Math CS 122A HW9

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Question 1 *Ahlfors Pg. 154 Problem 2:*
Hwo many roots of the equation $z^4 - 6z + 3 = 0$ *have their modulus between* 1 *and* 2*?*

Pf:

For the disk $|z| < 1$, consider the function $-6z + 3$: It has one zero in $|z| < 1$, namely $z = \frac{1}{2}$.

On the other hand, for circle $|z| = 1$, the following inequalities are true:

$$|(z^2 - 6z + 3) - (-6z + 3)| = |z|^4 = 1$$

$$|-6z + 3| \geq |6|z| - 3| = 6 - 3 = 3$$

So, since $|(z^2 - 6z + 3) - (-6z + 3)| = 1 \leq 3 \leq |-6z + 3|$ for all z on $|z| = 1$, then by Rouche's Theorem, the two polynomials have the same number of zeroes enclosed by the circle $|z| = 1$.

Since $-6z + 3$ only has one zero in this region, then $z^4 - 6z + 3$ also has one zero in this region.

Now, consider the disk $|z| < 2$, and the function z^4: It has four zeros in $|z| < 2$ counting multiplicity (namely $z = 0$).

On the other hand, for circle $|z| = 2$, the following inequalities are true:

$$|(z^2 - 6z + 3) - z^4| = |-6z + 3| \leq 6|z| + 3 = 15$$

$$|z^4| = |z|^4 = 16$$

So, since $|(z^4 - 6z + 3) - z^4| = 15 < 16 = |z^4|$ for all z on $|z| = 2$, by Rouche's Theorem again, the two polynomials have the same number of zeroes enclosed by the circle $|z| = 2$. Since z^4 has four zeroes in this region, then $z^4 - 6z + 3$ also has four zeroes in this region.

Then, since 4 zeroes has modulus less than 2, while 1 zero has modulus less than 1, counting the ones with modulus between 1 and 2, we have total of $4 - 1 = 3$ zeroes.

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Question 2 *Ahlfors Pg. 161 Problem 5:*
Complex integration can sometimes be used to evaluate area integrals. As an illustration, show that if $f(z)$ is analytic and bounded for $|z| < 1$ and if $|\zeta| < 1$, then

$$f(\zeta) = \frac{1}{\pi} \int\int_{|z|<1} \frac{f(z)dxdy}{(1-\bar{z}\zeta)^2}$$

Pf:
If convert the above integral to polar coordinates, we get the following:

$$\frac{1}{\pi} \int\int_{|z|<1} \frac{f(z)dxdy}{(1-\bar{z}\zeta)^2} = \frac{1}{\pi} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})r}{(1-\zeta re^{-i\theta})^2} d\theta dr = \frac{1}{\pi} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})(e^{i\theta})^2 \cdot r}{(e^{i\theta} - r\zeta)^2} d\theta dr$$

Then, define C to have parametrization $z = e^{i\theta}$ with $\theta \in [0, \pi]$, the inner part of the integral becomes:

$$\frac{1}{\pi} \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})(e^{i\theta})^2 r}{(e^{i\theta} - r\zeta)^2} d\theta = 2 \cdot \frac{1}{2\pi i} \int_{0}^{\pi} \frac{f(re^{i\theta})re^{i\theta}}{(e^{i\theta} - r\zeta)^2} ie^{i\theta} d\theta = 2 \cdot \frac{1}{2\pi i} \int_C \frac{f(rz)rz}{(z - r\zeta)^2} dz$$

Recall that for any analytic function $\phi(z)$ at a point a, by Cauchy's Integral formula, we get:

$$\phi'(a) = \frac{1!}{2\pi i} \int_C \frac{\phi(z)}{(z-a)^2} dz$$

So, given that $z = r\zeta$ (which with $r \in [0, 1]$ and $|\zeta| < 1$, $r\zeta$ is strictly in the unit disk, so integrate over C given above is valid), let $\phi(z) = f(rz)rz$ (which $\phi'(z) = f'(rz)r^2 z + f(rz)r$), we have:

$$f'(r(r\zeta))r^2(r\zeta) + f(r(r\zeta))r = \frac{1!}{2\pi i} \int_C \frac{f(rz)rz}{(z - r\zeta)^2} dz$$

$$2 \cdot \frac{1}{2\pi i} \int_C \frac{f(rz)rz}{(z - r\zeta)^2} dz = 2\left(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r\right)$$

Hence, the original integral can be rewrite as:

$$\frac{1}{\pi} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})(e^{i\theta})^2 \cdot r}{(e^{i\theta} - r\zeta)^2} d\theta dr = \int_{r=0}^{1} 2\left(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r\right) dr$$

Now, consider the function $f(\zeta r^2)r^2$, which has derivative $f'(\zeta r^2)r^2 \cdot 2r\zeta + f(\zeta r^2) \cdot 2r = 2\left(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r\right)$. Then, the above integral can be rewrite as:

$$\int_{r=0}^{1} 2\left(f'(r^2\zeta)r^3\zeta + f(r^2\zeta)r\right) dr = f(\zeta r^2)r^2 \Big|_{r=0}^{1} = f(\zeta)$$

Hence, we can claim that for $f(z)$ that's analytic and bounded on $|z| < 1$, and given $|\zeta| < 1$, the integral is true:

$$f(\zeta) = \frac{1}{\pi} \int\int_{|z|<1} \frac{f(z)dxdy}{(1-\bar{z}\zeta)^2}$$

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Question 3 *Stein and Shakarchi Pg. 64 Problem 1:*
Prove that
$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

Pf:

Consider the function e^{-z^2}, and the integration over a sector with origin at 0 and radius R. Which, this can be parametrized by three curves: γ_1 - a straight line on real axis with $x \in [0, R]$, γ_2 - a circular arc with radian $\frac{\pi}{4}$ and radius R (parametrized by $z = Re^{i\theta}$, where $\theta \in [0, \frac{\pi}{4}]$), and γ_3 - another straight line of $z = re^{i\frac{\pi}{4}}$ (where $r \in [0, R]$). The orientation is given as follow:

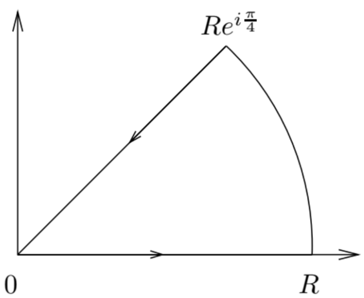

Figure 14. The contour in Exercise 1

If consider the integral over this closed curve, since e^{-z^2} is analytic on the whole plane, then the line integral is 0. So, $\int_{\gamma_1+\gamma_2+\gamma_3} e^{-z^2}dz = 0$.

For $\int_{\gamma_1} e^{-z^2}dz$, it is parametrized by $\int_0^R e^{-x^2}dx$, which $\lim_{R\to\infty}\int_0^R e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ (since e^{-x^2} is even, while $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$).

For $\int_{\gamma_2} e^{-z^2}dz$, it is parametrized by the following:

$$\int_{\gamma_2} e^{-z^2}dz = \int_0^{\frac{\pi}{4}} \exp\left(-(Re^{i\theta})^2\right) iRe^{i\theta}d\theta = \int_0^{\frac{\pi}{4}} \exp(-R^2e^{i2\theta})iRe^{i\theta}d\theta$$
$$= \int_0^{\frac{\pi}{4}} \exp(-R^2(\cos(2\theta) + i\sin(2\theta)))iRe^{i\theta}d\theta = \int_0^{\frac{\pi}{4}} e^{-R^2\cos(2\theta)}e^{-iR^2\sin(2\theta)}iRe^{i\theta}d\theta$$

Which, consider the modulus, the following inequality is true:

$$\left|\int_{\gamma_2} e^{-z^2}dz\right| \le \int_0^{\frac{\pi}{4}} |e^{-R^2\cos(2\theta)}|\cdot|e^{-iR^2\sin(2\theta)}|\cdot|iRe^{i\theta}|d\theta = \int_0^{\frac{\pi}{4}} Re^{-R^2\cos(2\theta)}d\theta$$
$$\left|\int_{\gamma_2} e^{-z^2}dz\right| \le \frac{1}{2}\int_0^{\frac{\pi}{2}} Re^{-R^2\cos(u)}du$$

(Note: the second line is done by the parametrization $u = 2\theta$).

Now, since in the domain $[0, \frac{\pi}{2}]$, $1 - \frac{2}{\pi}u \le \cos(u)$, then $e^{-R^2\cos(u)} \le e^{-R^2(1-\frac{2}{\pi}u)}$ (given that $-R^2 < 0$, while the two functions are positive on the given domain). Then, we can further bound the integral by:

$$\left|\int_{\gamma_2} e^{-z^2}dz\right| \le \frac{1}{2}\int_0^{\frac{\pi}{2}} Re^{-R^2\cos(u)}du \le \frac{1}{2}\int_0^{\frac{\pi}{2}} Re^{-R^2(1-\frac{2}{\pi}u)}du = \frac{1}{2}\int_0^{\frac{\pi}{2}} Re^{R^2\cdot\frac{2}{\pi}u - R^2}du$$

$$\leq \frac{R}{2} \cdot \frac{\pi}{2R^2} e^{R^2 \cdot \frac{2}{\pi} u - R^2} \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4R}(e^{R^2 \cdot \frac{2}{\pi} \cdot \frac{\pi}{2} - R^2} - e^{R^2 \cdot \frac{2}{\pi} \cdot 0 - R^2}) = \frac{\pi}{4R}(1 - e^{-R^2})$$

Then, since $\lim_{R\to\infty} \frac{\pi}{4R} = 0$, $\lim_{R\to\infty}(1 - e^{-R^2}) = 1$, then:

$$0 \leq \lim_{R\to\infty} \left| \int_{\gamma_2} e^{-z^2} dz \right| \leq \lim_{R\to\infty} \frac{\pi}{4R}(1 - e^{-R^2}) = 0$$

Hence, we can claim that $\lim_{R\to\infty} \int_{\gamma_2} e^{-z^2} dz = 0$.

Lastly, for $\int_{\gamma_3} e^{-z^2} dz$, it is parametrized by $\int_R^0 \exp(-(re^{i\frac{\pi}{4}})^2) e^{i\frac{\pi}{4}} dr$. Which, can be modified as:

$$\int_R^0 \exp(-r^2 e^{i\frac{\pi}{2}}) e^{i\frac{\pi}{4}} dr = e^{i\frac{\pi}{4}} \int_R^0 e^{-ir^2} dr = e^{i\frac{\pi}{4}} \left(\int_R^0 \cos(r^2) dr - i \int_R^0 \sin(r^2) dr \right)$$

$$= -e^{i\frac{\pi}{4}} \left(\int_0^R \cos(r^2) dr - i \int_0^R \sin(r^2) dr \right)$$

Now, because $\int_{\gamma_1+\gamma_2+\gamma_3} e^{-z^2} dz = 0$, then $\int_{\gamma_3} e^{-z^2} dz = -(\int_{\gamma_1} e^{-z^2} dz + \int_{\gamma_2} e^{-z^2} dz)$. Hence:

$$\lim_{R\to\infty} \int_{\gamma_3} e^{-z^2} dz = \lim_{R\to\infty} - \left(\int_{\gamma_1} e^{-z^2} dz + \int_{\gamma_2} e^{-z^2} dz \right) = -\frac{\sqrt{\pi}}{2}$$

$$\lim_{R\to\infty} -e^{i\frac{\pi}{4}} \left(\int_0^R \cos(r^2) dr - i \int_0^R \sin(r^2) dr \right) = -\frac{\sqrt{\pi}}{2}$$

Hence, we can claim the following:

$$\int_0^\infty \cos(r^2) dr - i \int_0^\infty \sin(r^2) dr = \frac{\sqrt{\pi}}{2} e^{-i\frac{\pi}{4}} = \frac{\sqrt{2\pi}}{4}(1 - i)$$

Then, take the real and imaginary part respectively, we get:

$$\int_0^\infty \cos(r^2) dr = Re\left(\frac{\sqrt{2\pi}}{4}(1 - i) \right) = \frac{\sqrt{2\pi}}{4}$$

$$\int_0^\infty \sin(r^2) dr = -Im\left(\frac{\sqrt{2\pi}}{4}(1 - i) \right) = \frac{\sqrt{2\pi}}{4}$$

Hence, the two integrals evaluated to be $\frac{\sqrt{2\pi}}{4}$.

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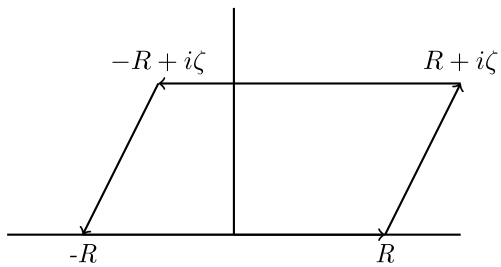
Question 4 *Stein and Shakarchi Pg. 65 Problem 4:*
Prove that for all $\zeta \in \mathbb{C}$ we have

$$e^{-\pi\zeta^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\zeta} dx$$

Pf:

Given $\zeta = u + iv$ for some $u, v \in \mathbb{R}$. Consider the function $e^{-\pi z^2}$ which is analytic on $\mathbb{C}$. There are three cases:

(1) **When** $u > 0$, given $R > 0$, consider the parallelagram generated by the points $-R, R, R+i\zeta, -R+i\zeta$ with counterclockwise orientation. The orientation is as follow:



Then, the integral of $e^{-\pi z^2}$ on the contour is 0 (since it is analytic on the whole plane). Which, it can be broken down into the sum of following integrals:

First, for the one on the real axis, it is parametrized by $\int_{-R}^{R} e^{-\pi x^2} dx$, where $\lim_{R\to\infty} \int_{-R}^{R} e^{-\pi x^2} dx = 1$ (Gauss Integral).

For the ones on the side, since the sides are parametrized by $R + i\zeta t$ and $-R + i\zeta t$ for $t \in [0, 1]$ respectively, then the first integral is given by:

$$\int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt = \int_0^1 e^{-\pi(R^2+i\cdot 2R\zeta t-\zeta^2 t^2)} \cdot i\zeta dt$$

$$= \int_0^1 e^{-\pi(R^2-2Rvt)} \cdot e^{-i\cdot 2\pi Rut+\pi\zeta^2t^2} \cdot i\zeta dt = e^{-\pi R^2} \int_0^1 e^{R\cdot 2\pi vt} \cdot e^{-i\cdot 2\pi Rut} \cdot e^{\pi\zeta^2t^2} \cdot i\zeta dt$$

Which, taking the modulus, it is bounded by:

$$\left|\int_0^1 e^{-\pi(R+i\zeta t)^2} \cdot i\zeta dt\right| \le e^{-\pi R^2} \int_0^1 e^{R\cdot 2\pi vt} \cdot |e^{-i\cdot 2\pi Rut}| \cdot |e^{\pi\zeta^2t^2}| \cdot |i\zeta| dt$$

$$\le e^{-\pi R^2} \int_0^1 e^{R\cdot|2\pi v|} \cdot |e^{\pi\zeta^2t^2}| \cdot |\zeta| dt = e^{\pi R^2+R\cdot|2\pi v|} \int_0^1 \cdot|e^{-\pi\zeta^2t^2}| \cdot |\zeta| dt$$

(Note: the above is given by $R \cdot 2\pi vt \le R \cdot |2\pi v| \cdot t \le R \cdot |2\pi v|$, since $t \in [0, 1]$).

Then, since $\lim_{R\to\infty} -\pi R^2 + R|2\pi v| = -\infty$, so $\lim_{R\to\infty} e^{-\pi R^2+R|2\pi v|} = 0$, then since $\int_0^1 \cdot|e^{\pi\zeta^2t^2}| \cdot |\zeta| dt$ is a constant, then:

$$\lim_{R\to\infty} e^{-\pi R^2+R\cdot|2\pi v|} \int_0^1 \cdot|e^{-\zeta^2t^2}| \cdot |\zeta| dt = 0$$

So, we can conclude the following:

$$\lim_{R\to\infty}\int_0^1 e^{-\pi(R+i\zeta t)^2}\cdot i\zeta dt = 0$$

Similar concepts applied on the line $-R+i\zeta t$ (since $-\pi(-R+i\zeta t)^2$ is then given by $-\pi(R^2+2Rvt)-\pi(-i\cdot 2Rut-\zeta^2t^2)$, so the same concept still applies, where the bound of the modulus is dominated by $e^{-\pi(R^2+2Rv)}$, which converges to 0 as $R\to\infty$). Hence:

$$\lim_{R\to\infty}\int_0^1 e^{-\pi(-R+i\zeta t)^2}\cdot i\zeta dt = 0$$

Lastly, the translated line is parametrized by $x+i\zeta$ for $x\in[-R,R]$, then the integral is given by:

$$\int_{-R}^{R} e^{-\pi(x+i\zeta)^2}dx = \int_{-R}^{R} e^{-\pi(x^2+2ix\zeta-\zeta^2)}dx = e^{\pi\zeta^2}\int_{-R}^{R} e^{-\pi x^2}\cdot e^{-2\pi ix\zeta}dx$$

Now, summing up all the path integrals with right orientation, we get the following:

$$\int_{-R}^{R} e^{-\pi x^2}dx + \int_0^1 e^{-\pi(R+i\zeta t)^2}\cdot i\zeta dt - \int_0^1 e^{-\pi(-R+i\zeta t)^2}\cdot i\zeta dt - \int_{-R}^{R} e^{-\pi(x+i\zeta)^2}dx = 0$$

$$\int_{-R}^{R} e^{-\pi(x+i\zeta)^2}dx = \int_{-R}^{R} e^{-\pi x^2}dx + \int_0^1 e^{-\pi(R+i\zeta t)^2}\cdot i\zeta dt - \int_0^1 e^{-\pi(-R+i\zeta t)^2}\cdot i\zeta dt$$

So, take $R\to\infty$, the first term on the right approaches 1, while the next two terms converges to 0 (from the above statements), then the limit becomes:

$$\lim_{R\to\infty}\int_{-R}^{R} e^{-\pi(x+i\zeta)^2}dx = \lim_{R\to\infty} e^{\pi\zeta^2}\int_{-R}^{R} e^{-\pi x^2}\cdot e^{-2\pi ix\zeta}dx = 1$$

So, we can conclude that $\int_{-\infty}^{\infty} e^{-\pi x^2}\cdot e^{-2\pi ix\zeta}dx = e^{-\pi\zeta^2}$.

Which, by doing substitution $u=-x$ ($-du=dx$), we get the following:

$$-\int_{\infty}^{-\infty} e^{-\pi u^2}\cdot e^{-2\pi i(-u)\zeta}du = \int_{-\infty}^{\infty} e^{-\pi u^2}\cdot e^{2\pi iu\zeta}du = e^{-\pi\zeta^2}$$

(2) **When** $u=0$, since $\zeta=iv$, then $\zeta^2=-v^2$. So, the proposed integral becomes:

$$\int_{-R}^{R} e^{-\pi x^2}\cdot e^{2\pi ix\zeta}dx = \int_{-R}^{R} e^{-\pi x^2}\cdot e^{2\pi ix\cdot iv}dx = \int_{-R}^{R} e^{-\pi x^2}\cdot e^{-2\pi vx}dx$$

Which, by completing the square, we get:

$$\int_{-R}^{R} e^{-\pi x^2}\cdot e^{2\pi ix\zeta}dx = e^{\pi v^2}\int_{-R}^{R} e^{-\pi x^2}\cdot e^{-2\pi vx}\cdot e^{-\pi v^2}dx = e^{\pi v^2}\int_{-R}^{R} e^{-\pi(x^2+2vx+v^2)}dx = e^{\pi v^2}\int_{-R}^{R} e^{-\pi(x+v)^2}dx$$

Then, as $R\to\infty$, we get:

$$\lim_{R\to\infty}\int_{-R}^{R} e^{-\pi x^2}\cdot e^{2\pi ix\zeta}dx = \lim_{R\to\infty} e^{\pi v^2}\int_{-R}^{R} e^{-\pi(x+v)^2}dx = e^{\pi v^2}\int_{-\infty}^{\infty} e^{-\pi(x+v)^2}dx = e^{\pi v^2}$$

And, since $e^{\pi v^2}=e^{-\pi\zeta^2}$, then:

$$e^{-\pi\zeta^2} = \int_{-\infty}^{\infty} e^{-\pi x^2}\cdot e^{2\pi ix\zeta}dx$$

So, regardless of the case, we can say that the following integral is true:

$$e^{-\pi\zeta^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{2\pi i x\zeta} dx$$

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Question 5 *Stein and Shakarchi Pg. 103 Problem 5:*
Use contour integration to show that for all ζ real

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\zeta}}{(1+x^2)^2} dx = \frac{\pi}{2}(1+2\pi|\zeta|)e^{-2\pi|\zeta|}$$

Pf:

Residue at $i, -i$:

Consider the function $f(z) = e^{-2\pi i\zeta z}/(1+z^2)^2 = e^{-2\pi i\zeta z}/((z-i)(z+i))^2$, which it has poles at $z = \pm i$, each with order 2 (since $(z^2+1)^2 = (z-i)^2(z+i)^2$).

Then, to show its residue at i, consider the derivative of $\phi_i(z) = e^{-2\pi i\zeta z}/(z+i)^2$:

$$\phi_i'(z) = \frac{-2\pi i\zeta e^{-2\pi i\zeta z} \cdot (z+i)^2 - 2(z+i)e^{-2\pi i\zeta z}}{(z+i)^4}, \quad \phi_i'(i) = \frac{-2\pi i\zeta e^{2\pi\zeta}(-4) - 2(2i)e^{2\pi\zeta}}{16} = -\frac{1}{4}(1-2\pi\zeta)ie^{2\pi\zeta}$$

Then, we can expand $\phi_i(z)$ as the following term:

$$\phi_i(z) = \phi_i(i) + \phi_i'(i)(z-i) + \phi_{i,2}(z)(z-i)^2$$

The above term has $\phi_{i,2}(z)$ being analytic at i. Hence, $f(z)$ can be represented as:

$$f(z) = \frac{\phi_i(z)}{(z-i)^2} = \frac{\phi_i(i)}{(z-i)^2} + \frac{\phi_i'(i)}{(z-i)} + \phi_{i,2}(z)$$

Because the first term has antiderivative, while the third term is analytic at i, then for sufficiently small circle C centered at i, the residue is given by:

$$Res_{z=i} f(z) = \frac{1}{2\pi i}\int_C \frac{\phi_i'(i)}{(z-i)} dz = n(C,i) \cdot \phi_i'(i) = -\frac{1}{4}(1-2\pi\zeta)ie^{2\pi\zeta}$$

Now, apply similar concept for $z = -i$, the derivative of $\phi_{-i}(z) = e^{-2\pi i\zeta z}/(z-i)^2$ is given as:

$$\phi_{-i}'(z) = \frac{-2\pi i\zeta e^{-2\pi i\zeta z} \cdot (z-i)^2 - 2(z-i)e^{-2\pi i\zeta z}}{(z-i)^4}, \quad \phi_{-i}'(-i) = \frac{-2\pi i\zeta e^{-2\pi\zeta}(-4) - 2(-2i)e^{-2\pi\zeta}}{16} = \frac{1}{4}(1+2\pi\zeta)ie^{-2\pi\zeta}$$

Then, expand $\phi_{-i}(z)$ as follow:

$$\phi_{-i}(z) = \phi_{-i}(-i) + \phi_{-i}'(-i)(z+i)^2 + \phi_{-i,2}(z)(z+i)^2$$

Then, the above term has $\phi_{-i,2}(z)$ being analytic at i. Hence, $f(z)$ can again be represented as:

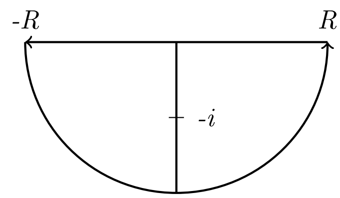
$$f(z) = \frac{\phi_{-i}(z)}{(z+i)^2} = \frac{\phi_{-i}(-i)}{(z+i)^2} + \frac{\phi_{-i}'(-i)}{(z+i)} + \phi_{-i,2}(z)$$

Therefore, based on similar reason as above (where the first and third terms are analytic or has antiderivative), with a sufficiently small circle C centered at $-i$, the residue at $-i$ is given as:

$$Res_{z=-i}f(z) = \frac{1}{2\pi i}\int_C \frac{\phi'_{-i}(-i)}{(z+i)}dz = n(C,-i)\phi'_{-i}(-i) = \frac{1}{4}(1+2\pi\zeta)ie^{-2\pi\zeta}$$

Integration for $\zeta \geq 0$:

Choose a radius $R > 1$, and consider a semicircle C_R in lower half plane parametrized by $z = Re^{-i\theta}$ with $\theta \in [0, \pi]$, and another straight line with $-R \leq x \leq R$ with the following orientation:



Since it encloses only $z = -i$, if we integrate $f(z)$ along the contour of the semicircle, we'll get:

$$\int_R^{-R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \cdot Res_{z=-i}f(z) = 2\pi i \cdot (\frac{1}{4}(1+2\pi\zeta)ie^{-2\pi\zeta}) = -\frac{\pi}{2}(1+2\pi\zeta)e^{-2\pi\zeta}$$

Now, consider the second integral above with the parametrization:

$$\int_{C_R} f(z)dz = \int_\pi^0 \frac{e^{-2\pi i\zeta Re^{-i\theta}}}{(1+(Re^{-i\theta})^2)^2} \cdot (-iR)e^{-i\theta}d\theta$$

Since $Re^{-i\theta} = R\cos(\theta) - i \cdot R\sin(\theta)$, then the exponential part could be rewrite as:

$$e^{-2\pi i\zeta Re^{-i\theta}} = e^{-2\pi i\zeta(R\cos(\theta) - i\cdot R\sin(\theta))} = e^{-2\pi R\zeta\sin(\theta)} \cdot e^{-i\cdot 2\pi R\zeta\cos(\theta)}$$

Hence, if we take the modulus, the following inequality is true:

$$\left|\int_{C_R} f(z)dz\right| = \left|-\int_0^\pi \frac{e^{-2\pi i\zeta Re^{-i\theta}}}{(1+(Re^{-i\theta})^2)^2} \cdot (-iR)e^{-i\theta}d\theta\right|$$

$$\leq \int_0^\pi \frac{|e^{-2\pi i\zeta Re^{-i\theta}}|}{|1+(Re^{-i\theta})^2|^2} \cdot |-iRe^{-i\theta}|d\theta \leq \int_0^\pi \frac{e^{-2\pi R\zeta\sin(\theta)}}{(R^2-1)^2}Rd\theta$$

Since $2\pi R\zeta\sin(\theta) \geq 0$ for $\theta \in [0, \pi]$ (since $\zeta \geq 0$ in this section), $e^{-2\pi R\zeta\sin(\theta)} \leq 1$. Then the above integral can then be bounded by:

$$\left|\int_{C_R} f(z)dz\right| \leq \int_0^\pi \frac{R}{(R^2-1)^2}d\theta = \frac{\pi R}{(R^2-1)^2}$$

So, as R grows indefinitely, we get:

$$0 \leq \lim_{R\to\infty}\left|\int_{C_R} f(z)dz\right| \leq \lim_{R\to\infty}\frac{\pi R}{(R^2-1)^2} = 0$$

Hence, $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$.

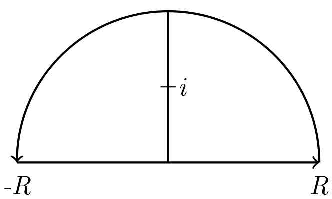
So, we can claim that $\lim_{R\to\infty}\int_R^{-R} f(x)dx + \int_{C_R} f(z)dz = \int_\infty^{-\infty} f(x)dx = -\frac{\pi}{2}(1+2\pi\zeta)e^{-2\pi\zeta}$, so $\int_{-\infty}^\infty f(x)dx = \frac{\pi}{2}(1+2\pi\zeta)e^{-2\pi\zeta}$.

Since $\zeta \geq 0$, then it can also be characterized as:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x\zeta}}{(1+x^2)^2}dx = \frac{\pi}{2}(1+2\pi|\zeta|)e^{-2\pi|\zeta|}$$

Integration for $\zeta < 0$:

Choose a radius $R > 1$, and the semicircle C_R in the upper half plane parametrized by $z = Re^{i\theta}$ with $\theta \in [0, \pi]$, and again consider a straight line with $-R \leq x \leq R$ with the following orientation:



Since it encloses only $z = i$, if integrate $f(z)$ along the contour of the semicircle, we'll get:

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \cdot Res_{z=i} f(z) = 2\pi i \cdot (-\frac{1}{4}(1-2\pi\zeta)ie^{2\pi\zeta}) = \frac{\pi}{2}(1-2\pi\zeta)e^{2\pi\zeta}$$

Then, using similar technique from previous part, we can prove that $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$.

Hence, $\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = -2\pi(1-2\pi\zeta)e^{2\pi\zeta}$.

Since $\zeta < 0$, then it is then characterized as:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x\zeta}}{(1+x^2)^2}dx = \frac{\pi}{2}(1+2\pi|\zeta|)e^{-2\pi|\zeta|}$$

So, regardless of the sign of ζ, the following integral is always true:

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\zeta}}{(1+x^2)^2}dx = \frac{\pi}{2}(1+2\pi|\zeta|)e^{-2\pi|\zeta|}$$

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Question 6 *Stein and Shakarchi Pg. 104 Problem 10:*
Show that if $a > 0$, then

$$\int_0^{\infty} \frac{\log x}{x^2+a^2}dx = \frac{\pi}{2a}\log a$$

Pf:

Choose $0 < \epsilon < a$, and $R > a$. Construct the semicircle C_ϵ and C_R for upper half plane, with C_r being characterized by $z = re^{i\theta}$ with $\theta \in [0, \pi]$. Along with two straight lines γ on real axis parametrized by $\epsilon \leq |x| \leq R$, we can create a contour with the following orientation:

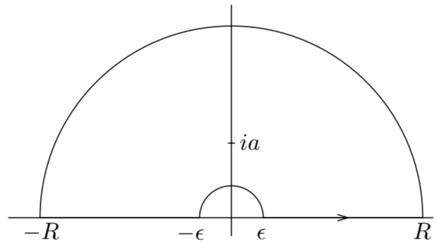

Figure 10. Contour in Exercise 10

Before starting, we need to redefine the logarithmic function, so that the region we're integrating over has a single-valued branch. Define the domain to be $\mathbb{C} \setminus \{ix \mid x \leq 0\}$, and for all z in the domain, $\log(z) = \ln|z| + i\arg(z)$, where $\arg(z) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ (so we can cover the whole real axis except for 0).

Then, for all $x < 0$, $\log(x) = \ln|x| + i\arg(x) = \ln|x| + i\pi$.

Now, if we consider the integral of $f(z) = \frac{\log(z)}{z^2+a^2} = \frac{\log(z)}{(z-ia)(z+ia)}$, the contour is enclosing the point ia. Notice that since $\frac{\log(z)}{(z+ia)}$ is analytic at ia, then choose a sufficiently small circle C centered at ia, the residue at ia is given as:

$$Res_{z=ia}f(z) = \frac{1}{2\pi i}\int_C \frac{\log(z)}{(z+ia)} \cdot \frac{1}{(z-ia)}dz = n(C, ia) \cdot \frac{\log(ia)}{(ia+ia)} = \frac{\ln(a) + i\frac{\pi}{2}}{2ia}$$

So, integrating over the contour with the chosen orientation, we get:

$$\int_{\gamma - C_\epsilon + C_R} f(z)dz = \left(\int_{-R}^{-\epsilon} f(x)dx + \int_{\epsilon}^{R} f(x)dx\right) - \int_{C_\epsilon} f(z)dz + \int_{C_R} f(z)dz$$

$$= 2\pi i \cdot Res_{z=ia}f(z) = 2\pi i \cdot \frac{\ln(a) + i\frac{\pi}{2}}{2ia} = \frac{\pi}{a}\ln(a) + i\frac{\pi^2}{2a}$$

Integral over C_R:

Given the parametrization $z = Re^{i\theta}$ with $\theta \in [0, \pi]$ for C_R, then the integral is given by:

$$\int_{C_R} f(z)dz = \int_0^\pi \frac{\log(Re^{i\theta})}{(Re^{i\theta}) + a^2} \cdot iRe^{i\theta}d\theta = \int_0^\pi \frac{\ln(R) + i\theta}{(Re^{i\theta})^2 + a^2} \cdot iRe^{i\theta}d\theta$$

Since $0 \leq \theta \leq \pi$ for variable θ, then the modulus of the integral can be bounded by:

$$\left|\int_{C_R} f(z)dz\right| \leq \int_0^\pi \frac{|\ln(R) + i\theta|}{|(Re^{i\theta})^2 + a^2|}|iRe^{i\theta}|d\theta \leq \int_0^\pi \frac{\sqrt{(\ln(R))^2 + \theta^2}}{|Re^{i\theta}|^2 - |a|^2}Rd\theta \leq \int_0^\pi \frac{\sqrt{(\ln(R))^2 + \pi^2}}{R^2 - a^2}Rd\theta$$

$$\leq \int_0^\pi \frac{|\ln(R)| + |\pi|}{R^2 - a^2}Rd\theta = \frac{\pi(|\ln(R)| + \pi)}{R^2 - a^2}R$$

WLOG, can assume the initial choice of $R \geq 1$, hence $\ln(R) \geq 0$, so $|\ln(R)| = \ln(R)$.

Then, as $R \to \infty$, we get:

$$0 \leq \lim_{R\to\infty}\left|\int_{C_R} f(z)dz\right| \leq \lim_{R\to\infty}\frac{\pi(\ln(R) + \pi)R}{R^2 - a^2} = \lim_{R\to\infty}\frac{\pi(\ln(R) + 1 + \pi)}{2R} = \lim_{R\to\infty}\frac{\pi/R}{2} = 0$$

(Note: the above limit is given by L'hopital's Rule).

Hence, $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$.

Integral over C_ϵ:

Given the parametrization $z = \epsilon e^{i\theta}$ with $\theta \in [0, \pi]$ for C_ϵ, then the integral is given by:

$$\int_{C_\epsilon} f(z)dz = \int_0^\pi \frac{\log(\epsilon e^{i\theta})}{(\epsilon e^{i\theta})^2 + a^2} i\epsilon e^{i\theta} d\theta = \int_0^\pi \frac{\ln(\epsilon) + i\theta}{(\epsilon e^{i\theta})^2 + a^2} i\epsilon e^{i\theta} d\theta$$

Based on similar argument, the modulus of the integral can be bounded by:

$$\left| \int_{C_\epsilon} f(z)dz \right| \le \int_0^\pi \frac{|\ln(\epsilon) + i\theta|}{|(\epsilon e^{i\theta})^2 + a^2|} |i\epsilon e^{i\theta}| d\theta \le \int_0^\pi \frac{\sqrt{(\ln(\epsilon))^2 + \theta^2}}{||\epsilon e^{i\theta}|^2 - a^2|} \epsilon d\theta \le \int_0^\pi \frac{|\ln(\epsilon)| + |\theta|}{a^2 - \epsilon^2} \epsilon d\theta$$

$$\le \int_0^\pi \frac{|\ln(\epsilon)| + |\pi|}{a^2 - \epsilon^2} \epsilon d\theta \le \frac{\pi(|ln(\epsilon)| + \pi)}{a^2 - \epsilon^2} \epsilon = \frac{\pi |\ln(\epsilon)| \epsilon + \pi^2 \epsilon}{a^2 - \epsilon^2}$$

WLOG, can assume $\epsilon < 1$, hence $\ln(\epsilon) < 0$, or $|\ln(\epsilon)| = -\ln(\epsilon)$ for simplicity.

Then, as $\epsilon \to 0$, the following limits are true:

$$\lim_{\epsilon \to 0^+} \frac{1}{a^2 - \epsilon^2} = \frac{1}{a^2}, \quad \lim_{\epsilon \to 0^+} \pi^2 \epsilon = 0, \quad \lim_{\epsilon \to 0^+} -\pi \ln(\epsilon)\epsilon = \lim_{\epsilon \to 0^+} -\pi \frac{\ln(\epsilon)}{1/\epsilon} = \lim_{\epsilon \to 0^+} -\pi \frac{1/\epsilon}{-1/\epsilon^2} = \lim_{\epsilon \to 0^+} \pi\epsilon = 0$$

Hence:

$$\lim_{\epsilon \to 0^+} \frac{\pi |\ln(\epsilon)| \epsilon + \pi^2 \epsilon}{a^2 - \epsilon^2} = \lim_{\epsilon \to 0^+} \frac{-\pi \ln(\epsilon)\epsilon + \pi^2\epsilon}{a^2 - \epsilon^2} = \frac{0 + 0}{a^2} = 0$$

So, $\lim_{\epsilon \to 0^+} \int_{C_\epsilon} f(z)dz = 0$.

Original Integral:

To retrieve the original integral $\int_0^\infty \frac{\log(x)}{x^2+a^2} dx$, we need $R \to \infty$ and $\epsilon \to 0^+$. So, the following is true:

$$\lim_{R \to \infty} \lim_{\epsilon \to 0^+} \left(\int_{-R}^{-\epsilon} f(x)dx + \int_\epsilon^R f(x)dx \right) - \int_{C_\epsilon} f(z)dz + \int_{C_R} f(z)dz$$

$$= \int_{-\infty}^{0^-} f(x)dx + \int_{0^+}^\infty f(x)dx - \lim_{\epsilon \to 0^+} \int_{C_\epsilon} f(z)dz + \lim_{R \to \infty} \int_{C_R} f(z)dz$$

$$= \int_{-\infty}^{0^-} f(x)dx + \int_{0^+}^\infty f(x)dx$$

Input the function $f(z)$, we get:

$$\int_{-\infty}^{0^-} f(x)dx + \int_{0^+}^\infty f(x)dx = \int_{-\infty}^{0^-} \frac{\log(x)}{x^2 + a^2} dx + \int_{0^+}^\infty \frac{\log(x)}{x^2 + a^2} dx$$

$$= \int_{-\infty}^{0^-} \frac{\ln|x| + i\pi}{x^2 + a^2} dx + \int_{0^+}^\infty \frac{\ln|x|}{x^2 + a^2} dx = \left(\int_{-\infty}^{0^-} \frac{\ln|x|}{x^2 + a^2} dx + \int_{0^+}^\infty \frac{\ln|x|}{x^2 + a^2} dx \right) + i \int_{-\infty}^{0^-} \frac{\pi}{x^2 + a^2} dx$$

Also, recall that the above quantity equals to $\frac{\pi}{a}\ln(a) + i\frac{\pi^2}{2a}$ by Residue Formula. Then:

$$\left(\int_{-\infty}^{0^-} \frac{\ln|x|}{x^2 + a^2} dx + \int_{0^+}^\infty \frac{\ln|x|}{x^2 + a^2} dx \right) = Re\left(\frac{\pi}{a}\ln(a) + i\frac{\pi^2}{2a} \right) = \frac{\pi}{a}\ln(a)$$

Lastly, since the function $\frac{ln|x|}{x^2+a^2}$ is in fact an even function, then $\int_{0^+}^\infty \frac{\ln|x|}{x^2+a^2} dx$ is half of the above quantity, or $\frac{\pi}{2a}\ln(a)$. Hence:

$$\int_0^\infty \frac{\ln(x)}{x^2 + a^2} dx = \frac{\pi}{2a}\ln(a)$$

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Question 7 *Stein and Shakarchi Pg. 104 Problem 11 :*
Show that if $|a| < 1$, then

$$\int_0^{2\pi} \log|1 - ae^{i\theta}| d\theta = 0$$

Then, prove that the above result remains true if we assume only that $|a| \leq 1$.

Pf:

When $|a| < 1$:

If $a = 0$, the case is trivial (since $\log|1 - ae^{i\theta}| = \log|1| = 0$, then the whole integral is 0). So, assume $a \neq 0$.

Consider the integral of $\log(1 - z)/iz$ along a circle C of radius $|a| < 1$ centered at 0. With the parametrization $z = ae^{i\theta}$ ($\theta \in [0, 2\pi]$), it can be expressed as:

$$I = \int_C \frac{\log(1 - z)}{iz} dz = \int_0^{2\pi} \frac{\log(1 - ae^{i\theta})}{iae^{i\theta}}(iae^{i\theta}) d\theta = \int_0^{2\pi} \log(1 - ae^{i\theta}) d\theta$$

Which, define the domain to be $\mathbb{C} \setminus \{x \geq 1\}$, and $\log(1 - z) = \ln|1 - z| + i \arg(1 - z)$, it can also be expressed as:

$$I = \int_0^{2\pi} \log(1 - ae^{i\theta}) d\theta = \int_0^{2\pi} \ln|1 - ae^{i\theta}| d\theta + i \int_0^{2\pi} \arg(1 - ae^{i\theta}) d\theta$$

Going back to the original integral, since the function $-\frac{\log(1-z)}{i}$ is analytic on the domain $\mathbb{C} \setminus \{x \geq 1\}$, so on the disk enclosed by C, the only Pole is generated by $\frac{1}{z}$ (at the origin). Hence, let $\phi(z) = \frac{\log(1-z)}{i}$, the integral is then characterized by Cauchy's Integral Formula:

$$\int_C \frac{\log(1 - z)}{iz} dz = \int_C \frac{\phi(z)}{z} dz = 2\pi i \cdot n(C, 0)\phi(0)$$

With $n(C, 0) = 1$ (winding number 1 by our construction), and $\phi(0) = \log(1 - 0)/(i \cdot 1) = 0$, then such integral is evaluated to be 0.

Now, since $Re(I) = \int_0^{2\pi} \log|1 - ae^{i\theta}| d\theta$, while $I = 0$, then this integral must also evaluated to be 0.

Case when $|a| = 1$ (Real Integral Approach):

For $|a| = 1$, WLOG, we can assume $a = 1$ (since other than that, any $e^{i\phi}$ is adding a phase of ϕ, which by changing the range of our integrand, we can modify it to be the same as $a = 1$).

Then, the integral can be rewrite as:

$$\int_0^{2\pi} \ln|1 - e^{i\theta}| d\theta = \int_0^{2\pi} \ln|1 - (\cos(\theta) + i\sin(\theta))| d\theta = \int_0^{2\pi} \ln\sqrt{(1 - \cos(\theta))^2 + (-\sin(\theta))^2} d\theta$$

$$= \int_0^{2\pi} \ln\sqrt{1 + \cos^2(\theta) - 2\cos(\theta) + \sin^2(\theta)} d\theta = \int_0^{2\pi} \ln\left((2 - 2\cos(\theta))^{\frac{1}{2}}\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \ln\left(4\left(\frac{1 - \cos(\theta)}{2}\right)\right) d\theta = \frac{1}{2}\int_0^{2\pi} \ln\left(2^2 \cdot \sin^2\left(\frac{\theta}{2}\right)\right) d\theta$$

$$= \frac{1}{2} \cdot 2 \int_0^{2\pi} \ln\left(2\sin\left(\frac{\theta}{2}\right)\right) d\theta$$

Using the substitution $u = \frac{\theta}{2}$ (with $2du = d\theta$), it can be modified to:

$$\frac{1}{2} \cdot 2 \int_0^{2\pi} \ln\left(2\sin\left(\frac{\theta}{2}\right)\right) d\theta = 2\int_0^{\pi} \ln(2\sin(u))du = 2\int_0^{\pi} (\ln(2) + \ln(\sin(u)))du$$

$$= 2\pi\ln(2) + 2\int_0^{\pi} \ln(\sin(u))du$$

So, it suffices to show the integral of $\ln(\sin(u))$ from 0 to π.

Notice that since $\sin(u) = \sin(\pi - u)$, then for the following integral, do the substitution $\theta = \pi - u$ (or $-d\theta = du$), we can do the following:

$$\int_{\frac{\pi}{2}}^{\pi} \ln(\sin(u))du = \int_{\frac{\pi}{2}}^{\pi} \ln(\sin(\pi - u))du = -\int_{\frac{\pi}{2}}^{0} \ln(\sin(\theta))d\theta = \int_0^{\frac{\pi}{2}} \ln(\sin(\theta))d\theta$$

So, the following is true:

$$\int_0^{\pi} \ln(\sin(u))du = 2\int_0^{\frac{\pi}{2}} \ln(\sin(u))du$$

Let $I = \int_0^{\frac{\pi}{2}} \ln(\sin(u))du$, using the fact that $\sin(u) = \cos(\frac{\pi}{2} - u)$, with substitution $\theta = \frac{\pi}{2} - u$ (or $-d\theta = du$), we can again get:

$$I = \int_0^{\frac{\pi}{2}} \ln\left(\cos\left(\frac{\pi}{2} - u\right)\right) du = -\int_{\frac{\pi}{2}}^{0} \ln(\cos(\theta))d\theta = \int_0^{\frac{\pi}{2}} \ln(\cos(\theta))d\theta$$

So, we can also get the following:

$$2I = \int_0^{\frac{\pi}{2}} \ln(\sin(u))du + \int_0^{\frac{\pi}{2}} \ln(\cos(u))du = \int_0^{\frac{\pi}{2}} \ln(\sin(u)\cos(u))du = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin(2u)\right) du$$

$$= \int_0^{\frac{\pi}{2}} \left(\ln\left(\frac{1}{2}\right) + \ln(\sin(2u))\right) du = -\frac{\pi}{2}\ln(2) + \int_0^{\frac{\pi}{2}} \ln(\sin(2u))du$$

Using the substitution $\theta = 2u$ (with $\frac{1}{2}d\theta = du$), we get the following:

$$2I = -\frac{\pi}{2}\ln(2) + \frac{1}{2}\int_0^{\pi} \ln(\sin(\theta))d\theta$$

But, recall the fact that $\int_0^{\pi} \ln(\sin(u))du = 2\int_0^{\frac{\pi}{2}} \ln(\sin(u))du = 2I$. So, we gt:

$$2I = -\frac{\pi}{2}\ln(2) + \frac{1}{2} \cdot 2I = -\frac{\pi}{2}\ln(2) + I, \quad I = -\frac{\pi}{2}\ln(2)$$

Substituting back to the original integral, we get the following:

$$\int_0^{\pi} \ln(\sin(u))du = 2I = -\pi\ln(2)$$

Finally, we get that the original integral is given by:

$$\int_0^{2\pi} \ln|1 - e^{i\theta}|d\theta = 2\pi\ln(2) + 2\int_0^{\pi} \ln(\sin(u))du = 2\pi\ln(2) + 2(-\pi\ln(2)) = 0$$

Hence, we can say it is possible to extend to the case $|a| = 1$, so the integral is true for all $|a| \leq 1$.