Math 118B HW6

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Question 1 *Rudin Chapter 5 Exercise 22:*
Suppose f is a real function on $\mathbb{R}$. Call x a fixed point of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

(b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \ldots$.

Pf:

(a) Given f is differentiable and $f'(t) \neq 1$ for all real t. Suppose the contrary that f has more than one fixed point, there exists distinct $x, y \in \mathbb{R}$ (and WLOG, assume $x < y$), with $f(x) = x$ and $f(y) = y$. However, by Mean Value Theorem, there exists $c \in (x, y)$, such that $f'(c) = \frac{f(y)-f(x)}{y-x} = \frac{y-x}{y-x} = 1$, which contradicts the assumption that all $t \in \mathbb{R}$ satisfies $f'(t) \neq 1$.

Hence, the assumption is wrong, f couldn't have more than one fixed point.

(b) Given $f(t) = t + (1 + e^t)^{-1}$, apply the differentiation rules, we get:

$$f'(t) = 1 - (1 + e^t)^{-1} \cdot e^t = 1 - \frac{e^t}{(1 + e^t)^2}$$

Since for all $t \in \mathbb{R}$, $e^t > 0$, so $(1 + e^t) > 1$ and $(1 + e^t) > e^t$. Hence, $0 < \frac{e^t}{(1+e^t)^2} < 1$ (since everything is positive, while $e^t < (1 + e^t) < (1 + e^t)^2$).

Yet, there doesn't exists a fixed point: If consider $f(t) - t$, we get $(1 + e^t)^{-1}$. Since $e^t > 0$ for all $t \in \mathbb{R}$, then $(1+e^t) > 0$, so does $(1+e^t)^{-1}$. Therefore, there doesn't exists $t \in \mathbb{R}$, with $(1+e^t)^{-1} = f(t)-t = 0$, so there doesn't exist any fixed point for this function.

(c) Suppose there exists $0 \leq A < 1$ such that $|f'(t)| \leq A$ for all real t. Then, for al distinct $x, y \in \mathbb{R}$ (WLOG, assume $x < y$), by Mean Value Theorem, there exists $c \in (x, y)$, with $f'(c)(x-y) = (f(x) - f(y))$. So, the following is true:

$$|f(x) - f(y)| = |f'(c)| \cdot |x - y| \leq A|x - y|$$

Now, for any $x_1 \in \mathbb{R}$, we'll prove by induction that all $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq A^{n-1}|x_2 - x_1|$.

For base case $n = 1$, it's clear that $|x_{1+1} - x_1| = |x_2 - x_1| \leq A^{1-1}|x_2 - x_1|$.

Now, suppose for given $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq A^{n-1}|x_2 - x_1|$, then for case $(n+1)$:

$$|x_{(n+1)+1} - x_{n+1}| = |f(x_{n+1}) - f(x_n)| \leq A|x_{n+1} - x_n| \leq A \cdot A^{n-1}|x_2 - x_1| = A^{(n+1)-1}|x_2 - x_1|$$

Which, this completes the induction, showing that all $n \in \mathbb{N}$ satisfies $|x_{n+1} - x_n| \leq A^{n-1}|x_2 - x_1|$.

Now, we can prove that the sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, therefore converges:

Given that $0 \leq A < 1$, then $\frac{1}{1-A} > 0$. Now, since $A^{n-1}|x_2 - x_1|$ defines a geometric sequence with ratio $0 \leq A < 1$, then $\lim_{n\to\infty} A^{n-1}|x_2 - x_1| = 0$. So, for all $\epsilon > 0$, since $\frac{1-A}{|x_2 - x_1|}\epsilon > 0$, there exists N, with $n \geq N$ implies $A^{n-1}|x_2 - x_1| < (1-A)\epsilon$.

Now, for all $m > n \geq N$, the following is true:

$$|x_m - x_n| = \left| \sum_{k=0}^{m-n-1} (x_{n+(k+1)} - x_{n+k}) \right| \leq \sum_{k=0}^{m-n-1} |x_{n+(i+1)} - x_{n+i}|$$

$$|x_m - x_n| \leq \sum_{k=0}^{m-n-1} |x_{n+(i+1)} - x_{n+i}| \leq \sum_{k=0}^{m-n-1} A^{n+k-1}|x_2 - x_1|$$

$$|x_m - x_n| \leq A^{n-1}|x_2 - x_1| \sum_{k=0}^{m-n-1} A^k \leq A^{n-1}|x_2 - x_1| \sum_{k=0}^{\infty} A^k$$

$$|x_m - x_n| \leq A^{n-1}|x_2 - x_1| \cdot \frac{1}{1-A} < (1-A)\epsilon \cdot \frac{1}{1-A} = \epsilon$$

Since for all $\epsilon > 0$, there exists N, with $m > n \geq N$ implies $|x_m - x_n| < \epsilon$, hence $(x_n)_{n \in \mathbb{N}}$ is a cauchy sequence, which converges to some $x \in \mathbb{R}$.

Then, since f is differentiable, then f is continuous; hence, the following is true:

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x), \quad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

Hence, $f(x) = x$, which any $x_1 \in \mathbb{R}$ with $x_{n+1} = f(x_n)$, has the sequential limit being a fixed point $x \in \mathbb{R}$.

Also, based on the previous part, since all $t \in \mathbb{R}$ has $|f'(t)| \leq A < 1$, then by part (a), since $f'(t) \neq 1$ for all t, f has at most one fixed point. Hence, this fixed point is unique, all such sequence $(x_n)_{n \in \mathbb{N}}$ converges to a unique fixed point $x \in \mathbb{R}$.

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Question 2 *For $f(x) = \cos(x)$, show that $x_{n+1} = f(x_n)$ defines a convergent sequence for arbitrary $x_0 \in \mathbb{R}$. Calculate the root $\alpha = \cos(\alpha)$, with an error less than 10^{-2}.*

Pf:

For all $x_0 \in \mathbb{R}$, since $|x_1| = |cos(x_0)| \leq 1$, then WLOG, we just need to consider the properties of $\cos(x)$ on the domain $[-1, 1]$.

For all distinct $x, y \in [-1, 1]$ (WLOG, assume $x < y$), since $\cos(x)$ is differentiable on $\mathbb{R}$ (with derivative $-\sin(x)$), by Mean Value Theorem, there exists $c \in (x, y)$, such that $(\cos(x) - \cos(y)) = -\sin(c)(x - y)$. Also, notice on $[-1, 1]$, $|\sin(x)|$ has a maximum at 1 (since $\sin(x)$ is strictly increasing on this domain, hence $-\sin(1) = \sin(-1) \leq \sin(x) \leq \sin(1) < 1$; so $|\sin(x)| \leq \sin(1)$ on $[-1, 1]$). Hence:

$$|\cos(x) - \cos(y)| = |-\sin(c)| \cdot |x - y| \leq \sin(1) \cdot |x - y|$$

Using similar from **Question 1**, with the above inequality, since $x_1 = \cos(x_0) \in [-1, 1]$ and $\sin(1) < 1$, then all $n \in \mathbb{N}$ satisfies $|x_{n+1} - x_n| \leq \sin(1)^{n-1} \cdot |x_2 - x_1|$.

Approximation:

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Question 3

Pf:

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Question 4

Pf:

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Question 5 *Let $K \subset \mathbb{R}^n$ be a compact set. Suppose that $T : K \to K$ satisfies*

$$\forall x, y \in K, \quad \|T(x) - T(y)\| < \|x - y\|$$

Show that there exists a unique $x_0 \in K$ such that $T(x_0) = x_0$.

Pf:

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Question 6 *Let $K \subset \mathbb{R}^n$ be a compact set and $f : K \to K$ be a function such that*

$$\|f(x) - f(y)\| = \|x - y\|, \quad \forall x, y \in K$$

Show that f is a bijection.

Pf:

f **is Injective:**

For all $x, y \in K$, suppose $f(x) = f(y)$, then since $0 = \|f(x) - f(y)\| = \|x - y\|$, then $x = y$ is enforced. Hence, this proves injectivity.

f **is Surjective:**

Suppose the contrary, that f is not surjective (so, $f(K) \subsetneq K$).

First, since for all $\epsilon > 0$, choose $\delta = \epsilon$, all $x, y \in K$ with $\|x - y\| < \delta = \epsilon$ satisfies $\|f(x) - f(y)\| = \|x - y\| < \epsilon$, hence f is uniformly continuous on K. Then, because K is compact, then $f(K)$ is also compact, which is closed and bounded.

Now, since $K \setminus f(K) \neq \emptyset$ based on assumption, there exists $x_0 \in K \setminus f(K)$. Which, because the sets $\{x_0\}$ and $f(K)$ are both compact (which are both closed), while the two sets are disjoint, then by **HW 1 Question 3** (part from **Rudin Chapter 4 Question 21**), in any metric space, disjoint closed set C and compact set K always have $\inf\{d(x, y) \mid x \in C, y \in K\} > 0$ (a positive distance between sets C and K). So, apply this to the two sets, there exists $\lambda > 0$, such that all $y \in f(K)$ satisfies $\|x_0 - y\| = d(x_0, y) \geq \lambda$.

Then, define $f_0(x) = x$, $f_1(x) = f(x)$, and for all integer $n \geq 1$, $f_{n+1}(x) = f(f_n(x))$. (Note: inductively, we can also prove that all $m, n \in \mathbb{N}$ satisfies $f_m(f_n(x)) = f_{m+n}(x) = f_n(f_m(x))$).

With this definition, we can prove by induction that for all $n \in \mathbb{N}$, all $y \in f_n(K)$ satisfies $\|f_n(x_0) - f(y)\| \geq \lambda$.

For base case $n = 1$, recall that for all $y \in f_1(K) = f(K)$, because all $y \in f(K)$ satisfies $\|x_0 - y\| \geq \lambda$, since f preserves distance, we have:

$$\|f_1(x_0) - f(y)\| = \|f(x_0) - f(y)\| = \|x_0 - y\| \geq \lambda$$

Hence, all $y \in f_1(K)$ satisfies $\|f_1(x_0) - f(y)\| \geq \lambda$, the claim is true for $n = 1$.

Now, suppose for given $n \in \mathbb{N}$, all $y \in f_n(K)$ satisfies $\|f_n(x_0) - f(y)\| \geq \lambda$. Then, for all $y \in f_{n+1}(K) = f(f_n(K))$, there exists $x \in f_n(K)$, with $f(x) = y$. Which, by induction hypothesis, $\|f_n(x_0) - y\| = \|f_n(x_0) - f(x)\| \geq \lambda$. Hence, the following inequality is true:

$$\|f_{n+1}(x_0) - f(y)\| = \|f(f_n(x_0)) - f(y)\| = \|f_n(x_0) - y\| \geq \lambda$$

Which, all $y \in f_{n+1}(K)$ satisfies $\|f_{n+1}(x_0) - f(y)\| \geq \lambda$, completing the induction.

Lastly, consider the sequence defined recursively as $x_n = f_n(x_0)$ for all $n \in \mathbb{N}$. Then, since f restrict the element to still be in K, then $(x_n)_{n\in\mathbb{N}} \subset K$, a compact set (which is closed and bounded). Hence, by Bolzano Weierstrass Theorem, since $(x_n)_{n\in\mathbb{N}}$ is bounded, there exists a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$, which this subsequence is Cauchy.

Then, given $\lambda > 0$, there exists $N \in \mathbb{N}$, such that all $p \geq N$ implies $\|x_{n_p} - x_{n_{p+1}}\| < \lambda$, by the definition of Cauchy Sequence.

However, since $n_{p+1} = n_p + k$ for some $k \in \mathbb{N}$, looking back at the definition, $x_{n_p} = f_{n_p}(x_0)$, while $x_{n_{p+1}} = x_{n_p+k} = f_{n_p+k}(x_0) = f_k(f_{n_p}(x_0))$.

Because $k \geq 1$, then $f_k(x) = f(f_{k-1}(x))$, so $x_{n_{p+1}} = f_k(f_{n_p}(x_0)) = f(f_{k-1}(f_{n_p}(x_0))) = f(f_{n_p}(f_{k-1}(x_0)))$.

So, let $y = f_{n_p}(f_{k-1}(x_0)) \in f_{n_p}(K)$, by the previous claim, the following inequality is true:

$$\|x_{n_p} - x_{n_{p+1}}\| = \|f_{n_p}(x_0) - f(f_{n_p}(f_{k-1}(x_0)))\| = \|f_{n_p}(x_0) - f(y)\| \geq \lambda$$

Yet, this contradicts the statement that $\|x_{n_p} - x_{n_{p+1}}\| < \lambda$.

Since we eventually reach a contradiction, then the assumption must be false, so f needs to be surjective.

The above two sections proved that f is in fact a bijection.