Braid Groups, and their Representations

University of California Santa Barbara, College of Creative Studies

Introduction

Braid Group formulates the algebraic / topological relation of braids. One center of studies is the Representations and their kernels. Here we'll briefly introduce two - Burau and Gassner Representation.

Braid Groups & Mapping Class Groups

Def: An n strands *Braid Group* B_n is generated by $\{\sigma_1, ..., \sigma_{n-1}\}$, satisfying *Braid Relations*:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$, if $|i-j| \geq 2$
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

Def: Let D_n be an n-punctured disk. Its *Mapping Class Group* $\mathfrak{M}(D_n)$ collects isotopic classes of self-homeomorphisms that fixes disk boundary ∂D.

Ex: The i^{th} *Half Twist* $\tau_i \in \mathfrak{M}(D_n)$ swaps the i^{th} and $(i+1)^{\text{th}}$ punctures, while fixing the remaining ones.

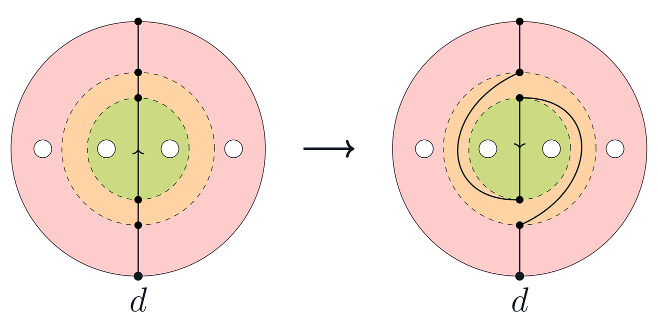

Figure: For $n=4$, Half Twist τ_2 Swapping Punctures 2 and 3

Property: $B_n \cong \mathfrak{M}(D_n)$, by $\sigma_i \mapsto \tau_i$.

Braid Automorphism

Fix $d \in \partial D$, the fundamental group $\pi_1(D_n, d)$ is generated by the n loops, each surrounding a puncture, which $\pi_1(D_n, d) = F_n(x_1, ..., x_n)$, the *Degree-$n$ Free Group*.

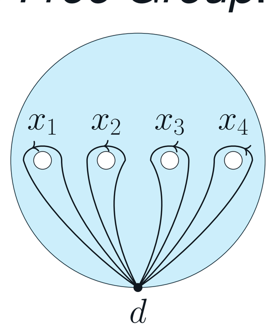

Figure: Loops Generating $\pi_1(D_4, d)$

Then, each homeomorphism in $\mathfrak{M}(D_n)$ generates a group automorphism on $\pi_1(D_n, d)$, called *Braid Automorphism*.

Ex: Half Twist's action on $\pi_1(D_n, d)$:

$$(\tau_i)_* \in \mathrm{Aut}(\pi_1(D_n, d)), \quad (\tau_i)_*(x_j) = \begin{cases} x_{i+1} & j = i \\ x_{i+1}x_ix_{i+1}^{-1} & j = i+1 \\ x_i & \text{Otherwise} \end{cases}$$

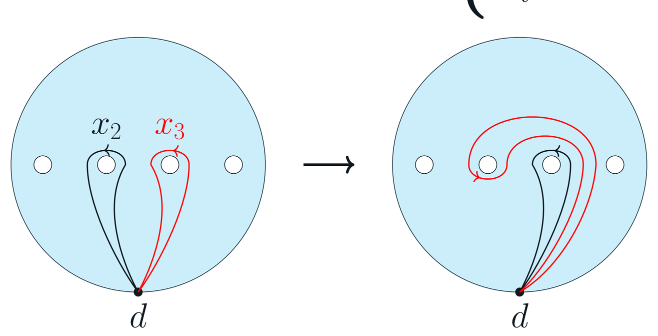

Figure: τ_2 Action on Loops in D_4

Reduced Burau Representation

$\psi_n^r : B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t^{\pm}])$ satisfies:

$$\sigma_1 \mapsto \left(\begin{array}{cc|c} -t & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & I_{n-3} \end{array}\right), \quad \sigma_{n-1} \mapsto \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & t \\ 0 & 0 & -t \end{array}\right), \quad \sigma_i \mapsto \left(\begin{array}{c|ccc|c} I_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n-i-2} \end{array}\right)$$

Ex: Homological Perspective on D_4

A 4-punctured disk D_4 can "continuously deform" into 4 circles joining at one point ($\bigvee_{i=1}^4 S^1$), $\implies$ Same Fundamental Group.

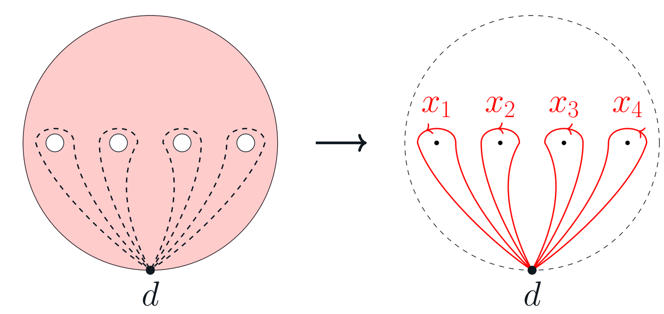

Figure: Deformation Retraction of D_4 to $\bigvee_{i=1}^4 S^1$

Let $S^{(4)} := \bigvee_{i=1}^4 S^1$, consider its *Covering Space* $\tilde{S}^{(4)}$ as below:

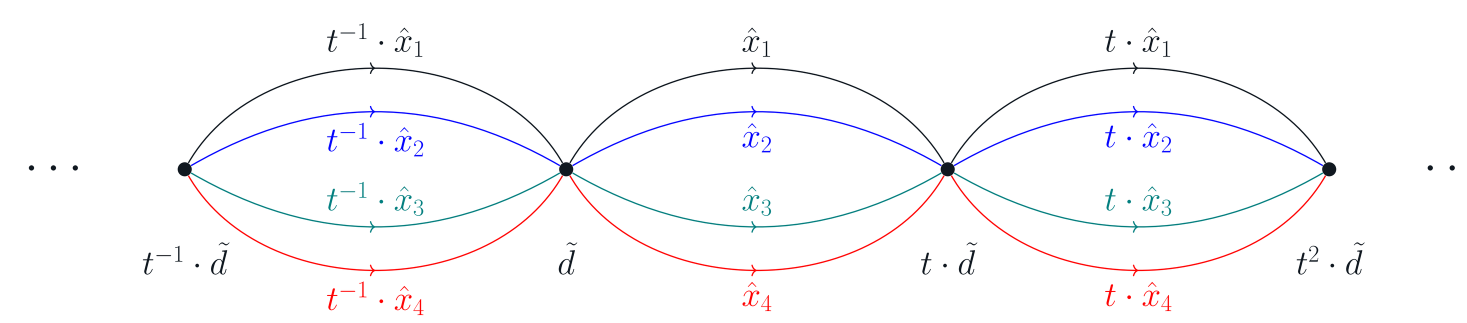

Figure: Infinite Cyclic Cover $\tilde{S}^{(4)}$

Here, t is a right shift of $\tilde{S}^{(n)}$ by degree 1:

- $t^k \cdot \tilde{d} =$ degree k right shift of $\tilde{d}$
- $t^k \cdot \hat{x}_i =$ degree k right shift of $\hat{x}_i$

There is a continuous covering map $p : \tilde{S}^{(4)} \to S^{(4)}$, each $p(t^k \cdot \hat{x}_i) = x_i$, and $p(t^k \cdot \tilde{d}) = d$.

Define the "Base Loops" $\ell_i := \hat{x}_{i+1} \cdot \hat{x}_i^{-1}$ (counterclockwise) for $1 \leq i \leq 3$:

- $-\ell_i =$ clockwise version of ℓ_i
- $t^k \cdot \ell_i =$ degree k right shift of ℓ_i

Then, all "Integer Laurent Polynomial" combination of ℓ_i forms the *First Homology* of $\tilde{S}^{(4)}$, $H_1(\tilde{S}^{(4)})$ as a free $\mathbb{Z}[t^{\pm}]$-module with basis ℓ_1, ℓ_2, ℓ_3.

Source

- Braid Groups (Christian Kassel, Vladimir Turaev)
- Algebraic Topology (Hatcher)
- Braids, Links, Mapping Class Groups (Joan Birman)

Braid Group Action on Homology

Recall: braid automorphism $(\tau_2)_*$ of $\pi_1(D_4, d)$ satisfies $(\tau_2)_*(x_2) = x_3$, and $(\tau_2)_*(x_3) = x_3 \cdot x_2 \cdot x_3^{-1}$. Which, it uniquely lifts to an transformation on the ℓ_i via p:

EX: $\ell_2 = \hat{x}_3 \cdot \hat{x}_2^{-1} \mapsto (\hat{x}_2 \cdot (t \cdot \hat{x}_2) \cdot (t \cdot \hat{x}_3^{-1})) \cdot \hat{x}_2^{-1} = -t \cdot \ell_2$.

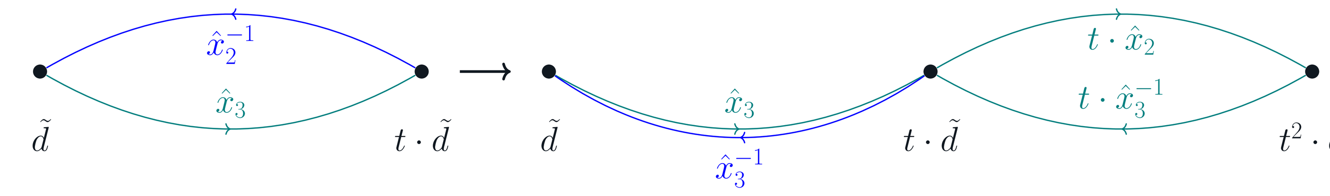

Figure: ℓ_2 (Counterclockwise) Maps to $-t \cdot \ell_2$ (Right Shift by degree 1, Clockwise)

Doing this for each ℓ_i, put into matrix form with basis $\{\ell_i\}$, we recover the Representation.

Gassner Representation (Need Diagram of Deformation Retract)

Def: The *Pure Braid Group* $P_n \subset B_n$, is the kernel of the map $B_n \to S_n$ by $\sigma_i \mapsto (i, i+1)$. Geometrically, it's the braids with every strand goes back to its original point.

If consider the *Integer Lattice* $\mathbb{Z}^n$ together with the shifts of connection segments $\hat{x}_i$ connecting $\overline{0}$ to e_i (the elementary basis of $\mathbb{Z}^n$), it again forms a Covering Space of $S^{(n)} := \bigvee_{i=1}^n S^1$, with covering map $\overline{d} \mapsto d$ for all $\overline{d} \in \mathbb{Z}^n$, and $\hat{x}_i \mapsto x_i$.

Then, each *Pure Braid* $\rho \in P_n$ (with $P_n \subset \mathfrak{M}(D_n) \cong B_n$) lifts to an action on the *First Homology* of the covering space, and forms the *Gassner Representation*.

Figure: Covering Space of $S^{(3)}$ corresponds to Gassner Representation

Conclusion & Future Directions

We've introduced some basics of Braid Groups, follow with the homological construction of Burau and Gassner Representations.

This project is intended to continue on with the study of relating kernels of the Representations, and/or other homological representations of Braid Groups.

Acknowledgement & Sources

We're genuinely thankful for the parent donors, Professor Cachadina, and Professor Casteels who made this program possible. We're also grateful for our mentor Choomno Moos with their effort and excellent guidance.