Math 118B HW6

Zih-Yu Hsieh

March 9, 2025

1

Question 1 *Rudin Chapter 5 Exercise 22:*
Suppose f is a real function on $\mathbb{R}$. Call x a fixed point of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

(b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, ...$

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to ...$$

Pf:

(a) Given f is differentiable and $f'(t) \neq 1$ for all real t. Suppose the contrary that f has more than one fixed point, there exists distinct $x, y \in \mathbb{R}$ (and WLOG, assume $x < y$), with $f(x) = x$ and $f(y) = y$. However, by Mean Value Theorem, there exists $c \in (x, y)$, such that $f'(c) = \frac{f(y)-f(x)}{y-x} = \frac{y-x}{y-x} = 1$, which contradicts the assumption that all $t \in \mathbb{R}$ satisfies $f'(t) \neq 1$.

Hence, the assumption is wrong, f couldn't have more than one fixed point.

(b) Given $f(t) = t + (1 + e^t)^{-1}$, apply the differentiation rules, we get:

$$f'(t) = 1 - (1 + e^t)^{-1} \cdot e^t = 1 - \frac{e^t}{(1 + e^t)^2}$$

Since for all $t \in \mathbb{R}$, $e^t > 0$, so $(1 + e^t) > 1$ and $(1 + e^t) > e^t$. Hence, $0 < \frac{e^t}{(1+e^t)^2} < 1$ (since everything is positive, while $e^t < (1 + e^t) < (1 + e^t)^2$).

Yet, there doesn't exists a fixed point: If consider $f(t)-t$, we get $(1+e^t)^{-1}$. Since $e^t > 0$ for all $t \in \mathbb{R}$, then $(1+e^t) > 0$, so does $(1+e^t)^{-1}$. Therefore, there doesn't exists $t \in \mathbb{R}$, with $(1+e^t)^{-1} = f(t)-t = 0$, so there doesn't exist any fixed point for this function.

(c) Suppose there exists $0 \le A < 1$ such that $|f'(t)| \le A$ for all real t. Then, for al distinct $x, y \in \mathbb{R}$ (WLOG, assume $x < y$), by Mean Value Theorem, there exists $c \in (x,y)$, with $f'(c)(x-y) = (f(x)-f(y))$. So, the following is true:

$$|f(x)-f(y)| = |f'(c)| \cdot |x-y| \le A|x-y|$$

Now, for any $x_1 \in \mathbb{R}$, we'll prove by induction that all $n \in \mathbb{N}$, $|x_{n+1}-x_n| \le A^{n-1}|x_2-x_1|$.

For base case $n=1$, it's clear that $|x_{1+1}-x_1| = |x_2-x_1| \le A^{1-1}|x_2-x_1|$.

Now, suppose for given $n \in \mathbb{N}$, $|x_{n+1}-x_n| \le A^{n-1}|x_2-x_1|$, then for case $(n+1)$:

$$|x_{(n+1)+1}-x_{n+1}| = |f(x_{n+1})-f(x_n)| \le A|x_{n+1}-x_n| \le A \cdot A^{n-1}|x_2-x_1| = A^{(n+1)-1}|x_2-x_1|$$

Which, this completes the induction, showing that all $n \in \mathbb{N}$ satisfies $|x_{n+1}-x_n| \le A^{n-1}|x_2-x_1|$.

Now, we can prove that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, therefore converges:

Given that $0 \le A < 1$, then $\frac{1}{1-A} > 0$. Now, since $A^{n-1}|x_2-x_1|$ defines a geometric sequence with ratio $0 \le A < 1$, then $\lim_{n\to\infty} A^{n-1}|x_2-x_1| = 0$. So, for all $\epsilon > 0$, since $(1-A)\epsilon > 0$, there exists N, with $n \ge N$ implies $A^{n-1}|x_2-x_1| < (1-A)\epsilon$.

Now, for all $m > n \ge N$, the following is true:

$$|x_m-x_n| = \left|\sum_{k=0}^{m-n-1}(x_{n+(k+1)}-x_{n+k})\right| \le \sum_{k=0}^{m-n-1}|x_{n+(k+1)}-x_{n+k}|$$

$$|x_m-x_n| \le \sum_{k=0}^{m-n-1}|x_{n+(k+1)}-x_{n+k}| \le \sum_{k=0}^{m-n-1}A^{n+k-1}|x_2-x_1|$$

$$|x_m-x_n| \le A^{n-1}|x_2-x_1|\sum_{k=0}^{m-n-1}A^k \le A^{n-1}|x_2-x_1|\sum_{k=0}^{\infty}A^k$$

$$|x_m-x_n| \le A^{n-1}|x_2-x_1| \cdot \frac{1}{1-A} < (1-A)\epsilon \cdot \frac{1}{1-A} = \epsilon$$

Since for all $\epsilon > 0$, there exists N, with $m > n \ge N$ implies $|x_m-x_n| < \epsilon$, hence $(x_n)_{n\in\mathbb{N}}$ is a cauchy sequence, which converges to some $x \in \mathbb{R}$.

Then, since f is differentiable, then f is continuous; hence, the following is true:

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x), \quad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

Hence, $f(x) = x$, which any $x_1 \in \mathbb{R}$ with $x_{n+1} = f(x_n)$, has the sequential limit being a fixed point $x \in \mathbb{R}$.

Also, based on the previous part, since all $t \in \mathbb{R}$ has $|f'(t)| \le A < 1$, then by **Part (a)**, since $f'(t) \ne 1$ for all t, f has at most one fixed point. Hence, this fixed point is unique, all such sequence $(x_n)_{n\in\mathbb{N}}$ converges to a unique fixed point $x \in \mathbb{R}$.

(d) Start with x_1, since x_1 gets mapped to x_2, the point on the graph is (x_1, x_2); then, to let x_2 be the next input, we need to match the x-coordinate to x_2, hence draw

2

Question 2 *For $f(x) = \cos(x)$, show that $x_{n+1} = f(x_n)$ defines a convergent sequence for arbitrary $x_0 \in \mathbb{R}$. Calculate the root $\alpha = \cos(\alpha)$, $\alpha > 0$, with an error less than 10^{-2}.*

Pf:

For all $x_0 \in \mathbb{R}$, since $|x_1| = |cos(x_0)| \leq 1$, then WLOG, we just need to consider the properties of $\cos(x)$ on the domain $[-1, 1]$.

For all distinct $x, y \in [-1, 1]$ (WLOG, assume $x < y$), since $\cos(x)$ is differentiable on $\mathbb{R}$ (with derivative $-\sin(x)$), by Mean Value Theorem, there exists $c \in (x, y)$, such that $(\cos(x) - \cos(y)) = -\sin(c)(x - y)$. Also, notice on $[-1, 1]$, $|\sin(x)|$ has a maximum at 1 (since $\sin(x)$ is strictly increasing on this domain, hence $-\sin(1) = \sin(-1) \leq \sin(x) \leq \sin(1) < 1$; so $|\sin(x)| \leq \sin(1)$ on $[-1, 1]$). Hence:

$$|\cos(x) - \cos(y)| = |-\sin(c)| \cdot |x - y| \leq \sin(1) \cdot |x - y|$$

Then, using this, we can prove inductively that all integer $n \geq 2$ has $|x_{n+1} - x_n| \leq (\sin(1))^{n-1}|x_2 - x_1|$:

For base case $n = 2$, $|x_{2+1} - x_2| = |\cos(x_2) - \cos(x_1)| \leq \sin(1) \cdot |x_2 - x_1|$.

Now, for give $n \geq 2$, if $|x_{n+1} - x_n| \leq (\sin(1))^{n-1}|x_2 - x_1|$, then for case $(n + 1)$:

$$|x_{(n+1)+1} - x_{(n+1)}| = |\cos(x_{n+1}) - \cos(x_n)| \leq \sin(1) \cdot |x_{n+1} - x_n| \leq \sin(1) \cdot (\sin(1))^{n-1}|x_2 - x_1|$$

So, this finishes the induction, all $n \geq 2$ satisfies $|x_{n+1} - x_n| \leq (\sin(1))^{n-1}|x_2 - x_1|$.

Since $0 < \sin(1) < 1$, then $\lim_{n\to\infty}(\sin(1))^{n-1}|x_2 - x_1| = 0$. Hence, for all $\epsilon > 0$ (which $(1 - \sin(1))\epsilon > 0$), there exists N, with $n \geq N$ implies $|x_{n+1} - x_n| \leq (\sin(1))^{n-1}|x_2 - x_1| < (1 - \sin(1))\epsilon$.

Then, for all $m > n \geq N$, the following summation is true:

$$|x_m - x_n| = \left|\sum_{k=0}^{m-n-1}(x_{n+k+1} - x_{n+k})\right| \leq \sum_{k=0}^{m-n-1}|x_{n+k+1} - x_{n+k}| \leq \sum_{k=0}^{m-n-1}(\sin(1))^{(n+k)-1}|x_2 - x_1|$$

$$\leq (\sin(1))^{n-1}|x_2 - x_1| \sum_{k=0}^{m-n-1}(\sin(1))^k \leq (\sin(1))^{n-1}|x_2 - x_1| \sum_{k=0}^{\infty}(\sin(1))^k$$

$$= (\sin(1))^{n-1}|x_2 - x_1| \cdot \frac{1}{1 - \sin(1)} < (1 - \sin(1))\epsilon \cdot \frac{1}{1 - \sin(1)} = \epsilon$$

Since all $\epsilon > 0$ has a corresponding N, with $m > n \geq N$ implies $|x_m - x_n| < \epsilon$, then the sequence defined by $x_{n+1} = \cos(x_n)$ is in fact a Cauchy Sequence, hence in the complete space $\mathbb{R}$, it is a convergent sequence.

So, regardless of the choice x_0, using the definition $x_{n+1} = \cos(x_n)$ for all $n \in \mathbb{N}$, the sequence $(x_n)_{n\in\mathbb{N}}$ converges.

Approximation of $\alpha = \cos(\alpha)$:

Since $\cos(x)$ is infinitely differentiable, while its derivatives are always $\pm\cos(x)$ or $\pm\sin(x)$, then the derivatives are always bounded by 1.

If do the 4^{th} degree taylor polynomial of $\cos(x)$ about 0, namely $P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, we know for all $y \in \mathbb{R}$, there exists $c \in \overline{xy}$, with $\cos(y) - P_4(y) = \frac{f^{(5)}(c)}{5!}y^5$. Hence:

$$|\cos(y) - P_4(y)| \leq \left|\frac{f^{(5)}(c)}{5!}y^5\right| \leq \frac{1}{5!}|y^5| \leq 10^{-2}|y^5|$$

(Note: $\frac{1}{5!} = \frac{1}{120} \leq 10^{-2}$).

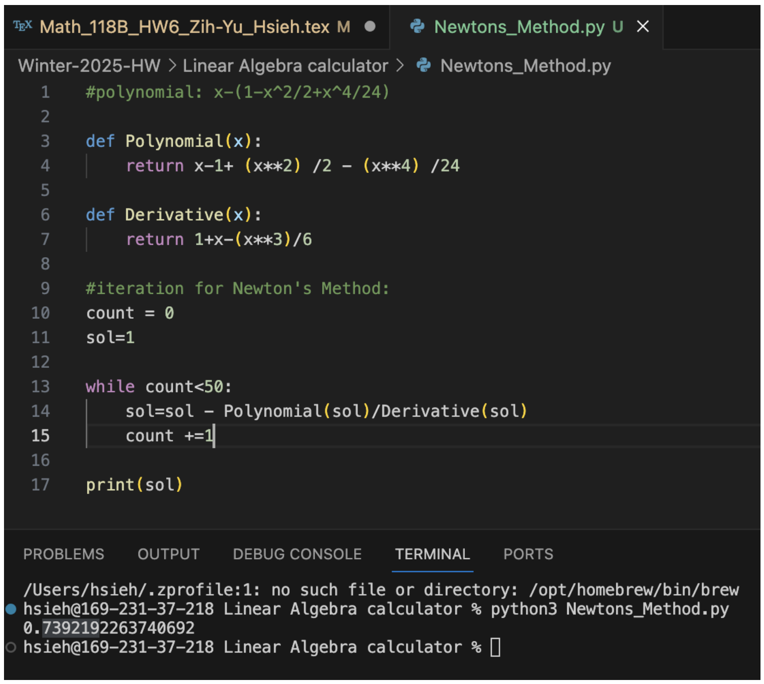
Then, given $\alpha = \cos(\alpha)$ (which $|\alpha| \leq 1$) and $\alpha > 0$, we have $|\alpha - P_4(\alpha)| = |\cos(\alpha) - P_4(\alpha)| \leq 10^{-2}|\alpha^5| \leq 10^{-2}$. If calculate the root of $x - P_4(x)$ on the domain $[0, 1]$, we'll be able to approximate the solution.

Running a Python Program using Newton's Method with 50 iterations, the root is approximately $r = 0.73922$.

Which, putting into $\cos(x)$, we get:

$$\cos(0.73922) \approx 0.73899$$

This root is nearly a fixed point for $\cos(x)$.



3

Question 3 *Let $A \subseteq \mathbb{R}^n$ be a convex and bounded set such that $\bar{0} \in A$. Let $T : \overline{A} \to \overline{A}$ be a function such that*

$$\forall x, y \in \overline{A}, \quad \|T(x) - T(y)\| \leq \|x - y\|$$

(a) Prove that the set of fixed point of T, $\{x \in \mathbb{R}^n \ : \ T(x) = x\}$ is convex and nonempty.

(b) Give examples showing that the hypotheses of convexity and boundedness of A are essential.

(c) Deduce a weaker condition than convexity under which the result still holds.

Pf:

(a) **Existence of Fixed Point:**

For all $\lambda \in (0, 1)$ (so $0 < \lambda < 1$), consider the function λT:

First, it is well-defined, since for all $x \in \overline{A}$, because $T(x), \bar{0} \in \overline{A}$, then by convexity, any $t \in [0, 1]$ has $t \cdot T(x) + (1 - t)\bar{0} = t \cdot T(x) \in \overline{A}$. Hence, since $\lambda \in [0, 1]$, then $\lambda T(x) \in \overline{A}$.

Since for all $x, y \in \overline{A}$, the following is satisfied:

$$\|\lambda T(x) - \lambda T(y)\| = \lambda \|T(x) - T(y)\| \leq \lambda \|x - y\|$$

Then, by Contraction Principle, λT has a unique fixed point in $\overline{A}$, each λ corresponds to a unique $x_\lambda \in \overline{A}$, with $\lambda T(x_\lambda) = x_\lambda$.

Now, consider a sequence $(\lambda_n)_{n \in \mathbb{N}} \subset (0, 1)$ that converges to 1. If we consider the sequence of fixed point $(x_{\lambda_n})_{n \in \mathbb{N}} \subset \overline{A}$, (with respect to each λ_n), since $\overline{A}$ is closed and bounded, then by Bolzano Weierstrass Theorem, there exists a convergent subsequence $(x_{\lambda_{n_k}})_{k \in \mathbb{N}}$ that converges to some x_1, and $x_1 \in \overline{A}$ since $\overline{A}$ is closed, it contains all its limit points.

Now, we can prove that x_1 is a fixed point of T: Because T is continuous (for all $\epsilon > 0$, let $\delta = \epsilon$, then all $x, y \in \overline{A}$ with $\|x - y\| < \delta = \epsilon$ has $\|T(x) - T(y)\| \leq \|x - y\| < \epsilon$), then since $\lim_{k \to \infty} x_{\lambda_{n_k}} = x_1$, then $\lim_{k \to \infty} T(x_{\lambda_{n_k}}) = T(x_1)$.

Also, recall that each $x_{\lambda_{n_k}}$ is a fixed point for the function $\lambda_{n_k} T$, so $x_{\lambda_{n_k}} = \lambda_{n_k} T(x_{\lambda_{n_k}})$, or $T(x_{\lambda_{n_k}}) = \frac{1}{\lambda_{n_k}} x_{\lambda_{n_k}}$. (Note: $\frac{1}{\lambda_{n_k}}$ is well-defined, since it is contained in $(0, 1)$, so it's never 0).

Then, because $(\lambda_n)_{n \in \mathbb{N}}$ converges to 1, so does its subsequence $(\lambda_{n_k})_{k \in \mathbb{N}}$. Then, since the sequence is never 0, while the limit is also nonzero, then:

$$\lim_{k \to \infty} \frac{1}{\lambda_{n_k}} = \frac{1}{\lim_{k \to \infty} \lambda_{n_k}} = 1$$

Because $\overline{A}$ is bounded, there exists $M > 0$ (choose $M > 2$, and sufficiently large), with all $x \in \overline{A}$, $\|x\| \leq M$.

So, by the above two limits, for all $\epsilon > 0$ (for simplicity, modify M from above such that $1 > \frac{\epsilon}{2M} > 0$), there exists N_1, N_2, such that:

$$k \geq N_1 \implies \|x_{\lambda_{n_k}} - x_1\| < \frac{\epsilon}{2M}, \quad k \geq N_2 \implies \left|\frac{1}{\lambda_{n_k}} - 1\right| < \frac{\epsilon}{2M}$$

Which, the second part above also implies that $0 < \frac{1}{\lambda_{n_k}} < 1 + \frac{\epsilon}{2M}$. So, for $N = \max\{N_1, N_2\}$, for all $k \geq N$ (so $k \geq N_1, N_2$), we have:

$$\begin{aligned}\left\|T(x_{\lambda_{n_k}}) - x_1\right\| &= \left\|\frac{1}{\lambda_{n_k}} x_{\lambda_{n_k}} - x_1\right\| = \left\|\left(\frac{1}{\lambda_{n_k}} x_{\lambda_{n_k}} - \frac{1}{\lambda_{n_k}} x_1\right) + \left(\frac{1}{\lambda_{n_k}} x_1 - x_1\right)\right\| \\ &\leq \left|\frac{1}{\lambda_{n_k}}\right| \cdot \left\|x_{\lambda_{n_k}} - x_1\right\| + \left|\frac{1}{\lambda_{n_k}} - 1\right| \cdot \|x_1\| < \left(1 + \frac{\epsilon}{2M}\right)\frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\ &\leq 2 \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2} = \frac{\epsilon}{M} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

(Note: For the second line, recall that $\frac{\epsilon}{2M} < 1$ by our choice; and for the last line, recall that $M > 2$). Hence, we can conclude that $\lim_{k \to \infty} T(x_{\lambda_{n_k}}) = x_1$ (since all $\epsilon > 0$, there exists N, with $\|T(x_{\lambda_{n_k}}) - x_1\| < \epsilon$).

Hence, $\lim_{k \to \infty} T(x_{\lambda_{n_k}}) = x_1$.

So, by the uniqueness of limit in metric space, $\lim_{k \to \infty} T(x_{\lambda_{n_k}}) = T(x_1) = x_1$, showing that x_1 is a fixed point of T. Hence, the set of fixed point of T is nonempty.

Convexity of the Set of Fixed Point:

Given that two points $x, y \in \overline{A}$ are fixed points of T (i.e. $T(x) = x$ and $T(y) = y$), then for all $t \in [0, 1]$, the point $z = tx + (1-t)y$ (on the line segment $\overline{xy}$), it satisfies the following:

$$\|T(z) - y\| = \|T(z) - T(y)\| \leq \|z - y\| = \|(tx + (1-t)y) - y\| = t\|x - y\|$$

$$\|T(z) - x\| = \|T(z) - T(x)\| \leq \|z - x\| = \|(tx + (1-t)y) - x\| = \|(t-1)x + (1-t)y\| = (1-t)\|x - y\|$$

Now, since distance between x, y satisfies:

$$\|x - y\| \leq \|T(z) - x\| + \|T(z) - y\| \leq (1-t)\|x - y\| + t\|x - y\| = \|x - y\|$$

Notice that if $T(z)$ is not colinear with x, y, then $\|x - y\| < \|T(z) - x\| + \|T(z) - y\|$ (since triangle inequality is equality iff the two components are nonnegative multiple of each other under vector space; if $T(z)$ is not in $\overline{xy}$, then $T(z) - x$ and $T(z) - y$ cannot be scalar nonnegative multiple of each other). Hence, $T(z) \in \overline{xy}$, or $T(z) = \lambda x + (1-\lambda)y$ for some $\lambda \in [0, 1]$. Which:

$$\|T(z) - y\| = \|(\lambda x + (1-\lambda)y) - y\| = \lambda\|x - y\|, \quad \|T(z) - x\| = \|(\lambda x + (1-\lambda)y) - x\| = (1-\lambda)\|x - y\|$$

On the other hand, we can also conclude:

$$(1-t)\|x - y\| \leq \|x - y\| - \|T(z) - y\| \leq \|T(z) - x\| \leq (1-t)\|x - y\|, \quad \|T(z) - x\| = (1-t)\|x - y\|$$

$$t\|x - y\| = (1 - (1-t))\|x - y\| \leq \|x - y\| - \|T(z) - x\| \leq \|T(z) - y\| \leq t\|x - y\|, \quad \|T(z) - y\| = t\|x - y\|$$

It implies that $\|T(z) - y\| = \lambda\|x - y\| = t\|x - y\|$, and $\|T(z) - x\| = (1-\lambda)\|x - y\| = (1-t)\|x - y\|$, which enforce the condition $\lambda = t$ (unless $\|x - y\| = 0$, but if $x = y$, $z = x = y$ is trivial). So, $T(z) = \lambda x + (1-\lambda)y = tx + (1-t)y = z$.

Hence, z is also a fixed point, which $\overline{xy}$ is a subset of this fixed point. Therefore, we can conclude that this set of fixed point is in fact convex.

(b) **Counterexample without convexity:**

Consider the unit circle $S^1 \subset \mathbb{R}^2$, which is not convex (since $(1,0), (-1,0) \in S^1$, yet $\frac{1}{2}(1,0) + (1 - \frac{1}{2})(-1,0) = (0,0) \notin S^1$).

Now, consider $T : S^1 \to S^1$ by $T(x,y) = -(x,y)$. For all $a, b \in S^1$, the given condition is true:

$$\|T(a) - T(b)\| = \|(-a) - (-b)\| \leq \|a - b\|$$

Yet, it has no fixed point, since the only point $v \in \mathbb{R}^2$ with $-v = v$ is $v = \bar{0}$, yet $\bar{0} \notin S^1$. So, this function has no fixed point.

Couterexample without Boundedness:

Consider any $n \in \mathbb{N}$, and any nonzero element $a \in \mathbb{R}^n$. Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = x + a$. For all $x, y \in \mathbb{R}^n$, the condition is satisfied:

$$\|T(x) - T(y)\| = \|(x + a) - (y + a)\| = \|x - y\|$$

Yet, since $a \neq \bar{0}$, all $x \in \mathbb{R}^n$ couldn't satisfy $T(x) = x$ (or else $a = \bar{0}$ is a contradiction). Hence, this function also has no fixed point.

(c) Instead of convexity, consider a Star-Shaped domain centered at $\bar{0}$: Given a nonempty set $A \subset \mathbb{R}^n$, it is Star-Shaped, if there exists a point $a_0 \in A$, such that for all $b \in A$, the line segmant $\overline{a_0 b} \subset A$. Which, call the point a_0 being a center of A.

If given a bounded Star-Shaped domain A that's centered at $\bar{0}$, and $T : \overline{A} \to \overline{A}$ satisfied $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in \overline{A}$. Then, for all $\lambda \in (0,1)$, λT is a well-defined function again (since for all $x \in \overline{A}$, $\lambda T(x) = \lambda T(x) + (1 - \lambda) \cdot \bar{0} \in \overline{A}$, due to the Star-Shaped assumption). Since λT satisfied $\|\lambda T(x) - \lambda T(y)\| = \lambda \|T(x) - T(y)\| \leq \lambda \|x - y\|$, then by contraction principle, there exists unique fixed point $x_\lambda \in \overline{A}$ (or $\lambda T(x_\lambda) = x_\lambda$).

Then, using the same method in **Part (a)** (since beside the Star-Shaped condition, closed and boundedness of $\overline{A}$ follows), choose a sequence $(\lambda_n)_{n \in \mathbb{N}} \subset (0,1)$ that converges to 1, the corresponding fixed point sequence $(x_{\lambda_n})_{n \in \mathbb{N}} \subset \overline{A}$, which has a convergent subsequence $(x_{\lambda_{n_k}})_{k \in \mathbb{N}}$ converging to some $x_1 \in \overline{A}$ due to the closed and boundedness of $\overline{A}$.

Again, by the continuity of T, we have $\lim_{k \to \infty} T(x_{\lambda_{n_k}}) = T(x_1)$, and since $\lambda_{n_k} T(x_{\lambda_{n_k}}) = x_{\lambda_{n_k}}$, then $T(x_{\lambda_{n_k}}) = \frac{1}{\lambda_{n_k}} x_{\lambda_{n_k}}$. Using the same method in **Part (a)**, we can again deduce:

$$\lim_{k \to \infty} T(x_{\lambda_{n_k}}) = \lim_{k \to \infty} \frac{1}{\lambda_{n_k}} x_{\lambda_{n_k}} = \left(\lim_{k \to \infty} \frac{1}{\lambda_{n_k}} \right) (\lim_{k \to \infty} x_{\lambda_{n_k}}) = x_1$$

Hence, $T(x_1) = x_1$, showing that the fixed point is nonempty.

So, given a bounded Star-Shaped domain centered at $\bar{0}$, we can still deduce the fact that the function has a nonempty collection of fixed point (though convexity is not guaranteed).

4

Question 4 *Let $X = C([0,1] : \mathbb{R})$ be the space of continuous real-valued functions defined in the interval $[0,1]$. Prove that for any $\lambda \in (0,1)$ the functional equation*

$$f(t) = \int_0^1 e^{-st}\cos(\lambda f(s))ds$$

has a unique solution in X. Extend this result to the case $\lambda = 1$.

Pf:

Unique Solution for $\lambda \in (0,1)$:

First, recall that $X = C([0,1] : \mathbb{R})$ is a Banach Space, a complete normed vector space, so the Contraction Principle works in here. Define a transformation $T_\lambda : X \to X$ by $(T_\lambda(f))(t) = \int_0^1 e^{-st}\cos(\lambda f(s))ds$ for all $\lambda \in (0,1)$ and $f \in X$.

Now, notice that by Mean Value Theorem, for all distinct $x, y \in \mathbb{R}$ (assume $x < y$), since the derivative of $\cos(t)$ is given by $-\sin(t)$, then there exists $c \in (x,y)$, such that $(\cos(x) - \cos(y)) = -\sin(c)(x-y)$. Hence, $|\cos(x) - \cos(y)| \leq |-\sin(c)| \cdot |x-y| \leq |x-y|$ (so this inequality can be generalized to all $x, y \in \mathbb{R}$).

Then, for any $f, g \in X$, any $t \in [0,1]$ satisfies the following:

$$|(T_\lambda(f))(t) - (T_\lambda(g))(t)| = \left|\int_0^1 e^{-st}\cos(\lambda f(s))ds - \int_0^1 e^{-st}\cos(\lambda g(s))ds\right|$$

$$= \left|\int_0^1 e^{-st}(\cos(\lambda f(s)) - \cos(\lambda g(s)))ds\right| \leq \int_0^1 |e^{-st}| \cdot |\cos(\lambda f(s)) - \cos(\lambda g(s))|ds$$

$$\leq \int_0^1 |\lambda f(s) - \lambda g(s)|ds = \lambda \int_0^1 |f(s) - g(s)|ds$$

Since the variable $s \in [0,1]$ (domain of all functions in the function space X), then $|f(s) - g(s)| \leq \|f - g\|_\infty$, hence the above inequality can be rewrite as:

$$|(T_\lambda(f))(t) - (T_\lambda(g))(t)| \leq \lambda \int_0^1 |f(s) - g(s)|ds \leq \lambda \int_0^1 \|f - g\|_\infty ds = \lambda \|f - g\|_\infty$$

And, since the above inequality is true for all $t \in [0,1]$, then in fact $\|T_\lambda(f) - T_\lambda(g)\| \leq \lambda \|f - g\|_\infty$.

Now, because all $f, g \in X$ satisfies $\|T_\lambda(f) - T_\lambda(g)\|_\infty \leq \lambda \|f - g\|_\infty$ while $\lambda < 1$, then by contraction principle, there exists a unique $f_\lambda \in X$, such that $T_\lambda(f_\lambda) = f_\lambda$. Or, $f_\lambda \in X$ is the unique equation satisfying:

$$f_\lambda(t) = \int_0^1 e^{-st}\cos(\lambda f_\lambda(s))ds$$

Extension to $\lambda = 1$:

Before starting, since for all $\lambda \in (0,1)$ and all $t \in [0,1]$, the following inequality is satisfied:

$$|f_\lambda(t)| = \left|\int_0^1 e^{-st}\cos(\lambda f_\lambda(s))ds\right| \leq \int_0^1 |e^{-st}| \cdot |\cos(\lambda f_\lambda(s))|ds \leq \int_0^1 1ds = 1$$

Then, we can conclude that $\|f_\lambda\|_\infty \leq 1$.

Now, consider a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset (0,1)$ that converges to 1, and consider the corresponding sequence of functions $\{f_{\lambda_n}\}_{n\in\mathbb{N}}$, our goal is to apply Arzela-Ascoli Theorem.

First, we know the domain $[0, 1]$ for all these functions are bounded, and the above statement proved that the sequence of function is uniformly bounded by 1, so the remaining condition is to prove that the sequence of function is equicontinuous.

Recall that the function e^x is a continuous function at $x = 0$, so for all $\epsilon > 0$, there exists $\delta > 0$, such that $|x| < \delta$ implies $|1 - e^x| < \epsilon$. Then, using the same δ, for all $n \in \mathbb{N}$, any $t_1, t_2 \in [0, 1]$ satisfying $|t_1 - t_2| < \delta$, we have:

$$|f_{\lambda_n}(t_1) - f_{\lambda_n}(t_2)| = \left| \int_0^1 e^{-st_1} \cos(\lambda_n f_{\lambda_n}(s))ds - \int_0^1 e^{-st_2} \cos(\lambda_n f_{\lambda_n}(s))ds \right|$$

$$= \left| \int_0^1 (e^{-st_1} - e^{-st_2}) \cos(\lambda_n f_{\lambda_n}(s))ds \right| \leq \int_0^1 |e^{-st_1} - e^{-st_2}| \cdot |\cos(\lambda_n f_{\lambda_n}(s))|ds$$

$$\leq \int_0^1 |e^{-st_1}| \cdot |1 - e^{st_1 - st_2}|ds \leq \int_0^1 |1 - e^{s(t_1 - t_2)}|ds$$

For all $s \in [0, 1]$, since $0 \leq s|t_1 - t_2| \leq |t_1 - t_2| < \delta$, then by continuity of e^x, we know $|1 - e^{s(t_1 - t_2)}| < \epsilon$. Hence, the above inequality becomes:

$$|f_{\lambda_n}(t_1) - f_{\lambda_n}(t_2)| \leq \int_0^1 |1 - e^{s(t_1 - t_2)}|ds < \int_0^1 \epsilon ds = \epsilon$$

Since regardless of $n \in \mathbb{N}$, every $\epsilon > 0$ has a corresponding $\delta > 0$, with $|t_1 - t_2| < \delta$ implies $|f_{\lambda_n}(t_1) - f_{\lambda_n}(t_2)| < \epsilon$, then this concludes that the sequence of functions $\{f_{\lambda_n}\}_{n\in\mathbb{N}}$ is in fact equicontinuous.

Since all three conditions are satisfied, by Arzela-Ascoli Theorem, there exists a convergent subsequence $\{f_{\lambda_{n_k}}\}_{k\in\mathbb{N}} \subset \{f_{\lambda_n}\}_{n\in\mathbb{N}}$, define $f \in X$ to be the subsequential limit of $\{f_{\lambda_{n_k}}\}_{k\in\mathbb{N}}$.

Finally, we can prove that $f(t) = \int_0^1 e^{-st} \cos(f(s))ds$ (a solution for $\lambda = 1$):

Since $\{f_{\lambda_{n_k}}\}_{k\in\mathbb{N}}$ converges to f uniformly, then for all $\epsilon > 0$ (which $\frac{\epsilon}{2} > 0$), there exists K, with $k \geq K$ implies $\|f - f_{\lambda_{n_k}}\|_\infty < \frac{\epsilon}{2}$.

Also, since $\{\lambda_n\}_{n\in\mathbb{N}}$ converges to 1, then the subseuqnce $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ also converges to 1. Hence, for the given $\epsilon > 0$ above, there exists K', with $k \geq K'$ implies $|1 - \lambda_{n_k}| < \frac{\epsilon}{2}$.

Now, let $g \in X$ be defined as $g(t) = \int_0^1 e^{-st} \cos(f(s))ds$.

Choose $N = \max\{K, K'\}$, for all $k \geq N$ (which $k \geq K', K$), then for all $t \in [0, 1]$, it satisfies:

$$|g(t) - f_{\lambda_{n_k}}(t)| = \left| \int_0^1 e^{-st} \cos(f(s))ds - \int_0^1 e^{-st} \cos(\lambda_{n_k} f_{\lambda_{n_k}}(s))ds \right|$$

$$= \left| \int_0^1 e^{-st} (\cos(f(s)) - \cos(\lambda_{n_k} f_{\lambda_{n_k}}(s)))ds \right| \leq \int_0^1 |e^{-st}| \cdot |\cos(f(s)) - \cos(\lambda_{n_k} f_{\lambda_{n_k}}(s))|ds$$

$$\leq \int_0^1 |f(s) - \lambda_{n_k} f_{\lambda_{n_k}} f(s)|ds \leq \int_0^1 \|f - \lambda_{n_k} f_{\lambda_{n_k}}\|_\infty ds = \|f - \lambda_{n_k} f_{\lambda_{n_k}}\|_\infty$$

Which, the above term can be rewrite as:

$$\|f - \lambda_{n_k} f_{\lambda_{n_k}}\|_\infty = \|(f - f_{\lambda_{n_k}}) + (f_{\lambda_{n_k}} - \lambda_{n_k} f_{\lambda_{n_k}})\|_\infty$$

$$\leq \|f - f_{\lambda_{n_k}}\|_\infty + \|(1 - \lambda_{n_k}) f_{\lambda_{n_k}}\|_\infty < \frac{\epsilon}{2} + |1 - \lambda_{n_k}| \cdot \|f_{\lambda_{n_k}}\|_\infty$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \cdot \|f_{\lambda_{n_k}}\|_\infty \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(Note: Recall that $\|f_{\lambda_{n_k}}\|_\infty \leq 1$).

Hence, the above combined to be the below inequality:

$$|g(t) - f_{\lambda_{n_k}}(t)| \leq \|f - \lambda_{n_k} f_{\lambda_{n_k}}\|_\infty < \epsilon$$

Which further implies that $\|g - f_{\lambda_{n_k}}\|_\infty \leq \epsilon$. Since all $\epsilon > 0$ has an N, such that $k \geq N$ implies $\|g - f_{\lambda_{n_k}}\|_\infty \leq \epsilon$, then g is the limit of the subsequence $\{f_{\lambda_{n_k}}\}_{k\in\mathbb{N}}$. Which, because under metric space, the limit is unique, therefore $g = f$.

Hence, we can conclude that $f(t) = g(t) = \int_0^1 e^{-st}\cos(f(s))ds$. For, $\lambda = 1$, f is a solution for the given functional equation.

5

Question 5 *Let $K \subset \mathbb{R}^n$ be a compact set. Suppose that $T : K \to K$ satisfies*

$$\forall x, y \in K, \quad \|T(x) - T(y)\| < \|x - y\|$$

Show that there exists a unique $x_0 \in K$ such that $T(x_0) = x_0$.

Pf:

Existence:

First, we'll verify that the map $D : K \to \mathbb{R}$ by $D(x) = \|x - T(x)\|$ is continuous:

For all $x, y \in K$, given any $\epsilon > 0$, if chosen $\delta = \frac{\epsilon}{2} > 0$, then for $\|x - y\| < \delta = \frac{\epsilon}{2}$, we have $\|T(x) - T(y)\| < \|x - y\| < \frac{\epsilon}{2}$. Hence, the above function D satisfies:

$$\|D(x) - D(y)\| = \|(x - T(x)) - (y - T(y))\| = \|(x - y) + (T(y) - T(x))\|$$

$$\|D(x) - D(y)\| \leq \|x - y\| + \|T(x) - T(y)\| < 2\|x - y\| < 2 \cdot \frac{\epsilon}{2} = \epsilon$$

Hence, we can conclude that D is continuous on K, a compact set. Which, the image $D(K) \subset \mathbb{R}$ is compact, therefore a minmum $\lambda = \min D(K)$ exists.

Notice that $\lambda \geq 0$, since $D(x) \geq 0$ for all $x \in K$; also, we know there exists $x_0 \in K$, with $D(x_0) = \lambda$ by the definition of minimum.

Now, to prove that $\lambda = 0$, suppose $\lambda \neq 0$ (which $\lambda > 0$) for the sake of contradiction. Then, notice that x_0 and $T(x_0)$ satisfies:

$$D(T(x_0)) = \|T(x_0) - T(T(x_0))\| < \|x_0 - T(x_0)\| = D(x_0) = \lambda$$

Which, $D(T(x_0)) < D(x_0)$, while $D(x_0) = \lambda$ is assumed to be the minimum of the set $D(K)$ (which $D(T(x_0))) \in D(K)$). So, this is a contradiction, hence the initial assumption must be false. Therefore, $\lambda = 0$. Hence, there exists $x_0 \in K$, with $D(x_0) = \|x_0 - T(x_0)\| = 0$, so $T(x_0) = x_0$.

Uniqueness:

Suppose the contrary that there exists more than one fixed point (let $x_0, y_0 \in K$ be two distinct fixed points). Then, $\|x_0 - y_0\| = \|T(x_0) - T(y_0)\| < \|x_0 - y_0\|$ is a contradiction. Therefore, the assumption is false, there must have at most one fixed point. And by the existence argument, we know there exists a unique fixed point.

6

Question 6 *Let $K \subset \mathbb{R}^n$ be a compact set and $f : K \to K$ be a function such that*

$$\|f(x) - f(y)\| = \|x - y\|, \quad \forall x, y \in K$$

Show that f is a bijection.

Pf:

f **is Injective:**

For all $x, y \in K$, suppose $f(x) = f(y)$, then since $0 = \|f(x) - f(y)\| = \|x - y\|$, then $x = y$ is enforced. Hence, this proves injectivity.

f **is Surjective:**

Suppose the contrary, that f is not surjective (so, $f(K) \subsetneq K$).

First, since for all $\epsilon > 0$, choose $\delta = \epsilon$, all $x, y \in K$ with $\|x - y\| < \delta = \epsilon$ satisfies $\|f(x) - f(y)\| = \|x - y\| < \epsilon$, hence f is uniformly continuous on K. Then, because K is compact, then $f(K)$ is also compact, which is closed and bounded.

Now, since $K \setminus f(K) \neq \emptyset$ based on assumption, there exists $x_0 \in K \setminus f(K)$. Which, because the sets $\{x_0\}$ and $f(K)$ are both compact (which are both closed), while the two sets are disjoint, then by **HW 1 Question 3** (part from **Rudin Chapter 4 Question 21**), in any metric space, disjoint closed set C and compact set K always have $\inf\{d(x, y) \mid x \in C, y \in K\} > 0$ (a positive distance between sets C and K). So, apply this to the two sets, there exists $\lambda > 0$, such that all $y \in f(K)$ satisfies $\|x_0 - y\| = d(x_0, y) \geq \lambda$.

Then, define $f_0(x) = x$, $f_1(x) = f(x)$, and for all integer $n \geq 1$, $f_{n+1}(x) = f(f_n(x))$. (Note: inductively, we can also prove that all $m, n \in \mathbb{N}$ satisfies $f_m(f_n(x)) = f_{m+n}(x) = f_n(f_m(x))$).

With this definition, we can prove by induction that for all $n \in \mathbb{N}$, all $y \in f_n(K)$ satisfies $\|f_n(x_0) - f(y)\| \geq \lambda$.

For base case $n = 1$, recall that for all $y \in f_1(K) = f(K)$, because all $y \in f(K)$ satisfies $\|x_0 - y\| \geq \lambda$, since f preserves distance, we have:

$$\|f_1(x_0) - f(y)\| = \|f(x_0) - f(y)\| = \|x_0 - y\| \geq \lambda$$

Hence, all $y \in f_1(K)$ satisfies $\|f_1(x_0) - f(y)\| \geq \lambda$, the claim is true for $n = 1$.

Now, suppose for given $n \in \mathbb{N}$, all $y \in f_n(K)$ satisfies $\|f_n(x_0) - f(y)\| \geq \lambda$. Then, for all $y \in f_{n+1}(K) = f(f_n(K))$, there exists $x \in f_n(K)$, with $f(x) = y$. Which, by induction hypothesis, $\|f_n(x_0) - y\| = \|f_n(x_0) - f(x)\| \geq \lambda$. Hence, the following inequality is true:

$$\|f_{n+1}(x_0) - f(y)\| = \|f(f_n(x_0)) - f(y)\| = \|f_n(x_0) - y\| \geq \lambda$$

Which, all $y \in f_{n+1}(K)$ satisfies $\|f_{n+1}(x_0) - f(y)\| \geq \lambda$, completing the induction.

Lastly, consider the sequence defined recursively as $x_n = f_n(x_0)$ for all $n \in \mathbb{N}$. Then, since f restrict the element to still be in K, then $(x_n)_{n\in\mathbb{N}} \subset K$, a compact set (which is closed and bounded). Hence, by Bolzano Weierstrass Theorem, since $(x_n)_{n\in\mathbb{N}}$ is bounded, there exists a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$, which this subsequence is Cauchy.

Then, given $\lambda > 0$, there exists $N \in \mathbb{N}$, such that all $p \geq N$ implies $\|x_{n_p} - x_{n_{p+1}}\| < \lambda$, by the definition of Cauchy Sequence.

However, since $n_{p+1} = n_p + k$ for some $k \in \mathbb{N}$, looking back at the definition, $x_{n_p} = f_{n_p}(x_0)$, while $x_{n_{p+1}} = x_{n_p+k} = f_{n_p+k}(x_0) = f_k(f_{n_p}(x_0))$.

Because $k \geq 1$, then $f_k(x) = f(f_{k-1}(x))$, so $x_{n_{p+1}} = f_k(f_{n_p}(x_0)) = f(f_{k-1}(f_{n_p}(x_0))) = f(f_{n_p}(f_{k-1}(x_0)))$.

So, let $y = f_{n_p}(f_{k-1}(x_0)) \in f_{n_p}(K)$, by the previous claim, the following inequality is true:

$$\|x_{n_p} - x_{n_{p+1}}\| = \|f_{n_p}(x_0) - f(f_{n_p}(f_{k-1}(x_0)))\| = \|f_{n_p}(x_0) - f(y)\| \geq \lambda$$

Yet, this contradicts the statement that $\|x_{n_p} - x_{n_{p+1}}\| < \lambda$.

Since we eventually reach a contradiction, then the assumption must be false, so f needs to be surjective.

The above two sections proved that f is in fact a bijection.