

Math CS 122A HW9

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Question 1 Ahlfors Pg. 154 Problem 2:

How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

Pf:

For the disk $|z| < 1$, consider the function $-6z + 3$: It has one zero in $|z| < 1$, namely $z = \frac{1}{2}$.

On the other hand, for circle $|z| = 1$, the following inequalities are true:

$$|(z^2 - 6z + 3) - (-6z + 3)| = |z|^4 = 1$$

$$|-6z + 3| \geq |6|z| - 3| = 6 - 3 = 3$$

So, since $|(z^2 - 6z + 3) - (-6z + 3)| = 1 \leq 3 \leq |-6z + 3|$ for all z on $|z| = 1$, then by Rouché's Theorem, the two polynomials have the same number of zeroes enclosed by the circle $|z| = 1$.

Since $-6z + 3$ only has one zero in this region, then $z^4 - 6z + 3$ also has one zero in this region.

Now, consider the disk $|z| < 2$, and the function z^4 : It has four zeros in $|z| < 2$ counting multiplicity (namely $z = 0$).

On the other hand, for circle $|z| = 2$, the following inequalities are true:

$$|(z^2 - 6z + 3) - z^4| = |-6z + 3| \leq 6|z| + 3 = 15$$

$$|z^4| = |z|^4 = 16$$

So, since $|(z^4 - 6z + 3) - z^4| = 15 < 16 = |z^4|$ for all z on $|z| = 2$, by Rouché's Theorem again, the two polynomials have the same number of zeroes enclosed by the circle $|z| = 2$. Since z^4 has four zeroes in this region, then $z^4 - 6z + 3$ also has four zeroes in this region.

Then, since 4 zeroes has modulus less than 2, while 1 zero has modulus less than 1, counting the ones with modulus between 1 and 2, we have total of $4 - 1 = 3$ zeroes.

Question 2 Ahlfors Pg. 161 Problem 5:

Complex integration can sometimes be used to evaluate area integrals. As an illustration, show that if $f(z)$ is analytic and bounded for $|z| < 1$ and if $|\zeta| < 1$, then

$$f(\zeta) = \frac{1}{\pi} \int \int_{|z| < 1} \frac{f(z) dx dy}{(1 - \bar{z}\zeta)^2}$$

Pf:

If convert the above integral to polar coordinates, we get the following:

$$\frac{1}{\pi} \int \int_{|z| < 1} \frac{f(z) dx dy}{(1 - \bar{z}\zeta)^2} = \frac{1}{\pi} \int_{r=0}^1 \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})r}{(1 - \zeta re^{-i\theta})^2} d\theta dr = \frac{1}{\pi} \int_{r=0}^1 \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})(e^{i\theta})^2 \cdot r}{(e^{i\theta} - r\zeta)^2} d\theta dr$$

Then, define C to have parametrization $z = e^{i\theta}$ with $\theta \in [0, \pi]$, the inner part of the integral becomes:

$$\frac{1}{\pi} \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})(e^{i\theta})^2 r}{(e^{i\theta} - r\zeta)^2} d\theta = 2 \cdot \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{i\theta})re^{i\theta}}{(e^{i\theta} - r\zeta)^2} ie^{i\theta} d\theta = 2 \cdot \frac{1}{2\pi i} \int_C \frac{f(rz)rz}{(z - r\zeta)^2} dz$$

Recall that for any analytic function $\phi(z)$ at a point a , by Cauchy's Integral formula, we get:

$$\phi'(a) = \frac{1!}{2\pi i} \int_C \frac{\phi(z)}{(z - a)^2} dz$$

So, given that $a = r\zeta$ (which with $r \in [0, 1]$ and $|\zeta| < 1$, $r\zeta$ is strictly in the unit disk, so integrate over C given above is valid), let $\phi(z) = f(rz)rz$ (which $\phi'(z) = f'(rz)r^2z + f(rz)r$), we have:

$$\begin{aligned} f'(r\zeta)r^2(r\zeta) + f(r\zeta)r &= \frac{1!}{2\pi i} \int_C \frac{f(rz)rz}{(z - r\zeta)^2} dz \\ 2 \cdot \frac{1}{2\pi i} \int_C \frac{f(rz)rz}{(z - r\zeta)^2} dz &= 2(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r) \end{aligned}$$

Hence, the original integral can be rewrite as:

$$\frac{1}{\pi} \int_{r=0}^1 \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})(e^{i\theta})^2 \cdot r}{(e^{i\theta} - r\zeta)^2} d\theta dr = \int_{r=0}^1 2(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r) dr$$

Now, consider the function $f(\zeta r^2)r^2$, which has derivative with respect to r being $f'(\zeta r^2)r^2 \cdot 2r\zeta + f(\zeta r^2) \cdot 2r = 2(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r)$. Then, the above integral can be rewrite as:

$$\int_{r=0}^1 2(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r) dr = f(\zeta r^2)r^2 \Big|_{r=0}^1 = f(\zeta)$$

Hence, we can claim that for $f(z)$ that's analytic and bounded on $|z| < 1$, and given $|\zeta| < 1$, the integral is true:

$$f(\zeta) = \frac{1}{\pi} \int \int_{|z| < 1} \frac{f(z) dx dy}{(1 - \bar{z}\zeta)^2}$$

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Question 3 Stein and Shakarchi Pg. 64 Problem 1:

Prove that

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

Pf:

Consider the function e^{-z^2} , and the integration over a sector with origin at 0 and radius R . Which, this can be parametrized by three curves: γ_1 - a straight line on real axis with $x \in [0, R]$, γ_2 - a circular arc with radian $\frac{\pi}{4}$ and radius R (parametrized by $z = Re^{i\theta}$, where $\theta \in [0, \frac{\pi}{4}]$), and γ_3 - another straight line of $z = re^{i\frac{\pi}{4}}$ (where $r \in [0, R]$). The orientation is given as follow:

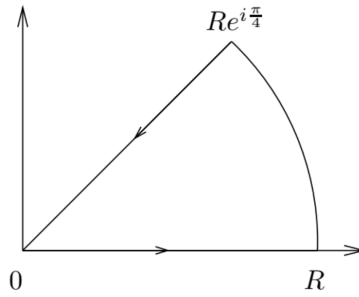


Figure 14. The contour in Exercise 1

If consider the integral over this closed curve, since e^{-z^2} is analytic on the whole plane, then the line integral is 0. So, $\int_{\gamma_1+\gamma_2+\gamma_3} e^{-z^2} dz = 0$.

For $\int_{\gamma_1} e^{-z^2} dz$, it is parametrized by $\int_0^R e^{-x^2} dx$, which $\lim_{R \rightarrow \infty} \int_0^R e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (since e^{-x^2} is even, while $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$).

For $\int_{\gamma_2} e^{-z^2} dz$, it is parametrized by the following:

$$\begin{aligned} \int_{\gamma_2} e^{-z^2} dz &= \int_0^{\frac{\pi}{4}} \exp(-(Re^{i\theta})^2) iRe^{i\theta} d\theta = \int_0^{\frac{\pi}{4}} \exp(-R^2 e^{i2\theta}) iRe^{i\theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \exp(-R^2(\cos(2\theta) + i\sin(2\theta))) iRe^{i\theta} d\theta = \int_0^{\frac{\pi}{4}} e^{-R^2 \cos(2\theta)} e^{-iR^2 \sin(2\theta)} iRe^{i\theta} d\theta \end{aligned}$$

Which, consider the modulus, the following inequality is true:

$$\begin{aligned} \left| \int_{\gamma_2} e^{-z^2} dz \right| &\leq \int_0^{\frac{\pi}{4}} |e^{-R^2 \cos(2\theta)}| \cdot |e^{-iR^2 \sin(2\theta)}| \cdot |iRe^{i\theta}| d\theta = \int_0^{\frac{\pi}{4}} Re^{-R^2 \cos(2\theta)} d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} Re^{-R^2 \cos(u)} du \end{aligned}$$

(Note: the second line is done by the parametrization $u = 2\theta$).

Now, since in the domain $[0, \frac{\pi}{2}]$, $1 - \frac{2}{\pi}u \leq \cos(u)$, then $e^{-R^2 \cos(u)} \leq e^{-R^2(1-\frac{2}{\pi}u)}$ (given that $-R^2 < 0$, while the two functions are positive on the given domain). Then, we can further bound the integral by:

$$\left| \int_{\gamma_2} e^{-z^2} dz \right| \leq \frac{1}{2} \int_0^{\frac{\pi}{2}} Re^{-R^2 \cos(u)} du \leq \frac{1}{2} \int_0^{\frac{\pi}{2}} Re^{-R^2(1-\frac{2}{\pi}u)} du = \frac{1}{2} \int_0^{\frac{\pi}{2}} Re^{R^2 \cdot \frac{2}{\pi}u - R^2} du$$

$$\leq \frac{R}{2} \cdot \frac{\pi}{2R^2} e^{R^2 \cdot \frac{2}{\pi} u - R^2} \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4R} (e^{R^2 \cdot \frac{2}{\pi} \cdot \frac{\pi}{2} - R^2} - e^{R^2 \cdot \frac{2}{\pi} \cdot 0 - R^2}) = \frac{\pi}{4R} (1 - e^{-R^2})$$

Then, since $\lim_{R \rightarrow \infty} \frac{\pi}{4R} = 0$, $\lim_{R \rightarrow \infty} (1 - e^{-R^2}) = 1$, then:

$$0 \leq \lim_{R \rightarrow \infty} \left| \int_{\gamma_2} e^{-z^2} dz \right| \leq \lim_{R \rightarrow \infty} \frac{\pi}{4R} (1 - e^{-R^2}) = 0$$

Hence, we can claim that $\lim_{R \rightarrow \infty} \int_{\gamma_2} e^{-z^2} dz = 0$.

Lastly, for $\int_{\gamma_3} e^{-z^2} dz$, it is parametrized by $\int_R^0 \exp(-(re^{i\frac{\pi}{4}})^2) e^{i\frac{\pi}{4}} dr$. Which, can be modified as:

$$\begin{aligned} \int_R^0 \exp(-r^2 e^{i\frac{\pi}{2}}) e^{i\frac{\pi}{4}} dr &= e^{i\frac{\pi}{4}} \int_R^0 e^{-ir^2} dr = e^{i\frac{\pi}{4}} \left(\int_R^0 \cos(r^2) dr - i \int_R^0 \sin(r^2) dr \right) \\ &= -e^{i\frac{\pi}{4}} \left(\int_0^R \cos(r^2) dr - i \int_0^R \sin(r^2) dr \right) \end{aligned}$$

Now, because $\int_{\gamma_1 + \gamma_2 + \gamma_3} e^{-z^2} dz = 0$, then $\int_{\gamma_3} e^{-z^2} dz = -(\int_{\gamma_1} e^{-z^2} dz + \int_{\gamma_2} e^{-z^2} dz)$. Hence:

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{\gamma_3} e^{-z^2} dz &= \lim_{R \rightarrow \infty} - \left(\int_{\gamma_1} e^{-z^2} dz + \int_{\gamma_2} e^{-z^2} dz \right) = -\frac{\sqrt{\pi}}{2} \\ \lim_{R \rightarrow \infty} -e^{i\frac{\pi}{4}} \left(\int_0^R \cos(r^2) dr - i \int_0^R \sin(r^2) dr \right) &= -\frac{\sqrt{\pi}}{2} \end{aligned}$$

Hence, we can claim the following:

$$\int_0^\infty \cos(r^2) dr - i \int_0^\infty \sin(r^2) dr = \frac{\sqrt{\pi}}{2} e^{-i\frac{\pi}{4}} = \frac{\sqrt{2\pi}}{4} (1 - i)$$

Then, take the real and imaginary part respectively, we get:

$$\begin{aligned} \int_0^\infty \cos(r^2) dr &= \operatorname{Re} \left(\frac{\sqrt{2\pi}}{4} (1 - i) \right) = \frac{\sqrt{2\pi}}{4} \\ \int_0^\infty \sin(r^2) dr &= -\operatorname{Im} \left(\frac{\sqrt{2\pi}}{4} (1 - i) \right) = \frac{\sqrt{2\pi}}{4} \end{aligned}$$

Hence, the two integrals evaluated to be $\frac{\sqrt{2\pi}}{4}$.

Question 4 *Stein and Shakarchi Pg. 65 Problem 4:*

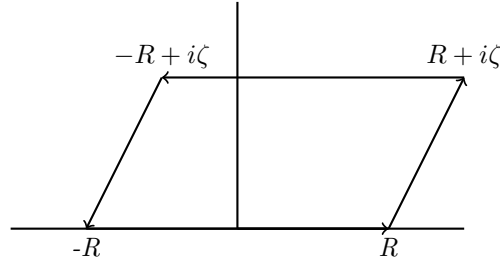
Prove that for all $\zeta \in \mathbb{C}$ we have

$$e^{-\pi\zeta^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \zeta} dx$$

Pf:

Given $\zeta = u + iv$ for some $u, v \in \mathbb{R}$. Consider the function $e^{-\pi z^2}$ which is analytic on \mathbb{C} . There are three cases:

- (1) **When** $u > 0$, given $R > 0$, consider the parallelogram generated by the points $-R, R, R + i\zeta, -R + i\zeta$ with counterclockwise orientation. The orientation is as follow:



Then, the integral of $e^{-\pi z^2}$ on the contour is 0 (since it is analytic on the whole plane). Which, it can be broken down into the sum of following integrals:

First, for the one on the real axis, it is parametrized by $\int_{-R}^R e^{-\pi x^2} dx$, where $\lim_{R \rightarrow \infty} \int_{-R}^R e^{-\pi x^2} dx = 1$ (Gauss Integral).

For the ones on the side, since the sides are parametrized by $R + i\zeta t$ and $-R + i\zeta t$ for $t \in [0, 1]$ respectively, then the first integral is given by:

$$\begin{aligned} \int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt &= \int_0^1 e^{-\pi(R^2+i\cdot 2R\zeta t-\zeta^2 t^2)} \cdot i\zeta dt \\ &= \int_0^1 e^{-\pi(R^2-2Rvt)} \cdot e^{-i\cdot 2\pi R u t + \pi \zeta^2 t^2} \cdot i\zeta dt = e^{-\pi R^2} \int_0^1 e^{R\cdot 2\pi vt} \cdot e^{-i\cdot 2\pi R u t} \cdot e^{\pi \zeta^2 t^2} \cdot i\zeta dt \end{aligned}$$

Which, taking the modulus, it is bounded by:

$$\begin{aligned} \left| \int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt \right| &\leq e^{-\pi R^2} \int_0^1 e^{R\cdot 2\pi vt} \cdot |e^{-i\cdot 2\pi R u t}| \cdot |e^{\pi \zeta^2 t^2}| \cdot |\zeta| dt \\ &\leq e^{-\pi R^2} \int_0^1 e^{R\cdot |2\pi v|} \cdot |e^{\pi \zeta^2 t^2}| \cdot |\zeta| dt = e^{\pi R^2 + R\cdot |2\pi v|} \int_0^1 |e^{-\pi \zeta^2 t^2}| \cdot |\zeta| dt \end{aligned}$$

(Note: the above is given by $R \cdot 2\pi vt \leq R \cdot |2\pi v| \cdot t \leq R \cdot |2\pi v|$, since $t \in [0, 1]$).

Then, since $\lim_{R \rightarrow \infty} -\pi R^2 + R|2\pi v| = -\infty$, so $\lim_{R \rightarrow \infty} e^{-\pi R^2 + R|2\pi v|} = 0$, then since $\int_0^1 |e^{\pi \zeta^2 t^2}| \cdot |\zeta| dt$ is a constant, then:

$$\lim_{R \rightarrow \infty} e^{-\pi R^2 + R\cdot |2\pi v|} \int_0^1 |e^{-\zeta^2 t^2}| \cdot |\zeta| dt = 0$$

So, we can conclude the following:

$$\lim_{R \rightarrow \infty} \int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt = 0$$

Similar concepts applied on the line $-R + i\zeta t$ (since $-\pi(-R + i\zeta t)^2$ is then given by $-\pi(R^2 + 2Rvt) - \pi(-i \cdot 2Rut - \zeta^2 t^2)$, so the same concept still applies, where the bound of the modulus is dominated by $e^{-\pi(R^2 + 2Rv)}$, which converges to 0 as $R \rightarrow \infty$). Hence:

$$\lim_{R \rightarrow \infty} \int_0^1 e^{-\pi(-R+i\zeta t)^2} \cdot i\zeta dt = 0$$

Lastly, the translated line is parametrized by $x + i\zeta$ for $x \in [-R, R]$, then the integral is given by:

$$\int_{-R}^R e^{-\pi(x+i\zeta)^2} dx = \int_{-R}^R e^{-\pi(x^2+2ix\zeta-\zeta^2)} dx = e^{\pi\zeta^2} \int_{-R}^R e^{-\pi x^2} \cdot e^{-2\pi ix\zeta} dx$$

Now, summing up all the path integrals with right orientation, we get the following:

$$\begin{aligned} \int_{-R}^R e^{-\pi x^2} dx + \int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt - \int_0^1 e^{-\pi(-R+i\zeta t)^2} \cdot i\zeta dt - \int_{-R}^R e^{-\pi(x+i\zeta)^2} dx &= 0 \\ \int_{-R}^R e^{-\pi(x+i\zeta)^2} dx &= \int_{-R}^R e^{-\pi x^2} dx + \int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt - \int_0^1 e^{-\pi(-R+i\zeta t)^2} \cdot i\zeta dt \end{aligned}$$

So, take $R \rightarrow \infty$, the first term on the right approaches 1, while the next two terms converges to 0 (from the above statements), then the limit becomes:

$$\lim_{R \rightarrow \infty} \int_{-R}^R e^{-\pi(x+i\zeta)^2} dx = \lim_{R \rightarrow \infty} e^{\pi\zeta^2} \int_{-R}^R e^{-\pi x^2} \cdot e^{-2\pi ix\zeta} dx = 1$$

So, we can conclude that $\int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi ix\zeta} dx = e^{-\pi\zeta^2}$.

Which, by doing substitution $u = -x$ ($-du = dx$), we get the following:

$$-\int_{\infty}^{-\infty} e^{-\pi u^2} \cdot e^{-2\pi i(-u)\zeta} du = \int_{-\infty}^{\infty} e^{-\pi u^2} \cdot e^{2\pi iu\zeta} du = e^{-\pi\zeta^2}$$

(2) **When** $u = 0$, since $\zeta = iv$, then $\zeta^2 = -v^2$. So, the proposed integral becomes:

$$\int_{-R}^R e^{-\pi x^2} \cdot e^{2\pi ix\zeta} dx = \int_{-R}^R e^{-\pi x^2} \cdot e^{2\pi ix \cdot iv} dx = \int_{-R}^R e^{-\pi x^2} \cdot e^{-2\pi vx} dx$$

Which, by completing the square, we get:

$$\int_{-R}^R e^{-\pi x^2} \cdot e^{2\pi ix\zeta} dx = e^{\pi v^2} \int_{-R}^R e^{-\pi x^2} \cdot e^{-2\pi vx} \cdot e^{-\pi v^2} dx = e^{\pi v^2} \int_{-R}^R e^{-\pi(x^2+2vx+v^2)} dx = e^{\pi v^2} \int_{-R}^R e^{-\pi(x+v)^2} dx$$

Then, as $R \rightarrow \infty$, we get:

$$\lim_{R \rightarrow \infty} \int_{-R}^R e^{-\pi x^2} \cdot e^{2\pi ix\zeta} dx = \lim_{R \rightarrow \infty} e^{\pi v^2} \int_{-R}^R e^{-\pi(x+v)^2} dx = e^{\pi v^2} \int_{-\infty}^{\infty} e^{-\pi(x+v)^2} dx = e^{\pi v^2}$$

And, since $e^{\pi v^2} = e^{-\pi\zeta^2}$, then:

$$e^{-\pi\zeta^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{2\pi ix\zeta} dx$$

So, regardless of the case, we can say that the following integral is true:

$$e^{-\pi\zeta^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{2\pi i x \zeta} dx$$

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Question 5 *Stein and Shakarchi Pg. 103 Problem 5:*

Use contour integration to show that for all ζ real

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \zeta}}{(1+x^2)^2} dx = \frac{\pi}{2} (1+2\pi|\zeta|) e^{-2\pi|\zeta|}$$

Pf:

Residue at $i, -i$:

Consider the function $f(z) = e^{-2\pi i \zeta z} / (1+z^2)^2 = e^{-2\pi i \zeta z} / ((z-i)(z+i))^2$, which it has poles at $z = \pm i$, each with order 2 (since $(z^2+1)^2 = (z-i)^2(z+i)^2$).

Then, to show its residue at i , consider the derivative of $\phi_i(z) = e^{-2\pi i \zeta z} / (z+i)^2$:

$$\phi'_i(z) = \frac{-2\pi i \zeta e^{-2\pi i \zeta z} \cdot (z+i)^2 - 2(z+i)e^{-2\pi i \zeta z}}{(z+i)^4}, \quad \phi'_i(i) = \frac{-2\pi i \zeta e^{2\pi \zeta}(-4) - 2(2i)e^{2\pi \zeta}}{16} = -\frac{1}{4}(1-2\pi\zeta)ie^{2\pi\zeta}$$

Then, we can expand $\phi_i(z)$ as the following term:

$$\phi_i(z) = \phi_i(i) + \phi'_i(i)(z-i) + \phi_{i,2}(z)(z-i)^2$$

The above term has $\phi_{i,2}(z)$ being analytic at i . Hence, $f(z)$ can be represented as:

$$f(z) = \frac{\phi_i(z)}{(z-i)^2} = \frac{\phi_i(i)}{(z-i)^2} + \frac{\phi'_i(i)}{(z-i)} + \phi_{i,2}(z)$$

Because the first term has antiderivative, while the third term is analytic at i , then for sufficiently small circle C centered at i , the residue is given by:

$$\text{Res}_{z=i} f(z) = \frac{1}{2\pi i} \int_C \frac{\phi'_i(i)}{(z-i)} dz = n(C, i) \cdot \phi'_i(i) = -\frac{1}{4}(1-2\pi\zeta)ie^{2\pi\zeta}$$

Now, apply similar concept for $z = -i$, the derivative of $\phi_{-i}(z) = e^{-2\pi i \zeta z} / (z-i)^2$ is given as:

$$\phi'_{-i}(z) = \frac{-2\pi i \zeta e^{-2\pi i \zeta z} \cdot (z-i)^2 - 2(z-i)e^{-2\pi i \zeta z}}{(z-i)^4}, \quad \phi'_{-i}(-i) = \frac{-2\pi i \zeta e^{-2\pi \zeta}(-4) - 2(-2i)e^{-2\pi \zeta}}{16} = \frac{1}{4}(1+2\pi\zeta)ie^{-2\pi\zeta}$$

Then, expand $\phi_{-i}(z)$ as follow:

$$\phi_{-i}(z) = \phi_{-i}(-i) + \phi'_{-i}(-i)(z+i)^2 + \phi_{-i,2}(z)(z+i)^2$$

Then, the above term has $\phi_{-i,2}(z)$ being analytic at i . Hence, $f(z)$ can again be represented as:

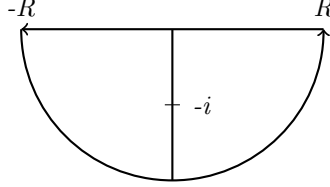
$$f(z) = \frac{\phi_{-i}(z)}{(z+i)^2} = \frac{\phi_{-i}(-i)}{(z+i)^2} + \frac{\phi'_{-i}(-i)}{(z+i)} + \phi_{-i,2}(z)$$

Therefore, based on similar reason as above (where the first and third terms are analytic or has antiderivative), with a sufficiently small circle C centered at $-i$, the residue at $-i$ is given as:

$$\text{Res}_{z=-i} f(z) = \frac{1}{2\pi i} \int_C \frac{\phi'_{-i}(-i)}{(z+i)} dz = n(C, -i) \phi'_{-i}(-i) = \frac{1}{4}(1 + 2\pi\zeta)ie^{-2\pi\zeta}$$

Integration for $\zeta \geq 0$:

Choose a radius $R > 1$, and consider a semicircle C_R in lower half plane parametrized by $z = Re^{-i\theta}$ with $\theta \in [0, \pi]$, and another straight line with $-R \leq x \leq R$ with the following orientation:



Since it encloses only $z = -i$, if we integrate $f(z)$ along the contour of the semicircle, we'll get:

$$\int_R^{-R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \cdot \text{Res}_{z=-i} f(z) = 2\pi i \cdot \left(\frac{1}{4}(1 + 2\pi\zeta)ie^{-2\pi\zeta}\right) = -\frac{\pi}{2}(1 + 2\pi\zeta)e^{-2\pi\zeta}$$

Now, consider the second integral above with the parametrization:

$$\int_{C_R} f(z)dz = \int_{\pi}^0 \frac{e^{-2\pi i\zeta Re^{-i\theta}}}{(1 + (Re^{-i\theta})^2)^2} \cdot (-iR)e^{-i\theta} d\theta$$

Since $Re^{-i\theta} = R\cos(\theta) - i \cdot R\sin(\theta)$, then the exponential part could be rewrite as:

$$e^{-2\pi i\zeta Re^{-i\theta}} = e^{-2\pi i\zeta (R\cos(\theta) - i \cdot R\sin(\theta))} = e^{-2\pi R\zeta \sin(\theta)} \cdot e^{-i \cdot 2\pi R\zeta \cos(\theta)}$$

Hence, if we take the modulus, the following inequality is true:

$$\begin{aligned} \left| \int_{C_R} f(z)dz \right| &= \left| - \int_0^{\pi} \frac{e^{-2\pi i\zeta Re^{-i\theta}}}{(1 + (Re^{-i\theta})^2)^2} \cdot (-iR)e^{-i\theta} d\theta \right| \\ &\leq \int_0^{\pi} \frac{|e^{-2\pi i\zeta Re^{-i\theta}}|}{|1 + (Re^{-i\theta})^2|^2} \cdot |-iRe^{-i\theta}| d\theta \leq \int_0^{\pi} \frac{e^{-2\pi R\zeta \sin(\theta)}}{(R^2 - 1)^2} R d\theta \end{aligned}$$

Since $2\pi R\zeta \sin(\theta) \geq 0$ for $\theta \in [0, \pi]$ (since $\zeta \geq 0$ in this section), $e^{-2\pi R\zeta \sin(\theta)} \leq 1$. Then the above integral can then be bounded by:

$$\left| \int_{C_R} f(z)dz \right| \leq \int_0^{\pi} \frac{R}{(R^2 - 1)^2} d\theta = \frac{\pi R}{(R^2 - 1)^2}$$

So, as R grows indefinitely, we get:

$$0 \leq \lim_{R \rightarrow \infty} \left| \int_{C_R} f(z)dz \right| \leq \lim_{R \rightarrow \infty} \frac{\pi R}{(R^2 - 1)^2} = 0$$

Hence, $\lim_{R \rightarrow \infty} \int_{C_R} f(z)dz = 0$.

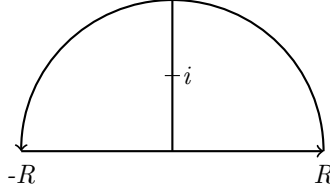
So, we can claim that $\lim_{R \rightarrow \infty} \int_R^{-R} f(x)dx + \int_{C_R} f(z)dz = \int_{-\infty}^{\infty} f(x)dx = -\frac{\pi}{2}(1 + 2\pi\zeta)e^{-2\pi\zeta}$, so $\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2}(1 + 2\pi\zeta)e^{-2\pi\zeta}$.

Since $\zeta \geq 0$, then it can also be characterized as:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x \zeta}}{(1+x^2)^2} dx = \frac{\pi}{2} (1 + 2\pi|\zeta|) e^{-2\pi|\zeta|}$$

Integration for $\zeta < 0$:

Choose a radius $R > 1$, and the semicircle C_R in the upper half plane parametrized by $z = Re^{i\theta}$ with $\theta \in [0, \pi]$, and again consider a straight line with $-R \leq x \leq R$ with the following orientation:



Since it encloses only $z = i$, if integrate $f(z)$ along the contour of the semicircle, we'll get:

$$\int_{-R}^R f(x)dx + \int_{C_R} f(z)dz = 2\pi i \cdot \text{Res}_{z=i} f(z) = 2\pi i \cdot \left(-\frac{1}{4}(1 - 2\pi\zeta)ie^{2\pi\zeta}\right) = \frac{\pi}{2}(1 - 2\pi\zeta)e^{2\pi\zeta}$$

Then, using similar technique from previous part, we can prove that $\lim_{R \rightarrow \infty} \int_{C_R} f(z)dz = 0$.

Hence, $\lim_{R \rightarrow \infty} \int_{-R}^R f(x)dx = \int_{-\infty}^{\infty} f(x)dx = -2\pi(1 - 2\pi\zeta)e^{2\pi\zeta}$.

Since $\zeta < 0$, then it is then characterized as:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x \zeta}}{(1+x^2)^2} dx = \frac{\pi}{2} (1 + 2\pi|\zeta|) e^{-2\pi|\zeta|}$$

So, regardless of the sign of ζ , the following integral is always true:

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \zeta}}{(1+x^2)^2} dx = \frac{\pi}{2} (1 + 2\pi|\zeta|) e^{-2\pi|\zeta|}$$

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Question 6 Stein and Shakarchi Pg. 104 Problem 10:

Show that if $a > 0$, then

$$\int_0^{\infty} \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a$$

Pf:

Choose $0 < \epsilon < a$, and $R > a$. Construct the semicircle C_ϵ and C_R for upper half plane, with C_r being characterized by $z = re^{i\theta}$ with $\theta \in [0, \pi]$. Along with two straight lines γ on real axis parametrized by $\epsilon \leq |x| \leq R$, we can create a contour with the following orientation:

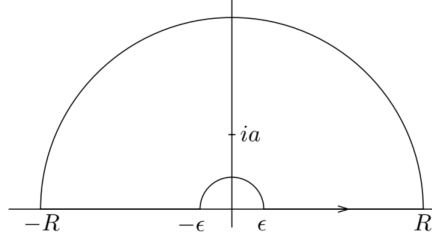


Figure 10. Contour in Exercise 10

Before starting, we need to redefine the logarithmic function, so that the region we're integrating over has a single-valued branch. Define the domain to be $\mathbb{C} \setminus \{ix \mid x \leq 0\}$, and for all z in the domain, $\log(z) = \ln|z| + i\arg(z)$, where $\arg(z) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ (so we can cover the whole real axis except for 0).

Then, for all $x < 0$, $\log(x) = \ln|x| + i\arg(x) = \ln|x| + i\pi$.

Now, if we consider the integral of $f(z) = \frac{\log(z)}{z^2 + a^2} = \frac{\log(z)}{(z-ia)(z+ia)}$, the contour is enclosing the point ia . Notice that since $\frac{\log(z)}{(z+ia)}$ is analytic at ia , then choose a sufficiently small circle C centered at ia , the residue at ia is given as:

$$\text{Res}_{z=ia} f(z) = \frac{1}{2\pi i} \int_C \frac{\log(z)}{(z+ia)} \cdot \frac{1}{(z-ia)} dz = n(C, ia) \cdot \frac{\log(ia)}{(ia+ia)} = \frac{\ln(a) + i\frac{\pi}{2}}{2ia}$$

So, integrating over the contour with the chosen orientation, we get:

$$\begin{aligned} \int_{\gamma - C_\epsilon + C_R} f(z) dz &= \left(\int_{-R}^{-\epsilon} f(x) dx + \int_{\epsilon}^R f(x) dx \right) - \int_{C_\epsilon} f(z) dz + \int_{C_R} f(z) dz \\ &= 2\pi i \cdot \text{Res}_{z=ia} f(z) = 2\pi i \cdot \frac{\ln(a) + i\frac{\pi}{2}}{2ia} = \frac{\pi}{a} \ln(a) + i\frac{\pi^2}{2a} \end{aligned}$$

Integral over C_R :

Given the parametrization $z = Re^{i\theta}$ with $\theta \in [0, \pi]$ for C_R , then the integral is given by:

$$\int_{C_R} f(z) dz = \int_0^\pi \frac{\log(Re^{i\theta})}{(Re^{i\theta})^2 + a^2} \cdot iRe^{i\theta} d\theta = \int_0^\pi \frac{\ln(R) + i\theta}{(Re^{i\theta})^2 + a^2} \cdot iRe^{i\theta} d\theta$$

Since $0 \leq \theta \leq \pi$ for variable θ , then the modulus of the integral can be bounded by:

$$\begin{aligned} \left| \int_{C_R} f(z) dz \right| &\leq \int_0^\pi \frac{|\ln(R) + i\theta|}{|(Re^{i\theta})^2 + a^2|} |iRe^{i\theta}| d\theta \leq \int_0^\pi \frac{\sqrt{(\ln(R))^2 + \theta^2}}{|Re^{i\theta}|^2 - |a|^2} R d\theta \leq \int_0^\pi \frac{\sqrt{(\ln(R))^2 + \pi^2}}{R^2 - a^2} R d\theta \\ &\leq \int_0^\pi \frac{|\ln(R)| + |\pi|}{R^2 - a^2} R d\theta = \frac{\pi(|\ln(R)| + \pi)}{R^2 - a^2} R \end{aligned}$$

WLOG, can assume the initial choice of $R \geq 1$, hence $\ln(R) \geq 0$, so $|\ln(R)| = \ln(R)$.

Then, as $R \rightarrow \infty$, we get:

$$0 \leq \lim_{R \rightarrow \infty} \left| \int_{C_R} f(z) dz \right| \leq \lim_{R \rightarrow \infty} \frac{\pi(\ln(R) + \pi)R}{R^2 - a^2} = \lim_{R \rightarrow \infty} \frac{\pi(\ln(R) + 1 + \pi)}{2R} = \lim_{R \rightarrow \infty} \frac{\pi/R}{2} = 0$$

(Note: the above limit is given by L'hôpital's Rule).

Hence, $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$.

Integral over C_ϵ :

Given the parametrization $z = \epsilon e^{i\theta}$ with $\theta \in [0, \pi]$ for C_ϵ , then the integral is given by:

$$\int_{C_\epsilon} f(z) dz = \int_0^\pi \frac{\log(\epsilon e^{i\theta})}{(\epsilon e^{i\theta})^2 + a^2} i\epsilon e^{i\theta} d\theta = \int_0^\pi \frac{\ln(\epsilon) + i\theta}{(\epsilon e^{i\theta})^2 + a^2} i\epsilon e^{i\theta} d\theta$$

Based on similar argument, the modulus of the integral can be bounded by:

$$\begin{aligned} \left| \int_{C_\epsilon} f(z) dz \right| &\leq \int_0^\pi \frac{|\ln(\epsilon) + i\theta|}{|(\epsilon e^{i\theta})^2 + a^2|} |\epsilon e^{i\theta}| d\theta \leq \int_0^\pi \frac{\sqrt{(\ln(\epsilon))^2 + \theta^2}}{||\epsilon e^{i\theta}|^2 - a^2|} \epsilon d\theta \leq \int_0^\pi \frac{|\ln(\epsilon)| + |\theta|}{a^2 - \epsilon^2} \epsilon d\theta \\ &\leq \int_0^\pi \frac{|\ln(\epsilon)| + |\pi|}{a^2 - \epsilon^2} \epsilon d\theta \leq \frac{\pi(|\ln(\epsilon)| + \pi)}{a^2 - \epsilon^2} \epsilon = \frac{\pi|\ln(\epsilon)|\epsilon + \pi^2\epsilon}{a^2 - \epsilon^2} \end{aligned}$$

WLOG, can assume $\epsilon < 1$, hence $\ln(\epsilon) < 0$, or $|\ln(\epsilon)| = -\ln(\epsilon)$ for simplicity.

Then, as $\epsilon \rightarrow 0$, the following limits are true:

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{a^2 - \epsilon^2} = \frac{1}{a^2}, \quad \lim_{\epsilon \rightarrow 0^+} \pi^2 \epsilon = 0, \quad \lim_{\epsilon \rightarrow 0^+} -\pi \ln(\epsilon) \epsilon = \lim_{\epsilon \rightarrow 0^+} -\pi \frac{\ln(\epsilon)}{1/\epsilon} = \lim_{\epsilon \rightarrow 0^+} -\pi \frac{1/\epsilon}{-1/\epsilon^2} = \lim_{\epsilon \rightarrow 0^+} \pi \epsilon = 0$$

Hence:

$$\lim_{\epsilon \rightarrow 0^+} \frac{\pi|\ln(\epsilon)|\epsilon + \pi^2\epsilon}{a^2 - \epsilon^2} = \lim_{\epsilon \rightarrow 0^+} \frac{-\pi \ln(\epsilon) \epsilon + \pi^2 \epsilon}{a^2 - \epsilon^2} = \frac{0 + 0}{a^2} = 0$$

So, $\lim_{\epsilon \rightarrow 0^+} \int_{C_\epsilon} f(z) dz = 0$.

Original Integral:

To retrieve the original integral $\int_0^\infty \frac{\log(x)}{x^2 + a^2} dx$, we need $R \rightarrow \infty$ and $\epsilon \rightarrow 0^+$. So, the following is true:

$$\begin{aligned} &\lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0^+} \left(\int_{-R}^{-\epsilon} f(x) dx + \int_{\epsilon}^R f(x) dx \right) - \int_{C_\epsilon} f(z) dz + \int_{C_R} f(z) dz \\ &= \int_{-\infty}^{0^-} f(x) dx + \int_{0^+}^{\infty} f(x) dx - \lim_{\epsilon \rightarrow 0^+} \int_{C_\epsilon} f(z) dz + \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz \\ &= \int_{-\infty}^{0^-} f(x) dx + \int_{0^+}^{\infty} f(x) dx \end{aligned}$$

Input the function $f(z)$, we get:

$$\begin{aligned} &\int_{-\infty}^{0^-} f(x) dx + \int_{0^+}^{\infty} f(x) dx = \int_{-\infty}^{0^-} \frac{\log(x)}{x^2 + a^2} dx + \int_{0^+}^{\infty} \frac{\log(x)}{x^2 + a^2} dx \\ &= \int_{-\infty}^{0^-} \frac{\ln|x| + i\pi}{x^2 + a^2} dx + \int_{0^+}^{\infty} \frac{\ln|x|}{x^2 + a^2} dx = \left(\int_{-\infty}^{0^-} \frac{\ln|x|}{x^2 + a^2} dx + \int_{0^+}^{\infty} \frac{\ln|x|}{x^2 + a^2} dx \right) + i \int_{-\infty}^{0^-} \frac{\pi}{x^2 + a^2} dx \end{aligned}$$

Also, recall that the above quantity equals to $\frac{\pi}{a} \ln(a) + i \frac{\pi^2}{2a}$ by Residue Formula. Then:

$$\left(\int_{-\infty}^{0^-} \frac{\ln|x|}{x^2 + a^2} dx + \int_{0^+}^{\infty} \frac{\ln|x|}{x^2 + a^2} dx \right) = \operatorname{Re} \left(\frac{\pi}{a} \ln(a) + i \frac{\pi^2}{2a} \right) = \frac{\pi}{a} \ln(a)$$

Lastly, since the function $\frac{\ln|x|}{x^2 + a^2}$ is in fact an even function, then $\int_{0^+}^{\infty} \frac{\ln|x|}{x^2 + a^2} dx$ is half of the above quantity, or $\frac{\pi}{2a} \ln(a)$. Hence:

$$\int_0^\infty \frac{\ln(x)}{x^2 + a^2} dx = \frac{\pi}{2a} \ln(a)$$

Question 7 Stein and Shakarchi Pg. 104 Problem 11 :

Show that if $|a| < 1$, then

$$\int_0^{2\pi} \log |1 - ae^{i\theta}| d\theta = 0$$

Then, prove that the above result remains true if we assume only that $|a| \leq 1$.

Pf:

When $|a| < 1$:

If $a = 0$, the case is trivial (since $\log |1 - ae^{i\theta}| = \log |1| = 0$, then the whole integral is 0). So, assume $a \neq 0$.

Consider the integral of $\log(1 - z)/iz$ along a circle C of radius $|a| < 1$ centered at 0. With the parametrization $z = ae^{i\theta}$ ($\theta \in [0, 2\pi]$), it can be expressed as:

$$I = \int_C \frac{\log(1 - z)}{iz} dz = \int_0^{2\pi} \frac{\log(1 - ae^{i\theta})}{iae^{i\theta}} (iae^{i\theta}) d\theta = \int_0^{2\pi} \log(1 - ae^{i\theta}) d\theta$$

Which, define the domain to be $\mathbb{C} \setminus \{x \geq 1\}$, and $\log(1 - z) = \ln |1 - z| + i \arg(1 - z)$, it can also be expressed as:

$$I = \int_0^{2\pi} \log(1 - ae^{i\theta}) d\theta = \int_0^{2\pi} \ln |1 - ae^{i\theta}| d\theta + i \int_0^{2\pi} \arg(1 - ae^{i\theta}) d\theta$$

Going back to the original integral, since the function $-\frac{\log(1-z)}{i}$ is analytic on the domain $\mathbb{C} \setminus \{x \geq 1\}$, so on the disk enclosed by C , the only Pole is generated by $\frac{1}{z}$ (at the origin). Hence, let $\phi(z) = \frac{\log(1-z)}{i}$, the integral is then characterized by Cauchy's Integral Formula:

$$\int_C \frac{\log(1 - z)}{iz} dz = \int_C \frac{\phi(z)}{z} dz = 2\pi i \cdot n(C, 0) \phi(0)$$

With $n(C, 0) = 1$ (winding number 1 by our construction), and $\phi(0) = \log(1 - 0)/(i \cdot 1) = 0$, then such integral is evaluated to be 0.

Now, since $Re(I) = \int_0^{2\pi} \log |1 - ae^{i\theta}| d\theta$, while $I = 0$, then this integral must also evaluated to be 0.

Case when $|a| = 1$ (Real Integral Approach):

For $|a| = 1$, WLOG, we can assume $a = 1$ (since other than that, any $e^{i\phi}$ is adding a phase of ϕ , which by changing the range of our integrand, we can modify it to be the same as $a = 1$).

Then, the integral can be rewrite as:

$$\begin{aligned} \int_0^{2\pi} \ln |1 - e^{i\theta}| d\theta &= \int_0^{2\pi} \ln |1 - (\cos(\theta) + i \sin(\theta))| d\theta = \int_0^{2\pi} \ln \sqrt{(1 - \cos(\theta))^2 + (-\sin(\theta))^2} d\theta \\ &= \int_0^{2\pi} \ln \sqrt{1 + \cos^2(\theta) - 2 \cos(\theta) + \sin^2(\theta)} d\theta = \int_0^{2\pi} \ln \left((2 - 2 \cos(\theta))^{\frac{1}{2}} \right) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \ln \left(4 \left(\frac{1 - \cos(\theta)}{2} \right) \right) d\theta = \frac{1}{2} \int_0^{2\pi} \ln \left(2^2 \cdot \sin^2 \left(\frac{\theta}{2} \right) \right) d\theta \\ &= \frac{1}{2} \cdot 2 \int_0^{2\pi} \ln \left(2 \sin \left(\frac{\theta}{2} \right) \right) d\theta \end{aligned}$$

Using the substitution $u = \frac{\theta}{2}$ (with $2du = d\theta$), it can be modified to:

$$\begin{aligned} \frac{1}{2} \cdot 2 \int_0^{2\pi} \ln \left(2 \sin \left(\frac{\theta}{2} \right) \right) d\theta &= 2 \int_0^\pi \ln(2 \sin(u)) du = 2 \int_0^\pi (\ln(2) + \ln(\sin(u))) du \\ &= 2\pi \ln(2) + 2 \int_0^\pi \ln(\sin(u)) du \end{aligned}$$

So, it suffices to show the integral of $\ln(\sin(u))$ from 0 to π .

Notice that since $\sin(u) = \sin(\pi - u)$, then for the following integral, do the substitution $\theta = \pi - u$ (or $-d\theta = du$), we can do the following:

$$\int_{\frac{\pi}{2}}^\pi \ln(\sin(u)) du = \int_{\frac{\pi}{2}}^\pi \ln(\sin(\pi - u)) du = - \int_0^{\frac{\pi}{2}} \ln(\sin(\theta)) d\theta = \int_0^{\frac{\pi}{2}} \ln(\sin(\theta)) d\theta$$

So, the following is true:

$$\int_0^\pi \ln(\sin(u)) du = 2 \int_0^{\frac{\pi}{2}} \ln(\sin(u)) du$$

Let $I = \int_0^{\frac{\pi}{2}} \ln(\sin(u)) du$, using the fact that $\sin(u) = \cos(\frac{\pi}{2} - u)$, with substitution $\theta = \frac{\pi}{2} - u$ (or $-d\theta = du$), we can again get:

$$I = \int_0^{\frac{\pi}{2}} \ln \left(\cos \left(\frac{\pi}{2} - u \right) \right) du = - \int_{\frac{\pi}{2}}^0 \ln(\cos(\theta)) d\theta = \int_0^{\frac{\pi}{2}} \ln(\cos(\theta)) d\theta$$

So, we can also get the following:

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \ln(\sin(u)) du + \int_0^{\frac{\pi}{2}} \ln(\cos(u)) du = \int_0^{\frac{\pi}{2}} \ln(\sin(u) \cos(u)) du = \int_0^{\frac{\pi}{2}} \ln \left(\frac{1}{2} \sin(2u) \right) du \\ &= \int_0^{\frac{\pi}{2}} \left(\ln \left(\frac{1}{2} \right) + \ln(\sin(2u)) \right) du = -\frac{\pi}{2} \ln(2) + \int_0^{\frac{\pi}{2}} \ln(\sin(2u)) du \end{aligned}$$

Using the substitution $\theta = 2u$ (with $\frac{1}{2}d\theta = du$), we get the following:

$$2I = -\frac{\pi}{2} \ln(2) + \frac{1}{2} \int_0^\pi \ln(\sin(\theta)) d\theta$$

But, recall the fact that $\int_0^\pi \ln(\sin(u)) du = 2 \int_0^{\frac{\pi}{2}} \ln(\sin(u)) du = 2I$. So, we get:

$$2I = -\frac{\pi}{2} \ln(2) + \frac{1}{2} \cdot 2I = -\frac{\pi}{2} \ln(2) + I, \quad I = -\frac{\pi}{2} \ln(2)$$

Substituting back to the original integral, we get the following:

$$\int_0^\pi \ln(\sin(u)) du = 2I = -\pi \ln(2)$$

Finally, we get that the original integral is given by:

$$\int_0^{2\pi} \ln |1 - e^{i\theta}| d\theta = 2\pi \ln(2) + 2 \int_0^\pi \ln(\sin(u)) du = 2\pi \ln(2) + 2(-\pi \ln(2)) = 0$$

Hence, we can say it is possible to extend to the case $|a| = 1$, so the integral is true for all $|a| \leq 1$.