

Math CS 122B HW1

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Question 1 Ahlfors Pg. 178 Problem 2:

Show that the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

converges for $\operatorname{Re}(z) > 1$, and represent its derivative in series form.

Pf:

The following proof would assume the domain of the above series is the half plane $\operatorname{Re}(z) > 1$.

The series converges pointwise:

For all $z \in \mathbb{C}$ satisfying $\operatorname{Re}(z) > 1$, $z = a + bi$ for $a, b \in \mathbb{R}$, and $a > 1$. Then, for any $n \in \mathbb{N}$, the number $n^{-z} = e^{-z \log(n)} = e^{-(a+bi) \ln(n)} = e^{-a \ln(n)} \cdot e^{i(-b \ln(n))} = n^{-a} \cdot e^{i(-b \ln(n))}$. Hence, if taken the modulus $e^{-a \ln(n)}$, since $a > 1$, by p-series test, the following series converges:

$$\sum_{n=1}^{\infty} n^{-a}$$

Which, since the original series satisfies the following:

$$\sum_{n=1}^{\infty} |n^{-z}| = \sum_{n=1}^{\infty} \left| n^{-a} \cdot e^{i(-b \ln(n))} \right| = \sum_{n=1}^{\infty} n^{-a}$$

Hence, the series absolutely converges, which $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ is defined on $\operatorname{Re}(z) > 1$.

Partial Sum converges uniformly on any Compact Subset:

Suppose $K \subset \mathbb{C}$ is a compact subset of the plane $\operatorname{Re}(z) > 1$, each component $n^{-z} = e^{-z \ln(n)} = n^{-a} \cdot e^{i(-b \ln(n))}$ is analytic on the half plane (also on K), hence there exists $z_0 \in K$, such that $|n^{-z_0}|$ yields the maximum.

Since for $z_0 = a + bi$, $|n^{-z_0}| = |n^{-a} \cdot e^{i(-b \ln(n))}| = n^{-a}$, and since z_0 is in the half plane, so $\operatorname{Re}(z_0) = a > 1$. Then, the series $\sum_{n=1}^{\infty} n^{-a}$ converges.

Now, notice that for each $n \in \mathbb{N}$, $M_n = \sup_{z \in K} |n^{-z}| = \max_{z \in K} |n^{-z}| = n^{-a}$ satisfies $\sum_{n=1}^{\infty} M_n$ converges, then by Weierstrass M-Test, the series $\sum_{n=1}^{\infty} n^{-z}$ in fact converges uniformly on K .

Then, because the series $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ converges absolutely on $\operatorname{Re}(z) > 1$, converging uniformly on all compact subset of the half plane, and each component is analytic on the half plane, then by the theorem in Ahlfors pg. 176, $\zeta(z)$ is analytic, and the partial sum $\sum_{n=1}^N n^{-z}$ (for $N \in \mathbb{N}$) has derivative converges to

$\zeta'(z)$ uniformly on all compact subsets of the half plane. Hence, based on the same theorem again, we can claim the folloing on the chosen half plane:

$$\zeta'(z) = \lim_{N \rightarrow \infty} \frac{d}{dz} \sum_{n=1}^N n^{-z} = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{d}{dz} (e^{-z \ln(n)}) = \lim_{N \rightarrow \infty} \sum_{n=1}^N -\ln(n) e^{-z \ln(n)} = - \sum_{n=1}^{\infty} \ln(n) n^{-z}$$

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Question 2 Ahlfors Pg. 184 Problem 5:

The Fibonacci numbers are defined by $c_0 = 0$, $c_1 = 1$, and $c_n = c_{n-1} + c_{n-2}$ for all $n \geq 2$.

Show that the c_n are Taylor Coefficients of a rational function, and determine a closed expression for c_n .

Pf:

Consider the generating function, a formal power series defined as $F(z) = \sum_{n=0}^{\infty} c_n z^n$.

Radius of Convergence of the Power Series:

Recall that radius of convergence of power series $R = \limsup(|c_n|^{\frac{1}{n}})^{-1} \in [0, \infty]$, where c_n is the coefficients of each degree.

First, we can verify that for all $n \in \mathbb{N}$, $c_n < 2^n$:

For base case $n = 1$, $c_1 = 1 < 2^1$.

Now, suppose for given $n \geq 1$, $c_n < 2^n$, then for case $(n+1) \geq 2$, since $c_{n+1} = c_n + c_{n-1} < 2 \cdot c_n$ (since $c_n > c_{n-1}$), then $c_{n+1} < 2 \cdot c_n < 2 \cdot 2^n = 2^{n+1}$ by induction hypothesis, which this completes the induction.

Since all $n \in \mathbb{N}$ has $0 < c_n < 2^n$, then $|c_n|^{\frac{1}{n}} = c_n^{\frac{1}{n}} < (2^n)^{\frac{1}{n}} < 2$, so $\limsup(|c_n|^{\frac{1}{n}}) \leq 2$, hence $R = \limsup(|c_n|^{\frac{1}{n}})^{-1} \geq \frac{1}{2}$. Thus, we can claim that the power series $F(z) = \sum_{n=0}^{\infty} c_n z^n$ in fact converges absolutely for disk $|z| < \frac{1}{2}$ (since $|z| < \frac{1}{2}$ is contained in the radius of convergence).

Closed Expression of c_n :

Now, consider power series $F(z)$ on $|z| < \frac{1}{2}$: Since $F(z)$ can be rewritten as $c_0 + c_1 z + \sum_{n=2}^{\infty} c_n z^n = 1 + z + \sum_{n=2}^{\infty} c_n z^n$. Then, based on the definition of Fibonnaci numbers, it can be rewritten as:

$$\begin{aligned} F(z) &= 1 + z + \sum_{n=2}^{\infty} (c_{n-1} + c_{n-2}) z^n = 1 + z + \sum_{n=2}^{\infty} c_{n-1} z^n + \sum_{n=2}^{\infty} c_{n-2} z^n \\ &= 1 + z + \sum_{n=1}^{\infty} c_n z^{n+1} + \sum_{n=0}^{\infty} c_n z^{n+2} = 1 + z \left(1 + \sum_{n=1}^{\infty} c_n z^n \right) + z^2 \sum_{n=0}^{\infty} c_n z^n \\ &= 1 + z \sum_{n=0}^{\infty} c_n z^n + z^2 F(z) = 1 + z F(z) + z^2 F(z) \end{aligned}$$

Then, we can yield the following:

$$F(z) = 1 + z F(z) + z^2 F(z), \quad F(z)(1 - z - z^2) = 1, \quad F(z) = \frac{1}{1 - z - z^2}$$

Now, if $1-z-z^2 = 0$ (or $z^2+z-1 = 0$), we have $z = \frac{-1 \pm \sqrt{5}}{2}$. Hence, $1-z-z^2 = -\left(\frac{-1+\sqrt{5}}{2} - z\right)\left(\frac{-1-\sqrt{5}}{2} - z\right)$. Then, $F(z)$ can be decomposed using partial fraction:

$$F(z) = \frac{A}{\frac{-1+\sqrt{5}}{2} - z} + \frac{B}{\frac{-1-\sqrt{5}}{2} - z} = \frac{1}{1-z-z^2}, \quad B\left(\frac{-1+\sqrt{5}}{2} - z\right) + A\left(\frac{-1-\sqrt{5}}{2} - z\right) = -1$$

So, from the above expression, we get:

$$\begin{aligned} B \cdot \frac{-1+\sqrt{5}}{2} + A \cdot \frac{-1-\sqrt{5}}{2} &= -1, \quad -B - A = 0 \\ \implies A &= -B, \quad B \cdot \frac{-1+\sqrt{5}}{2} - B \cdot \frac{-1-\sqrt{5}}{2} = -1 \\ \implies B \left(\frac{-1+\sqrt{5}}{2} - \frac{-1-\sqrt{5}}{2} \right) &= B \cdot \sqrt{5} = -1, \quad B = -\frac{1}{\sqrt{5}}, \quad A = \frac{1}{\sqrt{5}} \end{aligned}$$

So, $F(z)$ can be expressed as:

$$F(z) = \frac{1}{\sqrt{5}} \left(\frac{1}{\frac{-1+\sqrt{5}}{2} - z} - \frac{1}{\frac{-1-\sqrt{5}}{2} - z} \right)$$

Now, notice that for any $k \neq 0$, on $|z| < |k|$, since $|z/k| < 1$, then $\sum_{n=0}^{\infty} (z/k)^n$ converges absolutely to $\frac{1}{1-z/k} = \frac{k}{k-z}$, which $\frac{1}{k-z} = \frac{1}{k} \sum_{n=0}^{\infty} (z/k)^n$.

Because both $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$ has absolute values greater than $\frac{1}{2}$ (first one is approximately 0.618, the second one is approximately -1.618), hence, on the disk $|z| < \frac{1}{2}$, both equations below are true based on the above formula:

$$\frac{1}{\frac{-1+\sqrt{5}}{2} - z} = \frac{1}{\frac{-1+\sqrt{5}}{2}} \sum_{n=0}^{\infty} \left(\frac{z}{\frac{-1+\sqrt{5}}{2}} \right)^n, \quad \frac{1}{\frac{-1-\sqrt{5}}{2} - z} = \frac{1}{\frac{-1-\sqrt{5}}{2}} \sum_{n=0}^{\infty} \left(\frac{z}{\frac{-1-\sqrt{5}}{2}} \right)^n$$

Hence, $F(z)$ can be expressed as:

$$\begin{aligned} F(z) &= \frac{1}{\sqrt{5}} \left(\sum_{n=0}^{\infty} \left(\frac{1}{\frac{-1+\sqrt{5}}{2}} \right)^{n+1} z^n - \sum_{n=0}^{\infty} \left(\frac{1}{\frac{-1-\sqrt{5}}{2}} \right)^{n+1} z^n \right) \\ &= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{\left(\frac{-1-\sqrt{5}}{2} \right)^{n+1} - \left(\frac{-1+\sqrt{5}}{2} \right)^{n+1}}{\left(\frac{-1+\sqrt{5}}{2} \right)^{n+1} \left(\frac{-1-\sqrt{5}}{2} \right)^{n+1}} z^n = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{\left(\frac{-1-\sqrt{5}}{2} \right)^{n+1} - \left(\frac{-1+\sqrt{5}}{2} \right)^{n+1}}{\left(\frac{(-1)^2 - (\sqrt{5})^2}{4} \right)^{n+1}} z^n \\ &= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{\left(\frac{-1-\sqrt{5}}{2} \right)^{n+1} - \left(\frac{-1+\sqrt{5}}{2} \right)^{n+1}}{(-1)^{n+1}} z^n = \sum_{n=0}^{\infty} \frac{\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}}{\sqrt{5}} z^n \end{aligned}$$

Then, by the uniqueness of Taylor Series, the following is the closed expression of c_n :

$$\forall n \in \mathbb{N}, \quad c_n = \frac{\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}}{\sqrt{5}}$$

Question 3 Ahlfors Pg. 186 Problem 4:

Show that the Laurent development of $(e^z - 1)^{-1}$ at the origin is of the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$$

where the numbers B_k are known as the Bernoulli numbers. Calculate B_1, B_2, B_3 .

Pf:

Given the function $f(z) = (e^z - 1)^{-1}$, it is analytic on $\mathbb{C} \setminus \{0\}$ (with $0 < |z| < \infty$), hence there exists a laurent development that agrees on the whole $\mathbb{C} \setminus \{0\}$:

$$f(z) = \sum_{n=-\infty}^{\infty} A_n z^n$$

And, for all index n , the formula of A_n is given as follow:

$$n \geq 1, \quad A_{-n} = \frac{1}{2\pi i} \int_C f(\zeta) \zeta^{n-1} d\zeta \quad n \geq 0, \quad A_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{n+1}} d\zeta$$

Where C is a circle with radius $r > 0$, centered at $z = 0$.

Coefficients of Negative Degree:

First, for $n = 1$, the coefficient A_{-1} is given as:

$$A_{-1} = \frac{1}{2\pi i} \int_C f(\zeta) \zeta^{1-1} d\zeta = \frac{1}{2\pi i} \int_C \frac{1}{e^\zeta - 1} d\zeta = \frac{1}{2\pi i} \int_C \frac{1}{\zeta} \cdot \frac{\zeta}{e^\zeta - 1} d\zeta$$

Now, notice that for $\frac{\zeta}{e^\zeta - 1}$ with an isolated singularity at $\zeta = 0$, since $\lim_{\zeta \rightarrow 0} \frac{\zeta}{e^\zeta - 1} = 1$ (since the limit of its reciprocal is $\lim_{\zeta \rightarrow 0} \frac{e^\zeta - e^0}{\zeta} = 1$, which is the derivative of e^z at 0), then, $\lim_{\zeta \rightarrow 0} \zeta \cdot \frac{\zeta}{e^\zeta - 1} = 0$, which is a sufficient and necessary condition for $\zeta = 0$ to be a removable singularity of $\frac{\zeta}{e^\zeta - 1}$.

Hence, $\frac{\zeta}{e^\zeta - 1}$ has an analytic extension onto the whole \mathbb{C} , with the function being 1 at $\zeta = 0$. Which, by Cauchy's Integral Formula, A_{-1} of the above form, is the evaluation of $\frac{\zeta}{e^\zeta - 1}$ at 0 (more precisely, evaluation of its extension at 0), which provides $A_{-1} = 1$.

Then, for $n > 1$, the coefficient A_{-n} is given by:

$$A_{-n} = \frac{1}{2\pi i} \int_C f(\zeta) \zeta^{n-1} d\zeta = \frac{1}{2\pi i} \int_C \frac{1}{\zeta} \cdot \frac{\zeta^n}{e^\zeta - 1} d\zeta$$

Notice that for $n > 1$, the function $\frac{\zeta^n}{e^\zeta - 1}$ has isolated singularity at 0, since $\lim_{\zeta \rightarrow 0} \zeta \cdot \frac{\zeta^n}{e^\zeta - 1} = 0$, then the singularity at 0 is in fact removable. Hence, it has an analytic extension onto \mathbb{C} , where the evaluation at $\zeta = 0$ is $\lim_{\zeta \rightarrow 0} \frac{\zeta^n}{e^\zeta - 1} = 0$ (since it can be broken down into ζ^{n-1} and $\frac{\zeta}{e^\zeta - 1}$; the first one has limit 0 since $n - 1 \geq 1$, while the second one has limit 1, hence the product has limit 0).

Again, the integral form of A_{-n} is in fact the evaluation of the extension of $\frac{\zeta^n}{e^\zeta - 1}$ at $\zeta = 0$ by Cauchy's Integral Formula, hence $A_{-n} = 0$.

Coefficients of Nonnegative Degree:

For $n \geq 0$, the coefficient A_n is given as:

$$A_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{n+1}} d\zeta = \frac{1}{2\pi i} \int_C \frac{1}{\zeta^{n+1}(e^\zeta - 1)} d\zeta = \frac{1}{2\pi i} \int_C \frac{1}{\zeta^{n+2}} \cdot \frac{\zeta}{e^\zeta - 1} d\zeta$$

Since we've verified above, that $\frac{z}{e^z - 1}$ has an analytic extension onto \mathbb{C} , it has a power series expansion about 0, $\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} C_n z^n$, and it agrees with the function on the whole \mathbb{C} . We'll find the coefficient to help calculate A_n .

Notice that since $\frac{e^z - 1}{z} \cdot \frac{z}{e^z - 1} = 1$, while $\frac{e^z - 1}{z}$ also can be extended analytically onto \mathbb{C} , given the following power series expansion of $\frac{e^z - 1}{z}$:

$$e^z = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!}, \quad \frac{e^z - 1}{z} = \frac{1}{z} \sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

Which we can conclude the following:

$$\frac{e^z - 1}{z} \cdot \frac{z}{e^z - 1} = \left(\sum_{n=0}^{\infty} C_n z^n \right) \left(\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} \right) = 1 = 1 + \sum_{n=1}^{\infty} 0 \cdot z^n$$

Since regardless of $z \in \mathbb{C}$, the above equation is true, $\sum_{n=0}^{\infty} C_n z^n$ is in fact the inverse of the formal power series $\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} \in \mathbb{C}[[z]]$. Which, they satisfy the following relationship:

- For coefficient of degree 0, we have $C_0 \cdot \frac{1}{(0+1)!} = 1$, hence $C_0 = 1$.
- For coefficient of degree 1, we have $C_1 \cdot \frac{1}{(0+1)!} + C_0 \cdot \frac{1}{(1+1)!} = 0$, hence $C_1 + \frac{1}{2!} = 0$, $C_1 = -\frac{1}{2}$.
- For coefficient of degree $n \geq 2$, we have $\sum_{k=0}^n C_k \cdot \frac{1}{((n-k)+1)!} = 0$, hence:

$$C_n = C_n \cdot \frac{1}{(0+1)!} = - \sum_{k=0}^{n-1} C_k \cdot \frac{1}{((n-k)+1)!}$$

Since $g(z) = \frac{z}{e^z - 1}$ has its power series $\sum_{n=0}^{\infty} C_n z^n$ converge to itself on the whole \mathbb{C} , then its n^{th} derivative at 0 is given as $g^{(n)}(0) = n!C_n$. Which, A_n can be rewritten as the following using Cauchy's Integral Formula:

$$A_n = \frac{1}{2\pi i} \int_C \frac{1}{\zeta^{n+2}} \cdot \frac{\zeta}{e^\zeta - 1} d\zeta = \frac{1}{2\pi i} \int_C \frac{g(\zeta)}{\zeta^{n+2}} d\zeta = \frac{g^{(n+1)}(0)}{(n+1)!} = \frac{(n+1)!C_{n+1}}{(n+1)!} = C_{n+1}$$

Hence, with the previous section (where $A_{-1} = 1$, and $A_{-n} = 0$ for $n > 1$), we can express the laurent series as the following:

$$\sum_{n=-\infty}^{\infty} A_n z^n = \sum_{n=1}^{\infty} A_{-n} z^{-n} + \sum_{n=0}^{\infty} A_n z^n = \frac{1}{z} + \sum_{n=0}^{\infty} C_{n+1} z^n = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} C_{n+1} z^n$$

Question 4 *Stein and Shakarchi Pg. 86 Problem 2:*

Let

$$F(z) = \sum_{n=1}^{\infty} d(n)z^n$$

for $|z| < 1$, where $d(n)$ denotes the number of divisors of n . Observe that the radius of convergence of this series is 1. Verify the identity

$$\sum_{n=1}^{\infty} d(n)z^n = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$$

Using this identity, show that if $z = r$ with $0 < r < 1$, then

$$|F(r)| \geq c \frac{1}{1-r} \log(1/(1-r))$$

as $r \rightarrow 1$. Similarly, if $\theta = 2\pi p/q$ where p and q are positive integers and $z = re^{i\theta}$, then

$$|F(re^{i\theta})| \geq c_{p/q} \frac{1}{1-r} \log(1/(1-r))$$

as $r \rightarrow 1$. Conclude that F cannot be continued analytically past the unit disk.

Pf: