Math 118C HW4

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Question 1 *Rudin Pg. 242 Problem 27:*
Put $f(0,0) = 0$, *and*

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x,y) \neq (0,0)$. *Prove that*

(a) $f,\ D_1 f,\ D_2 f$ *are continuous in* $\mathbb{R}^2$.

(b) $D_{12}f$ *and* $D_{21}f$ *exist at every point of* $\mathbb{R}^2$, *and are continuous except at* $(0,0)$.

(c) $D_{12}f(0,0) = 1$, *and* $D_{21}f(0,0) = -1$.

Pf:

For all $(x,y) \in \mathbb{R}^2$ with $(x,y) \neq (0,0)$, using polar coordinates, $(x,y) = (r\cos(\theta), r\sin(\theta))$ for some $r > 0$ and $\theta \in [0, 2\pi)$. Which, $|(x,y)| = r$, when consider limit definition, we'll use polar coordinates instead.

(a) f **is continuous:**

For $(x,y) \neq (0,0)$, since f is a defined rational function, it is continuous, so it suffices to show f is continuous at 0. For all $\epsilon > 0$, choose $\delta = \sqrt{\frac{\epsilon}{2}} > 0$, then for all (x,y) satisfying $0 < |(x,y)| = r < \delta$, we get the following:

$$|f(x,y) - f(0,0)| = \left| \frac{(r\cos(\theta))(r\sin(\theta))((r\cos(\theta))^2 - (r\sin(\theta))^2)}{(r\cos(\theta))^2 + (r\sin(\theta))^2} - 0 \right|$$

$$= \left| \frac{r^4 \sin(\theta)\cos(\theta)(\cos^2(\theta) - \sin^2(\theta))}{r^2} \right| \leq r^2 |sin(\theta)| \cdot |\cos(\theta)| \cdot (|\cos(\theta)|^2 + |\sin(\theta)|^2)$$

$$\leq 2r^2 < 2\left(\sqrt{\frac{\epsilon}{2}}\right)^2 = 2 \cdot \frac{\epsilon}{2} = \epsilon$$

This shows that f is continuous at $(0,0)$, hence f in continuous in $\mathbb{R}^2$.

$D_1 f$ **is continuous:**

First, using basic differentiation rule, for $(x,y) \neq (0,0)$, we get the following:

$$D_1 f(x,y) = \frac{\partial}{\partial x}\left(\frac{xy(x^2 - y^2)}{x^2 + y^2}\right) = \frac{(3x^2 y - y^3)(x^2 + y^2) - xy(x^2 - y^2)2x}{(x^2 + y^2)^2} = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}$$

Which, at $(0,0)$, $D_1 f$ could be obtained through limit:

$$D_1 f(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h \cdot 0(h^2 - 0^2)}{(h^2 + 0^2)h} = \lim_{h \to 0} 0 = 0$$

Which, $D_1 f(x,y)$ for $(x,y) \neq (0,0)$ is again a rational function, which is continuous, so to verify continuity, it suffices to check $(0,0)$. For all $\epsilon > 0$, choose $\delta = \frac{\epsilon}{6} > 0$, then for all (x,y) satisfying $0 < |(x,y)| = r < \delta$, we get the following:

$$|D_1 f(x,y) - D_1 f(0,0)| = \left| \frac{(r\cos(\theta))^4 (r\sin(\theta)) + 4(r\cos(\theta))^2 (r\sin(\theta))^3 - (r\sin(\theta)^5)}{((r\cos(\theta))^2 + (r\sin(\theta))^2)^2} - 0 \right|$$

$$= \left| \frac{r^5(\cos^4(\theta)\sin(\theta) + 4\cos^2(\theta)\sin^3(\theta)) - \sin^5(\theta)}{r^4} \right| \leq r(|cos^4(\theta)\sin(\theta)| + 4|cos^2(\theta)\sin^3(\theta)| + |\sin^5(\theta)|)$$

$$\leq r(1 + 4 + 1) < 6 \cdot \frac{\epsilon}{6} = \epsilon$$

This proves the continuity of $D_1 f$ at $(0,0)$, so $D_1 f$ is continuous in $\mathbb{R}^2$.

$D_2 f$ is continuous:

Using differentiation rule, for $(x,y) \neq (0,0)$, we get the following:

$$D_2 f(x,y) = \frac{\partial}{\partial y} \left(\frac{xy(x^2 - y^2)}{x^2 + y^2} \right) = \frac{(x^3 - 3xy^2)(x^2 + y^2) - xy(x^2 - y^2)2y}{(x^2 + y^2)^2} = \frac{x^5 - xy^4 - 4x^3y^2}{(x^2 + y^2)^2}$$

Again, at $(0,0)$, $D_2 f$ could be obtained through limit:

$$D_2 f(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0 \cdot h(0^2 - h^2)}{(0^2 + h^2)h} = \lim_{h \to 0} 0 = 0$$

Notice that $D_2 f(x,y)$ for $(x,y) \neq (0,0)$ is a rational function, which is continuous, so to verify continuity, it suffices to check $(0,0)$. For all $\epsilon > 0$, choose $\delta = \frac{\epsilon}{6} > 0$, then for all (x,y) satisfying $0 < |(x,y)| = r < \delta$, we get the following:

$$|D_2 f(x,y) - D_2 f(0,0)| = \left| \frac{(r\cos(\theta))^5 - (r\cos(\theta))(r\sin(\theta))^4 - 4(r\cos(\theta))^3 (r\sin(\theta))^2}{((r\cos(\theta))^2 + (r\sin(\theta))^2)^2} - 0 \right|$$

$$= \left| \frac{r^5(\cos^5(\theta) - \cos(\theta)\sin^4(\theta) - 4\cos^3(\theta)\sin^2(\theta))}{r^4} \right| \leq r(|\cos^5(\theta)| + |\cos(\theta)\sin^4(\theta)| + 4|\cos^3(\theta)\sin^2(\theta)|)$$

$$\leq r(1 + 1 + 4) < 6 \cdot \frac{\epsilon}{6} = \epsilon$$

This proves the continuity of $D_2 f$ at $(0,0)$, hence $D_2 f$ is continuous in $\mathbb{R}^2$.

(b) **Function $D_{21} f$:**

Given that $D_1 f(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$ and $D_1 f(0,0) = 0$, apply differentiation rule for $(x,y) \neq (0,0)$, we get:

$$D_{21} f(x,y) = \frac{\partial}{\partial y} \left(\frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} \right) = \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2 + y^2)^2 - (x^4y + 4x^2y^3 - y^5)2(x^2 + y^2)2y}{(x^2 + y^2)^4}$$

Which, $D_{21}f(x,y)$ is continuous for $(x,y) \neq (0,0)$ (since it's a rational function).

Now, to get $D_{21}f(0,0)$, we'll use limit definition:

$$D_{21}f(0,0) = \lim_{h\to 0} \frac{D_1f(0,h) - D_1f(0,0)}{h} = \lim_{h\to 0} \frac{0^4 \cdot h + 4 \cdot 0^2 \cdot h^3 - h^5}{(0^2+h^2)^2 h} = \lim_{h\to 0} -\frac{h^5}{h^5} = -1$$

Hence, $D_{21}f$ exists on the whole $\mathbb{R}^2$, and is continuous at all $(x,y) \neq (0,0)$. But, it is not continuous at $(0,0)$, since choosing $x \neq 0$ and $y = 0$, $D_{21}f$ becomes:

$$D_{21}f(x,0) = \frac{x^8}{x^8} = 1$$

Hence, $\lim_{x\to 0} D_{21}f(x,0) = 1 \neq -1 = D_{21}f(0,0)$, showing the discontinuity at $(0,0)$.

So, $D_{21}f$ exists on $\mathbb{R}^2$, while being continuous on $\mathbb{R}^2 \setminus \{0\}$.

Function $D_{12}f$:

Given that $D_2f(x,y) = \frac{x^5 - xy^4 - 4x^3y^2}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$ and $D_2f(0,0) = 0$, apply differentiation rule for $(x,y) \neq (0,0)$, we get:

$$D_{12}f(x,y) = \frac{\partial}{\partial x}\left(\frac{x^5 - xy^4 - 4x^3y^2}{(x^2+y^2)^2}\right) = \frac{(5x^4 - y^4 - 12x^2y^2)(x^2+y^2)^2 - (x^5 - xy^4 - 4x^3y^2)2(x^2+y^2)2x}{(x^2+y^2)^4}$$

Hence, $D_{12}f$ is continuous for $(x,y) \neq (0,0)$, since it's also a rational function.

Now, to get $D_{12}f(0,0)$, we'll again use limit definition:

$$D_{12}f(0,0) = \lim_{h\to 0} \frac{D_2f(h,0) - D_2f(0,0)}{h} = \lim_{h\to 0} \frac{h^5 - h \cdot 0^4 - 4h^3 \cdot 0^2}{(h^2+0^2)^2 h} = \lim_{h\to 0} \frac{h^5}{h^5} = 1$$

Hence, $D_{12}f$ exists on the whole $\mathbb{R}^2$, and is continuous at all $(x,y) \neq (0,0)$. But again, it's not continuous at $(0,0)$, since choosing $x = 0$ and $y \neq 0$, $D_{12}f$ becomes:

$$D_{12}f(0,y) = \frac{-y^8}{y^8} = -1$$

Hence, $\lim_{y\to 0} D_{12}f(0,y) = -1 \neq 1 = D_{12}f(0,0)$, showing the discontinuity at $(0,0)$.

So, $D_{12}f$ exists on $\mathbb{R}^2$, while being continuous on $\mathbb{R}^2 \setminus \{0\}$.

(c) From **part (b)**, when verifying that the existence of $D_{12}f(0,0)$ and $D_{21}f(0,0)$, we've shown that $D_{12}f(0,0) = 1$, and $D_{21}f(0,0) = -1$.

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Question 2 *Rudin Pg. 242 Problem 28:*
For $t \geq 0$, put

$$\varphi(x,t) = \begin{cases} x & 0 \leq x \leq \sqrt{t} \\ -x + 2\sqrt{t} & \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0 & \textit{otherwise} \end{cases}$$

and put $\varphi(x,t) = -\varphi(x,|t|)$ if $t < 0$.
Show that φ is continuous on $\mathbb{R}^2$, and $D_2\varphi(x,0) = 0$ for all x. Define

$$f(t) = \int_{-1}^{1} \varphi(x,t)dx$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^{1} D_2\varphi(x,0)dx$$

Pf:
Continuity of φ:

Partial derivative of φ with respect to t when $t = 0$:
For all $x \in \mathbb{R}$, if $x = 0$, then regardless of the points,

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Question 3 *Rudin Pg. 243 Problem 30:*
Let $f \in \mathcal{C}^{(m)}(E)$, where E is an open subset of $\mathbb{R}^n$. Fix $a \in E$, and suppose $x \in \mathbb{R}^n$ is so close to 0 that the points $p(t) = a + tx$ lie in E whenever $0 \le t \le 1$. Define $h(t) = f(p(t))$ for all $t \in \mathbb{R}$ for which $p(t) \in E$.

(a) For $1 \le k \le m$, show (by repeated application of the chain rule) that

$$h^{(k)}(t) = \sum (D_{l_1 \dots l_k} f)(p(t)) x_{l_1} \dots x_{l_k}$$

The sum extends over all order k-tuples $(l_1, \dots, l_k)$ in which each l_j is one of the integers $1, \dots, n$.

Pf:

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Question 4 *Rudin Pg. 288 Problem 2:*
For $i = 1, 2, 3, ...,$ *let* $\varphi_i \in \mathcal{C}(\mathbb{R})$ *have support in* $(2^{-i}, 2^{1-i})$, *such that* $\int \varphi_i = 1$. *Put*

$$f(x, y) = \sum_{i=1}^{\infty} (\varphi_i(x) - \varphi_{i+1}(x))\varphi_i(y)$$

Then f *has compact support in* $\mathbb{R}^2$, f *is cotinuous except at* $(0, 0)$, *and*

$$\int dy \int f(x, y)dx = 0, \quad but \int dx \int f(x, y)dy = 1$$

Observe that f *is unbounded in every neighborhood of* $(0, 0)$.

Pf: