Math CS 122A HW1

Zih-Yu Hsieh

January 18, 2025

1

Question 1 *Ahlfors Pg. 33 Problem 4*

Pf:

Suppose $R(z)$ is a rational function such that the numerator and denominator have no common roots, and it satisfies $|R(z)| = 1$ whenever $|z| = 1$. For simplicity, $R(z)$ is in the following form:

$$R(z) = \frac{a_0 + a_1 z + ... + a_n z^n}{b_0 + b_1 z + ... + b_m z^m}, \quad m, n \in \mathbb{N}, \quad a_n, b_n \neq 0$$

Notice that without loss of generality, we can assume $m = n$: If the two are not equal, multiply $R(z)$ by z^{m-n} would form the same degree on both the numerator and denominator (if $m > n$, the numerator has highest degree of $z^{m-n} \cdot z^n = z^m$; else if $m < n$, it's the same as the denominator multiplied by z^{n-m}, which the highest degree of the denominator is $z^{n-m} \cdot z^m = z^n$).

Also, since for all $z \in \mathbb{C}$ with $|z| = 1$, $|z^{m-n}| = 1$, thus $R_1(z) = z^{m-n} R(z)$ still fulfills the given property (if $|z| = 1$, $|z^{m-n} R(z)| = |z|^{m-n} |R(z)| = 1$).

Now, for all $z \in \mathbb{C}$ with $|z| = 1$, $|z|^2 = z\bar{z} = 1$, thus $z = 1/\bar{z}$. Similarly, since $|R(z)| = 1$, then $|R(z)|^2 = R(z)\overline{R(z)} = 1$. Which, substitute z by $1/\bar{z}$ would get the following:

$$|z| = 1 \implies R(z)\overline{R(1/\bar{z})} = 1$$

Notice that $\overline{R(1/\bar{z})}$ itself is also a rational function:

$$\overline{R(1/\bar{z})} = \overline{\left(\frac{a_0 + a_1(1/\bar{z}) + ... + a_n(1/\bar{z})^n}{b_0 + b_1(1/\bar{z}) + ... + b_n(1/\bar{z})^n} \right)} = \overline{\left(\frac{a_0\bar{z}^n + a_1\bar{z}^{n-1} + ... + a_n}{b_0\bar{z}^n + b_1\bar{z}^{n-1} + ... + b_n} \right)}$$

$$\overline{R(1/\bar{z})} = \frac{\bar{a_0} z^n + \bar{a_1} z^{n-1} + ... + \bar{a_n}}{\bar{b_0} z^n + \bar{b_1} z^{n-1} + ... + \bar{b_n}}$$

Thus, the product $R(z)\overline{R(1/\bar{z})}$ is also a rational function.

Then, consider $R(z)\overline{R(1/\bar{z})} - 1$: From the above equation, every $z \in \mathbb{C}$ with $|z| = 1$ satifies the following:

$$R(z)\overline{R(1/\bar{z})} - 1 = 1 - 1 = 0$$

Thus, every z on the unit circle is a zero of the rational function $R(z)\overline{R(1/\bar{z})} - 1$, it has infinite zeroes; yet, suppose the rational function has order $m > 0$, it has at most m distinct zeroes, which is a contradiction. Therefore, $R(z)\overline{R(1/\bar{z})} - 1$ must have order 0, indicating that it is a constant function.

Also, since every z on the unit circle has $R(z)\overline{R(1/\bar{z})} - 1 = 0$, then the function itself (as a constant) must be 0, which implies the function $R(z)\overline{R(1/\bar{z})} = 1$.

Finally, since $R(z)\overline{R(1/\bar{z})} = 1$, then for all $\alpha \in \mathbb{C}$ that is a zero of $R(z)$ ($R(\alpha) = 0$), must also be the pole of $\overline{R(1/\bar{z})}$: Supopse $\alpha \in \mathbb{C}$ is a zero of $R(z)$, but not a pole of $\overline{R(1/\bar{z})}$, then $\overline{R(1/\bar{\alpha})} \in \mathbb{C}$ and $R(\alpha) = 0$. Then

$R(\alpha)\overline{R(1/\bar{\alpha})} = 0 \cdot \overline{R(1/\bar{\alpha})} = 0 \neq 1$. Which, the function $R(z)\overline{R(1/\bar{z})}$ is defined on α, but has an output of 0 instead of 1, indicating the function is not a constant function. Yet, this contradicts the previous statement, so α must a pole of $\overline{R(1/\bar{z})}$, or $1/\bar{\alpha}$ is a pole of $R(z)$.

Given the rational function with the condition $|z| = 1$ implies $|R(z)| = 1$, if $\alpha \neq 0$ is a zero of $R(z)$, then $1/\bar{\alpha}$ must be a pole of $R(z)$.

For the special case $\alpha = 0$ ($R(0) = 0$), since as z approaches 0, $\overline{R(1/\bar{z})} = 1/R(z)$ diverges, indicating that as $1/\bar{z}$ goes unbounded (approaching ∞ on extended complex plane), $\overline{R(1/\bar{z})}$ diverges, hence $R(z)$ has a pole at ∞.

And, for the other special case $\alpha = \infty$, the function $R(1/z)$ approaches 0 as z approaches 0 (or $\frac{1}{z}$ goes unbounded, approaching ∞ on the extended complex plane), which $\overline{R(1/\overline{(1/z)})} = \overline{R(\bar{z})}$ would diverge when z approaches 0, indicating that $R(z)$ has a pole at 0.

2

Question 2 *Ahlfors Pg. 37 Problem 2*

Pf:

Suppose $\lim_{n\to\infty} z_n = A$, then for all $\epsilon > 0$, there exists N, with $n \geq N \implies |z_n - A| < \epsilon$.

Also, because the sequence converges, it is also bounded. Thus, there exists $M > 0$, such that for every $n \in \mathbb{N}$, $|z_n - A| < M$.

Which, for all $\epsilon > 0$, since $\frac{\epsilon}{2} > 0$, there exists $N_1 \in \mathbb{N}$, with $n \geq N_1$ implies $|z_n - A| < \frac{\epsilon}{2}$.

Then, for the given ϵ, since $\frac{\epsilon}{2} > 0$, by Archimedean's Property, there exists $N_2 \in \mathbb{N}$, with $N_1 M < N_2 \frac{\epsilon}{2}$ (Or, $\frac{N_1 M}{N_2} < \frac{\epsilon}{2}$).

Now, let $N = \max\{N_1, N_2\} + 1$, for all $n \geq N$, it is clear that $n > N_1$ and $n > N_2$. Which, consider the following difference:

$$\left|\frac{\sum_{i=1}^n z_i}{n} - A\right| = \left|\frac{\sum_{i=1}^n (z_i - A)}{n}\right| = \left|\sum_{i=1}^{N_1} \frac{(z_i - A)}{n} + \sum_{i=N_1+1}^{n} \frac{(z_i - A)}{n}\right|$$

$$\left|\frac{\sum_{i=1}^n z_i}{n} - A\right| \leq \sum_{i=1}^{N_1} \frac{|z_i - A|}{n} + \sum_{i=N_1+1}^{n} \frac{|z_i - A|}{n}$$

Which, by the construction beforehand, for index $i \in \{1, ..., N_1\}$, $|z_i - A| < M$; and for index $j \in \{N_1 + 1, ..., n\}$, $|z_j - A| < \frac{\epsilon}{2}$ (since $j > N_1$). Thus, the above inequality can be expressed as:

$$\left|\frac{\sum_{i=1}^n z_i}{n} - A\right| \leq \sum_{i=1}^{N_1} \frac{M}{n} + \sum_{i=N_1+1}^{n} \frac{\epsilon/2}{n} = \frac{N_1 M}{n} + \frac{(n - N_1)\epsilon}{2n}$$

$$\left|\frac{\sum_{i=1}^n z_i}{n} - A\right| \leq \frac{N_1 M}{n} + \frac{n\epsilon}{2n} \leq \frac{N_1 M}{n} + \frac{\epsilon}{2}$$

(Note: the second inequality holds since $(n - N_1) < n$).

Now, since $n > N_2$, then $\frac{1}{n} < \frac{1}{N_2}$. So, $\frac{N_1 M}{n} < \frac{N_1 M}{N_2} < \frac{\epsilon}{2}$. Then, the above inequality becomes:

$$\left|\frac{\sum_{i=1}^n z_i}{n} - A\right| \leq \frac{N_1 M}{n} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, for any $\epsilon > 0$, there exists N, with $n \geq N$ implies $\left|\sum_{i=1}^n \frac{z_i}{n} - A\right| < \epsilon$, which implies:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{z_i}{n} = A$$

3

Question 3 *Ahlfors Pg. 41 Poblem 7*

Pf:

Given that $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R$ ($R \in [0, \infty]$). Without Loss of Generality, one can assume after some sufficiently large index n, $|a_n| > 0$ for the limit of ratio to be well defined, and all the proof below would assume for chosen index n, $|a_n| > 0$.

When $0 < R < \infty$**:**

Since $\frac{1}{R}$ is well-defined, then $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{|a_n|/|a_{n+1}|} = \frac{1}{R}$. Now, the goal is to prove $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}$:

(1) $\limsup\{\sqrt[n]{|a_n|}\} \leq \frac{1}{R}$: To approach this, consider any $U > \frac{1}{R}$. Since $(U - \frac{1}{R}) > 0$, by the definition of convergence, there exists N, with $n \geq N$ implies $\left|\frac{|a_{n+1}|}{|a_n|} - \frac{1}{R}\right| < (U - \frac{1}{R})$. Thus:

$$\left(\frac{1}{R} - U\right) < \frac{|a_{n+1}|}{|a_n|} - \frac{1}{R} < \left(U - \frac{1}{R}\right), \quad \frac{|a_{n+1}|}{|a_n|} < U$$

Then, for the fixed N and U constructed above, consider arbitrary $n > N$, the term $|a_n|$ could be expressed as:

$$|a_n| = \frac{|a_n|}{|a_{n-1}|} \cdot \ldots \cdot \frac{|a_{N+1}|}{|a_N|} \cdot |a_N| = |a_N| \cdot \prod_{k=N}^{n-1} \frac{|a_{k+1}|}{|a_k|}$$

Notice that for index $k \in \{N, N+1, ..., n-1\}$, since $k \geq N$, then $0 < \frac{|a_{k+1}|}{|a_k|} < U$, thus:

$$|a_n| = |a_N| \cdot \prod_{k=N}^{n-1} \frac{|a_{k+1}|}{|a_k|} < |a_N| \prod_{k=N}^{n-1} U = |a_N| U^{n-N}$$

Now, let $M = |a_N| U^{-N} > 0$, for all $n > N$, $|a_n| < U^n \cdot M$, or $\sqrt[n]{|a_n|} < \sqrt[n]{U^n \cdot M} = U \sqrt[n]{M}$.

Based on this inequality, define the two quantities as follow:

$$\alpha_n = \sup\{\sqrt[k]{|a_k|} \mid k \geq n\}, \quad \beta_n = \sup\{U \sqrt[k]{M} \mid k \geq n\}$$

Since for all $k \geq n$, $\sqrt[k]{|a_k|} < U \sqrt[k]{M} \leq \beta_n$, thus β_n is the upper bound of the set $\{\sqrt[k]{|a_k|} \mid k \geq n\}$, hence $\alpha_n \leq \beta_n$; and, since $\lim_{n\to\infty} \sqrt[n]{M} = 1$ for $M > 0$, then $\lim_{n\to\infty} U \sqrt[n]{M} = U$, which all subsequential limit is U. Thus, the following is true:

$$\lim_{n\to\infty} \beta_n = \limsup\{U \sqrt[n]{M}\} = \lim_{n\to\infty} U \sqrt[n]{M} = U$$

Which, since for all $n > N$, $\alpha_n \leq \beta_n$, the following is true:

$$\limsup\{\sqrt[n]{|a_n|}\} = \lim_{n\to\infty} \alpha_n \leq \lim_{n\to\infty} \beta_n = U$$

Thus, $\limsup\{\sqrt[n]{|a_n|}\} \leq U$ for all $U > \frac{1}{R}$, hence $\limsup\{\sqrt[n]{|a_n|}\} \leq \frac{1}{R}$.

(2) $\liminf\{\sqrt[n]{|a_n|}\} \geq \frac{1}{R}$: Similarly, consider any $0 < L < \frac{1}{R}$. Since $(\frac{1}{R} - L) > 0$, there exists N, with $n \geq N$ implies $\left|\frac{|a_{n+1}|}{|a_n|} - \frac{1}{R}\right| < (\frac{1}{R} - L)$. Thus:

$$\left(\text{Ł} - \frac{1}{R}\right) < \frac{|a_{n+1}|}{|a_n|} - \frac{1}{R} < \left(\frac{1}{R} - L\right), \quad 0 < L < \frac{|a_{n+1}|}{|a_n|}$$

Then, for the fixed N and L, any $n > N$ satisfies the following:

$$|a_n| = |a_N| \cdot \prod_{k=N}^{n-1} \frac{|a_{k+1}|}{|a_k|} > |a_N| \cdot \prod_{k=N}^{n-1} L = |a_N| \cdot L^{n-N}$$

Now, let $m = |a_N| \cdot L^{-N} > 0$, for all $n > N$, $|a_n| > L^n \cdot m$, thus $\sqrt[n]{|a_n|} > \sqrt[n]{L^n \cdot m} = L\sqrt[n]{m}$.
Again, define the following two quantities:

$$\gamma_n = \inf\{\sqrt[k]{|a_k|} \mid k \geq n\}, \quad \delta_n = \inf\{L\sqrt[k]{m} \mid k \geq n\}$$

Since for all $k \geq n$, $\sqrt[k]{|a_k|} > L\sqrt[k]{m} \geq \delta_n$, thus δ_n is a lower bound of $\{\sqrt[k]{|a_k|} \mid k \geq n\}$, hence $\gamma_n \geq \delta_n$. And, since $m > 0$, $\lim_{n\to\infty} \sqrt[n]{m} = 1$, thus $\lim_{n\to\not\infty} L\sqrt[n]{m} = L$. Thus:

$$\lim_{n\to\infty} \delta_n = \liminf\{L\sqrt[n]{m}\} = \lim_{n\to\not\infty} L\sqrt[n]{m} = L$$

Which, since for all $n > N$, $\gamma_n \geq \delta_n$, the following is true:

$$\liminf\{\sqrt[n]{|a_n|}\} = \lim_{n\to\infty} \gamma_n \geq \lim_{n\to\infty} \delta_n = L$$

Hence, $\liminf\{\sqrt[n]{|a_n|}\} \geq L$ for all L satisfying $0 < L < \frac{1}{R}$, which $\liminf\{\sqrt[n]{|a_n|}\} \geq \frac{1}{R}$.

From the above 2 statements, the following is true:

$$\frac{1}{R} \leq \liminf\{\sqrt[n]{|a_n|}\} \leq \limsup\{\sqrt[n]{|a_n|}\} \leq \frac{1}{R}$$

Thus, $\liminf\{\sqrt[n]{|a_n|}\} = \limsup\{\sqrt[n]{|a_n|}\} = \frac{1}{R}$, so the radius of convergence $\frac{1}{\limsup\{\sqrt[n]{|a_n|}\}} = R$.

When $R = \infty$:

Now, given that $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R = \infty$, which for all $M > 0$, there exists N, with $n \geq N$ implies $\frac{|a_n|}{|a_{n+1}|} > M$.

We'll prove by contradiciton. Suppose the radius of convergence $R' < R$, which $R' \in [0, \infty)$. Then, choose $r \in (R', \infty)$, and consider $\sum_{n=1}^{\infty} a_n r^n$:

For all $n \in \mathbb{N}$, the ratio $\frac{|a_{n+1}r^{n+1}|}{|a_n r^n|} = \frac{r}{|a_n|/|a_{n+1}|}$. Now, for all $\epsilon > 0$ (which $\frac{\epsilon}{r} > 0$), since there exists $M \in \mathbb{N}$ with $1 < M\frac{\epsilon}{r}$, then $\frac{1}{M} < \frac{\epsilon}{r}$. For the chosen M, there exists N, such that $n \geq N$ implies $\frac{|a_n|}{|a_{n+1}|} > M$, thus the ratio $\frac{1}{|a_n|/|a_{n+1}|} < \frac{1}{M}$. So:

$$\frac{|a_{n+1}r^{n+1}|}{|a_n r^n|} = \frac{r}{|a_n|/|a_{n+1}|} < \frac{r}{M} < r\frac{\epsilon}{r} = \epsilon$$

So, for all $\epsilon > 0$, there exists N with $n \geq N$ implies $\left|\frac{|a_{n+1}r^{n+1}|}{|a_n r^n|} - 0\right| < \epsilon$, thus $\lim_{n\to\infty} \frac{|a_{n+1}r^{n+1}|}{|a_n r^n|} = 0 < 1$. Then, by Ratio Test, we can conclude that $\sum_{n=1}^{\infty} a_n r^n$ converges. Yet, since $|r| = r > R'$, it is outside of the radius of convergence, so the given series should diverge, and this is a contradiction.

So, the radius of convergence $R' \geq R$, which since $R = \infty$, $R' = \infty$ is the radius of convergence.

When $R = 0$:

Now, given that $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R = 0$, which for all $\epsilon > 0$, there exists N, with $n \geq N$ implies $\left|\frac{|a_n|}{|a_{n+1}|} - 0\right| < \epsilon$.

We'll approach by contradiction again. Suppose the radius of convergence $R' > R = 0$, which $R' \in (0, \infty]$. Then, choose $r \in (0, R')$, and consider $\sum_{n=1}^{\infty} a_n r^n$:

Again, for all $n \in \mathbb{N}$, the ratio $\frac{|a_{n+1} r^{n+1}|}{|a_n r^n|} = \frac{r}{|a_n|/|a_{n+1}|}$. Which, for all $M > 0$ ($\frac{r}{M} > 0$), since there exists N, with $n \geq N$ implies $\left|\frac{|a_n|}{|a_{n+1}|} - 0\right| = \frac{|a_n|}{|a_{n+1}|} < \frac{r}{M}$. Then, $\frac{1}{|a_n|/|a_{n+1}|} > \frac{M}{r}$.

So, for any $n \geq N$, the following is true:

$$\frac{|a_{n+1} r^{n+1}|}{|a_n r^n|} = \frac{r}{|a_n|/|a_{n+1}|} > r\frac{M}{r} = M$$

Since the choice of $M > 0$ is arbitrary, then the sequence $\frac{|a_{n+1} r^{n+1}|}{|a_n r^n|}$ is not bounded, which according to ratio test, the series $\sum_{n=1}^{\infty} a_n r^n$ diverges.

Yet, since $0 < |r| = r < R'$, it is in the radius of convergence, $\sum_{n=1}^{\infty} a_n r^n$ should converge, which is a contradiction.

So, the radius of convergence $R' \leq R = 0$, which indicates that $R' = 0$ is the radius of convergence.

Regardless of the case, R is always the radius of convergence, thus we can also define radius of convergence as $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$, if the limit is well-defined.

4

Question 4 *Ahlfors Pg. 41 Problem 9*

Pf:

Given the following series $\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$, to indicate the values of $z \in \mathbb{C}$ that lead to convergence, there are three cases:

(i): When $|z| < 1$:

For all $z \in \mathbb{C}$ with $|z| < 1$, since $\lim_{n\to\infty} |z|^n = 0$, choose $\epsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$, such that $n \geq N$ implies $|z|^n < \frac{1}{2}$ (or $-|z|^n > -\frac{1}{2}$). Then, for all $n \geq N$ (which $2n \geq N$), the following is true:

$$|1 + z^{2n}| = |1 - (-z^{2n})| \geq \big||1| - |-z^{2n}|\big| = 1 - |z^{2n}| > 1 - \frac{1}{2} = \frac{1}{2}$$

Thus, for $n \geq N$, $\frac{1}{2} < |1 + z^{2n}|$, which indicates the following:

$$\frac{1}{|1 + z^{2n}|} < 2, \quad \left|\frac{z^n}{1 + z^{2n}}\right| = \frac{|z^n|}{|1 + z^{2n}|} < 2|z^n| = 2|z|^n$$

Now, consider $\sum_{n=N}^{\infty} \left|\frac{z^n}{1+z^{2n}}\right|$, since every term satisfies $0 \leq \left|\frac{z^n}{1+z^{2n}}\right| < 2|z|^n$, and the series $\sum_{n=N}^{\infty} 2|z|^n$ converges due to the assumption that $|z| < 1$, then by comparison test, the series $\sum_{n=N}^{\infty} \left|\frac{z^n}{1+z^{2n}}\right|$ converges, which implies $\sum_{n=N}^{\infty} \frac{z^n}{1+z^{2n}}$ absolutely converges.

Thus, given $|z| < 1$, the series $\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ converges.

(ii): When $|z| = 1$:

For all $z \in \mathbb{C}$ with $|z| = 1$, and any $n \in \mathbb{N}$, consider $\left|\frac{z^n}{1+z^{2n}}\right|$: Suppose $1 + z^{2n} \neq 0$, by the Triangle Inequality, since $|1 + z^{2n}| \leq |1| + |z^{2n}| = 2$, then:

$$\frac{1}{|1 + z^{2n}|} \geq \frac{1}{2}, \quad \left|\frac{z^n}{1 + z^{2n}}\right| \geq \frac{|z^n|}{2} = \frac{1}{2}$$

This indicates that $\lim_{n\to\infty} \frac{z^n}{1+z^{2n}} \neq 0$, since choosing $\epsilon = \frac{1}{2}$, every $n \in \mathbb{N}$ with $(1 + z^{2n}) \neq 0$, satisfies $\left|\frac{z^n}{1+z^{2n}} - 0\right| \geq \frac{1}{2} = \epsilon$.

Then, since the sequence $\frac{z^n}{1+z^{2n}}$ does not converge to 0 for all z with $|z| = 1$, the series $\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ diverges.

(iii): When $|z| > 1$:

For all $z \in \mathbb{C}$ with $|z| > 1$, then for all $n \in \mathbb{N}$, $|z|^{2n} > |z|^n > 1$. Also, since the sequence $|z|^n$ is strictly increasing and not bounded, there exists N, such that $n \geq N$ implies $|z|^{2n} > |z|^n > 2$, or $\frac{1}{2}|z|^{2n} > 1$.

Thus, the following is true:

$$|1 + z^{2n}| = |z^{2n} - (-1)| \geq \big||z^{2n}| - |-1|\big| = |z|^{2n} - 1 > |z|^{2n} - \frac{1}{2}|z|^{2n} = \frac{1}{2}|z|^{2n}$$

Which, the above inequality indicates the following:

$$\frac{1}{|1 + z^{2n}|} < \frac{1}{\frac{1}{2}|z|^{2n}} = \frac{2}{|z|^{2n}}, \quad \left|\frac{z^n}{1 + z^{2n}}\right| = \frac{|z|^n}{|1 + z^{2n}|} < \frac{2|z|^n}{|z|^{2n}} = \frac{2}{|z|^n} = 2\left|\frac{1}{z}\right|^n$$

If we consider the series $\sum_{n=N}^{\infty} \left|\frac{z^n}{1+z^{2n}}\right|$, since every term satisfies $0 \leq \left|\frac{z^n}{1+z^{2n}}\right| < 2\left|\frac{1}{z}\right|^n$, and the series $\sum_{n=N}^{\infty} 2\left|\frac{1}{z}\right|^n$ converges since $\left|\frac{1}{z}\right| < 1$, then by comparison test, the series $\sum_{n=N}^{\infty} \left|\frac{z^n}{1+z^{2n}}\right|$ converges.

Thus, the original series $\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ is absolutely converging.

5

Question 5 *Stein and Shakarchi Pg. 28 Problem 16 (e)*

Given the hypergeometric series as:

$$F(\alpha,\beta,\gamma;z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+n-1)\beta(\beta+1)...(\beta+n-1)}{n!\gamma(\gamma+1)...(\gamma+n-1)} z^n$$

With $\alpha,\beta \in \mathbb{C}$, and $\gamma \notin \{-n \mid n \in \mathbb{N}\}$.

For all positive integer n, define the coefficient a_n as follow:

$$a_n = \frac{\alpha(\alpha+1)...(\alpha+n-1)\beta(\beta+1)...(\beta+n-1))}{n!\gamma(\gamma+1)...(\gamma+n-1)}$$

(i): If α or β are non-positive integers:

Without Loss of Generality, can assume α is a non-positive integer (since interchanging α and β doesn't affect the coefficient). Then, $\alpha = -k$ for some $k \in \mathbb{N}$. Which, for all index $n > k$, the coefficient's numerator involves a term $(\alpha + k) = (-k + k) = 0$, which the coefficient $a_n = 0$.

For all $N > k$, $a_N = 0$, which the following partial sum can be expressed as:

$$\sum_{n=1}^{N} a_n z^n = \sum_{n=1}^{k} a_n z^n + \sum_{n=(k+1)}^{N} a_n z^n = \sum_{n=1}^{k} a_n z^n$$

Thus, the sequence of series $s_N = \sum_{n=1}^{N} a_n z^n = \sum_{n=1}^{k} a_n z^n = s_k$ for all $N > k$, which is eventually a constant sequence. So, the series $\sum_{n=1}^{\infty} a_n z^n$ converges.

Thus, for all $z \in \mathbb{C}$, $F(\alpha,\beta,\gamma;z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ is defined, which the radius of convergence is $R = \infty$.

(ii): If both α, β are not non-positive integers:

Now, in **Question 3** it has proven, if the limit $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R$ for some $R \in [0,\infty]$, then R is precisely the radius of convergence.

Which, for all $n \in \mathbb{N}$, the ratio $\frac{|a_n|}{|a_{n+1}|}$ is defined as follow:

$$\frac{\alpha(\alpha+1)...(\alpha+n-1)\beta(\beta+1)...(\beta+n-1)}{n!\gamma(\gamma+1)...(\gamma+n-1)} \cdot \frac{(n+1)!\gamma(\gamma+1)...(\gamma+n-1)(\gamma+n)}{\alpha(\alpha+1)...(\alpha+n-1)(\alpha+n)\beta(\beta+1)...(\beta+n-1)(\beta+n)}$$

$$= \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} = \frac{n^2+(\gamma+1)n+\gamma}{n^2+(\alpha+\beta)n+\alpha\beta} = \frac{1+(\gamma+1)/n+\gamma/n^2}{1+(\alpha+\beta)/n+\alpha\beta/n^2}$$

Then, since $\lim_{n\to\infty} \frac{1}{n} = 0$, then the following limit is defined as:

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{1+(\gamma+1)/n+\gamma/n^2}{1+(\alpha+\beta)/n+\alpha\beta/n^2} = \frac{1+(\gamma+1)\cdot 0+\gamma\cdot 0}{1+(\alpha+\beta)\cdot 0+\alpha\beta\cdot 0} = 1$$

Which, the radius of convergence of hypergeometric series is $R = 1$.

6

Question 6 *Stein and Shakarchi Pg. 29 Problem 19 (c)*

Pf:

For all $z \in \mathbb{C}$ ($z \neq 1$) satisfying $|z| = 1$, consider the following partial sum:

$$A_n = \sum_{i=0}^{n} z^n = \frac{1 - z^{n+1}}{1 - z}$$

Notice that for all $n \in \mathbb{N}$, the following inequality is true:

$$|A_n| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \leq \frac{|1| + |z^{n+1}|}{|1 - z|} = \frac{2}{|1 - z|}$$

Thus, for given z with $z \neq 1$ and $|z| = 1$, the geometric partial sum A_n is always bounded by $\frac{2}{|1-z|}$.

Summation by Part Formula:

Given sequence $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$, and let $A_N = \sum_{n=1}^{N} a_n$ (with $A_0 = 0$), then for all $p, q \in \mathbb{N}$ (with $p < q$), the following formula is true:

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} \left(\sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k\right) b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=(p-1)}^{(q-1)} A_n b_{n+1}$$

$$= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Convergence of Series of Products:

Now, suppose $(a_n)_{n\in\mathbb{N}}$ is a complex sequence, and $(b_n)_{n\in\mathbb{N}}$ is a real sequence, such that the partial sum of a_n are all bounded (i.e. there exists $M > 0$, such that every $N \in \mathbb{N}$ satisfies $A_N = \sum_{n=1}^{N} a_n$ has $|A_N| < M$), and b_n is a monotonic non-increasing sequence that converges to 0 (i.e. for all $n \in \mathbb{N}$, $b_n \geq b_{n+1}$, and $\lim_{n\to\infty} b_n = 0$; this also implies $b_n \geq 0$). Then, $\sum_{n=1}^{\infty} a_n b_n$ converges.

To prove this, let $s_N = \sum_{n=1}^{N} a_n b_n$, the goal is to prove that the sequence $(s_N)_{N\in\mathbb{N}}$ is Cauchy.

First, by the convergence of b_n, for all $\epsilon > 0$, since $\frac{\epsilon}{2M} > 0$, there exists N, with $n \geq N$ implies $|b_n - 0| = b_n < \frac{\epsilon}{2M}$. (Note: $M > 0$ is the bound of A_N).

Then, for the same ϵ given, any $p, q > N$ with $p < q$ (Note: with $(p-1) \geq N$) satisfy the following:

$$|s_q - s_{p-1}| = \left|\sum_{n=1}^{q} a_n b_n - \sum_{n=1}^{p-1} a_n b_n\right| = \left|\sum_{n=p}^{q} a_n b_n\right|$$

$$= \left|\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{n-1} b_n\right| \leq \sum_{n=p}^{q-1} |A_n (b_n - b_{n+1})| + |A_q b_q| + |A_{n-1} b_n|$$

Which, since every $n \in \mathbb{N}$ satisfies $|A_n| < M$, then:

$$|s_q - s_{p-1}| \leq \sum_{n=p}^{q-1} |A_n (b_n - b_{n+1})| + |A_q b_q| + |A_{p-1} b_p| \leq \sum_{n=p}^{q-1} M|(b_n - b_{n+1})| + M|b_q| + M|b_p|$$

Also, since $b_n \geq b_{n+1}$ for all $n \in \mathbb{N}$, thus $(b_n - b_{n+1}) \geq 0$; along with the condition that $b_n \geq 0$, the following is true:

$$|s_q - s_{p-1}| \leq \sum_{n=p}^{q-1} M|(b_n - b_{n+1})| + M|b_q| + M|b_p| = M\left(\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p\right)$$

$$|s_q - s_{p-1}| \leq M\left(\sum_{n=p}^{q-1} b_n - \sum_{n=p}^{q-1} b_{n+1} + b_q + b_p\right)$$

$$|s_q - s_{p-1}| \leq M\left(\sum_{n=p}^{q-1} b_n - \sum_{n=p+1}^{q} b_n + b_q + b_p\right)$$

$$|s_q - s_{p-1}| \leq M(b_p - b_q + b_q + b_p) = 2Mb_p$$

Now, since $p \geq N$, then by the convergence of b_n constructed beforehand, $b_p < \frac{\epsilon}{2M}$. Thus:

$$|s_q - s_{p-1}| \leq 2Mb_p < 2M\frac{\epsilon}{2M} = \epsilon$$

Hence, the sequence $(s_N)_{N\in\mathbb{N}}$ is Cauchy, thus converges.

Convergence of $\sum_{n=1}^{\infty} z^n/n$ on unit circle:

For any $z \neq 1$ with $|z| = 1$, let $a_n = z^n$ and $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$.

From the first part, the partial sum of a_n is bounded (proven that $|A_n| \leq \frac{2}{|1-z|}$), and $b_n = \frac{1}{n}$ is a nonincreasing sequence that converges to 0. Then, by the above statement, the series of product $\sum_{n=1}^{\infty} a_n b_n$ converges. Thus, the following series converges, given that $z \neq 1$ and $|z| = 1$:

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = L \in \mathbb{C}$$