Math 118B HW3 - Lebesgue Criterion of Riemann Integrability

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The goal of this Homework is to understand and prove the Lebesgue Criterion of Riemann Integrability.

Setup 1

Definition 1 *A set $E \subset \mathbb{R}$ is said to be of measure zero if given $\epsilon > 0$ there is a countable collection of open intervals $\{I_j\}_{j\in\mathbb{Z}_+}$ which covers E, i.e. $E \subseteq \bigcup_{j\in\mathbb{Z}_+} I_j$ and such that*

$$\sum_{j=1}^{\infty} |I_j| < \epsilon$$

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Question 1 *Prove that every countable set of $\mathbb{R}$ is a set of measure zero.*

Pf:

Given $E \subset \mathbb{R}$ that is countable, then there exists a bijection $f : E \to \mathbb{N}$, which generates an index for all element $a \in E$.

Then, given any $\epsilon > 0$, for all $a \in E$, let $j = f(a)$, consider the open inverval $I_j = (a - \frac{\epsilon}{2^{j+2}}, a + \frac{\epsilon}{2^{j+2}})$: the collection $\{I_j\}_{j\in\mathbb{Z}_+}$ is countable, and $E \subseteq \bigcup_{j\in\mathbb{Z}_+} I_j$, since for all $a \in E$, let $j = f(a) \in \mathbb{Z}_+$, we have $f(a) \in I_j = (a - \frac{\epsilon}{2^{j+2}}, a + \frac{\epsilon}{2^{j+2}})$.

On the other hand, the following is true for the length of the countable set:

$$\forall j \in \mathbb{Z}_+, \quad |I_j| = \left|(a + \frac{\epsilon}{2^{j+2}}) - (a - \frac{\epsilon}{2^{j+2}})\right| = \left|2 \cdot \frac{\epsilon}{2^{j+2}}\right| = \frac{\epsilon}{2^{j+1}}$$

$$\sum_{i=1}^{\infty} |I_j| = \sum_{i=1}^{\infty} \frac{\epsilon}{2^{j+1}} = \frac{\epsilon}{2} < \epsilon$$

The above series is converging since it's a geometric series with radius $\frac{1}{2} < 1$. Hence, for all $\epsilon > 0$, there is a countable collection of open intervals covering E with the series of length bounded by ϵ, proving that E the countable set has measure 0.

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Question 2 *Prove that the countable union of sets of measure zero has measure zero.*

Pf:

Let $\{E_n\}_{n\in\mathbb{Z}_+}$ be a countable collection of sets, each with measure 0. Then, for any given $\epsilon > 0$, for every $n \in \mathbb{Z}_+$, since $\frac{\epsilon}{2^n} > 0$, there exists a countable collection of open interval $\{I_j^n\}_{j\in\mathbb{Z}_+}$, with $E_n \subseteq \bigcup_{j\in\mathbb{Z}_+} I_j^n$, and $\sum_{j=1}^{\infty} |I_j^n| < \frac{\epsilon}{2^n}$.

Now, consider the collection $\mathcal{F} = \bigcup_{n\in\mathbb{Z}_+}\{I_j^n\}_{j\in\mathbb{Z}_+}$, a countable union of "countable collection of open intervals", which is again countable. Which, since $E_n \subseteq \bigcup_{j\in\mathbb{Z}_+} I_j^n$ for all $n \in \mathbb{Z}_+$, then:

$$\bigcup_{n\in\mathbb{Z}} E_n \subseteq \bigcup_{n\in\mathbb{Z}_+}\left(\bigcup_{j\in\mathbb{Z}_+} I_j^n\right)$$

Which the left side is countable union of sets with measure 0, while the right side is the union of open intervals in family $\mathcal{F}$.

Then, to consider the length of $\mathcal{F}$, since it is countable, there exists a bijection $f : \mathcal{F} \to \mathbb{N}$ that generates the index. Which, for the first k elements in this index of $\mathcal{F}$, the elements are $\{I_{j_1}^{n_1}, ..., I_{j_k}^{n_k}\}$. let $J = \max\{j_1, ..., j_k\}$ and $N = \max\{n_1, ..., n_k\}$, then these elements are in the collection $\bigcup_{n=1}^{N}\{I_j^n\}_{j=1}^{J}$. Which, the collection has the length being bounded:

$$\forall n \in \{1, ..., N\}, \quad \sum_{j=1}^{J} |I_j^n| \leq \sum_{j=1}^{\infty} |I_j^n| = \frac{\epsilon}{2^n}$$

$$\sum_{n=1}^{N}\left(\sum_{j=1}^{J} |I_j^n|\right) \leq \sum_{j=1}^{\infty} |I_j^n| \leq \sum_{n=1}^{N} \frac{\epsilon}{2^n} \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

The above two inequalities are true, since each partial sum is monotonically non-decreasing, and bounded above. Hence, the sum of length $s_k = \sum_{i=1}^{k} |I_{j_i}^{n_i}| \leq \sum_{n=1}^{N}\left(\sum_{j=1}^{J} |I_j^n|\right) \leq \epsilon$ for any positive integer k, while this partial sum of length s_k is also monotonically non-decreasing, hence the series of length converges, and the following is true:

$$\lim_{k\to\infty} s_k = \sum_{k=1}^{\infty} |I_{j_k}^{n_k}| = \sup\{s_k\} \leq \epsilon$$

(Note: since ϵ is the upper bound of the partial sums, hence the above inequality is true). Then, since $\mathcal{F}$ covers the $\bigcup_{n\in\mathbb{Z}_+} E_n$, and the series of $\mathcal{F}$ elements' length satisfy $\sum_{k=1}^{\infty} |I_{j_k}^{n_k}| \leq \epsilon$, then we can conclude that $\bigcup_{n\in\mathbb{Z}_+} E_n$ has measure 0.

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Question 3 *Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. We have seen that E is uncountable. Prove that E is a set of measure zero.*

Pf:

From the description, $E = \{x \in [0,1] \mid x = 0.a_1a_2...a_n...,\ \forall n \in \mathbb{N}, a_n \in \{4,7\}\}$. For each $n \in \mathbb{N}$, let $E_n = \{x \in [0,1] \mid x = 0.a_1a_2...a_n...,\ \forall i \in \{1,...,n\}, a_i \in \{4,7\}\}$ (set of reals in $[0,1]$ with the first n decimals being 4 or 7).

Notice that for $n \geq 2$, there are $2^{(n-1)}$ distinct cases for $0.a_1a_2...a_{(n-1)}$ (first $(n-1)$ decimals) in E_n, then for each case, if $x \in E_n$ has this arrangement for the first $(n-1)$ decimals:

$$0.a_1a_2...a_{(n-1)}3 < x < 0.a_1a_2...a_{(n-1)}8$$

Hence, for each arrangement, they're contained in the open interval $(0.a_1a_2...a_{(n-1)}3, 0.a_1a_2...a_{(n-1)}8)$, which has length $(0.a_1a_2...a_{(n-1)}8 - 0.a_1a_2...a_{(n-1)}3) = \frac{8-3}{10^n} = \frac{5}{10^n}$.

All $2^{(n-1)}$ collection of these open intervals would cover E_n, since the first $(n-1)$ decimals for each $x \in E_n$ must be some arrangement of 4 and 7. Hence, E_n can be covered by unions of $2^{(n-1)}$ open intervals, each with length $\frac{5}{10^n}$, hence the total length of the open cover is $2^{(n-1)} \cdot 5 \cdot \frac{1}{10^n} = \frac{1}{2} \cdot (\frac{2}{10})^{(n-1)} = \frac{1}{2}(\frac{1}{5})^{(n-1)}$.

Now, since for each $x \in E$, the first n decimals are consist of 4 and 7, hence $x \in E_n$, or $E \subseteq E_n$. From the previous part, since for each $n \in \mathbb{N}$, the set E_n could be covered by a finite collection of open intervals with sum of length $\frac{1}{2}(\frac{1}{5})^{(n-1)}$, so does the set E.

Then, becase $\lim_{n\to\infty} \frac{1}{2}(\frac{1}{5})^{(n-1)} = 0$, then for all $\epsilon > 0$, there exists n, with $\frac{1}{2}(\frac{1}{5})^{(n-1)} < \epsilon$. Hence, choose the collection of open intervals for E_n (along with countable empty sets), E can be covered with the given collection of open intervals, with total length $\frac{1}{2}(\frac{1}{5})^{(n-1)} < \epsilon$, showing that E in fact has measure 0.

Setup 2

Definition 2 *Let $f : [a, b] \to \mathbb{R}$ be a bounded function, $(b - a) < \infty$. For $x \in [a, b]$ and $\eta > 0$ define*

$$\Omega(f, x, \eta) = \sup\{|f(x_1) - f(x_2)| \ : \ x_1, x_2 \in (x - \eta, x + \eta) \cap [a, b]\}$$

and the oscillation of f at a point $x \in [a, b]$

$$\omega_f(x) = \lim_{\eta \to 0^+} \Omega(f, x, \eta) = \inf_{\eta > 0}\{\Omega(f, x, \eta)\}$$

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Question 4 *Prove that $\omega_f(x)$ is defined for any $x \in [a, b]$.*

Pf:

Given any $x \in [a, b]$, and $\eta_1, \eta_2 > 0$ with $\eta_1 > \eta_2$, since $(x - \eta_2, x + \eta_2) \subset (x - \eta_1, x + \eta_1)$, hence:

$$\{|f(x_1) - f(x_2)| \ : \ x_1, x_2 \in (x - \eta_2, x + \eta_2) \cap [a, b]\} \subseteq \{|f(x_1) - f(x_2)| \ : \ x_1, x_2 \in (x - \eta_1, x + \eta_1) \cap [a, b]\}$$

This implies $\Omega(f, x, \eta_2) \le \Omega(f, x, \eta_1)$, since the supremum of the set on the right, is also an upper bound of the set on the left.

On the other hand, since the set $\{|f(x_1) - f(x_2)| \ : \ x_1, x_2 \in (x - \eta, x + \eta) \cap [a, b]\}$ for all $\eta > 0$ is a collection of distance in $\mathbb{R}$, hence 0 is always a lower bound of the set, showing that $0 \le \Omega(f, x, \eta)$.

Now, since for all $\eta > 0$, the value $\Omega(f, x, \eta)$ is bounded below by 0 for all $\eta > 0$, this implies $\inf_{\eta > 0}\{\Omega(f, x, \eta)\}$ exists. Then, to prove that $\lim_{\eta \to 0^+} \Omega(f, x, \eta) = \inf_{\eta > 0}\{\Omega(f, x, \eta)\} = \omega$, for all $\epsilon > 0$, since $\omega + \epsilon$ is no longer a lower bound of the set, there exists $\eta > 0$, with $\omega \le \Omega(f, x, \eta) < \omega + \epsilon$. Then, choose $\delta = \eta > 0$, for all $\mu' > 0$ with $\mu' < \mu = \delta$, from the previous section, $\omega \le \Omega(f, x, \eta') \le \Omega(f, x, \eta) < \omega + \epsilon$, hence:

$$|\omega - \Omega(f, x, \eta')| < \epsilon$$

This demonstrates that $\lim_{\eta \to 0^+} \Omega(f, x, \eta) = \omega = \inf_{\eta > 0}\{\Omega(f, x, \eta)\}$, hence $\omega_f(x)$ is defined.

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Question 5 *Prove that f is continuous at x_0 if and only if $\omega_f(x_0) = 0$.*

Pf:

$\Longrightarrow$: Suppose f is continuous at x_0, for all $\epsilon > 0$ (since $\frac{\epsilon}{2} > 0$), there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. Then, choose $\eta = \delta$, consider $\Omega(f, x_0, \delta) = \sup\{|f(x_1) - f(x_2)| \ : \ x_1, x_2 \in (x_0 - \delta, x_0 + \delta) \cap [a, b]\}$:

For all $x_1, x_2 \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$, since $x_1, x_2 \in B_\delta(x_0)$, then by assumption $|f(x_1) - f(x_0)|, |f(x_2) - f(x_0)| < \frac{\epsilon}{2}$. Hence, the following is true:

$$|f(x_1) - f(x_2)| = |(f(x_1) - f(x_0)) + (f(x_0) - f(x_2))| \le |f(x_1) - f(x_0)| + |f(x_2) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, ϵ is an upper bound of the set $\{|f(x_1) - f(x_2)| \ : \ x_1, x_2 \in (x_0 - \delta, x_0 + \delta) \cap [a, b]\}$, showing that $\Omega(f, x_0, \delta) \le \epsilon$.

Because the choice $\epsilon > 0$ is arbitrary, then with the corresponding $\delta > 0$, $\omega_f(x_0)\Omega(f, x_0, \delta) \le \epsilon$. Then, $\omega_f(x_0) \le \epsilon$ for all $\epsilon > 0$, showing that $\omega_f(x_0) \le 0$; also, $\omega_f(x_0)$ is an infimum of all nonnegative numbers (in **4** we've proven $0 \le \Omega(f, x, \eta)$ for all $\eta > 0$), hence $\omega_f(x_0) \ge 0$. The two statements imply $\omega_f(x_0) = 0$.

$\Longleftarrow$: Suppose $\omega_f(x_0) = 0$. Then, by definition, for all $\epsilon > 0$, since $\epsilon = 0 + \epsilon$ is no longer a lower bound of the set $\{\Omega(f, x_0, \eta) \mid \eta > 0\}$, there exists $\delta = \eta > 0$, such that $0 \le \Omega(f, x_0, \delta) < \epsilon$.

Hence, for all $x \in B_\delta(x_0) \cap [a, b]$, $|f(x) - f(x_0)| \le \Omega(f, x_0, \delta) < \epsilon$, proving that f is continuous at x_0.

The above two implications shows that f is continuous at x_0 if and only if $\omega_f(x_0) = 0$.

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Question 6 *Prove that for any $\mu > 0$ the set $A_\mu = \{x \in [a, b] \ : \ \omega_f(x) \ge \mu\}$ is compact.*

Pf:

Since $A_\mu \subseteq [a, b]$ while $[a, b]$ is compact, then to prove that A_μ is compact, it suffices to show that A_μ is closed, or $A'_\mu \subseteq A_\mu$.

For all $x_0 \in A'_\mu$, for every radius $r > 0$, there exists $x_1 \in B_r(x_0) \setminus \{x_0\} \cap A_\mu$. Hence, $\omega_f(x_1) \ge \mu$. Now, take $\eta = r - |x_0 - x_1| > 0$, for all $x \in B_\eta(x_1)$, since $|x - x_1| < \eta = r - |x_0 - x_1|$, then:

$$|x - x_0| = |(x - x_1) + (x_1 - x_0)| \le |x - x_1| + |x_1 - x_0| < (r - |x_0 - x_1|) + |x_1 - x_0| = r$$

This indicates that $x \in B_r(x_0)$, or $B_\eta(x_1) \subseteq B_r(x_0)$.

Hence, for all $x_c, x_d \in (B_\eta(x_1) \cap [a,b]) \subseteq (B_r(x_0) \cap [a,b])$, since the following is true:

$$|f(x_c) - f(x_d)| \leq \Omega(f, x_0, r) = \sup\{|f(x) - f(x')| \ : \ x, x' \in B_r(x_0) \cap [a,b]\}$$

Hence, $\Omega(f, x_0, r)$ is an upper bound of the set $\{|f(x) - f(x')| \ : \ x, x' \in B_\eta(x_1) \cap [a,b]\}$, which implies the following:

$$\Omega(f, x_0, r) \geq \Omega(f, x_1, \eta) = \sup\{|f(x) - f(x')| \ : \ x, x' \in B_\eta(x_1) \cap [a,b]\}$$

Thus, we can further conclude that $\Omega(f, x_0, r) \geq \Omega(f, x_1, \eta) \geq \omega_f(x_1) \geq \mu$.

Now, because for all $r > 0$, $\Omega(f, x_0, r) \geq \mu$, then μ is the lower bound of the set $\{\Omega(f, x_0, r) \mid r > 0\}$, showing that $\mu \leq \omega_f(x_0) = \inf\{\Omega(f, x_0, r) \mid r > 0\}$. Hence, $x_0 \in A_\mu$, showing that $A'_\mu \subseteq A_\mu$.

This proves that A_μ is closed, and since $A_m u \subseteq [a,b]$ a compact set, then A_μ is also compact.

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Question 7 *Prove that the set of discontinuities of f can be written as*

$$D_f = \bigcup_{j \in \mathbb{Z}_+} A_{1/j} = \bigcup_{j \in \mathbb{Z}_+} \left\{ x \in [a,b] \ : \ \omega_f(x) \geq \frac{1}{j} \right\}$$

Pf:

In **Question 5**, we've proven the equivalence of continuity at x_0 and $\omega_f(x_0) = 0$, hence $x \in [a,b]$ is a discontinuity of f iff $\omega_f(x) \neq 0$ (which actually is $\omega_f(x) > 0$). Hence, $D_f = \{x \in [a,b] \mid \omega_f(x) > 0\}$.

Now, for all $x \in D_f$, since $\omega_f(x) > 0$, by Archimedean's Property, there exists $j \in \mathbb{Z}_+$, with $\omega_f(x) > \frac{1}{j} > 0$, this implies $x \in A_{1/j} = \left\{ x \in [a,b] \ : \ \omega_f(x) \geq \frac{1}{j} \right\}$, hence $x \in \bigcup_{j \in \mathbb{Z}_+} A_{1/j}$. This implies $D_f \subseteq \bigcup_{j \in \mathbb{Z}_+} A_{1/j}$.

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Question 8 *Prove that if for some $\epsilon > 0$, $\omega_f(x) < \epsilon$ for any $x \in [a,b]$, then there exists $\eta > 0$ such that for all $x \in [a,b]$,*

$$\Omega(f, x, \eta) < \epsilon$$

Pf:

Suppose there exists $\epsilon > 0$, with $\omega_f(x) < \epsilon$ for all $x \in [a,b]$, then since ϵ is not a lower bound of the set $\{\Omega(f,x,\eta) \mid \eta > 0\}$, then there exists $\eta_x > 0$, such that $\omega_f(x) \leq \Omega(f,x,\eta_x) < \epsilon$.

Now, consider the collection of open intervals $\mathcal{F} = \{(x - \eta_x/2, x + \eta_x/2) \mid x \in [a,b]\}$: Since $[a,b] \subseteq \bigcup \mathcal{F}$, then $\mathcal{F}$ is an open cover of $[a,b]$; hence, by the compactness of $[a,b]$, there exists $x_1, ..., x_n \in [a,b]$, such that $[a,b] \subseteq \bigcup_{i=1}^n (x_i - \mu_{x_i}/2, x_i + \mu_{x_i}/2)$.

Then, let $\eta = \min\{\frac{1}{2}\eta_{x_1}, ..., \frac{1}{2}\eta_{x_n}\} > 0$. For all $x \in [a,b]$, from the above construction, there exists $i \in \{1, ..., n\}$ with $x \in B_{\eta_{x_i}/2}(x_i)$. Now, consider the set $S = \{|f(x_c) - f(x_d)| \ : \ x_c, x_d \in (x-\eta, x+\eta) \cap [a,b]\}$: For all $x_c, x_d \in (x-\eta, x+\eta) \cap [a,b]$, they satisfy $|x_c - x|, |x_d - x| < \mu \leq \frac{1}{2}\eta_{x_i}$. Hence, the following inequalities are true:

$$|x_c - x_i| = |(x_c - x) + (x - x_i)| \leq |x_c - x| + |x - x_i| < \frac{1}{2}\eta_{x_i} + \frac{1}{2}\eta_{x_i} = \eta_{x_i}$$

$$|x_d - x_i| = |(x_d - x) + (x - x_i)| \leq |x_d - x| + |x - x_i| < \frac{1}{2}\eta_{x_i} + \frac{1}{2}\eta_{x_i} = \eta_{x_i}$$

These two inequalities imply $x_c, x_d \in (x_i - \eta_{x_i}, x_i + \eta_{x_i})$, which $|f(x_c) - f(x_d)| \leq \Omega(f, x_i, \eta_{x_i})$. Hence, $\Omega(f, x_i, \eta_{x_i})$ is an upper bound of the set S, showing that $\sup(S) = \Omega(f, x, \eta) \leq \Omega(f, x_i, \eta_{x_i}) < \epsilon$ regarding the initial construction.

So, this $\eta > 0$ satisfies the desired condition.

Proof of the Main Theorem

Theorem 1 *Let $f : [a, b] \to \mathbb{R}$ be a bounded function $(b - a) < \infty$. Then f is Riemann integrable on $[a, b]$ if and only if the set of discontinuities of f, D_f is a set of measure zero.*

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Question 9 *Prove the theorem.*

Pf:

$\implies$: We'll approach this by contradiction. Suppose $f : [a, b] \to \mathbb{R}$ is Riemann Integrable, yet D_f has measure greater than 0.

In **Quesion 7**, we've proven that $D_f = \bigcup_{j\in\mathbb{Z}_+} A_{1/j}$ (Countable union of A_j), with $A_{1/j}$ being defined in **Question 6**. With the assumption that D_f has measure 0, there exists $j_0 \in \mathbb{Z}_+$, with A_{1/j_0} having measure greater than 0: If all $j \in \mathbb{Z}_+$ has measure 0, then by the statement proven in **Question 2**, the countable union $D_f = \bigcup_{j\in\mathbb{Z}_+} A_{1/j}$ should also have measure 0, which contradicts the assumption.

With the given $j_0 \in \mathbb{Z}_+$, consider any partition $P = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ with $x_{i-1} < x_i$ for all index i. Since $A_{1/j_0} \subseteq [a, b]$, there exists some intervals in the partition $I_{n_1}, ..., I_{n_i}$ that covers A_{1/j_0} (here, assume every chosen interval covered some part of A_{1/j_0}, not disjoint with it).

WLOG, we can assume that for each interval I_{n_j}, there exists a point of discontinuity $x \in A_{1/j_0}$ that is an interior point of I_{n_j}: If $x = x_i$ for some $i \neq 0$ and $i \neq n$, then we can combine two intervals $I' = I_i \cup I_{i+1} = [x_{i-1}, x_{i+1}]$, which x_i becomes the interior point of the modified interval; else if $x = a$ or $x = b$, since all the definition only consider the cases in $[a, b]$, then a, b is could be considered as the interior point of $[a, b]$ under subspace topology.

Because A_{1/j_0} has nonzero measure, then there exists $\epsilon > 0$, such that for any open interval covering $\{I_j\}_{j\in\mathbb{Z}_+}$ of A_{1/j_0}, the sum of length of the intervals $\sum_{j=1}^{\infty} |I_j| \geq \epsilon$, regardless of the collection of open intervals (in particular, $\sum_{j=1}^{i} |I_{n_j}| \geq \epsilon$, since each interval has the same length with its interior).

Furthermore, from the previous assumption, there exists $x \in A_{1/j_0}$ that is an interior point of I_{n_j} for each $j \in \{1, ..., i\}$, hence there exists radius $r_j > 0$, with $B_{r_j}(x) \cap [a, b] \subseteq I_{n_j}$. Now, by definition, since for all $x_1, x_2 \in B_{r_j}(x) \cap [a, b] \subseteq I_{n_j}$ satisfies $\inf_{x\in I_{n_j}}\{f(x)\} \leq f(x_1), f(x_2) \leq \sup_{x\in I_{n_j}}\{f(x)\}$, hence:

$$|f(x_1) - f(x_2)| \leq \left(\sup_{x\in I_{n_j}} \{f(x)\} - \inf_{x\in I_{n_j}} \{f(x)\} \right)$$

This implies the following:

$$\sup\{|f(x_1) - f(x_2)| \ : \ x_1, x_2 \in B_{r_j}(x) \cap [a, b]\} = \Omega(f, x, r_j) \leq \left(\sup_{x\in I_{n_j}} \{f(x)\} - \inf_{x\in I_{n_j}} \{f(x)\} \right)$$

$$\frac{1}{j_0} \leq \omega_f(x) \leq \Omega(f, x, r_j) \leq \left(\sup_{x\in I_{n_j}} \{f(x)\} - \inf_{x\in I_{n_j}} \{f(x)\} \right)$$

Hence, consider the difference in upper and lower sum, we yield:

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \left(\sup_{x\in I_k}\{f(x)\} - \inf_{x\in I_k}\{f(x)\} \right) \cdot |I_k| \geq \sum_{j=1}^{i} \left(\sup_{x\in I_{n_j}}\{f(x)\} - \inf_{x\in I_{n_j}}\{f(x)\} \right) \cdot |I_{n_j}|$$

$$U(f,P) - L(f,P) \geq \sum_{j=1}^{i} \left(\sup_{x\in I_{n_j}}\{f(x)\} - \inf_{x\in I_{n_j}}\{f(x)\} \right) \cdot |I_{n_j}| \geq \sum_{j=1}^{i} \frac{1}{j_0}|I_{n_j}|$$

$$U(f,P) - L(f,P) \geq \frac{1}{j_0}\sum_{j=1}^{i} |I_{n_j}| \geq \frac{1}{j_0}\cdot \epsilon$$

(Note: The above is true, since the collection $I_{n_1}, ..., I_{n_j}$ is part of the partition).

Hence, the difference in the upper and lower sum for any P is at least $\epsilon/j_0 > 0$, showing that f is not Riemann Integrable. Yet, this contradicts our initial assumption; hence, the assumption is false, f is Riemann Integrable implies D_f has measure 0.

$\Longleftarrow$: Suppose D_f has measure 0. Then, for all $\epsilon > 0$, since $\frac{\epsilon}{2(b-a)} > 0$, by Archimedean's Property, there exists $j \in \mathbb{N}$, with $\frac{1}{j} < \frac{\epsilon}{2(b-a)}$. Then, consider the set $A_{1/j} \subseteq D_f$:

Partition for Points in $A_{1/j}$:

Before starting this, since the function f is bounded, there exists $M > 0$, such that $|f(x)| < M$ for all $x \in [a,b]$. In other word, for all $x \in [a,b]$, $-M < f(x) < M$.

Given that $A_{1/j} \subseteq D_f$ and D_f has measure 0, then $A_{1/j}$ has measure 0 (or else, if $A_{1/j}$ has nonzero measure, then ever open interval cover of D_f covers $A_{1/j}$, which must have some nonzero total length greater than some $r > 0$, showing that D_f also has nonzero measure).

Hence, for given $\epsilon > 0$, since $\frac{\epsilon}{4M} > 0$, there exists collection of open interval covering $\{I_n\}_{n\in\mathbb{Z}_+}$ for $A_{1/j}$, with $\sum_{n=1}^{\infty} |I_n| < \frac{\epsilon}{4M}$. Then, by statement proven in **Question 6**, since $A_{1/j}$ is compact, there exists $n_1, ..., n_k \in \mathbb{Z}_+$, with $A_{1/j} \subseteq \bigcup_{i=1}^{k} I_{n_i}$.

Let P_1 be a partition consists of the infimum and supremum of each interval $I_{n_i} \cap [a,b]$, which for each $x_i \in P_1$, it is the endpoints of the closure of $I_{n_i} \cap [a,b]$ (which, any point $x \in A_{1/j}$ must be contained in between some $x_{i-1}, x_i \in P_1$).

Then, for any interval described by P_1 (each is a closure of $I_{n_i} \cap [a,b]$), since for $x \in \overline{I_{n_i} \cap [a,b]}$, we have $-M < f(x) < M$. Hence:

$$-M \leq \inf_{x\in\overline{I_{n_i}\cap[a,b]}}(f(x)) \leq \sup_{x\in\overline{I_{n_i}\cap[a,b]}}(f(x)) \leq M$$

$$\left(\sup_{x\in\overline{I_{n_i}\cap[a,b]}}(f(x)) - \inf_{x\in\overline{I_{n_i}\cap[a,b]}}(f(x)) \right) \leq 2M$$

So, taking the difference of upper sum and lower sum over these intervals, we get:

$$U(f,P_1)-L(f,P_1)=\sum_{i=1}^{k}\left(\sup_{x\in\overline{I_{n_i}\cap[a,b]}}(f(x))-\inf_{x\in\overline{I_{n_i}\cap[a,b]}}(f(x))\right)\cdot|\overline{I_{n_i}\cap[a,b]}|$$

$$\leq\sum_{i=1}^{k}2M\cdot|I_{n_i}|\leq 2M\sum_{n=1}^{\infty}|I_n|<2M\cdot\frac{\epsilon}{4M}=\frac{\epsilon}{2}$$

So, consider the partition over the region covering $A_{1/j}$, the difference in upper and lower sum is bounded by $\frac{\epsilon}{2}$.

(Note: this upper and lower sum is evaluating over the intervals covering $A_{1/j}$ only, not the whole $[a,b]$).

Partition for Points outside $A_{1/j}$:

Consider the set $K=[a,b]\setminus(\bigcup_{i=1}^{k}I_{n_i})$: By definition of $A_{1/j}$ in **Question 6**, for every $x\in K$, since $x\notin\bigcup_{i=1}^{k}I_{n_i}$ while $A_{1/j}\subseteq\bigcup_{i=1}^{k}I_{n_i}$, then $x\notin A_{1/j}$, hence $\omega_f(x)<\frac{1}{j}$. Then, because $\bigcup_{i=1}^{k}I_{n_i}$ is open, then $\left(\bigcup_{i=1}^{k}I_{n_i}\right)^C$ is closed, hence $K=[a,b]\cap\left(\bigcup_{i=1}^{k}I_{n_i}\right)^C$ is closed and bounded, which is compact. Which, using the same proof in **Question 8**, since $\omega_f(x)<\frac{1}{j}$ for all $x\in K$ while K is compact, then there exists $\eta>0$, such that for all $x\in K$, $\Omega(f,x,\eta)<\frac{1}{j}$ (Note: the proof is by swapping $[a,b]$ with K; since K is compact, same proof relying on the compactness of $[a,b]$ would work on K also).

Then, Consider the partition P_2' of $[a,b]$ with $\|P_2'\|=\max\{(x_i-x_{i-1})\}<\eta$, and take $P_2=P_2'\setminus(\bigcup_{i=1}^{k}I_{n_i})$ (the goal here is to construct partitions for the rest of the part not covered by the open interval covering of $A_{1/j}$).

For any interval I_i described by P_2 (the interval disjoint from the previous part that cover $A_{1/j}$), since each $x\in I_i$ is not in $A_{1/j}$, then we know:

$$\sup\{|f(x_1)-f(x_2)|\ :\ x_1,x_2\in(x-\eta,x+\eta)\cap[a,b]\}=\Omega(f,x,\eta)<\frac{1}{j}$$

Also, notice that since $|I_i|\leq\|P_2'\|<\eta$, hence $I_i\subseteq(x-\eta,x+\eta)$, hence for all $x_1,x_2\in I_i$ (WLOG, assume $f(x_1)>f(x_2)$), we have $|f(x_1)-f(x_2)|=(f(x_1)-f(x_2))\leq\Omega(f,x,\eta)<\frac{1}{j}$, which $\frac{1}{j}$ is an upper bound of the difference.

Hence, if consider the fact that $(f(x_1)-f(x_2))\leq\left(\sup_{x\in I_i}(f(x))-\inf_{x\in I_i}(f(x))\right)$, and this quantity is the supremum of the difference, then $(\sup_{x\in I_i}(f(x))-\inf_{x\in I_i}(f(x)))\leq\frac{1}{j}$.

If there are total of m intervals described in P_2, then the following is true:

$$U(f,P_2)-L(f,P_2)=\sum_{i=1}^{m}(\sup_{x\in I_i}(f(x))-\inf_{x\in I_i}(f(x)))\cdot|I_i|\leq\sum_{i=1}^{m}\frac{1}{j}\cdot|I_i|$$

$$\leq\frac{\epsilon}{2(b-a)}\sum_{i=1}^{m}|I_i|\leq\frac{\epsilon}{2(b-a)}(b-a)=\frac{\epsilon}{2}$$

(Note: the initial construction states that $\frac{1}{j}<\frac{\epsilon}{2(b-a)}$, and the sum of all length of intervals in P_2, is at most $(b-a)$ due to the fact that $P_2\subseteq P_2'$, and P_2' is a partition of $[a,b]$, which the total length is at most $(b-a)$).

(Note 2: the upper and lower sum here is only for the regions not covered by the intervals used to cover $A_{1/j}$, which might not cover the whol $[a, b]$).

Hence, for both cases above, take $P = P_1 \cup P_2$, then evaluating the difference in upper sum and lower sum, it will be the sum of the difference from above, which is at most $\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Then, because we can choose a partition, such that the difference in upper and lower sum is at most ϵ for arbitrary ϵ, we can conclude that f is Riemann Integrable on $[a, b]$.

Combining both statements above, we can conclude that f is Riemann Integrable, iff the points of discontinuities, D_f has measure 0.