Math CS 122A HW4

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Question 1 *Ahlfors Pg. 96 Problem 2:*
Map the region between $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ on a half plane.

Pf:
Consider the following transformation $g : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$:

$$f(z) = \frac{z+1}{z-1} \cdot \frac{-i-1}{-i+1}, \quad g(z) = e^{\pi f(z)}$$

First, if consider the points $-i, -1, 1$ respectively on $|z| = 1$, linear transformation f maps the following:

$$f(-i) = \frac{-i+1}{-i-1} \cdot \frac{-i-1}{-i+1} = 1, \quad f(-1) = \frac{-1+1}{-1-1} \cdot \frac{-i=1}{-i+1} = 0, \quad f(1) = \infty$$

(Note: Since $f(1)$ is not defined under $\mathbb{C}$, it gets map to ∞).

Because the orientation of $|z| = 1$ is $-i$ to -1 to 1, going clockwise, and the orientation of the image is 1 to 0 to ∞, which on the right side is the half plane with positive imaginary parts. Hence, the right of $|z| = 1$ under this orientation (which is the interior of $|z| = 1$) gets mapped to the half plane $Im(z) > 0$.

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Now, consider the points $\frac{1}{2}(1-i), 0, 1$ on $|z - \frac{1}{2}| = \frac{1}{2}$, linear transformation f maps the following:

$$f\left(\frac{1}{2}(1-i)\right) = \frac{(\frac{1}{2} - \frac{1}{2}i) + 1}{(\frac{1}{2} - \frac{1}{2}i) - 1} \cdot \frac{-i-1}{-i+1} = \frac{(1-i)+2}{(1-i)-2} \cdot \frac{-i-1}{-i+1} = \frac{3-i}{-1-i} \cdot \frac{-1-i}{1-i}$$

$$= \frac{3-i}{1-i} = \frac{(3-i)(1+i)}{(1-i)(1+i)} = \frac{3+1-i+3i}{2} = \frac{4+2i}{2} = 2+i$$

$$f(0) = \frac{1}{-1} \cdot \frac{-i-1}{-i+1} = -\frac{-(1+i)^2}{(1-i)(1+i)} = \frac{2i}{2} = i, \quad f(1) = \infty$$

So, since the three points gets mapped to $(2+i), i, \infty$ respectively, and linear transformation maps circle to circle, hence this is a circle passing through ∞, or a straight line passing through i and $(2+i)$, which is the line $Im(z) = 1$. Then, with the orientation $\frac{1}{2}(1-i)$ to 0 to 1, the image has orientation $(2+i)$ to i to ∞, which the left side is the half plane $Im(z) < 1$. Hence, the left of $|z - \frac{1}{2}| = \frac{1}{2}$ under this orientation (the exterior of $|z - \frac{1}{2}| = 1$) gets mapped to the half plane $Im(z) < 1$.

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With the above statements, all points in the region between $|z|=1$ and $|z-\frac{1}{2}|=\frac{1}{2}$ are in the interior of $|z|=1$, and in the exterior of $|z-\frac{1}{2}|=\frac{1}{2}$. So, they are the intersection of $Im(z)>0$ and $Im(z)<1$.

Which, $\pi f(z)$ represents the region $0<Im(z)<\pi$.

So, for all z_0 in the given open region, $z_0=a+bi$, where $a\in\mathbb{R}$, and $0<b<\pi$. So:

$$e^{z_0}=e^{a+bi}=e^a\cdot e^{ib},\quad b\in(0,\pi)$$

Hence, e^{z_0} satisfies $\arg(e^{z_0})=b\in(0,\pi)$, and $|e^{z_0}|=e^a>0$, hence the image of the region $0<Im(z)<\pi$ is in the half plane $Im(z)>0$ (in fact, the image is the whole half plane, since the choice of $a\in\mathbb{R}$ and $b\in(0,\pi)$ are arbitrary, hence $e^a\in(0,\infty)$ could be any value in the given region).

Eventually, since $\pi f(z)$ maps the region between $|z|=1$ and $|z-\frac{1}{2}|=\frac{1}{2}$ onto the region $0<Im(z)<1$, while e_0^z maps this new region onto the half plane $Im(z)>0$, then the composition $e^{\pi f(z)}$ maps the desired region to the half plane $Im(z)>0$.

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Question 2 *Ahlfors Pg. 97 Problem 5:*
Map the inside of the right-hand branch of the hyperbola $x^2 - y^2 = a^2$ on the disk $|w| < 1$ so that the focus corresopnds to $w = 0$ and the vertex to $w = -1$.

Pf:

WLOG, assume $a > 0$ (Note: $a < 0$ can be replaced with $(-a)$ instead). Under this configuration, the vertex is when $y = 0$, or $x = a$ for the right hand branch (the vertex is $z = a$). Also, the focus is given by $(ka, 0)$ with $k = \sqrt{1 + \frac{b'^2}{a'^2}}$ when given the hyperbola $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$, which under this configuration, $a' = b' = a$, hence $k = \sqrt{2}$ (so the focus is $z = \sqrt{2}a$).

(Note 2: under the requirement, the focus and vertex needs to be two distinct points, hence $a \neq 0$).

Map of z^2:

Notice that for all $z \in \mathbb{C}$, since $z = x + iy$ for some $x, y \in \mathbb{R}$, then $z^2 = (x^2 - y^2) + i \cdot 2xy$.

If take the plane $Re(z) > 0$ (where $x > 0$), the map is injective: Suppoze $z^2 = z_1^2$ for $z, z_1 \in \mathbb{C}$, then $z^2 - z_1^2 = (z - z_1)(z + z_1) = 0$, hence $z = z_1$ or $z = -z_1$. However, if restrict onto the plane $Re(z) > 0$, then $z = -z_1$ is impossible for all values on this plane, hence $z = z_1$, showing it's injective.

Now, consider the inside of the right-hand branch of the hyperbola $x^2 - y^2 = a^2$, which is restricted by the condition $x^2 - y^2 \geq a^2$: For all $z = x + iy$ in the given region, $x^2 - y^2 \geq a^2$; hence, $z^2 = (x^2 - y^2) + i \cdot 2xy$ is in the half plane $Re(w) \geq a^2$. Also, for all w in the half plane $Re(w) \geq a^2$ ($a^2 > 0$), since it is in the domain of $\sqrt{z}$ (which is in $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$), then there exists $z = x + iy$ with $z^2 = w$, hence $Re(w) = Re(z^2) = (x^2 - y^2) \geq a^2$, showing that z is in the given region.

Hence, we can conclude that the function z^2 restricting onto the inside of the right-hand branch of the given hyperbola (with condition $x^2 - y^2 \geq a^2$), it is an injective function mapping the region onto the half plane $Re(z) \geq a^2$.

Mapping the Half Plane $Re(z) \geq a^2$ onto the Unit Disk:

Consider the following linear transformation:

$$f(w) = 1 - \frac{2a^2}{w}$$

For the points w_0 on the line $Re(w) = a^2$, $w_0 = a^2 + iv$ for some $v \in \mathbb{R}$, hence the following is true:

$$f(w_0) = 1 - \frac{2a^2}{w_0} = \frac{w_0 - 2a^2}{w_0} = \frac{(a^2 + iv) - 2a^2}{a^2 + iv} = \frac{-a^2 + iv}{a^2 + iv} = \frac{-(a^2 - iv)}{a^2 + iv} = -\frac{\bar{w_0}}{w_0}$$

Hence, $|f(w_0)| = \left|-\frac{\bar{w_0}}{w_0}\right| = \frac{|\bar{w_0}|}{|w_0|} = 1$, the boundary or the half plane gets mapped to the boundary of the unit disk $|w| < 1$;

Also, for all points w_1 in the plane $Re(w) > a^2$ (let $w = u + iv$ for $u, v \in \mathbb{R}$, hence $u > a^2$), there are two cases to conside. The following is what w_1 gets mapped to:

$$f(w_1) = 1 - \frac{2a^2}{w_1} = \frac{w_1 - 2a^2}{w_1} = \frac{(u - 2a^2) + iv}{u + iv}$$

First, if $u \leq 2a^2$, notice that since $0 \leq |u - 2a^2| = (2a^2 - u) < (2a^2 - a^2) = a^2 < u$, then, $|(u - 2a^2) + iv| = \sqrt{(u - 2a^2)^2 + v^2} < \sqrt{u^2 + v^2} = |w_1|$, hence $|f(w_1)| = \frac{|(u-2a^2)+iv|}{|u+iv|} < 1$.

Else, if $u > 2a^2$, then since $0 < (u - 2a^2) < u$, then again $|(u - 2a^2) + iv| = \sqrt{(u - 2a^2)^2 + v^2} < \sqrt{u^2 + v^2} = |w_1|$, hence $|f(w_1)| = \frac{|(u-2a^2)+iv|}{|u+iv|} < 1$ is still true.

So, we can conclude that the half plane $Re(w) \geq a^2$ gets mapped to the unit disk $|w| = 1$, and since this is a linear transformation, the map is bijective.

Mapping Inside of Hyperbola to Unit Disk:

If Compose the two functions above, consider the following transformation $\bar{f}(z) = f(z^2) = 1 - \frac{2a^2}{z^2}$: First, for all z in the inside of the given branch of hyperboala (in the region $x^2 - y^2 \geq a^2$), z^2 appears in the half plane $Re(w) \geq a^2$, and there is a one-to-one correspondence between the two regions under the map; furthermore, since f maps the half plane $Re(w) \geq a^2$ to the unit disk $|w| \leq 1$, and is also a one-to-one correspondence, then the composition $f(z^2)$ maps the interior of the hyperbola to the unit disk.

Also, computing the following, we get:

$$\bar{f}(a) = 1 - \frac{2a^2}{a^2} = 1 - 2 = -1, \quad \bar{f}(\sqrt{2}a) = 1 - \frac{2a^2}{(\sqrt{2}a)^2} = 1 - \frac{2a^2}{2a^2} = 1 - 1 = 0$$

Which, since given the right branch of hyperbola $x^2 - y^2 = a^2$, $z_0 = a$ is the vertex and $z_1 = \sqrt{2}a$ is the focus, then the vertex gets mapped to -1, and the focus gets mapped to 0, hence this conformal map satisfies the given condition.

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Question 3 *Ahlfors Pg. 78 Problem 4:*
Show that any linear transformation which transforms the real axis into itself can be written with real coefficient.

Pf:

Let $S : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be arbitrary linear transformation that transforms the real axis to itself, then if restricted onto $\mathbb{R}$, the image of the function is also the real axis.

Notice that since the transformation is bijective, there exists distinct points $z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}$, with $S(z_1) = 1$, $S(z_2) = 0$, and $S(z_3) = \infty$. Which, this indicates that z_1, z_2, z_3 is in fact on $\mathbb{R} \cup \{\infty\}$:

Suppose there exists a point not on $\mathbb{R} \cup \{\infty\}$, then the circle (or straight line if one of them is ∞) determined by z_1, z_2, z_3 is not on $\mathbb{R} \cup \{\infty\}$; yet, since the image of z_1, z_2, z_3 is on the straight line $\mathbb{R} \cup \{\infty\}$, that means the circle deteined by z_1, z_2, z_3 is mapped onto $\mathbb{R} \cup \{\infty\}$, which contradicts the fact that the preimage of the real axis should be the real axis, under the given condition.

Hence, $z_1, z_2, z_3 \in \mathbb{R} \cup \{\infty\}$. Then, based on the formula for cross ratio, the unique transformation S with $S(z_1) = 1, S(z_2) = 0$, and $S(z_3) = \infty$, has the following formula:

$$S(z) = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$

Hence, since $z_1, z_2, z_3 \in \mathbb{R} \cup \{\infty\}$, then the above transformation can be simplified to real coefficients.

For all three points being real:

$S(z)$ is in the given form above, where every coefficients are real.

For one points being ∞:

If $z_1 = \infty$:

$$S(z) = \lim_{z_1 \to \infty} \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} = \frac{z - z_2}{z - z_3}$$

If $z_2 = \infty$:

$$S(z) = \lim_{z_2 \to \infty} \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} = \frac{z_1 - z_3}{z - z_3}$$

Else if $z_3 = \infty$:

$$S(z) = \lim_{z_3 \to \infty} \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} = \frac{z - z_2}{z_1 - z_2}$$

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Question 4 *Ahlors Pg. 80 Problem 3:*
If the consecutive vertices z_1, z_2, z_3, z_4 of a quadrilateral lie on a circle, prove that

$$|z_1 - z_3| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4|$$

and interpret the result geometrically.

Pf:
First, consider the right hand side of the equation:

$$|z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4| = |z_2 - z_3| \cdot |z_1 - z_4| \cdot \left(\left| \frac{(z_1 - z_2) \cdot (z_3 - z_4)}{(z_2 - z_3) \cdot (z_1 - z_4)} \right| + 1 \right)$$

Then, recall that the cross ratio of (z_1, z_3, z_2, z_4) can be expressed as:

$$(z_1, z_3, z_2, z_4) = \frac{(z_1 - z_2) \cdot (z_3 - z_4)}{(z_1 - z_4) \cdot (z_3 - z_2)}$$

Hence, the above expression can be rewritten as:

$$|z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4| = |z_2 - z_3| \cdot |z_1 - z_4| \cdot \left(\left| - \frac{(z_1 - z_2) \cdot (z_3 - z_4)}{(z_3 - z_2) \cdot (z_1 - z_4)} \right| + 1 \right)$$

$$= |z_2 - z_3| \cdot |z_1 - z_4| \cdot (| - (z_1, z_3, z_2, z_4)| + 1)$$

Notice that since z_1, z_2, z_3, z_4 is consecutive vertices on a circle, then the cross ratio is real; furthermore, by the statement in **Question 6**, since z_1, z_3, z_4 and z_2, z_3, z_4 have the same orientation, hence the cross ratio $(z_1, z_2, z_3, z_4) > 0$.

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Similarly, when viewing in order of z_1, z_3, z_2, z_4, the orientation z_1, z_3, z_4 and z_3, z_2, z_4 are different, hence the cross ratio $(z_1, z_3, z_2, z_4) < 0$.

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Then, since $(z_1, z_3, z_2, z_4) < 0$, $-(z_1, z_3, z_2, z_4) > 0$, hence $|-(z_1, z_3, z_2, z_4)| = -(z_1, z_3, z_2, z_4)$. The above identity becomes:

$$|z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4| = |z_2 - z_3| \cdot |z_1 - z_4| \cdot (| - (z_1, z_3, z_2, z_4)| + 1)$$

$$|z_2 - z_3| \cdot |z_1 - z_4| \cdot |(-(z_1, z_3, z_2, z_4) + 1)|$$

Compute the third term in the equation, we get:

$$-(z_1, z_3, z_2, z_4) + 1 = -\frac{(z_1 - z_2) \cdot (z_3 - z_4)}{(z_3 - z_2) \cdot (z_1 - z_4)} + 1$$

$$= \frac{(z_3 - z_2)(z_1 - z_4) - (z_1 - z_2)(z_3 - z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

$$= \frac{(z_1 z_3 - z_1 z_2 - z_3 z_4 + z_2 z_4) - (z_1 z_3 - z_1 z_4 - z_2 z_3 + z_2 z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

$$= \frac{-z_1z_2 - z_3z_4 + z_1z_4 + z_2z_3}{(z_3 - z_2)(z_1 - z_4)} = \frac{z_1(z_4 - z_2) + z_3(z_2 - z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

$$= \frac{(z_3 - z_1)(z_2 - z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

Hence, plug back into the original equation, we get:

$$|z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4| = |z_2 - z_3| \cdot |z_1 - z_4| \cdot |(-(z_1, z_3, z_2, z_4) + 1)|$$

$$= |z_2 - z_3| \cdot |z_1 - z_4| \cdot \left| \frac{(z_3 - z_1)(z_2 - z_4)}{(z_3 - z_2)(z_1 - z_4)} \right| = |(z_3 - z_1)(z_2 - z_4)|$$

So, the original original formula is true:

$$|z_3 - z_1| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4|$$

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Question 5 *Ahlfors Pg. 83 Problem 4:*
Find the linear transformation which carries the circle $|z| = 2$ into $|z+1| = 1$, the point -2 into the origin, and the origin into i.

Pf:

The following map:

$$z \mapsto w = (\frac{1}{2}z + 1) \mapsto \frac{-(1-i)w}{2w - (1-i)}$$

Symmetric points:

First, for circle $|z| = 2$, since the origin 0 is not on the circle, then to find a precise map, we also need to consider its symmetric point, namely ∞. (Note: the symmetric point of the center of a circle is always ∞).

Now, consider the points they get mapped to: Since any linear transformation should preserve the symmetric points, then as 0 gets mapped to i, ∞ gets mapped to the symmetric point of i on the circle $|z+1| = 1$. The following is the computation based on the formula given in the textbook. Let $z_0 = i$, radius $r = 1$, and the center $a = -1$, then its symmetric point z_0^* is given by:

$$z_0^* = \frac{r^2}{(\bar{z}_0 - a)} + a = \frac{1}{-i - (-1)} - 1 = \frac{1}{1-i} - 1 = \frac{(1+i)}{(1-i)(1+i)} - 1$$
$$= \frac{1+i}{2} - 1 = -\frac{1}{2} + \frac{1}{2}i$$

Hence, under the desired linear transformation, ∞ gets mapped to $z_0^* = -\frac{1}{2} + \frac{1}{2}i$.

Formula for Linear Transformation:

Given that $-2 \mapsto 0$, $0 \mapsto i$, and $\infty \mapsto (-\frac{1}{2} + \frac{1}{2}i)$, consider the following map:

$$f(z) = \frac{-(1-i)z - 2(1-i)}{2z + 2(1+i)}$$

Which, it maps the given point as follow:

$$f(-2) = \frac{-(1-i)(-2) - 2(1-i)}{2(-2) + 2(1+i)} = \frac{0 \cdot (1-i)}{-4 + 2 + 2i} = 0$$

$$f(0) = \frac{-(1-i) \cdot 0 - 2(1-i)}{2 \cdot 0 + 2(1+i)} = \frac{-2(1-i)}{2(1+i)} = -\frac{(1-i)^2}{(1-i)(1+i)} = -\frac{1 - 1 - 2i}{1+1} = \frac{2i}{2} = i$$

$$f(\infty) = \lim_{z \to \infty} \frac{-(1-i)z - 2(1-i)}{2z + 2(1+i)} = \frac{-(1-i)}{2} = -\frac{1}{2} + \frac{1}{2}i$$

Hence, the given linear transformation maps the three points to the correct locations.

Circle Maps to Circle:

To verify that $|z| = 2$ gets mapped to $|z+1| = 1$, it suffices to show that three points on $|z| = 2$ get mapped onto $|z+1| = 1$.

First, we already have $-2 \mapsto 0$, which is a point satisfying the condition.

Now, consider the point $2i, -2i$ on the circle $|z| = 2$:

$$f(2i) = \frac{-(1-i)2i - 2(1-i)}{2 \cdot 2i + 2(1+i)} = \frac{-2(1+i)(1-i)}{2 + 6i} = \frac{-2}{1+3i} = \frac{-2(1-3i)}{(1+3i)(1-3i)} = \frac{-2+6i}{10} = \frac{-1+3i}{5}$$

$$f(-2i) = \frac{-(1-i)(-2i) - 2(1-i)}{2(-2i) + 2(1+i)} = \frac{-2(1-i)(1-i)}{2-2i} = -(1-i)$$

Then, consider the distance $|f(2i)+1|$ and $|f(-2i)+1|$, we get:

$$|f(2i)+1| = \left|\frac{-1+3i}{5} + 1\right| = \left|\frac{4+3i}{5}\right| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = 1$$

$$|f(-2i)+1| = |-(1-i)+1| = |i| = 1$$

Hence, $f(2i), f(-2i)$ are two points on the circle $|z+1| = 1$.

Since $-2, 2i, -2i$ are three points on the circle $|z| = 2$, and they get mapped to points on $|z+1| = 1$ by the linear transformation f, hence $|z| = 2$ is mapped to $|z+1| = 1$, showing that f is in fact the desired linear transformation.

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Question 6 *Ahlfors Pg. 84 Problem 1:*
If z_1, z_2, z_3, z_4 are points on a circle, show that z_1, z_3, z_4 and z_2, z_3, z_4 determine the same orientation if and only If $(z_1, z_2, z_3, z_4) > 0$.

Pf:

Given four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}$, to determine the cross ratio (z_1, z_2, z_3, z_4), it is given by the linear transformation that gives $z_2 \mapsto 1$, $z_3 \mapsto 0$, and $z_4 \mapsto \infty$.

If consider the orientation as z_2, z_3, z_4 respectively:

If z_1, z_3, z_4 has the same orientation as above, then z_1, z_2 needs to be on the same arc when the circle is separated by z_3 and z_4.

Which, this happens if the linear transformation would transform z_1, z_2 onto the same side of the real line, so z_1 gets mapped to a positive value. Hence, $(z_1, z_2, z_3, z_4) > 0$.

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Conversely, if $(z_1, z_2, z_3, z_4) > 0$, then z_1 gets mapped to a positive value on the real axis. Which, since the orienation is given by z_2, z_3, z_4 in order, and z_1, z_2 both get mapped to positive values while z_3 gets mapped to 0, hence the orientation z_1, z_3, z_4 must have the same orientation as z_2, z_3, z_4.

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Question 7 *Ahlfors Pg. 88 Problem 6:*
Find all circles which are orthogonal to $|z| = 1$ *and* $|z - 1| = 4$.

Pf:

Textbook Pg. 87, 88 were talking about this (about under conformal linear transformation, all the circles orthogonal to the two should correspond to a family of circles, all ampped to similar lines.)