Math CS 122A HW6

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Question 1 *Ahlfors Pg. 123 Problem 2:*
Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some nand all sufficiently large $|z|$ reduces to a polynomial.

Pf:

Given that for $z \in \mathbb{C}$ with $|z|$ being sufficiently large, $|f(z)| < |z|^n$ is satisfied, then there exists a radius $r > 0$, such that $|z| \geq r$ implies $|f(z)| < |z|^n$. Which, we'll consider the n^{th} derivative, $f^{(n)}(z)$. (Note: Since f is analytic on the whole plane, then all of its derivative exists, and is analytic on the whole plane).

First, consider the disk $D_{2r} = \{z \in \mathbb{C} \mid |z| \leq 2r\}$: Since it is a closed and bounded set, then it is compact. Hence, since $|f^{(n)}|$ is also continuous due to the analytic nature of $f^{(n)}$, then $|f^{(n)}|(D_{2r}) \subseteq \mathbb{R}$ is also compact, there exists $M > 0$, such that for all $z \in D_{2r}$, $|f^{(n)}(z)| \leq M$.

Then, for all $z \in \mathbb{C} \setminus D_{2r}$, we'll consider $f^{(n)}(z)$ using Cauchy's Integral Formula: Let γ be the curve of the circle $|z| = r$, then for all z not on the given circle, the following is true:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

Which, for $z \in \mathbb{C} \setminus D_{2r}$, since $|z| > 2r > r$, then for all $\zeta \in \gamma$ (which $|\zeta| = r$), the following is true:

$$|\zeta - z| \geq ||\zeta| - |z|| = |r - |z|| = |z| - r > 2r - r = r, \quad |\zeta - z|^{n+1} > r^{n+1}, \quad \frac{1}{|\zeta - z|^{n+1}} < \frac{1}{r^{n+1}}$$

Similarly, since $|\zeta| \geq r$, then based on the assumption, $|f(\zeta)| < |\zeta|^n = r^n$. Hence, the following inequality is true:

$$|f^{(n)}(z)| = \left| \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right| \leq \frac{n!}{2\pi} \int_\gamma \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} \cdot |d\zeta| < \frac{n!}{2\pi} \int_\gamma \frac{r^n}{r^{n+1}} \cdot |d\zeta|$$

$$|f^{(n)}(z)| < \frac{n!}{2\pi} \cdot \frac{1}{r} \cdot 2\pi r = n!$$

(Note: the first inequality is true, based on the statement that $|f(\zeta)| < r^n$ and $\frac{1}{|\zeta - z|^{n+1}} < \frac{1}{r^{n+1}}$).

Hence, take $M' = \max\{M, n!\}$, then for all $z \in \mathbb{C}$, if $z \in D_{2r}$, then $|f^{(n)}(z)| \leq M \leq M'$; else if $z \in \mathbb{C} \setminus D_{2r}$, then $|f^n(z)| \leq n! \leq M'$. So, the analytic function $f^{(n)}(z)$ is bounded on the whole plane, which by Liouville's Theorem, $f^{(n)}(z)$ must be a constant function.

Then, since the n^{th} derivative of f is a constant, then f must be a polynomial (in fact, with degree at most n).

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Question 2 *Ahlfors Pg. 123 Problem 5:*
Show that the successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(z)| > n!n^n$. Formulate a sharper theorem of the same kind.

Pf:

Let the analytic function f be defined on an open set Ω, which for all $z_0 \in \Omega$, there exists $r' > 0$, such that the open disk $|z - z_0| < r'$ is within Ω. If we let $r = \frac{r'}{2} > 0$, then the closed disk $|z - z_0| \leq r$ is fully contained in $|z - z_0| < r'$, which is within Ω.

Now, let γ be the circle $|z - z_0| = r$, since it is a compact set where $|f|$ is continuous since f is analytic, then $|f|(\gamma) \subseteq \mathbb{R}$ has a maximum, there exists $M > 0$, such that for all $z \in \gamma$, $|f(z)| \leq M$ (For simplicity, choose $M \geq 1$).

Hence, based on Cauchy's Integral Formula, for all $n \in \mathbb{N}$, the following formula is true:

$$f^{(n)}(z_0) = \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\right| \leq \frac{n!}{2\pi}\int_\gamma \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} \cdot |d\zeta| \leq \frac{n!}{2\pi}\int_\gamma \frac{M}{r^{n+1}} \cdot |d\zeta|$$

$$f^{(n)}(z_0) \leq \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{n!M}{r^n}$$

(Note: For all $\zeta \in \gamma$, $|\zeta - z_0| = r$, and $|f(\zeta)| \leq M$).

Notice that since $\frac{M}{r} > 0$, then by Archimedean's Property, there exists $k \in \mathbb{N}$, with $k > \frac{M}{r}$, which since $M \geq 1$ is assumed, the following inequality is true:

$$k^k > \left(\frac{M}{r}\right)^k = \frac{M^k}{r^k} \geq \frac{M}{r^k}, \quad |f^{(k)}(z_0)| \leq \frac{k!M}{r^k} < k!k^k$$

Also, for all integer $n \geq k$, the following is satisfied:

$$n^n \geq k^n > \left(\frac{M}{r}\right)^n = \frac{M^n}{r^n} \geq \frac{M}{r^n}, \quad |f^{(n)}(z_0)| \leq \frac{n!M}{r^n} < n!n^n$$

So, for all $z_0 \in \Omega$, there exists $k \in \mathbb{N}$, such that $n \geq k$ implies $|f^{(n)}(z_0)| < n!n^n$, showing that $|f^{(n)}(z)| > n!n^n$ can never be satisfied by any point z and for all but finitely many $n \in \mathbb{N}$.

Stronger Condition:

Recall that for all $r_0 > 0$, by Archimedean's Property, there exists $N \in \mathbb{N}$ with $N > r_0$. Therefore, for $n \geq N$, $\frac{r_0^{n+1}}{(n+1)!} = \frac{r_0}{(n+1)} \cdot \frac{r_0^n}{n!} < \frac{r_0}{N} \cdot \frac{r_0^n}{n!}$, which for all positive integer k, we can inductively prove that $\frac{r_0^{N+k}}{(N+k)!} < \left(\frac{r_0}{N}\right)^k \cdot \frac{r_0^N}{N!}$.

Hence, since $\frac{r_0}{N} < 1$, then the following is true:

$$0 < \frac{r_0^{N+k}}{(N+k)!} < \left(\frac{r_0}{N}\right)^k \cdot \frac{r_0^N}{N!}$$

$$0 \leq \lim_{k\to\infty} \frac{r_0^{N+k}}{(N+k)!} \leq \lim_{k\to\infty}\left(\frac{r_0}{N}\right)^k \cdot \frac{r_0^N}{N!} = 0$$

Which, $\lim_{n\to\infty} \frac{r_0^n}{n!} = 0$ based on the above inequality, so there exists $N \in \mathbb{N}$, such that $n \geq N$ implies $\frac{r_0^n}{n!} < 1$, or $r_0^n < n!$.

Then, looking back to the inequality $|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}$, since $\frac{M^{1/n}}{r} > 0$, there exists N, such that $n \geq N$ implies $\frac{M}{r^n} = \left(\frac{M^{1/n}}{r}\right)^n < n!$. Hence, the following inequality is true:

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n} < n! \cdot n! = (n!)^2$$

So, we can conclude that for some $N \in \mathbb{N}$, $n \geq N$ implies $|f^{(n)}(z_0)| < (n!)^2$, which is a stricter condition than $n!n^n$, since $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ (Because this implies that for all sufficiently large n, $n! < n^n$).

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Question 3 *Ahlfors Pg. 130 Problem 2:*
Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ reduces to a polynomial.

Pf:

Before starting the proof, we'll first prove a lemma for this problem:

Lemma 1 *Suppose f is analytic in the whole plane, and $\lim_{z\to\infty} f(z)$ converges to some value in $\mathbb{C}$, then f is a constant function.*

To show this, since $\lim_{z\to\infty} f(z) = L$ for some $L \in \mathbb{C}$, then for all $\epsilon > 0$, there exists $M_1 > 0$, such that $|z| > M_1$ implies $|f(z) - L| < \epsilon$, hence $|f(z)| = |(f(z) - L) + L| \leq |f(z) - L| + |L| < |L| + \epsilon$.

Also, consider the closed disk $\mathbb{D}_{M_1}$: Since it is a bounded closed set, then it is compact. Hence, since $|f|$ is continuous, $|f|(\mathbb{D}_{M_1}) \subseteq \mathbb{R}$ is also a compact set, hence there exists $M_2 > 0$, such that for all $z \in \mathbb{D}_{M_1}$, $|f(z)| \leq M_2$.

Then, fix $\epsilon > 0$ and take $M = \max\{|L| + \epsilon, M_2\}$, for all $z \in \mathbb{C}$, if $z \in \mathbb{D}_{M_1}$, then $|f(z)| \leq M_2 \leq M$; else if $z \notin \mathbb{D}_{M_1}$, then $|z| > M_1$, hence $|f(z)| < |L| + \epsilon \leq M$. This shows that f is bounded on the whole plane, which by Liouville's Theorem, f must be a constant function, and the Lemma has been proven.

Function is a Polynomial:

Given that f only has a nonessential singularity at ∞, while analytic on the whole plane, there are two cases to consider: First, if f is a removable singularity (which $\lim_{z\to\infty} f(z) = L$ for some $L \in \mathbb{C}$), or f has a pole at ∞.

For the first case, since f is analytic on the whole plane, while $lim_{z\to\infty} f(z)$ converges to some $L \in \mathbb{C}$, then apply **Lemma 1**, we get that f is a constant function, which is a polynomial.

For the second case, take $g(z) = f(\frac{1}{z})$ as the analytic function, then since $\lim_{z\to 0} g(z) = \lim_{z\to 0} f(\frac{1}{z}) = \infty$ ($f(z)$ has a pole at ∞, which $g(z) = f(\frac{1}{z})$ has a pole at 0), then suppose g has the pole at 0 with order $h \in \mathbb{N}$, there exists some analytic function $g_h(z)$ that is analytic at 0, with $g(z) = \frac{1}{z^h} g_h(z)$.

Notice that since $g(z) = f(\frac{1}{z}) = \frac{1}{z^h} g_h(z)$ is analytic for all $z \neq 0$ (because $\frac{1}{z} \in \mathbb{C}$, while f is analytic on the whole plane), then $g_h(z) = z^h g(z)$ is analytic at all $z \neq 0$, while also analytic at 0 from the above statement, hence $g_h(z)$ is analytic on the whole plane $\mathbb{C}$.

Also, $\lim_{z\to\infty} g(z)$ must also converge to some value in $\mathbb{C}$, since it is equivalent to $\lim_{z\to\infty} f(\frac{1}{z}) = \lim_{z\to 0} f(z) = f(0)$. Then, this implies for all $\epsilon > 0$, there exists $M > 0$, such that $|z| > M$ implies $|g(z) - f(0)| < \epsilon$, hence, the following is true:

$$|g(z)| = |(g(z) - f(0)) + f(0)| \leq |g(z) - f(0)| + |f(0)| < |f(0)| + \epsilon$$

$$|g(z)| = \left|\frac{1}{z^h} g_h(z)\right| < |f(0)| + \epsilon, \quad |g_h(z)| < (|f(0)| + \epsilon)|z|^h$$

Which, let $C = |f(0)| + \epsilon > 0$, for all z with $|z| > M$, we have $|g_h(z)| < C|z|^h$; which let $M' = \max\{M, C\}$, when $|z| > M'$, since $|z| > C$ and $|z| > M$, the following is true:

$$|g_h(z)| < C|z|^h < |z|^{h+1}$$

Now, apply the statement proven in **Question 1**, since $g_h(z)$ is analytic on the whole plane, while for sufficiently large $|z|$, $|g_h(z)| < |z|^{h+1}$ (with $(h+1) \in \mathbb{N}$ being fixed), then $g_h(z)$ reduces to a polynomial, and with degree at most $(h+1)$. Hence, $g_h(z) = a_0 + a_1 z + ... + a_{h+1} z^{h+1}$ for some $a_0, a_1, ..., a_{h+1} \in \mathbb{C}$.

Then, recall the fact that $f(\frac{1}{z}) = g(z)$, then $f(z) = g(\frac{1}{z}) = \frac{1}{(1/z)^h} g_h(\frac{1}{z})$, we get the following:

$$f(z) = z^h \cdot \left(a_0 + a_1 \frac{1}{z} + ... + a_{h+1} \frac{1}{z^{h+1}}\right) = a_0 z^h + a_1 z^{h-1} + ... + a_h + a_{h+1} \frac{1}{z}$$

Also, notice that since $f(0)$ is defined, then we must have $a_{h+1} = 0$ (or else the above function wouldn't be defined at 0).

Hence, $f(z) = a_0 z^h + a_1 z^{h-1} + ... + a_h$, which is a polynomial.

Regardless of the case, we always get the fact that $f(z)$ is a polynomial.

Hence, having f being analytic on the whole plane, while having a nonessential singularity at ∞ implies that $f(z)$ must be a polynomial.

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Question 4 *Ahflors Pg. 130 Problem 6:*
Show that an isolated singularity of $f(z)$ *cannot be a pole of* $\exp(f(z))$.

Pf:

Suppose $f(z)$ has an isolated singularity at $a \in \mathbb{C}$, there are three cases to consider: Whether a is a removable singularity, an essential singularity, or a pole at a.

When f has a Removable Singularity at a:

Since it's removable at a, $\lim_{z\to a} f(z) = L$ for some $L \in \mathbb{C}$, hence $\lim_{z\to a} e^{f(z)} = e^L \in \mathbb{C}$, so it is also a removable singularity for e^f, not a pole.

When f has an Essential Singularity at a:

Since it is an essential singularity at a, for all $z \in \mathbb{C}$ (except possibly finitely many points), every neighborhood of a has value of f being arbitrarily close to z, then take $\log(z) \in \mathbb{C}$ (except for $z = 0$), since f can get arbitrarily close to $\log(z)$ in any of the neighborhood of a, then e^f can get arbitrarily close to $e^{log(z)} = z$ for every neighborhood around a, showing that e^f also has an essential singularity at a, instead of a pole.

When f has a Pole at a:

Suppose f has a pole of order h at a, then $f(z) = \frac{1}{(z-a)^h} f_h(z)$ for some analytic function $f_h(z)$ that's analytic at a (so, $f_h(a)$ is defined, and $f_h(a) \neq 0$).

First, for all $w \in \mathbb{C}$, $w = |w| \cdot e^{i\cdot \arg(w)}$, for $k \in \mathbb{N}$, define $w^{\frac{1}{k}} = |w|^{\frac{1}{k}} \cdot e^{i\cdot \frac{\arg(w)}{k}}$ for simplicity.

Now, for all $z \in \mathbb{C}$ (where $z \neq 0$), since $e^{f_h(a)} \neq 0$, then $z \cdot e^{f_h(a)}$ can be arbitrary member of $\mathbb{C} \setminus \{0\}$. Consider the sequence $(z_n)_{n\in\mathbb{N}}$ defined by:

$$\forall n \in \mathbb{N}, \quad z_n = \left(\frac{1}{\ln|z| + i\cdot(\arg(z) + 2n\pi)}\right)^{\frac{1}{h}} + a$$

Then, since as $\lim_{n\to\infty} z_n = a$ (because the first fraction term converges to 0 due to an unbounded denominator), then $\lim_{n\to\infty} f(z_n) = \infty$.

However, when consider $e^{f(z_n)}$, we yield the following:

$$e^{f(z_n)} = \exp\left(\frac{1}{(z_n - a)^h} f_h(z_n)\right) = \exp\left(\frac{1}{\left(\left(\frac{1}{\ln|z|+i\cdot(\arg(z)+2n\pi)}\right)^{\frac{1}{h}} + a - a\right)^h} f_h(z_n)\right)$$

$$= \exp\left(\frac{1}{1/(\ln|z| + i\cdot(\arg(z) + 2n\pi))} f_h(z_n)\right) = \exp\left(\ln|z| + i\cdot(\arg(z) + 2n\pi)\right) \cdot e^{f_h(z_n)}$$
$$= z \cdot e^{f_h(z_n)}$$

Which, taking $n \to \infty$, we get the following:

$$\lim_{n\to\infty} e^{f(z_n)} = \lim_{n\to\infty} z e^{f_h(z_n)} = z \cdot e^{f_h(a)}$$

Since $z \cdot e^{f_h(a)}$ can be arbitrary within $\mathbb{C} \setminus \{0\}$, then for any neighborhood of a, the function $e^{f(z)}$ can be close to arbitrary values besides 0 (since there exists a sequence $(z_n)_{n\in\mathbb{N}}$ such that $e^{f(z_n)}$ converging to such arbitrary value), then e^f has an essential singularity at a, instead of having a pole.

Hence, in all cases, when f has an isolated singularity at a, e^f has either a removable, or essential singularity at a, showing that an isolated singularity of $f(z)$ cannot be a pole of $e^{f(z)}$.

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Question 5 *Stein and Shakarchi Pg. 66 Problem 7:*
Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies $2|f'(0)| \leq d$.
Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.

Pf:

Fix an $r \in (0, 1)$, and let γ be the circle $|z| = r$. Then, based on Cauchy's Integral Formula, $f'(0)$ can be extracted as:

$$f'(0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^2}d\zeta$$

Which, if consider $f(-z)$ instead, its derivative is given by $-f'(-z)$, hence with $z = 0$, the following is true:

$$-f'(0) = \frac{1}{2\pi i}\int_\gamma \frac{f(-\zeta)}{\zeta^2}d\zeta, \quad f'(0) = -\frac{1}{2\pi i}\int_\gamma \frac{f(-\zeta)}{\zeta^2}d\zeta$$

Which, adding the two equations together, we yield:

$$2f'(0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta) - f(-\zeta)}{\zeta^2}d\zeta$$

Given that $g(z) = f(z) - f(-z)$ is an analytic function (sum and composition of analytic functions), and γ is a compact set (boundary of a bounded circle), then since $|g|$ is continuous, $|g|(\gamma) \subseteq \mathbb{R}$ is a compact set. Hence, there exists a maximum on γ, for some $z_0 \in \gamma$, all $z \in \gamma$ satisfies:

$$|f(z) - f(-z)| = |g(z)| \leq |g(z_0)| = |f(z_0) - f(-z_0)|$$

And, notice that since $z_0, -z_0$ have modulus r (since they're on γ), then $z_0, -z_0 \in \mathbb{D}$. Therefore, $|g(z_0)| = |f(z_0) - f(-z_0)| \leq \sup_{z,w\in\mathbb{D}} |f(z) - f(w)| = d$, thus for all $z \in \gamma$, $|g(z)| \leq d$.

Now, going back to the third equation, since for all $\zeta \in \gamma$, it satisfies $|g(\zeta)| \leq d$ and $|\zeta| = r$, we can conclude the following inequality:

$$2|f'(0)| = |2f'(0)| = \left|\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta) - f(-\zeta)}{\zeta^2}d\zeta\right| \leq \frac{1}{2\pi}\int_\gamma \frac{|g(\zeta)|}{|\zeta|^2} \cdot |d\zeta| \leq \frac{1}{2\pi}\int_\gamma \frac{d}{r^2} \cdot |d\zeta|$$

$$2|f'(0)| \leq \frac{1}{2\pi} \cdot \frac{d}{r^2} \cdot 2\pi r = \frac{d}{r}$$

Also, notice that the above inequality is true for all $r \in (0, 1)$.

(Note: since for all $r \in (0,1)$, $\frac{1}{r} > 1$, then $\frac{d}{r} > d$; on the other hand, for all $\epsilon > 0$, since $d + \epsilon/2 = d(1 + \frac{\epsilon}{2d}) = \frac{d}{1/(1+\frac{\epsilon}{2d})}$, because $(1 + \frac{\epsilon}{2d}) > 1$, then $0 < 1/(1 + \frac{\epsilon}{2d}) < 1$, hence $(1 + \frac{\epsilon}{2d}) \in (0,1)$, or $d + \epsilon/2 = \frac{d}{(1+\frac{\epsilon}{2d})} \in \{\frac{d}{r} \mid r \in (0,1)\}$. However, since $d + \epsilon/2 < d + \epsilon$, then $d + \epsilon$ is not a lower bound of the set. Hence, $d = \inf\{\frac{d}{r} \mid r \in (0,1)\}$).

Since $d = \inf\{\frac{d}{r} \mid r \in (0,1)\}$,then, since $2|f'(0)| \leq \frac{d}{r}$ for all $r \in (0,1)$, then $2|f'(0)|$ is a lower bound of $\{\frac{d}{r} \mid r \in (0,1)\}$, hence $2|f'(0)| \leq \inf\{\frac{d}{r} \mid r \in (0,1)\} = d$.

Equality under Linear Functions:

Let $f(z) = a_0 + a_1 z$ for $a_0, a_1 \in \mathbb{C}$, then $f'(z) = a_1$, hence $2|f'(0)| = 2|a_1|$. (Note: if $|a_1| = 0$, then $f' = 0$, which f is a constant, hence $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)| = 0$ regardless; so, assume $|a_1| \neq 0$).

Now, for all $z, w \in \mathbb{D}$ (which $|z|, |w| < 1$), $f(z) - f(w) = (a_0 + a_1 z) - (a_0 + a_1 w) = a_1(z - w)$, which:

$$|f(z) - f(w)| = |a_1| \cdot |z - w| \leq |a_1| \cdot (|z| + |w|) < 2|a_1|$$

So, we can first conclude that $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)| \leq 2|a_1|$ (since $2|a_1|$ is an upper bound).

Then, for all $\epsilon > 0$ (and let $\epsilon < 4|a_1|$ for simplicity), consider $2|a_1| - \epsilon$: Let $z = (1 - \frac{\epsilon}{4|a_1|}) \in \mathbb{D}$ (which $0 < z < 1$, since $0 < \frac{\epsilon}{4|a_1|} < \frac{4|a_1|}{4|a_1|} = 1$), consider $|f(z) - f(-z)|$:

$$|f(z) - f(-z)| = |a_1| \cdot |z - (-z)| = |a_1| \cdot |2z| = 2|a_1| \cdot |z| = 2|a_1| \cdot z = 2|a_1| \left(1 - \frac{\epsilon}{4|a_1|}\right)$$

$$|f(z) - f(-z)| = 2|a_1| - \frac{\epsilon}{2} > 2|a_1| - \epsilon$$

Since $z, -z \in \mathbb{D}$, then $|f(z) - f(-z)| > (2|a_1| - \epsilon)$ indicates that $(2|a_1| - \epsilon)$ is no longer an upper bound. Hence, $2|a_1|$ is in fact $\sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$.

So, recall that $2|f'(0)| = 2|a_1|$, then $2|f'(0)| = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)| = d$, which the equality is true.

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Question 6 *Stein and Shakarchi Pg. 66 Problem 8:*
If f is a holomorphic function on the strip $-1 < y < 1$, $x \in \mathbb{R}$ with

$$|f(z)| \leq A(1+|z|)^\eta$$

η a fixed real number and for all z in that strip.
Show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that

$$|f^{(n)}(x)| \leq A_n(1+|x|)^\eta$$

Pf:

First, WLOG, we can assume $\eta \geq 0$: For all z in the given region, since $|z| \geq 0$, then $(1+|z|) \geq 1$. Hence, if $\eta < 0$ (which $|\eta| > 0$, and $\eta = -|\eta|$), since $(1+|z|)^{|\eta|} \geq (1+|z|) \geq 1$, the following inequality is true:

$$|f(z)| \leq A(1+|z|)^\eta \leq A(1+|z|)^{-|\eta|} \cdot (1+|z|)^{|\eta|} = A(1+|z|)^0$$

Hence, choose $\eta' = 0$, $|f(z)| \leq A(1+|z|)^{\eta'}$ is still valid. Also, since $|f(z)| \geq 0$, and $(1+|z|)^\eta > 0$ for all z in the given strip, then $A \geq 0$ is also enforced.

Now, given any $x \in \mathbb{R}$, it is also within the given region (since $y = Im(x) = 0$). Then, choose radius $r = \frac{1}{2}$, construct the circle γ_x by $|z-x| = r$: Since for all $z \in \gamma_x$, $|Im(z)| = |Im(z) - Im(x)| \leq |z-x| = \frac{1}{2}$, then z is also within the given region (since $-1 < -\frac{1}{2} < Im(z) < \frac{1}{2} < 1$), so it is a valid circle to use or Cauchy's Integral Formula.

Construction of A_n:

Based on Cauchy's Integral Formula, the n^{th} derivative can be governed by the following:

$$f^{(n)}(x) = \frac{n!}{2\pi i}\int_{\gamma_x} \frac{f(\zeta)}{(\zeta - x)^{n+1}} d\zeta$$

Which, for all $\zeta \in \gamma_x$, $|\zeta - x|^{n+1} = r^{n+1}$; furthermore, since $|f(\zeta)| \leq A(1+|\zeta|)^\eta$, while $\zeta = x + re^{i\theta}$ for some $\theta \in [0, 2\pi)$, hence by Triangle Inequality, the following is true:

$$|\zeta| = |x + re^{i\theta}| \leq |x| + |re^{i\theta}| = |x| + r$$

Hence, $(1+|\zeta|) \leq (1+|x|+r) = (\frac{3}{2}+|x|)$ (Note: $r = \frac{1}{2}$), and since $|x| \geq 0$, then $(\frac{3}{2}+|x|) \leq (\frac{3}{2}+\frac{3}{2}|x|) = \frac{3}{2}(1+|x|)$. Therefore, with $\eta \geq 0$, we have $(1+|\zeta|)^\eta \leq (\frac{3}{2})^\eta \cdot (1+|x|)^\eta$.

Now, apply these inequalities into the Integral Formula, we get the following inequality:

$$|f^{(n)}(x)| = \left|\frac{n!}{2\pi i}\int_{\gamma_x} \frac{f(\zeta)}{(\zeta - x)^{n+1}} d\zeta\right| \leq \frac{n!}{2\pi}\int_{\gamma_x} \frac{|f(\zeta)|}{|\zeta - x|^{n+1}} \cdot |d\zeta| \leq \frac{n!}{2\pi}\int_{\gamma_x} \frac{A(1+|\zeta|)^\eta}{r^{n+1}} \cdot |d\zeta|$$

$$|f^{(n)}(x)| \leq \frac{n!}{2\pi}\int_{\gamma_x} \frac{A(1+|\zeta|)^\eta}{r^{n+1}} \cdot |d\zeta| \leq \frac{n!}{2\pi r^{n+1}}\int_{\gamma_x} A(3/2)^\eta \cdot (1+|x|)^\eta \cdot |d\zeta|$$

$$|f^{(n)}(x)| \leq \frac{n! \cdot A(3/2)^\eta}{2\pi r^{n+1}} \cdot (1+|x|)^\eta \cdot 2\pi r = \frac{n! A(3/2)^\eta}{r^n}(1+|x|)^\eta$$

Recall that $r = \frac{1}{2}$, then the above expression can be simplified to $|f^{(n)}(x)| \leq 2^n n! A(3/2)^\eta \cdot (1+|x|)^\eta$. Hence, let $A_n = 2^n \cdot n! \cdot A(3/2)^\eta$, all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ satisfies $|f^{(n)}(x)| \leq A_n(1+|x|)^\eta$.