Commutative Algebra Chapter 1 Problems

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Question 1.1: Exercise 1.13 (unsolved);

1. $\sqrt{I} = R \iff I = R$.
2. If ideal P is prime, then $\sqrt{P^n} = P$ for all $n \in \mathbb{N}$.

Pf:

1. $\implies$: If $\sqrt{I} = R$, then since $R = \sqrt{I} = \varphi^{-1}(\text{Nil}(R/I))$ (where φ is the projection onto R/I), then we have $\text{Nil}(R/I) = R/I$. However, if ring $S \neq (0)$, then $\text{Nil}(S) \subsetneq S$, so since $\text{Nil}(R/I) = R/I$, we must have $R/I = (0)$, showing that $I = R$. $\impliedby$: If $I = R$, it follows that $\sqrt{I} = R$.

2. Given P is a prime ideal, then for any $n \in \mathbb{N}$, any $x \in \sqrt{P^n}$ satisfies $x^k \in P^n \subseteq P$, hence by induction one can prove that $x \in P$. So, $\sqrt{P^n} \subseteq P$. Also, for all $x \in P$, $x^n \in P^n$, hence $P \subseteq \sqrt{P^n}$, eventually proving that $\sqrt{P^n} = P$.

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Question 2.1: Let x be a nilpotent element of a ring R. Show that $1 + x$ is a unit is R. Deduce that the sum of a nilpotent element and a unit is a unit.

Pf:

Given that $x \in R$ is nilpotent, then $x^k = 0$ for some $k \in \mathbb{N}$ (also, this implies that $y = -x$ is also nilpotent with the same constant).

Then, $1 + x = 1 - (-x) = 1 - y$, which consider the following equation:

$$1 = 1 - 0 = 1 - y^k = (1 - y)\left(\sum_{i=0}^{k-1} y^i\right) \tag{2.1}$$

In other words, the above term is the inverse of $1 - y = 1 + x$, which implies it is invertible.

Now, for any unit $u \in R$ and nilpotent $x \in R$, since $u + x = u(1 + u^{-1}x)$, where $u^{-1}x$ is nilpotent, then $u + x$ is product of two units, hence is a unit.

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Question 3.1: Let R be a ring. Let $f = a_0 + a_1 x + ... + a_n x^n \in R[x]$. Prove that:

1. f is a unit $\iff$ a_0 is a unit in R and $a_1, ..., a_n$ are nilpotent.
2. f is nilpotent $\iff$ $a_0, ..., a_n$ are nilpotent.
3. f is a zero-divisor $\iff$ there exists $a \neq 0$ in R such that $af = 0$.
4. f is primitive if $(a_0, ..., a_n) = R$ (as an ideal). Prove that $f, g \in R[x]$, then fg is primitive $\iff$ f and g are primitive.

Pf:

1. $\Longrightarrow$: Given $f = a_0 + a_1 x + ... + a_n x^n$ is a unit, there exists $g = b_0 + b_1 x + ... + b_m x^m$, where $fg = 1$. Which, the constant coefficient is given by $a_0 b_0 = 1$, so a_0, b_0 are both units.

   Now, we'll use induction to prove that $a_n^{r+1} b_{m-r}$ is nilpotent, given $0 \leq r \leq m$: First consider the base case $r = 0$, the coefficient for degree $(n + m - r) = n + m$ is given by $a_n b_m = 0$. Then, for $r = 1$, the coefficient for $n + m - r$ is given by $a_{n-1} b_m + a_n b_{m-1} = 0$, multiply by a_n on both sides, we get:

   $$a_{n-1} b_m a_n + a_n^2 b_{m-1} = 0 \Longrightarrow a_n^2 b_{m-1} = 0 \tag{3.1}$$

   Now, suppose for given $0 \leq r < m$, the equation is true, then for $r + 1$, we get the coefficient of degree $(n + m - (r + 1))$ be as follow:

   $$\sum_{\max\{0, n-(r+1)\} \leq i \leq n} a_i b_{n+m-(r+1)-i} = 0 \tag{3.2}$$

   Which, multiply by a_n^{r+1}, since $n - (r + 1) \leq i \leq n$, then $n \leq r + 1 + i \leq n + r + 1$, hence the coefficient $b_{m-(r+1+i-n)}$ has $0 \leq r + 1 + i - n \leq r + 1$, which for ever index i with this expression being at most r, by induction hypothesis, $a_n^{r+1} b_{m-(r+1+i-n)} = 0$, hence every term (besides when the expression is $r + 1$) gets annihilated. So, eventually we get:

   $$r + 1 + i - n = r + 1 \Longrightarrow i = n \Rightarrow a_n \cdot a_n^{r+1} b_{n+m-(r+1-n)} = 0 \Longrightarrow a_n^{r+2} b_{m-(r+1)} = 0 \tag{3.3}$$

   This completes the induction.

   Hence, for $r = m$, we get $a_n^{m+1} b_0 = 0$, because b_0 is a unit, then a_n is in fact nilpotent, which $-a_n x^n$ is also nilpotent.

   By Question 2.1, $f - a_n x^n$ is still a unit, and with degree $n - 1$. Then, the other non-constnat coefficients can be proven to be nilpotent by induction.

2. $\Longrightarrow$: If f is nilpotent, then $f^k = (a_0 + a_1 x + ... + a_n x^n)^k = 0$ for some $k \in \mathbb{N}$. Which, the leading term is $a_n^k (x^n)^k = 0$, hence $a_n^k = 0$, or a_n is nilpotent. Since $a_n x^n$ is also nilpotent, then $f - a_n x^n$ is nilpotent (with $\deg(f - a_n x^n) = n - 1$). So, since the base case $f = a_0$ is nilpotent implies a_0 is nilpotent, by induction we can show that each a_i is nilpotent.

   $\Longleftarrow$: If each coefficient is nilpotent, it's obvious that each degree's component is nilpotent (based on the proof above), hence f is the sum of nilpotent elements, which is nilpotent.

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Question 4.1: Generalize the results in Question 3.1 to polynomial rings with several variables.

Pf:

All the setup can be done through induction. For base case $n = 1$ it is verified in Question 3.1. Now, if all the statements are true for $n - 1$ (where $n \in \mathbb{N}$), then since $R[x_1, ..., x_n] = K[x_n]$, where $K = R[x_1, ..., x_{n-1}]$. Then:

1. $f \in K[x_n]$ is a unit $\iff$ constant coefficient $f_0 \in K = R[x_1, ..., x_{n-1}]$ is unit, and the other coefficients $f_1, ..., f_k \in K$ are nilpotent. Which, since the constant of $f \in R[x_1, ..., x_n]$ is provided in f_0, while other non-constant terms' coefficients scattered in $f_1, ..., f_k$ (and also the non-constant coefficients in f_1 as a member of polynomial ring $R[x_1, ..., x_{n-1}]$), by induction hypothesis, this happens iff the constant coefficient of f (also the constant coefficient of f_0) is unit, while the other terms are nilpotent.

2. $f \in K[x_n]$ is nilpotent $\iff$ all coefficients $f_0, ..., f_k \in R[x_1, ..., x_{n-1}]$ is nilpotent. Again, by induction hypothesis, all the coefficients of $f_0, ..., f_k$ in R (also the coefficients of f) must be nilpotent.

3. $f \in K[x_n]$ is a zero divisor $\iff$ all its coefficients $f_0, ..., f_k \in R[x_1, ..., x_{n-1}]$ all have some $a_0, ..., a_k \in R$, such that for each index i, $a_i f_i = 0$; which, f multiplied by $a_0 ... a_k$ would make all coefficients $f_i \in R[x_1, ..., x_{n-1}]$ go to 0, hence $a = a_0 ... a_k$ is the desired element with $af = 0$.

4. $fg \in K[x_n]$ is primitive $\iff$ f and g are primitive in $K[x_n]$. Which, their coefficients in $R[x_1, ..., x_{n-1}]$ must have gcd being 1. However, the gcd of all its coefficients in R also divides all their coefficients in $R[x_1, ..., x_{n-1}]$, hence the gcd in R is limited to be 1.

5 D

Question 5.1: In the ring $R[x]$, the Jacobson radical is equal to the nilradical.

Pf: Let N be the nilradical, and J be the Jacobson radical of $R[x]$. Since J is the intersection of all maximal ideals, N is the intersection of all prime ideals, while maximal ideals are prime, then $N \subseteq J$ (N could be the intersection of more ideals, since prime is not necessarily maximal).

Now, if $f \in J$, by definition $1 - f$ is a unit. This happens $\iff$ every non-constant coefficients of $1 - f$ is nilpotent (they are given by $-a_1, ..., -a_n$, the negaitve non-constant coefficients of f), while the constant coefficient of f, say a_0 satisfies $1 - a_0$ being a unit (since $1 - a_0$ is the constant coefficient of $1 - f$). So, all the non-constant coefficients of f are nilpotent.

Then, since $1 - yf$ is also a unit for all $y \in R[x]$, consider $y = 1 + x$: The polynomial $(1 + x)f$ is given as follow:

$$(1 + x)f = a_0 + \sum_{i=1}^{n} (a_{i-1} + a_i)x^i + a_n x^{n+1} \tag{5.1}$$

Then, $1 - (1 + x)f$ has $-(a_0 + a_1)$ as the degree 1 coefficient. Since, $1 - (1 + x)f$ is a unit, this enforces $-(a_0 + a_1)$ to be nilpotent; and since a_1 is nilpotent, a_0 must also be nilpotent (since $\mathrm{Nil}(R)$ is an ideal, which forms a group under addition).

So, because every coefficients are nilpotent, f is nilpotent, hence $f \in N$, showing the other inclusion $J \subseteq N$.

6 ND

Question 6.1: Let R be a ring, and consider $R[[x]]$ (formal power series ring). Show that:

1. f is a unit in $R[[x]] \iff a_0$ is a unit in R.
2. If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true?
3. f belongs to the Jacobson radical of $R[[x]] \iff a_0$ belongs to the Jacobson radical of R.
4. The contraction of a maximal ideal M of $R[[x]]$ is a maximal ideal of R, and M i generated by M^c and x.
5. Every prime ideal of R is the contraction of a prime ideal of $R[[x]]$.

Pf:

1. $\Longrightarrow$: If f is a unit in $R[[x]]$, there exists $g \in R[[x]]$, with $fg = 1$. Then, the constant coefficient 1 is given by the multiplication ofconstant coefficients of f and g, showing that a_0 (constant coefficient of f) is a unit.

   $\Longleftarrow$: If a_0 is a unit in R, our goal is to find $g = \sum_{n=0}^{\infty} b_n x^n$, where $fg = 1$.

   First, it's clear that $b_0 = a_0^{-1}$. Now, for b_1, since we want the degree 1 coefficient of fg to be 0, and the degree 1 coefficient is given b $a_0 b_1 + a_1 b_0$, then set $b_1 = -a_0^{-1} a_1 b_0$, we get the desired result.

   Inductively, when $b_0, ..., b_{n-1}$ all have fixed expression using the collections of a_n, since degree n coefficient of fg is given by $\sum_{i=0}^{n} a_i b_{n-i}$, then if we want the expression to be 0, we can set b_n as follow:

$$a_0 b_n + \sum_{i=1}^{n} a_i b_{n-i} = 0, \quad b_n = -a_0^{-1} \sum_{i=1}^{n} a_i b_{n-i} \tag{6.1}$$

   So, there exists an expression of g, where $fg = 1$, showing that f is a unit.

2.

7 ND

Question 7.1: A ring R is such that every ideal not contained in the nilradical contains a nonzero idempotent (an elemenet e with $e^2 = e \neq 0$). Prove that the nilradical and the Jacobson radical of R are equal.

Pf:

Let N, J represent the niradical and Jacobson radical respectively. It is clear that $N \subseteq J$ by definition.

To prove that $J \subseteq N$ by contradiction, suppose the contrary that $J \nsubseteq N$, by assumption there exists $e \in J$ with $e^2 = e$. Now, consider the ideal (e):

8 ND

Question 8.1: Let R be a ring in which every element satisfies $x^n = x$ for some $n > 1$. Show that every prime ideal in R is maximal.

Pf:

First, $\mathrm{Nil}(R) = (0)$: If $x \in \mathrm{Nil}(R)$, then since there exist $n, k \in \mathbb{N}$, with $x^n = x$ and $x^k = 0$ (where we demand k to be the smallest, and $n > 1$ by assumption), there are two cases to consider:

1. If $k \leq n$, then $x^n = 0$, showing that $x = 0$.
2. if $k > n$, then $k = ln + r$ for some $l, r \in \mathbb{N}$, and $0 \leq r < n$. Which, the following is satisfiesd:

$$x^k = x^{ln+r} = (x^n)^l \cdot x^r = x^{l+r} = 0 \tag{8.1}$$

Notice that $l + r < ln + r = k$ by assumption that $n > 1$, so we reach a contradiction (since there exists $l + r < k$, with $x^{l+r} = 0$).

Hence, the second case doesn't exist, where the first case shows that $\mathrm{Nil}(R) = (0)$.

9 D

Question 9.1: Let $R \neq 0$ be a ring. Show that the set of prime ideals of R has minimal elements with respect to inclusion.

Pf:

We'll prove by Zorn's Lemma, where let A be the set of all prime ideals, and the Partial Order given by $P_1 \succeq P_2$ iff $P_1 \subseteq P_2$.

Let $C \subseteq A$ be a chain, and let $P_C = \bigcap_{P \in C} P$. It is clear that P_C is an ideal, and if $P_C \in A$, then P_C is an upper bound of C. So, it suffices to show that $P_C \in A$ (or P_C is a prime ideal).

Suppose $x, y \in R$ satisfies $xy \in P_C$, then since for any prime ideal $P \in C$, $xy \in P$, then either $x \in P$ or $y \in P$. If all $P \in C$ contains x (or y), then we're done. Now, if some contains x and some contains y, consider the subchain $C_x := \{P \in C \mid x \in P\}$:

- If C_x is comaximal in C (in a set theoretic), then for every $P \in C$, there exists $P_x \in C_x$, where $P_x \succeq P$, so $P_x \subseteq P$, hence $x \in P$, showing that $x \in P_C$.
- Else if C_x is not comaximal in C, then there exists $P \in C$, where all $P_x \in C_x$ has $P \succneq P_x$ (which $P \notin C_x$). Hence, $y \in P$, showing that all $P_x \in C_x$ has $P \subsetneq P_x$, or $y \in P_x$. So, given $P \in C$, regardless of its containment in C_x, we have $y \in P$, showing that $y \in P_C$.

The above statements show that P_C is prime, hence $P_C \in A$, every chain has an upper bound. Then, by Zorn's Lemma, this POset has a maximal element, which is the minimal elements with respect to inclusion.

10 D

Question 10.1: Let $I \subsetneq R$ be an ideal. Show that $I = \sqrt{I} \iff I$ is an intersection of prime ideals.

$\Longrightarrow$: If $\sqrt{I} = I$, since the projection map $\varphi : R \twoheadrightarrow R/I$ satisfies the following:

$$I = \sqrt{I} = \varphi^{-1}(\mathrm{Nil}(R/I)) = \bigcap_{\overline{P} \subset R/I \text{ prime}} \varphi^{-1}\left(\overline{P}\right) = \bigcap_{I \subseteq P \subset R \text{ prime}} P \tag{10.1}$$

Which is an intersection of prime ideals.

$\Longleftarrow$: Suppose $\{P_i\}_{i \in \mathrm{A}}$ is a collection of prime ideals, and define $I := \bigcap_{i \in \mathrm{A}} P_i$. Then, for all $x \in \sqrt{I}$, since there exists $n \in \mathbb{N}$, with $x^n \in I$, because $x^n \in P_i$ for all index $i \in \mathrm{A}$, then $x \in P_i$, hence $x \in I$, showing that $\sqrt{I} \subseteq I$. Since the other inclusion is trivially true, $\sqrt{I} = I$.

11 D

Question 11.1: Let R be a ring, $\mathrm{Nil}(R)$ be its nilradical. Show that the following are equivalent:

1. R has exactly one prime ideal.
2. Every element of R is either a unit or nilpotent.
3. $R/\mathrm{Nil}(R)$ is a field.

$1 \Longrightarrow 2$: Suppose R has precisely one prime ideal, then since $\mathrm{Nil}(R)$ is the intersection of all prime ideals, $\mathrm{Nil}(R) = P$ (the prime ideal). This also enforces $\mathrm{Nil}(R)$ to be maximal (since every commutative ring has a maximal ideal, and all maximal ideal is prime).

Now, suppose $u \in R \setminus \mathrm{Nil}(R)$ (i.e. not nilpotent), then since $\mathrm{Nil}(R) \subsetneq \mathrm{Nil}(R) + (u)$, then $\mathrm{Nil}(R) + (u) = R$, showing that $1 = ku + x$ for some $k \in R$ and $x \in \mathrm{Nil}(R)$. Notice that $-x$ is nilpotent, which $1 - x$ is a unit, hence $1 - x = ku$, showing that ku is a unit, which u is a unit.

Hence, every element of R is either a unit or nilpotent.

$2 \Longrightarrow 3$: Suppose every element is either a unit or nilpotent, then for all $\overline{u} \in R/\mathrm{Nil}(R)$ (with $\overline{u} \coloneqq u \bmod \mathrm{Nil}(R)$) that is nonzero, since u is a unit, then inherantly, $\overline{u}$ is also a unit in $R/\mathrm{Nil}(R)$, showing that it is a field.

$3 \Longrightarrow 1$: Suppose $R/\mathrm{Nil}(R)$ is a field, then $\mathrm{Nil}(R)$ is maximal. Now, suppose P is a prime ideal, then because $\mathrm{Nil}(R) \subseteq P \subsetneq R$, then this enforces $\mathrm{Nil}(R) = P$. Hence, there is only one prime ideal, namely $\mathrm{Nil}(R)$.

12 ND

Question 12.1: A ring R is a *Boolean Ring* if $x^2 = x$ for all $x \in R$. In a boolean ring R, show that:

1. $2x \coloneqq x + x = 0$ for all $x \in R$.
2. Every prime ideal P is maximal, and R/P is a field with two elements.
3. Every finitely generated ideal in R is principal.

Pf:

1. For all $x \in R$, since $x^2 = x$, we have $x + 1 = (x+1)^2 = x^2 + 2x + 1 = x + 2x + 1$, hence after cancellation, $2x = 0$.
2. Based on Question 8.1, since all element $x \in R$ has some $n > 1$, with $x^n = x$ (in this case, $n = 2$), then all prime ideal P is maximal, showing that R/P is a field.

   Now, suppose $x \in R$ satisfies $\overline{x} \in R/P$ is nonzero, then since $(\overline{x})^2 = \overline{x}$, then it is a root of the polynomial $y^2 - y \in R/P[y]$. Since this is a UFD, then there exists only two solution, namely 0 and 1. because $\overline{x} \neq 0$ by assumption, then $\overline{x} = 1$. Hence, $R/P \cong \mathbb{Z}_2$.
3. Suppose $I = (a_1, ..., a_n)$ is a finitely generated ideal, we claim that everything is generated by $a_1 + ... + a_n$.

13 ND

Question 13.1: A local ring contains no idempotent other than $0, 1$.

Pf:

Recall that a local ring R has exactly one maximal ideal, say M. Now, suppose $e \in R$ is idempotent, then in the quotient ring R/M (which is a field), since it is also a root of the polynomial $x^2 - x \in R/M[x]$, then $e \equiv 0 \bmod M$, or $e \equiv 1 \bmod M$.

For the first case, we have $(1+e)^2 = 1 + 2e + e^2 = 1 + 3e$

For the second case, we have $e = 1 + m$ for some $m \in M$, hence $m = e - 1$. Which, $m^2 = e^2 - 2e + 1 = -e + 1 = -(e-1) = -m$, showing that $\left(m^2\right)^2 = m^2$