

# Math 118C HW3

Zih-Yu Hsieh

May 4, 2025

1

**Question 1** *Rudin Pg. 241 Problem 19:*

*Show that the system of equations*

$$3x + y - z + u^2 = 0$$

$$x - y + 2z + u = 0$$

$$2x + 2y - 3z + 2u = 0$$

*can be solved for  $x, y, u$  in terms of  $z$ ; for  $x, z, u$  in terms of  $y$ ; for  $y, z, u$  in terms of  $x$ ; but not for  $x, y, z$  in terms of  $u$ .*

**Pf:**

## 2

**Question 2** *Rudin Pg. 242 Problem 23:*

Define  $f$  in  $\mathbb{R}^3$  by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2$$

Show that  $f(0, 1, -1) = 0$ ,  $(D_1 f)(0, 1, -1) \neq 0$ , and that there exists therefore a differentiable function  $g$  in some neighborhood of  $(1, -1)$  in  $\mathbb{R}^2$ , such that  $g(1, -1) = 0$  and

$$f(g(y_1, y_2), y_1, y_2) = 0$$

Find  $(D_1 g)(1, -1)$  and  $(D_2 g)(1, -1)$ .

**Pf:**

$f$  and  $Df$  at  $(0, 1, -1)$ :

**Question 3** Rudin Pg. 242 Problem 24:

For  $(x, y) \neq (0, 0)$ , define  $f = (f_1, f_2)$  by

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad f_2(x, y) = \frac{xy}{x^2 + y^2}$$

Compute the rank of  $f'(x, y)$ , and find the range of  $f$ .

**Pf:**

**Rank of  $f'(x, y) = Df(x, y)$ :**

Given  $f_1$  and  $f_2$ , their partial derivatives are given as follow:

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= \frac{2x(x^2 + y^2) - 2x(x^2 - y^2)}{(x^2 + y^2)^2} = \frac{4xy^2}{(x^2 + y^2)^2}, & \frac{\partial f_1}{\partial y} &= \frac{-2y(x^2 + y^2) - 2y(x^2 - y^2)}{(x^2 + y^2)^2} = \frac{-4x^2y}{(x^2 + y^2)^2} \\ \frac{\partial f_2}{\partial x} &= \frac{y(x^2 + y^2) - 2x(xy)}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2}, & \frac{\partial f_2}{\partial y} &= \frac{x(x^2 + y^2) - 2y(xy)}{(x^2 + y^2)^2} = \frac{x^3 - xy^2}{(x^2 + y^2)^2} \end{aligned}$$

Hence, the differential  $Df$  is given as:

$$Df(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} 4xy^2 & -4x^2y \\ y^3 - x^2y & x^3 - xy^2 \end{pmatrix}$$

- If  $x = 0$ , the matrix is given by:

$$Df(0, y) = \frac{1}{y^4} \begin{pmatrix} 0 & 0 \\ y^3 & 0 \end{pmatrix}$$

Which, with  $y \neq 0$  (due to the condition  $(x, y) \neq (0, 0)$ ), the above matrix has rank 1.

- If  $y = 0$ , the matrix is given by:

$$Df(x, 0) = \frac{1}{x^4} \begin{pmatrix} 0 & 0 \\ 0 & x^3 \end{pmatrix}$$

Which again, with  $x \neq 0$ , the above matrix has rank 1.

- If both  $x, y \neq 0$ , then consider any vector  $r(x, -y)$  for  $r \in \mathbb{R}$ , we get::

$$\begin{aligned} Df(x, y)r \begin{pmatrix} x \\ -y \end{pmatrix} &= \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} xy^2 & -x^2y \\ y^3 - x^2y & x^3 - xy^2 \end{pmatrix} r \begin{pmatrix} x \\ -y \end{pmatrix} \\ &= \frac{r}{(x^2 + y^2)^2} \begin{pmatrix} x^2y^2 - x^2y^2 \\ (xy^3 - x^3y) - (-x^3y + xy^3) \end{pmatrix} = \bar{0} \end{aligned}$$

So the  $\text{span}\{(x, -y)\}$  is within the null space of  $Df(x, y)$ , so the dimension of null space is at least 1. On the other hand, consider  $(1, 0) \in \mathbb{R}^2$  (that is linearly independent with  $(x, -y)$ , since  $(x, -y)$  has both entries being nonzero), it has the following:

$$Df(x, y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} xy^2 & -x^2y \\ y^3 - x^2y & x^3 - xy^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} xy^2 \\ y^3 - x^2y \end{pmatrix} \neq \bar{0}$$

Which, it shows that the range of  $Df(x, y)$  is nontrivial, hence it has dimension at least 1 also.

Because both the null space and the range have dimension  $\geq 1$ , while  $\mathbb{R}^2$  has dimension 2, by Rank Nullity Theorem, it enforces both the null space and the range must have dimension precisely 1 (since the sum of the dimension of the null space and the range must be 2). So, the rank of  $Df(x, y)$  is again 1.

So, regardless of the case,  $Df(x, y)$  has rank 1.

**Range of  $f$ :**

When fixing  $(x, y) \neq (0, 0)$  in  $\mathbb{R}^2$ , there exists  $r > 0$ , and  $\theta \in [0, 2\pi)$ , such that  $x = r \cos(\theta)$  and  $y = r \sin(\theta)$  (under polar coordinates). Then, we get the output of  $f_1, f_2$  as:

$$f_1(x, y) = \frac{(r \cos(\theta))^2 - (r \sin(\theta))^2}{(r \cos(\theta))^2 + (r \sin(\theta))^2} = \frac{r^2(\cos^2(\theta) - \sin^2(\theta))}{r^2} = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$$

$$f_2(x, y) = \frac{(r \cos(\theta))(r \sin(\theta))}{(r \cos(\theta))^2 + (r \sin(\theta))^2} = \frac{r^2 \cos(\theta) \sin(\theta)}{r^2} = \sin(\theta) \cos(\theta) = \frac{1}{2} \sin(2\theta)$$

Hence, for all  $(x, y) \neq (0, 0)$ ,  $f = (f_1, f_2)$  satisfies the following equation:

$$f_1^2 + 4f_2^2 = \cos^2(2\theta) + 4 \cdot \frac{1}{4} \sin^2(2\theta) = \cos^2(2\theta) + \sin^2(2\theta) = 1$$

Hence,  $(u, v) = f(x, y)$  is a solution to  $u^2 + 4v^2 = 1$ , so the range of  $f$  is contained in the ellipse characterized by  $u^2 + 4v^2 = 1$ .

On the other hand, for all point  $(u, v)$  satisfying  $u^2 + 4v^2 = 1$ , there exists  $\theta \in [0, 2\pi)$ , such that  $u = \cos(\theta)$  and  $v = \frac{1}{2} \sin(\theta)$ . Then, consider the point  $(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})) \in \mathbb{R}^2$ , we have:

$$f(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})) = (f_1(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})), f_2(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2}))) = \left( \cos\left(2 \cdot \frac{\theta}{2}\right), \frac{1}{2} \sin\left(2 \cdot \frac{\theta}{2}\right) \right)$$

$$= \left( \cos(\theta), \frac{1}{2} \sin(\theta) \right) = (u, v)$$

Hence,  $(u, v)$  is also in the range of  $f$ . This proves that  $f$  has the range precisely described by the ellipse  $u^2 + 4v^2 = 1$ .

**Question 4** Rudin Pg. 242 Problem 25:

Suppose  $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ , let  $r$  be the rank of  $A$ .

(a) Define  $S$  as in the proof of Theorem 9.32. Show that  $SA$  is a projection in  $\mathbb{R}^n$  whose null space is  $\text{null}(A)$  and whose range is  $\text{range}(S)$ .

(b) Use (a) to show that

$$\dim(\text{null}(A)) + \dim(\text{range}(A)) = n$$

**Pf:**

- (a) Given that  $A$  has rank  $r$ , then its range  $\text{range}(A) \subseteq \mathbb{R}^m$  is an  $r$ -dimensional linear subspace, hence there exists  $y_1, \dots, y_r \in \text{range}(A)$  that forms a basis of it.

Then, by the text in Rudin, choose  $z_1, \dots, z_r \in \mathbb{R}^n$ , so for each index  $i \in \{1, \dots, r\}$ ,  $Az_i = y_i$ . Which, the collection  $z_1, \dots, z_r \in \mathbb{R}^n$  is linearly independent, since if  $a_1, \dots, a_r \in \mathbb{R}$  satisfies  $\sum_{i=1}^r a_i z_i = \bar{0}$ , then the following is true:

$$A \left( \sum_{i=1}^r a_i z_i \right) = \sum_{i=1}^r a_i (Az_i) = \sum_{i=1}^r a_i y_i$$

By the linear independence of  $y_1, \dots, y_r \in \text{range}(A)$ , each  $a_i = 0$ , which proves the linear independence of  $z_1, \dots, z_r \in \mathbb{R}^n$ .

Now, define  $S \in \mathcal{L}(\text{range}(A), \mathbb{R}^n)$  the same as in the text, which has the following formula:

$$\forall c_1, \dots, c_r \in \mathbb{R}, \quad S \left( \sum_{i=1}^r c_i y_i \right) = \sum_{i=1}^r c_i z_i$$

Then, for all  $x \in \mathbb{R}^n$ , since  $Ax \in \text{range}(A)$ , it is spanned by  $y_1, \dots, y_r$ , hence there exists unique  $a_1, \dots, a_r \in \mathbb{R}$ , such that the following is true:

$$Ax = \sum_{i=1}^r a_i y_i$$

Hence, we get the following:

$$SAx = S \left( \sum_{i=1}^r a_i y_i \right) = \sum_{i=1}^r a_i z_i$$

Hence, applying  $SA$  twice, we get:

$$SA(SAx) = SA \left( \sum_{i=1}^r a_i z_i \right) = S \left( \sum_{i=1}^r a_i y_i \right) = \sum_{i=1}^r a_i z_i$$

This shows that  $SA(SAx) = SAx$  for all  $x \in \mathbb{R}^n$ , hence  $SA$  is a projection on  $\mathbb{R}^n$ .

Now, to find the null space and range, consider the following:

- For all  $x \in \text{null}(A)$ , since  $Ax = 0$ , then  $SAx = S(0) = 0$ , so  $x \in \text{null}(SA)$ , or  $\text{null}(A) \subseteq \text{null}(SA)$ .

On the other hand, for all  $x \in \text{null}(SA)$ , since  $S(Ax) = 0$ ,  $Ax \in \text{null}(S)$ . But, since  $Ax \in \text{range}(A)$ , there exists unique  $a_1, \dots, a_r \in \mathbb{R}$ , with  $Ax = \sum_{i=1}^r a_i y_i$ . Hence, we have the following:

$$0 = S(Ax) = S\left(\sum_{i=1}^r a_i y_i\right) = \sum_{i=1}^r a_i z_i$$

Hence, by linear independence of  $z_1, \dots, z_r \in \mathbb{R}^n$ , we must have  $a_i = 0$  for all index  $i \in \{1, \dots, r\}$ . This proves that  $Ax = \sum_{i=1}^r a_i y_i = 0$ , so  $x \in \text{null}(A)$ . Hence,  $\text{null}(SA) \subseteq \text{null}(A)$ , showing that  $\text{null}(SA) = \text{null}(A)$ .

- For all  $z \in \text{range}(SA)$ , there exists  $x \in \mathbb{R}^n$  with  $SAx = z$ . Since  $z = S(Ax) \in \text{range}(S)$ , then  $\text{range}(SA) \subseteq \text{range}(S)$ .

Similarly, for all  $z \in \text{range}(S)$ , there exists  $y \in \text{range}(A)$  (the domain of  $S$ ), with  $Sy = z$ ; then because there exists  $x \in \mathbb{R}^n$ , with  $Ax = y$  by the definition of range, we have  $SAx = S(Ax) = Sy = z$ , hence  $z \in \text{range}(SA)$ , proving that  $\text{range}(S) \subseteq \text{range}(SA)$ , or  $\text{range}(S) = \text{range}(SA)$ .

Hence, the above two cases prove that  $\text{null}(SA) = \text{null}(A)$ , while  $\text{range}(S) = \text{range}(SA)$ . So,  $SA$  is a projection in  $\mathbb{R}^n$  with null space being  $\text{null}(A)$ , and range being  $\text{range}(S)$ .

- (b) With the linearly independent set  $z_1, \dots, z_r \in \mathbb{R}^n$  given in **part (a)**, we'll consider an extra list  $x_1, \dots, x_k \in \text{null}(A) \subseteq \mathbb{R}^n$  that forms a basis of  $\text{null}(A)$ . Our goal is to prove that  $x_1, \dots, x_k, z_1, \dots, z_r$  forms a basis of  $\mathbb{R}^n$ .

First, consider  $a_1, \dots, a_k, b_1, \dots, b_r \in \mathbb{R}$ , suppose the vector  $\sum_{i=1}^k a_i x_i + \sum_{j=1}^r b_j z_j = \bar{0} \in \mathbb{R}^n$ , then we have the following:

$$0 = A(\bar{0}) = A\left(\sum_{i=1}^k a_i x_i + \sum_{j=1}^r b_j z_j\right) = A\left(\sum_{i=1}^k a_i x_i\right) + A\left(\sum_{j=1}^r b_j z_j\right) = \sum_{j=1}^r b_j (Az_j) = \sum_{j=1}^r b_j y_j$$

Which, by the linear independence of  $y_1, \dots, y_r \in \text{range}(A)$  assumed in **part (a)**, we must have  $b_j = 0$  for all  $j \in \{1, \dots, r\}$ . So,  $\sum_{i=1}^k a_i x_i + \sum_{j=1}^r b_j z_j = \sum_{i=1}^k a_i x_i = \bar{0}$ . But again, based on the linear independence of  $x_1, \dots, x_k$  by assumption, we get  $a_i = 0$  for all  $i \in \{1, \dots, k\}$ . This proves that all  $a_i, b_j = 0$ , which the collection  $x_1, \dots, x_k, z_1, \dots, z_r$  is linearly independent.

Then, for all  $x \in \mathbb{R}^n$ , since  $Ax \in \text{range}(A)$ , then there exists unique  $b_1, \dots, b_r \in \mathbb{R}$ , with  $Ax = \sum_{j=1}^r b_j y_j$ . Hence, we get the following:

$$Ax = \sum_{j=1}^r b_j y_j = \sum_{j=1}^r b_j (Az_j) = A\left(\sum_{j=1}^r b_j z_j\right)$$

So, we can reduce to the following:

$$Ax - A\left(\sum_{j=1}^r b_j z_j\right) = A\left(x - \sum_{j=1}^r b_j z_j\right) = 0, \quad x - \sum_{j=1}^r b_j z_j \in \text{null}(A)$$

Then, since  $x_1, \dots, x_k$  forms a basis of  $\text{null}(A)$ , then there exists unique  $a_1, \dots, a_k \in \mathbb{R}$ , with:

$$x - \sum_{j=1}^r b_j z_j = \sum_{i=1}^k a_i x_i, \quad x = \sum_{i=1}^k a_i x_i + \sum_{j=1}^r b_j z_j$$

So,  $x \in \text{span}\{x_1, \dots, x_k, z_1, \dots, z_r\}$ , proving that  $\mathbb{R}^n = \text{span}\{x_1, \dots, x_k, z_1, \dots, z_r\}$ .

Hence, since  $x_1, \dots, x_k, z_1, \dots, z_r$  spans  $\mathbb{R}^n$  while being linearly independent, it is a basis of  $\mathbb{R}^n$ . Hence, the length of the basis,  $k + r = \dim(\mathbb{R}^n) = n$ .

Which, since  $x_1, \dots, x_k$  is a basis of  $\text{null}(A)$ , then  $\dim(\text{null}(A)) = k$ .

On the other hand, since  $S \in \mathcal{L}(\text{range}(A), \mathbb{R}^n)$  is injective (since it maps  $y_1, \dots, y_r$  a basis of  $\text{range}(A)$ , to  $z_1, \dots, z_r \in \mathbb{R}^n$  a linearly independent set), then the domain  $\text{range}(A)$  and  $\text{range}$  are in fact isomorphic as vector spaces, while the range of  $S$  is precisely  $\text{span}\{z_1, \dots, z_r\}$  (since the definition of  $S$  is maps  $y_i$  to  $z_i$  for each  $i \in \{1, \dots, r\}$ , showing that the output value must be a linear combination of all  $z_i$ ). Hence,  $\dim(\text{range}(A)) = \dim(\text{span}\{z_1, \dots, z_r\}) = r$  (since  $z_1, \dots, z_r$  is linearly independent, it forms a basis of the span).

So, compile the information from above, we get:

$$\begin{aligned} k + r &= n, \quad k = \dim(\text{null}(A)), \quad r = \dim(\text{range}(A)) \\ \implies \dim(\text{null}(A)) + \dim(\text{range}(A)) &= n \end{aligned}$$

## 5

**Question 5** Rudin Pg. 242 Problem 26:

Show that the existence (and even the continuity) of  $D_{12}f$  does not imply the existence of  $D_1f$ . For example, let  $f(x, y) = g(x)$ , where  $g$  is nowhere differentiable.

**Pf:**

Consider the Weierstrass Function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , which is uniformly continuous, while being differentiable nowhere.

Then, given the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $f(x, y) = g(x)$ , since  $g$  is not differentiable with respect to its variable  $x$ , then  $D_1f$  does not exist; yet, since  $D_2f \equiv 0$  (due to the fact that  $g$  is a constant when  $x$  is fixed), then  $D_{12}f = D_1(D_2f) = 0$ .

Hence, even though  $D_{12}f$  is continuous,  $D_1f$  doesn't exist in this case.