

# Math 118B HW3 - Lebesgue Criterion of Riemann Integrability

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The goal of this Homework is to understand and prove the Lebesgue Criterion of Riemann Integrability.

## Setup 1

**Definition 1** A set  $E \subset \mathbb{R}$  is said to be of measure zero if given  $\epsilon > 0$  there is a countable collection of open intervals  $\{I_j\}_{j \in \mathbb{Z}_+}$  which covers  $E$ , i.e.  $E \subseteq \bigcup_{j \in \mathbb{Z}_+} I_j$  and such that

$$\sum_{j=1}^{\infty} |I_j| < \epsilon$$

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**Question 1** Prove that every countable set of  $\mathbb{R}$  is a set of measure zero.

**Pf:**

Given  $E \subset \mathbb{R}$  that is countable, then there exists a bijection  $f : E \rightarrow \mathbb{N}$ , which generates an index for all element  $a \in E$ .

Then, given any  $\epsilon > 0$ , for all  $a \in E$ , let  $j = f(a)$ , consider the open interval  $I_j = (a - \frac{\epsilon}{2^{j+2}}, a + \frac{\epsilon}{2^{j+2}})$ : the collection  $\{I_j\}_{j \in \mathbb{Z}_+}$  is countable, and  $E \subseteq \bigcup_{j \in \mathbb{Z}_+} I_j$ , since for all  $a \in E$ , let  $j = f(a) \in \mathbb{Z}_+$ , we have  $f(a) \in I_j = (a - \frac{\epsilon}{2^{j+2}}, a + \frac{\epsilon}{2^{j+2}})$ .

On the other hand, the following is true for the length of the countable set:

$$\forall j \in \mathbb{Z}_+, \quad |I_j| = \left| \left( a + \frac{\epsilon}{2^{j+2}} \right) - \left( a - \frac{\epsilon}{2^{j+2}} \right) \right| = \left| 2 \cdot \frac{\epsilon}{2^{j+2}} \right| = \frac{\epsilon}{2^{j+1}}$$

$$\sum_{i=1}^{\infty} |I_j| = \sum_{i=1}^{\infty} \frac{\epsilon}{2^{j+1}} = \frac{\epsilon}{2} < \epsilon$$

The above series is converging since it's a geometric series with radius  $\frac{1}{2} < 1$ . Hence, for all  $\epsilon > 0$ , there is a countable collection of open intervals covering  $E$  with the series of length bounded by  $\epsilon$ , proving that  $E$  the countable set has measure 0.

**Question 2** Prove that the countable union of sets of measure zero has measure zero.

**Pf:**

Let  $\{E_n\}_{n \in \mathbb{Z}_+}$  be a countable collection of sets, each with measure 0. Then, for any given  $\epsilon > 0$ , for every  $n \in \mathbb{Z}_+$ , since  $\frac{\epsilon}{2^n} > 0$ , there exists a countable collection of open interval  $\{I_j^n\}_{j \in \mathbb{Z}_+}$ , with  $E_n \subseteq \bigcup_{j \in \mathbb{Z}_+} I_j^n$ , and  $\sum_{j=1}^{\infty} |I_j^n| < \frac{\epsilon}{2^n}$ .

Now, consider the collection  $\mathcal{F} = \bigcup_{n \in \mathbb{Z}_+} \{I_j^n\}_{j \in \mathbb{Z}_+}$ , a countable union of "countable collection of open intervals", which is again countable. Which, since  $E_n \subseteq \bigcup_{j \in \mathbb{Z}_+} I_j^n$  for all  $n \in \mathbb{Z}_+$ , then:

$$\bigcup_{n \in \mathbb{Z}_+} E_n \subseteq \bigcup_{n \in \mathbb{Z}_+} \left( \bigcup_{j \in \mathbb{Z}_+} I_j^n \right)$$

Which the left side is countable union of sets with measure 0, while the right side is the union of open intervals in family  $\mathcal{F}$ .

Then, to consider the length of  $\mathcal{F}$ , since it is countable, there exists a bijection  $f : \mathcal{F} \rightarrow \mathbb{N}$  that generates the index. Which, for the first  $k$  elements in this index of  $\mathcal{F}$ , the elements are  $\{I_{j_1}^{n_1}, \dots, I_{j_k}^{n_k}\}$ . let  $J = \max\{j_1, \dots, j_k\}$  and  $N = \max\{n_1, \dots, n_k\}$ , then these elements are in the collection  $\bigcup_{n=1}^N \{I_j^n\}_{j=1}^J$ . Which, the collection has the length being bounded:

$$\begin{aligned} \forall n \in \{1, \dots, N\}, \quad \sum_{j=1}^J |I_j^n| &\leq \sum_{j=1}^{\infty} |I_j^n| = \frac{\epsilon}{2^n} \\ \sum_{n=1}^N \left( \sum_{j=1}^J |I_j^n| \right) &\leq \sum_{j=1}^{\infty} |I_j^n| \leq \sum_{n=1}^N \frac{\epsilon}{2^n} \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon \end{aligned}$$

The above two inequalities are true, since each partial sum is monotonically non-decreasing, and bounded above. Hence, the sum of length  $s_k = \sum_{i=1}^k |I_{j_i}^{n_i}| \leq \sum_{n=1}^N \left( \sum_{j=1}^J |I_j^n| \right) \leq \epsilon$  for any positive integer  $k$ , while this partial sum of length  $s_k$  is also monotonically non-decreasing, hence the series of length converges, and the following is true:

$$\lim_{k \rightarrow \infty} s_k = \sum_{k=1}^{\infty} |I_{j_k}^{n_k}| = \sup\{s_k\} \leq \epsilon$$

(Note: since  $\epsilon$  is the upper bound of the partial sums, hence the above inequality is true). Then, since  $\mathcal{F}$  covers the  $\bigcup_{n \in \mathbb{Z}_+} E_n$ , and the series of  $\mathcal{F}$  elements' length satisfy  $\sum_{k=1}^{\infty} |I_{j_k}^{n_k}| \leq \epsilon$ , then we can conclude that  $\bigcup_{n \in \mathbb{Z}_+} E_n$  has measure 0.

**Question 3** Let  $E$  be the set of all  $x \in [0, 1]$  whose decimal expansion contains only the digits 4 and 7. We have seen that  $E$  is uncountable. Prove that  $E$  is a set of measure zero.

**Pf:**

From the description,  $E = \{x \in [0, 1] \mid x = 0.a_1a_2...a_n..., \forall n \in \mathbb{N}, a_n \in \{4, 7\}\}$ . For each  $n \in \mathbb{N}$ , let  $E_n = \{x \in [0, 1] \mid x = 0.a_1a_2...a_n..., \forall i \in \{1, ..., n\}, a_i \in \{4, 7\}\}$  (set of reals in  $[0, 1]$  with the first  $n$  decimals being 4 or 7).

Notice that for  $n \geq 2$ , there are  $2^{(n-1)}$  distinct cases for  $0.a_1a_2...a_{(n-1)}$  (first  $(n-1)$  decimals) in  $E_n$ , then for each case, if  $x \in E_n$  has this arrangement for the first  $(n-1)$  decimals:

$$0.a_1a_2...a_{(n-1)}3 < x < 0.a_1a_2...a_{(n-1)}8$$

Hence, for each arrangement, they're contained in the open interval  $(0.a_1a_2...a_{(n-1)}3, 0.a_1a_2...a_{(n-1)}8)$ , which has length  $(0.a_1a_2...a_{(n-1)}8 - 0.a_1a_2...a_{(n-1)}3) = \frac{8-3}{10^n} = \frac{5}{10^n}$ .

All  $2^{(n-1)}$  collection of these open intervals would cover  $E_n$ , since the first  $(n-1)$  decimals for each  $x \in E_n$  must be some arrangement of 4 and 7. Hence,  $E_n$  can be covered by unions of  $2^{(n-1)}$  open intervals, each with length  $\frac{5}{10^n}$ , hence the total length of the open cover is  $2^{(n-1)} \cdot 5 \cdot \frac{1}{10^n} = \frac{1}{2} \cdot \left(\frac{2}{10}\right)^{(n-1)} = \frac{1}{2} \left(\frac{1}{5}\right)^{(n-1)}$ .

Now, since for each  $x \in E$ , the first  $n$  decimals are consist of 4 and 7, hence  $x \in E_n$ , or  $E \subseteq E_n$ . From the previous part, since for each  $n \in \mathbb{N}$ , the set  $E_n$  could be covered by a finite collection of open intervals with sum of length  $\frac{1}{2} \left(\frac{1}{5}\right)^{(n-1)}$ , so does the set  $E$ .

Then, because  $\lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{1}{5}\right)^{(n-1)} = 0$ , then for all  $\epsilon > 0$ , there exists  $n$ , with  $\frac{1}{2} \left(\frac{1}{5}\right)^{(n-1)} < \epsilon$ . Hence, choose the collection of open intervals for  $E_n$  (along with countable empty sets),  $E$  can be covered with the given collection of open intervals, with total length  $\frac{1}{2} \left(\frac{1}{5}\right)^{(n-1)} < \epsilon$ , showing that  $E$  in fact has measure 0.

## Setup 2

**Definition 2** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function,  $(b - a) < \infty$ . For  $x \in [a, b]$  and  $\eta > 0$  define

$$\Omega(f, x, \eta) = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in (x - \eta, x + \eta) \cap [a, b]\}$$

and the oscillation of  $f$  at a point  $x \in [a, b]$

$$\omega_f(x) = \lim_{\eta \rightarrow 0^+} \Omega(f, x, \eta) = \inf_{\eta > 0} \{\Omega(f, x, \eta)\}$$

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**Question 4** Prove that  $\omega_f(x)$  is defined for any  $x \in [a, b]$ .

**Pf:**

Given any  $x \in [a, b]$ , and  $\eta_1, \eta_2 > 0$  with  $\eta_1 > \eta_2$ , since  $(x - \eta_2, x + \eta_2) \subset (x - \eta_1, x + \eta_1)$ , hence:

$$\{|f(x_1) - f(x_2)| : x_1, x_2 \in (x - \eta_2, x + \eta_2) \cap [a, b]\} \subseteq \{|f(x_1) - f(x_2)| : x_1, x_2 \in (x - \eta_1, x + \eta_1) \cap [a, b]\}$$

This implies  $\Omega(f, x, \eta_2) \leq \Omega(f, x, \eta_1)$ , since the supremum of the set on the right, is also an upper bound of the set on the left.

On the other hand, since the set  $\{|f(x_1) - f(x_2)| : x_1, x_2 \in (x - \eta, x + \eta) \cap [a, b]\}$  for all  $\eta > 0$  is a collection of distance in  $\mathbb{R}$ , hence 0 is always a lower bound of the set, showing that  $0 \leq \Omega(f, x, \eta)$ .

Now, since for all  $\eta > 0$ , the value  $\Omega(f, x, \eta)$  is bounded below by 0 for all  $\eta > 0$ , this implies  $\inf_{\eta > 0} \{\Omega(f, x, \eta)\}$  exists. Then, to prove that  $\lim_{\eta \rightarrow 0^+} \Omega(f, x, \eta) = \inf_{\eta > 0} \{\Omega(f, x, \eta)\} = \omega$ , for all  $\epsilon > 0$ , since  $\omega + \epsilon$  is no longer a lower bound of the set, there exists  $\eta > 0$ , with  $\omega \leq \Omega(f, x, \eta) < \omega + \epsilon$ . Then, choose  $\delta = \eta > 0$ , for all  $\mu' > 0$  with  $\mu' < \mu = \delta$ , from the previous section,  $\omega \leq \Omega(f, x, \eta') \leq \Omega(f, x, \eta) < \omega + \epsilon$ , hence:

$$|\omega - \Omega(f, x, \eta')| < \epsilon$$

This demonstrates that  $\lim_{\eta \rightarrow 0^+} \Omega(f, x, \eta) = \omega = \inf_{\eta > 0} \{\Omega(f, x, \eta)\}$ , hence  $\omega_f(x)$  is defined.

**Question 5** Prove that  $f$  is continuous at  $x_0$  if and only if  $\omega_f(x_0) = 0$ .

**Pf:**

$\Rightarrow$  : Suppose  $f$  is continuous at  $x_0$ , for all  $\epsilon > 0$  (since  $\frac{\epsilon}{2} > 0$ ), there exists  $\delta > 0$  such that  $|x - x_0| < \delta$  implies  $|f(x) - f(x_0)| < \epsilon$ . Then, choose  $\eta = \delta$ , consider  $\Omega(f, x_0, \delta) = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in (x_0 - \delta, x_0 + \delta) \cap [a, b]\}$ :

For all  $x_1, x_2 \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$ , since  $x_1, x_2 \in B_\delta(x_0)$ , then by assumption  $|f(x_1) - f(x_0)|, |f(x_2) - f(x_0)| < \frac{\epsilon}{2}$ . Hence, the following is true:

$$|f(x_1) - f(x_2)| = |(f(x_1) - f(x_0)) + (f(x_0) - f(x_2))| \leq |f(x_1) - f(x_0)| + |f(x_2) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence,  $\epsilon$  is an upper bound of the set  $\{|f(x_1) - f(x_2)| : x_1, x_2 \in (x_0 - \delta, x_0 + \delta) \cap [a, b]\}$ , showing that  $\Omega(f, x_0, \delta) \leq \epsilon$ .

Because the choice  $\epsilon > 0$  is arbitrary, then with the corresponding  $\delta > 0$ ,  $\omega_f(x_0) \leq \Omega(f, x_0, \delta) \leq \epsilon$ . Then,  $\omega_f(x_0) \leq \epsilon$  for all  $\epsilon > 0$ , showing that  $\omega_f(x_0) \leq 0$ ; also,  $\omega_f(x_0)$  is an infimum of all nonnegative numbers (in 4 we've proven  $0 \leq \Omega(f, x, \eta)$  for all  $\eta > 0$ ), hence  $\omega_f(x_0) \geq 0$ . The two statements imply  $\omega_f(x_0) = 0$ .

$\Leftarrow$  : Suppose  $\omega_f(x_0) = 0$ . Then, by definition, for all  $\epsilon > 0$ , since  $\epsilon = 0 + \epsilon$  is no longer a lower bound of the set  $\{\Omega(f, x_0, \eta) \mid \eta > 0\}$ , there exists  $\delta = \eta > 0$ , such that  $0 \leq \Omega(f, x_0, \delta) < \epsilon$ .

Hence, for all  $x \in B_\delta(x_0) \cap [a, b]$ ,  $|f(x) - f(x_0)| \leq \Omega(f, x_0, \delta) < \epsilon$ , proving that  $f$  is continuous at  $x_0$ .

The above two implications shows that  $f$  is continuous at  $x_0$  if and only if  $\omega_f(x_0) = 0$ .

**Question 6** Prove that for any  $\mu > 0$  the set  $A_\mu = \{x \in [a, b] : \omega_f(x) \geq \mu\}$  is compact.

**Pf:**

Since  $A_\mu \subseteq [a, b]$  while  $[a, b]$  is compact, then to prove that  $A_\mu$  is compact, it suffices to show that  $A_\mu$  is closed, or  $A'_\mu \subseteq A_\mu$ .

For all  $x_0 \in A'_\mu$ , for every radius  $r > 0$ , there exists  $x_1 \in B_r(x_0) \setminus \{x_0\} \cap A_\mu$ . Hence,  $\omega_f(x_1) \geq \mu$ .

Now, take  $\eta = r - |x_0 - x_1| > 0$ , for all  $x \in B_\eta(x_1)$ , since  $|x - x_1| < \eta = r - |x_0 - x_1|$ , then:

$$|x - x_0| = |(x - x_1) + (x_1 - x_0)| \leq |x - x_1| + |x_1 - x_0| < (r - |x_0 - x_1|) + |x_1 - x_0| = r$$

This indicates that  $x \in B_r(x_0)$ , or  $B_\eta(x_1) \subseteq B_r(x_0)$ .

Hence, for all  $x_c, x_d \in (B_\eta(x_1) \cap [a, b]) \subseteq (B_r(x_0) \cap [a, b])$ , since the following is true:

$$|f(x_c) - f(x_d)| \leq \Omega(f, x_0, r) = \sup\{|f(x) - f(x')| : x, x' \in B_r(x_0) \cap [a, b]\}$$

Hence,  $\Omega(f, x_0, r)$  is an upper bound of the set  $\{|f(x) - f(x')| : x, x' \in B_\eta(x_1) \cap [a, b]\}$ , which implies the following:

$$\Omega(f, x_0, r) \geq \Omega(f, x_1, \eta) = \sup\{|f(x) - f(x')| : x, x' \in B_\eta(x_1) \cap [a, b]\}$$

Thus, we can further conclude that  $\Omega(f, x_0, r) \geq \Omega(f, x_1, \eta) \geq \omega_f(x_1) \geq \mu$ .

Now, because for all  $r > 0$ ,  $\Omega(f, x_0, r) \geq \mu$ , then  $\mu$  is the lower bound of the set  $\{\Omega(f, x_0, r) \mid r > 0\}$ , showing that  $\mu \leq \omega_f(x_0) = \inf\{\Omega(f, x_0, r) \mid r > 0\}$ . Hence,  $x_0 \in A_\mu$ , showing that  $A'_\mu \subseteq A_\mu$ .

This proves that  $A_\mu$  is closed, and since  $A_m u \subseteq [a, b]$  a compact set, then  $A_\mu$  is also compact.

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**Question 7** Prove that the set of discontinuities of  $f$  can be written as

$$D_f = \bigcup_{j \in \mathbb{Z}_+} A_{1/j} = \bigcup_{j \in \mathbb{Z}_+} \left\{ x \in [a, b] : \omega_f(x) \geq \frac{1}{j} \right\}$$

**Pf:**

In **Question 5**, we've proven the equivalence of continuity at  $x_0$  and  $\omega_f(x_0) = 0$ , hence  $x \in [a, b]$  is a discontinuity of  $f$  iff  $\omega_f(x) \neq 0$  (which actually is  $\omega_f(x) > 0$ ). Hence,  $D_f = \{x \in [a, b] \mid \omega_f(x) > 0\}$ .

Now, for all  $x \in D_f$ , since  $\omega_f(x) > 0$ , by Archimedean's Property, there exists  $j \in \mathbb{Z}_+$ , with  $\omega_f(x) > \frac{1}{j} > 0$ , this implies  $x \in A_{1/j} = \left\{ x \in [a, b] : \omega_f(x) \geq \frac{1}{j} \right\}$ , hence  $x \in \bigcup_{j \in \mathbb{Z}_+} A_{1/j}$ . This implies  $D_f \subseteq \bigcup_{j \in \mathbb{Z}_+} A_{1/j}$ .

**Question 8** Prove that if for some  $\epsilon > 0$ ,  $\omega_f(x) < \epsilon$  for any  $x \in [a, b]$ , then there exists  $\eta > 0$  such that for all  $x \in [a, b]$ ,

$$\Omega(f, x, \eta) < \epsilon$$

**Pf:**

Suppose there exists  $\epsilon > 0$ , with  $\omega_f(x) < \epsilon$  for all  $x \in [a, b]$ , then since  $\epsilon$  is not a lower bound of the set  $\{\Omega(f, x, \eta) \mid \eta > 0\}$ , then there exists  $\eta_x > 0$ , such that  $\omega_f(x) \leq \Omega(f, x, \eta_x) < \epsilon$ .

Now, consider the collection of open intervals  $\mathcal{F} = \{(x - \eta_x/2, x + \eta_x/2) \mid x \in [a, b]\}$ : Since  $[a, b] \subseteq \bigcup \mathcal{F}$ , then  $\mathcal{F}$  is an open cover of  $[a, b]$ ; hence, by the compactness of  $[a, b]$ , there exists  $x_1, \dots, x_n \in [a, b]$ , such that  $[a, b] \subseteq \bigcup_{i=1}^n (x_i - \mu_{x_i}/2, x_i + \mu_{x_i}/2)$ .

Then, let  $\eta = \min\{\frac{1}{2}\eta_{x_1}, \dots, \frac{1}{2}\eta_{x_n}\} > 0$ . For all  $x \in [a, b]$ , from the above construction, there exists  $i \in \{1, \dots, n\}$  with  $x \in B_{\eta_{x_i}/2}(x_i)$ . Now, consider the set  $S = \{|f(x_c) - f(x_d)| : x_c, x_d \in (x - \eta, x + \eta) \cap [a, b]\}$ : For all  $x_c, x_d \in (x - \eta, x + \eta) \cap [a, b]$ , they satisfy  $|x_c - x|, |x_d - x| < \mu \leq \frac{1}{2}\eta_{x_i}$ . Hence, the following inequalities are true:

$$|x_c - x_i| = |(x_c - x) + (x - x_i)| \leq |x_c - x| + |x - x_i| < \frac{1}{2}\eta_{x_i} + \frac{1}{2}\eta_{x_i} = \eta_{x_i}$$

$$|x_d - x_i| = |(x_d - x) + (x - x_i)| \leq |x_d - x| + |x - x_i| < \frac{1}{2}\eta_{x_i} + \frac{1}{2}\eta_{x_i} = \eta_{x_i}$$

These two inequalities imply  $x_c, x_d \in (x_i - \eta_{x_i}, x_i + \eta_{x_i})$ , which  $|f(x_c) - f(x_d)| \leq \Omega(f, x_i, \eta_{x_i})$ . Hence,  $\Omega(f, x_i, \eta_{x_i})$  is an upper bound of the set  $S$ , showing that  $\sup(S) = \Omega(f, x, \eta) \leq \Omega(f, x_i, \eta_{x_i}) < \epsilon$  regarding the initial construction.

So, this  $\eta > 0$  satisfies the desired condition.

## Proof of the Main Theorem

**Theorem 1** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function  $(b - a) < \infty$ . Then  $f$  is Riemann integrable on  $[a, b]$  if and only if the set of discontinuities of  $f$ ,  $D_f$  is a set of measure zero.

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**Question 9** Prove the theorem.

**Pf:**

$\Rightarrow$  : We'll approach this by contradiction. Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann Integrable, yet  $D_f$  has measure greater than 0.

In **Question 7**, we've proven that  $D_f = \bigcup_{j \in \mathbb{Z}_+} A_{1/j}$  (Countable union of  $A_j$ ), with  $A_{1/j}$  being defined in **Question 6**. With the assumption that  $D_f$  has measure 0, there exists  $j_0 \in \mathbb{Z}_+$ , with  $A_{1/j_0}$  having measure greater than 0: If all  $j \in \mathbb{Z}_+$  has measure 0, then by the statement proven in **Question 2**, the countable union  $D_f = \bigcup_{j \in \mathbb{Z}_+} A_{1/j}$  should also have measure 0, which contradicts the assumption.

With the given  $j_0 \in \mathbb{Z}_+$ , consider any partition  $P = \{x_0 = a, x_1, \dots, x_{n-1}, x_n = b\}$  with  $x_{i-1} < x_i$  for all index  $i$ . Since  $A_{1/j_0} \subseteq [a, b]$ , there exists some intervals in the partition  $I_{n_1}, \dots, I_{n_i}$  that covers  $A_{1/j_0}$  (here, assume every chosen interval covered some part of  $A_{1/j_0}$ , not disjoint with it).

WLOG, we can assume that for each interval  $I_{n_j}$ , there exists a point of discontinuity  $x \in A_{1/j_0}$  that is an interior point of  $I_{n_j}$ : If  $x = x_i$  for some  $i \neq 0$  and  $i \neq n$ , then we can combine two intervals  $I' = I_i \cup I_{i+1} = [x_{i-1}, x_{i+1}]$ , which  $x_i$  becomes the interior point of the modified interval; else if  $x = a$  or  $x = b$ , since all the definition only consider the cases in  $[a, b]$ , then  $a, b$  is could be considered as the interior point of  $[a, b]$  under subspace topology.

Because  $A_{1/j_0}$  has nonzero measure, then there exists  $\epsilon > 0$ , such that for any open interval covering  $\{I_j\}_{j \in \mathbb{Z}_+}$  of  $A_{1/j_0}$ , the sum of length of the intervals  $\sum_{j=1}^{\infty} |I_j| \geq \epsilon$ , regardless of the collection of open intervals (in particular,  $\sum_{j=1}^i |I_{n_j}| \geq \epsilon$ , since each interval has the same length with its interior).

Furthermore, from the previous assumption, there exists  $x \in A_{1/j_0}$  that is an interior point of  $I_{n_j}$  for each  $j \in \{1, \dots, i\}$ , hence there exists radius  $r_j > 0$ , with  $B_{r_j}(x) \cap [a, b] \subseteq I_{n_j}$ . Now, by definition, since for all  $x_1, x_2 \in B_{r_j}(x) \cap [a, b] \subseteq I_{n_j}$  satisfies  $\inf_{x \in I_{n_j}} \{f(x)\} \leq f(x_1), f(x_2) \leq \sup_{x \in I_{n_j}} \{f(x)\}$ , hence:

$$|f(x_1) - f(x_2)| \leq \left( \sup_{x \in I_{n_j}} \{f(x)\} - \inf_{x \in I_{n_j}} \{f(x)\} \right)$$

This implies the following:

$$\sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in B_{r_j}(x) \cap [a, b]\} = \Omega(f, x, r_j) \leq \left( \sup_{x \in I_{n_j}} \{f(x)\} - \inf_{x \in I_{n_j}} \{f(x)\} \right)$$

$$\frac{1}{j_0} \leq \omega_f(x) \leq \Omega(f, x, r_j) \leq \left( \sup_{x \in I_{n_j}} \{f(x)\} - \inf_{x \in I_{n_j}} \{f(x)\} \right)$$



Hence, consider the difference in upper and lower sum, we yield:

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{k=1}^n \left( \sup_{x \in I_k} \{f(x)\} - \inf_{x \in I_k} \{f(x)\} \right) \cdot |I_k| \geq \sum_{j=1}^i \left( \sup_{x \in I_{n_j}} \{f(x)\} - \inf_{x \in I_{n_j}} \{f(x)\} \right) \cdot |I_{n_j}| \\
U(f, P) - L(f, P) &\geq \sum_{j=1}^i \left( \sup_{x \in I_{n_j}} \{f(x)\} - \inf_{x \in I_{n_j}} \{f(x)\} \right) \cdot |I_{n_j}| \geq \sum_{j=1}^i \frac{1}{j_0} |I_{n_j}| \\
U(f, P) - L(f, P) &\geq \frac{1}{j_0} \sum_{j=1}^i |I_{n_j}| \geq \frac{1}{j_0} \cdot \epsilon
\end{aligned}$$

(Note: The above is true, since the collection  $I_{n_1}, \dots, I_{n_j}$  is part of the partition).

Hence, the difference in the upper and lower sum for any  $P$  is at least  $\epsilon/j_0 > 0$ , showing that  $f$  is not Riemann Integrable. Yet, this contradicts our initial assumption; hence, the assumption is false,  $f$  is Riemann Integrable implies  $D_f$  has measure 0.

$\Leftarrow$ : Suppose  $D_f$  has measure 0. Then, for all  $\epsilon > 0$ , since  $\frac{\epsilon}{2(b-a)} > 0$ , by Archimedean's Property, there exists  $j \in \mathbb{N}$ , with  $\frac{1}{j} < \frac{\epsilon}{2(b-a)}$ . Then, consider the set  $A_{1/j} \subseteq D_f$ :

#### Partition for Points in $A_{1/j}$ :

Before starting this, since the function  $f$  is bounded, there exists  $M > 0$ , such that  $|f(x)| < M$  for all  $x \in [a, b]$ . Hence, for all  $x_1, x_2 \in [a, b]$ ,  $|f(x_1) - f(x_2)| \leq |f(x_1)| + |f(x_2)| < 2M$ , or  $\Omega(f, x, \eta) \leq 2M$  for all  $x \in [a, b]$  and  $\eta > 0$ .

Given that  $A_{1/j} \subseteq D_f$  and  $D_f$  has measure 0, then  $A_{1/j}$  has measure 0 (or else, if  $A_{1/j}$  has nonzero measure, then ever open interval cover of  $D_f$  covers  $A_{1/j}$ , which must have some nonzero total length greater than some  $r > 0$ , showing that  $D_f$  also has nonzero measure).

Hence, for given  $\epsilon > 0$ , since  $\frac{\epsilon}{2M} > 0$ , there exists collection of open interval covering  $\{I_n\}_{n \in \mathbb{Z}_+}$  for  $A_{1/j}$ , with  $\sum_{n=1}^{\infty} |I_n| < \frac{\epsilon}{2M}$ . Then, by statement proven in **Question 6**, since  $A_{1/j}$  is compact, there exists  $n_1, \dots, n_k \in \mathbb{Z}_+$ , with  $A_{1/j} \subseteq \bigcup_{i=1}^k I_{n_i}$

(Note 1: WLOG, assume the open intervals in this collection is mutually disjoint; if not, we can take the union of the ones with common intersection instead, while the length is still bounded by  $\frac{\epsilon}{2M}$ ).

(Note 2: Assume each  $I_{n_i}$  has nonempty intersection with  $A_{1/j}$ ).

Let  $P_1$  be a partition consists of the infimum and supremum of each interval  $I_{n_i} \cap [a, b]$ , which will be used later.

#### Partition for Points outside $A_{1/j}$ :

Consider the set  $K = [a, b] \setminus (\bigcup_{i=1}^k I_{n_i})$ : By definition of  $A_{1/j}$  in **Question 6**, for every  $x \in K$ , since  $x \notin \bigcup_{i=1}^k I_{n_i}$  while  $A_{1/j} \subseteq \bigcup_{i=1}^k I_{n_i}$ , then  $x \notin A_{1/j}$ , hence  $\omega_f(x) < \frac{1}{j}$ . Then, because  $\bigcup_{i=1}^k I_{n_i}$  is open,

then  $\left(\bigcup_{i=1}^k I_{n_i}\right)^C$  is closed, hence  $K = [a, b] \cap \left(\bigcup_{i=1}^k I_{n_i}\right)^C$  is closed and bounded, which is compact. Which, using the same proof in **Question 8**, since  $\omega_f(x) < \frac{1}{j}$  for all  $x \in K$  while  $K$  is compact, then there exists  $\eta > 0$ , such that for all  $x \in K$ ,  $\Omega(f, x, \eta) < \frac{1}{j}$  (Note: the proof is by swapping  $[a, b]$  with  $K$ ; since  $K$  is compact, same proof relying on the compactness of  $[a, b]$  would work on  $K$  also).

Which, Consider the partition  $P_2$  of  $K$  with  $\|P_2\| = \max\{(x_i - x_{i-1}) : [x_{i-1}, x_i] \subseteq K\} < \eta$ .

### The Difference in Upper and Lower Sum:

Consider the partition  $P = P_1 \cup P_2$ : If consider the difference in Upper and Lower Sum based on this partition, it is as follow:

$$U(f, P) - L(f, P) = \sum_{i=1}^m \left( \sup_{x \in I'_i} \{f(x)\} - \inf_{x \in I'_i} \{f(x)\} \right) \cdot |I'_i|$$

Which, there are two cases to consider for each interval.

Suppose  $m_1, \dots, m_{k'}$  is a collection of indices with each  $I'_{m_i} \subseteq \overline{I_{n_i}}$  provided for  $A_{1/j}$ . Then,