Representation Theory Chap 1 Questions

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1 D

Question 1.1: Let $L = \mathbb{R}^3$. Define $[x, y] = x \times y$ the cross product, and verify that L is a Lie algebra. Write down the structure constnats relative to the usual basis of $\mathbb{R}^3$.

Pf:

Cross product is both bilinear and antisymmetric, hence the first two axioms are satisfied. It remains to check the Jacobi's Identity.

Given any $x, y, z \in \mathbb{R}^3$, they can be written as linear combinations of e_1, e_2, e_3 (the standard basis), hence it suffices to check the Jacobi's identity for any e_i, e_j, e_k (where $i, j, k \in \{1, 2, 3\}$).

If i, j, k are all distinct, then since the cross product of any two produces the third one, then, $[e_i, [e_j, e_k]] = \pm[e_i, e_i] = 0$, which satisfies the Jacobi's identity.

If i, j, k are all the same, then it is trivial (since one entry would provide 0, so Jacobi's identity is trivially truw).

If given 3-tuple i, i, j, we get:

$$[e_i, [e_i, e_j]] + [e_i, [e_j, e_i]] + [e_j, [e_i, e_i]] = [e_i, [e_i, e_j]] - [e_i, [e_i, e_j]] = 0 \tag{1.1}$$

Hence, the basis satisfies Jacobi's identity, which provides that in general cross product satisfies it.

Now, given standard basis e_1, e_2, e_3, the structure constant is given by:

$$a_{12}^3 = 1 = -a_{21}^3, \quad a_{23}^1 = 1 = -a_{32}^1, \quad a_{31}^2 = 1 = -a_{13}^2 \tag{1.2}$$

The reversion rule applies, and if a_{ij}^k has $k = i$ or $k = j$ or $i = j$, the constant is 0.

2 D

Question 2.1: Verify that the following equations and those implied by (L1) (L2) define a Lie algebra structure on a three dimensional vector space with basis (x, y, z): $[x, y] = z$, $[x, z] = y$, $[y, z] = 0$.

Given any $a, b, c, d, e, f \in F$, consider the following:

$$\begin{aligned}[ax+by+cz, dx+ey+fz] &= ae[x,y]+af[x,z]+bd[y,x]+bf[y,z]+cd[z,x]+ce[z,y] \\ &= (af-cd)y+(ae-bd)z\end{aligned} \tag{2.1}$$

Which, let a_1, a_2, a_3 be the components of u, b_1, b_2, b_3 be the components of v, and c_1, c_2, c_3 be the components of w, we get:

$$\begin{aligned}[u,[v,w]] &= [a_1x+a_2y+a_3z, (b_1c_3-b_3c_1)y+(b_1c_2-b_2c_1)z] \\ &= a_1(b_1c_2-b_2c_1)y+a_1(b_1c_3-b_3c_1)z\end{aligned} \tag{2.2}$$

$$[v,[w,u]] = b_1(c_1a_2-c_2a_1)y+b_1(c_1a_3-c_3a_1)z \tag{2.3}$$

$$[w,[u,v]] = c_1(a_1b_2-a_2b_1)y+c_1(a_1b_3-a_3b_1)z \tag{2.4}$$

Which, adding all three terms, it turns out to be 0. So, Jacobi's Identity is satisfied, it is a Lie algebra.

3 D

Question 3.1: Given ordered bases $\left\{x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$ be an ordered basis for $\mathfrak{sl}(2,F)$. Compute the matrices of $\mathrm{ad}x, \mathrm{ad}h$, and $\mathrm{ad}y$ relative to this basis.

Pf:

FIrst, for $\mathrm{ad}x$, havng the input of x, h, y provides the follow:

$$\mathrm{ad}x(x) = [x,x] = 0 \tag{3.1}$$

$$\mathrm{ad}x(h) = [x,h] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -2x \tag{3.2}$$

$$\mathrm{ad}x(y) = [x,y] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = h \tag{3.3}$$

Which, in the ordered basis, the matrix is given by:

$$\mathcal{M}(\mathrm{ad}x) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.4}$$

For $\mathrm{ad}h$, the input x, h, y provides:

$$\mathrm{ad}h(x) = [h,x] = -[x,h] = 2x \tag{3.5}$$

$$\mathrm{ad}h(h) = [h,h] = 0 \tag{3.6}$$

$$\mathrm{ad}h(y) = [h,y] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -2y \tag{3.7}$$

Hence, the matrix of ad$h$ is provided as:

$$\mathcal{M}(\mathrm{ad}h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{3.8}$$

For ad$y$, the input x, h, y provides:

$$\mathrm{ad}y(x) = [y, x] = -[x, y] = -h \tag{3.9}$$

$$\mathrm{ad}y(h) = [y, h] = -[h, y] = 2y \tag{3.10}$$

$$\mathrm{ad}y(y) = 0 \tag{3.11}$$

Which, it has the following matrix:

$$\mathcal{M}(\mathrm{ad}y) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \tag{3.12}$$

4 D

Question 4.1: Find a linear Lie algebra isomorphic to the nonabelian two dimensional algebra constructed in (1,4) (i.e. $[x, y] = x$, given the basis $x, y \in V$).

Given V a 2-dimensional lie algebra, consider its adjoint representation: It is a linear lie algebra, and for any $v = ax + by \in V$, for any $z \in V$, the adjoint $\mathrm{ad}v(z) = [v, z] = a[x, z] + b[y, z] = a(\mathrm{ad}x)(z) + b(\mathrm{ad}y)(z)$. Hence, ad$x$, ad$y$ span the adjoint representation.

First, to verify it's two-dimension, suppose some linear combination $a(\mathrm{ad}x) + b(\mathrm{ad}y) = 0$, then for all $z \in V$, we have $a[x, z] + b[y, z] = 0$, which plug in $z = x$ and $z = y$, we would get $b = 0$ and $a = 0$ respectively, so the two are linearly independent, which further shows that they're basis of the adjoint represention (which has 2-dimension).

Now, consider the commutator of ad$x$ and ad$y$: For all $z \in V$, the commutator acts on it as such:

$$\begin{aligned}(\mathrm{ad}x \circ \mathrm{ad}y - \mathrm{ad}y \circ \mathrm{ad}x)(z) &= [x, [y, z]] - [y, [x, z]] = [x, [y, z]] + [y, [z, x]] \\ &= -[z, [x, y]] = -[z, x] = [x, z] = \mathrm{ad}x(z)\end{aligned} \tag{4.1}$$

Hence, the commutator provides ad$x$, showing that the map $x \mapsto \mathrm{ad}x$, $y \mapsto \mathrm{ad}y$ actually defines an isomorphism between lie algebra.

5 D

Question 5.1: Verify the asertions made in (1,2) about $\mathfrak{t}(n, F), \mathfrak{d}(n, F), \mathfrak{n}(n, F)$, and compute the dimension of each algebra, by exhibiting bases.

Pf:

$\mathfrak{t}(n, F)$ as a set of all upper triangular matrices, is a lie algebra (since multiplication of two upper triangular is upper triangular), and it has dimension $\frac{n(n+1)}{2}$ (all upper triangular entries).

$\mathfrak{d}(n, F)$ as a set of all diagonal matrices, is a lie algebra (multiplication of two diagonal matrices is diagonal), and it has dimension n (all n diagonal entries).

$\mathfrak{n}(n, F)$ as a set of all strict upper triangular is also a lie algebra based on the same reason, and has dimension $\frac{n(n-1)}{2}$ (all strict upper triangular entries).

6 D

> **Question 6.1**: Let $x \in \mathfrak{gl}(n, F)$ have n distinct eigenvalues $a_1, ..., a_n$ in F. Prove that the eigenvalues of $\mathrm{ad}x$ are precisely the n^2 scalars $a_i - a_j$ (need not be distinct).

Pf:

Let $v_1, ..., v_n \in F^n$ be the distinct eigenvectors of x, and $u_1, ..., u_n \in F^n$ be the distinct eigenvectors of x^T corresponding to $a_1, ..., a_n$ respectively (in matrix representation).

This is well-defined, because having n distinct eigenvalues makes x diagonalizable, hence there exists invertible $T \in \mathfrak{gl}(n, F)$, such that $TxT^{-1} = D$ (diagonal consists of $a_1, ..., a_n$), which the transpose $\left(T^T\right)^{-1} x^T T^T = D$, showing that x^T is also diagonalizable, with the same eigenvalues.

Consider the set matrices $\lambda_{ij} := v_i u_j^T \in \mathfrak{gl}(n, F)$ (where $1 \leq i, j \leq n$): The action of $\mathrm{ad}x$ on them becomes:

$$\begin{aligned} \mathrm{ad}x(\lambda_{ij}) = [x, \lambda_{ij}] = x\left(v_i u_j^T\right) - \left(v_i u_j^T\right)x = a_i\left(v_i u_j^T\right)^T - \left(x^T u_j v_i^T\right) = a_i \lambda_{ij} - \left(a_j u_j v_i^T\right)^T \\ = a_i \lambda_{ij} - a_j\left(v_i u_j^T\right) = \left(a_i - a_j\right)\lambda_{ij} \end{aligned} \tag{6.1}$$

Hence, $a_i - a_j$ is an eigenvalue for all $1 \leq i, j \leq n$.

7 D

> **Question 7.1**: Let $\mathfrak{s}(n, F) \subset \mathfrak{gl}(n, F)$ denote the scalar matrices (set of scalar multiples of the identity). If $\mathrm{char}(F) = 0$ or else a prime not dividing n, prove that $\mathfrak{gl}(n, F) = \mathfrak{sl}(n, F) \oplus \mathfrak{s}(n, F)$, with $[\mathfrak{s}(n, F), \mathfrak{gl}(n, F)] = 0$.

Pf:

Since all scalar multiples of identity commutes with all matrices in $\mathfrak{gl}(n, F)$, it is clear that $[\mathfrak{s}(n, F), \mathfrak{gl}(n, F)] = 0$ (since $\mathfrak{s}(n, F)$ is in fact the center of $\mathfrak{gl}(n, F)$).

Then, the reason why $\mathfrak{gl}(n, F) = \mathfrak{sl}(n, F) + \mathfrak{s}(n, F)$, is because if $\mathrm{char}(F) = 0$, or a prime not dividng n, then $n \in F$ is nonzero. Hence, for any $x \in \mathfrak{gl}(n, F)$, $\frac{\mathrm{tr}(x)}{n} \in F$ exists, therefore x can be decomposed as:

$$x = \left(x - \frac{\mathrm{tr}(x)}{n} I\right) + \frac{\mathrm{tr}(x)}{n} I \tag{7.1}$$

Where, $\frac{\mathrm{tr}(x)}{n} I \in \mathfrak{s}(n, F)$, and $x - \frac{\mathrm{tr}(x)}{n} I \in \mathfrak{sl}(n, F)$ because the trace is given by $\mathrm{tr}(x) - n \cdot \frac{\mathrm{tr}(x)}{n} = 0$.

Finally, it is a direct sum, because given $aI \in \mathfrak{sl}(n, F) \cap \mathfrak{s}(n, F)$ (where $a \in F$), we have $\mathrm{tr}(aI) = a \cdot n = 0$, but since F is a field, and $n \neq 0$, we must have $a = 0$. Hence, the intersection is in fact only the zero matrix, proving that the two forms a direct sum.

8 D

Question 8.1: Verify the stated dimension of D_l (already done in the notes).

9 ND

Question 9.1: When $\mathrm{char}(F) = 0$, show that each classical algebra $L = A_l, B_l, C_l, D_l$ is equal to $[L, L]$.

Pf:

Given $A_l = \mathfrak{sl}(l+1, F)$ (sets of all matrices with trace 0),

10 ND

Question 10.1: Show that A_1, B_1, C_1 are all isomorphic, while D_1 is a 1-dimensional Lie algebra. Show that B_2 is isomorphic to C_2, and D_3 to A_3. What can you say about D_2?

11 D

Question 11.1: Verify that the commutator of two derivations of an F-algebra is again a derivation, whereas the ordinary product need not be.

Pf:

Suppose $\delta, \delta' \in \mathrm{Der}(L)$ are two derivations, then for all $u, v \in L$, the following is true:

$$\delta(\delta'(uv)) = \delta(\delta'(u)v + u\delta'(v)) = (\delta\delta'(u)v + \delta'(u)\delta(v)) + (\delta(u)\delta'(v) + u\delta\delta'(v)) \tag{11.1}$$

Then, the commutator has the following behavior:

$$\begin{gathered}(\delta\delta' - \delta'\delta)(uv) = (\delta\delta'(u)v + \delta'(u)\delta(v)) + (\delta(u)\delta'(v) + u\delta\delta'(v)) \\ -(\delta'\delta(u)v + \delta(u)\delta'(v)) - (\delta'(u)\delta(v) + u\delta'\delta(v))\end{gathered} \tag{11.2}$$

$$= (\delta\delta' - \delta'\delta)(u)v + u(\delta\delta' - \delta'\delta)(v) \tag{11.3}$$

Hence, the commutator of δ, δ' is again a derivation, showing that $\mathrm{Der}(L)$ is a Lie algebra with commutator.

As a counterexample of general product (composition), consider the polynomial ring $\mathbb{R}[x, y]$ together with the derivations $\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}$ acting on the polynomials x, y respectively:

$$\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}(xy)\right) = \frac{\partial}{\partial x}(x^2) = 2x \tag{11.4}$$

But, if consider the situation when product rule applies, we get:

$$x \cdot \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}(y)\right) + \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}(x)\right) \cdot y = x + 0 = x \tag{11.5}$$

Since the two doesn't match, this example doesn't satisfy product rule, hence general product of two derivations don't necessarily produce a derivation.

12 D

Question 12.1: Let L be a Lie algebra and let $x \in L$. Prove that the subspace of L spanned by the eigenvectors of $\mathrm{ad}x$ is a subalgebra.

Pf:

Let $K \subseteq L$ be the subspace of L spanned by the eigenvectors of $\mathrm{ad}x$. To show that it's closed under the bracket operation, it suffices to show for any two distinct eigenvectors u, v of $\mathrm{ad}x$ (with eigenvalues $a, b \in F$), $[u, v] \in K$ (since every vector in K is spanned by fiitely many eigenvectors of $\mathrm{ad}x$, using bilinearity it can be broken down into multiple brackets of pairs of eigenvectors).

Given that $\mathrm{ad}x(u) = [x, u] = au$, and $\mathrm{ad}x(v) = [x, v] = bv$. Then, if consider the following using Jacobi's Identity:

$$\begin{aligned} \mathrm{ad}x([u, v]) = [x, [u, v]] &= -[u, [v, x]] - [v, [x, u]] = [u, [x, v]] - [v, au] \\ &= [u, bv] + [au, v] = (a + b)[u, v] \end{aligned} \tag{12.1}$$

Hence, $[u, v]$ is also an eigenvector of $\mathrm{ad}x$, showing that K is closed under bracket operation, hence a subalgebra of L.