Math 118B HW6

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Question 1 *Rudin Chapter 5 Exercise 22:*
Suppose f is a real function on $\mathbb{R}$. Call x a fixed point of f if $f(x) = x$.

(a) *If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.*

(b) *Show that the function f defined by*

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

(c) *However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and*

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \ldots$.

Pf:

(a) Given f is differentiable and $f'(t) \neq 1$ for all real t. Suppose the contrary that f has more than one fixed point, there exists distinct $x, y \in \mathbb{R}$ (and WLOG, assume $x < y$), with $f(x) = x$ and $f(y) = y$. However, by Mean Value Theorem, there exists $c \in (x, y)$, such that $f'(c) = \frac{f(y)-f(x)}{y-x} = \frac{y-x}{y-x} = 1$, which contradicts the assumption that all $t \in \mathbb{R}$ satisfies $f'(t) \neq 1$.

Hence, the assumption is wrong, f couldn't have more than one fixed point.

(b) Given $f(t) = t + (1 + e^t)^{-1}$, apply the differentiation rules, we get:

$$f'(t) = 1 - (1 + e^t)^{-1} \cdot e^t = 1 - \frac{e^t}{(1 + e^t)^2}$$

Since for all $t \in \mathbb{R}$, $e^t > 0$, so $(1 + e^t) > 1$ and $(1 + e^t) > e^t$. Hence, $0 < \frac{e^t}{(1+e^t)^2} < 1$ (since everything is positive, while $e^t < (1 + e^t) < (1 + e^t)^2$).

Yet, there doesn't exists a fixed point: If consider $f(t) - t$, we get $(1 + e^t)^{-1}$. Since $e^t > 0$ for all $t \in \mathbb{R}$, then $(1+e^t) > 0$, so does $(1+e^t)^{-1}$. Therefore, there doesn't exists $t \in \mathbb{R}$, with $(1+e^t)^{-1} = f(t)-t = 0$, so there doesn't exist any fixed point for this function.

(c) Suppose there exists $0 \leq A < 1$ such that $|f'(t)| \leq A$ for all real t. Then, for al distinct $x, y \in \mathbb{R}$ (WLOG, assume $x < y$), by Mean Value Theorem, there exists $c \in (x, y)$, with $f'(c)(x - y) = (f(x) - f(y))$. So, the following is true:

$$|f(x) - f(y)| = |f'(c)| \cdot |x - y| \leq A|x - y|$$

Now, for any $x_1 \in \mathbb{R}$, we'll prove by induction that all $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq A^{n-1}|x_2 - x_1|$.

For base case $n = 1$, it's clear that $|x_{1+1} - x_1| = |x_2 - x_1| \leq A^{1-1}|x_2 - x_1|$.

Now, suppose for given $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq A^{n-1}|x_2 - x_1|$, then for case $(n+1)$:

$$|x_{(n+1)+1} - x_{n+1}| = |f(x_{n+1}) - f(x_n)| \leq A|x_{n+1} - x_n| \leq A \cdot A^{n-1}|x_2 - x_1| = A^{(n+1)-1}|x_2 - x_1|$$

Which, this completes the induction, showing that all $n \in \mathbb{N}$ satisfies $|x_{n+1} - x_n| \leq A^{n-1}|x_2 - x_1|$.

Now, we can prove that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, therefore converges:

Given that $0 \leq A < 1$, then $\frac{1}{1-A} > 0$. Now, since $A^{n-1}|x_2 - x_1|$ defines a geometric sequence with ratio $0 \leq A < 1$, then $\lim_{n\to\infty} A^{n-1}|x_2 - x_1| = 0$. So, for all $\epsilon > 0$, since $\frac{1-A}{|x_2-x_1|}\epsilon > 0$, there exists N, with $n \geq N$ implies $A^{n-1}|x_2 - x_1| < (1 - A)\epsilon$.

Now, for all $m > n \geq N$, the following is true:

$$|x_m - x_n| = \left|\sum_{k=0}^{m-n-1} (x_{n+(k+1)} - x_{n+k})\right| \leq \sum_{k=0}^{m-n-1} |x_{n+(i+1)} - x_{n+i}|$$

$$|x_m - x_n| \leq \sum_{k=0}^{m-n-1} |x_{n+(i+1)} - x_{n+i}| \leq \sum_{k=0}^{m-n-1} A^{n+k-1}|x_2 - x_1|$$

$$|x_m - x_n| \leq A^{n-1}|x_2 - x_1| \sum_{k=0}^{m-n-1} A^k \leq A^{n-1}|x_2 - x_1| \sum_{k=0}^{\infty} A^k$$

$$|x_m - x_n| \leq A^{n-1}|x_2 - x_1| \cdot \frac{1}{1-A} < (1 - A)\epsilon \cdot \frac{1}{1-A} = \epsilon$$

Since for all $\epsilon > 0$, there exists N, with $m > n \geq N$ implies $|x_m - x_n| < \epsilon$, hence $(x_n)_{n\in\mathbb{N}}$ is a cauchy sequence, which converges to some $x \in \mathbb{R}$.

Then, since f is differentiable, then f is continuous; hence, the following is true:

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x), \quad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

Hence, $f(x) = x$, which any $x_1 \in \mathbb{R}$ with $x_{n+1} = f(x_n)$, has the sequential limit being a fixed point $x \in \mathbb{R}$.

Also, based on the previous part, since all $t \in \mathbb{R}$ has $|f'(t)| \leq A < 1$, then by part (a), since $f'(t) \neq 1$ for all t, f has at most one fixed point. Hence, this fixed point is unique, all such sequence $(x_n)_{n\in\mathbb{N}}$ converges to a unique fixed point $x \in \mathbb{R}$.

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Question 2 *For* $f(x) = \cos(x)$*, show that* $x_{n+1} = f(x_n)$ *defines a convergent sequence for arbitrary* $x_0 \in \mathbb{R}$*. Calculate the root* $\alpha = \cos(\alpha)$*, with an error less than* 10^{-2}*.*

Pf:

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Question 3

Pf:

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Question 4

Pf:

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Question 5

Pf:

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Question 6

Pf: