Week 3 Problem

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Question 1 *Give A, B two abelian categories, an additive functor $F : \mathsf{A} \to \mathsf{B}$ is exact if it maps short exact sequences in A to short exact sequences in B.*
Prove that exact functors commute with cohomology: if F is exact and $L^\bullet$ is a cochain complex in A, then $H^\bullet(F(L^\bullet)) \cong F(H^\bullet(L^\bullet))$, where $F(L^\bullet)$ denotes the cochain complex in B obtained by applying F to all objects and morphisms in $L^\bullet$.

Pf:

First, we need to show an exact functor F preserves kernels and cokernels (or, the image of a kernel / cokernel of φ obtains the property of kernel / cokernel of $F\varphi$).

For any morphism $\varphi : M \to N$ in A, $0 \longrightarrow \mathrm{Ker}(\varphi) \xrightarrow{\ker(\varphi)} M \xrightarrow{\varphi} N \xrightarrow{\mathrm{coker}(\varphi)} \mathrm{Cok}(\varphi) \longrightarrow 0$ is an exact sequence. So, with F preserves exact sequences, the following forms an exact sequence:

$$0 \longrightarrow F(\mathrm{Ker}(\varphi)) \xrightarrow{F\ker(\varphi)} F(M) \xrightarrow{F\varphi} F(N) \xrightarrow{F\mathrm{coker}(\varphi)} F(\mathrm{Cok}(\varphi)) \longrightarrow 0$$

First, it provides $F\ker(\varphi)$ as a monomorphism, and $F\mathrm{coker}(\varphi)$ as an epimorphism, hence $F\ker(\varphi)$ is a kernel of $\mathrm{coker}(F\ker(\varphi))$, while $F\mathrm{coker}(\varphi)$ is a cokernel of $\ker(F\mathrm{coker}(\varphi))$.

1) The exactness at $F(M)$ first provides $F\varphi \circ F\ker(\varphi) = 0$, hence $F\ker(\varphi)$ factors uniquely through $\ker(F\varphi) : \mathrm{Ker}(F\varphi) \to F(M)$, or there exists a unique morphism $\alpha : F(\mathrm{Ker}(\varphi)) \to \mathrm{Ker}(F\varphi)$ such that $F\ker(\varphi) = \ker(F\varphi) \circ \alpha$.

   Then, it also provides $\mathrm{coker}(F\ker(\varphi)) \circ \ker(F\varphi) = 0$, hence $\ker(F\varphi)$ must factor uniquely through kernel of $\mathrm{coker}(F\ker(\varphi))$, which $F\ker(\varphi)$ satisfies it as claimed before. So, there exists a unique morphism $\beta : \mathrm{Ker}(F\varphi) \to F(\mathrm{Ker}(\varphi))$, where $F\ker(\varphi) \circ \beta = \ker(F\varphi)$. Which, it can be represented as the following commutative diagram:

$$\begin{tikzcd} & \mathrm{Ker}(F\varphi) \arrow[d, "\ker(F\varphi)"] \arrow[dl, dashed, bend left, "\exists!\beta"] & \\ F(\mathrm{Ker}(\varphi)) \arrow[r, "F\ker(\varphi)"'] \arrow[ur, dashed, bend left, "\exists!\alpha"] & F(M) \arrow[r, "F\varphi"'] \arrow[d, "\mathrm{coker}(F\ker(\varphi))"] & F(N) \\ & \mathrm{Cok}(F\ker(\varphi)) & \end{tikzcd}$$

   Which, one can verify the existence of unique α and β implies $F(\mathrm{Ker}(\varphi)) \cong \mathrm{Ker}(F\varphi)$, and $F\ker(\varphi)$ satisfies all properties as a kernel of $F\varphi$.

2) Then, the exactness at $F(N)$ first provides $F\mathrm{coker}(\varphi) \circ F\varphi = 0$, hence $F\mathrm{coker}(\varphi)$ factors uniquely through $\mathrm{coker}(F\varphi) : F(N) \to \mathrm{Cok}(F\varphi)$, or there exists a unique morphism $\gamma : \mathrm{Cok}(F\varphi) \to F(\mathrm{Cok}(\varphi))$, such that $F\mathrm{coker}(\varphi) = \gamma \circ \mathrm{coker}(F\varphi)$.

   Similarly, it also provides $\mathrm{coker}(F\varphi) \circ \ker(F\mathrm{coker}(\varphi)) = 0$, hence $\mathrm{coker}(F\varphi)$ factors uniquely through cokernel of $\ker(F\mathrm{coker}(\varphi))$, which $F\mathrm{coker}(\varphi)$ itself satisfies this condition. So, there exists a unique morphism $\epsilon : F(\mathrm{Cok}(\varphi)) \to \mathrm{Cok}(F\varphi)$, such that $\mathrm{coker}(F\varphi) = \epsilon \circ F\mathrm{coker}(\varphi)$. Diagramatically, we get:

$$\begin{tikzcd}
 & \mathrm{Ker}(F\mathrm{coker}(\varphi)) \arrow[d, "\ker(F\mathrm{coker}(\varphi))"'] & \\
F(M) \arrow[r, "F\varphi"] & F(N) \arrow[r, "F\mathrm{coker}(\varphi)"] \arrow[d, "\mathrm{coker}(F\varphi)"'] & F(\mathrm{Cok}(\varphi)) \arrow[dl, dashed, shift right, "\exists!\epsilon"'] \\
 & \mathrm{Cok}(F\varphi) \arrow[ur, dashed, shift right, "\exists!\gamma"'] &
\end{tikzcd}$$

   Which, existence of unique ϵ, γ guarantees $\mathrm{Cok}(F\varphi) \cong F(\mathrm{Cok}(\varphi))$, and $F\mathrm{coker}(\varphi)$ obtains all desired properties as a cokernel of $F\varphi$.

So, the above demonstrates how exact functor F preserves kernels and cokernels of morphisms.

Now, given cochain complex $L^\bullet$, since for any index i, $\mathrm{im}(\delta^{i-1})$ factors uniquely through $\ker(\delta^i)$ through some morphism α^i (take this as given), with $H^i(L^\bullet) := \mathrm{Cok}(\alpha^i)$, one obtains the following diagram:

$$\begin{tikzcd}
L^{i-1} \arrow[rr, "\delta^{i-1}"] & & L^i \arrow[rr, "\delta^i"] & & L^{i+1} & \\
 & \mathrm{Im}(\delta^{i-1}) \arrow[ur, "\mathrm{im}(\delta^{i-1})"] \arrow[rr, dashed, "\exists!\alpha^i"'] & & \mathrm{Ker}(\delta^i) \arrow[ul, "\ker(\delta^i)"'] \arrow[rr, "\mathrm{coker}(\alpha^i)"'] & & H^i(L^\bullet)
\end{tikzcd}$$

Applying functor F, we get:

$$\begin{tikzcd}
F(L^{i-1}) \arrow[rr, "F\delta^{i-1}"] & & L^i \arrow[rr, "F\delta^i"] & & F(L^{i+1}) & \\
 & F(\mathrm{Im}(\delta^{i-1})) \arrow[ur, "F\mathrm{im}(\delta^{i-1})"] \arrow[rr, "F\alpha^i"'] & & F(\mathrm{Ker}(\delta^i)) \arrow[ul, "F\ker(\delta^i)"'] \arrow[rr, "F\mathrm{coker}(\alpha^i)"'] & & F(H^i(L^\bullet))
\end{tikzcd}$$

Also, we've seen that kernel and cokernel of $F\varphi$ factor uniquely through the image of kernel and cokernel of φ (respectively) via invertible morphisms, so if apply the functor F first before consider its cohomology, together with the above diagrams, we get:

$$\begin{tikzcd}
F(L^{i-1}) \arrow[rr, "F\delta^{i-1}"] & & F(L^i) \arrow[rr, "F\delta^i"] & & F(L^{i+1}) \\
 & \mathrm{Im}(F\delta^{i-1}) \arrow[ur, "\mathrm{im}(F\delta^{i-1})"] \arrow[d, dashed, "\exists!f"', "\sim"] & & \mathrm{Ker}(F\delta^i) \arrow[ul, "\ker(F\delta^i)"'] \arrow[d, dashed, "\sim"', "\exists!g"] & \\
 & F(\mathrm{Im}(\delta^{i-1})) \arrow[uur, "F\mathrm{im}(\delta^{i-1})"'] \arrow[rr, "F\alpha^i"'] & & F(\mathrm{Ker}(\delta^i)) \arrow[uul, "F\ker(\delta^i)"] &
\end{tikzcd}$$

Hence, $\mathrm{im}(F\delta^{i-1})$ factors through $\ker(F\delta^i)$ via a morphism $\beta^i = g^{-1} \circ F\alpha^i \circ f : \mathrm{Im}(F\delta^{i-1}) \to \mathrm{Ker}(F\delta^i)$, or $\mathrm{im}(F\delta^{i-1}) = \ker(F\delta^i) \circ \beta^i$. With such factorization being unique, then β^i is the unique factorization of $\mathrm{im}(F\delta^{i-1})$ through $\ker(F\delta^i)$, so cohomology of $F(L^i)$ can be derived through $\mathrm{Cok}(\beta^i)$.

Finally, compile the diagrams above, we get the following diagram:

$$
\begin{tikzcd}
F(L^{i-1}) \arrow[rr, "F\delta^{i-1}"] & & F(L^i) \arrow[rr, "F\delta^i"] & & F(L^{i+1}) & \\
& \mathrm{Im}(F\delta^{i-1}) \arrow[ur, "\mathrm{im}(F\delta^{i-1})"] \arrow[rr, "\beta^i"] \arrow[d, "f"', "\sim"] & & \mathrm{Ker}(F\delta^i) \arrow[ul, "\ker(F\delta^i)"'] \arrow[rr, "\mathrm{coker}(\beta^i)"] \arrow[d, "g"', "\sim"] & & H^i(F(L^\bullet)) \\
& F(\mathrm{Im}(\delta^{i-1})) \arrow[rr, "F\alpha^i"'] & & F(\mathrm{Ker}(\delta^i)) \arrow[rr, "F\mathrm{coker}(\alpha^i)"'] & & F(H^i(L^\bullet))
\end{tikzcd}
$$

Which, based on the diagram, we get the following relation:

$$\mathrm{coker}(\beta^i) \circ (g^{-1} \circ F\alpha^i \circ f) = \mathrm{coker}(\beta^i) \circ \beta^i = 0$$

$$\implies (\mathrm{coker}(\beta^i) \circ g^{-1}) \circ F\alpha^i = 0 \circ f^{-1} = 0$$

This indicates $\mathrm{coker}(\beta^i) \circ g^{-1}$ factors uniquely through $\mathrm{coker}(F\alpha^i)$, where $F\mathrm{coker}(\alpha^i)$ satisfies such requirement (since in some intuitive sense, kernel / cokernel operation commutes with F, an exact functor). Hence, there exists unique morphism $\gamma : F(H^i(L^\bullet)) \to H^i(F(L^\bullet))$, where $\mathrm{coker}(\beta^i) \circ g^{-1} = \gamma \circ F\mathrm{coker}(\alpha^i)$, or $\mathrm{coker}(\beta^i) = \gamma \circ F\mathrm{coker}(\alpha^i) \circ g^{-1}$.

Similarly, another relation is given as below:

$$(F\mathrm{coker}(\alpha^i) \circ g) \circ \beta^i = (F\mathrm{coker}(\alpha^i) \circ g) \circ (g^{-1} \circ F\alpha^i \circ f) = (F\mathrm{coker}(\alpha^i) \circ F\alpha^i) \circ f = 0 \circ f = 0$$

This indicates $F\mathrm{coker}(\alpha^i) \circ g$ factors uniquely through $\mathrm{coker}(\beta^i)$, there exists unique morphism $\epsilon : H^i(F(L^\bullet)) \to F(H^i(L^\bullet))$, where $F\mathrm{coker}(\alpha^i) \circ g = \epsilon \circ \mathrm{coker}(\beta^i)$. Which, we end up with the following commutative diagram:

$$
\begin{tikzcd}
\mathrm{Im}(F\delta^{i-1}) \arrow[r, "\beta^i"] \arrow[d, "f"', "\sim"] & \mathrm{Ker}(F\delta^i) \arrow[r, "\mathrm{coker}(\beta^i)"] \arrow[d, "g"', "\sim"] & H^i(F(L^\bullet)) \arrow[d, dashed, shift right, "\exists!\epsilon"'] \\
F(\mathrm{Im}(\delta^{i-1})) \arrow[r, "F\alpha^i"'] & F(\mathrm{Ker}(\delta^i)) \arrow[r, "F\mathrm{coker}(\alpha^i)"'] & F(H^i(L^\bullet)) \arrow[u, dashed, shift right, "\exists!\gamma"']
\end{tikzcd}
$$

Where, the existence of such γ and ϵ guarantees $H^i(F(L^\bullet)) \cong F(H^i(L^\bullet))$, so the exact functor F preserves the i^{th} cohomology. Hence, if given the cohomology functor $H^\bullet$ on the category of cochain complexes, we get $H^\bullet(F(L^\bullet)) \cong F(H^\bullet(L^\bullet))$, due to the relation above isomorphic relation.