Braid Groups, and their Representations

Zih-Yu Hsieh Mentor: Choomno Moos
University of California Santa Barbara, College of Creative Studies

Introduction

Braid Group formulates the algebraic / topological relation of braids. One center of studies is the Representations and their kernels. Here we'll briefly introduce two - Burau and Gassner Representation.

Braid Groups & Mapping Class Groups

Def: An n strands *Braid Group* B_n is generated by $\{\sigma_1, ..., \sigma_{n-1}\}$, satisfying *Braid Relations*:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$, if $|i-j| \geq 2$
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

Def: Let D_n be an n-punctured disk. Its *Mapping Class Group* $\mathfrak{M}(D_n)$ collects isotopic classes of self-homeomorphisms that fixes disk boundary ∂D.

Ex: The i^{th} *Half Twist* $\tau_i \in \mathfrak{M}(D_n)$ swaps the i^{th} and $(i+1)^{\text{th}}$ punctures, while fixing the remaining ones.

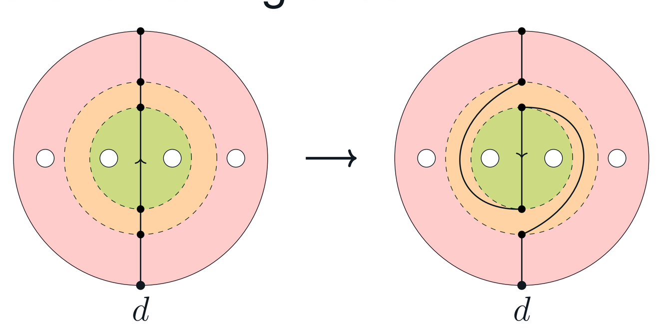

Figure: For $n = 4$, Half Twist τ_2 Swapping Punctures 2 and 3

Property: $B_n \cong \mathfrak{M}(D_n)$, by $\sigma_i \mapsto \tau_i$.

Braid Automorphism

Fix $d \in \partial D$, the fundamental group $\pi_1(D_n, d)$ is generated by the n loops, each surrounding a puncture, which $\pi_1(D_n, d) = F_n(x_1, ..., x_n)$, the *Degree-$n$ Free Group*.

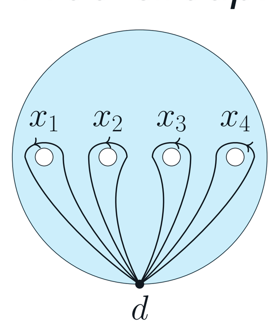

Figure: Loops Generating $\pi_1(D_4, d)$

Then, each homeomorphism in $\mathfrak{M}(D_n)$ generates a group automorphism on $\pi_1(D_n, d)$, called *Braid Automorphism*.

Ex: Half Twist's action on $\pi_1(D_n, d)$:

$$(\tau_i)_* \in \mathrm{Aut}(\pi_1(D_n, d)), \quad (\tau_i)_*(x_j) = \begin{cases} x_{i+1} & j = i \\ x_{i+1}x_ix_{i+1}^{-1} & j = i+1 \\ x_i & \text{Otherwise} \end{cases}$$

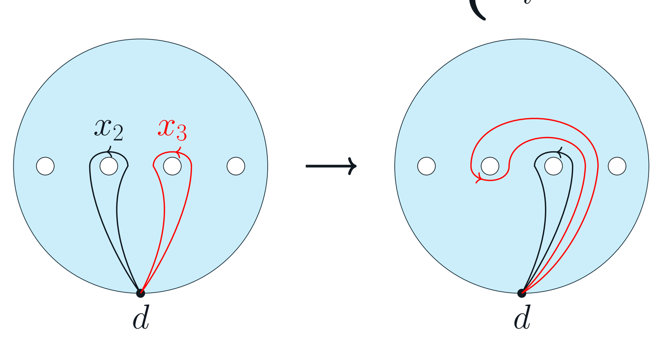

Figure: τ_2 Action on Loops in D_4

First Homology of Topological Space

Reduced Burau Representation

$\psi_n^r : B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t^{\pm}])$ satisfies:

$$\sigma_1 \mapsto \left(\begin{array}{cc|c} -t & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & I_{n-3} \end{array}\right), \quad \sigma_{n-1} \mapsto \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & t \\ 0 & 0 & -t \end{array}\right), \quad \sigma_i \mapsto \left(\begin{array}{c|ccc|c} I_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n-i-2} \end{array}\right)$$

Ex: Homological Perspective on D_4

A 4-punctured disk D_4 can "continuously deform" into 4 circles joining at one point ($\bigvee_{i=1}^{4} S^1$), $\implies$ Same Fundamental Group.

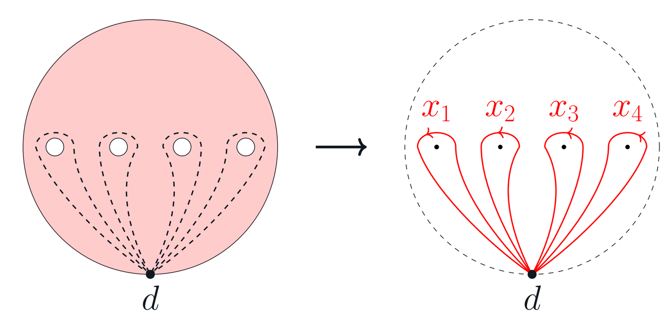

Figure: Deformation Retraction of D_4 to $\bigvee_{i=1}^{4} S^1$

Let $S^{(4)} := \bigvee_{i=1}^{4} S^1$, consider the space $\tilde{S}^{(4)}$ below, a *Covering Space* of $S^{(4)}$:

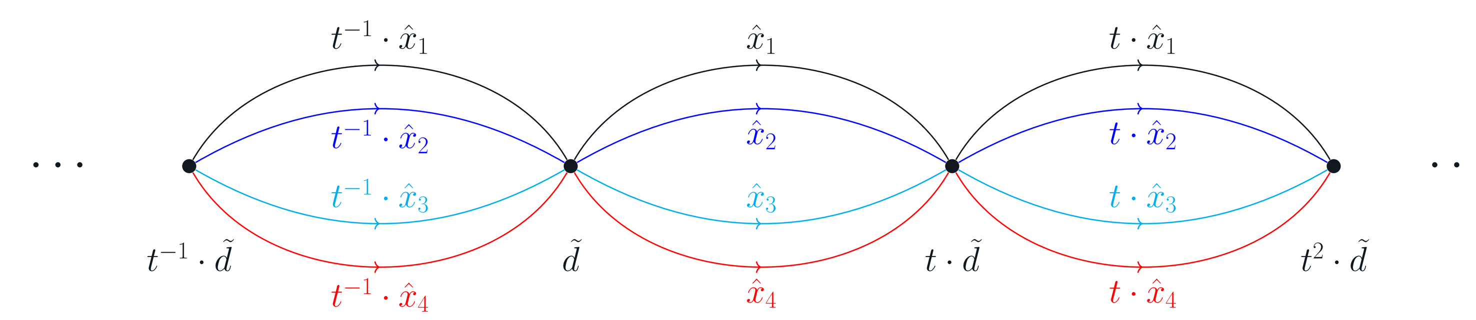

Figure: Infinite Cyclic Cover $\tilde{S}^{(4)}$

Here, t is a right shift of $\tilde{S}^{(n)}$ by degree 1:

- $t^k \cdot \tilde{d} =$ degree k right shift of $\tilde{d}$
- $t^k \cdot \hat{x}_i =$ degree k right shift of $\hat{x}_i$

There is a continuous covering map $p : \tilde{S}^{(4)} \to S^{(4)}$, each $p(t^k \cdot \hat{x}_i) = x_i$, and $p(t^k \cdot \tilde{d}) = d$.

Define the "Base Loops" $\ell_i := \hat{x}_{i+1} \cdot \hat{x}_i^{-1}$ (counterclockwise) for $1 \leq i \leq 3$:

- $-\ell_i =$ clockwise version of ℓ_i
- $t^k \cdot \ell_i =$ degree k right shift of ℓ_i

Then, all "Integer Laurent Polynomial" combination of ℓ_i forms $H_1(\tilde{S}^{(4)})$ as a free $\mathbb{Z}[t^{\pm}]$-module with basis ℓ_1, ℓ_2, ℓ_3.

Braid Group Action on Homology

Recall: braid automorphism $(\tau_2)_*$ of $\pi_1(D_4, d)$ satisfies $(\tau_2)_*(x_2) = x_3$, and $(\tau_2)_*(x_3) = x_3 \cdot x_2 \cdot x_3^{-1}$. Which, it uniquely lifts to an transformation on the ℓ_i via p:

EX: $\ell_2 = \hat{x}_3 \cdot \hat{x}_2^{-1} \mapsto \hat{x}_2 \cdot \left((t \cdot \hat{x}_2) \cdot (t \cdot \hat{x}_3^{-1}) \cdot \hat{x}_2^{-1}\right) = -t \cdot \ell_2$.

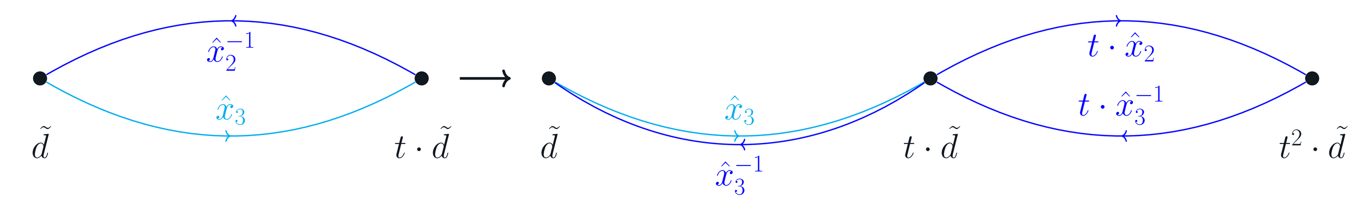

Figure: ℓ_2 (Counterclockwise) Maps to $-t \cdot \ell_2$ (Right Shift by degree 1, Clockwise)

Doing this for each ℓ_i, put into matrix form with basis $\{\ell_i\}$, we recover the Representation.

Gassner Representation

Instead of on braid groups B_n, this one is representing *Pure Braid Group* P_n: Given the map $B_n \to S_n$ (n^{th} Symmetry Group) by $\sigma_i \mapsto (i, i+1)$, P_n is the kernel of this morphism (Geometrically, it's the braids with the strand going from the i^{th} starting point to the i^{th} ending point, which forms identity as a permutation of the n endpoints).

If consider the covering map corresponding to the kernel of $\pi_1(S^{(n)}, d) \to \mathbb{Z}^n$ by $x_i \mapsto e_i$ (the i^{th} basis of $\mathbb{Z}^n$), it forms a representation $P_n \to \mathrm{GL}_n(\mathbf{Z}[t_1^{\pm}, ..., t_n^{\pm}])$.

Conclusion & Future Directions

Acknowledgement & Sources

We're genuinely thankful for the parent donors, Professor Cachadina, and Professor Casteels who made this program possible. We're also grateful for our mentor Choomno Moos with their effort and excellent guidance.

Source:

- Braid Groups (Christian Kassel, Vladimir Turaev)
- Algebraic Topology (Hatcher)
- Braids, Links, Mapping Class Groups (Joan Birman)