Math CS 122A HW4

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Question 1 *Ahlfors Pg. 96 Problem 2:*
Map the region between $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ on a half plane.

Pf:
Consider the following transformation $g : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$:

$$f(z) = \frac{z+1}{z-1} \cdot \frac{-i-1}{-i+1}, \quad g(z) = e^{\pi f(z)}$$

First, if consider the points $-i, -1, 1$ respectively on $|z| = 1$, linear transformation f maps the following:

$$f(-i) = \frac{-i+1}{-i-1} \cdot \frac{-i-1}{-i+1} = 1, \quad f(-1) = \frac{-1+1}{-1-1} \cdot \frac{-i=1}{-i+1} = 0, \quad f(1) = \infty$$

(Note: Since $f(1)$ is not defined under $\mathbb{C}$, it gets map to ∞).

Because the orientation of $|z| = 1$ is $-i$ to -1 to 1, going clockwise, and the orientation of the image is 1 to 0 to ∞, which on the right side is the half plane with positive imaginary parts. Hence, the right of $|z| = 1$ under this orientation (which is the interior of $|z| = 1$) gets mapped to the half plane $Im(z) > 0$.

Now, consider the points $\frac{1}{2}(1-i), 0, 1$ on $|z - \frac{1}{2}| = \frac{1}{2}$, linear transformation f maps the following:

$$f\left(\frac{1}{2}(1-i)\right) = \frac{(\frac{1}{2} - \frac{1}{2}i) + 1}{(\frac{1}{2} - \frac{1}{2}i) - 1} \cdot \frac{-i-1}{-i+1} = \frac{(1-i)+2}{(1-i)-2} \cdot \frac{-i-1}{-i+1} = \frac{3-i}{-1-i} \cdot \frac{-1-i}{1-i}$$

$$= \frac{3-i}{1-i} = \frac{(3-i)(1+i)}{(1-i)(1+i)} = \frac{3+1-i+3i}{2} = \frac{4+2i}{2} = 2+i$$

$$f(0) = \frac{1}{-1} \cdot \frac{-i-1}{-i+1} = -\frac{-(1+i)^2}{(1-i)(1+i)} = \frac{2i}{2} = i, \quad f(1) = \infty$$

So, since the three points gets mapped to $(2+i), i, \infty$ respectively, and linear transformation maps circle to circle, hence this is a circle passing through ∞, or a straight line passing through i and $(2+i)$, which is the line $Im(z) = 1$. Then, with the orientation $\frac{1}{2}(1-i)$ to 0 to 1, the image has orientation $(2+i)$ to i to ∞, which the left side is the half plane $Im(z) < 1$. Hence, the left of $|z - \frac{1}{2}| = \frac{1}{2}$ under this orientation (the exterior of $|z - \frac{1}{2}| = 1$) gets mapped to the half plane $Im(z) < 1$.

With the above statements, all points in the region between $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ are in the interior of $|z| = 1$, and in the exterior of $|z - \frac{1}{2}| = \frac{1}{2}$. So, they are the intersection of $Im(z) > 0$ and $Im(z) < 1$.

Which, $\pi f(z)$ represents the region $0 < Im(z) < \pi$.

So, for all z_0 in the given open region, $z_0 = a + bi$, where $a \in \mathbb{R}$, and $0 < b < \pi$. So:

$$e^{z_0} = e^{a+bi} = e^a \cdot e^{ib}, \quad b \in (0, \pi)$$

Hence, e^{z_0} satisfies $\arg(e^{z_0}) = b \in (0, \pi)$, and $|e^{z_0}| = e^a > 0$, hence the image of the region $0 < Im(z) < \pi$ is in the half plane $Im(z) > 0$ (in fact, the image is the whole half plane, since the choice of $a \in \mathbb{R}$ and $b \in (0, \pi)$ are arbitrary, hence $e^a \in (0, \infty)$ could be any value in the given region).

Eventually, since $\pi f(z)$ maps the region between $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ onto the region $0 < Im(z) < 1$, while e_0^z maps this new region onto the half plane $Im(z) > 0$, then the composition $e^{\pi f(z)}$ maps the desired region to the half plane $Im(z) > 0$.

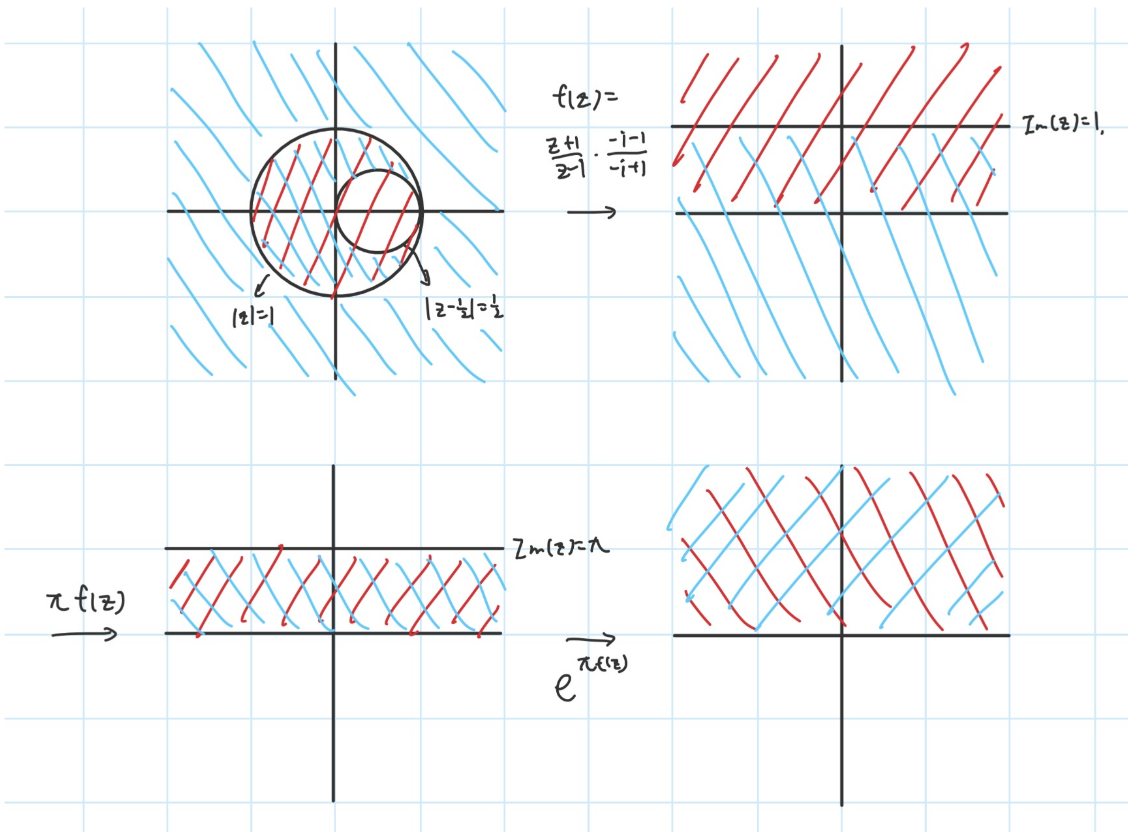

Figure 1: Transformation between regions

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Question 2 *Ahlfors Pg. 97 Problem 5:*
Map the inside of the right-hand branch of the hyperbola $x^2 - y^2 = a^2$ on the disk $|w| < 1$ so that the focus corresopnds to $w = 0$ and the vertex to $w = -1$.

Pf:

WLOG, assume $a > 0$ (Note: $a < 0$ can be replaced with $(-a)$ instead). Under this configuration, the vertex is when $y = 0$, or $x = a$ for the right hand branch (the vertex is $z = a$). Also, the focus is given by $(ka, 0)$ with $k = \sqrt{1 + \frac{b'^2}{a'^2}}$ when given the hyperbola $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$, which under this configuration, $a' = b' = a$, hence $k = \sqrt{2}$ (so the focus is $z = \sqrt{2}a$).

(Note 2: under the requirement, the focus and vertex needs to be two distinct points, hence $a \neq 0$).

Map of z^2:

Notice that for all $z \in \mathbb{C}$, since $z = x + iy$ for some $x, y \in \mathbb{R}$, then $z^2 = (x^2 - y^2) + i \cdot 2xy$.

If take the plane $Re(z) > 0$ (where $x > 0$), the map is injective: Suppoze $z^2 = z_1^2$ for $z, z_1 \in \mathbb{C}$, then $z^2 - z_1^2 = (z - z_1)(z + z_1) = 0$, hence $z = z_1$ or $z = -z_1$. However, if restrict onto the plane $Re(z) > 0$, then $z = -z_1$ is impossible for all values on this plane, hence $z = z_1$, showing it's injective.

Now, consider the inside of the right-hand branch of the hyperbola $x^2 - y^2 = a^2$, which is restricted by the condition $x^2 - y^2 \geq a^2$: For all $z = x + iy$ in the given region, $x^2 - y^2 \geq a^2$; hence, $z^2 = (x^2 - y^2) + i \cdot 2xy$ is in the half plane $Re(w) \geq a^2$. Also, for all w in the half plane $Re(w) \geq a^2$ $(a^2 > 0)$, since it is in the domain of $\sqrt{z}$ (which is in $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$), then there exists $z = x + iy$ with $z = \sqrt{w}$, hence $Re(w) = Re(z^2) = (x^2 - y^2) \geq a^2$, showing that z is in the given region.

Hence, we can conclude that the function z^2 restricting onto the inside of the right-hand branch of the given hyperbola (with condition $x^2 - y^2 \geq a^2$), it is an injective function mapping the region onto the half plane $Re(z) \geq a^2$.

Mapping the Half Plane $Re(z) \geq a^2$ onto the Unit Disk:

Consider the following linear transformation:

$$f(w) = 1 - \frac{2a^2}{w}$$

For the points w_0 on the line $Re(w) = a^2$, $w_0 = a^2 + iv$ for some $v \in \mathbb{R}$, hence the following is true:

$$f(w_0) = 1 - \frac{2a^2}{w_0} = \frac{w_0 - 2a^2}{w_0} = \frac{(a^2 + iv) - 2a^2}{a^2 + iv} = \frac{-a^2 + iv}{a^2 + iv} = \frac{-(a^2 - iv)}{a^2 + iv} = -\frac{\bar{w_0}}{w_0}$$

Hence, $|f(w_0)| = \left|-\frac{\bar{w_0}}{w_0}\right| = \frac{|\bar{w_0}|}{|w_0|} = 1$, the boundary or the half plane gets mapped to the boundary of the unit disk $|w| < 1$;

Also, for all points w_1 in the plane $Re(w) > a^2$ (let $w = u + iv$ for $u, v \in \mathbb{R}$, hence $u > a^2$), there are two cases to conside. The following is what w_1 gets mapped to:

$$f(w_1) = 1 - \frac{2a^2}{w_1} = \frac{w_1 - 2a^2}{w_1} = \frac{(u - 2a^2) + iv}{u + iv}$$

First, if $u \leq 2a^2$, notice that since $0 \leq |u - 2a^2| = (2a^2 - u) < (2a^2 - a^2) = a^2 < u$, then, $|(u - 2a^2) + iv| = \sqrt{(u - 2a^2)^2 + v^2} < \sqrt{u^2 + v^2} = |w_1|$, hence $|f(w_1)| = \frac{|(u - 2a^2) + iv|}{|u + iv|} < 1$.

Else, if $u > 2a^2$, then since $0 < (u - 2a^2) < u$, then again $|(u - 2a^2) + iv| = \sqrt{(u - 2a^2)^2 + v^2} < \sqrt{u^2 + v^2} = |w_1|$, hence $|f(w_1)| = \frac{|(u-2a^2)+iv|}{|u+iv|} < 1$ is still true.

So, we can conclude that the half plane $Re(w) \geq a^2$ gets mapped to the unit disk $|w| = 1$, and since this is a linear transformation, the map is bijective.

Mapping Inside of Hyperbola to Unit Disk:

If Compose the two functions above, consider the following transformation $\bar{f}(z) = f(z^2) = 1 - \frac{2a^2}{z^2}$: First, for all z in the inside of the given branch of hyperboala (in the region $x^2 - y^2 \geq a^2$), z^2 appears in the half plane $Re(w) \geq a^2$, and there is a one-to-one correspondence between the two regions under the map; furthermore, since f maps the half plane $Re(w) \geq a^2$ to the unit disk $|w| \leq 1$, and is also a one-to-one correspondence, then the composition $f(z^2)$ maps the interior of the hyperbola to the unit disk.

Also, computing the following, we get:

$$\bar{f}(a) = 1 - \frac{2a^2}{a^2} = 1 - 2 = -1, \quad \bar{f}(\sqrt{2}a) = 1 - \frac{2a^2}{(\sqrt{2}a)^2} = 1 - \frac{2a^2}{2a^2} = 1 - 1 = 0$$

Which, since given the right branch of hyperbola $x^2 - y^2 = a^2$, $z_0 = a$ is the vertex and $z_1 = \sqrt{2}a$ is the focus, then the vertex gets mapped to -1, and the focus gets mapped to 0, hence this conformal map satisfies the given condition.

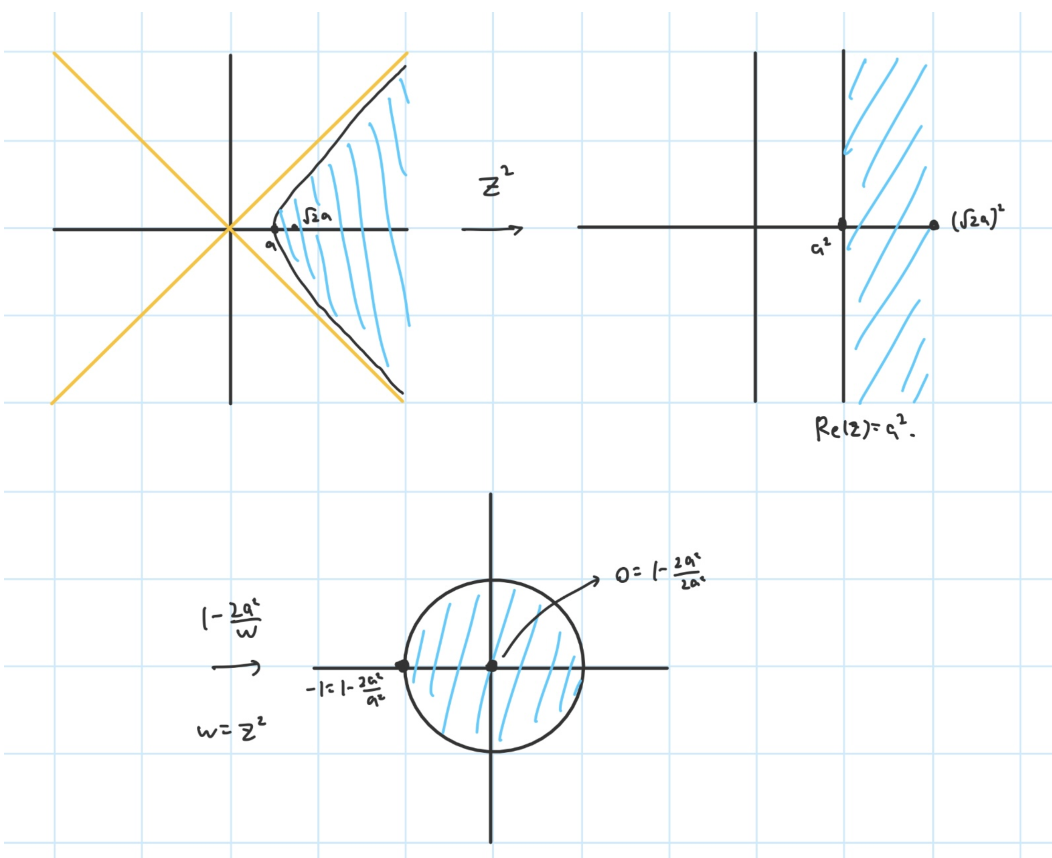

Figure 2: Transformation between Hyperbola and Circle

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Question 3 *Ahlfors Pg. 78 Problem 4:*
Show that any linear transformation which transforms the real axis into itself can be written with real coefficient.

Pf:

Let $S : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be arbitrary linear transformation that transforms the real axis to itself, then if restricted onto $\mathbb{R}$, the image of the function is also the real axis.

Notice that since the transformation is bijective, there exists distinct points $z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}$, with $S(z_1) = 1$, $S(z_2) = 0$, and $S(z_3) = \infty$. Which, this indicates that z_1, z_2, z_3 is in fact on $\mathbb{R} \cup \{\infty\}$:

Suppose there exists a point not on $\mathbb{R} \cup \{\infty\}$, then the circle (or straight line if one of them is ∞) determined by z_1, z_2, z_3 is not on $\mathbb{R} \cup \{\infty\}$; yet, since the image of z_1, z_2, z_3 is on the straight line $\mathbb{R} \cup \{\infty\}$, that means the circle determined by z_1, z_2, z_3 is mapped onto $\mathbb{R} \cup \{\infty\}$, which contradicts the fact that the preimage of the real axis should be the real axis, under the given condition.

Hence, $z_1, z_2, z_3 \in \mathbb{R} \cup \{\infty\}$. Then, based on the formula for cross ratio, the unique transformation S with $S(z_1) = 1, S(z_2) = 0$, and $S(z_3) = \infty$, has the following formula:

$$S(z) = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$

Hence, since $z_1, z_2, z_3 \in \mathbb{R} \cup \{\infty\}$, then the above transformation can be simplified to real coefficients.

For all three points being real:

$S(z)$ is in the given form above, where every coefficients are real.

For one points being ∞:

If $z_1 = \infty$:

$$S(z) = \lim_{z_1 \to \infty} \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} = \frac{z - z_2}{z - z_3}$$

If $z_2 = \infty$:

$$S(z) = \lim_{z_2 \to \infty} \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} = \frac{z_1 - z_3}{z - z_3}$$

Else if $z_3 = \infty$:

$$S(z) = \lim_{z_3 \to \infty} \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} = \frac{z - z_2}{z_1 - z_2}$$

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Question 4 *Ahlors Pg. 80 Problem 3:*
If the consecutive vertices z_1, z_2, z_3, z_4 of a quadrilateral lie on a circle, prove that

$$|z_1 - z_3| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4|$$

and interpret the result geometrically.

Pf:
First, consider the right hand side of the equation:

$$|z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4| = |z_2 - z_3| \cdot |z_1 - z_4| \cdot \left(\left| \frac{(z_1 - z_2) \cdot (z_3 - z_4)}{(z_2 - z_3) \cdot (z_1 - z_4)} \right| + 1 \right)$$

Then, recall that the cross ratio of (z_1, z_3, z_2, z_4) can be expressed as:

$$(z_1, z_3, z_2, z_4) = \frac{(z_1 - z_2) \cdot (z_3 - z_4)}{(z_1 - z_4) \cdot (z_3 - z_2)}$$

Hence, the above expression can be rewritten as:

$$|z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4| = |z_2 - z_3| \cdot |z_1 - z_4| \cdot \left(\left| - \frac{(z_1 - z_2) \cdot (z_3 - z_4)}{(z_3 - z_2) \cdot (z_1 - z_4)} \right| + 1 \right)$$
$$= |z_2 - z_3| \cdot |z_1 - z_4| \cdot (| - (z_1, z_3, z_2, z_4)| + 1)$$

Notice that since z_1, z_2, z_3, z_4 is consecutive vertices on a circle, then the cross ratio is real; furthermore, by the statement in **Question 6**, since z_1, z_3, z_4 and z_2, z_3, z_4 have the same orientation, hence the cross ratio $(z_1, z_2, z_3, z_4) > 0$.

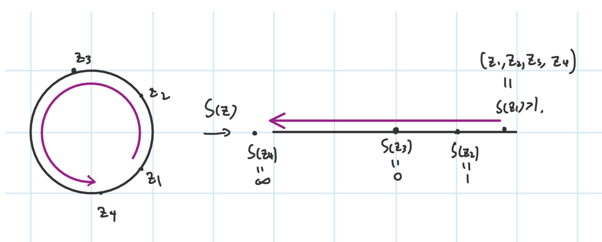

Figure 3: Cross Ratio of (z_1, z_2, z_3, z_4)

Similarly, when viewing in order of z_1, z_3, z_2, z_4, the orientation z_1, z_3, z_4 and z_3, z_2, z_4 are different, hence the cross ratio $(z_1, z_3, z_2, z_4) < 0$. (In **Figure 4**)

Then, since $(z_1, z_3, z_2, z_4) < 0$, $-(z_1, z_3, z_2, z_4) > 0$, hence $|-(z_1, z_3, z_2, z_4)| = -(z_1, z_3, z_2, z_4)$. The above identity becomes:

$$|z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4| = |z_2 - z_3| \cdot |z_1 - z_4| \cdot (| - (z_1, z_3, z_2, z_4)| + 1)$$

$$|z_2 - z_3| \cdot |z_1 - z_4| \cdot | - (z_1, z_3, z_2, z_4) + 1|$$

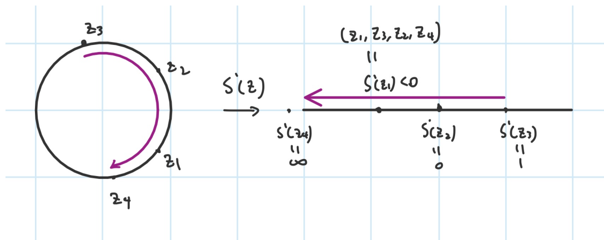

Figure 4: Cross Ratio of (z_1, z_3, z_2, z_4)

Compute the third term in the equation, we get:

$$-(z_1, z_3, z_2, z_4) + 1 = -\frac{(z_1 - z_2) \cdot (z_3 - z_4)}{(z_3 - z_2) \cdot (z_1 - z_4)} + 1$$

$$= \frac{(z_3 - z_2)(z_1 - z_4) - (z_1 - z_2)(z_3 - z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

$$= \frac{(z_1 z_3 - z_1 z_2 - z_3 z_4 + z_2 z_4) - (z_1 z_3 - z_1 z_4 - z_2 z_3 + z_2 z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

$$= \frac{-z_1 z_2 - z_3 z_4 + z_1 z_4 + z_2 z_3}{(z_3 - z_2)(z_1 - z_4)} = \frac{z_1(z_4 - z_2) + z_3(z_2 - z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

$$= \frac{(z_3 - z_1)(z_2 - z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

Hence, plug back into the original equation, we get:

$$|z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4| = |z_2 - z_3| \cdot |z_1 - z_4| \cdot |-(z_1, z_3, z_2, z_4) + 1|$$

$$= |z_2 - z_3| \cdot |z_1 - z_4| \cdot \left| \frac{(z_3 - z_1)(z_2 - z_4)}{(z_3 - z_2)(z_1 - z_4)} \right| = |(z_3 - z_1)(z_2 - z_4)|$$

So, the original original formula is true:

$$|z_3 - z_1| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4|$$

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Question 5 *Ahlfors Pg. 83 Problem 4:*
Find the linear transformation which carries the circle $|z| = 2$ into $|z+1| = 1$, the point -2 into the origin, and the origin into i.

Pf:

Symmetric points:

First, for circle $|z| = 2$, since the origin 0 is not on the circle, then to find a precise map, we also need to consider its symmetric point, namely ∞. (Note: the symmetric point of the center of a circle is always ∞).

Now, consider the points they get mapped to: Since any linear transformation should preserve the symmetric points, then as 0 gets mapped to i, ∞ gets mapped to the symmetric point of i on the circle $|z+1| = 1$. The following is the computation based on the formula given in the textbook. Let $z_0 = i$, radius $r = 1$, and the center $a = -1$, then its symmetric point z_0^* is given by:

$$z_0^* = \frac{r^2}{(\bar{z_0} - \bar{a})} + a = \frac{1}{-i-(-1)} - 1 = \frac{1}{1-i} - 1 = \frac{(1+i)}{(1-i)(1+i)} - 1$$
$$= \frac{1+i}{2} - 1 = -\frac{1}{2} + \frac{1}{2}i$$

Hence, under the desired linear transformation, ∞ gets mapped to $z_0^* = -\frac{1}{2} + \frac{1}{2}i$.

Formula for Linear Transformation:

Given that $-2 \mapsto 0$, $0 \mapsto i$, and $\infty \mapsto (-\frac{1}{2} + \frac{1}{2}i)$, consider the following map:

$$f(z) = \frac{-(1-i)z - 2(1-i)}{2z + 2(1+i)}$$

Which, it maps the given point as follow:

$$f(-2) = \frac{-(1-i)(-2) - 2(1-i)}{2(-2) + 2(1+i)} = \frac{0 \cdot (1-i)}{-4+2+2i} = 0$$

$$f(0) = \frac{-(1-i)\cdot 0 - 2(1-i)}{2\cdot 0 + 2(1+i)} = \frac{-2(1-i)}{2(1+i)} = -\frac{(1-i)^2}{(1-i)(1+i)} = -\frac{1-1-2i}{1+1} = \frac{2i}{2} = i$$

$$f(\infty) = \lim_{z\to\infty} \frac{-(1-i)z - 2(1-i)}{2z + 2(1+i)} = \frac{-(1-i)}{2} = -\frac{1}{2} + \frac{1}{2}i$$

Hence, the given linear transformation maps the three points to the correct locations.

Circle Maps to Circle:

To verify that $|z| = 2$ gets mapped to $|z+1| = 1$, it suffices to show that three points on $|z| = 2$ get mapped onto $|z+1| = 1$.

First, we already have $-2 \mapsto 0$, which is a point satisfying the condition.

Now, consider the point $2i, -2i$ on the circle $|z| = 2$:

$$f(2i) = \frac{-(1-i)2i - 2(1-i)}{2\cdot 2i + 2(1+i)} = \frac{-2(1+i)(1-i)}{2+6i} = \frac{-2}{1+3i} = \frac{-2(1-3i)}{(1+3i)(1-3i)} = \frac{-2+6i}{10} = \frac{-1+3i}{5}$$

$$f(-2i) = \frac{-(1-i)(-2i) - 2(1-i)}{2(-2i) + 2(1+i)} = \frac{-2(1-i)(1-i)}{2-2i} = -(1-i)$$

Then, consider the distance $|f(2i)+1|$ and $|f(-2i)+1|$, we get:

$$|f(2i)+1| = \left|\frac{-1+3i}{5}+1\right| = \left|\frac{4+3i}{5}\right| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = 1$$

$$|f(-2i)+1| = |-(1-i)+1| = |i| = 1$$

Hence, $f(2i), f(-2i)$ are two points on the circle $|z+1| = 1$.

Since $-2, 2i, -2i$ are three points on the circle $|z| = 2$, and they get mapped to points on $|z+1| = 1$ by the linear transformation f, hence $|z| = 2$ is mapped to $|z+1| = 1$, showing that f is in fact the desired linear transformation.

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Question 6 *Ahlfors Pg. 84 Problem 1:*
If z_1, z_2, z_3, z_4 are points on a circle, show that z_1, z_3, z_4 and z_2, z_3, z_4 determine the same orientation if and only If $(z_1, z_2, z_3, z_4) > 0$.

Pf:

Given four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}$, to determine the cross ratio (z_1, z_2, z_3, z_4), it is given by the linear transformation that gives $z_2 \mapsto 1$, $z_3 \mapsto 0$, and $z_4 \mapsto \infty$.

If consider the orientation as z_2, z_3, z_4 respectively:

If z_1, z_3, z_4 has the same orientation as above, then z_1, z_2 needs to be on the same arc when the circle is separated by z_3 and z_4.

Which, this happens if the linear transformation would transform z_1, z_2 onto the same side of the real line, so z_1 gets mapped to a positive value. Hence, $(z_1, z_2, z_3, z_4) > 0$.

Conversely, if $(z_1, z_2, z_3, z_4) > 0$, then z_1 gets mapped to a positive value on the real axis. Which, since the orienation is given by z_2, z_3, z_4 in order, and z_1, z_2 both get mapped to positive values while z_3 gets mapped to 0, hence z_1, z_2 must be on the same side of the circle when the circle is separated by z_3, z_4, the orientation z_1, z_3, z_4 must have the same orientation as z_2, z_3, z_4.

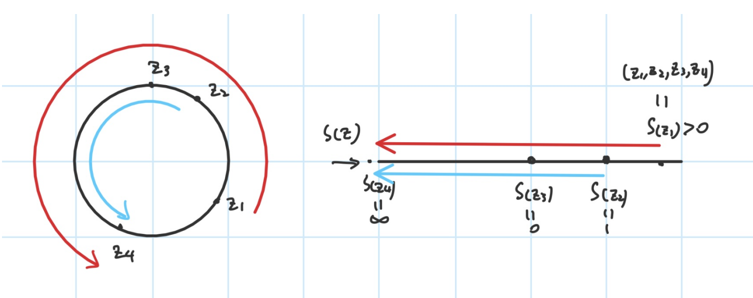

Figure 5: Cross Ratio and Orientation

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Question 7 *Ahlfors Pg. 88 Problem 6:*
Find all circles which are orthogonal to $|z| = 1$ and $|z-1| = 4$.

Pf:

Given the two circles $|z| = 1$ and $|z-1| = 4$, notice that the relationship is similar to the Circle of Apollonius (which there exists a fixed ratio for any circle, such that the distance from any points on the circle to some two points k, k_0 always form that fixed ratio). Then, the two limit points k, k_0 of the given circles, every circle in the system has a center collinear to the two limit points. Hence, with both circles $|z| = 1$ and $|z-1| = 4$ that have center 0 and 1, we can conclude that the limit points both lie on the real axis ($k, k_0 \in \mathbb{R} \cup \{\infty\}$).

Also, the two limit points are in fact symmetric points under any circle in the given system, so they must satisfy the following relation:

$$r_1 = 1, \quad c_1 = 0, \quad k_0 = \frac{r_1^2}{\bar{k} - \bar{c_1}} + c_1 = \frac{1}{k}$$

$$r_2 = 4, \quad c_2 = 1, \quad k_0 = \frac{r_2^2}{\bar{k} - \bar{c_2}} + c_2 = \frac{16}{k-1} + 1$$

(Note: the above equations are based on the relation of symmetric points with respect to each circle, where r is the radius and c is the center; and, since $k \in \mathbb{R} \cup \{\infty\}$, can assume $\bar{k} = k$).

Hence, we can deduce the following:

$$\frac{1}{k} = \frac{16}{k-1} + 1, \quad (k-1) = 16k + k(k-1), \quad k^2 + 14k + 1 = 0$$

$$k = -7 \pm 4\sqrt{3}$$

Which, take $k = -7 + 4\sqrt{3}$, $k_0 = \frac{1}{k} = -7 - 4\sqrt{3}$, so the two points satisfy the given condition.

(Note: in **HW 2, Ahlfors Pg. 33 Problem 4**, if the modulus of a circle after a rational function is constant, then the pole w^* and the zero w satisfy the relation $w^* = \frac{1}{\bar{w}}$, the two points are the symmetric points under linear transformation. Conversely, we can also conclude that since the given circles have k, k_0 being the symmetric points, the distance from the circle to the two points form a constrant ratio).

Now, if we consider the linear transformation $w = \frac{z-k}{z-k_0}$, based on the previous logic, the circles $|z| = 1$ and $|z-1| = 4$ get mapped to concentric circles around origin (since every point on the same circle has a fixed ratio between distances from k and k_0). Hence, every circle that's orthogonal to both circles, must get mapped to circles that are orthogonal to the concentric circles through the origin in the image.

This could happen only if the image of the circle is a straight line through the origin, which is parametrized by $w = w_0 t$ for nonzero number $w_0 \in \mathbb{C}$, and $t \in \mathbb{R}$.

To reconstruct the preimage, consider the inverse of the above linear transformation, given by $z = \frac{k_0 w - k}{w-1}$. Then, each straight line through the origin $w = w_0 t$, has the preimage $z = \frac{k_0 w_0 t - k}{w_0 t - 1}$, which is a circle passing through both k and k_0 (Take $t = 0$, $z = \frac{-k}{-1} = k$; else, take $t = \infty$, then $z = \lim_{t \to \infty} \frac{k_0 w_0 t - k}{w_0 t - 1} = \frac{k_0 w_0}{w_0} = k_0$).

So, any circle passing through both $k_0 = -7 + 4\sqrt{3}$ and $k = -7 - 4\sqrt{3}$ is a circle orthogonal to both circles $|z| = 1$ and $|z-1| = 4$.

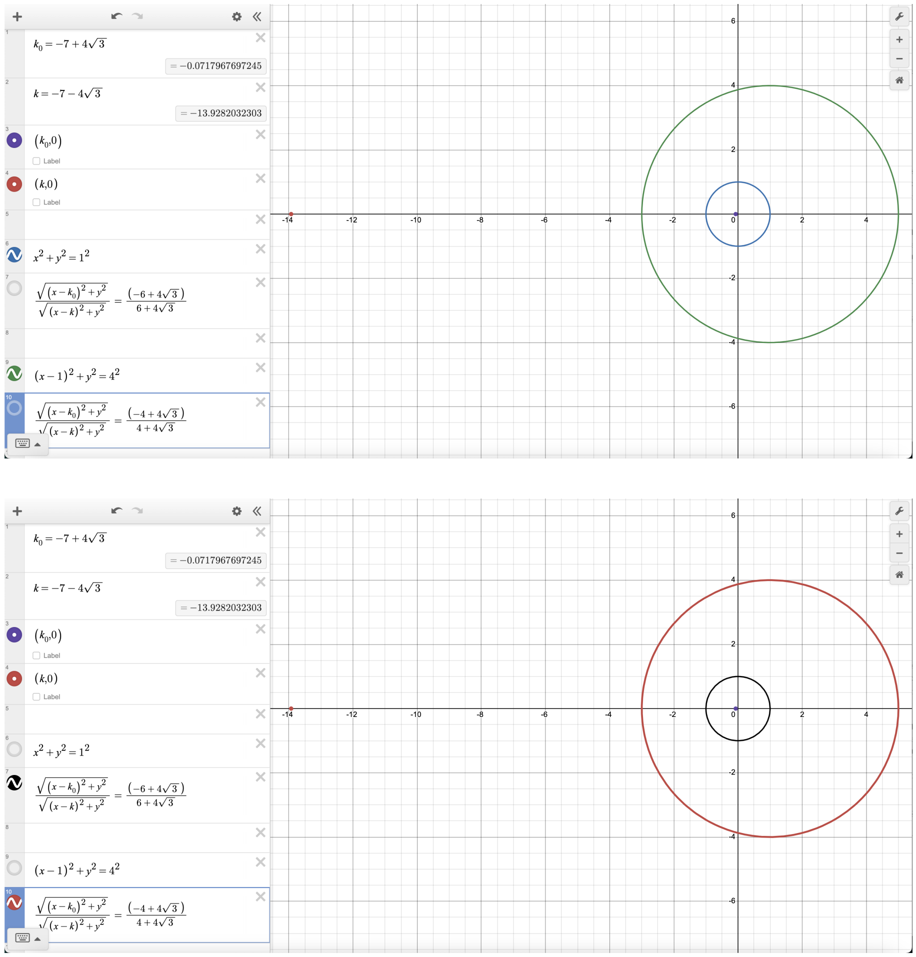

Figure 6: Comparison of Circles with Fixed Distance Ratio to k, k_0

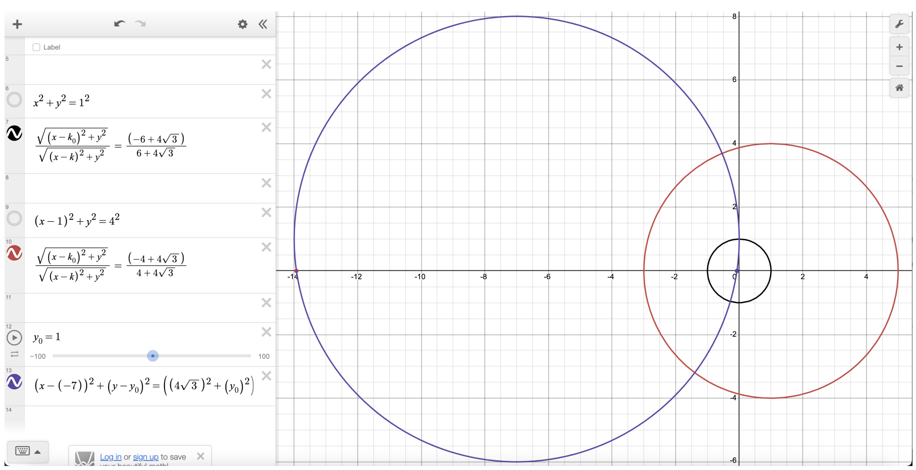

Figure 7: Parametrization of Circles orthogonal to given circles