Latex Template

Zih-Yu Hsieh

February 11, 2025

1

Question 1 *Prove or disprove the statement that if k is any field, then (X^2+1) is a maximal ideal of $k[X]$.*

Pf:

Consider $k = \mathbb{C}$. Then, notice that the following is true:

$$(X+i)(X-i) = X(X-i) + i(X-i) = (X^2 - ix) + (iX - i^2) = (X^2 - (-1)) = (X^2+1)$$

So, $(X^2+1) \subset (X+i)$, since $X^2+1 \in (X+i)$.

However, $(X^2+1) \subsetneq (X+i)$, since $X+i \notin (X^2+1)$: Suppose $X+i = (X^2+1)h(X)$ for some $h(X) \in \mathbb{C}[X]$, then since $X+i \neq 0$, then $h(X) \neq 0$; also, since $\mathbb{C}[X]$ is an integral domain, hence $1 = \deg(X+i) = \deg(X^2+1) + \deg(h(X)) \geq \deg(X^2+1) = 2$ (Note: since $\deg(h(x)) \geq 0$). However, this is a contradiction. Hence, $X+i \neq (X^2+1)h(X)$ for all $h(X) \in \mathbb{C}[X]$, showing that $X+i \notin (X^2+1)$.

Furthermore, $(X+i) \neq \mathbb{C}[X]$: Suppose $(X+i) = \mathbb{C}[X]$, then $1 \in (X+i)$, which there exists $h(X) \in \mathbb{C}[X]$, such that $(X+i)h(X) = 1$. However, since $1 \neq 0$, then $h(X) \neq 0$; also, since $\mathbb{C}[X]$ is an integral domain, then $0 = \deg(1) = \deg(X+i) + \deg(h(X)) \geq \deg(X+i) = 1$, which is a contradiction. Hence, $(X+i) \neq \mathbb{C}[X]$.

So, $(X+i)$ is an ideal strictly containing (X^2+1), while $(X+i) \neq \mathbb{C}[X]$, showing that (X^2+1) is not a maximal ideal of $\mathbb{C}[X]$.

2

Question 2 *Prove that if k is a field, then the map $\phi : k[X] \to k$ given by $\phi(f(X)) = f(a)$ $(a \in k)$ defines an isomorphism of rings, $\phi' : \frac{k[X]}{(X-a)} \tilde{\to} k$.*

Pf:

(Note: Possibly need to show that it is a kernel)

$(X-a)$ **is the kernel:**

Given the ring homomorphism $\phi : k[X] \to k$ defined as $\phi(f(X)) = f(a)$ ($a \in k$), for all $f(X) \in (X-a)$, since there exists $h(X) \in k[X]$, with $f(X) = (X-a)h(X)$. Hence:

$$\phi(f(X)) = f(a) = (a-a)h(a) = 0 \cdot h(a) = 0$$

This implies that $f(X) \in \ker(\phi)$, hence $(X-a) \subseteq \ker(\phi)$.

Similarly, for all $f(X) \in \ker(\phi)$ (which $f(X) = f_0 + f_1X + ... + f_nX^n$ for some $f_0, f_1, ..., f_n \in k$), since $\phi(f(X)) = f(a) = 0$, then the following is true:

$$f(a) = f_0 + f_1a + ... + f_na^n, \quad f(X) = f(X) - 0 = f(X) - f(a) = \sum_{j=0}^{n} f_jX^j - \sum_{j=0}^{n} f_ja^j$$

$$f(X) = \sum_{j=0}^{n} f_j(X^j - a^j)$$

(Note: the above equation is true, since $k[X]$ is commutative).

Now, notice that for all $m \in \mathbb{N}$ ($m \geq 2$), the following is true:

$$(X-a)\left(\sum_{j=0}^{m-1} X^ja^{(m-1)-j}\right) = X\sum_{j=0}^{m-1} X^ja^{(m-1)-j} - a\sum_{j=0}^{m-1} X^ja^{(m-1)-j}$$

$$= \sum_{j=0}^{m-1} X^{j+1}a^{(m-1)-j} - \sum_{j=0}^{m-1} X^ja^{(m-1)-j+1}$$

$$= X^ma^{(m-1)-(m-1)} + \sum_{j=0}^{m-2} X^{j+1}a^{(m-1)-j} - \sum_{j=1}^{m-1} X^ja^{(m-1)-j+1} - X^0a^{(m-1)-0+1}$$

$$= X^m + \sum_{j=1}^{m-1} X^ja^{(m-1)-(j-1)} - \sum_{j=1}^{m-1} X^ja^{m-j} - a^m$$

$$= X^m + \sum_{j=1}^{m-1} X^ja^{m-j} - \sum_{j=1}^{m-1} X^ja^{m-j}a^m$$

$$= X^m - a^m$$

Hence, for $m \geq 2$, $X^m - a^m = (X-a)h_m(X)$ for some $h_m(X) \in k[X]$. (And, for $m = 1$, $(X-a) = (X-a)\cdot 1$, and for $m = 0$, since $(X^0 - a^0) = (1-1) = 0$, which let $h_1(X) = 1$ and $h_0(X) = 0$, we can generalize it to $m = 1$ and $m = 0$ case).

So, the original function $f(X)$ can be rewrite as:

$$f(X) = \sum_{j=0}^{n} f_j(X^j - a^j) = \sum_{j=0}^{n} f_j(X-a)h_j(X) = (X-a)\left(\sum_{j=0}^{n} f_jh_j(X)\right)$$

Hence, $f(X) \in (X-a)$, showing that in fact $\ker(\phi) = (X-a)$.

Image of the map is k:

For all $c \in k$, since $c \in k[X]$, then $\phi(c) = c$, showing that ϕ is surjective.

Then, by First Isomorphism Theorem of Rings, we can conclude that $\frac{k[X]}{(X-a)} = \frac{k[X]}{\ker(\phi)} \cong \phi(k[X]) = k$, which the ring homomorphism ϕ defines a ring isomorphism ϕ' (projection map) between $\frac{k[X]}{(X-a)}$ and k.

3

Question 3 *Let $R = \mathbb{R}[X_1, X_2]$. Prove or disprove that $(X_1^2 + 1)$ is a maximal ideal of R.*

Pf:

Consider the ideal $(X_1^2 + 1, X_2)$: Notice that since $X_1^2 + 1 \in (X_1^2 + 1, X_2)$, so $(X_1^2 + 1) \subset (X_1^2 + 1, X_2)$; yet, $X_2 \notin (X_1^2 + 1)$:

Suppose $X_2 \in (X_1^2 + 1)$, then $X_2 = (X_1^2 + 1)h(X_1, X_2)$ for some $h(X_1, X_2) \in \mathbb{R}[X_1, X_2]$. However, if evaluate $X_2 = 1$, then we get the following:

$$1 = (X_1^2 + 1)h(X_1, 1)$$

Notice that since $1 \neq 0$, then $h(X_1, 1) \neq 0$; hence, with $h(X_1, 1), (X_1^2 + 1) \in \mathbb{R}[X_1]$, the following is true:

$$0 = \deg(1) = \deg(X_1^2 + 1) + \deg(h(X_1, 1)) \geq \deg(X_1^2 + 1) = 2$$

Which is a contradiction. Hence, the assumption is false, $X_2 \notin (X_1^2 + 1)$.

Hence, we can conclude that $(X_1^2 + 1) \subsetneq (X_1^2 + 1, X_2)$.

Also, notice that $(X_1^2 + 1, X_2) \neq \mathbb{R}[X_1, X_2]$: Suppose the two are the same, then $1 \in (X_1^2 + 1, X_2)$, so there exists $h_1(X_1, X_2), h_2(X_1, X_2) \in \mathbb{R}[X_1, X_2]$ with $1 = (X_1^2 + 1)h_1(X_1, X_2) + X_2 h_2(X_1, X_2)$.

Yet, if evaluate $X_2 = 0$, we'll get the following:

$$1 = (X_1^2 + 1)h_1(X_1, 0)$$

Which $h_1(X_1, 0) \in \mathbb{R}[X_1]$. Then, since $1 \neq 0$, then $h_1(X_1, 0) \neq 0$; then again, based on the degree of the polynomial, we yield:

$$0 = \deg(1) = \deg(X_1^2 + 1) + \deg(h_1(X_1, 0)) \geq \deg(X_1^2 + 1) = 2$$

Which again is a contradiction. Hence, we can't have $(X_1^2 + 1, X_2) = \mathbb{R}[X_1, X_2]$.

So, the above shows that $(X_1^2 + 1) \subsetneq (X_1^2 + 1, X_2) \subsetneq \mathbb{R}[X_1, X_2]$, showing that $(X_1^2 + 1)$ is not a maximal ideal.

4

Question 4 *Let n be a positive integer with decimal representation $a_k a_{k-1} ... a_1 a_0$. Show that n is divisible by 9 if and only if $\sum_{i=0}^{k} a_i$ is divisible by 9.*

Pf:

Powers of 10 modulo 9:

Notice that since $10 \equiv 1 \ (mod \ 9)$, then for all $n \in \mathbb{N}$, $10^n \equiv 1^n \ (mod \ 9)$, hence $10^n \equiv 1 \ (mod \ 9)$.

Now, notice that for any $n \in \mathbb{N}$, if the decimal representation is $a_k a_{k-1} ... a_1 a_0$ (with $a_0, a_1, ..., a_k \in \{0, 1, ..., 9\}$), it can also be rewritten as:

$$n = \sum_{j=0}^{k} a_j 10^j$$

Hence, n is divisible by 9, if and only if $\sum_{j=0}^{k} a_j 10^j$ is divisible by 9, or $\sum_{j=0}^{k} a_j 10^j \equiv 0 \ (mod \ 9)$.

Then, based on the ring property of $\mathbb{Z}_9$, the following is true:

$$\left(\sum_{j=0}^{k} a_j 10^j\right) \ mod \ 9 = \sum_{j=0}^{k} (a_j \ mod \ 9)(10^j \ mod \ 9) = \sum_{j=0}^{k} (a_j \ mod \ 9) = \left(\sum_{j=0}^{k} a_j\right) \ mod \ 9$$

(Note: the above is true, since $10^j \equiv 1 \ (mod \ 9)$ for all $j \in \mathbb{N}$).

Hence, we can conclude that $\sum_{j=0}^{k} a_j 10^j \equiv 0 \ (mod \ 9)$ if and only if $\left(\sum_{j=0}^{k} a_j\right) \equiv 0 \ (mod \ 9)$, or $\left(\sum_{j=0}^{k} a_j\right)$ is divisible by 9.

Therefore, we can conclude that n is divisible by 9, if and only if $\left(\sum_{j=0}^{k} a_j\right)$ is divisible by 9.

5

Question 5 *Let m and n be positive integers which are relative prime. Prove or disprove that the rings $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ are isomorphic.*

Pf:

Consider the following map $\phi : \mathbb{Z} \to \mathbb{Z}_m \times \mathbb{Z}_n$, by $\phi(k) = (k \ mod \ m, k \ mod \ n)$.

It is a ring homomorphism, because for all $a, b \in \mathbb{Z}$, the following is true:

$$\phi(a+b) = ((a+b) \ mod \ m, (a+b) \ mod \ n) = (a \ mod \ m, a \ mod \ n) + (b \ mod \ m, b \ mod \ n) = \phi(a) + \phi(b)$$

$$\phi(ab) = ((ab) \ mod \ m, (ab) \ mod \ n) = (a \ mod \ m, a \ mod \ n) \cdot (b \ mod \ m, b \ mod \ n) = \phi(a) \cdot \phi(b)$$

(Note: the addition and multiplication is defined coordinate wise).

So, the map is in fact a ring homomorphism.

Kernel of ϕ:

Now, consider $\ker(\phi)$: For all $k \in \ker(\phi)$, since $(k \ mod \ m, k \ mod \ n) = (0, 0)$, the $m \mid k$ and $n \mid k$, hence $lcm(m, n) \mid k$; however, since m, n are assumed to be coprime, then $lcm(m, n) = \frac{mn}{\gcd(m,n)} = mn$ (since $\gcd(m, n) = 1$). Hence, $mn \mid k$, showing that $\ker(\phi) \subseteq mn\mathbb{Z}$.

The converse is also true, since for all $k \in mn\mathbb{Z}$, $k = l \cdot mn$ for some $l \in \mathbb{Z}$, which:

$$\phi(k) = (l \cdot mn \ mod \ m, l \cdot mn \ mod \ n) = (0, 0)$$

Hence, $k \in \ker(\phi)$, or $mn\mathbb{Z} \subseteq \ker(\phi)$. Then, $\ker(\phi) = mn\mathbb{Z}$.

ϕ is Surjective:

Since m, n are coprime, the by Bezout's Lemma, there exists $s, t \in \mathbb{Z}$, with $ms + tn = 1$. Then, $ms = -tn+1$, which $\phi(ms) = (ms \bmod m, ms \bmod n) = (0, -tn+1 \bmod n) = (0, 1)$; also, since $tn = -ms+1$, which $\phi(tn) = (tn \ mod \ m, tn \ mod \ n) = (-ms + 1 \ mod \ m, 0) = (1, 0)$.

Then, for all $(a, b) \in \mathbb{Z}_m \times \mathbb{Z}_n$, the following is true:

$$(a, b) = a(1, 0) + b(0, 1) = a \cdot \phi(tn) + b \cdot \phi(ms) = \phi(atn + bms)$$

Hence, we can conclude that ϕ is a surjective ring homomorphism.

Now, with the above conditions, by First Isomorphism of Rings, we can conclude the following:

$$\mathbb{Z}_{mn} \cong \mathbb{Z}/mn\mathbb{Z} = \mathbb{Z}/\ker(\phi) \cong \phi(\mathbb{Z}) = (\mathbb{Z}_m \times \mathbb{Z}_n)$$

Which, $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ are isomorphic as rings.

6

Question 6

Pf: