

BRAID YOUR BRAID GROUP REPRESENTATIONS

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What's a Braid Group? Why Care?

Def: n -strand *Braid Group* B_n is generated by $\{\sigma_1, \dots, \sigma_{n-1}\}$, satisfying **Braid Relations**:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2; \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

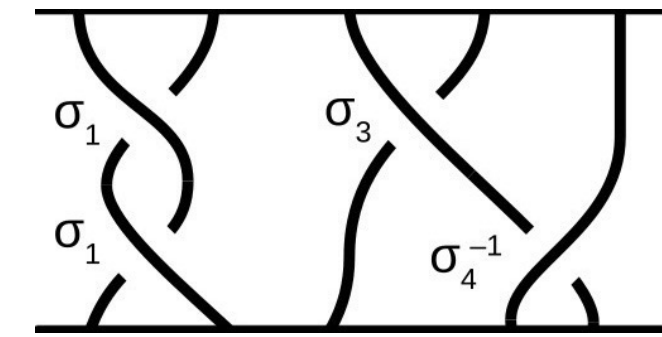


Figure: Geometric Braid of 4 Strands

Significance: Application in *Topological Quantum Computing*.

B_n as Mapping Class Groups

Def: Let D_n be an n -punctured disk. Its *Mapping Class Group* $\mathfrak{M}(D_n)$ collects isotopic classes of homeomorphisms fixing ∂D .

Ex: The i^{th} *Half Twist* $\tau_i \in \mathfrak{M}(D_n)$ swaps i^{th} , $(i+1)^{\text{th}}$ punctures.

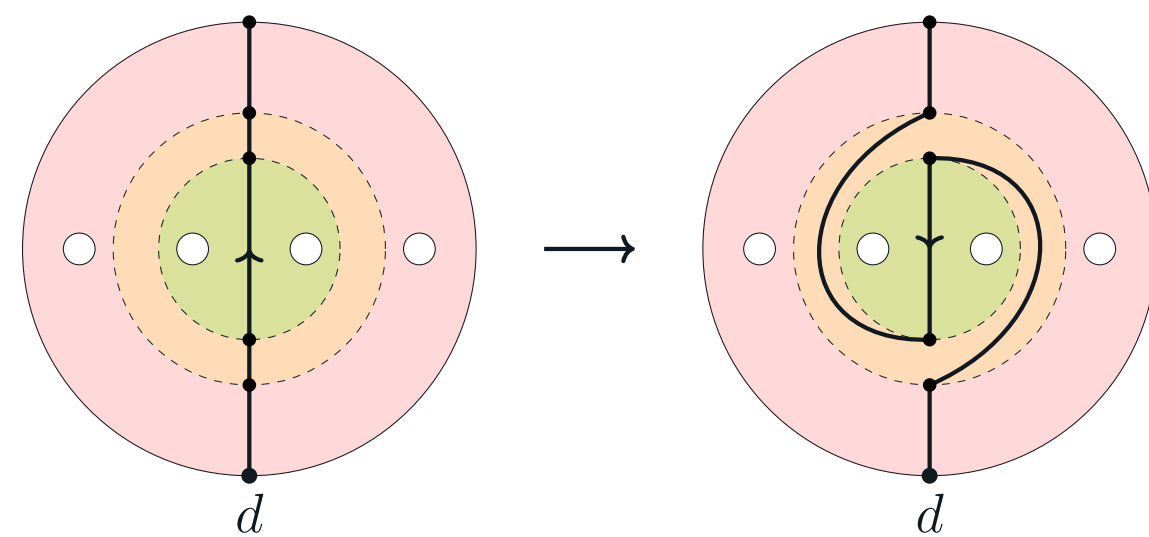


Figure: $n = 4$, Half Twist τ_2 's Action on D_4

Property: $B_n \cong \mathfrak{M}(D_n)$, by $\sigma_i \mapsto \tau_i$.

Braid Automorphism

Fix $d \in \partial D$, the **Fundamental Group** $\pi_1(D_n, d) \cong F_n$, a *Degree- n Free Group* generated by the n loops.

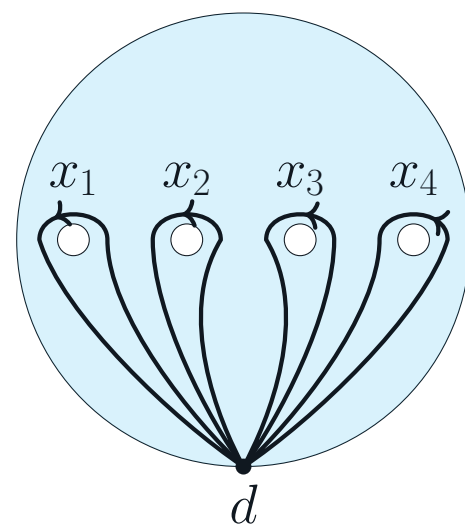


Figure: Loops Generating $\pi_1(D_4, d)$

Maps in $\mathfrak{M}(D_n)$ generate **Braid Automorphisms** on $\pi_1(D_n, d)$.

Ex: Half Twist's action on $\pi_1(D_n, d)$:

$$(\tau_i)_*(x_j) = \begin{cases} x_{i+1} & j = i \\ x_{i+1}x_i x_{i+1}^{-1} & j = i+1 \\ x_i & \text{Otherwise} \end{cases}$$

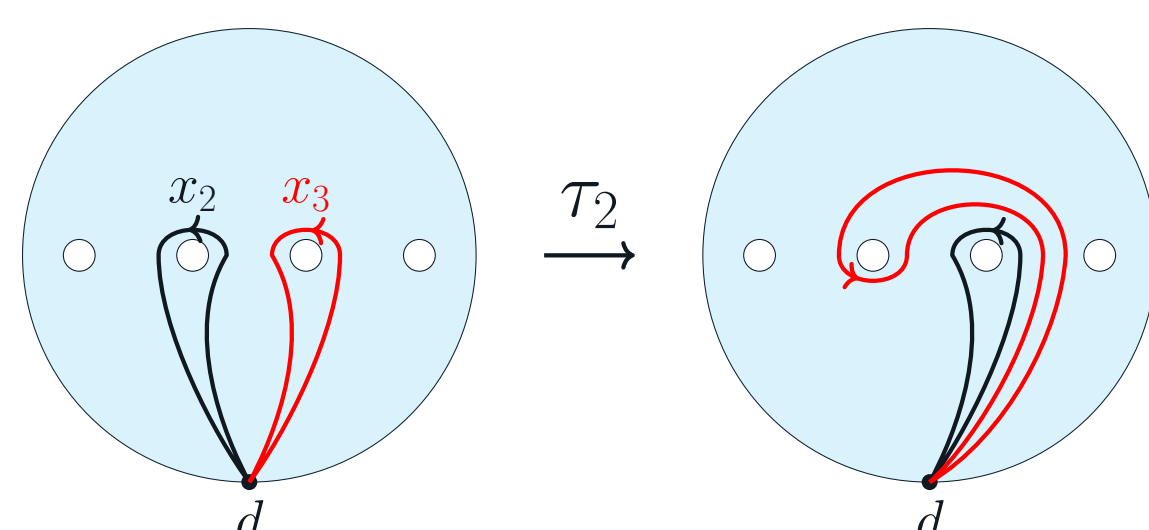


Figure: τ_2 Action on $\pi_1(D_4, d)$

Reduced Burau Representation

$\psi_n^r : B_n \rightarrow \text{GL}_{n-1}(\mathbb{Z}[t^{\pm}])$ satisfies:

$$\sigma_1 \mapsto \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}, \quad \sigma_{n-1} \mapsto \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix}, \quad \sigma_i \mapsto \begin{pmatrix} I_{i-2} & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{pmatrix}$$

EX: Burau Representation on D_4

D_4 **Continuously Deforms** into 4 circles join at a point ($\bigvee_{i=1}^4 S^1$).

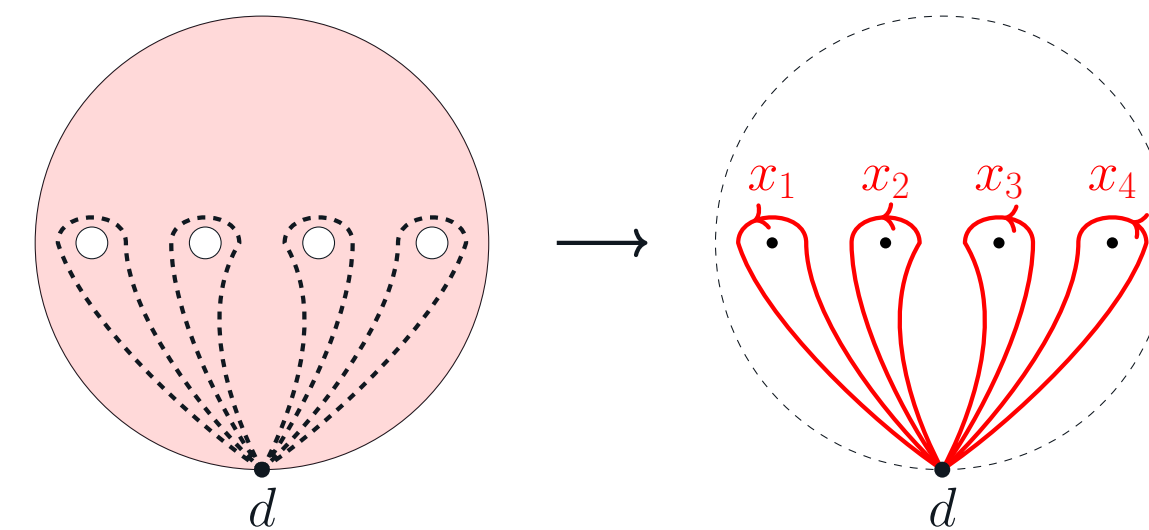


Figure: Deformation Retraction of D_4 to $\bigvee_{i=1}^4 S^1$

Let $S^{(4)} := \bigvee_{i=1}^4 S^1$, below is its *Infinite Cyclic Covering Space* $\tilde{S}^{(4)}$:

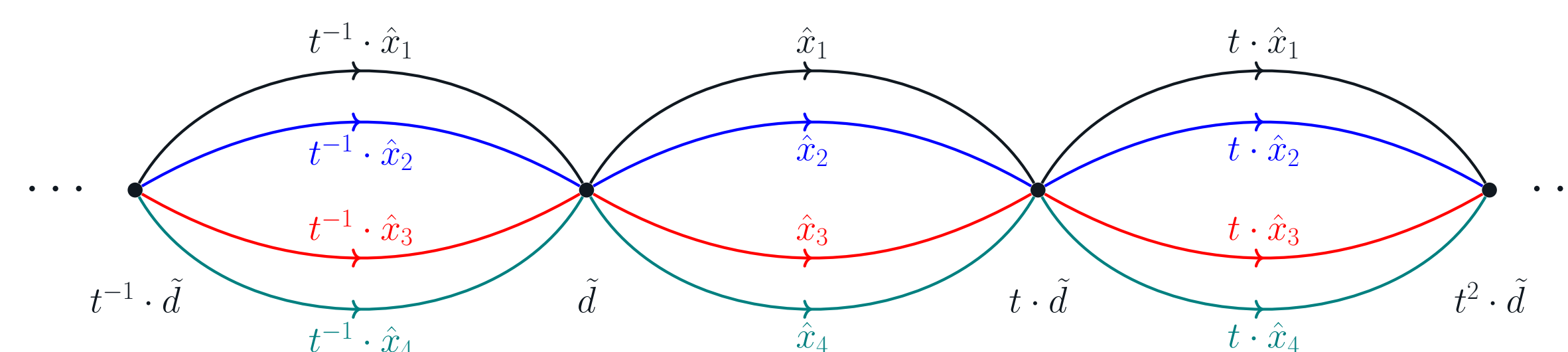


Figure: Infinite Cyclic Cover $\tilde{S}^{(4)}$

Rmk 1: t shifts $\tilde{S}^{(n)}$ by Degree 1.

Rmk 2: \exists *Covering Map* $p : \tilde{S}^{(4)} \rightarrow S^{(4)}$, each $p(t^k \cdot \hat{x}_i) = x_i$, and $p(t^k \cdot \tilde{d}) = d$.

Rmk 3: "Base Loops" $\ell_i := \hat{x}_{i+1} \cdot \hat{x}_i^{-1}$ (counterclockwise) for $1 \leq i \leq 3$:

- $-\ell_i$ = clockwise version of ℓ_i
- $t^k \cdot \ell_i$ = degree k right shift of ℓ_i

Property: all ℓ_i 's **Integer Laurent Polynomial** combinations form $\tilde{S}^{(4)}$'s *First Homology*, $H_1(\tilde{S}^{(4)})$ as a free $\mathbb{Z}[t^{\pm}]$ -module with basis $\{\ell_1, \ell_2, \ell_3\}$.

Action on Homology $H_1(\tilde{S}^{(4)})$

Recall: Braid Automorphism $(\tau_2)_*$ satisfies $(\tau_2)_*(x_2) = x_3$, and $(\tau_2)_*(x_3) = x_3 \cdot x_2 \cdot x_3^{-1}$. Which, it acts on base loops of $\tilde{S}^{(4)}$:

EX: $\ell_2 = \hat{x}_3 \cdot \hat{x}_2^{-1} \mapsto (\hat{x}_3 \cdot (t \cdot \hat{x}_2) \cdot (t \cdot \hat{x}_3^{-1})) \cdot \hat{x}_3^{-1} = -t \cdot \ell_2$.

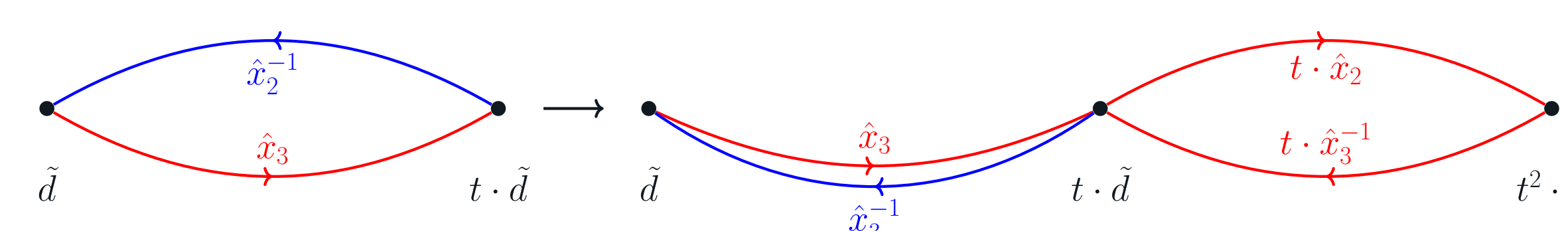


Figure: ℓ_2 Maps to $-t \cdot \ell_2$ via τ_2

Doing this for each ℓ_i , put into matrix with basis $\{\ell_i\}$, we recover the Representation.

Gassner Representation

Def: The *Pure Braid Group* $P_n \subset B_n$, is the kernel of the map $B_n \rightarrow S_n$ by $\sigma_i \mapsto (i, i+1)$.

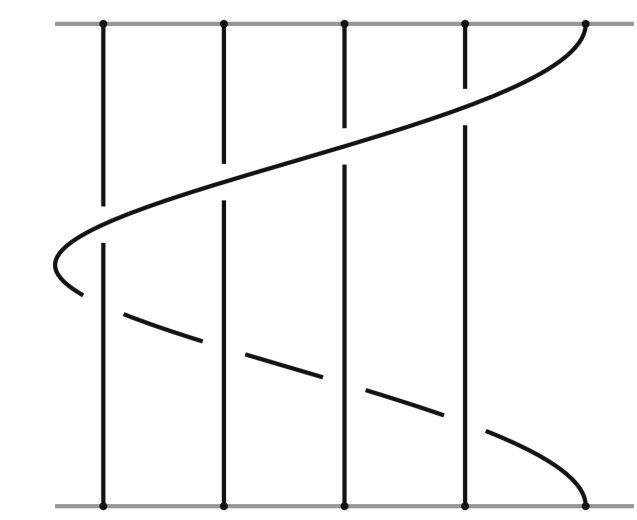


Figure: Example of a Pure Braid in P_5

If consider the *Integer Lattice* \mathbb{Z}^n together with the shifts of connection segments \hat{x}_i connecting $\bar{0}$ to e_i (the elementary basis of \mathbb{Z}^n), it again forms a *Covering Space* of $S^{(n)} := \bigvee_{i=1}^n S^1$, with covering map $\bar{d} \mapsto d$ for all $\bar{d} \in \mathbb{Z}^n$, and $\hat{x}_i \mapsto x_i$.

Then, each *Pure Braid* $\rho \in P_n$ (with $P_n \subset \mathfrak{M}(D_n) \cong B_n$) lifts to an action on the *First Homology* of the covering space, and forms the *Gassner Representation*.

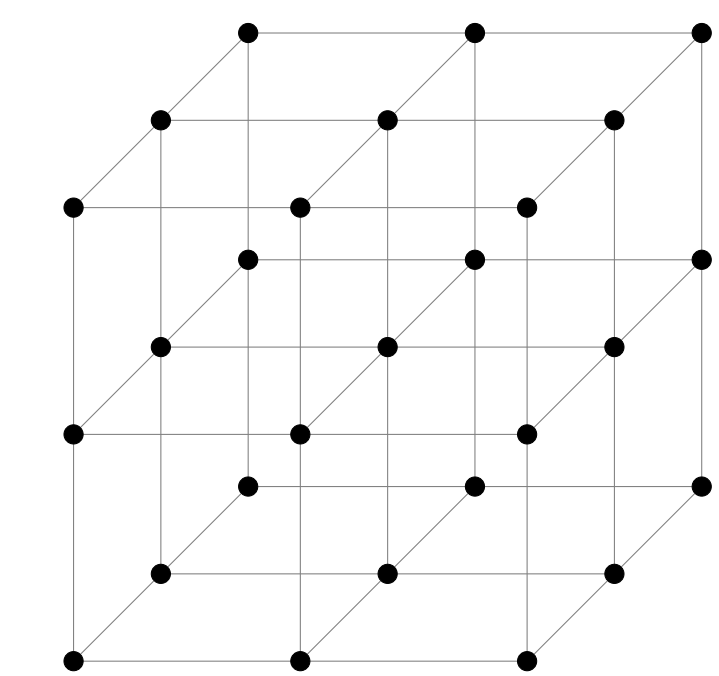


Figure: Gassner Representation's Covering Space of $S^{(3)}$

Future Directions

Open Problems:

1. For $n = 4$, is Burau Representation Faithful?
2. For $n \geq 4$, is Gassner Representation Faithful?

Future Direction: Continue on Studying kernels of the Representations, or other homological representations of Braid Groups.

Acknowledgement & Sources

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Source:

- Braid Groups (Christian Kassel, Vladimir Turaev)
- Algebraic Topology (Hatcher)
- Braids, Links, Mapping Class Groups (Joan Birman)