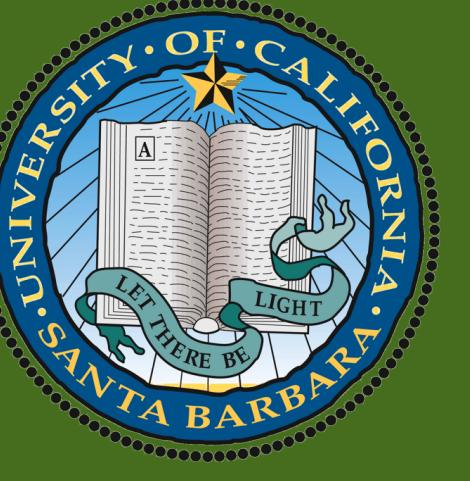


# BRAID YOUR BRAID GROUP REPRESENTATIONS

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## What's a Braid Group? Why Care?

**Def:**  $n$ -strand Braid Group  $B_n$  is generated by  $\{\sigma_1, \dots, \sigma_{n-1}\}$ , satisfying **Braid Relations**:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2; \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

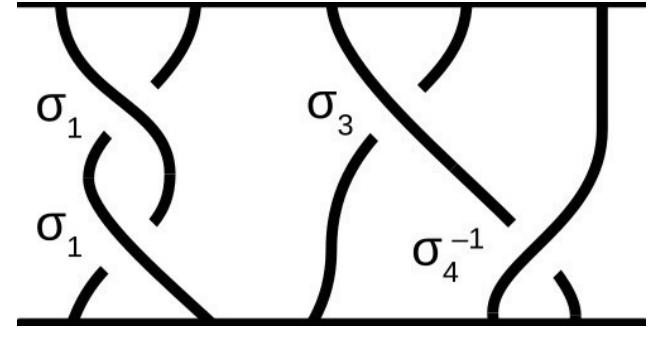


Figure: Geometric Braid of 4 Strands

**Significance:** Application in *Topological Quantum Computing*.

## $B_n$ as Mapping Class Groups

**Def:** Let  $D_n$  be an  $n$ -punctured disk. Its *Mapping Class Group*  $\mathfrak{M}(D_n)$  collects isotopic classes of homeomorphisms fixing  $\partial D$ .  
**Ex:** The  $i^{\text{th}}$  Half Twist  $\tau_i \in \mathfrak{M}(D_n)$  swaps  $i^{\text{th}}, (i+1)^{\text{th}}$  punctures.

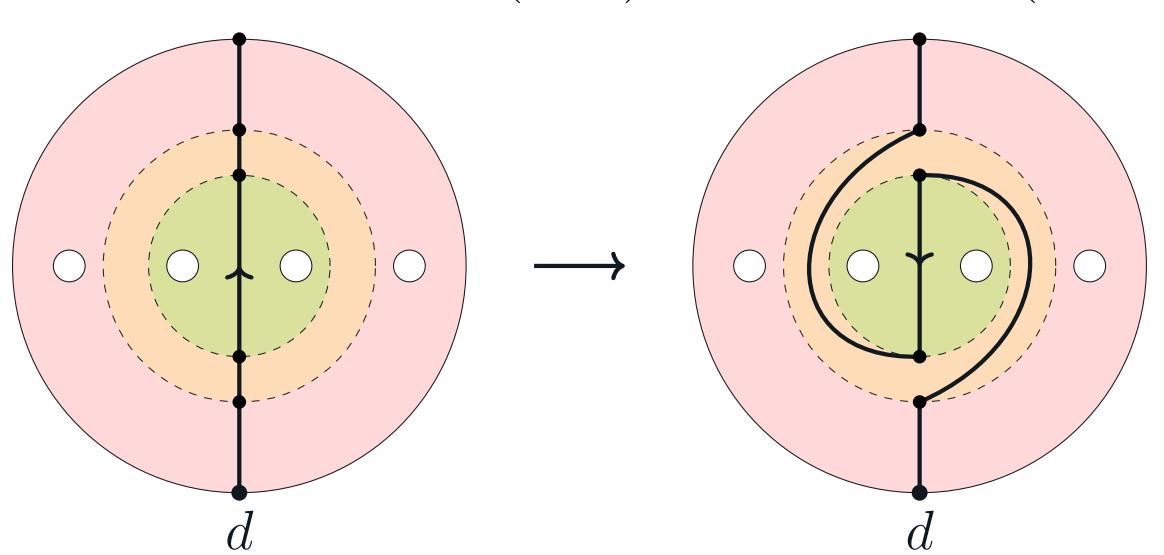


Figure:  $n = 4$ , Half Twist  $\tau_2$ 's Action on  $D_4$

**Property:**  $B_n \cong \mathfrak{M}(D_n)$ , by  $\sigma_i \mapsto \tau_i$ .

## Braid Automorphism

Fix  $d \in \partial D$ , the **Fundamental Group**  $\pi_1(D_n, d) \cong F_n$ , a *Degree- $n$  Free Group* generated by the  $n$  loops.

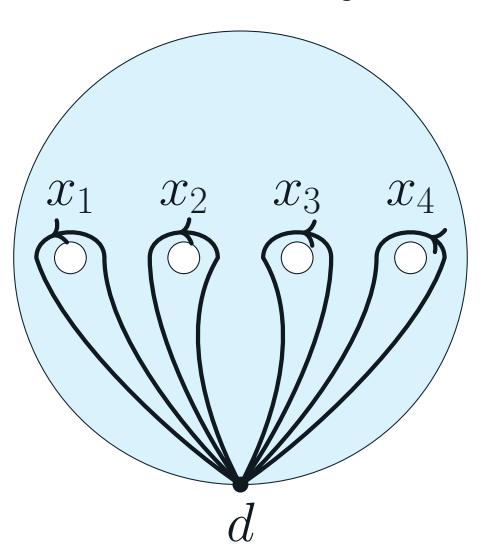


Figure: Loops Generating  $\pi_1(D_4, d)$

Maps in  $\mathfrak{M}(D_n)$  generate **Braid Automorphisms** on  $\pi_1(D_n, d)$ .

**Ex:** Half Twist's action on  $\pi_1(D_n, d)$ :

$$(\tau_i)_*(x_j) = \begin{cases} x_{i+1} & j = i \\ x_{i+1}x_i x_{i+1}^{-1} & j = i+1 \\ x_i & \text{Otherwise} \end{cases}$$

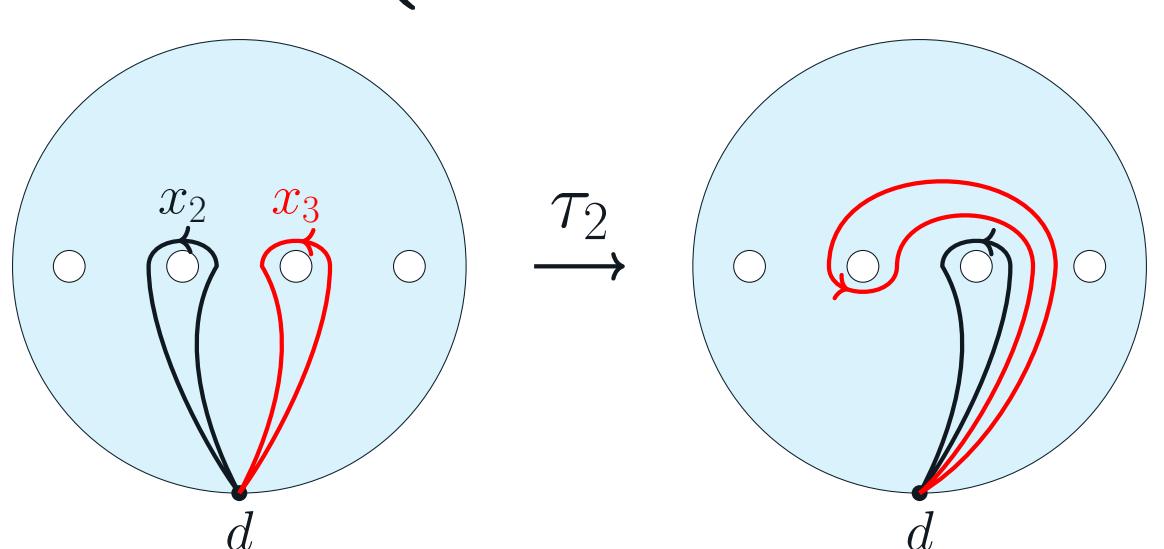


Figure:  $\tau_2$  Action on  $\pi_1(D_4, d)$

## Reduced Burau Representation

$\psi_n^r : B_n \rightarrow \mathrm{GL}_{n-1}(\mathbb{Z}[t^\pm])$  satisfies:

$$\sigma_1 \mapsto \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}, \quad \sigma_{n-1} \mapsto \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix}, \quad \sigma_i \mapsto \begin{pmatrix} I_{i-2} & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & I_{n-i-2} \end{pmatrix}$$

## EX: Burau Representation on $D_4$

$D_4$  Continuously Deforms into 4 circles join at a point ( $\bigvee_{i=1}^4 S^1$ ).

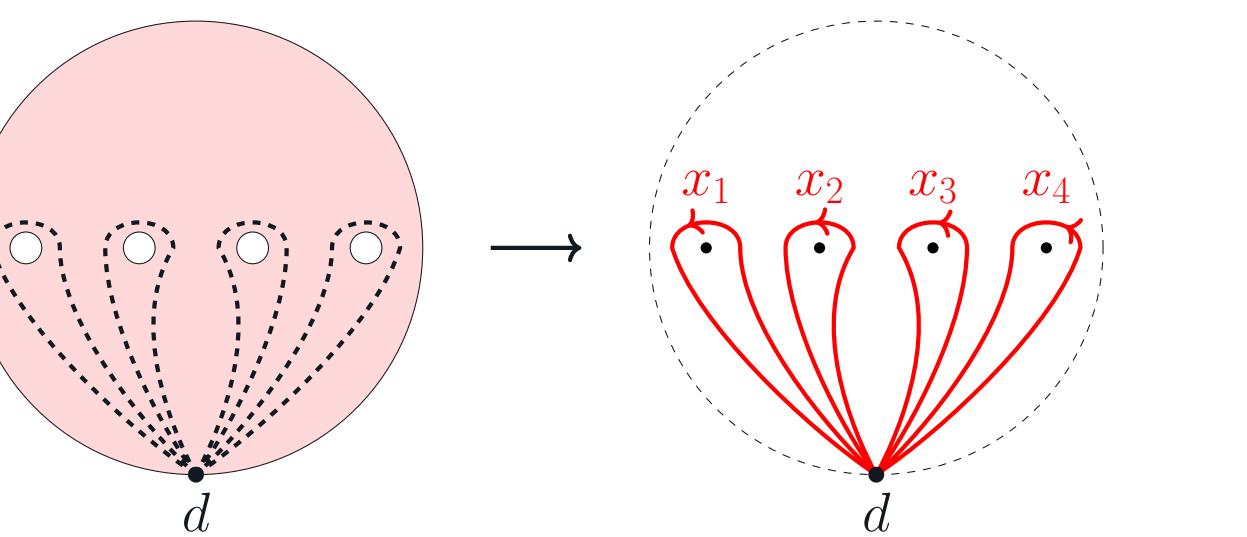


Figure: Deformation Retraction of  $D_4$  to  $\bigvee_{i=1}^4 S^1$

Let  $S^{(4)} := \bigvee_{i=1}^4 S^1$ , below is its *Infinite Cyclic Covering Space*  $\tilde{S}^{(4)}$ :

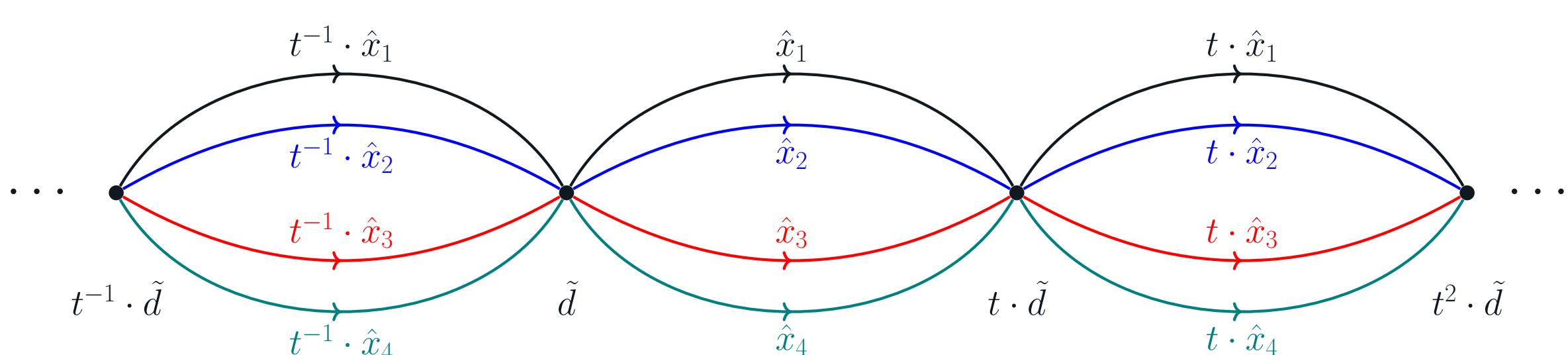


Figure: Infinite Cyclic Cover  $\tilde{S}^{(4)}$

**Rmk 1:**  $t$  shifts  $\tilde{S}^{(n)}$  by Degree 1.

**Rmk 2:**  $\exists$  Covering Map  $p : \tilde{S}^{(4)} \rightarrow S^{(4)}$ , each  $p(t^k \cdot \hat{x}_i) = x_i$ , and  $p(t^k \cdot \tilde{d}) = d$ .

**Rmk 3:** "Base Loops"  $\ell_i := \hat{x}_{i+1} \cdot \hat{x}_i^{-1}$  (counterclockwise) for  $1 \leq i \leq 3$ :

- $-\ell_i$  = clockwise version of  $\ell_i$
- $t^k \cdot \ell_i$  = degree  $k$  right shift of  $\ell_i$

**Property:** all  $\ell_i$ 's **Integer Laurent Polynomial** combinations form  $\tilde{S}^{(4)}$ 's *First Homology*,  $H_1(\tilde{S}^{(4)})$  as a free  $\mathbb{Z}[t^\pm]$ -module with basis  $\{\ell_1, \ell_2, \ell_3\}$ .

## Action on Homology $H_1(\tilde{S}^{(4)})$

**Recall:** Braid Automorphism  $(\tau_2)_*$  satisfies  $(\tau_2)_*(x_2) = x_3$ , and  $(\tau_2)_*(x_3) = x_3 \cdot x_2 \cdot x_3^{-1}$ . Which, it acts on base loops of  $\tilde{S}^{(4)}$ :

$$\text{EX: } \ell_2 = \hat{x}_3 \cdot \hat{x}_2^{-1} \mapsto (\hat{x}_3 \cdot (t \cdot \hat{x}_2) \cdot (t \cdot \hat{x}_3^{-1})) \cdot \hat{x}_3^{-1} = -t \cdot \ell_2.$$

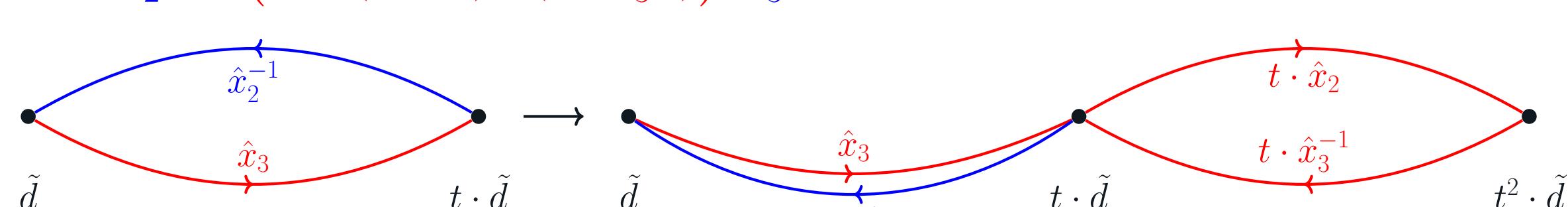


Figure:  $\ell_2$  Maps to  $-t \cdot \ell_2$  via  $\tau_2$

Doing this for each  $\ell_i$ , put into matrix with basis  $\{\ell_i\}$ , we recover the Representation.

## Gassner Representation

**Def:** The *Pure Braid Group*  $P_n \subset B_n$ , is the kernel of the map  $B_n \rightarrow S_n$  by  $\sigma_i \mapsto (i, i+1)$ .

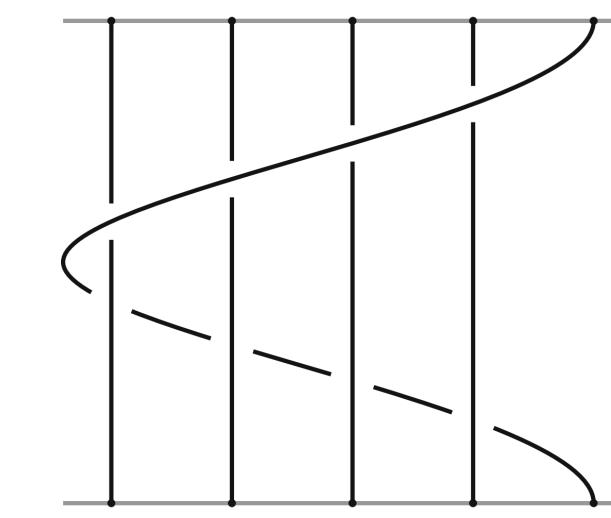


Figure: Example of a Pure Braid in  $P_5$

If consider the *Integer Lattice*  $\mathbb{Z}^n$  together with the shifts of connection segments  $\hat{x}_i$  connecting  $\bar{0}$  to  $e_i$  (the elementary basis of  $\mathbb{Z}^n$ ), it again forms a Covering Space of  $S^{(n)} := \bigvee_{i=1}^n S^1$ , with covering map  $\bar{d} \mapsto d$  for all  $\bar{d} \in \mathbb{Z}^n$ , and  $\hat{x}_i \mapsto x_i$ .

Then, each *Pure Braid*  $\rho \in P_n$  (with  $P_n \subset \mathfrak{M}(D_n) \cong B_n$ ) lifts to an action on the *First Homology* of the covering space, and forms the *Gassner Representation*.

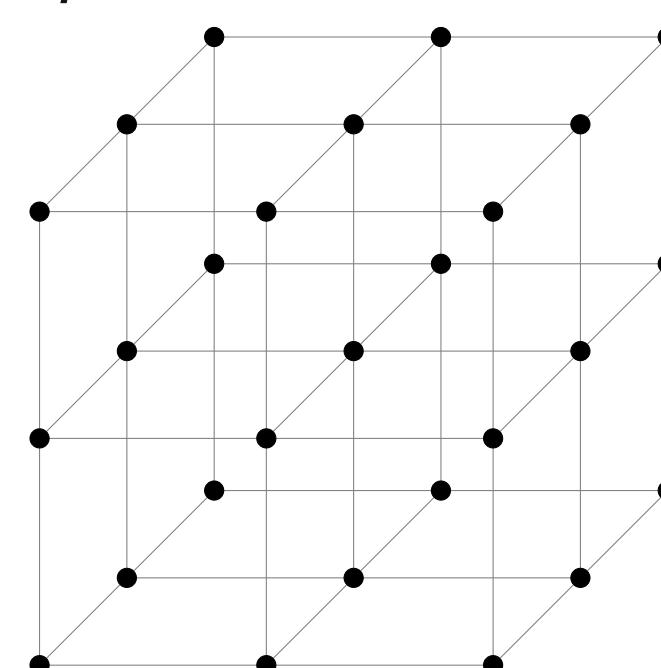


Figure: Gassner Representation's Covering Space of  $S^{(3)}$

## Future Directions

### Open Problems:

1. For  $n = 4$ , is Burau Representation Faithful?
2. For  $n \geq 4$ , is Gassner Representation Faithful?

**Future Direction:** Continue on Studying kernels of the Representations, or other homological representations of Braid Groups.

## Acknowledgement & Sources

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### Source:

- Braid Groups (Christian Kassel, Vladimir Turaev)
- Algebraic Topology (Hatcher)
- Braids, Links, Mapping Class Groups (Joan Birman)