

Commutative Algebra Chapter 1 Problems

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Question 1.1: Exercise 1.13 (unsolved);

1. $\sqrt{I} = R \iff I = R$.
2. If ideal P is prime, then $\sqrt{P^n} = P$ for all $n \in \mathbb{N}$.

Pf:

1. \implies : If $\sqrt{I} = R$, then since $R = \sqrt{I} = \varphi^{-1}(\text{Nil}(R/I))$ (where φ is the projection onto R/I), then we have $\text{Nil}(R/I) = R/I$. However, if ring $S \neq (0)$, then $\text{Nil}(S) \subsetneq S$, so since $\text{Nil}(R/I) = R/I$, we must have $R/I = (0)$, showing that $I = R$. \impliedby : If $I = R$, it follows that $\sqrt{I} = R$.
2. Given P is a prime ideal, then for any $n \in \mathbb{N}$, any $x \in \sqrt{P^n}$ satisfies $x^k \in P^n \subseteq P$, hence by induction one can prove that $x \in P$. So, $\sqrt{P^n} \subseteq P$. Also, for all $x \in P$, $x^n \in P^n$, hence $P \subseteq \sqrt{P^n}$, eventually proving that $\sqrt{P^n} = P$.

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Question 2.1: Let x be a nilpotent element of a ring R . Show that $1 + x$ is a unit in R . Deduce that the sum of a nilpotent element and a unit is a unit.

Pf:

Given that $x \in R$ is nilpotent, then $x^k = 0$ for some $k \in \mathbb{N}$ (also, this implies that $y = -x$ is also nilpotent with the same constant).

Then, $1 + x = 1 - (-x) = 1 - y$, which consider the following equation:

$$1 = 1 - 0 = 1 - y^k = (1 - y) \left(\sum_{i=0}^{k-1} y^i \right) \quad (2.1)$$

In other words, the above term is the inverse of $1 - y = 1 + x$, which implies it is invertible.

Now, for any unit $u \in R$ and nilpotent $x \in R$, since $u + x = u(1 + u^{-1}x)$, where $u^{-1}x$ is nilpotent, then $u + x$ is product of two units, hence is a unit.

3 (Not done)

Question 3.1: Let R be a ring. Let $f = a_0 + a_1x + \dots + a_nx^n \in R[x]$. Prove that:

1. f is a unit $\iff a_0$ is a unit in R and a_1, \dots, a_n are nilpotent.
2. f is nilpotent $\iff a_0, \dots, a_n$ are nilpotent.
3. f is a zero-divisor \iff there exists $a \neq 0$ in R such that $af = 0$.
4. f is primitive if $(a_0, \dots, a_n) = R$ (as an ideal). Prove that $f, g \in R[x]$, then fg is primitive $\iff f$ and g are primitive.

Pf:

1. \implies : Given $f = a_0 + a_1x + \dots + a_nx^n$ is a unit, there exists $g = b_0 + b_1x + \dots + b_mx^m$, where $fg = 1$. Which, the constant coefficient is given by $a_0b_0 = 1$, so a_0, b_0 are both units.

Now, we'll use induction to prove that $a_n^{r+1}b_{m-r}$ is nilpotent, given $0 \leq r \leq m$: First consider the base case $r = 0$, the coefficient for degree $(n + m - r) = n + m$ is given by $a_nb_m = 0$. Then, for $r = 1$, the coefficient for $n + m - r$ is given by $a_{n-1}b_m + a_nb_{m-1} = 0$, multiply by a_n on both sides, we get:

$$a_{n-1}b_ma_n + a_n^2b_{m-1} = 0 \implies a_n^2b_{m-1} = 0 \quad (3.1)$$

Now, suppose for given $0 \leq r < m$, the equation is true, then for $r + 1$, we get the coefficient of degree $(n + m - (r + 1))$ be as follow:

$$\sum_{\max\{0, n-(r+1)\} \leq i \leq n} a_ib_{n+m-(r+1)-i} = 0 \quad (3.2)$$

Which, multiply by a_n^{r+1} , since $n - (r + 1) \leq i \leq n$, then $n \leq r + 1 + i \leq n + r + 1$, hence the coefficient $b_{m-(r+1+i-n)}$ has $0 \leq r + 1 + i - n \leq r + 1$, which for ever index i with this expression being at most r , by induction hypothesis, $a_n^{r+1}b_{m-(r+1+i-n)} = 0$, hence every term (besides when the expression is $r + 1$) gets annihilated. So, eventually we get:

$$r + 1 + i - n = r + 1 \implies i = n \implies a_n \cdot a_n^{r+1}b_{n+m-(r+1)-n} = 0 \implies a_n^{r+2}b_{m-(r+1)} = 0 \quad (3.3)$$

This completes the induction.

Hence, for $r = m$, we get $a_n^{m+1}b_0 = 0$, because b_0 is a unit, then a_n is in fact nilpotent, which $-a_nx^n$ is also nilpotent.

By Question 2.1, $f - a_nx^n$ is still a unit, and with degree $n - 1$. Then, the other non-constant coefficients can be proven to be nilpotent by induction.

2. \implies : If f is nilpotent, then $f^k = (a_0 + a_1x + \dots + a_nx^n)^k = 0$ for some $k \in \mathbb{N}$. Which, the leading term is $a_n^k(x^n)^k = 0$, hence $a_n^k = 0$, or a_n is nilpotent. Since a_nx^n is also nilpotent, then $f - a_nx^n$ is nilpotent (with $\deg(f - a_nx^n) = n - 1$). So, since the base case $f = a_0$ is nilpotent implies a_0 is nilpotent, by induction we can show that each a_i is nilpotent.

\Leftarrow : If each coefficient is nilpotent, it's obvious that each degree's component is nilpotent (based on the proof above), hence f is the sum of nilpotent elements, which is nilpotent.

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Question 4.1: Generalize the results in Question 3.1 to polynomial rings with several variables.

Pf:

All the setup can be done through induction. For base case $n = 1$ it is verified in Question 3.1. Now, if all the statements are true for $n - 1$ (where $n \in \mathbb{N}$), then since $R[x_1, \dots, x_n] = K[x_n]$, where $K = R[x_1, \dots, x_{n-1}]$. Then:

1. $f \in K[x_n]$ is a unit \iff constant coefficient $f_0 \in K = R[x_1, \dots, x_{n-1}]$ is unit, and the other coefficients $f_1, \dots, f_k \in K$ are nilpotent. Which, since the constant of $f \in R[x_1, \dots, x_n]$ is provided in f_0 , while other non-constant terms' coefficients scattered in f_1, \dots, f_k (and also the non-constant coefficients in f_1 as a member of polynomial ring $R[x_1, \dots, x_{n-1}]$), by induction hypothesis, this happens iff the constant coefficient of f (also the constant coefficient of f_0) is unit, while the other terms are nilpotent.
2. $f \in K[x_n]$ is nilpotent \iff all coefficients $f_0, \dots, f_k \in R[x_1, \dots, x_{n-1}]$ is nilpotent. Again, by induction hypothesis, all the coefficients of f_0, \dots, f_k in R (also the coefficients of f) must be nilpotent.
3. $f \in K[x_n]$ is a zero divisor \iff all its coefficients $f_0, \dots, f_k \in R[x_1, \dots, x_{n-1}]$ all have some $a_0, \dots, a_k \in R$, such that for each index i , $a_i f_i = 0$; which, f multiplied by $a_0 \dots a_k$ would make all coefficients $f_i \in R[x_1, \dots, x_{n-1}]$ go to 0, hence $a = a_0 \dots a_k$ is the desired element with $af = 0$.
4. $fg \in K[x_n]$ is primitive \iff f and g are primitive in $K[x_n]$. Which, their coefficients in $R[x_1, \dots, x_{n-1}]$ must have gcd being 1. However, the gcd of all its coefficients in R also divides all their coefficients in $R[x_1, \dots, x_{n-1}]$, hence the gcd in R is limited to be 1.

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Question 5.1: In the ring $R[x]$, the Jacobson radical is equal to the nilradical.

Pf: Let N be the nilradical, and J be the Jacobson radical of $R[x]$. Since J is the intersection of all maximal ideals, N is the intersection of all prime ideals, while maximal ideals are prime, then $N \subseteq J$ (N could be the intersection of more ideals, since prime is not necessarily maximal).

Now, if $f \in J$, by definition $1 - f$ is a unit. This happens \iff every non-constant coefficients of $1 - f$ is nilpotent (they are given by $-a_1, \dots, -a_n$, the negative non-constant coefficients of f), while the constant coefficient of f , say a_0 satisfies $1 - a_0$ being a unit (since $1 - a_0$ is the constant coefficient of $1 - f$). So, all the non-constant coefficients of f are nilpotent.

Then, since $1 - yf$ is also a unit for all $y \in R[x]$, consider $y = 1 + x$: The polynomial $(1 + x)f$ is given as follow:

$$(1 + x)f = a_0 + \sum_{i=1}^n (a_{i-1} + a_i)x^i + a_n x^{n+1} \quad (5.1)$$

Then, $1 - (1 + x)f$ has $-(a_0 + a_1)$ as the degree 1 coefficient. Since, $1 - (1 + x)f$ is a unit, this enforces $-(a_0 + a_1)$ to be nilpotent; and since a_1 is nilpotent, a_0 must also be nilpotent (since $\text{Nil}(R)$ is an ideal, which forms a group under addition).

So, because every coefficients are nilpotent, f is nilpotent, hence $f \in N$, showing the other inclusion $J \subseteq N$.

6 (Not done)

Question 6.1: Let R be a ring, and consider $R[[x]]$ (formal power series ring). Show that:

1. f is a unit in $R[[x]] \iff a_0$ is a unit in R .
2. If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true?
3. f belongs to the Jacobson radical of $R[[x]] \iff a_0$ belongs to the Jacobson radical of R .
4. The contraction of a maximal ideal M of $R[[x]]$ is a maximal ideal of R , and M is generated by M^c and x .
5. Every prime ideal of R is the contraction of a prime ideal of $R[[x]]$.

Pf:

1. \implies : If f is a unit in $R[[x]]$, there exists $g \in R[[x]]$, with $fg = 1$. Then, the constant coefficient 1 is given by the multiplication of constant coefficients of f and g , showing that a_0 (constant coefficient of f) is a unit.

\Leftarrow : If a_0 is a unit in R , our goal is to find $g = \sum_{n=0}^{\infty} b_n x^n$, where $fg = 1$.

First, it's clear that $b_0 = a_0^{-1}$. Now, for b_1 , since we want the degree 1 coefficient of fg to be 0, and the degree 1 coefficient is given by $a_0 b_1 + a_1 b_0$, then set $b_1 = -a_0^{-1} a_1 b_0$, we get the desired result.

Inductively, when b_0, \dots, b_{n-1} all have fixed expression using the collections of a_n , since degree n coefficient of fg is given by $\sum_{i=0}^n a_i b_{n-i}$, then if we want the expression to be 0, we can set b_n as follow:

$$a_0 b_n + \sum_{i=1}^n a_i b_{n-i} = 0, \quad b_n = -a_0^{-1} \sum_{i=1}^n a_i b_{n-i} \quad (6.1)$$

So, there exists an expression of g , where $fg = 1$, showing that f is a unit.

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7 (Not done)

Question 7.1: A ring R is such that every ideal not contained in the nilradical contains a nonzero idempotent (an element e with $e^2 = e \neq 0$). Prove that the nilradical and the Jacobson radical of R are equal.

Pf:

Let N, J represent the nilradical and Jacobson radical respectively. It is clear that $N \subseteq J$ by definition.

To prove that $J \subseteq N$ by contradiction, suppose the contrary that $J \not\subseteq N$, by assumption there exists $e \in J$ with $e^2 = e$. Now, consider the ideal (e) :

8 (Not done)

Question 8.1: Let R be a ring in which every element satisfies $x^n = x$ for some $n \in \mathbb{N}$. Show that every prime ideal in R is maximal.

Pf:

Can try to prove that R is a PID. Given an ideal $I \neq (0)$, choose

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Question 9.1: Let $R \neq 0$ be a ring. Show that the set of prime ideals of R has minimal elements with respect to inclusion.

Pf:

We'll prove by Zorn's Lemma, where let A be the set of all prime ideals, and the Partial Order given by $P_1 \succeq P_2$ iff $P_1 \subseteq P_2$.

Let $C \subseteq A$ be a chain, and let $P_C = \bigcap_{P \in C} P$. It is clear that P_C is an ideal, and if $P_C \in A$, then P_C is an upper bound of C . So, it suffices to show that $P_C \in A$ (or P_C is a prime ideal).

Suppose $x, y \in R$ satisfies $xy \in P_C$, then since for any prime ideal $P \in C$, $xy \in P$, then either $x \in P$ or $y \in P$. If all $P \in C$ contains x (or y), then we're done. Now, if some contains x and some contains y , consider the subchain $C_x := \{P \in C \mid x \in P\}$:

- If C_x is comaximal in C (in a set theoretic), then for every $P \in C$, there exists $P_x \in C_x$, where $P_x \succeq P$, so $P_x \subseteq P$, hence $x \in P$, showing that $x \in P_C$.
- Else if C_x is not comaximal in C , then there exists $P \in C$, where all $P_x \in C_x$ has $P \succ P_x$ (which $P \notin C_x$). Hence, $y \in P$, showing that all $P_x \in C_x$ has $P \subsetneq P_x$, or $y \in P_x$. So, given $P \in C$, regardless of its containment in C_x , we have $y \in P$, showing that $y \in P_C$.

The above statements show that P_C is prime, hence $P_C \in A$, every chain has an upper bound. Then, by Zorn's Lemma, this POset has a maximal element, which is the minimal elements with respect to inclusion.

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Question 10.1: Let $I \subsetneq R$ be an ideal. Show that $I = \sqrt{I} \iff I$ is an intersection of prime ideals.

\implies : If $\sqrt{I} = I$, since the projection map $\varphi : R \twoheadrightarrow R/I$ satisfies the following:

$$I = \sqrt{I} = \varphi^{-1}(\text{Nil}(R/I)) = \bigcap_{\overline{P} \subset R/I \text{ prime}} \varphi^{-1}(\overline{P}) = \bigcap_{I \subseteq P \subset R \text{ prime}} P \quad (10.1)$$

Which is an intersection of prime ideals.

\impliedby : Suppose $\{P_i\}_{i \in A}$ is a collection of prime ideals, and define $I := \bigcap_{i \in A} P_i$. Then, for all $x \in \sqrt{I}$, since there exists $n \in \mathbb{N}$, with $x^n \in I$, because $x^n \in P_i$ for all index $i \in A$, then $x \in P_i$, hence $x \in I$, showing that $\sqrt{I} \subseteq I$. Since the other inclusion is trivially true, $\sqrt{I} = I$.

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Question 11.1: Let R be a ring, $\text{Nil}(R)$ be its nilradical. Show that the following are equivalent:

1. R has exactly one prime ideal.
2. Every element of R is either a unit or nilpotent.
3. $R/\text{Nil}(R)$ is a field.

$1 \implies 2$: Suppose R has precisely one prime ideal, then since $\text{Nil}(R)$ is the intersection of all prime ideals, $\text{Nil}(R) = P$ (the prime ideal). This also enforces $\text{Nil}(R)$ to be maximal (since every commutative ring has a maximal ideal, and all maximal ideal is prime).

Now, suppose $u \in R \setminus \text{Nil}(R)$ (i.e. not nilpotent), then since $\text{Nil}(R) \subsetneq \text{Nil}(R) + (u)$, then $\text{Nil}(R) + (u) = R$, showing that $1 = ku + x$ for some $k \in R$ and $x \in \text{Nil}(R)$. Notice that $-x$ is nilpotent, which $1 - x$ is a unit, hence $1 - x = ku$, showing that ku is a unit, which u is a unit.

Hence, every element of R is either a unit or nilpotent.

$2 \implies 3$: Suppose every element is either a unit or nilpotent, then for all $\bar{u} \in R/\text{Nil}(R)$ (with $\bar{u} := u \bmod \text{Nil}(R)$) that is nonzero, since u is a unit, then inherently, \bar{u} is also a unit in $R/\text{Nil}(R)$, showing that it is a field.

$3 \implies 1$: Suppose $R/\text{Nil}(R)$ is a field, then $\text{Nil}(R)$ is maximal. Now, suppose P is a prime ideal, then because $\text{Nil}(R) \subseteq P \subsetneq R$, then this enforces $\text{Nil}(R) = P$. Hence, there is only one prime ideal, namely $\text{Nil}(R)$.