

# Math CS 122B HW8 Part 2

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**Question 1** *Stein and Shakarchi Pg. 200-201 Exercise 4:*

Suppose  $\{a_n\}_{n \in \mathbb{N}}$  is a sequence of complex numbers such that  $a_n = a_m$  iff  $n \equiv m \pmod{q}$  for some positive integer  $q$ . Define the **Dirichlet L-series** associated to  $\{a_n\}$  by

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{for } \operatorname{Re}(s) > 1$$

Also, with  $a_0 = a_q$ , let

$$Q(x) = \sum_{m=0}^{q-1} a_{q-m} e^{mx}$$

Show, as in Exercises 15 and 16 of the previous chapter, that

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{Q(x)x^{s-1}}{e^{qx} - 1} dx \quad \text{for } \operatorname{Re}(s) > 1$$

Prove as a result that  $L(s)$  is continuable into the complex plane, with the only possible singularity a pole at  $s = 1$ . In fact,  $L(s)$  is regular at  $s = 1$  if and only if  $\sum_{m=0}^{q-1} a_m = 0$ . Note the connection with the Dirichlet  $L(s, \chi)$  series, taken up to Book I Chapter 8, and that as a consequence,  $L(s, \chi)$  is regular at  $s = 1$  if and only if  $\chi$  is a non-trivial character.

**Pf:**

## 1.1 Integral Representation of $L(s)$ :

Given  $\operatorname{Re}(s) > 1$ , and  $x \in (0, \infty)$ , notice that  $\frac{1}{e^{qx} - 1} = \frac{e^{-qx}}{1 - e^{-qx}}$ , with the fact that  $-qx < 0$ , then  $e^{-qx} < 1$ . Hence, the following expression is absolutely convergent, and converging normally for any compact subset of  $(0, \infty)$ :

$$\frac{1}{e^{qx} - 1} = \frac{e^{-qx}}{1 - e^{-qx}} = \sum_{n=1}^{\infty} (e^{-qx})^n \tag{1}$$

Since it converges normally within any compact subset of  $(0, \infty)$  (the domain of integration), then the integral expression in the question can be rewritten as:

$$\begin{aligned}
\frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx} - 1} dx &= \frac{1}{\Gamma(s)} \int_0^\infty Q(x)x^{s-1} \left( \sum_{n=1}^\infty e^{-nx} \right) dx \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \int_0^\infty \left( \sum_{m=0}^{q-1} a_{q-m} e^{mx} \right) x^{s-1} \cdot e^{-nx} dx \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \sum_{m=0}^{q-1} a_{q-m} \int_0^\infty x^{s-1} e^{-(nq-m)x} dx
\end{aligned} \tag{2}$$

Which, by swapping  $r = q - m$  (where  $r$  ranges from 1 to  $q$ ), extending from (2), we get the following:

$$\begin{aligned}
\frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx} - 1} dx &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \sum_{r=1}^q a_r \int_0^\infty x^{s-1} e^{-(nq-(q-r))x} dx \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \sum_{r=1}^q a_r \int_0^\infty x^{s-1} e^{-((n-1)q+r)x} dx \\
&= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^q a_r \int_0^\infty x^{s-1} e^{-(nq+r)x} dx
\end{aligned} \tag{3}$$

Then, performing substitution  $u = (nq + r)x$  for each index  $n$  and  $r$ ,  $du = (nq + r)dx$ , which (3) becomes:

$$\begin{aligned}
\frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx} - 1} dx &= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^q a_r \int_0^\infty \left( \frac{u}{nq + r} \right)^{s-1} \cdot e^{-u} \frac{du}{nq + r} \\
&= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^q a_r \cdot \frac{1}{(nq + r)^s} \int_0^\infty u^{s-1} e^{-u} du \\
&= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \sum_{r=1}^q \frac{a_r}{(nq + r)^s} \cdot \Gamma(s) = \sum_{n=0}^\infty \sum_{r=1}^q \frac{a_r}{(nq + r)^s}
\end{aligned} \tag{4}$$

Now, in terms of the original  $L(s)$ , recall that  $a_n = a_m$  iff  $n \equiv m \pmod{q}$ , so the original series expression can be rearranged as:

$$\begin{aligned}
L(s) &= \sum_{k=1}^\infty \frac{a_k}{k^s} = \sum_{n=1}^\infty \frac{a_{nq}}{(nq)^s} + \sum_{n=0}^\infty \sum_{r=1}^{q-1} \frac{a_{nq+r}}{(nq+r)^s} \\
&= \sum_{n=0}^\infty \frac{a_q}{(nq+q)^s} + \sum_{n=0}^\infty \sum_{r=1}^{q-1} \frac{a_r}{(nq+r)^s} = \sum_{n=0}^\infty \sum_{r=1}^q \frac{a_r}{(nq+r)^s}
\end{aligned} \tag{5}$$

Then, combining the results in (4) and (5), we get  $L(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx} - 1} dx$ .

## 1.2 Continuation to $\mathbb{C} \setminus \{1\}$ :

**Question 2** *Stein and Shakarchi Pg. 204 Problem 4:*

*One can combine ideas from the prime number theorem with the proof of Dirichlet's Theorem about primes in arithmetic progression (given in Book I) to prove the following: Let  $q$  and  $l$  be relatively prime integers. We consider the primes belonging to the arithmetic progression  $\{qk + l\}_{k \in \mathbb{N}}$ , and let  $\pi_{q,l}(x)$  denote the number of such primes  $\leq x$ . Then one has*

$$\pi_{q,l}(x) \sim \frac{x}{\varphi(q) \log(x)} \quad \text{as } x \rightarrow \infty$$

*where  $\varphi(q)$  denotes the number of positive integers less than  $q$  and relatively prime to  $q$  (i.e. the Euler Totient Function).*

**Pf:**