Math CS Topology HW4

Zih-Yu Hsieh

February 15, 2025

1 (Not done)

Question 1 *Let* $f : X \to Y$ *be a continuous map between topological spaces, and suppose* Y *is Hausdorff. Prove that the graph* $\{(x, f(x)) \mid x \in X\}$ *is a closed subset of* $X \times Y$*.*

Pf:

Let $G = \{(x, f(x)) \mid x \in X\}$ be the graph.

2

Question 2 *Prove that if X and Y are nonempty topological spaces then X is homeomorphic to a subspace of $X \times Y$.*

Pf:

Since Y is not empty, there exists $y_0 \in Y$. Consider the following map $f : X \to X \times Y$, such that for all $x \in X$, $f(x) = (x, y_0)$, which $f(X) = X \times \{y_0\}$ (since for all $x \in X$, $f(x) = (x, y_0) \in X \times \{y_0\}$, and for all $(x, y_0) \in X \times \{y_0\}$, $f(x) = (x, y_0)$). So, we'll restrict the codomain to the set $X \times \{y_0\}$, letting $f : X \to X \times \{y_0\}$.

f is Bijective:

First, we've verified that $f(X) = X \times \{y_0\}$, hence restricting the codomain to the image had made the map surjective.

To verify injectivity, consider $x_1, x_2 \in X$: If $f(x_1) = f(x_2)$, then $(x_1, y_0) = (x_2, y_0)$, so $x_1 = x_2$, proving that it's injective.

So, the map f is bijective, and $f^{-1} : X \times \{y_0\} \to X$ satisfies $f(x, y_0) = x$.

f is Continuous:

For all ope subset $U' \subseteq X \times \{y_0\}$, there exists open subset $U \subseteq X \times Y$, with $U \cap (X \times \{y_0\}) = U'$. Now, consider the preimage $f^{-1}(U')$: For all $x \in f^{-1}(U')$, since $f(x) = (x, y_0) \in U' \subseteq U$, there exists a basis element $A \times B$ (where $A \subseteq X$ and $B \subseteq Y$ are both open), such that $(x, y_0) \in A \times B \subseteq U$. Which:

$$A \times \{y_0\} = (A \cap X) \times (B \cap \{y_0\}) = (A \times B) \cap (X \times \{y_0\}) \subseteq U \cap (X \times \{y_0\}) = U'$$

So, $A \times \{y_0\}$ is an open subset of $X \times \{y_0\}$ under subspace topology, and $(A \times \{y_0\}) \subseteq U'$.

Now, consider all $a \in A \subseteq X$: Since $f(a) = (a, y_0) \in (A \times \{y_0\}) \subseteq U'$, then $a \in f^{-1}(U')$. Hence, $A \subseteq f^{-1}(U')$. Also, recall that $x \in A$, hence $x \in A \subseteq f^{-1}(U')$.

So, for every $x \in f^{-1}(U')$, there is an open subset $A \subseteq X$, with $x \in A \subseteq f^{-1}(U')$, showing that $f^{-1}(U') \subseteq X$ is open.

Therefore, we can conclude that f is continuous, since every open subset of $X \times \{y_0\}$ the image, the preimage in X is open.

f^{-1} is Continuous:

For all open subset $U \subseteq X$, notice that for all $(x, y_0) \in X \times \{y_0\}$, $f^{-1}(x, y_0) = x \in U$ if and only if $x \in U$, hence the preimage $(f^{-1})^{-1}(U) = U \times \{y_0\}$. Which, consider $U \times Y$ an open subset of $X \times Y$, the following is true:

$$(U \times Y) \cap (X \times \{y_0\}) = (U \cap X) \times (Y \cap \{y_0\}) = U \times \{y_0\}$$

Hence, $U \times \{y_0\}$ is an intersection of $X \times \{y_0\}$ and $(U \times Y)$, proving that $U \times \{y_0\}$ is an open subset of $X \times \{y_0\}$ under subspace topology, so the preimage of U under f^{-1}, $(f^{-1})^{-1}(U) = U \times \{y_0\}$ is open, showing that f^{-1} is continuous, since all open subset of X has a preimage being open.

Because f^{-1} exists when restricting the codomain to $X \times \{y_0\}$, and both f and f^{-1} are continuous using the given topology, hence f is a homeomorphism, showing that X and $X \times \{y_0\}$ (as a subspace of $X \times Y$) are homeomorphic.

3

Question 3 *If X is a metric space, prove that the distance function $d : X \times X \to \mathbb{R}$ is continuous, where $X \times X$ has the product of the metric topologies.*

Pf:

For all open subset $U \subseteq \mathbb{R}$, consider the preimage $d^{-1}(U) \subseteq X \times X$:

For all $(x_1, x_2) \in d^{-1}(U)$, since $y = d(x_1, x_2) \in U$ while U is open under standard topology of $\mathbb{R}$, then there exists $r > 0$, such that $(y - \frac{r}{3}, y + \frac{r}{3}) \subseteq (y - r, y + r) \subseteq U$.

Now, consider the basis element $\big(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\big) \subseteq X \times X$ under product topology: For all $(a, b) \in \big(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\big)$, the following is true:

$$d(a, b) \leq d(a, x_1) + d(x_1, b) \leq d(a, x_1) + d(x_1, x_2) + d(x_2, b) < \frac{r}{3} + y + \frac{r}{3} = y + \frac{2r}{3}$$

(Note: the above is true, since $a \in B_d(x_1, \frac{r}{3})$ and $b \in B_d(x_2, \frac{r}{3})$).

Hence, since $y < y + \frac{2r}{3} < y + r$, then $d(a, b) = (y + \frac{2r}{3}) \in (y - r, y + r) \subseteq U$, showing that $(a, b) \in d^{-1}(U)$. And, since the choice of $(a, b) \in \big(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\big)$ is arbitrary, $\big(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\big) \subseteq f^{-1}(U)$.

So, for all $(x_1, x_2) \in d^{-1}(U) \subseteq X \times X$, there exists a basis element $\big(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\big)$, such that $(x_1, x_2) \in \big(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\big) \subseteq d^{-1}(U)$, hence $d^{-1}(U)$ is open.

Which, we can conclude that $d : X \times X \to \mathbb{R}$ is continuous under product topology of $X \times X$ (based on metric topology of X), and standard topology of $\mathbb{R}$.

4

Question 4

Pf: