Braid Groups, and their Representations

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Introduction

Braid Groups & Mapping Class Groups

Def: Braid group of n strands B_n is generated by $n-1$ elements $\{\sigma_1, ..., \sigma_{n-1}\}$, satisfying *Braid Relations*:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$, if $|i-j| \geq 2$
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

Def: Let D_n be an n-punctured disk. The *Mapping Class Group* $\mathfrak{M}(D_n)$ collects classes of isotopic self-homeomorphisms on D_n that fixes disk boundary ∂D.

Ex: The i^{th} *Half Twist* $\tau_i \in \mathfrak{M}(D_n)$ swaps the i^{th} and $(i+1)^{\text{th}}$ punctures, while fixing the remaining ones.

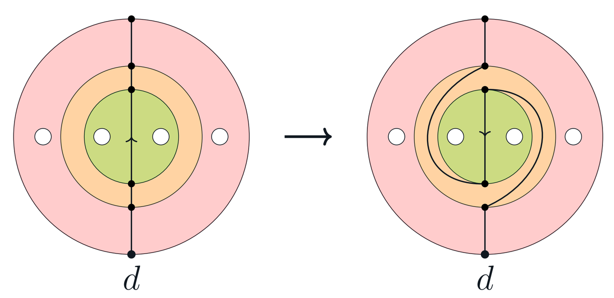

Figure: For $n=4$, Half Twist τ_2 Swapping Punctures 2 and 3

Property: *Half Twists* $\tau_1, ..., \tau_{n-1}$ generates $\mathfrak{M}(D_n)$ and satisfies *Braid Relations*; in fact, $B_n \cong \mathfrak{M}(D_n)$, by $\sigma_i \mapsto \tau_i$.

Fundamental Group of D_n & Braid Automorphism

For n-punctured disk D_n, fix $d \in \partial D$, the fundamental group $\pi_1(D_n, d)$ is generated by the n loops, each surrounding a puncture, which $\pi_1(D_n, d) = F_n(x_1, ..., x_n)$, the *Degree-$n$ Free Group*.

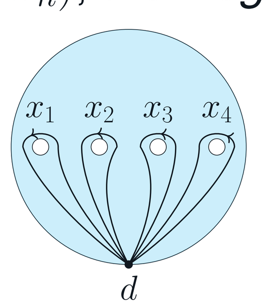

Figure: Fundamental Group of D_4

Then, each homeomorphism in $\mathfrak{M}(D_n)$ generates a group automorphism on $\pi_1(D_n, d)$, called *Braid Automorphism*.

Ex: Half Twist's action on $\pi_1(D_n, d)$:

$$(\tau_i)_* \in \operatorname{Aut}(\pi_1(D_n, d)), \quad (\tau_i)_*(x_j) = \begin{cases} x_{i+1} & j = i \\ x_{i+1}x_ix_{i+1}^{-1} & j = i+1 \\ x_i & \text{Otherwise} \end{cases}$$

Figure: τ_2 Action on Loops in D_4

Reduced Burau Representation

$\psi_n^r : B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t^{\pm}])$ satisfies:

$$\sigma_1 \mapsto \left(\begin{array}{cc|c} -t & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & I_{n-3} \end{array}\right), \quad \sigma_{n-1} \mapsto \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & t \\ 0 & 0 & -t \end{array}\right), \quad \sigma_i \mapsto \left(\begin{array}{c|ccc|c} I_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n-i-2} \end{array}\right)$$

Ex: Homological Perspective on D_4

A 4-punctured disk D_4 can "continuously deform" into 4 circles joining at one point ($\bigvee_{i=1}^4 S^1$), $\implies$ Same Fundamental Group.

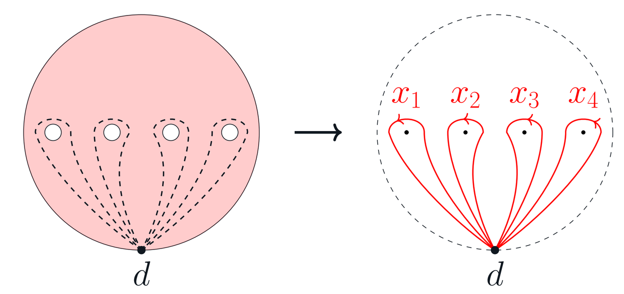

Figure: Deformation Retraction of D_4 to $\bigvee_{i=1}^4 S^1$

Let $S^{(4)} := \bigvee_{i=1}^4 S^1$, consider the space $\tilde{S}^{(4)}$ below, a *Covering Space* of $S^{(4)}$:

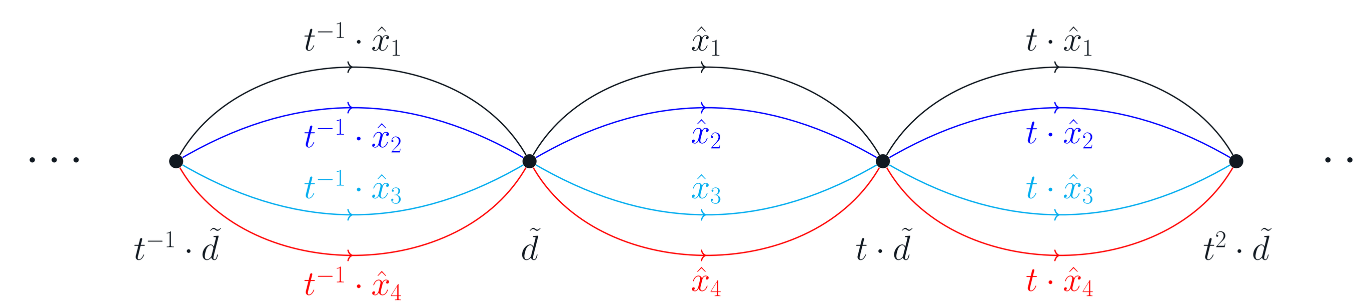

Figure: Infinite Cyclic Cover $\tilde{S}^{(4)}$

Here, t is a right shift of $\tilde{S}^{(n)}$ by degree 1:

- $t^k \cdot \tilde{d} =$ degree k right shift of $\tilde{d}$
- $t^k \cdot \hat{x}_i =$ degree k right shift of $\hat{x}_i$

There is a continuous covering map $p : \tilde{S}^{(4)} \to S^{(4)}$, each $p(t^k \cdot \hat{x}_i) = x_i$, and $p(t^k \cdot \tilde{d}) = d$.

Define the "Base Loops" $\ell_i := \hat{x}_{i+1} \cdot \hat{x}_i^{-1}$ (counterclockwise) for $1 \leq i \leq 3$:

- $-\ell_i =$ clockwise version of ℓ_i
- $t^k \cdot \ell_i =$ degree k right shift of ℓ_i

Then, all "Integer Laurent Polynomial" combination of ℓ_i forms $H_1(\tilde{S}^{(4)})$ as a free $\mathbb{Z}[t^{\pm}]$-module with basis ℓ_1, ℓ_2, ℓ_3.

Braid Group's Action on Covering Space

Recall: braid automorphism $(\tau_2)_*$ of $\pi_1(D_4, d)$ satisfies $(\tau_2)_*(x_2) = x_3$, and $(\tau_2)_*(x_3) = x_3 \cdot x_2 \cdot x_3^{-1}$. Which, it uniquely lifts to an transformation on the ℓ_i via p:

EX: $\ell_2 = \hat{x}_3 \cdot \hat{x}_2^{-1} \mapsto \hat{x}_2 \cdot \left((t \cdot \hat{x}_2) \cdot (t \cdot \hat{x}_3^{-1}) \cdot \hat{x}_2^{-1}\right) = -t \cdot \ell_2$.

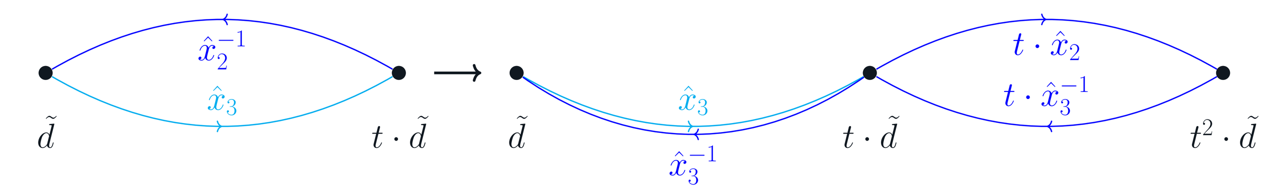

Figure: ℓ_2 (Counterclockwise) Maps to $-t \cdot \ell_2$ (Right Shift by degree 1, Clockwise)

Doing this for each ℓ_i, put into matrix form with basis $\{\ell_i\}$, we recover the Representation.

Conclusion & Future Directions

Acknowledgement & Sources

We're genuinely thankful for the parent donors, Professor Cachadina and Professor Casteels who made this program possible. We also want to thank our mentor Choomno Moos for their great guidance.

- Braids, Links, Mapping Class Groups (Joan Birman)
- Briad Groups (Christian Kassel, Vladimir Turaev)
- Category Theory in Context (Emily Riehl)
- Algebra Chapter 0 (Paolo Aluffi)