# Lie Algebra of a Lie Group

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## Tangent Space, Tangent Vectors and Derivations

In simplest case, if embedd manifold $M^n$ into $\mathbb{R}^m$, for any chart $(U, \phi)$ of $M$, since $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ has its inverse $\phi^{-1}$ being smooth, for any $u \in U \subseteq M$, a tangent vector $v_u$ associates with vector $v \in \mathbb{R}^n$, is characterized by differential of $\phi^{-1}$:

$$v_u := D\phi^{-1}(\phi(u))(v) = \lim_{t\to 0} \frac{\phi^{-1}(\phi(u)+tv) - \phi^{-1}(\phi(u))}{t}$$

A collection of all such vector is the **Geometric Tangent Space** of $u$, denoted as $T_u(M)$.

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Notice that for any smooth function $f \in C^\infty(M)$, it has a notion of directional derivative at $u$ depending on the tangent vector $v_u \in T_u(M)$, and such derivative satisfies genral differentiation rules (for instance, product rule).

To generalize such notion into abstract manifold (space with no definition of vectors), we need a notion of **Derivation**: For any point $u \in M$, a **Derivation at** $u$, is a linear map $v_u : C^\infty(M) \to \mathbb{R}$, that satisfies the product rule:

$$\forall f, g \in C^\infty(M), \quad v_u(fg) = f(u)(v_u g) + g(u)(v_u f)$$

Which, the set of all derivations at $u$, denoted as $T_u(M)$, is the **Tangent Space** of $M$ at $u$, and each derivation $v_u \in T_u(M)$ is called the **Tangent Vector** of $u$.

## Vector Fields & Smooth Conditions

Given smooth manifold $M$, a vector field $X$ is a function associating each point $u \in M$ with a tangent vector of $u$, so $X(u) \in T_u(M)$. More precisely, a vector field is a map $X : M \to TM$ (where $TM$ denotes the **Tangent Bundle** of $M$), such that with the canonical projection map $\pi : TM \to M$, $\pi \circ X : M \to M$ is an identity.

Which, $X$ is a **Smooth Vector Field**, if $X : M \to TM$ is a smooth map. And, a collection of smooth vector fields on $M$ is denoted as $\mathfrak{X}(M)$.

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An equivalent condition of saying a vector field $X$ is smooth, is through smooth functions $f \in C^\infty(M)$: Since for all $u \in M$, $X(u) = X_u \in T_u(M)$ is a derivation at $u$, define $Xf : M \to \mathbb{R}$ by $Xf(u) = X_u(f)$. Then, $X$ is a smooth vector field iff $Xf \in C^\infty(M)$.

## Vector Fields of Different Manifolds

Given $M, N$ two smooth manifolds, and smooth map $F : M \to N$. Let $X \in \mathfrak{X}(M)$, it would be ideal if we can send vector field $X$ to be a vector field of $N$. Yet, this requires both injectivity and surjectivity, which is too much to assume.

**Insert an example**

So, we'll consider a weaker notion, called an $F$**-Relation**: Given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, the two are $F$-related, if for all $u \in M$, the following is true:

$$dF_u(X_u) = Y_{F(u)}$$

Simply speaking, $F$ maps the tangent vectors collected by $X$, to be compatible with tangent vectors collected by $Y$.

**Insert another example**

## Lie Brackets on Vector Fields

The initial motivation is to combine two vector fields $X, Y \in \mathfrak{X}(M)$ to be another vector field. Which, for all $f \in C^\infty(M)$, since $Yf \in C^\infty(M)$ from previous characterization, then $XYf = X(Yf) \in C^\infty(M)$.

## Surfaces

## Acknowledgements

## References