Braid out your Braid Group Representations

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What's a Braid Group? Why Care?

Def: An n strands *Braid Group* B_n is generated by $\{\sigma_1, ..., \sigma_{n-1}\}$, satisfying **Braid Relations**:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$, if $|i-j| \geq 2$
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

B_n as Mapping Class Groups

Def: Let D_n be an n-punctured disk. Its *Mapping Class Group* $\mathfrak{M}(D_n)$ collects isotopic classes of self-homeomorphisms that fixes disk boundary ∂D.

Ex: The i^{th} *Half Twist* $\tau_i \in \mathfrak{M}(D_n)$ swaps the i^{th} and $(i+1)^{\text{th}}$ punctures, while fixing the remaining ones.

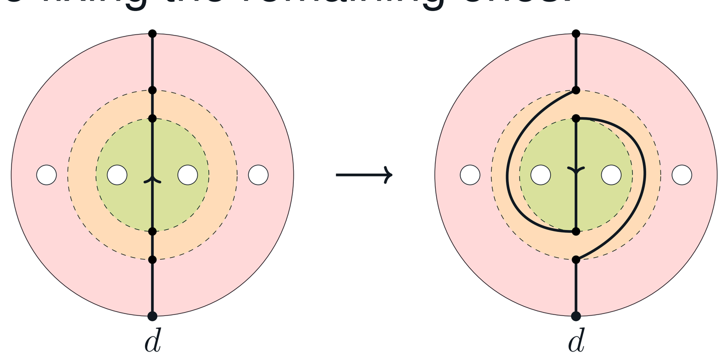

Figure: For $n = 4$, Half Twist τ_2 Swapping Punctures 2 and 3

Property: $B_n \cong \mathfrak{M}(D_n)$, by $\sigma_i \mapsto \tau_i$.

Braid Automorphism

Fix $d \in \partial D$, the **Fundamental Group** $\pi_1(D_n, d) \cong F_n(x_1, ..., x_n)$, a *Degree-$n$ Free Group* generated by the n loops.

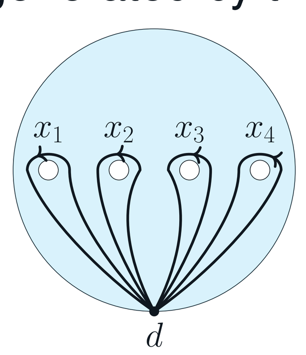

Figure: Loops Generating $\pi_1(D_4, d)$

Maps in $\mathfrak{M}(D_n)$ generate **Braid Automorphisms** on $\pi_1(D_n, d)$.

Ex: Half Twist's action on $\pi_1(D_n, d)$:

$$(\tau_i)_* \in \mathrm{Aut}(\pi_1(D_n, d)), \quad (\tau_i)_*(x_j) = \begin{cases} x_{i+1} & j = i \\ x_{i+1}x_i x_{i+1}^{-1} & j = i+1 \\ x_i & \text{Otherwise} \end{cases}$$

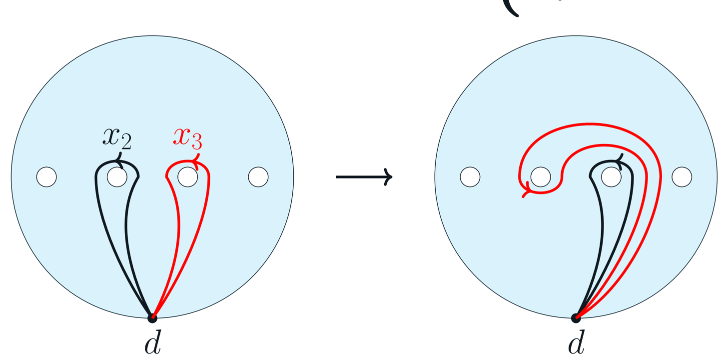

Figure: τ_2 Action on Loops in D_4

EX: Burau Representation on D_4

D_4 **Continuously Deforms** into 4 circles join at a point ($\bigvee_{i=1}^4 S^1$), $\implies$ Same Fundamental Group.

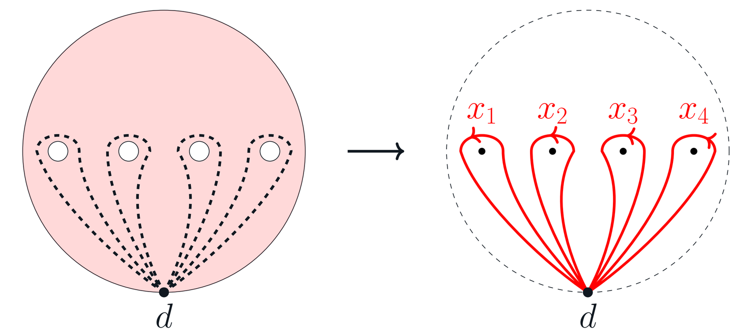

Figure: Deformation Retraction of D_4 to $\bigvee_{i=1}^4 S^1$

Let $S^{(4)} := \bigvee_{i=1}^4 S^1$, below is its *Infinite Cyclic Covering Space* $\tilde{S}^{(4)}$:

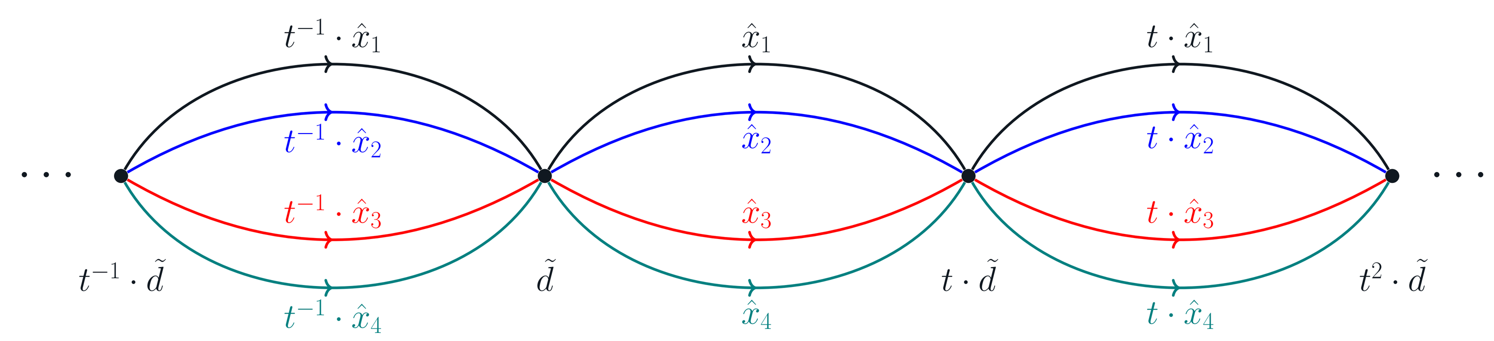

Figure: Infinite Cyclic Cover $\tilde{S}^{(4)}$

Rmk 1: t shifts $\tilde{S}^{(n)}$ by Degree 1.

Rmk 2: $\exists$ *Covering Map* $p : \tilde{S}^{(4)} \to S^{(4)}$, each $p(t^k \cdot \hat{x}_i) = x_i$, and $p(t^k \cdot \tilde{d}) = d$.

Rmk 3: "Base Loops" $\ell_i := \hat{x}_{i+1} \cdot \hat{x}_i^{-1}$ (counterclockwise) for $1 \leq i \leq 3$:

- $-\ell_i =$ clockwise version of ℓ_i
- $t^k \cdot \ell_i =$ degree k right shift of ℓ_i

Then, all **Integer Laurent Polynomial** combinations of ℓ_i forms the *First Homology* of $\tilde{S}^{(4)}$, $H_1(\tilde{S}^{(4)})$ as a free $\mathbb{Z}[t^{\pm}]$-module with basis ℓ_1, ℓ_2, ℓ_3.

Action on Homology $H_1(\tilde{S}^{(4)})$

Recall: Braid Automorphism $(\tau_2)_*$ satisfies $(\tau_2)_*(x_2) = x_3$, and $(\tau_2)_*(x_3) = x_3 \cdot x_2 \cdot x_3^{-1}$. Which, it acts on base loops of $\tilde{S}^{(4)}$:

EX: $\ell_2 = \hat{x}_3 \cdot \hat{x}_2^{-1} \mapsto \left(\hat{x}_2 \cdot (t \cdot \hat{x}_2) \cdot (t \cdot \hat{x}_3^{-1})\right) \cdot \hat{x}_2^{-1} = -t \cdot \ell_2$.

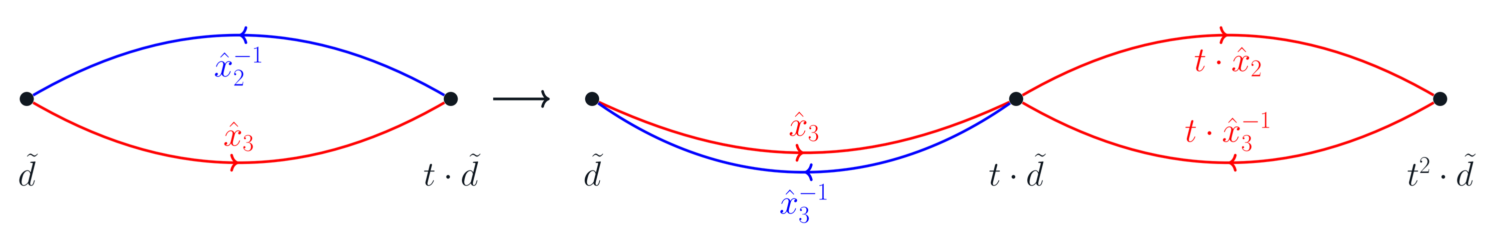

Figure: ℓ_2 (Counterclockwise) Maps to $-t \cdot \ell_2$ (Right Shift by degree 1, Clockwise)

Doing this for each ℓ_i, put into matrix with basis $\{\ell_i\}$, we recover the Representation.

Reduced Burau Representation

$\psi_n^r : B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t^{\pm}])$ satisfies:

$$\sigma_1 \mapsto \left(\begin{array}{cc|c} -t & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & I_{n-3} \end{array}\right), \quad \sigma_{n-1} \mapsto \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & t \\ 0 & 0 & -t \end{array}\right), \quad \sigma_i \mapsto \left(\begin{array}{c|ccc|c} I_{i-2} & 0 & 0 & 0 & \\ \hline 0 & 1 & t & 0 & \\ 0 & 0 & -t & 0 & \\ 0 & 0 & 1 & 1 & \\ \hline 0 & 0 & 0 & 0 & I_{n-} \end{array}\right.$$

Conclusion & Future Directions

We've introduced some basics of Braid Groups, follow with the homological construction of Burau and Gassner Representations.

This project is intended to continue on with the study of relating kernels of the Representations, and/or other homological representations of Braid Groups.

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