Math CS Topology HW2

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Question 1 *Let A^* denote the set of limit points of A. Prove this satisfies:*

- $\emptyset^* = \emptyset$
- $x \notin \{x\}^*$
- $A^{**} \subseteq A \cup A^*$
- $(A \cup B)^* = A^* \cup B^*$

Pf:
All the below proofs are based on a nonempty topological space X.

1. To prove that $\emptyset^* = \emptyset$, we'll use contradiction: Suppose there exists $x \in X$ with $x \in \emptyset^*$, then by definition, every open neighborhood U of x, the intersection $\emptyset \cap (U \setminus \{x\}) \neq \emptyset$.

   However, since every set A satisfies $\emptyset \cap A = \emptyset$, the above condition is a contradiction. Therefore, there's no such $x \in X$ satisfying $x \in \emptyset^*$, thus $\emptyset^* = \emptyset$.

2. To prove that $x \notin \{x\}^*$, consider any arbitrary open neighborhood U of x: Since $x \notin U \setminus \{x\}$, then $\{x\} \cap (U \setminus \{x\}) = \emptyset$. Thus, x is not a limit point of $\{x\}$, or $x \notin \{x\}^*$.

3. To prove that $A^{**} \subseteq A \cup A^*$, consider any $x \in A^{**}$:

   If $x \in A$, then $x \in A \cup A^*$.

   Else, if $x \notin A$, by definition, for every open neightborhood U of x, there exists $y \in A^* \cap (U \setminus \{x\})$, which $y \in A^*$ and $y \in U$, thus U is an open neighborhood of y.

   Then, since y is a limit point of A, then there exists $a \in A \cap (U \setminus \{y\})$, which $a \in A$ and $a \in U$.

   Yet, since $x \notin A$, so $a \neq x$, thus $a \in U \setminus \{x\}$, proving that $A \cap (U \setminus \{x\}) \neq \emptyset$.

   Since every open neighborhood of x satisfies $A \cap (U \setminus \{x\}) \neq \emptyset$, then x is a limit point of A, thus $x \in A^* \subseteq A \cup A^*$.

   So, regardless of the case, $x \in A^{**}$ implies $x \in A \cup A^*$, thus $A^{**} \subseteq A \cup A^*$.

4. To prove that $(A \cup B)^* = A^* \cup B^*$, consider the following:

   First, $A^* \cup B^* \subseteq (A \cup B)^*$: Since $A, B \subseteq (A \cup B)$, then if $x \in A^*$, every open neighborhood U of x satisfies $A \cap (U \setminus \{x\}) \neq \emptyset$, thus $(A \cup B) \cap (U \setminus \{x\}) \neq \emptyset$, showing that $x \in (A \cup B)^*$, or $A^* \subseteq (A \cup B)^*$. Applying the same logic on B^*, we'll get $B^* \subseteq (A \cup B)^*$, hence $(A^* \cup B^*) \subseteq (A \cup B)^*$.

Now, to prove that $(A \cup B)^* \subseteq (A^* \cup B^*)$, we'll approach by contradiction:

Suppose $(A \cup B)^* \nsubseteq (A^* \cup B^*)$, there exists $x \in (A \cup B)^*$, while $x \notin (A^* \cup B^*)$.

Then, since $x \notin A^*$, there exists open neighborhood U_1 of x, with $A \cap (U_1 \setminus \{x\}) = \emptyset$; similarly, since $x \notin B^*$, there exists open neighborhood U_2 of x, with $B \cap (U_2 \setminus \{x\}) = \emptyset$.

Now, consider $U = U_1 \cap U_2$: It is an open set, and since $x \in U_1$ and $x \in U_2$, then $x \in (U_1 \cap U_2) = U$, thus U is an open neighborhood of x.

However, since $U \subseteq U_1$, then $A \cap (U \setminus \{x\}) \subseteq A \cap (U_1 \setminus \{x\}) = \emptyset$; similarly, $U \subseteq U_2$ implies $B \cap (U \setminus \{x\}) \subseteq B \cap (U_2 \setminus \{x\}) = \emptyset$.

So, $(A \cup B) \cap (U \setminus \{x\}) = (A \cap (U \setminus \{x\})) \cup (B \cap (U \setminus \{x\})) = \emptyset$. Yet, if $x \in (A \cup B)^*$, then every open neighborhood of x should have nonempty intersection with $(A \cup B)$, while not including x.

So, this is a contradiction. Hence, $(A \cup B)^* \subseteq (A^* \cup B^*)$.

With the above two statements, $(A \cup B)^* = A^* \cup B^*$.

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Question 2 *Prove that the boundary operation satisfies:*

- $\partial A = \partial(X \setminus A)$
- $\partial\partial A \subseteq \partial A$
- $\partial(A \cup B) \subseteq \partial A \cup \partial B$
- $A \subseteq B \implies \partial A \subseteq (B \cup \partial B)$

Pf:

1. Given any set $A \subseteq X$, since $\partial A = \overline{A} \cap \overline{X \setminus A}$ and $\partial(X \setminus A) = \overline{X \setminus A} \cap \overline{X \setminus (X \setminus A)} = \overline{X \setminus A} \cap \overline{A}$, thus $\partial A = \partial(X \setminus A)$.

2. Given $\partial A = \overline{A} \cap \overline{X \setminus A}$, since $\overline{A}$ and $\overline{X \setminus A}$ are both closed, the ∂A is closed (intersection of arbitrary closed set is closed). Which, $\overline{\partial A} = \partial A$. Hence, $\partial\partial A = \overline{\partial A} \cap \overline{X \setminus \partial A} \subseteq \overline{\partial A} = \partial A$. So, $\partial\partial A \subseteq \partial A$.

3. For all $x \in \partial(A \cup B)$, $x \in \overline{A \cup B} = \overline{A} \cup \overline{B}$, and $x \in \overline{X \setminus (A \cup B)} = \overline{(X \setminus A) \cap (X \setminus B)} \subseteq (\overline{X \setminus A} \cap \overline{X \setminus B})$. Then, there are two cases to consider:

   First, if $x \in \overline{A}$, since $x \in (\overline{X \setminus A} \cap \overline{X \setminus B}) \subseteq \overline{X \setminus A}$, then $x \in \partial A = \overline{A} \cap \overline{X \setminus A}$.

   Else, if $x \in \overline{B}$, since $x \in (\overline{X \setminus A} \cap \overline{X \setminus B}) \subseteq \overline{X \setminus B}$, then $x \in \partial B = \overline{B} \cap \overline{X \setminus B}$.

   In either case, $x \in \partial A \cup \partial B$, thus we can conclude that $\partial(A \cup B) \subseteq \partial A \cup \partial B$.

4. Suppose $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$. Thus, $\partial A = \overline{A} \cap \overline{X \setminus A} \subseteq \overline{A} \subseteq \overline{B}$. Now, for all $x \in \partial A \subseteq \overline{B}$, there are two cases:

   If $x \in B$, then $x \in (B \cup \partial B)$.

   Else, if $x \notin B$, then $x \in X \setminus B \subseteq \overline{X \setminus B}$. With the statement that $x \in \overline{B}$, $x \in \overline{B} \cap \overline{X \setminus B} = \partial B$. Hence again, $x \in (B \cup \partial B)$.

   So, regardless of the case, $x \in \partial A$ implies $x \in (B \cup \partial B)$, thus $\partial A \subseteq (B \cup \partial B)$.

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Question 3 *Let A be a set in a topological space. Prove that the closure of the interior of the closure of the interior of A equals the closure of the interior of A.*

Pf:

Let $B = \overline{A^\circ}$ (the closure of the interior of A), which $\overline{B^\circ}$ is the closure of the interior of closure of the interior of A.

Notice that since $B^\circ \subseteq B$, then $\overline{B^\circ} \subseteq \overline{B}$; and since $B = \overline{A^\circ}$, which is already closed, then $\overline{B} = B$. Thus, $\overline{B^\circ} \subseteq \overline{B} = B$.

Also, since $A^\circ \subseteq \overline{A^\circ} = B$ while A° is open, then $A^\circ \subseteq B^\circ$; hence, $B = \overline{A^\circ} \subseteq \overline{B^\circ}$.

Combining both criteria, $B = \overline{B^\circ}$, so the Closure of the Interior of A, equals to the Closure of the Interior of the Closure of the Interior of A.

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Question 4 *Give an example of a topological space that is not Hausdorff, but still has the property that every sequence converges to at most one point. Prove your answer is correct. I suggest using the "countable complement" (also called "cocountable") topology.*

Pf:

Given not countable set X, and consider the Countable Complement Topology on X (which, $U \subseteq X$ is open iff either $X \setminus U$ is at most countable, or $U = \emptyset$).

The Cocountable Topology on X is not Hausdorff:

We'll prove by contradiction. Suppose the given topology is Hausdorff, then for all $x, y \in X$ with $x \neq y$, there exists disjoint open neighborhood $U, V \subseteq X$, with $x \in U$ and $y \in V$ (which $y \notin U$ and $x \notin V$).

First, since U is open, then $X \setminus U$ is at most countable, according to the definition of cocountable topology.

Then, since U, V are disjoint, then every point $z \in V$ satisfies $z \notin U$, or $z \in X \setminus U$. Hence, $V \subseteq X \setminus U$, which implies V is also countable (subset of at most countable set is at most countable).

However, this implies that $X \setminus V$ is not countable: If $X \setminus V$ is countable, then $V \cup (X \setminus V) = X$ is countable, which contradicts the assumption that X is uncountable.

Yet, if $X \setminus V$ is not countable, V is no longer open, which again contradicts our assumption that V is open.

Thus, the initial assumption is false, the Cocountable Topology on X is not Hausdorff.

Type of convergent sequences in Cocountable Topology:

We'll prove that the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges iff it is eventually constant (which, after some index k, $n \geq k$ implies $x_n = x$ for some $x \in X$).

To prove the forward implication, we'll use contradiction.

Suppose there exists a sequence $(x_n)_{n\in\mathbb{N}}$ that's not eventually constant, but still converges to $x \in X$.

Since it's not eventually constant, for all $N > 0$, there exists index $n \geq N$, with $x_n \neq x$.

Which, consider an arbitrary open neighborhood U of x, and consider the set $V = \{x\} \cup (U \setminus (x_n)_{n\in\mathbb{N}})$:

Taking the complement:

$$X \setminus V = X \setminus (\{x\} \cup (U \setminus (x_n)_{n\in\mathbb{N}})) = (X \setminus \{x\}) \cap (X \setminus (U \setminus (x_n)_{n\in\mathbb{N}}))$$

$$= (X \setminus \{x\}) \cap ((x_n)_{n\in\mathbb{N}} \cup (X \setminus U))$$

Notice that the set $(x_n)_{n\in\mathbb{N}}$ is at most countable, and since U is open, $X \setminus U$ is also at most countable. Thus, the set $(x_n)_{n\in\mathbb{N}} \cup (X \setminus U)$ is at most countable, which $X \setminus V$ as a subset of $(x_n)_{n\in\mathbb{N}} \cup (X \setminus U)$, must also be at most countable.

Hence, V is actually an open set, which since $x \in V$, it is an open neighborhood of x.

However, for all $N > 0$, there exists $n \geq N$, with $x_n \neq x$, which $x_n \notin (U \setminus (x_n)_{n\in\mathbb{N}})$, and $x_n \notin \{x\}$, hence $x_n \notin V = \{x\} \cup (U \setminus (x_n)_{n\in\mathbb{N}})$. This contradictts with the fact that $(x_n)_{n\in\mathbb{N}}$ converges to x, since there should exist N, with $n \geq N$ implies $x_n \in V$.

So, the assumption must be false, $(x_n)_{n\in\mathbb{N}} \subset X$ converges implies it is eventually constant.

To prove the converse, suppose the sequence $(x_n)_{n\in\mathbb{N}}$ is eventually constant, there exists $x \in X$ and index N, with $n \geq N$ implies $x_n = x$.

Which, for all open neighborhood U of x, choose the index N, for all $n \geq N$ satisfies $x_n = x \in U$, thus by the definition of convergence, $\lim_{n\to\infty} x_n = x$.

Limit of Converging Sequence has at most one limit:

In previous section, we've proven that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges iff it is eventually constant. Then, there exists an index N, such that $n \geq N$ implies $x_n = x$. This implies that the limit is actually unique:

For all $x' \in X$ with $x' \neq x$, consider any open neighborhood U of x': Since $X \setminus U$ is at most countable, then $X \setminus (U \setminus \{x\}) = \{x\} \cup (X \setminus U)$ is also at most countable. Thus, the set $U \setminus \{x\}$ is open under cocountable topology, and $x' \in U \setminus \{x\}$ (since $x' \in U$ and $x' \neq x$). Hence, $U \setminus \{x\}$ is an open neighborhood of x'.

However, if consider all index $n \geq N$, since $x_n = x$, then $x_n \notin U \setminus \{x\}$, this indicates that $(x_n)_{n\in\mathbb{N}}$ is not converging to x'.

Since for all $x' \in X$ with $x' \neq x$, it is not a limit of $(x_n)_{n\in\mathbb{N}}$, then the only limit is x, indicating that there is at most one limit for $(x_n)_{n\in\mathbb{N}}$.

So, under Cocountable Topology for an uncountable set X, even though the space is not Hausdorff, but the limit is still unique.