Math CS 122A HW8

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Question 1 *Ahlfors Pg. 148 Problem 2:*
Prove that the region obtained from a simply connected region by removing m points has the connectivity $m+1$, and find a homology basis.

Pf:

Given the open region Ω is simply connected, which the complement Ω^C is connected in the extended complex plane. Now, let $z_1, ..., z_m \in \Omega$ denote the m distinct points being reomoved. Then, the new region $\Omega' = \Omega \setminus \{z_1, ..., z_m\}$, and $(\Omega')^C = \Omega^C \cup \{z_1, ..., z_m\}$.

First, all $i \in \{1, ..., m\}$ has $\{z_i\}$ being disjoint from Ω^C: Since $z_i \in \Omega$, while Ω is open, then there exists $\epsilon_i > 0$, with $B_{\epsilon_i}(z_i) \subseteq \Omega$. Hence, for all $a \in \Omega^C$, $d(z_i, a) \geq \epsilon_i$, showing that z_i and Ω^C are disjoint.

Then, since $\{z_i\}$ and Ω^C are both closed under standard topology, the two being disjoint implies the two are not connected, hence belong to different connected components.

Furthermore, all distinct $i, j \in \{1, ..., m\}$ have $\{z_i\}, \{z_j\}$ being disjoint, since they're distinct points by assumption. Then, the two sets are also not connected, hence they belong to different connected components.

Then, because each set $\Omega^C, \{z_1\}, ..., \{z_m\}$ all belong to distinct connected component, while each set is connected (Ω^C is connected in the extended complex plane by assumption, while a singleton is always connected under standard topology), then, there are $m+1$ connected components for the above collection. Hence, $(\Omega')^C$ has $m+1$ connected components, showing that the new region Ω' has connectivity $m+1$.

Homology Basis:

Recall the above definition, each $i \in \{1, ..., m\}$ exists $\epsilon_i > 0$, with $d(z_i, a) \geq \epsilon_i$ for all $a \in \Omega^C$; also, since for $j \neq i$, $d(z_i, z_j) > 0$, then let $d = \min\{\epsilon_i \mid 1 \leq i \leq m\} \cup \{d(z_i, z_j) \mid 1 \leq i < j \leq m\}$. Which, $d > 0$, and $B_d(z_i)$ contains no points from other connected components (for $a \in \Omega^C$, $d(z_i, a) \geq \epsilon_i \geq d$, showing that $a \notin B_d(z_i)$; and for $j \neq i$, $d(z_i, z_j) \geq d$, so $z_j \notin B_d(z_i)$ also). Therefore, $B_d(z_i) \setminus \{z_i\}$ is disjoint from $(\Omega')^C = \Omega^C \cup \{z_1, ..., z_m\}$.

Then, for each index i, let cycle γ_i be defined as the parametrization $z_i + \frac{d}{2}e^{i\theta}$ for $\theta \in [0, 2\pi]$. Then, since for all $a \in \gamma_i$, $|a - z_i| = |\frac{d}{2}e^{i\theta}| = \frac{d}{2} < d$, then $a \in B_d(z_i) \setminus \{z_i\}$. showing that $\gamma_i \subset B_d(z_i) \setminus \{z_i\}$. Therefore, γ_i is disjoint from $(\Omega')^C$, which is contained fully in Ω'.

Also, since $\gamma_i \subset B_d(z_i)$ (which is simply connected), for all $j \neq i$, since $z_j \notin B_d(z_i)$, then $n(\gamma_i, z_j) = 0$; the same applies for all $a \in \Omega^C$ also. And, if do the following integration, we get:

$$n(\gamma_i, z_i) = \frac{1}{2\pi i}\int_{\gamma_i} \frac{dz}{z - z_i} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{(z_i + \frac{d}{2}e^{i\theta}) - z_i} i\frac{d}{2}e^{i\theta} d\theta = \frac{1}{2\pi i}\int_0^{2\pi} i d\theta = 1$$

Hence, γ_i has winding number 1 for component $\{z_i\}$, while winding number 0 for other components for the complement. Therefore, the collection $\gamma_1, ..., \gamma_m$ forms a homology basis for Σ'.

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Question 2 *Ahlfors Pg. 148 Problem 4:*
Show that single-valued analytic branches of $\log z$, z^α *and* z^z *can be defined in any simply connected region which does not contain the origin.*

Pf:

Single-Valued Branch of $\log(z)$**:**

Given that Ω is a simply connected region that doesn't contain 0, then since $z \neq 0$ in the region, then based on **Corollary 2** of Generalized Cauchy's Theorem in the textbook (Ahlfors Pg. 142), one can define a single-valued branch of $\log(z)$ in this region.

Single-Valued Branch of z^α **and** z^z**:**

From the above part, since the single-valued branch of $\log(z)$ can be defined, then for all $\alpha \in \mathbb{R}$, $\alpha \log(z)$ and $z \log(z)$ both have single-valued branch.

Hence, $z^\alpha = e^{\alpha \log(z)}$ and $z^z = e^{z \log(z)}$ are also well-defined.

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Question 3 *Ahlfors Pg. 148 Problem 5:*
Show that a single-valued analytic branch of $\sqrt{1-z^2}$ *can be defined in any region such that the points* ± 1 *are in the same component of the complement. What are the possible values of*

$$\int \frac{dz}{\sqrt{1-z^2}}$$

over a closed curve in the region?

Pf:

Assume Ω is the open region, where $1, -1$ are in the same connected component of the complement. Which, for all cycle $\gamma \subset \Omega$, the winding number $n(\gamma, 1) = n(\gamma, -1)$.

Single-Valued Branch:

First, consider the analytic function $\left(\frac{1}{z+1} - \frac{1}{z-1}\right)$ on Ω: For all cycle $\gamma \subset \Omega$, the following integral is true:

$$\int_\gamma \left(\frac{1}{z+1} - \frac{1}{z-1}\right) dz = \left(\int_\gamma \frac{1}{z+1} dz - \int_\gamma \frac{1}{z-1} dz\right) = 2\pi i (n(\gamma, 1) - n(\gamma, -1)) = 0$$

Then, this implies that an antiderivative $F(z)$ exists on Ω (with $F'(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z-1}\right)$). Then, since the original function can be rewrite as:

$$F'(z) = \left(\frac{1}{z+1} - \frac{1}{z-1}\right) = \frac{(z-1)-(z+1)}{(z+1)(z-1)} = \frac{-2}{z^2-1} = \frac{2}{1-z^2}$$

Hence, $F(z)$ is an antiderivative of $\frac{2}{1-z^2}$.

(Note: In real numbers, the antiderivative of the above equation is also written as $\ln(x+1) - \ln(x-1) = \ln \frac{x+1}{x-1}$, which will be a tool for guess here).

Now, consider the equation $\frac{z+1}{z-1} e^{-F(z)}$, and its derivative:

$$\frac{z+1}{z-1} = 1 + \frac{2}{z-1}, \quad \frac{d}{dz}\left(\frac{z+1}{z-1}\right) = \frac{d}{dz}\left(\frac{2}{z-1}\right) = \frac{-2}{(z-1)^2}$$

$$\begin{aligned}
\frac{d}{dz}\left(\frac{z+1}{z-1} e^{-F(z)}\right) &= \frac{-2}{(z-1)^2} e^{-F(z)} + \frac{z+1}{z-1}\left(-\frac{dF}{dz}\right) e^{-F(z)} \\
&= \frac{-2}{(z-1)^2} e^{-F(z)} + \frac{z+1}{z-1}\left(\frac{-2}{1-z^2}\right) e^{-F(z)} \\
&= \frac{-2}{(z-1)^2} e^{-F(z)} + \frac{z+1}{z-1}\left(\frac{-2}{(1-z)(1+z)}\right) e^{-F(z)} \\
&= \frac{-2}{(z-1)^2} e^{-F(z)} + \frac{2}{(z-1)^2} e^{-F(z)} = 0
\end{aligned}$$

Then, since the derivative is 0, then the function is in fact a constant over Ω; and, since Ω excludes both ± 1, then the value of $\frac{z+1}{z-1} e^{-F(z)}$ is always nonzero. Hence, $C = \frac{z+1}{z-1} e^{-F(z)} \neq 0$.

Which, rewrite the function, we get:

$$C = \frac{z+1}{z-1} e^{-F(z)} = \frac{(z+1)^2}{(z^2-1)} e^{-F(z)} = -\frac{(z+1)^2}{1-z^2} e^{-F(z)}$$

Hence, we can rewrite it as the follow:

$$1 - z^2 = -\frac{1}{C}(z+1)^2 e^{-F(z)}$$

Then, define the function $\frac{i}{\sqrt{C}}(z+1)e^{-\frac{F(z)}{2}}$, we get:

$$\left(\frac{i}{\sqrt{C}}(z+1)e^{-\frac{F(z)}{2}}\right)^2 = -\frac{1}{C}(z+1)^2 e^{-F(z)} = 1 - z^2$$

Hence, define this branch as $\sqrt{1-z^2} = \frac{i}{\sqrt{C}}(z+1)e^{-\frac{F(z)}{2}}$, it is a well-defined single-valued branch.

Integral of $1/\sqrt{1-z^2}$: