Braid Groups, and their Representations

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Introduction

Braid Groups & Mapping Class Groups

Def: Braid group of n strands B_n is generated by $n-1$ elements $\{\sigma_1, ..., \sigma_{n-1}\}$, satisfying *Braid Relations*:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$, if $|i-j| \geq 2$
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

Def: Let D_n be a disk with n holes. The *Mapping Class Group* $\mathfrak{M}(D_n)$ collects classes isotopic self-homeomorphisms on D_n that fixes disk boundary ∂D, and sends punctures to punctures.
Ex: The i^{th} *Half Twist* is a Self-Homeomorphism of D_n, swapping the i^{th} and $(i+1)^{\text{th}}$ punctures, while fixing the remaining ones.

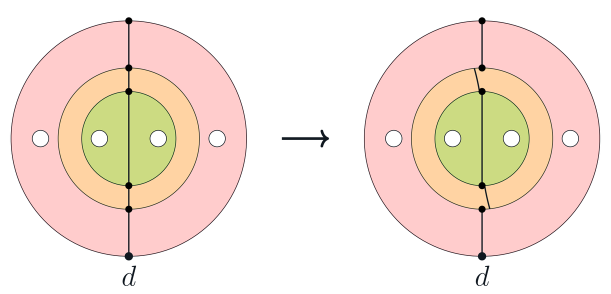

Figure: For $n = 4$, Half Twist τ_2 Swapping Punctures 2 and 3

Property: Collection of *Half Twists* $\tau_1, ..., \tau_{n-1}$ generates $\mathfrak{M}(D_n)$ and satisfies *Braid Relations*; in fact, $B_n \cong \mathfrak{M}(D_n)$, by $\sigma_i \mapsto \tau_i$.

Fundamental Group of D_n & Braid Automorphism

For n-punctured disk $D - Q_n$, fixing a base point $d \in \partial D$, the fundamental group $\pi_1(D - Q_n, d)$ is generated by the n nontrivial loops, each surrounding a specific puncture, which $\pi_1(D - Q_n, d) = F_n(x_1, ..., x_n)$ (the *Free Group* of n generators, each x_i is a loop around the i^{th} puncture).

Then, each self-homeomorphism in the mapping class group $\mathfrak{M}(D - Q_n)$ generates a group automorphism on the fundamental group $\pi_1(D - Q_n, d)$, called *Braid Automorphism*. Specifically, each τ_i acts on the fundamental group.
Which, the corresponding Braid Automorphism is:

$$(\tau_i)_* \in \mathrm{Aut}(\pi_1(D - Q_n, d)), \quad (\tau_i)_*(x_j) = \begin{cases} x_i x_{i+1} x_i^{-1} & j = i \\ x_i & j = i+1 \\ x_j & \text{Otherwise} \end{cases}$$

The collection of $(\tau_i)_*$ again satisfies the Braid Relations, which generates a group homomorphism $B_n \to \mathrm{Aut}(\pi_1(D - Q_n, d))$ by $\sigma_i \mapsto (\tau_i)_*$; this homomorphism is in fact injective, so the braid group of n strands B_n can be viewed as a subgroup of $\mathrm{Aut}(\pi_1(D - Q_n, d))$.

Burau Representation

Definition - Reduced Burau Representation: $\psi_n^r : B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t^{\pm}])$ satisfies:

$$\psi_n^r(\sigma_1) = \left(\begin{array}{cc|c} -t & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & I_{n-3} \end{array}\right), \quad \psi_n^r(\sigma_{n-1}) = \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & t \\ 0 & 0 & -t \end{array}\right), \quad \psi_n^r(\sigma_i) = \left(\begin{array}{c|ccc|c} I_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n-i-2} \end{array}\right)$$

EX / Homological Perspective: A 4-punctured disk D_4 can "continuously deform" into 4 circles joining at a single point ($\bigvee_{i=1}^4 S^1$), $\implies$ Same loop structures (Fundamental Group).

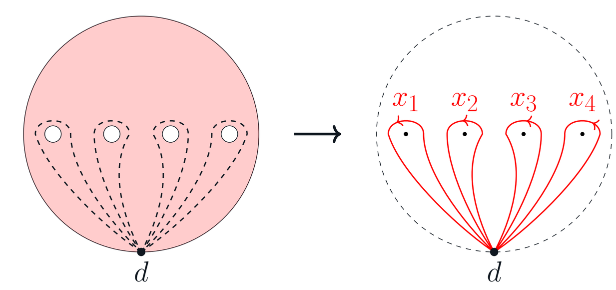

Figure: Deformation Retraction of D_4 to $\bigvee_{i=1}^4 S^1$

Let $S^{(4)} := \bigvee_{i=1}^4 S^1$, consider the following space $\tilde{S}^{(4)}$:

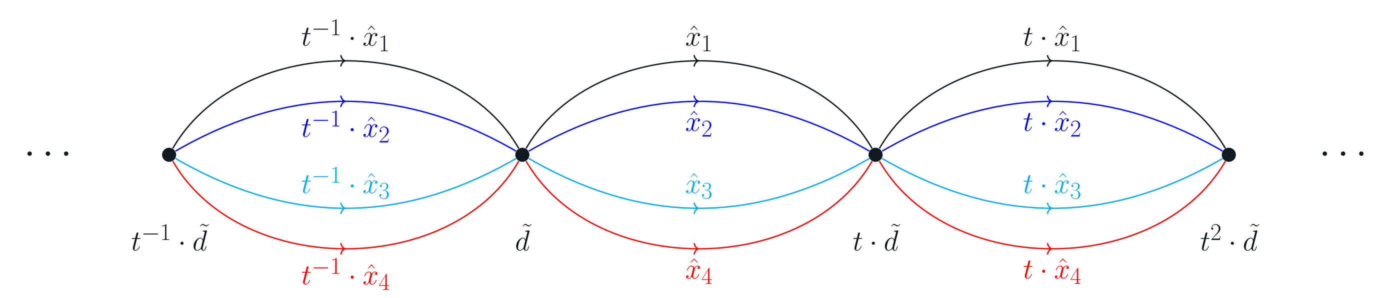

Figure: Infinite Cyclic Cover $\tilde{S}^{(4)}$

Here, t is a right shift of $\tilde{S}^{(n)}$ by degree 1:

- $t^k \cdot \tilde{d} =$ degree k right shift of $\tilde{d}$
- $t^k \cdot \hat{x}_i =$ degree k right shift of $\hat{x}_i$

There is a continuous map $p : \tilde{S}^{(4)} \to S^{(4)}$, each $p(t^k \cdot \hat{x}_i) = x_i$, and $p(t^k \cdot \tilde{d}) = d$.
Define the "Base Loops" $\ell_i := \hat{x}_{i+1} \cdot \hat{x}_i^{-1}$ (counterclockwise) for $1 \leq i \leq 3$:

- $-\ell_i =$ counterclockise version of ℓ_i
- $t^k \cdot \ell_i =$ degree k right shift of ℓ_i

Then, all "Integer Laurent Polynomial" combination of ℓ_i forms $H_1(\tilde{S}^{(4)})$ as a free $\mathbb{Z}[t^{\pm}]$-module with basis ℓ_1, ℓ_2, ℓ_3.
Recall: braid automorphism $(\tau_2)_*$ of $\pi_1(D_4, d)$ satisfies $(\tau_2)_*(x_2) = x_2 \cdot x_3 \cdot x_2^{-1}$, and $(\tau_2)_*(x_3) = x_2$. Which, it uniquely lifts to an transformation on the ℓ_i via p:
EX: $\ell_2 = \hat{x}_3 \cdot \hat{x}_2^{-1} \mapsto \hat{x}_2 \cdot \left((t \cdot \hat{x}_2) \cdot (t \cdot \hat{x}_3^{-1}) \cdot \hat{x}_2^{-1}\right) = -t \cdot \ell_2$.

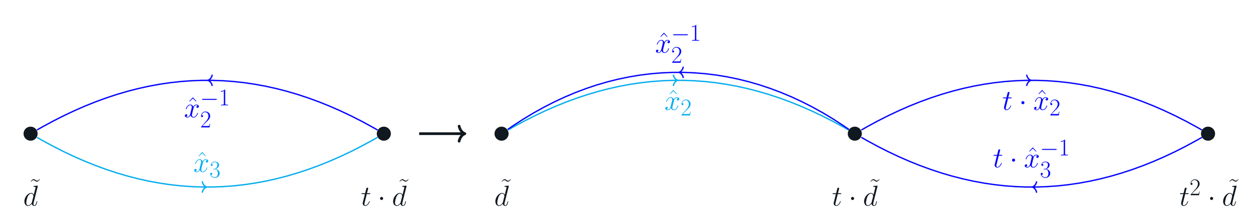

Figure: ℓ_2 (Counterclockwise) Maps to $-t \cdot \ell_2$ (Right Shift by degree 1, Clockwise)

Doing this for each ℓ_i, and write into matrix form with basis ℓ_i, we recover the Representation.

Conclusion & Future Directions

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- Braids, Links, Mapping Class Groups (Joan Birman)
- Briad Groups (Christian Kassel, Vladimir Turaev)
- Category Theory in Context (Emily Riehl)
- Algebra Chapter 0 (Paolo Aluffi)