N \in \mathbb{N}$, since $\lim_{r\to 1^-}\frac{1}{1-r^N} = \infty$, for all $M > 0$, there exists $0 < \delta < 1$, with $1-\delta < r < 1$ implies $\frac{1}{1-r^N} > M$. Hence, let $M = H_N$, choose the corresponding δ_N, any r with $1-\delta_N < r < 1$ satisfies $\frac{1}{1-r^N} > H_N$.

Now, for all $1 \leq n \leq N$, since $|r| < 1$, then $r^n \geq r^N$, $1-r^n \leq 1-r^N$, or $\frac{1}{1-r^n} \geq \frac{1}{1-r^N}$; similarly, since H_n is an increasing sequence, then $H_N \geq H_n$. Hence:

$$\frac{1}{1-r^n} \geq \frac{1}{1-r^N} > H_N \geq H_n$$

Hence, this choice of δ_N also guarantees that all index $n \leq N$ has $\frac{1}{1-r^n} > H_n$.

Which, the following inequality is true for $1-\delta_N < r < 1$:

$$\sum_{n=1}^{N}\frac{r^n}{1-r^n} > \sum_{n=1}^{N} H_n r^n$$

Hence, as $N \to \infty$ (which $H_N \to \infty$, for $\frac{1}{1-r^N} \geq H_N$, we need $r \to 1^-$), we can claim that $\sum_{n=1}^{N} \frac{r^n}{1-r^n} > \sum_{n=1}^{N} H_n r^n$, which the left side is bounded above by $F(r)$, while the right side is bounded by $\frac{1}{1-r}\log(1/(1-r))$.

Therefore, we can claim that as $r \to 1$, $|F(r)| \geq \frac{1}{1-r}\log(1/(1-r))$, where the unknown constant c in the question can be chosen as $c = 1$.

The second inequality for $z = re^{i\theta}$, $0 < r < 1$, and $\theta = 2\pi p/q$:

Given $0 < r < 1$, and $\theta = 2\pi p/q$ for positive integers p, q (WLOG, assume $\gcd(p, q) = 1$). Notice that for any $n \in \mathbb{N}$ that's divisible by q, $e^{i\theta}n = 1$. Then, by separating out all multiples of q, the value $F(re^{i\theta})$ can be expressed as:

$$F(re^{i\theta}) = \sum_{n=1}^{\infty} \frac{(re^{i\theta})^n}{1-(re^{i\theta})^n} = \sum_{k=1}^{\infty} \frac{(re^{i\theta})^{kq}}{1-(re^{i\theta})^{kq}} + \sum_{n\in\mathbb{N},\ q\nmid n} \frac{(re^{i\theta})^n}{1-(re^{i\theta})^n}$$

$$= \sum_{k=1}^{\infty} \frac{(r^q)^k}{1-(r^q)^k} + \sum_{n\in\mathbb{N},\ q\nmid n} \frac{(re^{i\theta})^n}{1-(re^{i\theta})^n}$$

Now, given the following equation, we can derive an inequality:

$$\sum_{k=1}^{\infty} \frac{(r^q)^k}{1-(r^q)^k} = F(re^{i\theta}) - \sum_{n\in\mathbb{N},\ q\nmid n} \frac{(re^{i\theta})^n}{1-(re^{i\theta})^n}$$

$$\left|F(re^{i\theta})\right| + \left|\sum_{n\in\mathbb{N},\ q\nmid n} \frac{(re^{i\theta})^n}{1-(re^{i\theta})^n}\right| \geq \left|F(re^{i\theta}) - \sum_{n\in\mathbb{N},\ q\nmid n} \frac{(re^{i\theta})^n}{1-(re^{i\theta})^n}\right| = \left|\sum_{k=1}^{\infty} \frac{(r^q)^k}{1-(r^q)^k}\right|$$

$$\left|F(re^{i\theta})\right| \geq \left|\sum_{k=1}^{\infty} \frac{(r^q)^k}{1-(r^q)^k}\right| - \left|\sum_{n\in\mathbb{N},\ q\nmid n} \frac{(re^{i\theta})^n}{1-(re^{i\theta})^n}\right| \geq |F(r^q)| - \sum_{n\in\mathbb{N},\ q\nmid n} \frac{|(re^{i\theta})^n|}{|1-(re^{i\theta})^n|}$$

Now, since for all $n \in \mathbb{N}$ with $q \nmid n$, $(e^{i\theta})^n = (e^{i\theta})^k$, where $k \in \mathbb{Z}_q$ satisfies $k \equiv n\ (mod\ q)$, and $k \neq 0$. Then, for all $n \in \mathbb{N}$, there exists nonzero $k \in \mathbb{Z}_q$, where $(re^{i\theta})^n \in R_k = \{r(e^{i\theta})^k \mid r \in \mathbb{R}\}$.

For each nonzero $k \in \mathbb{Z}_q$, since the set $R_k \subset \mathbb{C}$ is closed (a straight line passing through the origin), while $1 \notin R_k$ (since $(e^{i\theta})^k \notin \mathbb{R}$), then since $\{1\}$ is a compact set and R_k is closed, there is a nonzero distance $d_k > 0$, with all $r \in \mathbb{R}$, $|1 - (re^{i\theta})^k| \geq d_k$ (i.e. the two set has nonzero distance).

Which, let $d_q = \min\{d_1, ., , , d_{q-1}\} > 0$, for all nonzero $n \in \mathbb{N}$ with $q \nmid n$ and $r \in \mathbb{R}$, since $n \equiv k\ (mod\ q)$ for some nonzero $k \in \mathbb{Z}_q$, we get the following:

$$|1 - (re^{i\theta})^n| = |1 - r^n(e^{i\theta})^k| \geq d_k \geq d_q > 0$$

Hence, $\frac{1}{|1-(re^{i\theta})^n|} \leq \frac{1}{d_q}$, or $-\frac{1}{|1-(re^{i\theta})^n|} \geq -\frac{1}{d_q}$.

So, the original inequality can then be converted to:

$$|F(re^{i\theta})| \geq |F(r^q)| - \sum_{n\in\mathbb{N},\ q\nmid n} \frac{|(re^{i\theta})^n|}{|1-(re^{i\theta})^n|} \geq |F(r^q)| - \sum_{n\in\mathbb{N},\ q\nmid n} \frac{r^n}{d_q}$$

$$|F(re^{i\theta})| \geq |F(r^q)| - \frac{1}{d_q}\sum_{n=0}^{\infty} r^n = |F(r^q)| - \frac{1}{d_q} \cdot \frac{1}{1-r}$$

Then, based on the previous inequality, as $r \to 1$, $r^q \to 1$, hence $|F(r^q)| \geq \frac{1}{1-r^q}\log(1/(1-r^q))$.

Now, for $0 < r < 1$, since the function $q(1-r) \geq (1-r^q)$ (since by Bernoulli's Inequality, as $(r-1) > -1$, we have $r^q = (1+(r-1))^q \geq 1 + q(r-1)$, hence $q(1-r) = -q(r-1) \geq 1 - r^q$), then we get the following:

$$\frac{1}{q(1-r)} \leq \frac{1}{1-r^q}, \quad r > \frac{q-1}{q} \implies q(1-r) < q \cdot \frac{1}{q} = 1 \implies \frac{1}{q(1-r)} > 1$$

Which, the second impliciation above guarantees the following:

$$0 = \log(1) < \log\left(\frac{1}{q(1-r)}\right) \leq \log\left(\frac{1}{1-r^q}\right)$$

Hence, the following is true:

$$\frac{1}{q} \cdot \frac{1}{1-r} \cdot \left(\log\left(\frac{1}{1-r}\right) + \log\left(\frac{1}{q}\right)\right) = \frac{1}{q(1-r)} \log\left(\frac{1}{q(1-r)}\right) \leq \frac{1}{1-r^q} \log\left(\frac{1}{1-r^q}\right)$$

Combining all the inequalities above, as $r \to 1$ (can assume $r > \frac{q-1}{q}$), we get:

$$|F(re^{i\theta})| \geq |F(r^q)| - \frac{1}{d_q} \cdot \frac{1}{1-r} \geq \frac{1}{1-r^q} \log(1/(1-r^q)) - \frac{1}{d_q} \cdot \frac{1}{1-r}$$

$$|F(re^{i\theta})| \geq \frac{1}{q} \cdot \frac{1}{1-r} \log\left(\frac{1}{1-r}\right) - \frac{\log(q)}{q} \cdot \frac{1}{1-r} - \frac{1}{d_q} \cdot \frac{1}{1-r}$$

$$|F(re^{i\theta})| \geq \frac{1}{1-r}\left(\frac{1}{q}\log(1/(1-r)) - \frac{\log(q)}{q} - \frac{1}{d_q}\right)$$

Since both $\frac{\log(q)}{q}, \frac{1}{d_q}$ are constant for fixed q, and $\lim_{r\to 1^-} \log(1/(1-r)) = \infty$, then there exists $R \in (0,1)$, such that $R < r < 1$ implies $\frac{1}{2q}\log(1/(1-r)) > \frac{\log(q)}{q} + \frac{1}{d_q}$, then as $r \to 1$, we can conclude the following;

$$|F(re^{i\theta})| \geq \frac{1}{1-r}\left(\frac{1}{q}\log(1/(1-r)) - \frac{\log(q)}{q} - \frac{1}{d_q}\right) > \frac{1}{1-r}\left(\frac{1}{q}\log(1/(1-r)) - \frac{1}{2q}\log(1/(1-r))\right)$$

$$|F(re^{i\theta})| \geq \frac{1}{2q} \cdot \frac{1}{1-r}\log(1/(1-r))$$

Choose $c_{p/q} = \frac{1}{2q}$, as $r \to 1$, $|F(re^{i\theta})| \geq c_{p/q}\frac{1}{1-r}\log(1/(1-r))$, hence the second inequality is true.

The function can't be continued analytically past the disk:

Suppose the contrary, that F can be continued analytically past the disk. To continue analytically past the unit disk, there must exists an open connected domain $V \subseteq \mathbb{C}$ that strictly contains the open unit disk $\mathbb{D}$, and an analytic function $\overline{F}$ on V, where for all $|z| < 1$, $\overline{F}(z) = F(z)$ (i.e. they must agree on $|z| < 1$).

Which, this new open connected domain V must contain some part of the unit disk boundary, $\partial\mathbb{D}$ with all $|z| = 1$. Suppose $V \cap \partial\mathbb{D} = \emptyset$, then let $A_1 = \{z \in \mathbb{C} \mid |z| > 1\}$ (an open subset of $\mathbb{C}$), we have $A_1 \cap V$ and $\mathbb{D}$ be two open sets with no intersection, while the union becomes V (since V strictly contains $\mathbb{D}$, while contains non of its boundary, then all $z \in V$ satisfies $|z| < 1$ or $|z| > 1$). Hence, $A_1 \cap V$ and $\mathbb{D}$ becomes a separation of V, which is a contradiction if $A_1 \cap V \neq \emptyset$. But, if $A_1 \cap V = \emptyset$, then $V = \mathbb{D}$, which violates the assumption that it strictly contains the unit disk $\mathbb{D}$. Therefore, the assumption is false, we need $V \cap \partial D \neq \emptyset$.

Then, since $V \cap \partial D$ is nonempty, while $\mathbb{Q}$ is dense in $\mathbb{R}$, then on ∂D, the collection of all $e^{i\theta}$ with $\theta = 2\pi p/q$ (for $p/q \in \mathbb{Q}$) is also dense on S^1. Hence, for some $p/q \in \mathbb{Q}$, $\theta = 2\pi p/q$ satisfies $e^{i\theta} \in (V \cap \partial D)$, which $\overline{F}(e^{i\theta})$ should be defined.

However, since $\overline{F}$ agrees with F on $\mathbb{D}$, while being analtic on V, we need the following to be true:

$$\lim_{r\to 1^-} F(re^{i\theta}) = \overline{F}(e^{i\theta})$$

Yet, the previous inequality shows that as $r \to 1$, $|F(re^{i\theta})| \geq c_{p/q}\frac{1}{1-r}\log(1/(1-r))$, showing that its modulus in fact is not bounded, so $\lim_{r\to 1^-} F(e^{i\theta})$ diverges. However, this contradicts the assumption that $\overline{F}(e^{i\theta})$ is defined, hence the assumption is wrong, F can't be continued analytically past the disk.