Phys 103 HW2

Zih-Yu Hsieh

July 7, 2025

1

Question 1 *Consider an underdamped oscillator. Technically, because the amplitude decreases, the motion is not periodic and there is thus no period. We can, however, something that's enough like the period that we often just call it "the period".*

(a) If the oscillator starts at $x(0) = 0$ (but with some initial velocity), show that subsequent zeroes are located at $t = \frac{n\pi}{w_1}$, $n \in \mathbb{Z}$. Defining the period as the amount of time to get two zeroes thus gives $\frac{2\pi}{w_1}$.

(b) Againg setting $x(0) = 0$, show that the local maxima in $x(t)$ do not occur at $t = \frac{(2n+1/2)\pi}{w_1}$; the maxima do not occur a quarter period after a zero, as would be the case for sinusoidal motion. On the other hand, show that the time between successive maxima is $\frac{2\pi}{w_1}$; defining the period as the amount of time between successive local maxima thus gives $\frac{2\pi}{w_1}$ as before.

Pf:

(a)

Given the oscillator has the motion $x(t) = Ae^{-\beta t}\cos(w_1 t - \phi)$, since $x(0) = A\cos(\phi) = 0$, WLOG, can guess $\phi = \frac{\pi}{2}$ (if its $n\pi + \frac{\pi}{2}$, one can always change the constant A accordingly). So, $x(t) = Ae^{-\beta t}\cos(w_1 t - \frac{\pi}{2}) = Ae^{-\beta t}\sin(w_1 t)$ (here, assume $A > 0$).

Which, if $x(t) = 0$, we have $Ae^{-\beta t}\sin(w_1 t) = 0$. Since $A, e^{-\beta t} \neq 0$, we must have $\sin(w_1 t) = 0$. As a result, $t = \frac{n\pi}{w_1}$ for $n \in \mathbb{Z}$. So, if the period is defined to be the amount of time to get two zeros, the period is $\frac{2\pi}{w_1}$ (since subsequent zeros apart each other with time $\frac{\pi}{w_1}$).

(b)

If using the same equation $x(t) = Ae^{-\beta t}\sin(w_1 t)$, to calculate where the local maxima is, we first consider up to its derivative:

$$x'(t) = -A\beta e^{-\beta t}\sin(w_1 t) + Ae^{-\beta t} w_1 \cos(w_1 t) \tag{1}$$

Which, for any $n \in \mathbb{Z}$, if plug in $t = \frac{(2n+1/2)\pi}{w_1}$ to the derivative, we get:

$$\begin{aligned} x'\left(\frac{(2n+1/2)\pi}{w_1}\right) &= -Ae^{-\beta t}\left(\beta \sin\left(w_1 \frac{(2n+1/2)\pi}{w_1}\right) - w_1 \cos\left(w_1 \frac{(2n+1/2)\pi}{w_1}\right)\right) &(2) \\ &= -Ae^{-\beta t} \cdot \beta \neq 0 &(3) \end{aligned}$$

This shows that $t = \frac{(2n+1/2)\pi}{w_1}$ is no longer local maxima, since $x'(t) \neq 0$ at this point.

However, if consider where $x'(t) = 0$, suppose $t_0 \in [0, 2\pi)$ satisfies $x'(t_0) = 0$, we get the following relation:

$$\begin{aligned}
x'(t_0) = -Ae^{-\beta t_0}(-\beta \sin(w_1 t_0) + w_1 \cos(w_1 t_0)) = 0, \quad Ae^{-\beta t_0} \neq 0 \qquad (4)\\
\implies -\beta \sin(w_1 t_0) + w_1 \cos(w_1 t_0) = 0 \qquad (5)\\
\implies \tan(w_1 t_0) = \frac{w_1}{\beta} \qquad (6)
\end{aligned}$$

Hence, for any $n \in \mathbb{Z}$, $t = \frac{n\pi}{w_1} + t_0$ all satisfies $\tan(w_1 t) = \frac{w_1}{\beta}$,which are all zeros of $x'(t_0)$.

On the other hand, if consider the second derivative, we get the following:

$$x''(t) = A(\beta^2 - w_1^2)e^{-\beta t} \sin(w_1 t) - 2A\beta w_1 e^{-\beta t} \cos(w_1 t) \qquad (7)$$

Given that t_0 is local maximuj, $x''(t_0) < 0$; however, for all $n \in \mathbb{Z}$, if n is odd ($n = 2k+1$ for some $k \in \mathbb{Z}$), the time $t = \frac{n\pi}{w_1} + t_0$ satisfies:

$$\begin{aligned}
x''\left(\frac{n\pi}{w_1} + t_0\right) &= A(\beta^2 - w_1^2)e^{-\beta t} \sin\left(w_1 \frac{n\pi}{w_1} + w_1 t_0\right) - 2A\beta w_1 e^{-\beta t} \cos\left(w_1 \frac{n\pi}{w_1} + w_1 t_0\right) \qquad (8)\\
&= A(\beta^2 - w_1^2)e^{-\beta(\frac{n\pi}{w_1} + t_0)} \sin((2k+1)\pi + w_1 t_0) - 2A\beta w_1 e^{-\beta(\frac{n\pi}{w_1} + t_0)} \cos((2k+1)\pi + w_1 t_0) \qquad (9)\\
&= -e^{-\beta\frac{n\pi}{w_1}} \left(A(\beta^2 - w_1^2)e^{-\beta t_0} \sin(w_1 t_0) - 2A\beta w_1 e^{-\beta t_0} \cos(w_1 t_0)\right) \qquad (10)\\
&= -e^{-\beta\frac{n\pi}{w_1}} \cdot x''(t_0) > 0 \qquad (11)
\end{aligned}$$

Which, showing that $t = \frac{n\pi}{w_1} + t_0$ are all local minimum (since second derivatives are all positive).

If n is even instead ($n = 2k$ for some $k \in \mathbb{Z}$), the above equation has no negative in the front, hence the second derivative remains negative, showing that $t = \frac{2k\pi}{w_1} + t_0$ are all local maxima. So, any subsequent local maxima have a time difference of $\frac{2\pi}{w_1}$, showing that the period $\frac{2\pi}{w_1}$ can also be determined by the time between subsequent local maxima.

2

Question 2

(a) *Show that the energy E of an underdamped oscillator (with $x = Ae^{-\beta t}\cos(w_1 t + \phi)$) is*

$$E = \frac{1}{2}kA^2 e^{-2\beta t}\left(1 + \frac{1}{2Q}\cos\left(2w_1 t + 2\phi - \arccos\frac{1}{2Q}\right)\right)$$

(b) *The overall exponential is easy to explain: the amplitude of the motion decreases as the damping dissipates energy, so the energy corresponding decays. Explain the physical origin of the cosine term.*

(c) $\frac{dE}{dt}$ *tells us the rate at which the oscillator is losing energy. However, we can make a couople of improvements. First, rather than expressing the rate of energy loss as* per unit time *we can also express it* per oscillation, *as the oscillation itself provides a natural timescale. Similarly, rather than the absolute amount of energy lost, we can express the fractional energy loss by dividing by E. Next, the oscillatory terms are because of an actual physical effect: the energy does wobble a little throughout a cycle. If the oscillator is only weakly damped ($Q >> 1$), we likely care much more about the overall decay after many cycles than about the slight wobble within each cycle. By time averaging over one period, we can do awway with the wobbles. These two together give us a dimensionless measure of how fast the oscillator loses on average. Show that*

$$-\left\langle \frac{1}{E}\frac{dE}{d\tilde{t}} \right\rangle = \frac{2\pi}{Q}\left(1 - \frac{1}{4Q^2}\right)^{-1/2}$$

where $\tilde{t} = \frac{t}{2\pi/w_1}$ and the angle brackets indicate that we're averaging over a duration of $\frac{2\pi}{w_1}$. One interpretation of a "high quality" oscillator is the one that loses energy very slowly; if Q is large, doubling the quality factor causes the oscillator to lose energy half as quickly.

Pf:

(a)

Recall that the elastic potential energy $U = \frac{1}{2}kx^2$, the kinetic energy is $K = \frac{1}{2}mv^2$, the natural frequency square $w_0^2 = \frac{k}{m}$, and the frequency square $w_1^2 = w_0^2 - \beta^2$.

With $x(t) = Ae^{-\beta t}\cos(w_1 t + \phi)$, we have the derivative $x'(t) = v(t) = -Ae^{-\beta t}(\beta\cos(w_1 t + \phi) + w_1\sin(w_1 t + \phi))$. Then plug in above, we get the energy as:

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 e^{-2\beta t}\cos^2(w_1 t + \phi) \tag{12}$$

$$\begin{aligned}
K = \frac{1}{2}mv^2 &= \frac{1}{2}mA^2 e^{-2\beta t}(\beta^2\cos^2(w_1 t + \phi) + w_1^2\sin^2(w_1 t + \phi) + 2\beta w_1\sin(w_1 t + \phi)\cos(w_1 t + \phi)) && (13)\\
&= \frac{1}{2}mA^2 e^{-2\beta t}(w_0^2\sin^2(w_1 t + \phi) + \beta^2(\cos^2(w_1 t + \phi) - \sin^2(w_1 t + \phi)) + \beta w_1\sin(2w_1 t + 2\phi)) && (14)\\
&= \frac{1}{2}kA^2 e^{-2\beta t}\sin^2(w_1 t + \phi) + \frac{1}{2}mA^2 e^{-2\beta t}(\beta^2\cos(2w_1 t + 2\phi) + \beta w_1\sin(2w_1 t + 2\phi)) && (15)\\
&= \frac{1}{2}kA^2 e^{-2\beta t}\sin^2(w_1 t + \phi) + \frac{1}{2}kA^2 e^{-2\beta t}\cdot\frac{1}{w_0^2}\left(\beta^2\cos(2w_1 t + 2\phi) + \beta w_1\sin(2w_1 t + 2\phi)\right) && (16)
\end{aligned}$$

Recall that the linear combination of sin, cos can be given as follow, for all $A, B \in \mathbb{R}$:

$$A\sin(x) + B\cos(x) = \sqrt{A^2 + B^2}\cos\left(x - \arctan\left(\frac{B}{A}\right)\right) \tag{17}$$

So, the kinetic energy can then be expressed as:

$$K = \frac{1}{2}kA^2e^{-2\beta t}\sin^2(w_1t + \phi) + \frac{1}{2}kA^2e^{-2\beta t}\cdot\left(\frac{\beta}{w_0}\right)^2\cdot\sqrt{1 + \left(\frac{w_1}{\beta}\right)^2}\cos\left(2w_1t + 2\phi - \arctan\frac{w_1}{\beta}\right) \tag{18}$$

$$= \frac{1}{2}kA^2e^{-2\beta t}\sin^2(w_1t + \phi) + \frac{1}{2}kA^2e^{-2\beta t}\cdot\left(\frac{\beta}{w_0}\right)^2\cdot\sqrt{\frac{\beta^2 + (w_0^2 - \beta^2)}{\beta^2}}\cos\left(2w_1t + 2\phi - \arctan\frac{\sqrt{w_0^2 - \beta^2}}{\beta}\right) \tag{19}$$

$$= \frac{1}{2}kA^2e^{-2\beta t}\sin^2(w_1t + \phi) + \frac{1}{2}kA^2e^{-2\beta t}\cdot\left(\frac{\beta}{w_0}\right)^2\cdot\frac{w_0}{\beta}\cos\left(2w_1t + 2\phi - \arccos\frac{\beta}{w_0}\right) \tag{20}$$

(Note: $\arctan(\sqrt{w_0^2 - \beta^2}/\beta) = \arccos(\beta/w_0)$ can be derived through right triangle's side relations).

Then, the total energy is given as follow:

$$E = K + U = \frac{1}{2}kA^3e^{-2\beta t}(\cos^2(w_1t + \phi) + \sin^2(w_1t + \phi)) + \frac{1}{2}kA^2e^{-2\beta t}\frac{2\beta}{2w_0}\cos\left(2w_1t + 2\phi - \arccos\frac{2\beta}{2w_0}\right) \tag{21}$$

$$= \frac{1}{2}kA^2e^{-2\beta t}\left(1 + \frac{1}{2Q}\cos\left(2w_1t + 2\phi - \arccos\frac{1}{2Q}\right)\right) \tag{22}$$

(Note: recall that $Q = \frac{w_0}{2\beta}$, so $\frac{2\beta}{2w_0} = \frac{1}{2Q}$).

(b)

When consider a physical (1-dimensional) simple harmonic oscillator (SHO), its amplitude represents its displacement away from the equilibrium, while the first derivative is the velocity. In this case, the damping force $-b\dot{x} = -bv$ is a non-conservative force proportional to velocity (which is always against the motion), in contrast to the force $-kx$ that is conservative, so when the object in SHO has nonzero velocity, the damping force will be dissipating energy, while the force $-kx$ is not disspating any energy (only transfers it between potential and kinetic energy).

Since the velocity is oscillating also (due to SHO's oscillating nature), then the power dissipated by damping force, $P_{\text{damp}} = (-bv)v = -bv^2$ is also oscillating, which the energy E over time (as an antiderivative of P, in this case P_{damp}) is also going to be oscillatory. (Note: since other forces, like $-kx$ is not dissipating any mechanical energy, then the only rate of change of energy is coming from the damping force, hence $P = P_{\text{damp}}$).

(c)

Given that $\tilde{t} = \frac{t}{2\pi/w_1}$, then $\frac{dE}{d\tilde{t}} = \frac{dE}{dt}\frac{dt}{d\tilde{t}} = \frac{2\pi}{w_1}\frac{dE}{dt}$. Which, define $\phi_1 = 2\phi - \arccos\frac{1}{2Q}$, calculating the derivative, we get:

$$\frac{dE}{d\tilde{t}} = \frac{2\pi}{w_1}\cdot\frac{1}{2}kA^2\left(-2\beta e^{-2\beta t}\left(1 + \frac{1}{2Q}\cos\left(2w_1t + 2\phi - \arccos\frac{1}{2Q}\right)\right) - \frac{w_1}{Q}e^{-2\beta t}\sin\left(2w_1t + 2\phi - \arccos\frac{1}{2Q}\right)\right) \tag{23}$$

$$= -\frac{2\pi}{w_1}\cdot 2\beta E - \frac{2\pi}{w_1}\cdot\frac{w_1}{Q}\cdot\frac{1}{2}kA^2e^{-2\beta t}\sin(2w_1t + \phi_1) \tag{24}$$

Which, consider the term $-\frac{1}{E}\frac{dE}{d\tilde{t}}$, we get:

$$-\frac{1}{E}\frac{dE}{d\tilde{t}} = \frac{2\pi}{w_1}\cdot 2\beta + \frac{2\pi}{w}\cdot\frac{w_1}{Q}\cdot\frac{\frac{1}{2}kA^2e^{-2\beta t}\sin(2w_1t+\phi_1)}{\frac{1}{2}kA^2e^{-2\beta t}\left(1+\frac{1}{2Q}\cos\left(2w_1t+\phi_1\right)\right)} \tag{25}$$

$$= \frac{2\pi}{w_1}\cdot 2\beta + \frac{2\pi}{w_1}\cdot\frac{2w_1}{2Q}\cdot\frac{\sin(2w_1t+\phi_1)}{1+\frac{1}{2Q}\cos(2w_1t+\phi_1)} \tag{26}$$

For the first term, taking the average over duration $\frac{2\pi}{w_1}$, we get $\frac{2\pi}{w_1}\cdot 2\beta$ (since it is a constant). For the second ter, since $u = 1+\frac{1}{2Q}\cos(2w_1t+\phi_1)$ has derivative $-\frac{2w_1}{2Q}\sin(2w_1t+\phi_1)$, taking the average, we get:

$$\frac{1}{2\pi/w_1}\int_{t=0}^{\frac{2\pi}{w_1}}\frac{2\pi}{w_1}\cdot\frac{2w_1}{2Q}\cdot\frac{\sin(2w_1t+\phi_1)}{1+\frac{1}{2Q}\cos(2w_1t+\phi_1)}dt = -\int_{t=0}^{\frac{2\pi}{w_1}}\frac{\frac{d}{dt}\left(1+\frac{1}{2Q}\cos(2w_1t+\phi_1)\right)}{1+\frac{1}{2Q}\cos(2w_1t+\phi_1)}dt \tag{27}$$

$$= -\ln\left(1+\frac{1}{2Q}\cos(2w_1t+\phi_1)\right)\Bigg|_0^{\frac{2\pi}{w_1}} \tag{28}$$

$$= 0 \tag{29}$$

(Note:recall that $\cos(2w_1t+\phi_1)$ has period $\frac{2\pi}{w_1}$).

As a result, since average can be distributed over addition, we get the following:

$$-\left\langle\frac{1}{E}\frac{dE}{d\tilde{t}}\right\rangle = \left\langle-\frac{1}{E}\frac{dE}{d\tilde{t}}\right\rangle = \frac{2\pi}{w_1}\cdot 2\beta \tag{30}$$

With $w_1 = \sqrt{w_0^2-\beta^2} = w_0\sqrt{1-\left(\frac{\beta}{w_0}\right)^2}$, and $Q = \frac{w_0}{2\beta}$, such average becomes:

$$-\left\langle\frac{1}{E}\frac{dE}{d\tilde{t}}\right\rangle = \frac{2\pi}{w_0\sqrt{1-\left(\frac{2\beta}{2w_0}\right)^2}}\cdot 2\beta = \frac{2\pi}{Q}\left(1-\frac{1}{4Q^2}\right)^{-1/2} \tag{31}$$

3

Question 3 *Consider a damped oscillator with natural frequency w_0 and damping constant β, driven by a force $F_0\cos(wt)$.*

(a) Show that the average power delivered to the oscillator by the driving force is

$$\langle P\rangle = m\beta w^2 A^2$$

and that the average power dissipated by the damping force is also the same.

(b) Find the driving frequency which maximizes the power, assuming F_0, w_0, β are all fixed.

Pf:

If the differential equation is $\ddot{x} + 2\beta\dot{x} + w_0^2 x = \frac{\tilde{F_0}}{m}e^{iwt}$ (where $\tilde{F_0} = F_0 e^{i\delta}$ for phase δ; since $\delta = 0$ here, $\tilde{F_0} = F_0$; divide by m is because the initial setup for force is $m\ddot{x}$, the whole differential equation is divided by m), then the solution is $\frac{F_0/m}{(w_0^2-w^2)+2i\beta w}e^{iwt} = \frac{F_0}{m((w_0^2-w^2)^2+(2\beta w)^2)}((w_0^2-w^2)-2\beta wi)(\cos(wt)+i\sin(wt))$. Which, taken the real part as the solution (which corresponds to the solution of $F_0\cos(wt)$), we get:

$$x(t) = \frac{F_0}{m((w_0^2-w^2)^2+(2\beta w)^2)}\big((w_0^2-w^2)\cos(wt) + 2\beta w\sin(wt)\big) \tag{32}$$

Which, this is the stable state solution of the driven oscillator, which we can use tfor calculation. (Note: based on the expression, since the linear combination of $B\sin(wt)+C\cos(wt) = A\cos(wt+\phi)$ has amplitude $A = \sqrt{B^2+C^2}$, then $x(t)$ in fact has amplitude $A = \frac{F_0}{m\sqrt{(w_0^2-w^2)^2+(2\beta w)^2}}$).

(a)

Power of Driving force:

Under 1-dimension, the power $P = Fv$ (where v is the velocity), hence we first need to find the velocity:

$$v(t) = x'(t) = \frac{F_0 w}{m((w_0^2-w^2)^2+(2\beta w)^2)}(2\beta w\cos(wt) - (w_0^2-w^2)\sin(wt)) \tag{33}$$

Then, with driving force $F_{\text{drive}}(t) = F_0\cos(wt)$, the power is given by:

$$\begin{aligned} P_{\text{drive}}(t) = F_{\text{drive}}(t)v(t) &= \frac{F_0^2 w}{m((w_0^2-w^2)^2+(2\beta w)^2)}(2\beta w\cos^2(wt) - (w_0^2-w^2)\sin(wt)\cos(wt)) && (34) \\ &= \frac{F_0^2 w}{m((w_0^2-w^2)^2+(2\beta w)^2)}\left(2\beta w\cdot\frac{1+\cos(2wt)}{2} - \frac{w_0^2-w^2}{2}\sin(2wt)\right) && (35) \end{aligned}$$

Which, notice that taking the average over period $\frac{2\pi}{w}$, since both $\cos(2wt), \sin(2wt)$ would provide an integral of 0, the only term left is the constant (in the paranthesis, it's provided by $2\beta w\cdot\frac{1}{2} = \beta w$). Hence, the average driving power is:

$$\langle P_{\text{drive}}\rangle = \frac{F_0^2\cdot\beta w^2}{m((w_0^2-w^2)^2+(2\beta w)^2)} = m\beta w^2\left(\frac{F_0}{m\sqrt{(w_0^2-w^2)^2+(2\beta w)^2}}\right)^2 = m\beta w^2 A^2 \tag{36}$$

Power of Damping force:

Recall that $\beta = \frac{b}{2m}$, and the damping force is provided by $F_{\text{damp}} = -bv$. Then, the power dissipated $P_{\text{damp}} = F_{\text{damp}}v = -bv^2$. Using the formula derived above, we get:

$$P_{\text{damp}}(t) = -bv^2(t) = -b \cdot \frac{F_0^2 w^2}{m^2((w_0^2 - w^2)^2 + (2\beta w)^2)^2}((2\beta w)^2\cos^2(wt) + (w_0^2 - w^2)^2\sin^2(wt) - 4\beta w(w_0^2 - w^2)\sin(wt)\cos(wt)) \tag{37}$$

Which, recall that $\cos^2(wt) = \frac{1+\cos(2wt)}{2}$, $\sin^2(wt) = \frac{1-\cos(2wt)}{2}$, and $2\sin(wt)\cos(wt) = \sin(2wt)$. Taking the average over a time duration of $\frac{2\pi}{w}$, with integer multiples of w being the frequency, $\sin(2wt), \cos(2wt)$ all provide average of 0, then only the constant terms are left (which are the ones multiplied by $\frac{1}{2}$, in the $\cos(2wt)$ form of $\sin^2(wt)$ and $\cos^2(wt)$). Which, the average power dissipated by damping force is given by the constant in the above function:

$$\langle P_{\text{damp}} \rangle = -b \cdot \frac{F_0^2 w^2}{m^2((w_0^2 - w^2)^2 + (2\beta w)^2)^1}\left(\frac{(2\beta w)^2}{2} + \frac{(w_0^2 - w^2)^2}{2}\right) \tag{38}$$

$$= -2m\beta w^2 \cdot \frac{1}{2}\left(\frac{F_0}{m\sqrt{(w_0^2 - w^2)^2 + (2\beta w)^2}}\right)^2 = -m\beta w^2 A^2 \tag{39}$$

This shows that the driving and the damping force provides the power with same magnitude, but different sign, so the average power provided is still 0 (i.e. the input energy from driving force is dissipated by the damping force).

(b)

Given that the average power (of the driving force) is $m\beta w^2 A^2$, the full form of the power is:

$$P_{\text{ave}} = m\beta w^2 \cdot \frac{F_0^2}{m^2((w_0^2 - w^2)^2 + (2\beta w)^2)} = \frac{\beta F_0^2}{m\left(\left(\frac{w_0^2}{w} - w\right)^2 + (2\beta)^2\right)} \tag{40}$$

Which, given that F_0, β, w_0 are all fixed (and nonzero), the above term has strictly psitive denominator. Hence, finding the maximum is equivalent to find the minimum of the denominator.

Given $\left(\frac{w_0^2}{w} - w\right)^2 + (2\beta)^2$, its minimum occurs when the quadratic term is 0. Hence, minimum occurs when $\frac{w_0^2}{w} - w = 0$, $w^2 = w_0^2$, or $w = \pm w_0$; and with $w > 0$, we have the minimum occurs at $w = w_0$.

Finally, this shows that P_{ave} has a maximum when $w = w_0$.

4

Question 4 *Consider an oscillator driven by a sawtooth wave:*

$$f(t) = f_0\left(\frac{t}{T} - \left\lfloor \frac{1}{2} + \frac{t}{T} \right\rfloor\right)$$

(a) *Sketch a sawtooth wave. What is the period?*

(b) *Calculate the Fourier series for a sawtooth wave. Plot a sawtooth wave and a Fourier series approximation on the same figure, keeping enough terms in the sum to get a reasonably good approximation.*

(c) *Calculate the response of an oscillator (with natural frequency w_0 and damping parameter β) driven by a sawtooth wave. make plots for several different choices of the dimensionless parameters Tw_0 and Q, making sure to include examples that are both on and off resonance. Qualitatively describe your result.*

Pf:

(a)

First, for all $t \in \mathbb{R}$, we have $f(t+T) = f_0\big(\frac{t+T}{T} - \big\lfloor \frac{1}{2} + \frac{t+T}{T} \big\rfloor\big) = f_0\big(1 + \frac{t}{T} - \big\lfloor \frac{1}{2} + \frac{t}{T} + 1 \big\rfloor\big)$, with $\lfloor x+1 \rfloor = \lfloor x \rfloor + 1$, we have $f(t+T) = f(t)$. Also, for all $n \in \mathbb{Z}$, if consider the region $[nT - \frac{T}{2}, nT + \frac{T}{2})$, the following is true:

$$\forall t \in \left[nT - \frac{T}{2}, nT + \frac{T}{2}\right), \quad \frac{1}{2} + \frac{nT - T/2}{T} \leq \frac{1}{2} + \frac{t}{T} < \frac{1}{2} + \frac{nT + T/2}{T} \tag{41}$$

$$\implies n \leq \frac{1}{2} + \frac{t}{T} < n + 1 \implies \left\lfloor \frac{1}{2} + \frac{t}{T} \right\rfloor = n \tag{42}$$

Hence, we have $f(t) = f_0\big(\frac{t}{T} - n\big) = f_0\big(\frac{t_0}{T}\big)$, where $t_0 \in [-\frac{T}{2}, \frac{T}{2})$ satisfies $t = Tn + t_0$ in this case.

Combining these relations, this determines that the period is T, and we get the following graph:

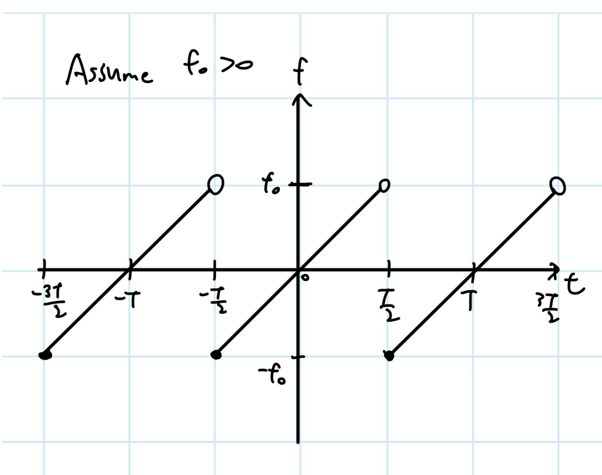


Figure 1: Graph of Sawtooth Function

(b)

Within the interval $(-\frac{T}{2}, \frac{T}{2})$, since $-\frac{1}{2} < \frac{t}{T} < \frac{1}{2}$, we have $0 < \frac{1}{2} + \frac{t}{T} < 1$, hence the floor function of this expression constantly provides 0. Therefore, $f(t) = f_0 \frac{t}{T}$.

Which, when calculating the Fourier coefficients, for all $n \in \mathbb{Z}$, we get the following:

$$n = 0, \quad a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{f_0 t}{T} dt = 0 \tag{43}$$

(Note: the above function is odd).

$$n \neq 0, \quad a_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i2\pi nt/T} dt = \frac{f_0}{T^2} \int_{-\frac{T}{2}}^{\frac{T}{2}} t e^{-i2\pi nt/T} dt \tag{44}$$

$$= \frac{-f_0}{T \cdot i2\pi n} t e^{-i2\pi nt/T} \Big|_{-\frac{T}{2}}^{\frac{T}{2}} + \frac{f_0}{T \cdot i2\pi n} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i2\pi nt/T} dt \tag{45}$$

$$= -\frac{f_0}{i2\pi n} e^{i\pi n} + \frac{f_0}{(2\pi n)^2} e^{-i2\pi nt/T} \Big|_{-\frac{T}{2}}^{\frac{T}{2}} = (-1)^n \frac{f_0 i}{2\pi n} \tag{46}$$

Hence, the Fourier Series of the function is:

$$f(t) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n f_0 i}{2\pi n} e^{i2\pi nt/T} \tag{47}$$

To plot the function, we'll convert it back to a real function. By pairing up the terms disregarding the sign, we get:

$$f(t) = \sum_{n=1}^{\infty} \frac{(-1)^n f_0 i}{2\pi} \left(\frac{e^{i2\pi nt/T}}{n} + \frac{e^{-i2\pi nt/T}}{-n} \right) = \sum_{n=1}^{\infty} \frac{(-1)^n f_0 i}{n\pi} \cdot i \sin\left(\frac{2n\pi}{T} t \right) \tag{48}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} f_0}{n\pi} \sin\left(\frac{2n\pi}{T} t \right) \tag{49}$$

Using $N = 10$ as the approximation, we get the following graph:

Here's the provided code (source: python):

```
import matplotlib.pyplot as plt
import numpy as np
import math

#define constant (Note: both need to be positive)
T = 4
f_0 = 4

#define original function
def f(t):
   return f_0*(t/T - math.floor(1/2+t/T))

#define fourier series approximation, with n=10
```

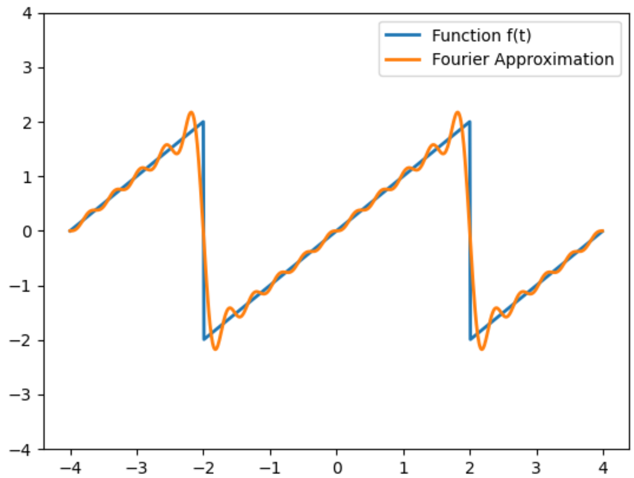


Figure 2: Fourier Approximation with index $n = 10$

```
def fourier(t):
    output = 0

    for n in range(1,11):
        output += (-1)**(n+1) * f_0 / (n* math.pi) * math.sin(2*n* math.pi * t/T)

    return output

#plot function
t = np.arange(-f_0, f_0, 0.01)

Function = []
Fourier = []
for i in range(len(t)):
    Function.append(f(t[i]))
    Fourier.append(fourier(t[i]))

plt.plot(t,Function, lw=2, label='Function f(t)')
plt.plot(t,Fourier, lw=2, label = 'Fourier Approximation')
plt.ylim(-T,T)

plt.legend()
plt.show()
```

(c)

(Note: for this part we'll focus on stable state solutions).

Recall that with periodic driven force $F_0 \sin(wt)$, using methods similar to Question 3, if develop a complex differential equation $\ddot{x} + 2\beta\dot{x} + w_0^2 x = \frac{F_0}{m} e^{iwt}$ (divided by mass to get acceleration), the solution is $\tilde{x}(t) = \frac{F_0/m}{(w_0^2 - w^2) + (2\beta w)i} e^{iwt} = \frac{F_0}{m((w_0^2 - w^2)^2 + (2\beta w)^2)}((w_0^2 - w^2) - (2\beta w)i)(\cos(wt) + i\sin(wt))$. Then, if take the imaginary part, since $\text{Im}\big(\frac{F_0}{m} e^{iwt}\big) = \frac{F_0}{m}\sin(wt)$ (the driven force over mass), then $\text{Im}(\tilde{x})$ would provide a solution to our driven force. Which, we get the corresponding solution as follow:

$$x(t) = \frac{F_0}{m((w_0^2 - w^2)^2 + (2\beta w)^2)} \big((w_0^2 - w^2)\sin(wt) - (2\beta w)\cos(wt)\big) \tag{50}$$

With $w = \frac{2\pi n}{T}$ for the case of Fourier Series, it can be simplified as follow:

$$x_n(t) = \frac{F_n}{m w_0^4 \left(\left(1 - \left(\frac{2\pi n}{T w_0}\right)^2\right)^2 + \left(\frac{2\beta}{w_0}\frac{2\pi n}{T w_0}\right)^2 \right)} \cdot w_0^2 \left(\left(1 - \left(\frac{2\pi n}{T w_0}\right)^2\right) \sin\left(\frac{2\pi n}{T} t\right) - \left(\frac{2\beta}{w_0}\frac{2\pi n}{T w_0}\right) \cos\left(\frac{2\pi n}{T} t\right) \right) \tag{51}$$

$$= \frac{F_n}{m w_0^2 \left(\left(1 - \left(\frac{2\pi n}{T w_0}\right)^2\right)^2 + \left(\frac{2\pi n}{Q \cdot T w_0}\right)^2 \right)} \left(\left(1 - \left(\frac{2\pi n}{T w_0}\right)^2\right) \sin\left(\frac{2\pi n}{T} t\right) - \left(\frac{2\pi n}{Q \cdot T w_0}\right) \cos\left(\frac{2\pi n}{T} t\right) \right) \tag{52}$$

Which, since the driven force is given by $f(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} f_0}{n\pi} \sin\big(\frac{2n\pi}{T} t\big)$, utilize linearity of differential equation, we get the following expression for the force:

$$x(t) = \sum_{n=1}^{\infty} x_n(t), \quad \forall n \in \mathbb{N},\ F_n = \frac{(-1)^{n+1} f_0}{n\pi} \tag{53}$$

(Note: each F_n represents the fourier coefficient of the $\sin(\frac{2n\pi}{T} t)$ term, since that is the "F_0" in the initial assumption of the sinusoidal driven force; here because each term is relatively complicated, we use some simplified notation).

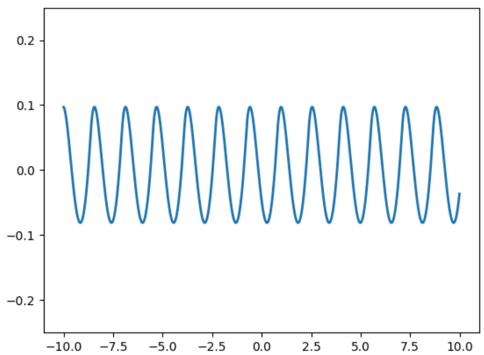
For the plots, I chose $w_0 = 2$, $T w_0 = \{\pi, 4\}$, and $Q = \{2, 20\}$ as parameters, and $n = 30$ as approximations. Here are the results:

(Note: rest of the graphs are on the next page).

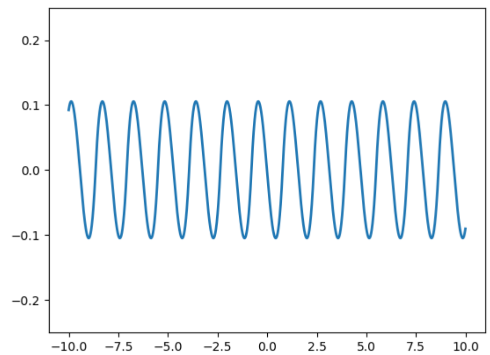
So qualitatively, after getting to stable state, when quality factor Q is higher, the amplitude of the system is higher (not too obvious though); on the other hand, being in phase or not affects the amplitude even more (where in phase provides a larger amplitude). One can see that when $T w_0 = \pi$ (when the system is not in phase with natural frequency), the amplitude with provided constants are explicitly lower than the amplitude when $T w_0 = 4$ (with $w_0 = 2$, the system is in phase with some force components).

Here's the code used to generate the graphs (source: python):

```
import matplotlib.pyplot as plt
import numpy as np
import math
```

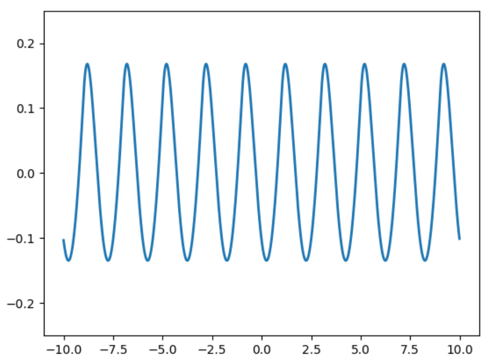


(a) $Q = 1$

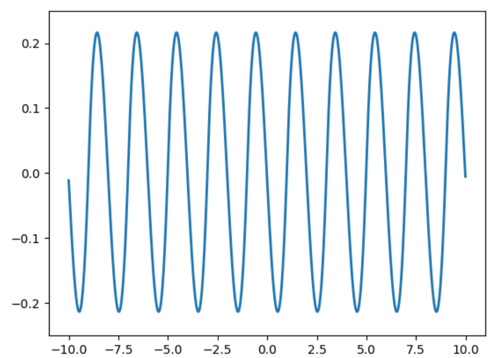


(b) $Q = 20$

Figure 3: When $Tw_0 = \pi$ (Not in phase with any term)



(a) $Q = 1$



(b) $Q = 20$

Figure 4: When $Tw_0 = 4$ (in phase with some term)

```
#define set of constants (Note: all need to be positive)
#fixed constant
w_0 = 2
f_0 = 4
m=1
#array of constants
Tw_0 = [math.pi,4] #the first is never in phase, while the second always
                    in phase with some term
Q = [1,20] #provide two different quality factors

#define the sum of solutions approximation, with n=10
def x(Tw, q, t): #here, Tw represents the Tw_0 chosen
   output = 0

   for n in range(1,31):
```

```
        w = 2* math.pi * n/(Tw)*w_0 #the frequency
        B = 1-(2 * math.pi * n/(Tw))**2 #coefficient of sin term
        C = 2* math.pi * n/(q * Tw) #coefficient of cos term

        A = (-1)**(n+1) * f_0 / (n * math.pi) * 1/(m * w_0**2 * (B**2+C**2)) #amplitude
        output += A*(B*math.sin(w*t)-C*math.cos(w*t))

    return output

#plot functions
for i in range(2): #for Tw_0
    for j in range(2): #for Q
        t = np.arange(-10, 10, 0.01)

        Tw = Tw_0[i]
        q = Q[j]

        #add X list
        X = []
        for k in range(len(t)):
            X.append(x(Tw, q, t[k]))

        #plot function on graph
        plt.plot(t,X, lw=2)
        plt.ylim(-0.25,0.25)

        plt.savefig(str(i)+',' +str(j)+'.png') #save graph
        plt.clf() #clear graph
```

5 Need revision (final explanation)

Question 5 *Consider an* undamped oscillator *driven exactly on resonance,*

$$\ddot{x} + w_0^2 x = Re(f_0 e^{i(w_0 t + \delta)})$$

If we naively plug into our formula for C, ϕ from lecture, we get that the amplitude is infinite, which (at least at first glance) doesn't make much sense. Solve the differential equation from scratch to get $x(t)$. Explain (with the benefit of hindsight) both how your solution is consistent with C being infinite, and why (physically) the response to being driven on-resonance with no damping has infinite amplitude.

Pf:

Temporarily, assume a complex solution exists (not considering just the real part), then using the expression given in class, since with linear operator $\frac{d}{dt}$, the differential equation can be written as:

$$\left(\frac{d}{dt} - iw_0\right)\left(\frac{d}{dt} + iw_0\right)x = f_0 e^{i(w_0 t + \delta)} \tag{54}$$

we get the following expression:

$$e^{iw_0 t}\frac{d}{dt}\left(e^{-iw_0 t} \cdot e^{-iw_0 t}\frac{d}{dt}(e^{iw_0 t}x)\right) = f_0 e^{i(w_0 t + \delta)} \implies \frac{d}{dt}\left(e^{-iw_0 t} \cdot e^{-iw_0 t}\frac{d}{dt}(e^{iw_0 t}x)\right) = f_0 e^{i\delta} \tag{55}$$

$$\implies e^{-2iw_0 t}\frac{d}{dt}(e^{iw_0 t}x) = f_0 e^{i\delta} t + C \implies \frac{d}{dt}(e^{iw_0 t}x) = f_0 t e^{2iw_0 t + i\delta} + Ce^{2iw_0 t} \tag{56}$$

$$\implies e^{iw_0 t}x = \frac{f_0}{2iw_0} t e^{2iw_0 t + i\delta} - \frac{f_0}{(2iw_0)^2} e^{2iw_0 t + i\delta} + \frac{C}{2iw_0} e^{2iw_0 t} + D \tag{57}$$

$$\implies x = \frac{f_0}{2iw_0} t e^{i(w_0 t + \delta)} + \frac{f_0}{4w_0} e^{i(w_0 t + \delta)} + C' e^{iw_0 t} + De^{-iw_0 t} \tag{58}$$

Given initial conditions of positions and velocity, $C', D \in \mathbb{C}$ can be solved. Notice that as $t \to \infty$, the term $te^{i(w_0 t + \delta)}$ goes unbounded, while the other terms governed purely by $e^{iw_0 t}$ remain bounded, showing that if taken the real part to get a physical solution for an oscillator, the amplitude of the oscillator would progressively grow, and eventually goes unbounded (i.e. infinite amplitude).

This is consistent with amplitude of C being infinite, since infinity is not a well-defined number in $\mathbb{R}$ or $\mathbb{C}$, so a solution just can't exist.

Physically, for a on-resonance undamped oscillator, then its constantly providing power to the system; however, in contrast to the case in Question 3, now since it's undamped, then there's no other forces dissipating the input power, hence the energy of the system would accumulate, and eventually goes unbounded.

6 Need more detailed explanation for (a)

Question 6 *Consider the equation of motion for the simple pendulum,*

$$\ddot{\phi} + w_0^2 \sin(\phi) = 0, \quad w_0 = \sqrt{\frac{g}{L}}$$

If we take the small-angle approximation $\sin(\phi) \approx \phi$, we get a simple harmonic oscillator. However, the pendulum is not quite a SHO: If the amplitude of oscillation (the angle) is large enough we should instead approximate $\sin(\phi) \approx \phi - \phi^3/6$; the equation of motion is then

$$\ddot{\phi} + w_0^2 \phi = \frac{1}{6} w_0^2 \phi^3$$

It turns out that this means the period of a pendulum depends on the maximum amplitude,

$$T \approx \frac{2\pi}{w_0}\left(1 \pm \frac{\phi_{\max}^2}{16}\right)$$

Where either the plus or minus sign is correct.
Additionally, any actual pendulum has some damping. For a pendulum of length l, the period is thus shifted by both the damping and the amplitude dependence. If you had a desired period T_d, you might think you coudl just adjust the length of your pendulum from $g(T/(2\pi))^2$ based on knowing the value of the two corrections. However, there's a problem: the damping causes the amplitude to decrease, so the amplitude-dependent shift in the period is not constant.

(a) Suppose you set your pendulum up to have an (initial) period of 2 seconds and count one second every time it passes through vertical. Does the pendulum (eventually) run fast or slow (i.e. is the plus or minus sign correct?)

(b) With $Q \approx 10,000$ (a high-quality but entirely achievable pendulum) and $\phi_{\max} = 0.1$ radians (initially), and assuming the pendulum has a period of 2 seconds with $\phi_{\max} = 0.1$ radians, about how many seconds is the pendulum ahead or behind after one day?

Pf:

(a)

When the angle is getting larger (where $\phi > 0$), if we modeled the angle by $\ddot{\phi} = -w_0^2\phi + \frac{1}{6}w_0^2\phi^3$, we get that this term has a smaller magnitude than $-w_0^2\phi$ (since for $\phi \in (0, \frac{\pi}{2})$ the normal range of oscillation, we have $-\phi < -\phi + \frac{1}{6}\phi^3 < 0$, due to the fact that $\frac{\pi}{2} < 2 < \sqrt{6}$). Since the corresponding acceleration is smaller than the case of a simple harmonic oscillator, it takes a longer time to complete one cycle (because when angle gets large, angular acceleration $\ddot{\phi}$ has smaller magnitude, which takes longer to accelerate, or get to a higher speed). So, the period $T \approx \frac{2\pi}{w_0}\left(1 + \frac{\phi_{\max}^2}{16}\right)$.

(b)

A simple harmonic oscillator in general has the amplitude decrease with a factor of $e^{-\beta t}$. With $Q = \frac{w_0}{2\beta}$, $\beta = \frac{w_0}{2Q}$, so the amplitude is decreasing with a factor of $e^{-\frac{w_0}{2Q}t}$.

Initially, with $\phi_{\max} = 0.1$ rad, and $T = 2$, we get the following relation:

$$2 = \frac{2\pi}{w_0}\left(1 + \frac{1}{1600}\right) \implies w_0 = \frac{1601\pi}{1600}\text{rad/s} \tag{59}$$

To calculate the difference in cycle, for a Simple Harmonic Oscillator, with period $T = \frac{2\pi}{w_0}$, and $86,400$ seconds a day, it goes through $\frac{86,400 \cdot w_0}{2\pi}$ cycles in total.

Now, for each small change in time Δt, assuming the oscillator iswith period T, then it approximately goes through $\frac{\Delta t}{T}$ cycles, and the total amount of cycle is approximately summing up all of these small pieces over the duration of $86,400$ seconds. Taking the limit, we get:

$$\#\text{ cycles} = \int_{t=0}^{86,400} \frac{dt}{T}, \quad T(t) = \frac{2\pi}{w_0}\left(1 + \frac{\phi_{\max}(t)^2}{16}\right) \tag{60}$$

Which, wit the give decreasing factor above, $\phi_{\max}(t) = 0.1 \cdot e^{-\frac{w_0}{2Q}t}$, so the integral becomes:

$$\int_0^{86,400} \frac{dt}{\frac{2\pi}{w_0}\left(1 + \frac{e^{-\frac{w_0}{Q}t}}{1600}\right)} = \frac{w_0}{2\pi}\int_0^{86,400} \frac{1600}{1600 + e^{-\frac{w_0}{Q}t}}dt = \frac{w_0}{2\pi}\int_0^{86,400}\left(1 - \frac{e^{-\frac{w_0}{Q}t}}{1600 + e^{-\frac{w_0}{Q}t}}\right)dt \tag{61}$$

$$= \frac{86,400 \cdot w_0}{2\pi} + \frac{w_0}{2\pi} \cdot \frac{Q}{w_0}\int_0^{86,400} \frac{-\frac{w_0}{Q}e^{-\frac{w_0}{Q}t}}{1600 + e^{-\frac{w_0}{Q}t}}dt \tag{62}$$

$$= \frac{86,400 \cdot w_0}{2\pi} + \frac{Q}{2\pi}\int_0^{86,400} \frac{\frac{d}{dt}(1600 + e^{-\frac{w_0}{Q}t})}{1600 + e^{-\frac{w_0}{Q}t}}dt \tag{63}$$

$$= \frac{86,400 \cdot w_0}{2\pi} + \frac{Q}{2\pi}\ln(1600 + e^{-\frac{w_0}{Q}t})\Big|_0^{86,400} \tag{64}$$

$$= \frac{86,400 \cdot w_0}{2\pi} + \frac{Q}{2\pi}\ln\left(\frac{1600 + e^{-\frac{w_0}{Q}\cdot 86,400}}{1601}\right) \tag{65}$$

Which, to calculate the number of cycles behind a Simple Harmonic Oscillator, we get:

$$\#\text{ cycles behind} = \frac{86,400 \cdot w_0}{2\pi} - \#\text{ cycles} = -\frac{Q}{2\pi}\ln\left(\frac{1600 + e^{-\frac{w_0}{Q}\cdot 86,400}}{1601}\right) \tag{66}$$

Then, the amount of seconds behind is given by (# cycles behind) $\cdot \frac{2\pi}{w_0}$ (number of cycles times the natural period), which is given by:

$$\Delta t = -\frac{Q}{w_0}\ln\left(\frac{1600 + e^{-\frac{w_0}{Q}\cdot 86,400}}{1601}\right) \tag{67}$$

Plugin $Q = 10,000$, $w_0 = \frac{1601\pi}{1600}$, we get that $\Delta t \approx 1.988$.

7 Extra Credit (not done)

Question 7 *Too long.*

Pf:

(a)

Given the ininite series expression, **assume the function is smooth and converges uniformly for simplicity**, then taking differentiation, the second derivative becomes:

$$\ddot{\phi} = \sum_{n=0}^{\infty} \epsilon^n \ddot{\phi}_n \tag{68}$$

Plug into the differential equation, we get:

$$\sum_{n=0}^{\infty} \epsilon^n \ddot{\phi}_n + w_0^2 \sum_{n=0}^{\infty} \epsilon^n \phi_n = \epsilon w_0^2 \left(\phi_0 + \epsilon \phi_1 + \sum_{n=2}^{\infty} \epsilon^n \phi_n \right)^3 \tag{69}$$

Taking the terms of ϵ^0 and ϵ^1 on both sides, we get the following two equations:

$$\begin{cases} \ddot{\phi}_0 + w_0^2 \phi_0 = 0 \\ \epsilon \ddot{\phi}_1 + w_0^2 \epsilon \phi_1 = \epsilon w_0^2 \phi_0^3 \end{cases} \implies \begin{cases} \ddot{\phi}_0 + w_0^2 \phi_0 = 0 \\ \ddot{\phi}_1 + w_0^2 \phi_1 = w_0^2 \phi_0^3 \end{cases} \tag{70}$$

The first equation is an undamped Single Harmonic Oscillator, which has solution $\phi_0(t) = A\cos(w_0 t + \varphi)$ for some amplitude A and phase φ. Now, for the second equation, using sum of angle formula, we get the following identity:

$$\cos^3(x) = \cos(x) \cdot \frac{1 + \cos(2x)}{2} = \frac{1}{2}(\cos(x) + (\cos(x)\cos(2x) - \sin(x)\sin(2x)) + \sin(x)\sin(2x)) \tag{71}$$

$$= \frac{1}{2}(\cos(x) + \cos(3x) + 2\sin^2(x)\cos(x)) = \frac{1}{2}(\cos(x) + \cos(3x) + 2\cos(x) - 2\cos^3(x)) \tag{72}$$

$$= \frac{3}{2}\cos(x) + \frac{1}{2}\cos(3x) - \cos^3(x) \tag{73}$$

So, $\cos^3(x) = \frac{3}{4}\cos(x) + \frac{1}{4}\cos(3x)$. Plug into the second differential equation, we get:

$$\ddot{\phi}_1 + w_0^2 \phi_1 = w_0^2 \cdot A^3 \cos^3(w_0 t + \varphi) = w_0^2 A^3 \left(\frac{3}{4}\cos(w_0 t + \delta) + \frac{1}{4}\cos(3w_0 t + 3\varphi) \right) \tag{74}$$

Using linearity, ϕ_1 can be expressed as a linear combination of the solution corresponds to $\cos(w_0 t + \varphi)$, and $\cos(3w_0 t + 3\varphi)$; However, for $\cos(w_0 t + \varphi)$, since the nonhomogeneous equation has the same frequency as the homogeneous equation (both uses frequency w_0), then based on Question 5 (if taken the real part of the complex solution), it has unbounded solution (or, there's no steady state, because the amplitude eventually goes unbounded).

(b)

Based on the expression in (a), ϕ_1 has no solution (physically, it means the system for ϕ_1 goes unbounded), and it's because there is a term of $\ddot{\phi}_1 + w_0^2 \phi_1 = K\cos(w_0 t + \varphi)$, which has the same situation as what happened in Question 5. Since we've proven that its energy would eventually diverge to infinity, then a solution can't exist.

(c)

Given the following expression of $\frac{d}{dt}$:

$$\frac{d}{dt} = \sum_{n=0}^{\infty} \epsilon^n \frac{\partial}{\partial T_n} \tag{75}$$

Assuming it's a linear operator on functions that converge under such operation, such that partial derivatives commute, then we get:

$$\frac{d^2}{dt^2} = \left(\sum_{n=0}^{\infty} \epsilon^n \frac{\partial}{\partial T_n}\right)\left(\sum_{m=0}^{\infty} \epsilon^m \frac{\partial}{\partial T_m}\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \epsilon^{n+m} \frac{\partial^2}{\partial T_n \partial T_m} = \sum_{k=0}^{\infty} \epsilon^k \left(\sum_{i=0}^{k} \frac{\partial^2}{\partial T_i \partial T_{k-i}}\right) \tag{76}$$

Up to second order in ϵ, we get the operator:

$$\frac{\partial^2}{\partial T_0^2} + \epsilon\left(\frac{\partial^2}{\partial T_0 \partial T_1} + \frac{\partial^2}{\partial T_1 \partial T_0}\right) + \epsilon^2\left(\frac{\partial^2}{\partial T_2 \partial T_0} + \frac{\partial^2}{\partial T_1^2} + \frac{\partial^2}{\partial T_0 \partial T_2}\right) \tag{77}$$

$$= \frac{\partial^2}{\partial T_0^2} + 2\epsilon \frac{\partial^2}{\partial T_0 \partial T_1} + 2\epsilon^2 \frac{\partial^2}{\partial T_0 \partial T_2} + \epsilon^2 \frac{\partial^2}{\partial T_1^2} \tag{78}$$

(d)

If just regarding up to ϵ^0 and ϵ^1, the derivative operator only need up to ϵ term, while the function only goes up to ϕ_1 (afterward all have order $\epsilon^n,\ n \geq 2$). Apply to $\phi = \sum_{n=0}^{\infty} \epsilon^n \phi_n$ and plug into the differential equation, we get:

$$\left(\frac{\partial^2}{\partial T_0^2} + 2\epsilon \frac{\partial^2}{\partial T_0 \partial T_1}\right)(\phi_0 + \epsilon\phi_1) + w_0^2(\phi_0 + \epsilon\phi_1) = \epsilon w_0^2(\phi_0 + \epsilon\phi_1)^3 \quad \text{(truncate up to } \epsilon^1) \tag{79}$$

$$\implies \frac{\partial^2 \phi_0}{\partial T_0^2} + \epsilon \frac{\partial^2 \phi_1}{\partial T_0^2} + 2\epsilon \frac{\partial^2 \phi_0}{\partial T_0 \partial T_1} + w_0^2 \phi_0 + \epsilon w_0^2 \phi_1 = \epsilon w_0^2 \phi_0^3 \tag{80}$$

Which, combine the terms based on ϵ, we get the following equation:

$$\begin{cases} \frac{\partial^2}{\partial T_0^2}\phi_0 + w_0^2\phi_0 = 0 & \epsilon^0 \text{ term} \\ \frac{\partial^2}{\partial T_0^2}\phi_1 + 2\frac{\partial^2}{\partial T_0 \partial T_1}\phi_0 + w_0^2\phi_1 = w_0^2\phi_0^3 & \epsilon^1 \text{ term} \end{cases} \tag{81}$$

The equation involving ϕ_0 only becomes a Simple Harmonic Oscillator (a homogeneous equation in this case), which we know the solution is given by:

$$\phi_0 = Ae^{iw_0T_0} + A^* e^{-iw_0T_0} \tag{82}$$

Where $A = A(T_1, ...)$ and $A^* = A^*(T_1, ...)$ (since they're terms non-dependent to variable T_0).

(e)

If assume there's no on-resonance term (for $e^{\pm iw_0T_0}$, the coefficients are 0), plug the solution of ϕ_0 to the ϵ^1-term differential equation from (d), we get:

$$\frac{\partial^2}{\partial T_0^2}\phi_1 + 2\frac{\partial}{\partial T_1}\left(\frac{\partial}{\partial T_0}\phi_0\right) + w_0^2\phi_1 = w_0^2\phi_0^3 \tag{83}$$

$$\implies \frac{\partial^2}{\partial T_0^2}\phi_1 + 2\frac{\partial}{\partial T_1}\left(iw_0(Ae^{iw_0T_0} - A^*e^{-iw_0T_0})\right) + w_0^2\phi_1 \tag{84}$$

$$= w_0^2\left(A^3e^{3iw_0T_0} + 3A^2A^*e^{iw_0T_0} + 3A(A^*)^2e^{-iw_0T_0} + (A^*)^3e^{-3iw_0T_0}\right) \tag{85}$$

$$\Longrightarrow \frac{\partial^2}{\partial T_0^2}\phi_1 - 2iw_0\left(-\frac{\partial A}{\partial T_1}e^{iw_0T_0} + \frac{\partial A^*}{\partial T_1}e^{-iw_0T_0}\right) + w_0^2\phi_1 \tag{86}$$

$$= w_0^2\big(A^3e^{3iw_0T_0} + 3A^2A^*e^{iw_0T_0} + 3A(A^*)^2e^{-iw_0T_0} + (A^*)^3e^{-3iw_0T_0}\big) \tag{87}$$

$$\Longrightarrow \frac{\partial^2}{\partial T_0^2}\phi_1 + w_0^2\phi_1 \tag{88}$$

$$= w_0^2A^3e^{3iw_0T_0} + w_0^2(A^*)^3e^{-3iw_0T_0} + \left(3w_0^2A^2A^* - 2iw_0\frac{\partial A}{\partial T_1}\right)e^{iw_0T_0} + \left(3w_0^2A(A^*)^2 + 2iw_0\frac{\partial A^*}{\partial T_1}\right)e^{-iw_0T_0} \tag{89}$$

Which, with the assumption that $e^{\pm iw_0T_0}$ need 0 coefficient, we get:

$$\begin{cases} 3w_0^2A^2A^* - 2iw_0\frac{\partial A}{\partial T_1} = 0 \\ 3w_0^2A(A^*)^2 + 2iw_0\frac{\partial A^*}{\partial T_1} = 0 \end{cases} \Longrightarrow \begin{cases} 3w_0A^2A^* - 2i\frac{\partial A}{\partial T_1} = 0 \\ 3w_0A(A^*)^2 + 2i\frac{\partial A^*}{\partial T_1} = 0 \end{cases} \tag{90}$$

Assume $A \neq 0$ in general, the first equation in (80) implies that $A^* = \frac{2i}{3w_0A^2}\frac{\partial A}{\partial T_1}$, plug into the second equation, we get:

$$-\frac{4}{3w_0A^3}\left(\frac{\partial A}{\partial T_1}\right)^2 + \frac{8}{3w_0A^3}\left(\frac{\partial A}{\partial T_1}\right)^2 - \frac{4}{3w_0A^2}\frac{\partial^2 A}{\partial T_1^2} = 0 \tag{91}$$

$$\Longrightarrow \frac{4}{3w_0A^2}\left(\frac{1}{A}\left(\frac{\partial A}{\partial T_1}\right)^2 - \frac{\partial^2 A}{\partial T_1^2}\right) = 0 \Longrightarrow \frac{1}{A}\left(\frac{\partial A}{\partial T_1}\right)^2 = \frac{\partial^2 A}{\partial T_1^2} \tag{92}$$

A possible solution to the above differential equation is $A = ke^{lT_1}$, where $k, l \in \mathbb{C}$, and $k \neq 0$.