

# Math CS 122A HW9

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1

**Question 1** Ahlfors Pg. 154 Problem 2:

*How many roots of the equation  $z^4 - 6z + 3 = 0$  have their modulus between 1 and 2?*

**Pf:**

For the disk  $|z| < 1$ , consider the function  $-6z + 3$ : It has one zero in  $|z| < 1$ , namely  $z = \frac{1}{2}$ .

On the other hand, for circle  $|z| = 1$ , the following inequalities are true:

$$|(z^2 - 6z + 3) - (-6z + 3)| = |z|^4 = 1$$

$$|-6z + 3| \geq |6|z| - 3| = 6 - 3 = 3$$

So, since  $|(z^2 - 6z + 3) - (-6z + 3)| = 1 \leq 3 \leq |-6z + 3|$  for all  $z$  on  $|z| = 1$ , then by Rouché's Theorem, the two polynomials have the same number of zeroes enclosed by the circle  $|z| = 1$ .

Since  $-6z + 3$  only has one zero in this region, then  $z^4 - 6z + 3$  also has one zero in this region.

Now, consider the disk  $|z| < 2$ , and the function  $z^4$ : It has four zeros in  $|z| < 2$  counting multiplicity (namely  $z = 0$ ).

On the other hand, for circle  $|z| = 2$ , the following inequalities are true:

$$|(z^2 - 6z + 3) - z^4| = |-6z + 3| \leq 6|z| + 3 = 15$$

$$|z^4| = |z|^4 = 16$$

So, since  $|(z^4 - 6z + 3) - z^4| = 15 < 16 = |z^4|$  for all  $z$  on  $|z| = 2$ , by Rouché's Theorem again, the two polynomials have the same number of zeroes enclosed by the circle  $|z| = 2$ . Since  $z^4$  has four zeroes in this region, then  $z^4 - 6z + 3$  also has four zeroes in this region.

Then, since 4 zeroes has modulus less than 2, while 1 zero has modulus less than 1, counting the ones with modulus between 1 and 2, we have total of  $4 - 1 = 3$  zeroes.

**Question 2** Ahlfors Pg. 161 Problem 5:

Complex integration can sometimes be used to evaluate area integrals. As an illustration, show that if  $f(z)$  is analytic and bounded for  $|z| < 1$  and if  $|\zeta| < 1$ , then

$$f(\zeta) = \frac{1}{\pi} \int \int_{|z| < 1} \frac{f(z) dx dy}{(1 - \bar{z}\zeta)^2}$$

**Pf:**

If convert the above integral to polar coordinates, we get the following:

$$\frac{1}{\pi} \int \int_{|z| < 1} \frac{f(z) dx dy}{(1 - \bar{z}\zeta)^2} = \frac{1}{\pi} \int_{r=0}^1 \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})r}{(1 - \zeta re^{-i\theta})^2} d\theta dr = \frac{1}{\pi} \int_{r=0}^1 \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})(e^{i\theta})^2 \cdot r}{(e^{i\theta} - r\zeta)^2} d\theta dr$$

Then, define  $C$  to have parametrization  $z = e^{i\theta}$  with  $\theta \in [0, \pi]$ , the inner part of the integral becomes:

$$\frac{1}{\pi} \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})(e^{i\theta})^2 r}{(e^{i\theta} - r\zeta)^2} d\theta = 2 \cdot \frac{1}{2\pi i} \int_0^\pi \frac{f(re^{i\theta})re^{i\theta}}{(e^{i\theta} - r\zeta)^2} i e^{i\theta} d\theta = 2 \cdot \frac{1}{2\pi i} \int_C \frac{f(rz)rz}{(z - r\zeta)^2} dz$$

Recall that for any analytic function  $\phi(z)$  at a point  $a$ , by Cauchy's Integral formula, we get:

$$\phi'(a) = \frac{1!}{2\pi i} \int_C \frac{\phi(z)}{(z - a)^2} dz$$

So, given that  $z = r\zeta$  (which with  $r \in [0, 1]$  and  $|\zeta| < 1$ ,  $r\zeta$  is strictly in the unit disk, so integrate over  $C$  given above is valid), let  $\phi(z) = f(rz)rz$  (which  $\phi'(z) = f'(rz)r^2z + f(rz)r$ ), we have:

$$\begin{aligned} f'(r\zeta)r^2(r\zeta) + f(r\zeta)r &= \frac{1!}{2\pi i} \int_C \frac{f(rz)rz}{(z - r\zeta)^2} dz \\ 2 \cdot \frac{1}{2\pi i} \int_C \frac{f(rz)rz}{(z - r\zeta)^2} dz &= 2(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r) \end{aligned}$$

Hence, the original integral can be rewrite as:

$$\frac{1}{\pi} \int_{r=0}^1 \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})(e^{i\theta})^2 \cdot r}{(e^{i\theta} - r\zeta)^2} d\theta dr = \int_{r=0}^1 2(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r) dr$$

Now, consider the function  $f(\zeta r^2)r^2$ , which has derivative  $f'(\zeta r^2)r^2 \cdot 2r\zeta + f(\zeta r^2) \cdot 2r = 2(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r)$ .

Then, the above integral can be rewrite as:

$$\int_{r=0}^1 2(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r) dr = f(\zeta r^2)r^2 \Big|_{r=0}^1 = f(\zeta)$$

Hence, we can claim that for  $f(z)$  that's analytic and bounded on  $|z| < 1$ , and given  $|\zeta| < 1$ , the integral is true:

$$f(\zeta) = \frac{1}{\pi} \int \int_{|z| < 1} \frac{f(z) dx dy}{(1 - \bar{z}\zeta)^2}$$

### 3

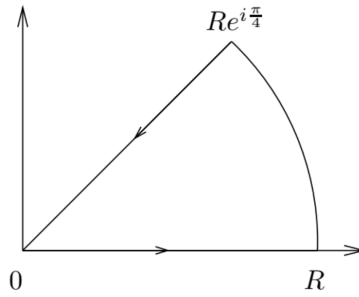
**Question 3** Stein and Shakarchi Pg. 64 Problem 1:

Prove that

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

**Pf:**

Consider the function  $e^{-z^2}$ , and the integration over a sector with origin at 0 and radius  $R$ . Which, this can be parametrized by three curves:  $\gamma_1$  - a straight line on real axis with  $x \in [0, R]$ ,  $\gamma_2$  - a circular arc with radian  $\frac{\pi}{4}$  and radius  $R$  (parametrized by  $z = Re^{i\theta}$ , where  $\theta \in [0, \frac{\pi}{4}]$ ), and  $\gamma_3$  - another straight line of  $z = re^{i\frac{\pi}{4}}$  (where  $r \in [0, R]$ ). The orientation is given as follow:



**Figure 14.** The contour in Exercise 1

If consider the integral over this closed curve, since  $e^{-z^2}$  is analytic on the whole plane, then the line integral is 0. So,  $\int_{\gamma_1+\gamma_2+\gamma_3} e^{-z^2} dz = 0$ .

For  $\int_{\gamma_1} e^{-z^2} dz$ , it is parametrized by  $\int_0^R e^{-x^2} dx$ , which  $\lim_{R \rightarrow \infty} \int_0^R e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$  (since  $e^{-x^2}$  is even, while  $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$ ).

For  $\int_{\gamma_2} e^{-z^2} dz$ , it is parametrized by the following:

$$\begin{aligned} \int_{\gamma_2} e^{-z^2} dz &= \int_0^{\frac{\pi}{4}} \exp(-(Re^{i\theta})^2) iRe^{i\theta} d\theta = \int_0^{\frac{\pi}{4}} \exp(-R^2 e^{i2\theta}) iRe^{i\theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \exp(-R^2(\cos(2\theta) + i\sin(2\theta))) iRe^{i\theta} d\theta = \int_0^{\frac{\pi}{4}} e^{-R^2 \cos(2\theta)} e^{-iR^2 \sin(2\theta)} iRe^{i\theta} d\theta \end{aligned}$$

Which, consider the modulus, the following inequality is true:

$$\begin{aligned} \left| \int_{\gamma_2} e^{-z^2} dz \right| &\leq \int_0^{\frac{\pi}{4}} |e^{-R^2 \cos(2\theta)}| \cdot |e^{-iR^2 \sin(2\theta)}| \cdot |iRe^{i\theta}| d\theta = \int_0^{\frac{\pi}{4}} Re^{-R^2 \cos(2\theta)} d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} Re^{-R^2 \cos(u)} du \end{aligned}$$

(Note: the second line is done by the parametrization  $u = 2\theta$ ).

Now, since in the domain  $[0, \frac{\pi}{2}]$ ,  $1 - \frac{2}{\pi}u \leq \cos(u)$ , then  $e^{-R^2 \cos(u)} \leq e^{-R^2(1-\frac{2}{\pi}u)}$  (given that  $-R^2 < 0$ , while the two functions are positive on the given domain). Then, we can further bound the integral by:

$$\left| \int_{\gamma_2} e^{-z^2} dz \right| \leq \frac{1}{2} \int_0^{\frac{\pi}{2}} Re^{-R^2 \cos(u)} du \leq \frac{1}{2} \int_0^{\frac{\pi}{2}} Re^{-R^2(1-\frac{2}{\pi}u)} du = \frac{1}{2} \int_0^{\frac{\pi}{2}} Re^{R^2 \cdot \frac{2}{\pi}u - R^2} du$$

$$\leq \frac{R}{2} \cdot \frac{\pi}{2R^2} e^{R^2 \cdot \frac{2}{\pi} u - R^2} \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4R} (e^{R^2 \cdot \frac{2}{\pi} \cdot \frac{\pi}{2} - R^2} - e^{R^2 \cdot \frac{2}{\pi} \cdot 0 - R^2}) = \frac{\pi}{4R} (1 - e^{-R^2})$$

Then, since  $\lim_{R \rightarrow \infty} \frac{\pi}{4R} = 0$ ,  $\lim_{R \rightarrow \infty} (1 - e^{-R^2}) = 1$ , then:

$$0 \leq \lim_{R \rightarrow \infty} \left| \int_{\gamma_2} e^{-z^2} dz \right| \leq \lim_{R \rightarrow \infty} \frac{\pi}{4R} (1 - e^{-R^2}) = 0$$

Hence, we can claim that  $\lim_{R \rightarrow \infty} \int_{\gamma_2} e^{-z^2} dz = 0$ .

Lastly, for  $\int_{\gamma_3} e^{-z^2} dz$ , it is parametrized by  $\int_R^0 \exp(-(re^{i\frac{\pi}{4}})^2) e^{i\frac{\pi}{4}} dr$ . Which, can be modified as:

$$\begin{aligned} \int_R^0 \exp(-r^2 e^{i\frac{\pi}{2}}) e^{i\frac{\pi}{4}} dr &= e^{i\frac{\pi}{4}} \int_R^0 e^{-ir^2} dr = e^{i\frac{\pi}{4}} \left( \int_R^0 \cos(r^2) dr - i \int_R^0 \sin(r^2) dr \right) \\ &= -e^{i\frac{\pi}{4}} \left( \int_0^R \cos(r^2) dr - i \int_0^R \sin(r^2) dr \right) \end{aligned}$$

Now, because  $\int_{\gamma_1 + \gamma_2 + \gamma_3} e^{-z^2} dz = 0$ , then  $\int_{\gamma_3} e^{-z^2} dz = -(\int_{\gamma_1} e^{-z^2} dz + \int_{\gamma_2} e^{-z^2} dz)$ . Hence:

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{\gamma_3} e^{-z^2} dz &= \lim_{R \rightarrow \infty} - \left( \int_{\gamma_1} e^{-z^2} dz + \int_{\gamma_2} e^{-z^2} dz \right) = -\frac{\sqrt{\pi}}{2} \\ \lim_{R \rightarrow \infty} -e^{i\frac{\pi}{4}} \left( \int_0^R \cos(r^2) dr - i \int_0^R \sin(r^2) dr \right) &= -\frac{\sqrt{\pi}}{2} \end{aligned}$$

Hence, we can claim the following:

$$\int_0^\infty \cos(r^2) dr - i \int_0^\infty \sin(r^2) dr = \frac{\sqrt{\pi}}{2} e^{-i\frac{\pi}{4}} = \frac{\sqrt{2\pi}}{4} (1 - i)$$

Then, take the real and imaginary part respectively, we get:

$$\begin{aligned} \int_0^\infty \cos(r^2) dr &= \operatorname{Re} \left( \frac{\sqrt{2\pi}}{4} (1 - i) \right) = \frac{\sqrt{2\pi}}{4} \\ \int_0^\infty \sin(r^2) dr &= -\operatorname{Im} \left( \frac{\sqrt{2\pi}}{4} (1 - i) \right) = \frac{\sqrt{2\pi}}{4} \end{aligned}$$

Hence, the two integrals evaluated to be  $\frac{\sqrt{2\pi}}{4}$ .

**Question 4** *Stein and Shakarchi Pg. 65 Problem 4:*

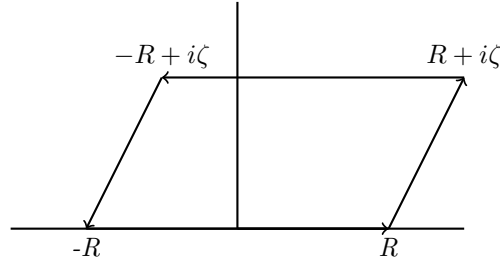
Prove that for all  $\zeta \in \mathbb{C}$  we have

$$e^{-\pi\zeta^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \zeta} dx$$

**Pf:**

Given  $\zeta = u + iv$  for some  $u, v \in \mathbb{R}$ . Consider the function  $e^{-\pi z^2}$  which is analytic on  $\mathbb{C}$ . There are three cases:

- (1) **When  $u > 0$ ,** given  $R > 0$ , consider the parallelogram generated by the points  $-R, R, R + i\zeta, -R + i\zeta$  with counterclockwise orientation. The orientation is as follow:



Then, the integral of  $e^{-\pi z^2}$  on the contour is 0 (since it is analytic on the whole plane). Which, it can be broken down into the sum of following integrals:

First, for the one on the real axis, it is parametrized by  $\int_{-R}^R e^{-\pi x^2} dx$ , where  $\lim_{R \rightarrow \infty} \int_{-R}^R e^{-\pi x^2} dx = 1$  (Gauss Integral).

For the ones on the side, since the sides are parametrized by  $R + i\zeta t$  and  $-R + i\zeta t$  for  $t \in [0, 1]$  respectively, then the first integral is given by:

$$\begin{aligned} \int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt &= \int_0^1 e^{-\pi(R^2 + i \cdot 2R\zeta t - \zeta^2 t^2)} \cdot i\zeta dt \\ &= \int_0^1 e^{-\pi(R^2 - 2Rvt)} \cdot e^{i \cdot 2Rut - \zeta^2 t^2} \cdot i\zeta dt = e^{-\pi R^2} \int_0^1 e^{R \cdot 2\pi vt} \cdot e^{i \cdot 2Rut} \cdot e^{-\zeta^2 t^2} \cdot i\zeta dt \end{aligned}$$

Which, taking the modulus, it is bounded by:

$$\begin{aligned} \left| \int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt \right| &\leq e^{-\pi R^2} \int_0^1 e^{R \cdot 2\pi vt} \cdot |e^{i \cdot 2Rut}| \cdot |e^{-\zeta^2 t^2}| \cdot |\zeta| dt \\ &\leq e^{-\pi R^2} \int_0^1 e^{R \cdot |2\pi v|} \cdot |e^{-\zeta^2 t^2}| \cdot |\zeta| dt = e^{-\pi R^2 + R \cdot |2\pi v|} \int_0^1 |e^{-\zeta^2 t^2}| \cdot |\zeta| dt \end{aligned}$$

(Note: the above is given by  $R \cdot 2\pi vt \leq R \cdot |2\pi v| \cdot t \leq R \cdot |2\pi v|$ , since  $t \in [0, 1]$ ).

Then, since  $\lim_{R \rightarrow \infty} -\pi R^2 + R|2\pi v| = -\infty$ , so  $\lim_{R \rightarrow \infty} e^{-\pi R^2 + R|2\pi v|} = 0$ , then since  $\int_0^1 |e^{-\zeta^2 t^2}| \cdot |\zeta| dt$  is a constant, then:

$$\lim_{R \rightarrow \infty} e^{-\pi R^2 + R \cdot |2\pi v|} \int_0^1 |e^{-\zeta^2 t^2}| \cdot |\zeta| dt = 0$$

So, we can conclude the following:

$$\lim_{R \rightarrow \infty} \int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt = 0$$

Similar concepts applied on the line  $-R + i\zeta t$  (since  $-\pi(-R + i\zeta t)^2$  is then given by  $-\pi(R^2 + 2Rvt) - \pi(-i \cdot 2Rut - \zeta^2 t^2)$ , so the same inequality still applies). Hence:

$$\lim_{R \rightarrow \infty} \int_0^1 e^{-\pi(-R+i\zeta t)^2} \cdot i\zeta dt = 0$$

Lastly, the translated line is parametrized by  $x + i\zeta$  for  $x \in [-R, R]$ , then the integral is given by:

$$\int_{-R}^R e^{-\pi(x+i\zeta)^2} dx = \int_{-R}^R e^{-\pi(x^2+2ix\zeta-\zeta^2)} dx = e^{\pi\zeta^2} \int_{-R}^R e^{-\pi x^2} \cdot e^{-2\pi ix\zeta} dx$$

Now, summing up all the path integrals with right orientation, we get the following:

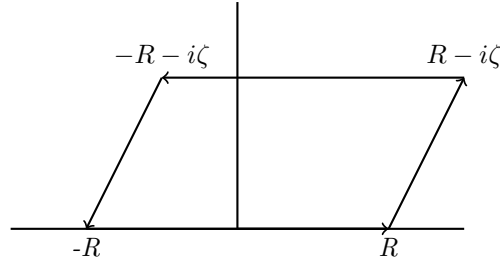
$$\begin{aligned} \int_{-R}^R e^{-\pi x^2} dx + \int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt - \int_0^1 e^{-\pi(-R+i\zeta t)^2} \cdot i\zeta dt - \int_{-R}^R e^{-\pi(x+i\zeta)^2} dx &= 0 \\ \int_{-R}^R e^{-\pi(x+i\zeta)^2} dx &= \int_{-R}^R e^{-\pi x^2} dx + \int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt - \int_0^1 e^{-\pi(-R+i\zeta t)^2} \cdot i\zeta dt \end{aligned}$$

So, take  $R \rightarrow \infty$ , the first term on the right approaches 1, while the next two terms converges to 0 (from the above statements), then the limit becomes:

$$\lim_{R \rightarrow \infty} \int_{-R}^R e^{-\pi(x+i\zeta)^2} dx = \lim_{R \rightarrow \infty} e^{\pi\zeta^2} \int_{-R}^R e^{-\pi x^2} \cdot e^{-2\pi ix\zeta} dx = 1$$

So, we can conclude that  $\int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi ix\zeta} dx = e^{-\pi\zeta^2}$ .

- (2) **When**  $u < 0$ , similar constructions can be made, but instead using  $-R, R, R - i\zeta, -R - i\zeta$  as the four points instead.



Using similar concept, we can prove the exact same result as when  $u > 0$ .

(Note: the reason using  $-i$ , is because we want the integration of  $e^{az}$  to have negative real parts in the exponent. So, instead of doing  $+i\zeta$ , do  $-i\zeta$  so as  $R \rightarrow \infty$  the side integrals actually converge).

(3) **When**  $u = 0$ , since  $\zeta = iv$ , then  $\zeta^2 = -v^2$ . So, the proposed integral becomes:

$$\int_{-R}^R e^{-\pi x^2} \cdot e^{2\pi i x \zeta} dx = \int_{-R}^R e^{-\pi x^2} \cdot e^{2\pi i x \cdot iv} dx = \int_{-R}^R e^{-\pi x^2} \cdot e^{-2\pi v x} dx$$

Which, by completing the square, we get:

$$\int_{-R}^R e^{-\pi x^2} \cdot e^{2\pi i x \zeta} dx = e^{\pi v^2} \int_{-R}^R e^{-\pi x^2} \cdot e^{-2\pi v x} \cdot e^{-\pi v^2} dx = e^{\pi v^2} \int_{-R}^R e^{-\pi(x^2 + 2vx + v^2)} dx = e^{\pi v^2} \int_{-R}^R e^{-\pi(x+v)^2} dx$$

Then, as  $R \rightarrow \infty$ , we get:

$$\lim_{R \rightarrow \infty} \int_{-R}^R e^{-\pi x^2} \cdot e^{2\pi i x \zeta} dx = \lim_{R \rightarrow \infty} e^{\pi v^2} \int_{-R}^R e^{-\pi(x+v)^2} dx = e^{\pi v^2} \int_{-\infty}^{\infty} e^{-\pi(x+v)^2} dx = e^{\pi v^2}$$

And, since  $e^{\pi v^2} = e^{-\pi \zeta^2}$ , then:

$$e^{-\pi \zeta^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{2\pi i x \zeta} dx$$

So, regardless of the case, we can say that the following integral is true:

$$e^{-\pi \zeta^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{2\pi i x \zeta} dx$$

## 5

**Question 5** *Stein and Shakarchi Pg. 103 Problem 5:*

*Use contour integration to show that for all  $\zeta$  real*

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \zeta}}{(1+x^2)^2} dx = \frac{\pi}{2} (1 + 2\pi|\zeta|) e^{-2\pi|\zeta|}$$

**Pf:**

**Residue at  $i, -i$ :**

Consider the function  $f(z) = e^{-2\pi i \zeta z} / (1+z^2)^2 = e^{-2\pi i \zeta z} / ((z-i)(z+i))^2$ , which it has poles at  $z = \pm i$ , each with order 2 (since  $(z^2+1)^2 = (z-i)^2(z+i)^2$ ).

Then, to show its residue at  $i$ , consider the derivative of  $\phi_i(z) = e^{-2\pi i \zeta z} / (z+i)^2$ :

$$\phi'_i(z) = \frac{-2\pi i \zeta e^{-2\pi i \zeta z} \cdot (z+i)^2 - 2(z+i) e^{-2\pi i \zeta z}}{(z+i)^4}, \quad \phi'_i(i) = \frac{-2\pi i \zeta e^{2\pi \zeta} (-4) - 2(2i) e^{2\pi \zeta}}{16} = -\frac{1}{4} (1 - 2\pi \zeta) i e^{2\pi \zeta}$$

Then, we can expand  $\phi_i(z)$  as the following term:

$$\phi_i(z) = \phi_i(i) + \phi'_i(i)(z-i) + \phi_{i,2}(z)(z-i)^2$$

The above term has  $\phi_{i,2}(z)$  being analytic at  $i$ . Hence,  $f(z)$  can be represented as:

$$f(z) = \frac{\phi_i(z)}{(z-i)^2} = \frac{\phi_i(i)}{(z-i)^2} + \frac{\phi'_i(i)}{(z-i)} + \phi_{i,2}(z)$$

Because the first term has antiderivative, while the third term is analytic at  $i$ , then for sufficiently small circle  $C$  centered at  $i$ , the residue is given by:

$$Res_{z=i}f(z) = \frac{1}{2\pi i} \int_C \frac{\phi'_i(i)}{(z-i)} dz = n(C, i) \cdot \phi'_i(i) = -\frac{1}{4}(1-2\pi\zeta)ie^{2\pi\zeta}$$

Now, apply similar concept for  $z = -i$ , the derivative of  $\phi_{-i}(z) = e^{-2\pi i\zeta z}/(z-i)^2$  is given as:

$$\phi'_{-i}(z) = \frac{-2\pi i\zeta e^{-2\pi i\zeta z} \cdot (z-i)^2 - 2(z-i)e^{-2\pi i\zeta z}}{(z-i)^4}, \quad \phi'_{-i}(-i) = \frac{-2\pi i\zeta e^{-2\pi\zeta}(-4) - 2(-2i)e^{-2\pi\zeta}}{16} = \frac{1}{4}(1+2\pi\zeta)ie^{-2\pi\zeta}$$

Then, expand  $\phi_{-i}(z)$  as follow:

$$\phi_{-i}(z) = \phi_{-i}(-i) + \phi'_{-i}(-i)(z+i)^2 + \phi_{-i,2}(z)(z+i)^2$$

Then, the above term has  $\phi_{-i,2}(z)$  being analytic at  $i$ . Hence,  $f(z)$  can again be represented as:

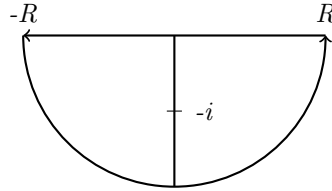
$$f(z) = \frac{\phi_{-i}(z)}{(z+i)^2} = \frac{\phi_{-i}(-i)}{(z+i)^2} + \frac{\phi'_{-i}(-i)}{(z+i)} + \phi_{-i,2}(z)$$

Therefore, based on similar reason as above (where the first and third terms are analytic or has antiderivative), with a sufficiently small circle  $C$  centered at  $-i$ , the residue at  $-i$  is given as:

$$Res_{z=-i}f(z) = \frac{1}{2\pi i} \int_C \frac{\phi'_{-i}(-i)}{(z+i)} dz = n(C, -i)\phi'_{-i}(-i) = \frac{1}{4}(1+2\pi\zeta)ie^{-2\pi\zeta}$$

#### Integration for $\zeta \geq 0$ :

Choose a radius  $R > 1$ , and consider a semicircle  $C_R$  in lower half plane parametrized by  $z = Re^{-i\theta}$  with  $\theta \in [0, \pi]$ , and another straight line with  $-R \leq x \leq R$  with the following orientation:



Since it encloses only  $z = -i$ , if we integrate  $f(z)$  along the contour of the semicircle, we'll get:

$$\int_R^{-R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \cdot Res_{z=-i}f(z) = 2\pi i \cdot \left(\frac{1}{4}(1+2\pi\zeta)ie^{-2\pi\zeta}\right) = -\frac{\pi}{2}(1+2\pi\zeta)e^{-2\pi\zeta}$$

Now, consider the second integral above with the parametrization:

$$\int_{C_R} f(z)dz = \int_\pi^0 \frac{e^{-2\pi i\zeta Re^{-i\theta}}}{(1+(Re^{-i\theta})^2)^2} \cdot (-iR)e^{-i\theta} d\theta$$

Since  $Re^{-i\theta} = R\cos(\theta) - i \cdot R\sin(\theta)$ , then the exponential part could be rewrite as:

$$e^{-2\pi i\zeta Re^{-i\theta}} = e^{-2\pi i\zeta(R\cos(\theta) - i \cdot R\sin(\theta))} = e^{-2\pi R\zeta \sin(\theta)} \cdot e^{-i \cdot 2\pi R\zeta \cos(\theta)}$$

Hence, if we take the modulus, the following inequality is true:

$$\left| \int_{C_R} f(z)dz \right| = \left| - \int_0^\pi \frac{e^{-2\pi i\zeta Re^{-i\theta}}}{(1+(Re^{-i\theta})^2)^2} \cdot (-iR)e^{-i\theta} d\theta \right|$$



$$\leq \int_0^\pi \frac{|e^{-2\pi i \zeta R e^{-i\theta}}|}{|1 + (R e^{-i\theta})^2|^2} \cdot |-i R e^{-i\theta}| d\theta \leq \int_0^\pi \frac{e^{-2\pi R \zeta \sin(\theta)}}{(R^2 - 1)^2} R d\theta$$

Since  $2\pi R \zeta \sin(\theta) \geq 0$  for  $\theta \in [0, \pi]$  (since  $\zeta \geq 0$  in this section),  $e^{-2\pi R \zeta \sin(\theta)} \leq 1$ . Then the above integral can then be bounded by:

$$\left| \int_{C_R} f(z) dz \right| \leq \int_0^\pi \frac{R}{(R^2 - 1)^2} d\theta = \frac{\pi R}{(R^2 - 1)^2}$$

So, as  $R$  grows indefinitely, we get:

$$0 \leq \lim_{R \rightarrow \infty} \left| \int_{C_R} f(z) dz \right| \leq \lim_{R \rightarrow \infty} \frac{\pi R}{(R^2 - 1)^2} = 0$$

Hence,  $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$ .

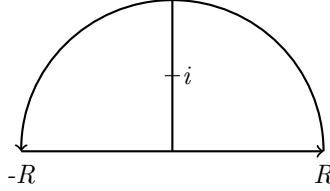
So, we can claim that  $\lim_{R \rightarrow \infty} \int_R^{-R} f(x) dx + \int_{C_R} f(z) dz = \int_{-\infty}^{\infty} f(x) dx = -\frac{\pi}{2}(1 + 2\pi\zeta)e^{-2\pi\zeta}$ , so  $\int_{-\infty}^{\infty} f(x) dx = \frac{\pi}{2}(1 + 2\pi\zeta)e^{-2\pi\zeta}$ .

Since  $\zeta \geq 0$ , then it can also be characterized as:

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x \zeta}}{(1 + x^2)^2} dx = \frac{\pi}{2}(1 + 2\pi|\zeta|)e^{-2\pi|\zeta|}$$

#### Integration for $\zeta < 0$ :

Choose a radius  $R > 1$ , and the semicircle  $C_R$  in the upper half plane parametrized by  $z = R e^{i\theta}$  with  $\theta \in [0, \pi]$ , and again consider a straight line with  $-R \leq x \leq R$  with the following orientation:



Since it encloses only  $z = i$ , if integrate  $f(z)$  along the contour of the semicircle, we'll get:

$$\int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = 2\pi i \cdot \text{Res}_{z=i} f(z) = 2\pi i \cdot \left(-\frac{1}{4}(1 - 2\pi\zeta)ie^{2\pi\zeta}\right) = \frac{\pi}{2}(1 - 2\pi\zeta)e^{2\pi\zeta}$$

Then, using similar technique from previous part, we can prove that  $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$ .

Hence,  $\lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx = \int_{-\infty}^{\infty} f(x) dx = -2\pi(1 - 2\pi\zeta)e^{2\pi\zeta}$ .

Since  $\zeta < 0$ , then it is then characterized as:

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x \zeta}}{(1 + x^2)^2} dx = \frac{\pi}{2}(1 + 2\pi|\zeta|)e^{-2\pi|\zeta|}$$

So, regardless of the sign of  $\zeta$ , the following integral is always true:

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \zeta}}{(1 + x^2)^2} dx = \frac{\pi}{2}(1 + 2\pi|\zeta|)e^{-2\pi|\zeta|}$$

**Question 6** Stein and Shakarchi Pg. 104 Problem 10:

Show that if  $a > 0$ , then

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a$$

**Pf:**

Choose  $0 < \epsilon < a$ , and  $R > a$ . Construct the semicircle  $C_\epsilon$  and  $C_R$  for upper half plane, with  $C_r$  being characterized by  $z = re^{i\theta}$  with  $\theta \in [0, \pi]$ . Along with two straight lines  $\gamma$  on real axis parametrized by  $\epsilon \leq |x| \leq R$ , we can create a contour with the following orientation:

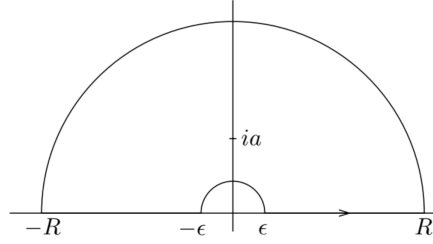


Figure 10. Contour in Exercise 10

Before starting, we need to redefine the logarithmic function, so that the region we're integrating over has a single-valued branch. Define the domain to be  $\mathbb{C} \setminus \{ix \mid x \leq 0\}$ , and for all  $z$  in the domain,  $\log(z) = \ln|z| + i \arg(z)$ , where  $\arg(z) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$  (so we can cover the whole real axis except for 0).

Then, for all  $x < 0$ ,  $\log(x) = \ln|x| + i \arg(x) = \ln|x| + i\pi$ .

Now, if we consider the integral of  $f(z) = \frac{\log(z)}{z^2 + a^2} = \frac{\log(z)}{(z-ia)(z+ia)}$ , the contour is enclosing the point  $ia$ . Notice that since  $\frac{\log(z)}{(z+ia)}$  is analytic at  $ia$ , then choose a sufficiently small circle  $C$  centered at  $ia$ , the residue at  $ia$  is given as:

$$\text{Res}_{z=ia} f(z) = \frac{1}{2\pi i} \int_C \frac{\log(z)}{(z+ia)} \cdot \frac{1}{(z-ia)} dz = n(C, ia) \cdot \frac{\log(ia)}{(ia+ia)} = \frac{\ln(a) + i\frac{\pi}{2}}{2ia}$$

So, integrating over the contour with the chosen orientation, we get:

$$\begin{aligned} \int_{\gamma - C_\epsilon + C_R} f(z) dz &= \left( \int_{-R}^{-\epsilon} f(x) dx + \int_{\epsilon}^R f(x) dx \right) - \int_{C_\epsilon} f(z) dz + \int_{C_R} f(z) dz \\ &= 2\pi i \cdot \text{Res}_{z=ia} f(z) = 2\pi i \cdot \frac{\ln(a) + i\frac{\pi}{2}}{2ia} = \frac{\pi}{a} \ln(a) + i\frac{\pi^2}{2a} \end{aligned}$$

**Integral over  $C_R$ :**

Given the parametrization  $z = Re^{i\theta}$  with  $\theta \in [0, \pi]$  for  $C_R$ , then the integral is given by:

$$\int_{C_R} f(z) dz = \int_0^\pi \frac{\log(Re^{i\theta})}{(Re^{i\theta})^2 + a^2} \cdot iRe^{i\theta} d\theta = \int_0^\pi \frac{\ln(R) + i\theta}{(Re^{i\theta})^2 + a^2} \cdot iRe^{i\theta} d\theta$$

Since  $0 \leq \theta \leq \pi$  for variable  $\theta$ , then the modulus of the integral can be bounded by:

$$\left| \int_{C_R} f(z) dz \right| \leq \int_0^\pi \frac{|\ln(R) + i\theta|}{|(Re^{i\theta})^2 + a^2|} |iRe^{i\theta}| d\theta \leq \int_0^\pi \frac{\sqrt{(\ln(R))^2 + \theta^2}}{|Re^{i\theta}|^2 - |a|^2} R d\theta \leq \int_0^\pi \frac{\sqrt{(\ln(R))^2 + \pi^2}}{R^2 - a^2} R d\theta$$

$$\leq \int_0^\pi \frac{|\ln(R)| + |\pi|}{R^2 - a^2} R d\theta = \frac{\pi(|\ln(R)| + \pi)}{R^2 - a^2} R$$

WLOG, can assume the initial choice of  $R \geq 1$ , hence  $\ln(R) \geq 0$ , so  $|\ln(R)| = \ln(R)$ .

Then, as  $R \rightarrow \infty$ , we get:

$$0 \leq \lim_{R \rightarrow \infty} \left| \int_{C_R} f(z) dz \right| \leq \lim_{R \rightarrow \infty} \frac{\pi(\ln(R) + \pi)R}{R^2 - a^2} = \lim_{R \rightarrow \infty} \frac{\pi(\ln(R) + 1 + \pi)}{2R} = \lim_{R \rightarrow \infty} \frac{\pi/R}{2} = 0$$

(Note: the above limit is given by L'hospital's Rule).

Hence,  $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$ .

### Integral over $C_\epsilon$ :

Given the parametrization  $z = \epsilon e^{i\theta}$  with  $\theta \in [0, \pi]$  for  $C_\epsilon$ , then the integral is given by:

$$\int_{C_\epsilon} f(z) dz = \int_0^\pi \frac{\log(\epsilon e^{i\theta})}{(\epsilon e^{i\theta})^2 + a^2} i\epsilon e^{i\theta} d\theta = \int_0^\pi \frac{\ln(\epsilon) + i\theta}{(\epsilon e^{i\theta})^2 + a^2} i\epsilon e^{i\theta} d\theta$$

Based on similar argument, the modulus of the integral can be bounded by:

$$\begin{aligned} \left| \int_{C_\epsilon} f(z) dz \right| &\leq \int_0^\pi \frac{|\ln(\epsilon) + i\theta|}{|(\epsilon e^{i\theta})^2 + a^2|} |i\epsilon e^{i\theta}| d\theta \leq \int_0^\pi \frac{\sqrt{(\ln(\epsilon))^2 + \theta^2}}{||\epsilon e^{i\theta}|^2 - a^2|} \epsilon d\theta \leq \int_0^\pi \frac{|\ln(\epsilon)| + |\theta|}{a^2 - \epsilon^2} \epsilon d\theta \\ &\leq \int_0^\pi \frac{|\ln(\epsilon)| + |\pi|}{a^2 - \epsilon^2} \epsilon d\theta \leq \frac{\pi(|\ln(\epsilon)| + \pi)}{a^2 - \epsilon^2} \epsilon = \frac{\pi|\ln(\epsilon)|\epsilon + \pi^2\epsilon}{a^2 - \epsilon^2} \end{aligned}$$

WLOG, can assume  $\epsilon < 1$ , hence  $\ln(\epsilon) < 0$ , or  $|\ln(\epsilon)| = -\ln(\epsilon)$  for simplicity.

Then, as  $\epsilon \rightarrow 0$ , the following limits are true:

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{a^2 - \epsilon^2} = \frac{1}{a^2}, \quad \lim_{\epsilon \rightarrow 0^+} \pi^2 \epsilon = 0, \quad \lim_{\epsilon \rightarrow 0^+} -\pi \ln(\epsilon) \epsilon = \lim_{\epsilon \rightarrow 0^+} -\pi \frac{\ln(\epsilon)}{1/\epsilon} = \lim_{\epsilon \rightarrow 0^+} -\pi \frac{1/\epsilon}{-1/\epsilon^2} = \lim_{\epsilon \rightarrow 0^+} \pi \epsilon = 0$$

Hence:

$$\lim_{\epsilon \rightarrow 0^+} \frac{\pi|\ln(\epsilon)|\epsilon + \pi^2\epsilon}{a^2 - \epsilon^2} = \lim_{\epsilon \rightarrow 0^+} \frac{-\pi \ln(\epsilon) \epsilon + \pi^2 \epsilon}{a^2 - \epsilon^2} = \frac{0 + 0}{a^2} = 0$$

So,  $\lim_{\epsilon \rightarrow 0^+} \int_{C_\epsilon} f(z) dz = 0$ .

### Original Integral:

To retrieve the original integral  $\int_0^\infty \frac{\log(x)}{x^2 + a^2} dx$ , we need  $R \rightarrow \infty$  and  $\epsilon \rightarrow 0^+$ . So, the following is true:

$$\begin{aligned} &\lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0^+} \left( \int_{-R}^{-\epsilon} f(x) dx + \int_{\epsilon}^R f(x) dx \right) - \int_{C_\epsilon} f(z) dz + \int_{C_R} f(z) dz \\ &= \int_{-\infty}^{0^-} f(x) dx + \int_{0^+}^{\infty} f(x) dx - \lim_{\epsilon \rightarrow 0^+} \int_{C_\epsilon} f(z) dz + \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz \\ &= \int_{-\infty}^{0^-} f(x) dx + \int_{0^+}^{\infty} f(x) dx \end{aligned}$$

Input the function  $f(z)$ , we get:

$$\begin{aligned} &\int_{-\infty}^{0^-} f(x) dx + \int_{0^+}^{\infty} f(x) dx = \int_{-\infty}^{0^-} \frac{\log(x)}{x^2 + a^2} dx + \int_{0^+}^{\infty} \frac{\log(x)}{x^2 + a^2} dx \\ &= \int_{-\infty}^{0^-} \frac{\ln|x| + i\pi}{x^2 + a^2} dx + \int_{0^+}^{\infty} \frac{\ln|x|}{x^2 + a^2} dx = \left( \int_{-\infty}^{0^-} \frac{\ln|x|}{x^2 + a^2} dx + \int_{0^+}^{\infty} \frac{\ln|x|}{x^2 + a^2} dx \right) + i \int_{-\infty}^{0^-} \frac{\pi}{x^2 + a^2} dx \end{aligned}$$

Also, recall that the above quantity equals to  $\frac{\pi}{a} \ln(a) + i\frac{\pi^2}{2a}$  by Residue Formula. Then:

$$\left( \int_{-\infty}^{0^-} \frac{\ln|x|}{x^2+a^2} dx + \int_{0^+}^{\infty} \frac{\ln|x|}{x^2+a^2} dx \right) = \operatorname{Re} \left( \frac{\pi}{a} \ln(a) + i\frac{\pi^2}{2a} \right) = \frac{\pi}{a} \ln(a)$$

Lastly, since the function  $\frac{\ln|x|}{x^2+a^2}$  is in fact an even function, then  $\int_{0^+}^{\infty} \frac{\ln|x|}{x^2+a^2} dx$  is half of the above quantity, or  $\frac{\pi}{2a} \ln(a)$ . Hence:

$$\int_0^{\infty} \frac{\ln(x)}{x^2+a^2} dx = \frac{\pi}{2a} \ln(a)$$

7

**Question 7** Stein and Shakarchi Pg. 104 Problem 11 :

Show that if  $|a| < 1$ , then

$$\int_0^{2\pi} \log|1 - ae^{i\theta}| d\theta = 0$$

Then, prove that the above result remains true if we assume only that  $|a| \leq 1$ .

**Pf:**

**When  $|a| < 1$ :**

If  $a = 0$ , the case is trivial (since  $\log|1 - ae^{i\theta}| = \log|1| = 0$ , then the whole integral is 0). So, assume  $a \neq 0$ .

Consider the integral of  $\log(1 - z)/iz$  along a circle  $C$  of radius  $|a| < 1$  centered at 0. With the parametrization  $z = ae^{i\theta}$  ( $\theta \in [0, 2\pi]$ ), it can be expressed as:

$$I = \int_C \frac{\log(1 - z)}{iz} dz = \int_0^{2\pi} \frac{\log(1 - ae^{i\theta})}{ia e^{i\theta}} (ia e^{i\theta}) d\theta = \int_0^{2\pi} \log(1 - ae^{i\theta}) d\theta$$

Which, define the domain to be  $\mathbb{C} \setminus \{x \geq 1\}$ , and  $\log(1 - z) = \ln|1 - z| + i \arg(1 - z)$ , it can also be expressed as:

$$I = \int_0^{2\pi} \log(1 - ae^{i\theta}) d\theta = \int_0^{2\pi} \ln|1 - ae^{i\theta}| d\theta + i \int_0^{2\pi} \arg(1 - ae^{i\theta}) d\theta$$

Going back to the original integral, since the function  $-\frac{\log(1-z)}{i}$  is analytic on the domain  $\mathbb{C} \setminus \{x \geq 1\}$ , so on the disk enclosed by  $C$ , the only Pole is generated by  $\frac{1}{z}$  (at the origin). Hence, let  $\phi(z) = \frac{\log(1-z)}{i}$ , the integral is then characterized by Cauchy's Integral Formula:

$$\int_C \frac{\log(1 - z)}{iz} dz = \int_C \frac{\phi(z)}{z} dz = 2\pi i \cdot n(C, 0) \phi(0)$$

With  $n(C, 0) = 1$  (winding number 1 by our construction), and  $\phi(0) = \log(1 - 0)/(i \cdot 1) = 0$ , then such integral is evaluated to be 0.

Now, since  $\operatorname{Re}(I) = \int_0^{2\pi} \log|1 - ae^{i\theta}| d\theta$ , while  $I = 0$ , then this integral must also evaluated to be 0.

**Case when  $|a| = 1$  (Real Integral Approach):**

For  $|a| = 1$ , WLOG, we can assume  $a = 1$  (since other than that, any  $e^{i\phi}$  is adding a phase of  $\phi$ , which by changing the range of our integrand, we can modify it to be the same as  $a = 1$ ).

Then, the integral can be rewrite as:

$$\begin{aligned}\int_0^{2\pi} \ln |1 - e^{i\theta}| d\theta &= \int_0^{2\pi} \ln |1 - (\cos(\theta) + i \sin(\theta))| d\theta = \int_0^{2\pi} \ln \sqrt{(1 - \cos(\theta))^2 + (-\sin(\theta))^2} d\theta \\ &= \int_0^{2\pi} \ln \sqrt{1 + \cos^2(\theta) - 2\cos(\theta) + \sin^2(\theta)} d\theta = \int_0^{2\pi} \ln \left( (2 - 2\cos(\theta))^{\frac{1}{2}} \right) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \ln \left( 4 \left( \frac{1 - \cos(\theta)}{2} \right) \right) d\theta = \frac{1}{2} \int_0^{2\pi} \ln \left( 2^2 \cdot \sin^2 \left( \frac{\theta}{2} \right) \right) d\theta \\ &= \frac{1}{2} \cdot 2 \int_0^{2\pi} \ln \left( 2 \sin \left( \frac{\theta}{2} \right) \right) d\theta\end{aligned}$$

Using the substitution  $u = \frac{\theta}{2}$  (with  $2du = d\theta$ ), it can be modified to:

$$\begin{aligned}\frac{1}{2} \cdot 2 \int_0^{2\pi} \ln \left( 2 \sin \left( \frac{\theta}{2} \right) \right) d\theta &= 2 \int_0^{\pi} \ln(2 \sin(u)) du = 2 \int_0^{\pi} (\ln(2) + \ln(\sin(u))) du \\ &= 2\pi \ln(2) + 2 \int_0^{\pi} \ln(\sin(u)) du\end{aligned}$$

So, it suffices to show the integral of  $\ln(\sin(u))$  from 0 to  $\pi$ .

Notice that since  $\sin(u) = \sin(\pi - u)$ , then for the following integral, do the substitution  $\theta = \pi - u$  (or  $-d\theta = du$ ), we can do the following:

$$\int_{\frac{\pi}{2}}^{\pi} \ln(\sin(u)) du = \int_{\frac{\pi}{2}}^{\pi} \ln(\sin(\pi - u)) du = - \int_{\frac{\pi}{2}}^0 \ln(\sin(\theta)) d\theta = \int_0^{\frac{\pi}{2}} \ln(\sin(\theta)) d\theta$$

So, the following is true:

$$\int_0^{\pi} \ln(\sin(u)) du = 2 \int_0^{\frac{\pi}{2}} \ln(\sin(u)) du$$

Let  $I = \int_0^{\frac{\pi}{2}} \ln(\sin(u)) du$ , using the fact that  $\sin(u) = \cos(\frac{\pi}{2} - u)$ , with substitution  $\theta = \frac{\pi}{2} - u$  (or  $-d\theta = du$ ), we can again get:

$$I = \int_0^{\frac{\pi}{2}} \ln \left( \cos \left( \frac{\pi}{2} - u \right) \right) du = - \int_{\frac{\pi}{2}}^0 \ln(\cos(\theta)) d\theta = \int_0^{\frac{\pi}{2}} \ln(\cos(\theta)) d\theta$$

So, we can also get the following:

$$\begin{aligned}2I &= \int_0^{\frac{\pi}{2}} \ln(\sin(u)) du + \int_0^{\frac{\pi}{2}} \ln(\cos(u)) du = \int_0^{\frac{\pi}{2}} \ln(\sin(u) \cos(u)) du = \int_0^{\frac{\pi}{2}} \ln \left( \frac{1}{2} \sin(2u) \right) du \\ &= \int_0^{\frac{\pi}{2}} \left( \ln \left( \frac{1}{2} \right) + \ln(\sin(2u)) \right) du = -\frac{\pi}{2} \ln(2) + \int_0^{\frac{\pi}{2}} \ln(\sin(2u)) du\end{aligned}$$

Using the substitution  $\theta = 2u$  (with  $\frac{1}{2}d\theta = du$ ), we get the following:

$$2I = -\frac{\pi}{2} \ln(2) + \frac{1}{2} \int_0^{\pi} \ln(\sin(\theta)) d\theta$$

But, recall the fact that  $\int_0^{\pi} \ln(\sin(u)) du = 2 \int_0^{\frac{\pi}{2}} \ln(\sin(u)) du = 2I$ . So, we get:

$$2I = -\frac{\pi}{2} \ln(2) + \frac{1}{2} \cdot 2I = -\frac{\pi}{2} \ln(2) + I, \quad I = -\frac{\pi}{2} \ln(2)$$

Substituting back to the original integral, we get the following:

$$\int_0^\pi \ln(\sin(u))du = 2I = -\pi \ln(2)$$

Finally, we get that the original integral is given by:

$$\int_0^{2\pi} \ln|1 - e^{i\theta}|d\theta = 2\pi \ln(2) + 2 \int_0^\pi \ln(\sin(u))du = 2\pi \ln(2) + 2(-\pi \ln(2)) = 0$$

Hence, we can say it is possible to extend to the case  $|a| = 1$ , so the integral is true for all  $|a| \leq 1$ .