Braid Groups and their Representations

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Summer 2025

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1 Introduction

Braid group as a mathematical structure initiated by the study of geometric braids, has hidden in multiple fields of math: From the association with braid automorphisms of a free group, the class of isotopic self-homeomorphism of punctured disks, even to its usage of formulating interactions of anyons in Quantum Mechanics - it is embedded in multiple aspectts of both math and physics.

A specific direction of research involves the study of Braid Group Representation, and one longstanding question is about the faithfulness of two representations - Burau and Gassner Representations: For $n \geq 6$ Burau Representation is proven to be not faithful in [Professor D. Long's article] and refined down to the case of $n \geq 5$ in [Professor S. Bigelow's article]. In [Birman's Book] a theorem states the kernel of Gassner Representation is a subgroup of the kernel of Burau Representation, which another work proving faithfulness of Burau Representation for $n \leq 3$ in [One of the articles] also implies the faithfulness of Gassner Representation for same cases. Now, the faithfulness of Burau Representation for $n = 4$ is unknown, and similar case for Gassner Representation in case $n \geq 4$.

In this survey we'll aim to understand the construction of Burau and Gassner Representation. In **Section 2** we'll cover some fundamentals of Algebraic Topology such as Homotopy, Fundamental Group, Deformation Retract, Covering Space, and Homology.; in **Section 3** we'll introduce Braid Groups, including its algebraic definition, the isomorphism to the *Mapping Class Group* of punctured disks, and its realization as braid automorphisms on free groups - which turns out to be analogous to the Mapping Class Group's action on punctured disk's fundamental group. In **Section 4** and **5** we'll cover Burau and Gassner Representation - both definition wise, and their homological construction.

Hope you enjoy the text :) **(Yeah please delete this line, please.)**

2 Topological / Algebraic Preliminaries

2.1 Homotopy

First, we'll land on the formulation of "Continuous Deformation" - which is usefule when classifying maps between topological spaces.

Definition 1. Given X, Y as topological spaces, $f, g : X \to Y$ are continuous maps, a *Homotopy* between f and g is a continuous map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$, and $H(x, 1) = g(x)$.

Also, given subset $A \subseteq X$, H is a Homotopy between f and g *Relative to* A, if $H(a, t) = f(a) = g(a)$ for all $a \in A$ and $t \in [0, 1]$.

If treated $[0, 1]$ as the time interval, one can visualize $H(x, t)$ for each $t \in [0, 1]$ as a slice of the deformation from f to g, and having it relative to subset A implies such deformation is not changing A.

It is well-known tha homotopy is an equivalence relation on the collection of continuous functions between two spaces. With this in mind, if fixing a base point $x_0 \in X$, we can classify the classes of

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