Math CS 122A HW6

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1

Question 1 *Ahlfors Pg. 123 Problem 2:*
Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some nand all sufficiently large $|z|$ reduces to a polynomial.

Pf:

Given that for $z \in \mathbb{C}$ with $|z|$ being sufficiently large, $|f(z)| < |z|^n$ is satisfied, then there exists a radius $r > 0$, such that $|z| \geq r$ implies $|f(z)| < |z|^n$. Which, we'll consider the n^{th} derivative, $f^{(n)}(z)$. (Note: Since f is analytic on the whole plane, then all of its derivative exists, and is analytic on the whole plane).

First, consider the disk $D_{2r} = \{z \in \mathbb{C} \mid |z| \leq 2r\}$: Since it is a closed and bounded set, then it is compact. Hence, since $|f^{(n)}|$ is also continuous due to the analytic nature of $f^{(n)}$, then $|f^{(n)}|(D_{2r}) \subseteq \mathbb{R}$ is also compact, hence there exists $M > 0$, such that for all $z \in D_{2r}$, $|f^{(n)}(z)| \leq M$.

Then, for all $z \in \mathbb{C} \setminus D_{2r}$, we'll consider $f^{(n)}(z)$ using Cauchy's Integral Formula: Let γ be the curve of the circle $|z| = r$, then for all z not on the given circle, the following is true:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

Which, for $z \in \mathbb{C} \setminus D_{2r}$, since $|z| > 2r > r$, then for all $\zeta \in \gamma$ (which $|\zeta| = r$), the following is true:

$$|\zeta - z| \geq ||\zeta| - |z|| = |r - |z|| = |z| - r > 2r - r = r, \quad |\zeta - z|^{n+1} > r^{n+1}, \quad \frac{1}{|\zeta - z|^{n+1}} < \frac{1}{r^{n+1}}$$

Similarly, since $|\zeta| \geq r$, then based on the assumption, $|f(\zeta)| < |\zeta|^n = r^n$. Hence, the following inequality is true:

$$|f^{(n)}(z)| = \left| \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right| \leq \frac{n!}{2\pi} \int_\gamma \left| \frac{f(\zeta)}{(\zeta - z)^{n+1}} \right| \cdot |d\zeta| < \frac{n!}{2\pi} \int_\gamma \frac{r^n}{r^{n+1}} \cdot |d\zeta|$$

$$|f^{(n)}(z)| < \frac{n!}{2\pi} \cdot \frac{1}{r} \cdot 2\pi r = n!$$

(Note: the first inequality is true, based on the statement that $|f(\zeta)| < r^n$ and $\frac{1}{|\zeta - z|^{n+1}} < \frac{1}{r^{n+1}}$).

Hence, take $M' = \max\{M, n!\}$, then for all $z \in \mathbb{C}$, if $z \in D_{2r}$, then $|f^{(n)}(z)| \leq M \leq M'$; else if $z \in \mathbb{C} \setminus D_{2r}$, then $|f^n(z)| \leq n! \leq M'$. So, the analytic function $f^{(n)}(z)$ is bounded on the whole plane, which by Liouville's Theorem, $f^{(n)}(z)$ must be a constant function.

Then, since the n^{th} derivative of f is a constant, then f must be a polynomial (in fact, with degree at most n).

2

Question 2 *Ahlfors Pg. 123 Problem 5:*
Show that the successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(z)| > n!n^n$. *Formulate a sharper theorem of the same kind.*

Pf:

Let the analytic function f be defined on an open set Ω, which for all $z_0 \in \Omega$, there exists $r' > 0$, such that the open disk $|z - z_0| < r'$ is within Ω. If we let $r = \frac{r'}{2} > 0$, then the closed disk $|z - z_0| \leq r$ is fully contained in $|z - z_0| < r'$, which is within Ω.

Now, let γ be the circle $|z - z_0| = r$, since it is a compact set where $|f|$ is defined while f is continuous, then $|f|(\gamma) \subseteq \mathbb{R}$ has a maximum, there exists $M > 0$, such that for all $z \in \gamma$, $|f(z)| \leq M$ (For simplicity, choose $M \geq 1$).

Hence, based on Cauchy's Integral Formula, for all $n \in \mathbb{N}$, the following formula is true:

$$f^{(n)}(z_0) = \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\right| \leq \frac{n!}{2\pi}\int_\gamma \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} \cdot |d\zeta| \leq \frac{n!}{2\pi}\int_\gamma \frac{M}{r^{n+1}} \cdot |d\zeta|$$

$$f^{(n)}(z_0) \leq \frac{n!}{2\pi \cdot \frac{M}{r^{n+1}}} \cdot 2\pi r = \frac{n!M}{r^n}$$

(Note: For all $\zeta \in \gamma$, $|\zeta - z_0| = r$, and $|f(\zeta)| \leq M$).

Notice that since $\frac{M}{r} > 0$, then by Archimedean's Property, there exists $k \in \mathbb{N}$, with $k > \frac{M}{r}$, which since $M \geq 1$ is assumed, the following inequality is true:

$$k^k > \left(\frac{M}{r}\right)^k = \frac{M^k}{r^k} \geq \frac{M}{r^k}, \quad |f^{(k)}(z_0)| \leq \frac{k!M}{r^k} < k!k^k$$

Also, for all integer $n \geq k$, the following is satisfied:

$$n^n \geq k^n > \left(\frac{M}{r}\right)^n = \frac{M^n}{r^n} \geq \frac{M}{r^n}, \quad |f^{(n)}(z_0)| \leq \frac{n!M}{r^n} < n!n^n$$

So, for all $z_0 \in \Omega$, there exists $k \in \mathbb{N}$, such that $n \geq k$ implies $|f^{(n)}(z_0)| < n!n^n$, showing that $|f^{(n)}(z)| > n!n^n$ can never be satisfied by any point z and for all but finitely many $n \in \mathbb{N}$.

Stronger Condition:

Recall that for all $r_0 > 0$, by Archimedean's Property, there exists $N \in \mathbb{N}$ with $N > r_0$. Therefore, for $n \geq N$, $\frac{r_0^{n+1}}{(n+1)!} = \frac{r_0}{(n+1)} \cdot \frac{r_0^n}{n!} < \frac{r_0}{N} \cdot \frac{r_0^n}{n!}$, which for all positive integer k, we can inductively prove that $\frac{r_0^{N+k}}{(N+k)!} < \left(\frac{r_0}{N}\right)^k \cdot \frac{r_0^N}{N!}$.

Hence, since $\frac{r_0}{N} < 1$, then the following is true:

$$0 < \frac{r_0^{N+k}}{(N+k)!} < \left(\frac{r_0}{N}\right)^k \cdot \frac{r_0^N}{N!}$$

$$0 \leq \lim_{k\to\infty} \frac{r_0^{N+k}}{(N+k)!} \leq \lim_{k\to\infty}\left(\frac{r_0}{N}\right)^k \cdot \frac{r_0^N}{N!} = 0$$

Which, $\lim_{n\to\infty} \frac{r_0^n}{n!} = 0$ based on the above inequality, so there exists $N \in \mathbb{N}$, such that $n \geq N$ implies $\frac{r_0^n}{n!} < 1$, or $r_0^n < n!$.

Then, looking back to the inequality $|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}$, since $\frac{M^{1/n}}{r} > 0$, there exists N, such that $n \geq N$ implies $\frac{M}{r^n} = \left(\frac{M^{1/n}}{r}\right)^n < n!$. Hence, the following inequality is true:

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n} < n! \cdot n! = (n!)^2$$

So, we can conclude that for some $N \in \mathbb{N}$, $n \geq N$ implies $|f^{(n)}(z_0)| < (n!)^2$, which is a stricter condition than $n!n^n$, since $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ (so for all sufficiently large n, $n! < n^n$).

3

Question 3 *Ahlfors Pg. 130 Problem 2:*

Pf:

4

Question 4 *Ahflors Pg. 130 Problem 6:*

Pf:

5

Question 5 *Stein and Shakarchi Pg. 66 Problem 7:*
Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies $2|f'(0)| \le d$.
Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.

Pf:

Fix an $r \in (0, 1)$, and let γ be the circle $|z| = r$. Then, based on Cauchy's Integral Formula, $f'(0)$ can be extracted as:

$$f'(0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^2}d\zeta$$

Which, if consider $f(-z)$ instead, its derivative is given by $-f'(-z)$, hence with $z = 0$, the following is true:

$$-f'(0) = \frac{1}{2\pi i}\int_\gamma \frac{f(-\zeta)}{\zeta^2}d\zeta, \quad f'(0) = -\frac{1}{2\pi i}\int_\gamma \frac{f(-\zeta)}{\zeta^2}d\zeta$$

Which, adding the two equations together, we yield:

$$2f'(0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta) - f(-\zeta)}{\zeta^2}d\zeta$$

Given that $g(z) = f(z) - f(-z)$ is an analytic function (sum and composition of analytic functions), and γ is a compact set (boundary of a bounded circle), then since $|g|$ is continuous, $|g|(\gamma)$ is a compact set. Hence, there exists a maximum on γ, for some $z_0 \in \gamma$, all $z \in \gamma$ satisfies:

$$|f(z) - f(-z)| = |g(z)| \le |g(z_0)| = |f(z_0) - f(-z_0)|$$

And, notice that since $z_0, -z_0$ have modulus r (since they're on γ), then $z_0, -z_0 \in \mathbb{D}$. Therefore, $|g(z_0)| = |f(z_0) - f(-z_0)| \le \sup_{z,w\in\mathbb{D}} |f(z) - f(w)| = d$, thus for all $z \in \gamma$, $|g(z)| \le d$.

Now, going back to the third equation, since for all $\zeta \in \gamma$, it satisfies $|g(\zeta)| \le d$ and $|\zeta| = r$, we can conclude the following inequality:

$$2|f'(0)| = |2f'(0)| = \left|\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta) - f(-\zeta)}{\zeta^2}d\zeta\right| \le \frac{1}{2\pi}\int_\gamma \frac{|g(\zeta)|}{|\zeta|^2}\cdot|d\zeta| \le \frac{1}{2\pi}\int_\gamma \frac{d}{r^2}\cdot|d\zeta|$$

$$2|f'(0)| \le \frac{1}{2\pi}\cdot\frac{d}{r^2}\cdot 2\pi r = \frac{d}{r}$$

Also, notice that the above inequality is true for all $r \in (0, 1)$, since $d = \inf\{\frac{d}{r} \mid r \in (0, 1)\}$,

(Note: since for all $r \in (0, 1)$, $\frac{1}{r} > 1$, then $\frac{d}{r} > d$; on the other hand, for all $\epsilon > 0$, since $d + \epsilon/2 = d(1 + \frac{\epsilon}{2d}) = \frac{d}{1/(1+\frac{\epsilon}{2d})}$, because $(1 + \frac{\epsilon}{2d}) > 1$, then $0 < 1/(1 + \frac{\epsilon}{2d}) < 1$, hence $(1 + \frac{\epsilon}{2d}) \in (0, 1)$, or $d + \epsilon/2 = \frac{d}{(1+\frac{\epsilon}{2d})} \in \{\frac{d}{r} \mid r \in (0, 1)\}$. However, since $d + \epsilon/2 < d + \epsilon$, then $d + \epsilon$ is not a lower bound of the set. Hence, $d = \inf\{\frac{d}{r} \mid r \in (0, 1)\}$).

then, since $2|f'(0)| \le \frac{d}{r}$ for all $r \in (0, 1)$, then $2|f'(0)|$ is a lower bound of $\{\frac{d}{r} \mid r \in (0, 1)\}$, hence $2|f'(0)| \le \inf\{\frac{d}{r} \mid r \in (0, 1)\} = d$.

6

Question 6 *Stein and Shakarchi Pg. 66 Problem 8:*
If f is a holomorphic function on the strip $-1 < y < 1$, $x \in \mathbb{R}$ with

$$|f(z)| \leq A(1 + |z|)^{\eta}$$

η a fixed real number and for all z in that strip.
Show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that

$$|f^{(n)}(x)| \leq A_n(1 + |x|)^{\eta}$$

Pf: