Braid Groups, and their Representations

Zih-Yu Hsieh Mentor: Choomno Moos

University of California Santa Barbara, College of Creative Studies

Introduction

Braid Groups & Mapping Class Groups

Def: Braid group of n strands B_n is generated by $n-1$ elements $\{\sigma_1, ..., \sigma_{n-1}\}$, satisfying *Braid Relations*:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$, if $|i-j| \geq 2$
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

Def: Let D_n be an n-punctured disk. The *Mapping Class Group* $\mathfrak{M}(D_n)$ collects classes of isotopic self-homeomorphisms on D_n that fixes disk boundary ∂D.

Ex: The i^{th} *Half Twist* $\tau_i \in \mathfrak{M}(D_n)$ swaps the i^{th} and $(i+1)^{\text{th}}$ punctures, while fixing the remaining ones.

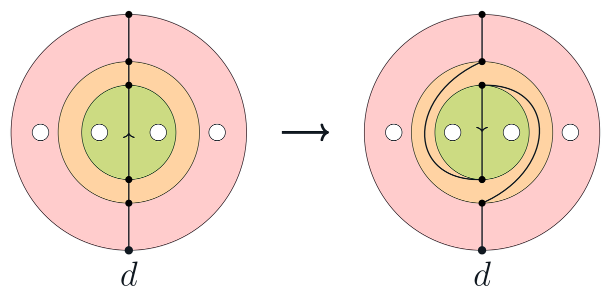

Figure: For $n=4$, Half Twist τ_2 Swapping Punctures 2 and 3

Property: *Half Twists* $\tau_1, ..., \tau_{n-1}$ generates $\mathfrak{M}(D_n)$ and satisfies *Braid Relations*; in fact, $B_n \cong \mathfrak{M}(D_n)$, by $\sigma_i \mapsto \tau_i$.

Fundamental Group of D_n & Braid Automorphism

For n-punctured disk D_n, fix $d \in \partial D$, the fundamental group $\pi_1(D_n, d)$ is generated by the n loops, each surrounding a puncture, which $\pi_1(D_n, d) = F_n(x_1, ..., x_n)$, the *Degree-$n$ Free Group*.

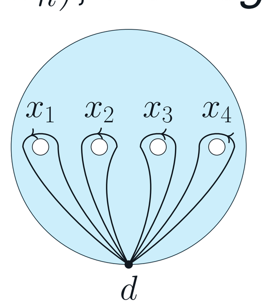

Figure: Fundamental Group of D_4

Then, each homeomorphism in $\mathfrak{M}(D_n)$ generates a group automorphism on $\pi_1(D_n, d)$, called *Braid Automorphism*.

Ex: Half Twist's action on $\pi_1(D_n, d)$:

$$(\tau_i)_* \in \mathrm{Aut}(\pi_1(D_n, d)), \quad (\tau_i)_*(x_j) = \begin{cases} x_{i+1} & j = i \\ x_{i+1}x_ix_{i+1}^{-1} & j = i+1 \\ x_i & \text{Otherwise} \end{cases}$$

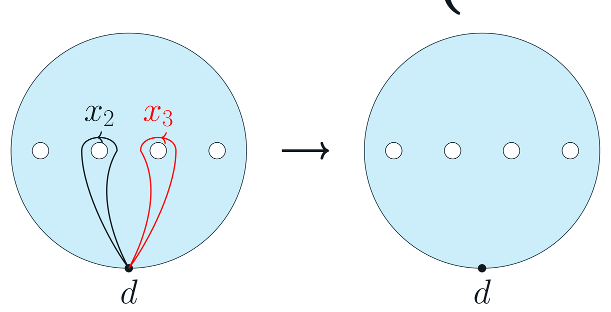

Figure: τ_2 Action on Loops in D_4

Reduced Burau Representation

$\psi_n^r : B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t^{\pm}])$ satisfies:

$$\sigma_1 \mapsto \left(\begin{array}{cc|c} -t & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & I_{n-3} \end{array}\right), \quad \sigma_{n-1} \mapsto \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & t \\ 0 & 0 & -t \end{array}\right), \quad \sigma_i \mapsto \left(\begin{array}{c|ccc|c} I_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n-i-2} \end{array}\right)$$

Ex: Homological Perspective on D_4

A 4-punctured disk D_4 can "continuously deform" into 4 circles joining at one point ($\bigvee_{i=1}^4 S^1$), $\implies$ Same Fundamental Group.

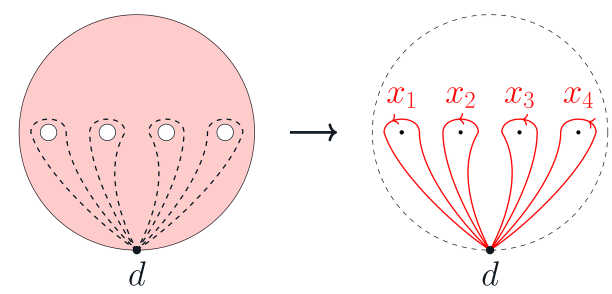

Figure: Deformation Retraction of D_4 to $\bigvee_{i=1}^4 S^1$

Let $S^{(4)} := \bigvee_{i=1}^4 S^1$, consider the space $\tilde{S}^{(4)}$ below, a *Covering Space* of $S^{(4)}$:

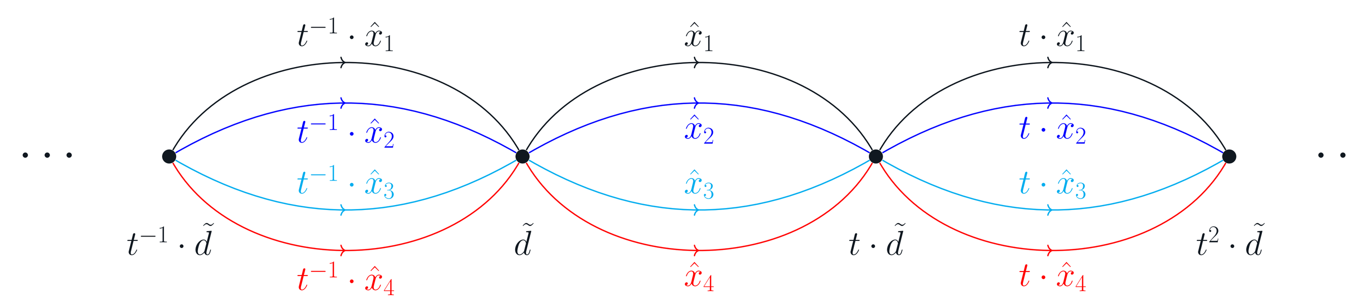

Figure: Infinite Cyclic Cover $\tilde{S}^{(4)}$

Here, t is a right shift of $\tilde{S}^{(n)}$ by degree 1:

- $t^k \cdot \tilde{d} =$ degree k right shift of $\tilde{d}$
- $t^k \cdot \hat{x}_i =$ degree k right shift of $\hat{x}_i$

There is a continuous covering map $p : \tilde{S}^{(4)} \to S^{(4)}$, each $p(t^k \cdot \hat{x}_i) = x_i$, and $p(t^k \cdot \tilde{d}) = d$.

Define the "Base Loops" $\ell_i := \hat{x}_{i+1} \cdot \hat{x}_i^{-1}$ (counterclockwise) for $1 \leq i \leq 3$:

- $-\ell_i =$ clockwise version of ℓ_i
- $t^k \cdot \ell_i =$ degree k right shift of ℓ_i

Then, all "Integer Laurent Polynomial" combination of ℓ_i forms $H_1(\tilde{S}^{(4)})$ as a free $\mathbb{Z}[t^{\pm}]$-module with basis ℓ_1, ℓ_2, ℓ_3.

Braid Group's Action on Covering Space

Recall: braid automorphism $(\tau_2)_*$ of $\pi_1(D_4, d)$ satisfies $(\tau_2)_*(x_2) = x_3$, and $(\tau_2)_*(x_3) = x_3 \cdot x_2 \cdot x_3^{-1}$. Which, it uniquely lifts to an transformation on the ℓ_i via p:

EX: $\ell_2 = \hat{x}_3 \cdot \hat{x}_2^{-1} \mapsto \hat{x}_2 \cdot \left((t \cdot \hat{x}_2) \cdot (t \cdot \hat{x}_3^{-1}) \cdot \hat{x}_2^{-1}\right) = -t \cdot \ell_2$.

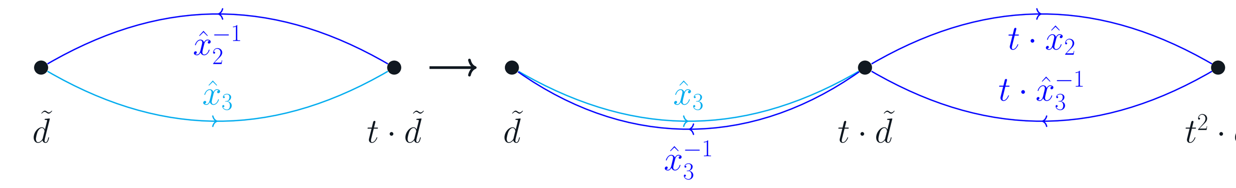

Figure: ℓ_2 (Counterclockwise) Maps to $-t \cdot \ell_2$ (Right Shift by degree 1, Clockwise)

Doing this for each ℓ_i, put into matrix form with basis $\{\ell_i\}$, we recover the Representation.

Gassner Representation

Instead of on braid groups B_n, this one is representing *Pure Braid Group* P_n: Given the map $B_n \to S_n$ (n^{th} Symmetry Group) by $\sigma_i \mapsto (i, i+1)$, P_n is the kernel of this morphism (Geometrically, it's the braids with the strand going from the i^{th} starting point to the i^{th} ending point, which forms identity as a permutation of the n endpoints).

If consider the covering map corresponding to the kernel of $\pi_1(S^{(n)}, d) \to \mathbb{Z}^n$ by $x_i \mapsto e_i$ (the i^{th} basis of $\mathbb{Z}^n$), it forms a representation $P_n \to \mathrm{GL}_n(\mathbf{Z}[t_1^{\pm}, ..., t_n^{\pm}])$.

Conclusion & Future Directions

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- Braid Groups (Christian Kassel, Vladimir Turaev)
- Algebraic Topology (Hatcher)
- Braids, Links, Mapping Class Groups (Joan Birman)