Phys 103 HW1 Pass 2

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Question 1

Pf:

For this question, I didn't interpret the question right: **I didn't include the effect of the merry-go-round being frictionless, and thought the trajectory of the puck would rotate also at the same rate as the angular velocity of the merry-go-round.**

On the other hand, I didn't notice that the traveling direction of the puck is toward the center, so I also need to adjust the radius function.

Regardless of the case, if just consider the fact that the puck has velocity v in negative radial direction (toward the center initially), with the initial radius $r(0) = R$, and the fact that the derivative of the radius function is $-v$ (since it is constantly going toward the center), we can construct the radius $r(t) = R - vt$.

(Note: If radius r and angle ϕ can have the range of $\mathbb{R}$, then the position of the puck can be expressed as continuous functions; if we limit radius to be nonnegative, there are cases where ϕ would instantly jump to $\pi + \phi$, causing the angle to be not continuous. If we want to verify that the velocity, i.e. derivative of position, is constantly 0 or not, then we need the position to be differentiable. As long as ϕ is not continuous, taking derivative is invalid, so for mathematical rigor, assuming radius r can be negative is more convenient).

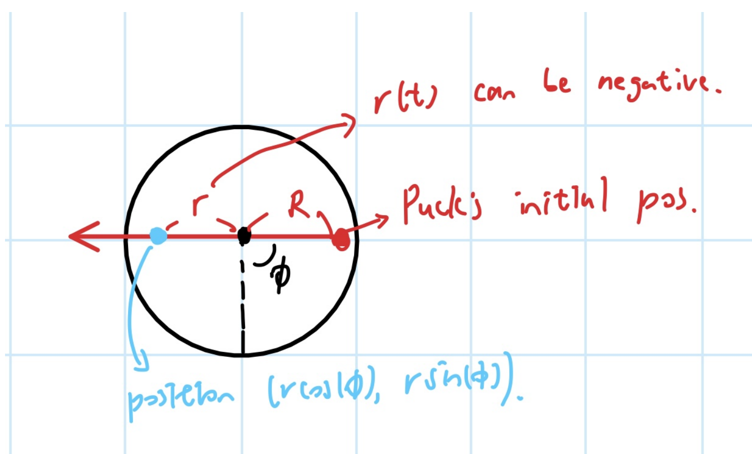

Figure 1: An illustraation of Position in r and ϕ

If the puck is not rotating with the merry-go-round, then here are the actual angles:

(a)

Since the merry-go-round is frictionless, then the puck's angle wouldn't be affected by the merry-go-round (here, keep track of the puck's initial). And, since the observer is next to the merry-go-round (not rotating with it), the relative angle between the observer and the puck's initial position never changes. Which, if the angle ϕ records the puck's initial position's angle relative to the observer, we have $\phi(t) \cong 0$ (since the initial position never changes).

So, the puck's position is given by:

$$\overline{x}(t) = (r\cos(\phi), r\sin(\phi)) = ((R - vt), 0) \tag{1}$$

$$\implies \overline{a} = \ddot{\overline{x}} = \overline{0} \tag{2}$$

Since the acceleration is $\overline{0}$, then in this frame the puck (assumed to be an isolated object) is not accelerating, hence the observer is in an inertial frame.

(b)

If the observer is constantly rotating with angular velocity w (assuming it's counterclockwise), then to the observer, the puck's radius is still changing in the same way, so $r'(t) = R - vt$ again, while the puck's initial position is rotating with angular velocity w in clockwise direction (or $-w$ in counterclockwise direction). With $\phi'(0) = 0$. Then, we get that $\phi'(t) = -wt$ (since $\frac{d\phi'}{dt} = -w$, the angular velocity of the puck in the observer's frame).

So, the puck's position is given by:

$$\overline{x'}(t) = (r'\cos(\phi'), r'\sin(\phi')) = (R - vt)(\cos(-wt), \sin(-wt)) = (R - vt)(\cos(wt), -\sin(wt)) \tag{3}$$

$$\implies \overline{a'}(t) = \frac{d^2\overline{x'}}{dt^2} = (-2wv\sin(wt) - w^2(R + vt)\cos(wt), -2wv\cos(wt) + w^2(R + vt)\sin(wt)) \tag{4}$$

Since the acceleration $\overline{a'} \neq \overline{0}$ in general, then the puck (the isolated object) is accelerating in this frame. Hence, the observer in (b) is not in an inertial frame.

Question 2

Pf:

In my original attempt, after doing massive calculation, I got to the part showing the following result:

$$\sin^2(\theta) \approx \frac{1}{4}\left(1 + \sqrt{1 - \frac{4gh}{v_0^2}}\right) \tag{5}$$

However, in my final solution, I didn't use Taylor Series approximation to get an approximated results for θ, which isn't the desired form of answer.

As a result, for $\sqrt{1-x}$, its first derivative is given as follow:

$$\frac{d}{dx}\sqrt{1-x} = -\frac{1}{2\sqrt{1-x}} \tag{6}$$

Which, with the center about $x = 0$, $\sqrt{1-x} \approx 1 - \frac{1}{2}x$ when x is small. Using this as a result, we get:

$$\sin^2(\theta) \approx \frac{1}{4}\left(1 + 1 - \frac{1}{2}\cdot\frac{4gh}{v_0^2}\right) = \frac{1}{2} - \frac{gh}{2v_0^2} \tag{7}$$

$$\implies \sin(\theta) \approx \frac{\sqrt{2}}{2}\sqrt{1 - \frac{gh}{2v_0^2}} \approx \frac{\sqrt{2}}{2}\left(1 - \frac{1}{2}\cdot\frac{gh}{v_0^2}\right) = \frac{\sqrt{2}}{2}\left(1 - \frac{gh}{2v_0^2}\right) \tag{8}$$

Now, since we can assume θ is relatively close to $\frac{\pi}{4}$ (due to the fact that h is small, so the maximum launch angle wouldn't be too far from $\frac{\pi}{4}$ - the solution corresponding to $h = 0$), then we can do an taylor series approximation of $\sin(\theta)$ about $\theta = \frac{\pi}{4}$:

Since $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, and $(\sin(\theta))'\Big|_{\theta=\frac{\pi}{4}} = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, we get $\sin(\theta) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(\theta - \frac{\pi}{4})$ when $\theta \approx \frac{\pi}{4}$. Plug into the above equation, we get:

$$\sin(\theta) \approx \frac{\sqrt{2}}{2}\left(1 + \left(\theta - \frac{\pi}{4}\right)\right) \approx \frac{\sqrt{2}}{2}\left(1 - \frac{gh}{2v_0^2}\right) \tag{9}$$

$$\implies 1 + \left(\theta - \frac{\pi}{4}\right) \approx 1 - \frac{gh}{2v_0^2} \tag{10}$$

$$\implies \theta \approx \frac{\pi}{4} - \frac{gh}{2v_0^2} \tag{11}$$

Which above is the desired form of approximation.

Question 4

Pf:

For this question, **even though the approximation of $\frac{r}{R}$ (with $r << R$) is applied, but my final answer isn't in the approximation form up to linear term of $\frac{r}{R}$**, which is wrong in this context.

After lengthy calculation, the final answer is approximated as follow:

$$\Delta t \approx \sqrt{\frac{R}{g}} \ln\left(\cot\left(\frac{r}{2R}\right) + \csc\left(\frac{r}{2R}\right)\right) = \sqrt{\frac{R}{g}} \ln\left(\frac{\cos(r/(2R)) + 1}{\sin(r/(2R))}\right) \tag{12}$$

With $\cos(x) \approx 1$ and $\sin(x) \approx x$ when x is small, plug into the relation, we get:

$$\Delta t \approx \sqrt{\frac{R}{g}} \ln\left(\frac{1+1}{r/(2R)}\right) = \sqrt{\frac{R}{g}} \ln\left(\frac{4R}{r}\right) \tag{13}$$

This is the final form of approximation (in terms of natural log).

As a side note, the previous comments of this question can be explained:

1. The first comment asking why choose $\cos(\theta) = \frac{1}{3}$, it's because here the purpose is to verify that the given term $4\cos(\theta) - 6\cos^2(\theta)$ can be positive (if it's not positive, there's no way the inequality proposed could work, because then we would have $\frac{m}{M}$ bounded by some negative value, which is not possible with the masses being positive).

2. The actual definition of infimum is as follow:

   Definition 1 *Given a collection of real numbers $S \subseteq \mathbb{R}$, the infimum of S is defined as follow:*

   $$m := \inf(S), \quad \textit{for all } x \in S,\ m \leq x$$

   $$\textit{For all } m' \in \mathbb{R},\ \textit{such that every } x \in S \textit{ satisfies } m' \leq x, \quad \textit{it satisfies } m' \leq m$$

   In other words, m is the "greatest lower bound" of the set of real numbers S.

   Here, the infimum of $\frac{m}{M}$ means th greatest lower bound of $\frac{m}{M}$, which is the smallest number $\frac{m}{M}$ can be arbitrarily close to (it's just a more rigorous definition of what "infinitesimally small" in $\mathbb{R}$ could be).

Question 5

Pf:

As of the first mistake, since here the propelled fuel has a speed of u relative to the rocket, and propelled in the negative direction (for the rocket to go forward), then the actual relative velocity is $-u$. **So, the actual formula is in fact as below:**

$$\frac{dv}{dt} = -u \cdot \frac{1}{m}\frac{dm}{dt} - g \tag{14}$$

Since the original equation, the position of (-u) is the relative velocity of the propellant in the frame of the rocket. So, the solution of the velocity now becomes:

$$\begin{aligned} \Delta v &= \int_0^{\Delta t} \frac{dv}{dt} dt = \int_0^{\frac{m_f \cdot \tau}{m_e}} \left(-u \cdot \frac{1}{m}\frac{dm}{dt} - g \right) dt = -u \ln(m(t)) \Big|_0^{\Delta t} - g\Delta t & (15) \\ &= -u \ln\left(\frac{m_{\text{fin}}}{m_0}\right) - \frac{g \cdot m_f \cdot \tau}{m_e} = -u \ln\left(\frac{m_t + m_e + m_p}{m_f + m_t + m_e + m_p}\right) - \frac{g \cdot m_f \cdot \tau}{m_e} & (16) \\ &= u \ln\left(\frac{m_f + m_t + m_e + m_p}{m_t + m_e + m_p}\right) - \frac{g \cdot m_f \cdot \tau}{m_e} & (17) \end{aligned}$$

The rest of the problems (when u appears) need to add an extra negative sign.

As of the second mistake (calculation error),**my initial answer is $a_0 \approx 0.0334 - g$ (the first number in km/s^2; convert back in g we get $a_0 \approx 2.4g$ as my initial answer)**. Now, when calculating the initial acceleration given that $\tilde{m}_p = 0.1$, I calculated $\tilde{m}_e \approx 0.0931$. And, the acceleration is given as follow:

$$\frac{dv}{dt} = -u \cdot \frac{1}{\tilde{m}}\frac{d\tilde{m}}{dt} - g = -u \cdot \frac{1}{\tilde{m}}\frac{-\tilde{m}_e}{\tau} - g = u \cdot \frac{\tilde{m}_e}{\tilde{m}\tau} - g \tag{18}$$

Which, with $u = 3$km/s, $\tilde{m}_e = 0.0931$, $\tau = 7$s, $\tilde{m}_t = 1/10$, and $g = 9.8$m/s, the initial acceleration is given as:

$$a_0 \approx 3 \cdot \frac{0.0931}{7 \cdot (1 + 0.1 + 0.0931 + 0.1)} - g \approx 0.0309 - g \tag{19}$$

Where because of the unit of u, 0.0308561 is in km/s^2; convert it back to m/s^2, we get:

$$a_0 \approx 30.856 - g = 30.856 - 9.8 = 21.056 \approx 2.149g \tag{20}$$

Which is close to the given answer $2.1g$ for $\tilde{m}_p = 0.1$.