Braid your Braid Group Representations

UC Santa Barbara, College of Creative Studies

What's a Braid Group?

Def: n-strand *Braid Group* B_n is generated by $\{\sigma_1, ..., \sigma_{n-1}\}$, satisfying **Braid Relations**:

$\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$; $\quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

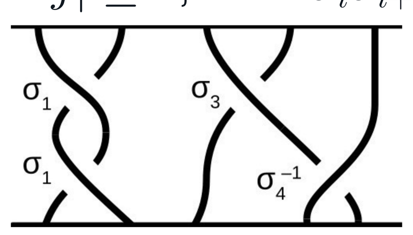

Figure: Geometric Braid of 4 strands

B_n as Mapping Class Groups

Def: Let D_n be an n-punctured disk. Its *Mapping Class Group* $\mathfrak{M}(D_n)$ collects isotopic classes of self-homeomorphisms that fixes disk boundary ∂D.
Ex: The i^{th} *Half Twist* $\tau_i \in \mathfrak{M}(D_n)$ swaps i^{th}, $(i+1)^{\text{th}}$ punctures.

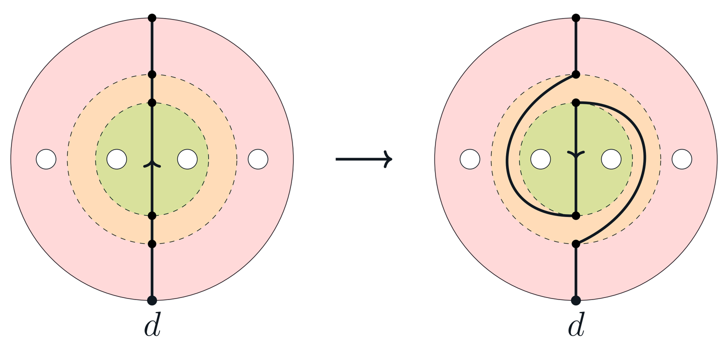

Figure: $n=4$, Half Twist τ_2's Action on D_4

Property: $B_n \cong \mathfrak{M}(D_n)$, by $\sigma_i \mapsto \tau_i$.

Braid Automorphism

Fix $d \in \partial D$, the **Fundamental Group** $\pi_1(D_n, d) \cong F_n$, a *Degree-n Free Group* generated by the n loops.

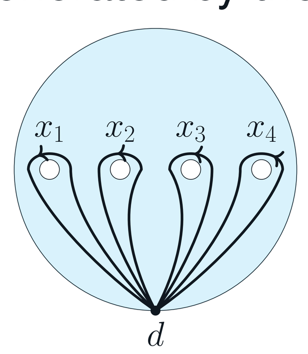

Figure: Loops Generating $\pi_1(D_4, d)$

Maps in $\mathfrak{M}(D_n)$ generate **Braid Automorphisms** on $\pi_1(D_n, d)$.
Ex: Half Twist's action on $\pi_1(D_n, d)$:

$$(\tau_i)_*(x_j) = \begin{cases} x_{i+1} & j = i \\ x_{i+1}x_ix_{i+1}^{-1} & j = i+1 \\ x_i & \text{Otherwise} \end{cases}$$

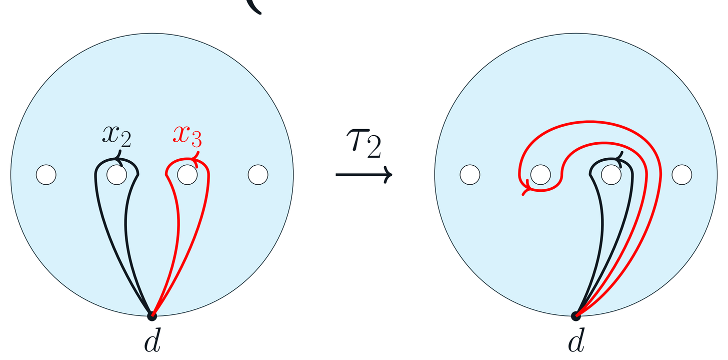

Figure: τ_2 Action on $\pi_1(D_4, d)$

Reduced Burau Representation

$\psi_n^r : B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t^{\pm}])$ satisfies:

$$\sigma_1 \mapsto \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}, \ \sigma_{n-1} \mapsto \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix}, \ \sigma_i \mapsto \begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{pmatrix}$$

EX: Burau Representation on D_4

D_4 **Continuously Deforms** into 4 circles join at a point ($\bigvee_{i=1}^4 S^1$).

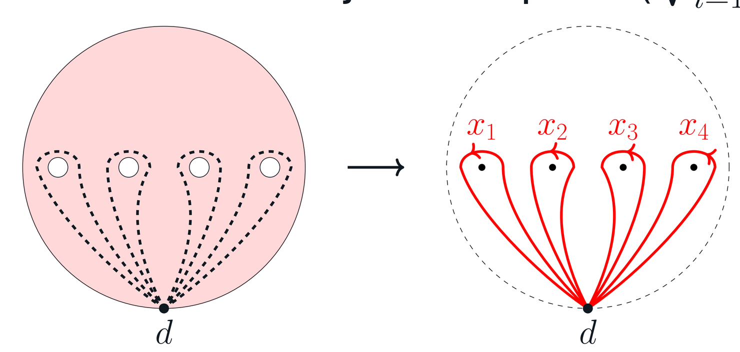

Figure: Deformation Retraction of D_4 to $\bigvee_{i=1}^4 S^1$

Let $S^{(4)} := \bigvee_{i=1}^4 S^1$, below is its *Infinite Cyclic Covering Space* $\tilde{S}^{(4)}$:

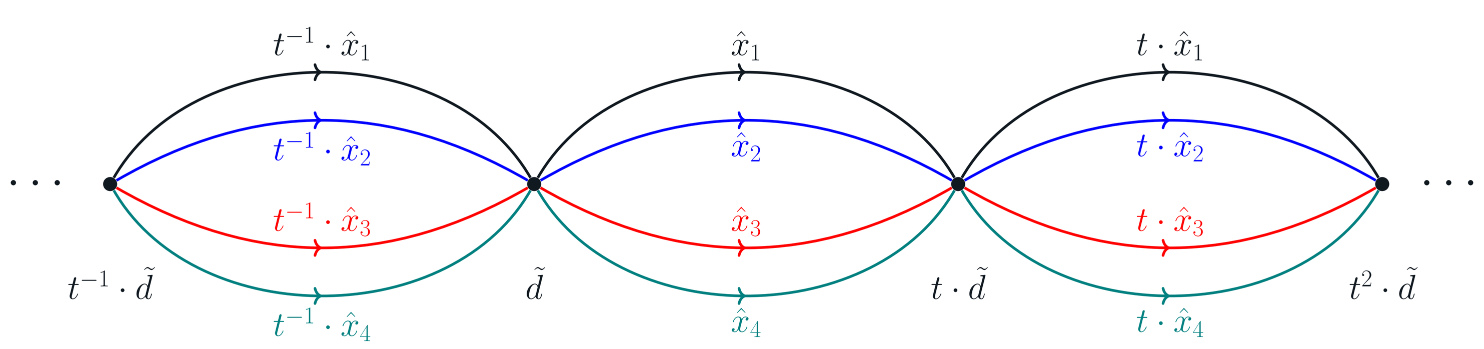

Figure: Infinite Cyclic Cover $\tilde{S}^{(4)}$

Rmk 1: t shifts $\tilde{S}^{(n)}$ by Degree 1.
Rmk 2: $\exists$ *Covering Map* $p : \tilde{S}^{(4)} \to S^{(4)}$, each $p(t^k \cdot \hat{x}_i) = x_i$, and $p(t^k \cdot \tilde{d}) = d$.
Rmk 3: "Base Loops" $\ell_i := \hat{x}_{i+1} \cdot \hat{x}_i^{-1}$ (counterclockwise) for $1 \leq i \leq 3$:

- $-\ell_i$ = clockwise version of ℓ_i
- $t^k \cdot \ell_i$ = degree k right shift of ℓ_i

Property: all ℓ_is' **Integer Laurent Polynomial** combinations form $\tilde{S}^{(4)}$'s *First Homology*, $H_1(\tilde{S}^{(4)})$ as a free $\mathbb{Z}[t^{\pm}]$-module with basis $\{\ell_1, \ell_2, \ell_3\}$.

Action on Homology $H_1(\tilde{S}^{(4)})$

Recall: Braid Automorphism $(\tau_2)_*$ satisfies $(\tau_2)_*(x_2) = x_3$, and $(\tau_2)_*(x_3) = x_3 \cdot x_2 \cdot x_3^{-1}$. Which, it acts on base loops of $\tilde{S}^{(4)}$:

EX: $\ell_2 = \hat{x}_3 \cdot \hat{x}_2^{-1} \mapsto (\hat{x}_2 \cdot (t \cdot \hat{x}_2) \cdot (t \cdot \hat{x}_3^{-1})) \cdot \hat{x}_2^{-1} = -t \cdot \ell_2$.

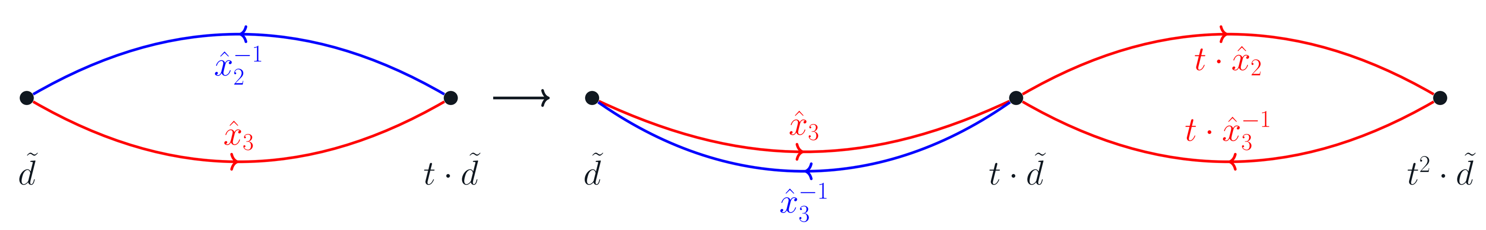

Figure: ℓ_2 Maps to $-t \cdot \ell_2$ via τ_2

Doing this for each ℓ_i, put into matrix with basis $\{\ell_i\}$, we recover the Representation.

Gassner Representation

Def: The *Pure Braid Group* $P_n \subset B_n$, is the kernel of the map $B_n \to S_n$ by $\sigma_i \mapsto (i, i+1)$.

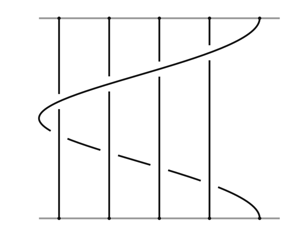

Figure: Example of a Pure Braid in P_5

If consider the *Integer Lattice* $\mathbb{Z}^n$ together with the shifts of connection segments $\hat{x}_i$ connecting $\bar{0}$ to e_i (the elementary basis of $\mathbb{Z}^n$), it again forms a Covering Space of $S^{(n)} := \bigvee_{i=1}^n S^1$, with covering map $\bar{d} \mapsto d$ for all $\bar{d} \in \mathbb{Z}^n$, and $\hat{x}_i \mapsto x_i$.

Then, each *Pure Braid* $\rho \in P_n$ (with $P_n \subset \mathfrak{M}(D_n) \cong B_n$) lifts to an action on the *First Homology* of the covering space, and forms the *Gassner Representation*.

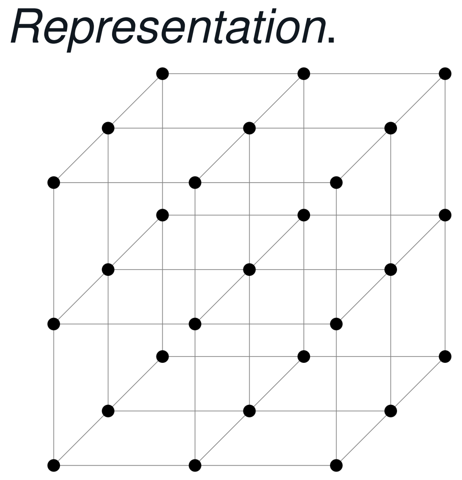

Figure: Gassner Representation's Covering Space of $S^{(3)}$

Significance & Future Directions

Future Direction: Continue on Studying kernels of the Representations, or other homological representations of Braid Groups.

Acknowledgement & Sources

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Source:

- Braid Groups (Christian Kassel, Vladimir Turaev)
- Algebraic Topology (Hatcher)
- Braids, Links, Mapping Class Groups (Joan Birman)
- Introduction to Topological Manifold (John Lee)