Math 118B HW5

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Question 1

(a) *Show that there exists a sequence of polynomials* $q_m : [0,1] \to \mathbb{R}$ *such that for each* $x \in [0,1]$

$$\lim_{m\to\infty} q_m(x) = 0$$

(pointwise convergence) but it does not converge uniformly.

(b) *Prove that if a sequence of polynomial* $p_m : [0,1] \to \mathbb{R}$ *converges pointwise to* 0 *and for all* $m \in \mathbb{N}$ *one has that* $\deg(p_m) \leq 100$*, then the* p_m *converges uniformly to* 0.

Pf:

Theorem 1

(a) **Continuous Functions Converging to** 0 **Pointwise, but not Uniformly:**

We'll first construct a sequence of continuous functions converging to 0 pointwise, but not uniformly. For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined as:

$$f_n(x) = \begin{cases} 4nx - 2 & x \in [\frac{2}{4n}, \frac{3}{4n}] \\ -4nx + 4 & x \in (\frac{3}{4n}, \frac{4}{4n}] \\ 0 & x \notin [\frac{2}{4n}, \frac{4}{4n}] \end{cases}$$

This is a continuous function for all $n \in \mathbb{N}$, since the limit at $\frac{3}{4n}$, $\frac{2}{4n}$, and $\frac{4}{4n}$ all agrees with the function f_n's actual values.

However, since at $x = \frac{3}{4n} \in [0,1]$, $f_n(x) = 4n \cdot \frac{3}{4n} - 2 = 3 - 2 = 1$, then $\|f_n\|_\infty = \sup_{x\in[0,1]} |f_n(x)| \geq 1$, showing that f_n doesn't converge to 0 uniformly (since the norm $\|\cdot\|_\infty$ is at least 1 for all $n \in \mathbb{N}$).

Sequence of Polynomials:

Now, since f_n is continuous on $[0,1]$, by Stone-Weierstrass Theorem, there exists a sequence of polynomials $\{q_{n,k}\}_{k\in\mathbb{N}}$ that converges to f_n uniformly.

For all $n \in \mathbb{N}$, since $\frac{1}{n} > 0$, by the uniform convergence of $\{q_{n,k}\}_{k\in\mathbb{N}}$ onto f_n, there exists N_n, such that $k_n \geq N_n$ implies $\|f_n - q_{n,k_n}\|_\infty < \frac{1}{n}$ (for simplicity, fix k_n to be the smallest integer with $k_n \geq N_n$). For the rest of the proof of **Part (a)**, consider the sequence of polynomials $\{q_{n,k_n}\}_{n\in\mathbb{N}}$.

The Sequence Pointwise Converges to 0:

For all $x \in [0,1]$, there are two cases to consider:

- First, if $x = 0$, for all $n \in \mathbb{N}$, we have $f_n(0) = 0$. Then, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$, with $\frac{1}{N} < \epsilon$ based on Archimedean's Property. For all $n \geq N$ (which $\frac{1}{n} \leq \frac{1}{N} < \epsilon$), the previous choice of polynomials satisfy:

$$|q_{n,k_n}(0)| = |q_{n,k_n}(0) - f_n(0)| \leq \|q_{n,k_n} - f_n\|_\infty < \frac{1}{n} < \epsilon$$

  Hence, this states that $\lim_{n\to\infty} q_{n,k_n}(0) = 0$.

- Else if $x \neq 0$ (which $x > 0$ since $x \in [0,1]$), there exists $N \in \mathbb{N}$, such that $\frac{1}{N} < x$ based on Archimedean's Property. Then, for all $n \geq N$, since $\frac{4}{4n} = \frac{1}{n} \leq \frac{1}{N} < x$, $f_n(x) = 0$ (since $x \notin [\frac{2}{4n}, \frac{4}{4n}]$).

  Again, for all $\epsilon > 0$, there exists $M \in \mathbb{N}$, with $\frac{1}{M} < \epsilon$ again based on Archimedean's Property. Choose $K = \max\{M, N\}$, for all $n \geq K$ (which $n \geq N$, showing that $f_n(x) = 0$; and $n \geq M$, showing that $\frac{1}{n} \leq \frac{1}{M} < \epsilon$), the previous choice of polynomials satisfy:

$$|q_{n,k_n}(x)| = |q_{n,k_n}(x) - f_n(x)| \leq \|q_{n,k_n} - f_n\|_\infty < \frac{1}{n} < \epsilon$$

  Hence, this states that $\lim_{n\to\infty} q_{n,k_n}(x) = 0$.

So, regardless of the case, $\lim_{n\to\infty} q_{n,k_n}(x) = 0$, showing that $\{q_{n,k_n}\}_{n\in\mathbb{N}}$ converges pointwise to 0.

The Convergence is not Uniform:

Recall that for all $n \in \mathbb{N}$, $\|f_n\|_\infty \geq 1$, and $\|f_n - q_{n,k_n}\|_\infty < \frac{1}{n}$. Hence, for $n \geq 2$ (which $\frac{1}{n} \leq \frac{1}{2}$), the following inequality is true:

$$\|q_{n,k_n}\|_\infty = \|(q_{n,k_n} - f_n) - (-f_n)\|_\infty \geq \Big| \|q_{n,k_n} - f_n\|_\infty - \|-f_n\|_\infty \Big| = \|f_n\|_\infty - \|q_{n,k_n} - f_n\|_\infty$$

$$\|q_{n,k_n}\|_\infty \geq \|f_n\|_\infty - \|q_{n,k_n} - f_n\|_\infty \geq 1 - \|q_{n,k_n} - f_n\|_\infty > 1 - \frac{1}{n} \geq 1 - \frac{1}{2} = \frac{1}{2}$$

So, since $\|q_{n,k_n}\|_\infty \geq \frac{1}{2}$ for all $n \geq 2$, the $\lim_{n\to\infty} \|q_{n,k_n}\|_\infty \neq 0$, showing that $\{q_{n,k_n}\}_{n\in\mathbb{N}}$ doesn't converge to 0 uniformly.

In Conclusion, $\{q_{n,k_n}\}_{n\in\mathbb{N}}$ constructed above, is a sequence of polynomial that converges pointwise to 0, yet it doesn't converge uniformly to 0. Which, it is a desired sequence for the question.

(b) Let $\mathcal{P}_{100}([0,1])$ be the real vector space of polynomial defined on $[0,1]$ with degree at most 100 (which $\dim(\mathcal{P}_{100}([0,1])) = 101$). For this part, the sequence $\{p_m\}_{m\in\mathbb{N}} \subset \mathcal{P}_{100}([0,1])$, and they converges pointwise to 0. Which, for each $m \in \mathbb{N}$, $p_m(x) = a_{0,m} + a_{1,m}x + ... + a_{100,m}x^{100}$ for some $\overline{a_m} = (a_{0,m}, a_{1,m}, ..., a_{100,m}) \in \mathbb{R}^{101}$.

Now, as a tool for problem solving, choose distinct points $x_0, x_1, ..., x_{100} \in [0,1]$. For all $(a_0, a_1, ..., a_{100}) \in \mathbb{R}^{101}$, let $p \in \mathcal{P}_{100}([0,1])$ satisfy $p(x) = a_0 + a_1x + ... + a_{100}x^{100}$. Define the map $T : \mathbb{R}^{101} \to \mathbb{R}^{101}$ as follow:

$$T(a_0, a_1, ..., a_{101}) = (p(x_0), p(x_1), ..., p(x_{100}))$$

T is a Linear Map:

For all $u = (u_0, u_1, ..., u_{100}), v = (v_0, v_1, ..., v_{100}) \in \mathbb{R}^{101}$ and $a, b \in \mathbb{R}$. Let $p, q \in \mathcal{P}_{100}([0,1])$ be defined as:

$$p(x) = u_0 + u_1x + ... + u_{100}x^{100}, \quad q(x) = v_0 + v_1x + ... + v_{100}x^{100}$$

Hence, $au + bv = (au_0 + bv_0, au_1 + bv_1, ..., au_{100} + bv_{100})$ corresponds to the following polynomial:

$$(au_0 + bv_0) + (au_1 + bv_1)x + ... + (au_{100} + bv_{100})x^{100}$$

$$= (au_0 + au_1x + ... + au_{100}x^{100}) + (bv_0 + bv_1x + ... + bv_{100}x^{100})$$

$$= a(u_0 + u_1x + ... + u_{100}x^{100}) + b(v_0 + v_1x + ... + v_{100}x^{100})$$

$$= ap(x) + bq(x)$$

Now, for $\bar{0} \in \mathbb{R}^{101}$, since it corresponds to the zero polynomial $0 + 0x + ... + 0x^{100}$, then $T(\bar{0}) = \bar{0}$.

Also, the linearity is satisfied:

$$T(au + bv) = (ap(x_0) + bq(x_0), ap(x_1) + bq(x_1), ..., ap(x_{100}) + bq(x_{100}))$$

$$= a(p(x_0), p(x_1), ..., p(x_{100})) + b(q(x_0), q(x_1), ..., q(x_{100}))$$

$$= aT(u) + bT(v)$$

The above statements showed that T is a linear map.

T is Bijective, hence T^{-1} Exists:

Since T is a linear operator on $\mathbb{R}^{101}$ (which is finite dimensional), it suffices to show that T is injective.

Suppose $v = (v_0, v_1, ..., v_{100}) \in \mathbb{R}^{101}$ with the corresponding polynomial $q(x) = v_0 + v_1x + ... + v_{100}x^{100}$ satisfies $T(v) = \bar{0}$ (or $v \in \ker(T)$). Then, $T(v) = (q(x_0), q(x_1), ..., q(x_{100})) = \bar{0}$, showing that q as a polynomial has 101 distinct roots. However, since by assumption, if $q \neq 0$, then since its degree is at most 100, by Fundamental Theorem of Algebra, it could have at most 100 distinct roots. Hence, this enforces $q = 0$ (with all coefficients being 0), showing that $v = \bar{0}$.

Hence, $\ker(T) = \{\bar{0}\}$, showing that T is injective, which is equivalent to T is bijective. Then, T^{-1} exists, and it is also bijective.

T^{-1} is Continuous:

For $\mathbb{R}^{101}$ both the domain and codomain of T^{-1}, use the usual Euclidean Inner Product to define the usual norm. Then, since T^{-1} is a bijective linear operator between inner product space, by Singular Value Decomposition, there exists two orthonormal bases $\{e_0, e_1, ..., e_{100}\} \subset \mathbb{R}^{101}$, $\{f_0, f_1, ..., f_{100}\} \subset \mathbb{R}^{101}$, and positive real numbers $s_0, s_1, ..., s_{100} > 0$, such that the following is true:

$$\forall v \in \mathbb{R}^{101}, \quad T^{-1}(v) = \sum_{i=0}^{100} s_i \langle v, e_i \rangle f_i$$

Which, let $s = \max\{s_0, s_1, ..., s_{100}\} > 0$. Based on the property of orthonormal basis, the following equations and inequalities are true for the norm:

$$\|v\|^2 = \sum_{i=0}^{100} |\langle v, e_i \rangle|^2$$

$$\|T^{-1}(v)\|^2 = \left\| \sum_{i=0}^{100} s_i \langle v, e_i \rangle f_i \right\|^2 = \sum_{i=0}^{100} \|s_i \langle v, e_i \rangle f_i\|^2 = \sum_{i=0}^{100} |s_i \langle v, e_i \rangle|^2$$

$$\|T^{-1}(v)\|^2 = \sum_{i=0}^{100} s_i^2 |\langle v, e_i \rangle|^2 \leq \sum_{i=0}^{100} s^2 |\langle v, e_i \rangle|^2 = s^2 \|v\|^2$$

Hence, $\|T^{-1}(v)\| \leq s\|v\|$.

Then, for all $\epsilon > 0$, define $\delta = \frac{\epsilon}{s} > 0$ For all $u, v \in \mathbb{R}^{101}$, if $\|u - v\| < \delta = \frac{\epsilon}{s}$, the following is true:

$$\|T^{-1}(u) - T^{-1}(v)\| = \|T^{-1}(u - v)\| \leq s\|u - v\| < s \cdot \frac{\epsilon}{s} = \epsilon$$

This shows that T^{-1} is uniformly continuous.

The sequence $\{T(\overline{a_m})\}_{m \in \mathbb{N}}$ converges to $\bar{0}$:

Recall initially that for each $p_m(x) = a_{0,m} + a_{1,m}x + ... + a_{100,m}x^{100}$, the vector $\overline{a_m} = (a_{0,m}, a_{1,m}, ..., a_{100,m})$ satisfies:

$$T(\overline{a_m}) = (p_m(x_0), p_m(x_1), ..., p_m(x_{100}))$$

Then, since the sequence p_m converges to 0 pointwise, for all $\epsilon > 0$ (with $\frac{\epsilon}{\sqrt{101}} > 0$), each $j \in \{0, 1, ..., 100\}$ has a corresponding N_j, such that $m \geq N_j$ implies $|p_m(x_j)| < \frac{\epsilon}{\sqrt{101}}$.

Now, choose $N = \max\{N_0, N_1, ..., N_{100}\}$. For all $m \geq N$ (which $m \geq N_j$ for each individual $j \in \{0, 1, ..., 100\}$), then $|p_m(x_j)| < \frac{\epsilon}{\sqrt{101}}$ for each index j. Hence, the following is true:

$$\|T(\overline{a_m})\| = \|(p_m(x_0), p_m(x_1), ..., p_m(x_{100}))\| = \sqrt{\sum_{j=0}^{100} |p_m(x_j)|^2} < \sqrt{\sum_{j=0}^{100} \left(\frac{\epsilon}{\sqrt{101}}\right)^2} = \sqrt{\epsilon^2} = \epsilon$$

This shows that $\lim_{m\to\infty} T(\overline{a_m}) = \bar{0}$.

The sequence $\{\overline{a_m}\}_{m \in \mathbb{N}}$ converges to $\bar{0}$: Since T^{-1} is continuous, and $\lim_{m\to\infty} T(\overline{a_m}) = \bar{0}$, then:

$$\lim_{m\to\infty} \overline{a_m} = \lim_{m\to\infty} T^{-1}(T(\overline{a_m})) = T^{-1}(\bar{0}) = \bar{0}$$

Hence, for all $\epsilon > 0$, there exists N, with $m \geq N$ implies $\|\overline{a_m}\| < \epsilon$.

The Sequence p_m converges uniformly to 0:

From the previous statement, for all $\epsilon > 0$ (which $\frac{\epsilon}{101} > 0$), there exists N, with $m \geq N$ implies $\|\overline{a_m}\| < \frac{\epsilon}{101}$.

Then, with $\overline{a_m} = (a_{0,m}, a_{1,m}, ..., a_{100,m})$, the following is true:

$$\forall j \in \{0, 1, ..., 100\}, \quad |a_{j,m}| = \sqrt{|a_{j,m}|^2} \leq \sqrt{\sum_{j=0}^{100} |a_{j,m}|^2} = \|\overline{a_m}\| < \frac{\epsilon}{101}$$

Hence, for all $x \in [0, 1]$, the following is true:

$$|p_m(x)| = \left|\sum_{j=0}^{100} a_{j,m} x^j\right| \leq \sum_{j=1}^{100} |a_{j,m}| \cdot |x^j| \leq \sum_{j=0}^{100} |a_{j,m}| < \sum_{j=0}^{100} \frac{\epsilon}{101} = \epsilon$$

This shows that $\|p_m\|_\infty = \sup_{x\in[0,1]} |p_m(x)| \leq \epsilon$. Which, the above statement proves that p_m converges uniformly to 0.

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Question 2 *Let $f : [0, 1] \to \mathbb{R}$ be a function such that $f', f'', f^{(3)}$ are defined and continuous in $[0, 1]$. Prove that for any $\epsilon > 0$ there exists a polynomial P such that*

$$\sum_{j=0}^{3} \|f^{(j)} - P^{(j)}\|_\infty = \sum_{j=0}^{3} \sup_{x\in[0,1]} |(f^{(j)} - P^{(j)})(x)| < \epsilon$$

Pf:

Before starting the prove, recall that the antiderivatives of a polynomial $p : [0, 1] \to \mathbb{R}$ is a collection of polynomials $\{P(x) + C \mid C \in \mathbb{R}\}$, where $P : [0, 1] \to \mathbb{R}$ is a polynomial satisfying $P' = p$.

When taking the antiderivative of any polynomial in the following steps, we'll explicitly state the initial condition to prevent ambiguity about the constant coefficients of the antiderivative.

Generalized Statement:

We'll prove a more general version recursively: For all $n \in \mathbb{N}$, let $f : [0, 1] \to \mathbb{R}$ be a function such that $f', ..., f^{(n)}$ are all defined and continuous on $[0, 1]$, then there exists a sequence of polynomials $\{P_m\}_{m\in\mathbb{N}}$, such that for all $j \in \{0, 1, ..., n\}$, $P_m^{(j)}$ converges to $f^{(j)}$ uniformly.

For base case, since $f^{(n)}$ is defined and continuous on $[0, 1]$, by Stone-Weierstrass Theorem, there exists a sequence of polynomials $\{p_{n,m}\}$ converging to $f^{(n)}$ uniformly.

Then as **Step (1)**, for all $m \in \mathbb{N}$, let polynomial $p_{(n-1),m} : [0, 1] \to \mathbb{R}$ be an antiderivative of $p_{n,m}$ ($p'_{(n-1),m} = p_{n,m}$) such that $p_{(n-1),m}(0) = f^{(n-1)}(0)$.

Which, since the sequence of polynomials $\{p_{(n-1),m}\}_{m\in\mathbb{N}}$ satisfies: $p'_{(n-1),m} = p_{n,m}$ converges to $(f^{(n-1)})' = f^{(n)}$ uniformly, and $\lim_{m\to\infty} p_{(n-1),m}(0) = f^{(n-1)}(0)$. Then, the sequence $p_{(n-1),m}$ converges to $f^{(n-1)}$ uniformly.

Now, for given $k \in \{1, ..., n-1\}$, at **Step (k)** we constructed a sequence of k^{th} antiderivative of the sequence of polynomials $\{p_{n,m}\}_{m\in\mathbb{N}}$ (denoted as $\{p_{(n-k),m}\}_{m\in\mathbb{N}}$), such that $p_{(n-k),m}$ converges to $f^{(n-k)}$ uniformly:

At **Step (k+1)**, for each $m \in \mathbb{N}$, let polynomial $p_{(n-(k+1)),m} : [0,1] \to \mathbb{R}$ be an antiderivative of $p_{(n-k),m}$ (which $p'_{(n-(k+1)),m} = p_{(n-k),m}$) such that $p_{(n-(k+1)),m}(0) = f^{(n-(k+1))}(0)$.

Which, since the new sequence of polynomials $\{p_{(n-(k+1)),m}\}_{m\in\mathbb{N}}$ satisfies: $p'_{(n-(k+1)),m} = p_{(n-k),m}$ converges to $(f^{(n-(k+1))})' = f^{(n-k)}$, and $\lim_{m\to\infty} p_{(n-(k+1)),m}(0) = f^{(n-(k+1))}(0)$. Then, the sequence $p_{(n-(k+1)),m}$ converges to $f^{(n-(k+1))}$ uniformly.

From the above process, since for all $k \in \{1, ..., n\}$, we can find a sequence of k^{th} antiderivative of polynomials $\{p_{n,m}\}_{m\in\mathbb{N}}$, denoted as $\{p_{(n-k),m}\}_{m\in\mathbb{N}}$, that converges to $f^{(n-k)}$ uniformly.

Then, $\{p_{0,m}\}_{m\in\mathbb{N}}$ is a sequence of polynomial that converges to $f^{(0)} = f$ uniformly. Which, for $j \in \{1, ..., n\}$, the sequence of j^{th} derivative $\{p_{j,m}\}_{m\in\mathbb{N}}$ converges uniformly to the j^{th} derivative of f, namely $f^{(j)}$. (Note: Recall that for all $j \in \{1, ..., n\}$ and all $m \in \mathbb{N}$, $p_{(j-1),m}$ is defined as an antiderivative of $p_{j,m}$).

Hence, the sequence of polynomials $\{p_{0,m}\}_{m\in\mathbb{N}}$ has its j^{th} derivative converges to $f^{(j)}$ uniformly for all given $f^{(j)}$, satisfying the desired condition stated initially.

The Original Problem:

From the above Generalized Statement, given $f : [0,1] \to \mathbb{R}$ such that $f', f'', f^{(3)}$ that are all defined and continuous on $[0,1]$, there exists a sequence of polynomials $\{P_m\}_{m\in\mathbb{N}}$, such that for $j \in \{0,1,2,3\}$, its j^{th} derivative $P_m^{(j)}$ converges to $f^{(j)}$ uniformly.

Hence, given arbitrary $\epsilon > 0$ (which $\frac{\epsilon}{4} > 0$), for each $j \in \{0,1,2,3\}$, there is a corresponding N_j, such that the following is true:

$$\forall m \in \mathbb{N}, \quad m \geq N_j \implies \|f^{(j)} - P_m^{(j)}\|_\infty < \frac{\epsilon}{4}$$

Then, choose $N = \max_{j\in\{0,1,2,3\}} N_j$, for any index $m \geq N$, since $m \geq N_j$ for all $j \in \{0,1,2,3\}$, the above statement guarantees $\|f^{(j)} - P_m^{(j)}\|_\infty < \frac{\epsilon}{4}$ for each j. Hence, the following inequality is true:

$$\sum_{j=0}^{3} \|f^{(j)} - P_m^{(j)}\|_\infty < \sum_{j=0}^{3} \frac{\epsilon}{4} = \epsilon$$

Therefore, for every $\epsilon > 0$, we can find a corresponding polynomial P, such that $\sum_{j=0}^{3} \|f^{(j)} - P^{(j)}\|_\infty < \epsilon$.

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Question 3 *Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that*

$$\int_0^1 f(x)x^j dx = 0, \qquad j = 0, 1, 2, ...$$

Prove that $f(x) = 0$, $\forall x \in [0, 1]$.

Pf:

Since $f(x)$ is continuous on $[0, 1]$ a bounded closed interval, by Stone-Weierstrass Theorem, there exists a sequence of polynomial $\{p_n\}_{n\in\mathbb{N}}$, such that p_n converges to f uniformly.

Now, notice that for all polynomial $p(x) = a_0 + a_1x + ... + a_mx^m$ (where $a_0, a_1, ..., a_m \in \mathbb{R}$), the following integral is true based on the Linearity of Riemann Integrable functions:

$$\int_0^1 f(x)p(x)dx = \int_0^1 f(x)\sum_{k=0}^{m} a_kx^k dx = \sum_{k=0}^{m} a_k \int_0^1 f(x)x^k dx = 0$$

Hence, for all $n \in \mathbb{N}$, we have $\int_0^1 f(x)p_n(x)dx = 0$.

fp_n Converges Uniformly to f^2:

Because f is continuous on $[0, 1]$ a compact set, hence f is bounded, there exists $M > 0$, such that all $x \in [0, 1]$ satisfies $|f(x)| < M$.

Also, since p_n converges to f uniformly, for all $\epsilon > 0$ (which $\frac{\epsilon}{M} > 0$), there exists N, such that $n \geq N$ implies $\|f - p_n\|_\infty < \frac{\epsilon}{M}$.

Hence, for all $n \geq N$, every $x \in [0, 1]$ satisfies the following:

$$|f(x)p_n(x) - (f(x))^2| = |f(x)| \cdot |p_n(x) - f(x)| < M \cdot |p_n(x) - f(x)| \leq M \cdot \|f - p_n\|_\infty < M \cdot \frac{\epsilon}{M} < \epsilon$$

Hence, ϵ is an upper bound of the set $\{|f(x)p_n(x) - (f(x))^2| \ : \ x \in [0, 1]\}$, showing that $\|fp_n - f^2\|_\infty = \sup_{x\in[0,1]} |f(x)p_n(x) - (f(x))^2| \leq \epsilon$. Based on the above statement, we can conclude that fp_n converges uniformly to f^2.

Integral of fp_n converges to Integral of f^2:

For all $n \in \mathbb{N}$, we have fp_n being continuous on $[0, 1]$ (since both f and p_n are continuous on $[0, 1]$), and fp_n converges to f^2 uniformly, hence the following is true:

$$\lim_{n\to\infty} \int_0^1 f(x)p_n(x)dx = \int_0^1 \lim_{n\to\infty} f(x)p_n(x)dx = \int_0^1 (f(x))^2 dx$$

Since $\int_0^1 f(x)p_n(x)dx = 0$, then the limit above is 0, hence $\int_0^1 (f(x))^2 dx = 0$.

Integral of f^2 is 0 implies $f = 0$:

Since f is continuous on $[0, 1]$, so does f^2; then, since for all $x \in [0, 1]$, $(f(x))^2 \geq 0$, together with the statement $\int_0^1 (f(x))^2 dx = 0$, this implies that $(f(x))^2 = 0$ for all $x \in [0, 1]$.

Therefore, $f(x) = 0$ for all $x \in [0, 1]$.