Math CS 122A HW1

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Question 1 ***Ahlfors Pg. 16 Problem 4***

Pf: Given that $w = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$ and $h \in \mathbb{Z}$ is not a multiple of n. Then, the term $\frac{h}{n}$ is not an integer, which consider w^h:

$$w^h = \cos\left(\frac{h \cdot 2\pi}{n}\right) + i\sin\left(\frac{h \cdot 2\pi}{n}\right) \neq 1$$

Since $\frac{h}{n}$ is not an integer, $\frac{h \cdot 2\pi}{n}$ is not an integer multiple of 2π, thus $\cos\left(\frac{h \cdot 2\pi}{n}\right) \neq 1$, or $w^h \neq 1$.

Hence, $(1 - w^h) \neq 0$, division with this number is defined. Now, consider the following:

$$1 + w^h + ... + w^{(n-1)h} = \frac{(1 - w^h)(1 + w^h + ... + w^{(n-1)h})}{(1 - w^h)} = \frac{1 - w^{nh}}{1 - w^h}$$

Which, since $w^n = \cos(\frac{n \cdot 2\pi}{n}) + i\sin(\frac{n \cdot 2\pi}{n}) = \cos(2\pi) + i\sin(2\pi) = 1$, then $w^{nh} = (w^n)^h = 1^h = 1$. Thus:

$$1 + w^h + ... + w^{(n-1)h} = \frac{1 - w^{nh}}{1 - w^h} = \frac{1 - 1}{1 - w^h} = 0$$

Which, the given equality is true.

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Question 2 ***Ahlfors Pg. 17 Problem 5***

Pf:

The sum $1 - w^h + w^{2h} - ... + (-1)^{(n-1)}w^{(n-1)h} = \sum_{i=0}^{(n-1)}(-w^h)^i$. Which, there are two cases to consider:

First, if $h \neq \frac{(2k+1)}{2}n$ for all $k \in \mathbb{Z}$. Thus, $h \cdot \frac{2\pi}{n} \neq \frac{(2k+1)n}{2} \cdot \frac{2\pi}{n} = (2k+1)\pi$ for all $k \in \mathbb{Z}$, $\cos\left(\frac{h \cdot 2\pi}{n}\right) \neq \cos((2k+1)\pi) = -1$, so $w^h = \cos\left(\frac{h \cdot 2\pi}{n}\right) + i \sin\left(\frac{h \cdot 2\pi}{n}\right) \neq -1$. Hence, $(1 - (-w^h)) = (1 + w^h) \neq 0$.

Then, the sum could be expressed as:

$$\sum_{i=0}^{(n-1)}(-w^h)^i = \frac{(1-(-w^h))(\sum_{i=0}^{(n-1)}(-w^h)^i)}{(1-(-w^h))} = \frac{1-(-w^h)^n}{1+w^h}$$

Which, there are two possibilities:

- If n is odd, then $(-w^h)^n = (-1)^n w^{nh} = -w^{nh}$, while $w^{nh} = 1$ (proven in **Question 1**). Thus, $(-w^h)^n = -1$, and the sum is as follow:

$$\sum_{i=0}^{(n-1)}(-w^h)^i = \frac{1-(-w^h)^n}{1+w^h} = \frac{1-(-1)}{1+w^h} = \frac{2}{1+w^h}$$

- Else if n is even, then $(-w^h)^n = (-1)^n w^{nh} = w^{nh}$, while $w^{nh} = 1$. Thus, the sum is as follow:

$$\sum_{i=0}^{(n-1)}(-w^h)^i = \frac{1-(-w^h)^n}{1+w^h} = \frac{1-1}{1+w^h} = 0$$

Else, if $h = \frac{(2k+1)}{2}n$ for some $k \in \mathbb{Z}$, then $\frac{h \cdot 2\pi}{n} = \frac{(2k+1)n}{2} \cdot \frac{2\pi}{n} = (2k+1)\pi$.

Thus, $w^h = \cos\left(\frac{h \cdot 2\pi}{n}\right) + i \sin\left(\frac{h \cdot 2\pi}{n}\right) = \cos((2k+1)\pi) + i\sin((2k+1)\pi) = -1$. So, the sum is expressed as:

$$\sum_{i=0}^{(n-1)}(-w^h)^i = \sum_{i=0}^{(n-1)}(-(-1))^i = \sum_{i=0}^{(n-1)} 1^i = \sum_{i=0}^{(n-1)} 1 = n$$

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Question 3 ***Ahlfors Pg. 28 Problem 4***

Pf:

Suppose $f(z)$ is an analytic function that has constant norm. Which, let $z = x + iy$ for any $x, y \in \mathbf{R}$, and $f(x + iy) = u(x, y) + iv(x, y)$ for first-order differentiable real-valued functions u, v.

Since f is analytic, u and v satisfy the Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Now, consider $|f| = |u + iv| = \sqrt{u^2 + v^2}$, which is assumed to be a constant. Then, there are two cases to consider:

First, if $|f| = 0$ for all $z \in Dom(f)$, then $\sqrt{u^2 + v^2} = 0$, which $u^2 + v^2 = 0$ while u, v are real-valued function. This only happens if $u, v = 0$, thus $f(z) = u(x, y) + iv(x, y) = 0$, which f is a constant function.

Else if $|f| = c$ for some $c > 0$ for all $z \in Dom(f)$. Then, consider the partial derivative of $|f|$:

$$\frac{\partial}{\partial x}(|f|) = \frac{\partial}{\partial x}(\sqrt{u^2 + v^2}) = \frac{1}{2\sqrt{u^2 + v^2}}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right) = \frac{1}{c}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial}{\partial y}(|f|) = \frac{\partial}{\partial y}(\sqrt{u^2 + v^2}) = \frac{1}{2\sqrt{u^2 + v^2}}\left(2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right) = \frac{1}{c}\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)$$

Since $|f| = c$ is a constant, then the partial derivatives are all 0. Thus:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0, \quad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

Which multiplying the second equation by i, we get:

$$iu\frac{\partial u}{\partial y} + iv\frac{\partial v}{\partial y} = 0, \quad iu\frac{\partial u}{\partial y} + iv\frac{\partial v}{\partial y} = u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}$$

Reorganize the equation, we get:

$$v\left(-\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right) = u\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$$

Based on Cauchy-Riemann Equation $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, substitute the variables containing these two terms, we get:

$$v\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = u\left(\frac{\partial u}{\partial x} - i\left(-\frac{\partial v}{\partial x}\right)\right)$$

Which, since $\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i\left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$, the following is true:

$$iv\left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = u\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

Now, recall that for analytic function, $\frac{\partial f}{\partial z} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)$. Thus the above equation could also be written as:

$$iv\frac{\partial f}{\partial z} = u\frac{\partial f}{\partial z}, \quad (u - iv)\frac{\partial f}{\partial z} = 0$$

However, since $|f(z)| = |u + iv| = c > 0$, then $(u + iv) \neq 0$, which its conjugate $(u - iv) \neq 0$ also. Thus, in case for the above equation to be true, $\frac{\partial f}{\partial z} = 0$, showing that f is a constant function.

Since for all analytic function, having constant norm implies the function itself is constant, then any analytic function that's not constant cannot have constant norm.

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Question 4 ***Stein and Shakarchi Pg. 26 Problem 7***

Pf:

(a) Given $z, w \in \mathbb{C}$ such that $\bar{z}w \neq 1$ (which implies that $\overline{\bar{z}w} = z\bar{w} \neq 1$, since $\overline{\bar{z}w} \neq \bar{1} = 1$).

First, for all $u, v \in \mathbb{C}$, the following identity is true:

$$|u - v|^2 = |u|^2 + |v|^2 - 2Re(\bar{u}v)$$

Which, apply it to $(w - z)$ and $(1 - \bar{w}z)$, we get:

$$|w - z|^2 = |w|^2 + |z|^2 - 2Re(\bar{w}z)$$

$$|1 - \bar{w}z|^2 = |1|^2 + |\bar{w}z|^2 - 2Re(\bar{1} \cdot (\bar{w}z)) = 1 + |w|^2 \cdot |z|^2 - 2Re(\bar{w}z)$$

When $|z|, |w| < 1$:

Given that $|z|, |w| < 1$, we just need to compare $|w|^2 + |z|^2$ and $1 + |w|^2 \cdot |z|^2$. Which, if we take the difference, it is as follow:

$$(1 + |w|^2 \cdot |z|^2) - (|w|^2 + |z|^2) = |w|^2(|z|^2 - 1) + (1 - |z|^2)$$

$$= -|w|^2(1 - |z|^2) + (1 - |z|^2) = (1 - |w|^2)(1 - |z|^2)$$

Since both $|z|, |w| < 1$, then $|z|^2, |w|^2 < 1$, which $0 < (1 - |z|^2), (1 - |w|^2)$, thus $(1 - |w|^2)(1 - |z|^2) > 0$.

From this, we can conclude the following:

$$0 < (1 - |w|^2)(1 - |z|^2) = (1 + |w|^2 \cdot |z|^2) - (|w|^2 + |z|^2), \quad (|w|^2 + |z|^2) < (1 + |w|^2 \cdot |z|^2)$$

$$|w|^2 + |z|^2 - 2Re(\bar{w}z) < 1 + |w|^2 \cdot |z|^2 - 2Re(\bar{w}z)$$

Substitute the original modulus form, we get:

$$|w - z|^2 < |1 - \bar{w}z|^2$$

$$\frac{|w - z|^2}{|1 - \bar{w}z|^2} < 1, \quad \left|\frac{w - z}{1 - \bar{w}z}\right|^2 < 1, \quad \left|\frac{w - z}{1 - \bar{w}z}\right| < 1$$

When $|z| = 1$ or $|w| = 1$:

Suppose $|z| = 1$ or $|w| = 1$.

If $|z| = 1$ (or $|z|^2 = 1$), the following is true:

$$|w - z|^2 = |w|^2 + |z|^2 - 2Re(\bar{w}z) = |w|^2 \cdot |z|^2 + 1 - 2Re(\bar{w}z) = |1 - \bar{w}z|^2$$

Else if $|w| = 1$ (or $|w|^2 = 1$), the following is true:

$$|w - z|^2 = |w|^2 + |z|^2 - 2Re(\bar{w}z) = 1 + |w|^2 \cdot |z|^2 - 2Re(\bar{w}z) = |1 - \bar{w}z|^2$$

Which, regardless of the case, $|w - z|^2 = |1 - \bar{w}z|^2$, or $|w - z| = |1 - \bar{w}z|$. Hence, since $\bar{w}z \neq 1$ by assumption ($1 - \bar{w}z \neq 0$, or $|1 - \bar{w}z| \neq 0$), the following is true:

$$\frac{|w - z|}{|1 - \bar{w}z|} = 1, \quad \left|\frac{w - z}{1 - \bar{w}z}\right| = 1$$

(b) Given a fixed complex number $w \in \mathbb{D}$, where $\mathbb{D} \subset \mathbb{C}$ is the unit disc, and the map $F : z \mapsto \frac{w-z}{1-\bar{w}z}$.

In case for the function F to be defined on w, we need $|w| < 1$: Since w is in the unit disc, then $|w| \leq 1$; yet, if $|w| = 1$, then $(1 - \bar{w}w) = (1 - |w|^2) = (1 - 1) = 0$, which the function is not defined since $(1 - \bar{w}w)$ is in the denominator. So, we need $|w| < 1$.

Then, there are some conditions to check:

(i) For all $z \in \mathbb{D}$, $|z| \leq 1$. And, since $|w| < 1$, then $|\bar{w}z| = |w| \cdot |z| < 1$, thus $\bar{w}z \neq 1$, or $(1 - \bar{w}z) \neq 0$. Thus, the value $F(z) = \frac{w-z}{1-\bar{w}z}$ is defined.

First, if $|z| = 1$, by the statement proven in part (a), the following is true:

$$|F(z)| = \left|\frac{w - z}{1 - \bar{w}z}\right| = 1$$

Thus, $F(z) \in \mathbb{D}$.

Else, if $|z| < 1$, then since $|w| < 1$ is proven beforehand, again by the statement proven in part (a), the following is true:

$$|F(z)| = \left|\frac{w - z}{1 - \bar{w}z}\right| < 1$$

Thus again, $F(z) \in \mathbb{D}$.

Regardless of the case, for all $z \in \mathbb{D}$, $F(z) \in \mathbb{D}$, thus restricting the domain to $\mathbb{D}$, $F(\mathbb{D}) \subseteq \mathbb{D}$. So, $F : \mathbb{D} \to \mathbb{D}$.

To prove that it is analytic (holomorphic), recall that the function z is holomorphic, which $(w - z)$ and $(1 - \bar{w}z)$ are both holomorphic functions (while given that $(1 - \bar{w}z) \neq 0$ for all $z \in \mathbb{D}$). Thus, the quotient of two functions $\frac{(w-z)}{(1-\bar{w}z)}$ is holomorphic.

(ii) Consider $F(0)$ and $F(w)$:

$$F(0) = \frac{w-0}{1-\bar{w}0} = \frac{w}{1-0} = w$$

$$F(w) = \frac{w-w}{1-\bar{w}w} = \frac{0}{1-|w|^2} = 0$$

Note: the second equation is defined, since we've proven that $|w| < 1$, which $|w|^2 < 1$, or $(1-|w|^2) > 0$, the function is defined.

(iii) If $|z| = 1$, then from what we've proven in part (a):

$$|F(z)| = \left|\frac{w-z}{1-\bar{w}z}\right| = 1$$

(iv) Consider $F \circ F$: For all $z \in \mathbb{D}$, the following is true:

$$F \circ F(z) = F\left(\frac{w-z}{1-\bar{w}z}\right) = \frac{w - \frac{w-z}{1-\bar{w}z}}{1-\bar{w}\frac{w-z}{1-\bar{w}z}} = \frac{w(1-\bar{w}z)-(w-z)}{1(1-\bar{w}z)-\bar{w}(w-z)}$$

$$= \frac{w - w\bar{w}z - w + z}{1-\bar{w}z - \bar{w}w + \bar{w}z} = \frac{-|w|^2 z + z}{1-|w|^2} = \frac{z(1-|w|^2)}{1-|w|^2} = z$$

(Note: since $(1-|w|^2) \neq 0$, the above equation is defined).

Which, $F \circ F$ is actually an Identity map from $\mathbb{D}$ to $\mathbb{D}$, it is bijective.

Thus, F is surjective: If F is not surjective, then there exists $u \in \mathbb{D}$, such that any $z \in \mathbb{D}$ cannot satisfy $F(z) = u$. However, that means for all $z \in \mathbb{D}$, since $F(z) \in \mathbb{D}$, $F \circ F(z) \neq u$, which $F \circ F(u) = u$ is a contradiction. Therefore, F must be surjective.

Also, F is injective: For all $z_1, z_2 \in \mathbb{D}$ with $F(z_1) = F(z_2)$, since $z_1 = F \circ F(z_1) = F(F(z_1)) = F(F(z_2)) = F \circ F(z_2) = z_2$, then $z_1 = z_2$, showing that F is injective.

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Question 5 ***Stein and Shakarchi Pg. 26 Problem 9***

Pf:

Given that $u(x,y), v(x,y)$ are two real-valued functions, with $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then, the following is true:

$$\frac{\partial x}{\partial r} = \cos(\theta), \quad \frac{\partial x}{\partial \theta} = -r\sin(\theta), \quad \frac{\partial y}{\partial r} = \sin(\theta), \quad \frac{\partial y}{\partial \theta} = r\cos(\theta)$$

Then, the partial derivative of u and v can be rewritten as:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin(\theta) + \frac{\partial u}{\partial y}r\cos(\theta)$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos(\theta) + \frac{\partial v}{\partial y}\sin(\theta)$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial v}{\partial x}r\sin(\theta) + \frac{\partial v}{\partial y}r\cos(\theta)$$

If Cauchy-Riemann Equation is satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Substitute into the previous equation, we can yield:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta) = \frac{\partial v}{\partial y}\cos(\theta) - \frac{\partial v}{\partial x}\sin(\theta)$$

$$= \frac{1}{r}\left(\frac{\partial v}{\partial y}r\cos(\theta) - \frac{\partial v}{\partial x}r\sin(\theta)\right) = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{1}{r}\left(-\frac{\partial u}{\partial x}r\sin(\theta) + \frac{\partial u}{\partial y}r\cos(\theta)\right) = -\frac{\partial u}{\partial x}\sin(\theta) + \frac{\partial u}{\partial y}\cos(\theta)$$

$$= -\frac{\partial v}{\partial y}\sin(\theta) - \frac{\partial v}{\partial x}\cos(\theta) = -\left(\frac{\partial v}{\partial y}\sin(\theta) + \frac{\partial v}{\partial x}\cos(\theta)\right) = -\frac{\partial v}{\partial r}$$

Which, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, and $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$.

Now, given logarithm function $\ln(z) = \ln(r) + i\theta$, with $r > 0$ and $-\pi < \theta < \pi$. Then, let $u(r,\theta) = \ln(r)$, and $v(r,\theta) = \theta$, then $\ln(z) = u + iv$.

Consider the first-order partial derivative of u and v:

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial u}{\partial \theta} = 0$$

$$\frac{\partial v}{\partial r} = 0, \quad \frac{\partial v}{\partial \theta} = 1$$

Which, the following are true:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot 1 = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{1}{r} \cdot 0 = 0 = -0 = -\frac{\partial v}{\partial r}$$

Thus, the Cauchy-Riemann Equation in polar coordinates is satisfied, proving that the logarithmic function is holomorphic on $r > 0$ and $-\pi < \theta < \pi$.