Math CS 122A HW3

Zih-Yu Hsieh

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Question 1 *Ahlfors Pg. 44 Problem 2*

Pf:

Expression of $\sinh, \cosh$**:**

Given that $\cosh(z) = \frac{e^z+e^{-z}}{2}$ and $\sinh(z) = \frac{e^z-e^{-z}}{2}$. Then, given that $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$, the following identities are true:

$$\cos(iz) = \frac{e^{i(iz)}+e^{-i(iz)}}{2} = \frac{e^{-z}+e^{z}}{2} = \cosh(z)$$

$$\sin(iz) = \frac{e^{i(iz)}-e^{-i(iz)}}{2i} = -i\frac{e^{-z}-e^{z}}{2} = i\left(\frac{e^{z}-e^{-z}}{2}\right) = i\sinh(z)$$

Thus, $\cosh(z) = \cos(iz)$, and $\sinh(z) = \frac{1}{i}\sin(iz) = -i\sin(iz)$.

Addition Formula:

Then, according to the original trigonometry addition formulas, for all $a, b \in \mathbb{C}$, the following is true:

$$\sinh(a+b) = -i\sin(i(a+b)) = -i\sin(ia+ib) = -i(\sin(ia)\cos(ib)+\sin(ib)\cos(ia))$$

$$= (-i\sin(ia))\cos(ib) + (-i\sin(ib))\cos(ia) = \sinh(a)\cosh(b)+\sinh(b)\cosh(a)$$

$$\cosh(a+b) = \cos(i(a+b)) = \cos(ia+ib) = \cos(ia)\cos(ib) - \sin(ia)\sin(ib)$$

$$= \cosh(a)\cosh(b) + (-i\sin(ia))(-i\sin(ib)) = \cosh(a)\cosh(b)+\sinh(a)\sinh(b)$$

Thus, the addition formula is given as:

$$\sinh(a+b) = \sinh(a)\cosh(b)+\sinh(b)\cosh(a), \quad \cosh(a+b) = \cosh(a)\cosh(b)+\sinh(a)\sinh(b)$$

Double Angle Formula: With the above formulas, for all $z \in \mathbb{C}$, $\sinh(2z), \cosh(2z)$ can be given as:

$$\sinh(2z) = \sinh(z+z) = \sinh(z)\cosh(z)+\sinh(z)\cosh(z) = 2\sinh(z)\cosh(z)$$

$$\cosh(2z) = \cosh(z+z) = \cosh(z)\cosh(z)+\sinh(z)\sinh(z) = \cosh(z)^2+\sinh(z)^2$$

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Question 2 *Ahlfors Pg. 47 Problem 6*

Pf:

Case for 2^i:

Consider $2^i = e^{i\cdot\log(2)}$, which $\log(2) = \{\ln(2) + i(\arg(2) + k\cdot 2\pi) \mid k \in \mathbb{Z}\}$ (where $\arg(2) = 0$, since $2 \in \mathbb{R}$). Thus:

$$2^i = e^{i\cdot\log(2)} = e^{i(\ln(2)+ik\cdot 2\pi)} = e^{-k\cdot 2\pi + i\ln(2)} = e^{-k\cdot 2\pi}\cdot e^{i\ln(2)}$$

So, $2^i = \{e^{-k\cdot 2\pi}\cdot e^{iln(2)} \mid k \in \mathbb{Z}\}$.

Case for i^i:

Consider $i^i = e^{i\cdot\log(i)}$, which $\log(i) = \{\ln|i| + i(\arg(i) + k\cdot 2\pi) \mid k \in \mathbb{Z}\}$ (where $\ln|i| = \ln(1) = 0$, and $\arg(i) = \frac{\pi}{2}$). Thus:

$$i^i = e^{i\cdot\log(i)} = e^{i\cdot i(\frac{\pi}{2}+k\cdot 2\pi)} = e^{-\frac{\pi}{2}-k\cdot 2\pi}$$

So, $i^i = \{e^{-\frac{\pi}{2}-k\cdot 2\pi} \mid k \in \mathbb{Z}\}$.

Case for $(-1)^{2i}$:

Consider $(-1)^{2i} = e^{2i\cdot\log(-1)}$, which $\log(-1) = \{\ln|-1| + i(\arg(-1) + k\cdot 2\pi) \mid k \in \mathbb{Z}\}$ (where $\ln|-1| = \ln(1) = 0$, and $\arg(-1) = \pi$). Thus:

$$(-1)^{2i} = e^{2i\cdot\log(-1)} = e^{2i\cdot i(\pi+k\cdot 2\pi)} = e^{-2(2k+1)\pi} = e^{-(4k+2)\pi}$$

So, $(-1)^{2i} = \{e^{-(4k+2)\pi} \mid k \in \mathbb{Z}\}$.

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Question 3 *Ahlfors Pg. 72 Problem 1*

Pf:

Define the region $\Omega = \mathbb{C}\setminus(-\infty,-1]\cup[1,\infty)$ (region excluding real numbers except for ones in between $(-1,1)$). Which, since $\sqrt{z}$ is defined as a single-valued function on $\mathbb{C}\setminus(-\infty,0]$, by $\sqrt{z} = \sqrt{|z|}(\cos(\arg(z)/2) + i\sin(\arg(z)/2))$ (which, $\arg(z) \in (-\pi,\pi)$).

Then, given the function $f(z) = \sqrt{1+z}+\sqrt{1-z}$, for all $z \in \Omega$, since $z \notin (-\infty,-1]$, then $(1+z) \notin (-\infty,0]$, thus $\sqrt{1+z}$ is well-defined; similarly, since $z \notin [1,\infty)$, thus $-z \notin (-\infty,-1]$, or $(1-z) \notin (-\infty,0]$. Hence, $\sqrt{1-z}$ is also well-defined.

Now, with the function $f(z)$ being well-defined on Ω an open subset, based on the definition of square root above, the following is true:

$$\forall z \in \Omega, \quad f(z) = \sqrt{1+z} + \sqrt{1-z}$$

$$= \sqrt{|1+z|}\left(\cos\left(\frac{\arg(1+z)}{2}\right) + i\sin\left(\frac{\arg(1+z)}{2}\right)\right) + \sqrt{|1-z|}\left(\cos\left(\frac{\arg(1-z)}{2}\right) + i\sin\left(\frac{\arg(1-z)}{2}\right)\right)$$

Then, since $\sqrt{z}$ is analytic, and the polynomial $(1+z), (1-z)$ are both analytic, because the composition of analytic functions are analytic, hence $\sqrt{1+z}$ and $\sqrt{1-z}$ are analytic; and since the sum of analytic function is again analytic, then $f(z) = \sqrt{1+z} + \sqrt{1-z}$ is again analytic.