

# Math CS 122B HW3

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1

**Question 1** Freitag Chap. IV.3 Exercise 3:

Show:

$$\frac{\pi}{\cos(\pi z)} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 - 4z^2}$$

and derive from this

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

**Pf:**

We'll complete this by the following trigonometric identity, and the expression of  $\frac{\pi}{\sin(\pi\zeta)}$  under partial fraction series:

$$\cos(\zeta) = \sin\left(\frac{\pi}{2} - \zeta\right)$$

$$\frac{\pi}{\sin(\pi\zeta)} = \frac{1}{\zeta} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{\zeta - n} + \frac{1}{\zeta + n} \right)$$

Then,  $\frac{\pi}{\cos(\pi z)}$  can be expressed as:

$$\begin{aligned} \frac{\pi}{\cos(\pi z)} &= \frac{\pi}{\sin(\pi/2 - \pi z)} = \frac{\pi}{\sin(\pi(1/2 - z))} = \frac{1}{1/2 - z} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{(1/2 - z) - n} + \frac{1}{(1/2 - z) + n} \right) \\ &= \frac{2}{1 - 2z} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{2}{(1 - 2z) - 2n} + \frac{2}{(1 - 2z) + 2n} \right) \\ &= \frac{2}{1 - 2z} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{2}{-(2n - 1) - 2z} + \frac{2}{(2n + 1) - 2z} \right) \end{aligned}$$

Which, for all  $z \notin \frac{1}{2} + \mathbb{Z}$ , if we view the partial sum of the above series, we get:

$$\begin{aligned} \forall N \in \mathbb{N}, N \geq 2, \quad S_N &= \frac{2}{1 - 2z} + \sum_{n=1}^N (-1)^n \left( \frac{2}{-(2n - 1) - 2z} + \frac{2}{(2n + 1) - 2z} \right) \\ &= \frac{2}{1 - 2z} + \frac{(-1)^1 \cdot 2}{-(2 \cdot 1 - 1) - 2z} + \sum_{n=2}^N \frac{(-1)^n \cdot 2}{-(2n - 1) - 2z} + \sum_{n=1}^{N-1} \frac{(-1)^n \cdot 2}{(2n + 1) - 2z} + \frac{(-1)^N \cdot 2}{(2N + 1) - 2z} \\ &= \frac{2}{1 - 2z} - \frac{2}{-1 - 2z} + \sum_{n=1}^{N-1} \frac{(-1)^{n+1} \cdot 2}{-(2(n+1) - 1) - 2z} + \sum_{n=1}^{N-1} \frac{(-1)^n \cdot 2}{(2n + 1) - 2z} + \frac{(-1)^N \cdot 2}{(2N + 1) - 2z} \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{2}{1-2z} - \frac{2}{-1-2z} \right) + \sum_{n=1}^{N-1} (-1)^n \left( \frac{2}{(2n+1)-2z} - \frac{2}{-(2n+1)-2z} \right) + \frac{(-1)^N \cdot 2}{(2N+1)-2z} \\
&= \sum_{n=0}^{N-1} (-1)^n \left( \frac{2}{(2n+1)-2z} - \frac{2}{-(2n+1)-2z} \right) + \frac{(-1)^N \cdot 2}{(2N+1)-2z} \\
&= \sum_{n=0}^{N-1} 2 \cdot (-1)^n \cdot \frac{(-(2n+1)-2z) - ((2n+1)-2z)}{4z^2 - (2n+1)^2} + \frac{(-1)^N \cdot 2}{(2N+1)-2z} \\
&= 2 \sum_{n=0}^{N-1} (-1)^n \cdot \frac{-2(2n+1)}{4z^2 - (2n+1)^2} + \frac{(-1)^N \cdot 2}{(2N+1)-2z} \\
&= 4 \sum_{n=0}^{N-1} (-1)^n \cdot \frac{(2n+1)}{(2n+1)^2 - 4z^2} + \frac{(-1)^N \cdot 2}{(2N+1)-2z}
\end{aligned}$$

So, we get:

$$\begin{aligned}
\lim_{N \rightarrow \infty} s_N &= \lim_{N \rightarrow \infty} 4 \sum_{n=0}^{N-1} (-1)^n \cdot \frac{(2n+1)}{(2n+1)^2 - 4z^2} + \frac{(-1)^N \cdot 2}{(2N+1)-2z} \\
&= 4 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 - 4z^2}
\end{aligned}$$

(Note: The above series converges, because before modifying the series, the partial sum already converges, and our modification provides the same sum for each  $N \in \mathbb{N}$ ).

Hence, we can conclude the following:

$$\frac{\pi}{\cos(\pi z)} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 - 4z^2}$$

Now, based on this formula, plugging in  $z = 0$ , we get the following:

$$\pi = \frac{\pi}{\cos(\pi \cdot 0)} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 - 4 \cdot 0^2} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}$$

Hence, we get the following expression of  $\frac{\pi}{4}$ :

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}$$

**Question 2** Freitag Chap. IV.3 Exercise 4:

Find a meromorphic function  $f$  in  $\mathbb{C}$  which has simple poles in

$$S = \{\sqrt{n} \mid n \in \mathbb{N}\}$$

with corresponding residues  $\text{Res}(f; \sqrt{n}) = \sqrt{n}$ , and is analytic in  $\mathbb{C} \setminus S$ .

**Pf:**

With the given condition, one could guess that for each  $n \in \mathbb{N}$ , at  $z = \sqrt{n}$ , the principal part is described using  $\frac{\sqrt{n}}{z - \sqrt{n}}$  (which is a simple pole, and has residue  $\lim_{z \rightarrow \sqrt{n}} (z - \sqrt{n}) \frac{\sqrt{n}}{z - \sqrt{n}} = \sqrt{n}$ ). However, the series of such function potentially diverges, hence we need to do some modification.

For all  $z \in \mathbb{C} \setminus \{\sqrt{n} \mid n \in \mathbb{N}\}$ , there exists  $N \in \mathbb{N}$ , such that  $n \geq N$  implies  $\frac{|z|}{\sqrt{n}} \leq \frac{1}{2}$  (which, we're working within the compact disk  $|z| \leq \frac{\sqrt{N}}{2}$ ). Then, for  $n \geq N$ , since  $\frac{z}{\sqrt{n}}$  is within the radius of convergence of the geometric series (since  $\left| \frac{z}{\sqrt{n}} \right| < 1$ ), we get:

$$\frac{\sqrt{n}}{z - \sqrt{n}} = \frac{-1}{1 - z/\sqrt{n}} = - \sum_{k=0}^{\infty} \left( \frac{z}{\sqrt{n}} \right)^k$$

Then, if we subtract out the terms up to degree 3, we get the following:

$$\frac{\sqrt{n}}{z - \sqrt{n}} + \sum_{k=0}^3 \left( \frac{z}{\sqrt{n}} \right)^k = - \sum_{k=0}^{\infty} \left( \frac{z}{\sqrt{n}} \right)^k + \sum_{k=0}^3 \left( \frac{z}{\sqrt{n}} \right)^k = - \sum_{k=4}^{\infty} \left( \frac{z}{\sqrt{n}} \right)^k = - \left( \frac{z}{\sqrt{n}} \right)^4 \sum_{k=0}^{\infty} \left( \frac{z}{\sqrt{n}} \right)^k$$

Which, compare the modulus, we get the following inequality:

$$\begin{aligned} \left| \frac{\sqrt{n}}{z - \sqrt{n}} + \sum_{k=0}^3 \left( \frac{z}{\sqrt{n}} \right)^k \right| &= \left| \left( \frac{z}{\sqrt{n}} \right)^4 \sum_{k=0}^{\infty} \left( \frac{z}{\sqrt{n}} \right)^k \right| \leq \frac{|z|^4}{n^2} \sum_{k=0}^{\infty} \left| \frac{z}{\sqrt{n}} \right|^k \\ &\leq \frac{(\sqrt{N}/2)^4}{n^2} \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k = \frac{N^2/(16)}{n^2} \cdot 2 = \frac{N^2}{8n^2} \end{aligned}$$

Hence, the following series of functions converges uniformly within the compact disk  $|z| \leq \frac{\sqrt{N}}{2}$ :

$$\left| \sum_{n=N}^{\infty} \left( \frac{\sqrt{n}}{z - \sqrt{n}} + \sum_{k=0}^3 \left( \frac{z}{\sqrt{n}} \right)^k \right) \right| \leq \sum_{n=N}^{\infty} \left| \frac{\sqrt{n}}{z - \sqrt{n}} + \sum_{k=0}^3 \left( \frac{z}{\sqrt{n}} \right)^k \right| \leq \sum_{n=N}^{\infty} \frac{N^2}{8n^2} < \infty$$

So, we can conclude that  $\sum_{n=1}^{\infty} \left( \frac{\sqrt{n}}{z - \sqrt{n}} + \sum_{k=0}^3 \left( \frac{z}{\sqrt{n}} \right)^k \right)$  converges normally on  $\mathbb{C} \setminus \{\sqrt{n}\}_{n \in \mathbb{N}}$ .

Since each  $\sqrt{n}$ ,  $n \in \mathbb{N}$  has the principal part given by  $\frac{\sqrt{n}}{z - \sqrt{n}}$ , while this principal part satisfies the desired properties, then this partial fraction series (which is a meromorphic function in this case) is a solution of the principal part distribution (simple poles at each  $\sqrt{n}$ , while having residue  $\sqrt{n}$ ).

**Question 3** Freitag Chap. IV.3 Exercise 5:

Prove the following refinement of the Mittag-Leffler Theorem:

**Theorem 1** Mittag-Leffler Let  $S \subset \mathbb{C}$  be a discrete subset. Then one can construct an analytic function  $f : \mathbb{C} \setminus S \rightarrow \mathbb{C}$  which has at any  $s \in S$ , not only given principal parts but also finitely many Laurent coefficients for nonnegative indices.

i.e. For each point, finitely many Laurent coefficients with nonnegative indices are predetermined.

**Pf:**

For every  $s \in S$ , if we want to construct a function with the given principal parts and finitely many Laurent coefficients for nonnegative indices being predetermined, then the goal is to create  $h$  (a partial fraction series) and  $g$  (a Weierstrass product), such that their product  $f = hg$  provides a Laurent series at  $s$ , with the first several determined coefficients being  $a_N, a_{N+1}, \dots, a_M$ , where  $N$  is the order of the pole of  $f$  at  $s$  (so every coefficient  $a_n$  with  $n < N$  is 0), and  $N, M$  are dependent on  $s$ .

**The Weierstrass Product  $g$ :**

For each  $s \in S$ , the largest index of the predetermined coefficient is  $M$  (dependent on  $s$ ).

If  $M < 0$ , we simply don't include this point as a zero for  $g$  (so the Taylor Series of  $g$  about  $s$  has nonzero constant term).

Else if  $M \geq 0$ , include  $s$  as a zero of  $g$ , with order being  $(M + 1)$  (provide higher degrees for the product  $hg$  to construct all the predetermined  $a_n$  with  $n \geq 0$ ).

**Construction of Principal Parts for  $h$ :**

For fixed  $s \in S$ ,  $g$  has a Taylor Series about  $s$  being  $\sum_{k=m}^{\infty} b_k(z-s)^k$ , where  $b_m$  (with  $m \geq 0$ ) is the first nonzero coefficient (so from the previous part, if  $M < 0$  for given  $s$ , then  $m = 0$ ; else if  $M \geq 0$ , then  $m = M + 1$ . Which,  $m > M$  for each  $s \in S$ ).

Now, we can construct the principal part of  $h$  at  $s$ , described by  $\sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n}$  (where  $\{c_n\}_{n=N-m}^{-1}$  are yet to be determined).

Our goal is to let the product  $(\sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n})(\sum_{k=m}^{\infty} b_k(z-s)^k)$  (which has all coefficients  $n \geq N$ ) to produce  $a_N, a_{N+1}, \dots, a_M$  as the first several coefficients. Which, for  $N \leq n \leq M$ , it suffices to solve the following equation (with all  $c_u$  being unknown variables):

$$\sum_{u+v=n} c_u b_v = a_n, \quad m \leq v \leq n-u, \quad N-m \leq u \leq n-m$$

For  $n = N$ , the only choice is  $u = (N - m)$  and  $v = m$  (since all other  $u > (N - m)$  and  $v > m$ , hence  $u + v > N$ ), so  $c_{N-m} b_m = a_N$ , or  $c_{N-m} = a_N / b_m$ .

Then, for  $N < n \leq M$ , we can recursively solve the expression for each  $c_{n-m}$  (since each equation about  $a_n$  only has finitely many  $b_v$  involved, while dependent on  $c_{N-m}, \dots, c_{n-m}$ , while the coefficients before  $c_{n-m}$  are solved by previous steps).

The remaining argument to make is why this determines  $a_N, \dots, a_M$  as the first several Laurent coefficients of  $f = hg$  when expanding about  $s$ .

For each  $s \in S$ , the principal part for  $s$  is  $\sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n}$  provided above, hence  $h(z) - \sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n}$  can be extended analytically to  $s$ , which has Taylor Series (within some radius of convergence) as follow:

$$\begin{aligned} h(z) - \sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n} &= \sum_{v=0}^{\infty} c_v (z-s)^v \\ \implies h(z) &= \left( \sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n} \right) + \left( \sum_{v=0}^{\infty} c_v (z-s)^v \right) \end{aligned}$$

Then, with  $g(z) = \sum_{k=m}^{\infty} b_k (z-s)^k$ , the Laurent Series of  $f = hg$  about  $s$  is given as:

$$\begin{aligned} f(z) &= \left[ \left( \sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n} \right) + \left( \sum_{v=0}^{\infty} c_v (z-s)^v \right) \right] \left( \sum_{k=m}^{\infty} b_k (z-s)^k \right) \\ &= \left( \sum_{n=N-m}^{-1} \frac{c_n}{(z-s)^n} \right) \left( \sum_{k=m}^{\infty} b_k (z-s)^k \right) + \left( \sum_{v=0}^{\infty} c_v (z-s)^v \right) \left( \sum_{k=m}^{\infty} b_k (z-s)^k \right) \end{aligned}$$

Which, the product on the left provides the first several coefficients to be  $a_N, \dots, a_M$  based on our construction, while the product on the right provides coefficients for degree  $v+k$ , with  $v \geq 0$  and  $k \geq m$  (so  $v+k \geq m > M$ ).

So, the product on the right only affects coefficients with index  $n > M$ , hence the coefficients with index  $n \leq M$  are all determined by the product on the left, showing that the first several coefficients are indeed  $a_N, \dots, a_M$ .

Hence, it's possible to construct such function  $f : \mathbb{C} \setminus S \rightarrow \mathbb{C}$ , such that at each  $s \in S$ , finitely many laurent coefficients are determined.

**Question 4** *Stein and Shakarchi Chap. 8 Problem 7: (Too long I don't want to copy it)*

**Pf:**

(a) **Expansion satisfies  $r_{f(K)} \geq r_K$ :**

For all radius  $0 < r < r_K$ , the circle  $c_r$  (with radius  $r$ ) is fully contained in  $K$  (since  $c_r \subset \mathbb{D}(0, r_K) \subseteq K$ ). Since  $f$  is an expansion, then for all  $z \neq 0$ ,  $f(z) \neq 0$  (since it is injective, and  $f(0) = 0$ ). Hence, by argument principle, the following integral shows the number of zeros enclosed by curve  $c_r$ :

$$\frac{1}{2\pi i} \int_{c_r} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{f(c_r)} \frac{1}{w} dw$$

Since  $c_r$  encloses only one zero (enclosing the origin, the only point that gets mapped to 0 by  $f$ ), then the above integral yields value 1. This also implies that  $f(c_r)$  is a closed curve satisfying  $n(f(c_r), 0) = 1$  (same argument applies to all points enclosed by  $c_r$ , hence  $f(c_r)$  is a simple closed curve enclosing region with 0).

On the other hand,  $f(c_r)$  is fully contained in the range  $f(K)$ , while  $f(K)$  is also simply connected, hence the curve  $f(c_r)$  is homologous to 0, the open region enclosed by  $f(c_r)$  (denoted as  $D$ ) is also fully contained in  $f(K)$ .

Then, since  $f(c_r)$  is compact, there exists  $w_0 \in f(c_r)$  that yields a minimum modulus; which, if consider  $|w_0|$  as a radius, since  $\mathbb{D}(0, |w_0|)$  again contains no points in  $f(c_r)$  (since  $z \in \mathbb{D}(0, |w_0|)$  satisfies  $|z| < |w_0|$ , while all  $w \in f(c_r)$  satisfies  $|w_0| \leq |w|$ ), then  $\mathbb{D}(0, |w_0|) \subseteq D \subseteq f(K)$ , hence  $|w_0| \leq r_{f(K)}$ .

However, if consider the point  $z_0 \in c_r$  that satisfies  $f(z_0) = w_0$  (which,  $|z_0| = r$ ), then we have the following inequality:

$$r = |z_0| < |f(z_0)| = |w_0| \leq r_{f(K)}$$

Hence, all  $0 < r < r_K$  satisfies  $r < r_{f(K)}$ , which implies that  $r_K \leq r_{f(K)}$  (since  $r_K$  is the supremum of  $(0, r_K)$ ).

**Expansion satisfies  $|f'(0)| > 1$ :**

Since  $f : K \rightarrow \mathbb{D}$  is an expansion, implies that  $f(0) = 0$ , then  $f(z) = zg(z)$  for some analytic  $g : K \rightarrow \mathbb{C}$ .

Now, if consider the fact that all  $z \in K \setminus \{0\}$  satisfies  $|f(z)| > |z|$ , we get the following inequality:

$$|g(z)| = \frac{|zg(z)|}{|z|} = \frac{|f(z)|}{|z|} > \frac{|z|}{|z|} = 1$$

Hence, if consider  $f'(0)$  using limit definition, we get:

$$|f'(0)| = \left| \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z - 0} \right| = \left| \lim_{z \rightarrow 0} \frac{zg(z)}{z} \right| = \left| \lim_{z \rightarrow 0} g(z) \right| \geq 1$$

Now, we'll prove that  $|f'(0)| = |g(0)| \neq 1$ : Suppose the contrary that  $|g(0)| = 1$ , since the above statement implies that all  $z \in K$  (including  $z = 0$ ) satisfies  $|g(z)| \geq 1$ , then  $g(z) \neq 0$  in  $K$ , hence  $1/g : K \rightarrow \mathbb{C}$  is a well-defined analytic function, satisfying  $|1/g(z)| \leq 1$ .

However, since  $K$  is an open set, while  $g$  is nonconstant (if  $g$  is constant, and  $|g(0)| = 1$ , then  $f(z) = zg(z) = g(0)z$ , which  $|f(z)| = |g(0)z| = |z|$ , contradicting the fact that  $f$  is an expansion),

then  $|1/g(z)|$  shouldn't obtain a maximum on any point  $z \in K$ . Yet, since we assume  $g(0) = 1$ , while  $|1/g(z)| \leq 1$ , hence  $|1/g(z)| \leq |1/g(0)|$  for all  $z \in K$ , showing that  $0 \in K$  is in fact a maximum of  $1/g$  on  $K$ , which violates the maximum principle.

Hence, our assumption must be false,  $|g(0)| \neq 1$ , showing that  $|g(0)| = |f'(0)| > 1$ .

- (b) Given Koebe domain  $K_0$ , and a sequence of expansion  $\{f_0, f_1, \dots\}$  satisfying  $K_{n+1} = f_n(K_n) \subseteq \mathbb{D}$ , define  $F_n : K_0 \rightarrow \mathbb{D}$  by  $F_n = f_n \circ \dots \circ f_0$ .

**$F_n$  is an expansion:**

We'll show this by induction (Note: expansion  $f$  satisfies  $f(0) = 0$ ).

First, for  $n = 1$ ,  $F_1 = f_1 \circ f_0$ , which for all  $z \in K_0 \setminus \{0\}$ , based on the fact that  $f_0, f_1$  are expansions, it satisfies:

$$|F_1(z)| = |f_1(f_0(z))| > |f_0(z)| > |z|$$

Also,  $F_1(0) = f_1(f_0(0)) = f_1(0) = 0$ . Which,  $F_1$  is an expansion.

Now, for given  $n \in \mathbb{N}$ , suppose  $F_n$  is an expansion. Then,  $F_{n+1} = f_{n+1} \circ (f_n \circ \dots \circ f_0) = f_{n+1} \circ F_n$ . Again, since both  $f_{n+1}, F_n$  are expansions, all  $z \in K_0 \setminus \{0\}$  satisfies:

$$|F_{n+1}(z)| = |f_{n+1}(F_n(z))| > |F_n(z)| > |z|$$

Also,  $F_{n+1}(0) = f_{n+1}(F_n(0)) = f_{n+1}(0) = 0$ . Hence,  $F_{n+1}$  is an expansion, and this completes the induction. (Note: since each  $f_n$  is injective, their finite composition is also injective, which completes the injectivity of all  $F_n$ ).

**Formula for  $F'_n(0)$ :**

Again, we can show by induction, that  $F'_n(0) = \prod_{k=0}^n f'_k(0)$ .

First, for  $n = 1$ ,  $F_1 = f_1 \circ f_0$ , then by chain rule,  $F'_1(0) = f'_1(f_0(0)) \cdot f'_0(0) = f'_1(0) \cdot f'_0(0)$ .

Now, suppose for given  $n \in \mathbb{N}$ ,  $F'_n(0) = \prod_{k=0}^n f'_k(0)$ , then for  $F_{n+1} = f_{n+1} \circ F_n$  satisfies:

$$F'_{n+1}(0) = f'_{n+1}(F_n(0)) \cdot F'_n(0) = f'_{n+1}(0) \cdot \prod_{k=0}^n f'_k(0) = \prod_{k=0}^{n+1} f'_k(0)$$

Which, this proves the case for  $(n+1)$ , and it completes the induction.

**Limit of  $|f'_n(0)|$ :**

First, based on the above formula of  $F'_n(0)$ , since for all  $n \in \mathbb{N}$ ,  $f_{n+1}$  is an expansion, then  $|f'_{n+1}(0)| > 1$  (based on **part (a)**). Hence, the following is true:

$$|F'_{n+1}(0)| = \left| \prod_{k=0}^{n+1} f'_k(0) \right| = |f'_{n+1}(0)| \cdot \prod_{k=0}^n |f'_k(0)| > \prod_{k=0}^n |f'_k(0)| = \left| \prod_{k=0}^n f'_k(0) \right| = |F'_n(0)|$$

This proves that  $\{|F'_n(0)|\}_{n \in \mathbb{N}}$  is a strictly increasing sequence.

Also, recall that since  $\mathbb{D}(0, r_{K_0}) \subseteq K_0$ , for each  $n$ , define  $\bar{F}_n : \mathbb{D} \rightarrow \mathbb{D}$  by  $\bar{F}_n(z) = F_n(r_{K_0}z)$  (Note: each  $z \in \mathbb{D}$ , since  $|z| < 1$ , then  $|r_{K_0}z| < r_{K_0}$ , hence  $r_{K_0}z \in \mathbb{D}(0, r_{K_0}) \subseteq K_0$ ). Since  $\bar{F}_n$  is an analytic map from  $\mathbb{D}$  to  $\mathbb{D}$ , and it satisfies  $\bar{F}_n(0) = F_n(r_{K_0} \cdot 0) = 0$ , then by Schwarz Lemma,  $|\bar{F}'_n(0)| \leq 1$ . So, we get the following:

$$\bar{F}'_n(z) = r_{K_0} F'_n(r_{K_0}z), \quad |\bar{F}'_n(0)| = r_{K_0} |F'_n(0)| \leq 1, \quad |F'_n(0)| \leq \frac{1}{r_{K_0}}$$

This proves that  $\{|F'_n(0)|\}_{n \in \mathbb{N}}$  is bounded above by  $\frac{1}{r_{K_0}} > 0$  (Note: since  $K_0$  is open,  $r_{K_0} > 0$ ).

Hence, since the sequence is strictly increasing while bounded from above,  $\lim_{n \rightarrow \infty} |F'_n(0)| = L \in \mathbb{R}$ .

Then, since the limit exists, while  $|F'_n(0)|$  is based on products of  $|f'_k(0)|$ , then:

$$\lim_{n \rightarrow \infty} |F'_n(0)| = \lim_{n \rightarrow \infty} \prod_{k=0}^n |f'_k(0)| = L \in \mathbb{R} \implies \lim_{n \rightarrow \infty} |f'_n(0)| = 1$$

- (c) Given the family of expansions  $\{F_n : K_0 \rightarrow \mathbb{D} \mid n \in \mathbb{N}\}$  with  $\lim_{n \rightarrow \infty} r_{K_n} = 1$ , unfortunately we cannot conclude that the sequence of functions converges (if  $F_n$  converges to  $F$ , since one can add a constant rotation of radians  $\pi/2$  in between  $f_n$  and  $f_{n+1}$  for each  $n \in \mathbb{N}$ , which for all  $z \in K_0 \setminus \{0\}$ , since  $F_n(z)$  now becomes a sequence that's constantly rotating, while the modulus  $|F_n(z)|$  is still increasing, then the modified sequence no longer converges).

However, based on **Montel's Theorem**, since the family of expansions are uniformly bounded (for all  $z \in K_0$ , all  $n \in \mathbb{N}$  satisfies  $|F_n(z)| < 1$ , because  $F_n(z) \in \mathbb{D}$ ), then there exists a subsequence  $\{F_{n_k}\}_{k \in \mathbb{N}}$  that converges locally uniformly to some function  $F : K_0 \rightarrow \mathbb{D}$  (i.e. on any compact subsets of  $K_0$ , the sequence of functions converges uniformly).

Now, since  $\lim_{k \rightarrow \infty} r_{K_{n_k}} = 1$  (subsequential limit agrees with the sequential limit if the original sequence converges), then  $r_{f(K_0)} \geq 1$ : For all  $0 < r < 1$ , because of the limit, there exists  $K \in \mathbb{N}$ , such that  $k \geq K$  implies  $r < r_{K_{n_k}} \leq 1$ . Then, based on the result in **part (a)**, we know  $r_{K_{n_k}} \leq r_{f_{n_k}(K_{n_k})}$ , which  $\mathbb{D}(0, r) \subseteq \mathbb{D}(0, r_{f_{n_k}(K_{n_k})}) = \mathbb{D}(0, r_{F_{n_k}(K_0)}) \subseteq F_{n_k}(K_0)$  (Note: recall that  $f_{n_k}$  has the image being the same as  $F_{n_k}$ ). Hence, as  $F_{n_k}$  converges to  $F$ , this implies that  $\mathbb{D}(0, r) \subseteq F(K_0)$ , showing that  $r \leq r_{F(K_0)}$ . Since for all  $0 < r < 1$ ,  $r \leq r_{F(K_0)}$ , then  $1 \leq r_{F(K_0)}$ . So, this implies that  $\mathbb{D} \subseteq \mathbb{D}(0, r_{F(K_0)}) \subseteq F(K_0)$ , showing that  $F$  is surjective.

Moreover, since the collection  $\{F_{n_k}\}_{k \in \mathbb{N}}$  are a sequence of expansions (analytic injective functions) that converges locally uniformly, then by **Hurwitz's Theorem**, the limit is either constant or injective; however, since it is surjective onto  $\mathbb{D}$ , the map  $F$  is not constant, hence it must be injective.

Because  $F$  is both injective and surjective while being analytic, it is a conformal map.

- (d) Given Koebe domain  $K$ , with  $\alpha \in \partial K$  such that  $|\alpha| = r_K > 0$ , and  $\beta \in \mathbb{D}$  such that  $\beta^2 = \alpha$ .

If  $|\alpha| = 1$ , this implies that  $r_K = 1$ , or  $\mathbb{D} \subseteq K$ , which  $K = \mathbb{D}$ , so there's no need for constructing a conformal map. Hence, can assume  $|\alpha| < 1$  (consequently, since  $\beta^2 = \alpha$ , then  $|\beta| < 1$  also).

Then, the following transformation is a conformal map from  $\mathbb{D} \rightarrow \mathbb{D}$ :

$$\psi_\alpha : \mathbb{D} \rightarrow \mathbb{D}, \quad \psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$

If we restrict the domain to be  $K$ , since  $K$  is open, then  $K = K^\circ$ . Which  $K^\circ \cap \partial K = \emptyset$ , showing that  $\alpha \notin K$ .

Then, if consider  $\psi_\alpha(K)$ , since for  $\psi_\alpha(z) = 0$ , we need  $\alpha - z = 0$ , or  $z = \alpha$ , then since  $\alpha \notin K$ , then  $0 \notin \psi_\alpha(K)$ .

Hence, because  $\psi_\alpha$  is a conformal map,  $K$  is open and simply connected, implies that  $\psi_\alpha(K)$  is also open and simply connected. Therefore, it's possible to define a single-valued branch of square root on  $\psi_\alpha(K)$ , or  $S : \psi_\alpha(K) \rightarrow \mathbb{D}$  that satisfies:

$$\forall w \in \psi_\alpha(K), \quad (S(w))^2 = w, \quad S(\alpha) = \beta$$



(Note: Since  $\psi_\alpha(0) = \alpha$ , then  $\alpha \in \psi_\alpha(K)$ , hence we can define the branch such that  $S(\alpha) = \beta$ ).

(Note 2: Because all  $w \in \psi_\alpha(K) \subseteq \mathbb{D}$ , then  $|w| < 1$ ; hence,  $|S(w)|^2 = |w| < 1$  implies  $|S(w)| < 1$ , showing that  $S(\psi_\alpha(K)) \subseteq \mathbb{D}$ ).

Also, since  $\alpha \in \partial K$  is a limit point of  $K$  (since  $|\alpha| = r_K$ , all  $0 < r < 1$  satisfies  $|r\alpha| < |\alpha| = r_K$ , which  $r\alpha \in K$ ; hence for all  $\epsilon > 0$ , choose  $r > 0$  such that  $1 - \epsilon < r < 1$ , then  $|\alpha - r\alpha| = |\alpha| \cdot |1 - r| < r_K \epsilon < \epsilon$ , showing that  $r\alpha \in B_\epsilon(\alpha)$ , hence  $\alpha$  is a limit point), then we can find a sequence  $(c_n)_{n \in \mathbb{N}} \subset K$  converging to  $\alpha$ . By continuity of  $\psi_\alpha$  on  $\mathbb{D}$ , we get:

$$\lim_{n \rightarrow \infty} \psi_\alpha(c_n) = \psi_\alpha(\alpha) = 0, \quad \lim_{n \rightarrow \infty} |S(\psi_\alpha(c_n))|^2 = \lim_{n \rightarrow \infty} |\psi_\alpha(c_n)| = 0$$

Hence,  $\lim_{n \rightarrow \infty} \sqrt{|S(\psi_\alpha(c_n))|^2} = \lim_{n \rightarrow \infty} |S(\psi_\alpha(c_n))| = 0$ , showing that  $\lim_{n \rightarrow \infty} S(\psi_\alpha(c_n)) = 0$ . So, without considering if the extension is analytic or not, we can define  $S(0) = 0$ . Which, the desired square root is defined.

### The expansion based on $\alpha$ and $\beta$ :

Consider  $f = \psi_\beta \circ S \circ \psi_\alpha : K \rightarrow \mathbb{D}$ .

To prove that it is an expansion, we'll first prove that  $f(0) = 0$ :

$$f(0) = \psi_\beta \circ S \circ \psi_\alpha(0) = \psi_\beta \circ S(\alpha) = \psi_\beta(\beta) = 0$$

(Note: the mobius transformation  $\psi_\alpha$  swaps  $\alpha$  and 0, while we define  $S(\alpha) = \beta$ ).

Then, we'll consider its inverse: For  $\psi_\alpha$  as an automorphism on  $\mathbb{D}$ , its inverse is itself (the property of mobius transformation given that  $|\alpha| < 1$ ), and the same logic applies to  $\psi_\beta$ . Now, if we consider  $g(z) = z^2$ , the composition  $h = \psi_\alpha \circ g \circ \psi_\beta : f(K) \rightarrow \mathbb{C}$  becomes a left inverse of  $f$ :

$$\begin{aligned} \forall z \in K, \quad h(f(z)) &= \psi_\alpha \circ g \circ \psi_\beta \circ \psi_\beta \circ S \circ \psi_\alpha(z) = \psi_\alpha \circ g \circ S(\psi_\alpha(z)) = \psi_\alpha((S(\psi_\alpha(z)))^2) \\ &= \psi_\alpha(\psi_\alpha(z)) = z \end{aligned}$$

Also, since  $h = \psi_\alpha \circ g \circ \psi_\beta$  is in fact a map from  $\mathbb{D} \rightarrow \mathbb{D}$  (since  $\psi_\alpha, \psi_\beta$  are automorphisms of  $\mathbb{D}$ , while  $g(z) = z^2$  has all  $z \in \mathbb{D}$  with  $|g(z)| = |z|^2 < |z| < 1$ , hence  $g(z) \in \mathbb{D}$ ), and it satisfies  $h(0) = 0$  (since  $\psi_\alpha \circ g \circ \psi_\beta(0) = \psi_\alpha \circ g(\beta) = \psi_\alpha(\beta^2) = \psi_\alpha(\alpha) = 0$ ), then apply Schwarz Lemma, we know all  $w \in f(K) \subseteq \mathbb{D}$  satisfies  $|h(w)| \leq |w|$ .

To prove that it is a strict inequality above, recall that Schwarz Lemma also states that if any nonzero  $w \in \mathbb{D}$  satisfies  $|h(w)| = |w|$ , then  $h$  must be a rotation (i.e. all  $w \in \mathbb{D}$  satisfies  $|h(w)| = |w|$ ). However, this is not the case for  $\beta \in \mathbb{D}$ :

$$h(\beta) = \psi_\alpha \circ g \circ \psi_\beta(\beta) = \psi_\alpha \circ g(0) = \psi_\alpha(0) = \alpha, \quad |h(\beta)| = |\alpha| = |\beta|^2 < |\beta|$$

Since  $\beta \in \mathbb{D}$  satisfies a strict inequality  $|h(\beta)| < |\beta|$ , then  $h$  cannot be a rotation, hence all nonzero  $w \in \mathbb{D}$  cannot satisfy  $|h(w)| = |w|$ , which enforces  $|h(w)| < |w|$ .

Hence, for all nonzero  $z \in K$ , we have:

$$|z| = |h(f(z))| < |f(z)|$$

Lastly, because both  $\psi_\alpha, \psi_\beta$  are injective (automorphisms of  $\mathbb{D}$ ), and  $S$  as a square root on  $\psi_\alpha(K)$  is also injective (if  $z, w \in \psi_\alpha(K)$  satisfies  $S(z) = S(w)$ , then  $z = S(z)^2 = S(w)^2 = w$ ), then  $f$  as a composition of them is also injective.

Because the function  $f$  satisfies  $f(0) = 0$ ,  $|z| < |f(z)|$  for all  $z \in K \setminus \{0\}$ , and is injective, then  $f$  is an expansion.

**Formula of  $|f'(0)|$ :**

First, the derivative of  $\psi_\alpha$  is given as follow:

$$\psi'_\alpha(z) = \frac{-(1 - \bar{\alpha}z) - (\alpha - z)(-\bar{\alpha})}{(1 - \bar{\alpha}z)^2}, \quad \psi'_\alpha(0) = \frac{-1 - \alpha(-\bar{\alpha})}{1^2} = |\alpha|^2 - 1$$

Similarly, derivative of  $\psi_\beta$  is given as follow:

$$\psi'_\beta(z) = \frac{-(1 - \bar{\beta}z) - (\beta - z)(-\bar{\beta})}{(1 - \bar{\beta}z)^2}, \quad \psi'_\beta(\beta) = \frac{-(1 - |\beta|^2) - 0}{(1 - |\beta|^2)^2} = \frac{-1}{(1 - |\beta|^2)} = \frac{-1}{1 - |\alpha|}$$

Then, derivative of  $S$  is given as follow:

$$\forall w \in \psi_\alpha(K), \quad (S(w))^2 = w, \quad 2S(w) \cdot S'(w) = 1, \quad S'(w) = \frac{1}{2S(w)}, \quad S'(\alpha) = \frac{1}{2S(\alpha)} = \frac{1}{2\beta}$$

Then, the derivative of  $f$  at 0 is given as:

$$\begin{aligned} f'(0) &= (\psi_\beta \circ S \circ \psi_\alpha)'(0) = \psi'_\beta(S \circ \psi_\alpha(0)) \cdot S'(\psi_\alpha(0)) \cdot \psi'_\alpha(0) \\ &= \psi'_\beta(S(\alpha)) \cdot S'(\alpha) \cdot (|\alpha|^2 - 1) = \psi'_\beta(\beta) \cdot \frac{1}{2\beta} \cdot (|\alpha|^2 - 1) = \frac{-1}{1 - |\alpha|} \cdot \frac{1}{2\beta} \cdot (|\alpha|^2 - 1) \\ &= \frac{(1 - |\alpha|^2)}{2\beta(1 - |\alpha|)} = \frac{1 + |\alpha|}{2\beta} \end{aligned}$$

Which, it has modulus given by:

$$|f'(0)| = \frac{|1 + |\alpha||}{|2\beta|} = \frac{1 + |\alpha|}{2\sqrt{|\beta|^2}} = \frac{1 + r_K}{2\sqrt{|\alpha|}} = \frac{1 + r_K}{2\sqrt{r_K}}$$

(e) To construct the expansion, we'll do this in an iterative manner (and we'll assume initially given Koebe Domain  $K_0$ ,  $r_{K_0} < 1$ , since the case  $r_{K_0} \geq 1$  implies  $K_0 = \mathbb{D}$ ).

0. First, find an  $\alpha_0 \in \partial K_0$  and its corresponding  $\beta_0 \in \mathbb{D}$  satisfying  $|\alpha_0| = r_{K_0}$  and  $\beta_0^2 = \alpha_0$ , then  $f_0 = \psi_{\beta_0} \circ S_{\alpha_0} \circ \psi_{\alpha_0} : K_0 \rightarrow \mathbb{D}$  is an expansion. Let  $K_1 = f_0(K_0) \subseteq \mathbb{D}$ . (Note:  $S_{\alpha_0}$  is the described square root in **part (d)**).
- n. With integer  $n \geq 1$ , in the previous step, we have new Koebe Domain  $K_n$ . Repeat the same process, choose  $\alpha_n \in \partial K_n$  with  $|\alpha_n| = r_{K_{n-1}}$  and  $\beta_n \in \mathbb{D}$  with  $\beta_n^2 = \alpha_n$ . Then,  $f_n = \psi_{\beta_n} \circ S_{\alpha_n} \circ \psi_{\alpha_n} : K_n \rightarrow \mathbb{D}$  is again an expansion. Let  $K_{n+1} = f_n(K_n) \subseteq \mathbb{D}$ .

The above constructs a sequence of expansion described in the problem. Then, for all  $n \in \mathbb{N}$ ,  $F_n = f_n \circ \dots \circ f_0$  is an expansion, and it satisfies  $\lim_{n \rightarrow \infty} |f'_n(0)| = 1$  (both proven in **part (b)**).

By the statement proven in **part (c)**, the sequence  $\{F_n\}_{n \in \mathbb{N}}$  has a subsequence converges locally uniformly onto some injective analytic function  $F : K \rightarrow \mathbb{D}$ ; also, since the sequence  $r_{K_n}$  is strictly increasing (proven in **part (a)**) while bounded above by 1, then  $\lim_{n \rightarrow \infty} r_{K_n} = d \leq 1$ . Which, based on the formula of  $|f'_n(0)|$  give in **part (d)**, we have the following:

$$1 = \lim_{n \rightarrow \infty} |f'_n(0)| = \lim_{n \rightarrow \infty} \frac{1 + r_{K_n}}{2\sqrt{r_{K_n}}} = \frac{1 + d}{2\sqrt{d}}$$

Hence,  $2\sqrt{d} = 1 + d$ ,  $4d = (1 + d)^2 = 1 + 2d + d^2$ ,  $(1 - 2d + d^2) = (1 - d)^2 = 0$ , so  $(1 - d) = 0$ , or  $d = 1$ .

Because  $\lim_{n \rightarrow \infty} r_{K_n} = 1$ , then based on the statement in **part (c)**, the function  $F$  that the subsequence converges to, is in fact conformal. Which, this is the result we want to show.