Math CS Topology HW4

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Question 1 *Let $f : X \to Y$ be a continuous map between topological spaces, and suppose Y is Hausdorff. Prove that the graph $\{(x, f(x)) \mid x \in X\}$ is a closed subset of $X \times Y$.*

Pf:

Let $G = \{(x, f(x)) \mid x \in X\}$ be the graph. To prove that G is closed, it is equivalent to show that $X \setminus G$ is open.

For all $(x, y) \in X \setminus G$, since the element is not in G, then $y \neq f(x)$. Then, by the Hausdorff Property of Y, there exists disjoint open subsets $U, V \subseteq Y$, such that $f(x) \in U$ and $y \in V$.

Notice that because f is continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are open; furthermore, since $U \cap V = \emptyset$, then $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

Now, consider the basis element $f^{-1}(U) \times V$: First, it is an open neighborhood of (x, y), since $y \in V$ and $f(x) \in U$ (which implies $x \in f^{-1}(U)$); futhermore, for all $a \in f^{-1}(U)$, since $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, then $f(a) \notin V$. Hence, for all $(a, b) \in f^{-1}(U) \times V$, since $f(a) \notin V$, $(a, f(a)) \notin f^{-1}(U) \times V$, hence $(a, b) \neq (a, f(a))$ (which $b \neq f(a)$), showing that $(a, b) \notin G$.

Therefore, $(a, b) \in X \setminus G$, implying that $f^{-1}(U) \times V \subseteq G$.

So, for all $(x, y) \in X \setminus G$, there exists a basis element B (Note: $B = f^{-1}(U) \times V$ in the above construction), such that $(x, y) \in B \subseteq X \setminus G$, hence $X \setminus G$ is open, showing that its complement G is closed.

Hence, the graph $G = \{(x, f(x)) \mid x \in X\}$ is closed.

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Question 2 *Prove that if X and Y are nonempty topological spaces then X is homeomorphic to a subspace of $X \times Y$.*

Pf:

Since Y is not empty, there exists $y_0 \in Y$. Consider the following map $f : X \to X \times Y$, such that for all $x \in X$, $f(x) = (x, y_0)$, which $f(X) = X \times \{y_0\}$ (since for all $x \in X$, $f(x) = (x, y_0) \in X \times \{y_0\}$, and for all $(x, y_0) \in X \times \{y_0\}$, $f(x) = (x, y_0)$). So, we'll restrict the codomain to the set $X \times \{y_0\}$, letting $f : X \to X \times \{y_0\}$.

f is Bijective:

First, we've verified that $f(X) = X \times \{y_0\}$, hence restricting the codomain to the image had made the map surjective.

To verify injectivity, consider $x_1, x_2 \in X$: If $f(x_1) = f(x_2)$, then $(x_1, y_0) = (x_2, y_0)$, so $x_1 = x_2$, proving that it's injective.

So, the map f is bijective, and $f^{-1} : X \times \{y_0\} \to X$ satisfies $f(x, y_0) = x$.

f is Continuous:

For all ope subset $U' \subseteq X \times \{y_0\}$, there exists open subset $U \subseteq X \times Y$, with $U \cap (X \times \{y_0\}) = U'$. Now, consider the preimage $f^{-1}(U')$: For all $x \in f^{-1}(U')$, since $f(x) = (x, y_0) \in U' \subseteq U$, there exists a basis element $A \times B$ (where $A \subseteq X$ and $B \subseteq Y$ are both open), such that $(x, y_0) \in A \times B \subseteq U$. Which:

$$A \times \{y_0\} = (A \cap X) \times (B \cap \{y_0\}) = (A \times B) \cap (X \times \{y_0\}) \subseteq U \cap (X \times \{y_0\}) = U'$$

So, $A \times \{y_0\}$ is an open subset of $X \times \{y_0\}$ under subspace topology, and $(A \times \{y_0\}) \subseteq U'$.

Now, consider all $a \in A \subseteq X$: Since $f(a) = (a, y_0) \in (A \times \{y_0\}) \subseteq U'$, then $a \in f^{-1}(U')$. Hence, $A \subseteq f^{-1}(U')$. Also, recall that $x \in A$, hence $x \in A \subseteq f^{-1}(U')$.

So, for every $x \in f^{-1}(U')$, there is an open subset $A \subseteq X$, with $x \in A \subseteq f^{-1}(U')$, showing that $f^{-1}(U') \subseteq X$ is open.

Therefore, we can conclude that f is continuous, since every open subset of $X \times \{y_0\}$ the image, the preimage in X is open.

f^{-1} is Continuous:

For all open subset $U \subseteq X$, notice that for all $(x, y_0) \in X \times \{y_0\}$, $f^{-1}(x, y_0) = x \in U$ if and only if $x \in U$, hence the preimage $(f^{-1})^{-1}(U) = U \times \{y_0\}$. Which, consider $U \times Y$ an open subset of $X \times Y$, the following is true:

$$(U \times Y) \cap (X \times \{y_0\}) = (U \cap X) \times (Y \cap \{y_0\}) = U \times \{y_0\}$$

Hence, $U \times \{y_0\}$ is an intersection of $X \times \{y_0\}$ and $(U \times Y)$, proving that $U \times \{y_0\}$ is an open subset of $X \times \{y_0\}$ under subspace topology, so the preimage of U under f^{-1}, $(f^{-1})^{-1}(U) = U \times \{y_0\}$ is open, showing that f^{-1} is continuous, since all open subset of X has a preimage being open.

Because f^{-1} exists when restricting the codomain to $X \times \{y_0\}$, and both f and f^{-1} are continuous using the given topology, hence f is a homeomorphism, showing that X and $X \times \{y_0\}$ (as a subspace of $X \times Y$) are homeomorphic.

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Question 3 *If X is a metric space, prove that the distance function $d : X \times X \to \mathbb{R}$ is continuous, where $X \times X$ has the product of the metric topologies.*

Pf:

For all open subset $U \subseteq \mathbb{R}$, consider the preimage $d^{-1}(U) \subseteq X \times X$:

For all $(x_1, x_2) \in d^{-1}(U)$, since $y = d(x_1, x_2) \in U$ while U is open under standard topology of $\mathbb{R}$, then there exists $r > 0$, such that $(y - \frac{r}{3}, y + \frac{r}{3}) \subseteq (y - r, y + r) \subseteq U$.

Now, consider the basis element $\left(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\right) \subseteq X \times X$ under product topology: For all $(a, b) \in \left(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\right)$, the following is true:

$$d(a, b) \leq d(a, x_1) + d(x_1, b) \leq d(a, x_1) + d(x_1, x_2) + d(x_2, b) < \frac{r}{3} + y + \frac{r}{3} = y + \frac{2r}{3}$$

(Note: the above is true, since $a \in B_d(x_1, \frac{r}{3})$ and $b \in B_d(x_2, \frac{r}{3})$).

Hence, since $y < y + \frac{2r}{3} < y + r$, then $d(a, b) = (y + \frac{2r}{3}) \in (y - r, y + r) \subseteq U$, showing that $(a, b) \in d^{-1}(U)$. And, since the choice of $(a, b) \in \left(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\right)$ is arbitrary, $\left(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\right) \subseteq f^{-1}(U)$.

So, for all $(x_1, x_2) \in d^{-1}(U) \subseteq X \times X$, there exists a basis element $\left(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\right)$, such that $(x_1, x_2) \in \left(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\right) \subseteq d^{-1}(U)$, hence $d^{-1}(U)$ is open.

Which, we can conclude that $d : X \times X \to \mathbb{R}$ is continuous under product topology of $X \times X$ (based on metric topology of X), and standard topology of $\mathbb{R}$.

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Question 4 *Let x be a point in a metric space X. Prove that $\{y \in X \mid d(x,y) \leq 1\}$ is closed, but is not necessarily equal to the closure of the unit open ball $B(x,1)$. (This is contrary to Exercise 5.14b in my copy of the textbook.)*

Pf:

To prove that $C = \{y \in X \mid d(x,y) \leq 1\}$ is closed, it suffices to prove that $X \setminus C = \{y \in X \mid d(x,y) > 1\}$ is open.

For all $y \in X \setminus C$, $d(x,y) > 1$. Which, consider $r = d(x,y) - 1 > 0$, and the open ball $B(y,r)$: For all $z \in B(y,r)$, $d(y,z) < r = d(x,y) - 1$. Which, consider $d(x,z)$, the following is true:

$$d(x,y) \leq d(x,z) + d(y,z) < d(x,z) + d(x,y) - 1$$

$$0 < d(x,z) - 1, \quad 1 < d(x,z)$$

Hence, we can conclude that $z \in X \setminus C$, which $y \in B(y,r) \subseteq (X \setminus C)$.

Since for all points in $X \setminus C$, there exists a basis element containing the point, that is a subset of $X \setminus C$, then $X \setminus C$ is open, hence C is closed.

Closure of Open Ball and Closed Ball could be Different:

For any nonempty set X with more than one element, consider the discrete metric $d : X \times X \to \mathbb{R}$ defined as follow:

$$d(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

For all $x \in X$, the ball $B(x,1) = \{x\}$, since for all $y \in X$ with $y \neq x$, $d(x,y) = 1$, so $y \notin B(x,1)$ (since the distance is strictly smaller than 1).

Which, if we take the closed ball of distance 1 around x, $CB(x,1) = \{y \in X \mid d(x,y) \leq 1\} = X$ (since everything has distance at most 1 from x).

Yet, the closure of open ball with radius 1, is $\overline{B(x,1)} = \{x\}$, since under discrete metric, $\{x\}$ is also a closed set containing itself, hence the closure (which is the intersection of closed set containing $\{x\}$) must be $\{x\}$, because it is the smallest closed set containing itself.

Hence, $CB(x,1) \neq \overline{B(x,1)}$, showing that under extreme cases (like discrete metric), the two may not be the same.