Math 111B HW2

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Question 1 *Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b).*
Prove : if $\forall x \in (a, b),\ f'(x) \neq 0$, then f is one-to-one on (a, b).
Give an example showing that the converse statement is in general not true.

Pf:
Suppose $\forall x \in (a, b),\ f'(x) \neq 0$:

(1) $f'(x)$ is strictly less than or greater than 0 on (a, b):
We'll prove by contradiction: Suppose $f'(x)$ is neither strictly less than 0 nor strictly greater than 0 on (a, b), then there exists $x_0, x_1 \in (a, b)$, with $f'(x_0) \leq 0$ and $f'(x_1) \geq 0$, and by the assumption that $f'(x) \neq 0$, the strict inequality $f'(x_0) < 0$ and $f'(x_1) > 0$ is applied. (This also implies $x_0 \neq x_1$, since derivatives are different at the two points).

Recall that for function $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b), if $a < c < d < b$ and $f'(c) \neq f'(d)$, for any λ strictly in between $f'(c)$ and $f'(d)$ (either $f'(c) < \lambda < f'(d)$ or $f'(c) > \lambda > f'(d)$), there exists $x \in (c, d)$ with $f'(x) = \lambda$.
Then, first suppose $x_0 < x_1$: f is differentiable on (a, b) and $f'(x_0) < 0 < f'(x_1)$ implies there exists $x \in (x_0, x_1)$ with $f'(x) = 0$, which contradicts the assumption;
then suppose $x_1 < x_0$: again, f is differentiable on (a, b) and $f'(x_1) > 0 > f'(x_0)$ implies there exists $x \in (x_1, x_0)$ with $f'(x) = 0$, which again contradicts the assumption.
So, the assumption is false, $f'(x)$ must be strictly greater than 0 or less than 0 for all $x \in (a, b)$.

(2) f is strictly increasing or decreasing on (a, b):
Based on **(1)**, $f'(x)$ is strictly less than 0 or strictly greater than 0.
Suppose $f'(x) > 0$ for all $x \in (a, b)$, then for any $x, y \in (a, b)$ with $x < y$, by the Mean Value Theorem, there exists $c \in (x, y) \subseteq (a, b)$, such that the following is true:

$$\frac{f(y) - f(x)}{y - x} = f'(c) > 0, \quad (f(y) - f(x)) = f'(c)(y - x)$$

Since $(y - x), f'(c) > 0$ by assumption, the $(f(y) - f(x)) = f'(c)(y - x) > 0$, thus $f(y) > f(x)$, showing that f is strictly increasing.
Similarly, suppose $f'(x) < 0$ for all $x \in (a, b)$, with the same x, y above, by Mean Value Theorem, there exists $c \in (x, y) \subseteq (a, b)$, such that the following is true:

$$\frac{f(y) - f(x)}{y - x} = f'(c) < 0, \quad (f(y) - f(x)) = f'(c)(y - x)$$

Since $(y - x) > 0$ and $f'(c) < 0$, then $(f(y) - f(x)) = f'(c)(y - x) < 0$, of $f(y) < f(x)$, showing that f is strictly decreasing.

With the above condition, since f is either strictly increasing or strictly decreasing on $[a, b]$, then for all $x, y \in (a, b)$, $x \neq y$ implies $f(x) \neq f(y)$ (or else it's no longer strictly increasing or decreasing). Thus, f is in fact one-to-one on (a, b).

Counterexample of Converse:

Let $f : [-1, 1] \to \mathbb{R}$ be $f(x) = x^3$, which $f'(x) = 3x^2$, which $f'(0) = 0$. Yet, suppose $x, y \in (-1, 1)$ has $x^3 = y^3$, then:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2) = 0$$

Which, the only real solution is $x = y$ (since if treating y as constant, $y^2 - 4y^2 = -3y^2 \leq 0$; the only time with real solution is when $y = 0$, which implies $x^3 = 0$, or $x = 0$).

So, $f(x) = x^3$ is one-to-one on the region $(-1, 1)$, but still has $f'(0) = 0$, which is a counterexample.

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Question 2 *Let $f : (a, b) \to R$ be a function such that:*

$$\exists M > 0, \exists \alpha > 0,\ \forall x, y \in (a, b),\ |f(x) - f(y)| < M|x - y|^\alpha$$

If $\alpha \in (0, 1)$, then f is Holder of order α in (a, b). If α = 1, then f is Lipschitz. Prove :
(a) If $\alpha > 1$, then f is constant.
(b) If $\alpha \in (0, 1]$, then f is uniformly continuous on (a, b).
(c) Give an example such that f is Lipschitz, but not differentiable.
(d) If f is differentiable on (a, b) and $f(x)$ is bounded on (a, b), then f is Lipschitz.

Pf:

(a) Suppose $\alpha > 1$, then there exists $\epsilon > 0$, such that $\alpha = 1 + \epsilon$. Which, for all $x, y \in (a, b)$ (with $x \neq y$), the following is true:

$$|f(x) - f(y)| < M|x - y|^\alpha = M|x - y|^{1+\epsilon} = M|x - y| \cdot |x - y|^\epsilon$$

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \frac{|f(x) - f(y)|}{|x - y|} < M|x - y|^\epsilon$$

Which, fix arbitrary $x_0 \in (a, b)$, for all $y \in (a, b)$ with $y \neq x_0$, the following is true:

$$0 \leq \left|\frac{f(x_0) - f(y)}{x_0 - y}\right| < M|x_0 - y|^\epsilon, \quad -M|x_0 - y|^\epsilon < \frac{f(x_0) - f(y)}{x_0 - y} < M|x_0 - y|^\epsilon$$

Since $\epsilon > 0$, then $\lim_{y \to x_0} |x_0 - y|^\epsilon = 0$. Which, by Squeeze Theorem, the following is true:

$$0 = \lim_{y \to x_0} -M|x_0 - y|^\epsilon \leq \lim_{y \to x_0} \frac{f(x_0) - f(y)}{x_0 - y} \leq \lim_{y \to x_0} M|x_0 - y|^\epsilon = 0$$

Thus, $\lim_{y \to x_0} \frac{f(x_0) - f(y)}{x_0 - y} = 0$, or $f'(x_0) = 0$.

This implies that $f(x)$ is a constant function: Suppose $f(x)$ is not a constant function, then there exists $c, d \in (a, b)$ with $c < d$, such that $f(c) \neq f(d)$.

Notice that since $f'(x_0)$ exists for all $x_0 \in (a, b)$, then by Mean Value Theorem, there exists $x \in (c, d)$, such that $f'(x)(d - c) = f(d) - f(c)$.

Yet, since $f'(x) = 0$, while $f(d) - f(c) \neq 0$, $0 = f'(x)(d - c) \neq f(d) - f(c)$, which it is a contradiction.

Thus, $f(x)$ must be a constant function.

(b) Suppose $\alpha \in (0,1]$, notice that for all $x, y \in (a,b)$, the following is true:

$$a < x < b, \quad -b < -y < -a, \quad (a-b) = -(b-a) < (x-y) < (b-a), \quad |x-y| < |b-a|$$

Which, since $\alpha > 0$, then $|x-y|^\alpha < |b-a|^\alpha$. Now, for any $\epsilon > 0$, define $\delta = (\frac{\epsilon}{M})^{\frac{1}{\alpha}} > 0$, then for all $x, y \in (a,b)$, if $|x-y| < \delta$, the following is true:

$$|f(x) - f(y)| < M|x-y|^\alpha < M \cdot \delta^\alpha$$

(Note: the above inequality is true, since $\alpha > 0$, then $0 \leq |x-y| < |b-a|$ implies $|x-y|^\alpha < |b-a|^\alpha$). Thus, it can be rewritten as:

$$|f(x) - f(y)| < M \cdot \delta^\alpha = M \cdot \left(\left(\frac{\epsilon}{M}\right)^{\frac{1}{\alpha}}\right)^\alpha = M \cdot \frac{\epsilon}{M} = \epsilon$$

Thus, since for all $\epsilon > 0$, there exists $\delta > 0$ with $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, f is uniformly continuous.

(c) Consider the function $f : (-1,1) \to \mathbb{R}$ by $f(x) = |x|$.

Choose $M = 1.01$ and $\alpha = 1$, then the following is true:

$$\forall x, y \in (-1,1), \quad |f(x) - f(y)| = ||x| - |y|| \leq |x-y| = |x-y|^\alpha < 1.01|x-y|^\alpha = M|x-y|^\alpha$$

Thus, f is Lipschitz continuous.

Yet, f is not differentiable at $x = 0$: For all $x < 0$ and $y > 0$ (with $x, y \in (-1,1)$), the following is true:

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x| - 0}{x - 0} = \frac{-x}{x} = -1$$

$$\frac{f(y) - f(0)}{y - 0} = \frac{|y| - 0}{y - 0} = \frac{y}{y} = 1$$

Which, $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ is not defined, since the left and right limits as x approaches 0 are different.

(d) Suppose f is differentiable on (a,b) and $f'(x)$ is bounded on (a,b), then there exists $M > 0$, with $|f'(x)| < M$ for all $x \in (a,b)$. Which, for all $x, y \in (a,b)$ with $x < y$, by the Mean Value Theorem, there exists $c \in (x,y)$, such that $f(y) - f(x) = f'(c)(y-x)$. Which, the following is true:

$$|f(y) - f(x)| = |f'(c)| \cdot |y - x| < M|y - x|$$

Thus, f is Lipschitz continuous.

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Question 3 *For any $a \geq 0$, define $f_a : \mathbb{R} \to \mathbb{R}$ as:*

$$f_a(x) = \begin{cases} x^a sin(\frac{1}{x}) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(a) For which values of a is f_a continuous at 0.
(b) For which values of a is $f_a'(0)$ defined.
(c) For which values of a is f_a' continuous at 0.
(d) For which values of a is $f_a''(0)$ defined.

Pf:

(a) **Ans:** $a > 0$. For $a = 0$, the function $f_a(x)$ is not continuous: Choose the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_n = \frac{1}{(2n+1/2)\pi} > 0$, then $\lim_{n\to\infty} \frac{1}{(2n+1/2)\pi} = 0$, thus x_n converges to 0; but, consider $(f_a(x_n))_{n\in\mathbb{N}}$:

$$\forall n \in \mathbb{N}, \quad f_a(x_n) = x_n^0 \sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{1}{1/(2n+1/2)\pi}\right) = \sin((2n+1/2)\pi) = 1$$

Which, $\lim_{n\to\infty} f_a(x_n) = 1 \neq 0 = f_a(0)$, thus $f_a(x_n)$ doesn't converge to $f_a(0)$, showing it's not continuous.

Now, for all $a > 0$, for any $x > 0$, since $x^a > 0$, it satisfies the following:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1, \quad -x^a \leq f_a(x) = x^a \sin\left(\frac{1}{x}\right) \leq x^a$$

Which, take the right limit of x^a of 0, $\lim_{x\to 0^+} x^a = 0$, then by Squeeze Theorem, the following is true:

$$0 = \lim_{x\to 0^+} -x^a \leq \lim_{x\to 0^+} x^a \sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0^+} x^a = 0$$

Thus, $\lim_{x\to 0^+} f_a(x) = 0$.

Also, since $\lim_{x\to 0^-} f_a(x) = 0$ (since for $x < 0$, $f_a(x) = 0$), then the left and right limits both agree with $f_a(0) = 0$, showing it's continuous at 0. Every $a > 0$ has $f_a(x)$ being continuous at 0.

(b) **Ans:** $a > 1$**.** In case for $f_a'(0)$ to be defined, f_a must be continuous at 0. Thus, $a > 0$ is required.

Consider the slope $\frac{f_a(x)-f_a(0)}{x-0}$ for all $x \neq 0$. If $x < 0$, then since $f_a(x) = 0$, then the slope is 0. Thus, the left limit of the slope $\lim_{x\to 0^-} \frac{f_a(x)-f_a(0)}{x-0} = 0$.

Now, consider the slope from the right:

$$x > 0, \quad \frac{f_a(x) - f_a(0)}{x - 0} = \frac{x^a \sin(1/x) - 0}{x - 0} = x^{a-1} \sin\left(\frac{1}{x}\right)$$

Since the left limit is evaluated as 0, in case for $f'(0)$ to be defined, the right limit also needs to converge to 0.

First, notice that if $a \leq 1$, the right limit doesn't exist:

Consider the same sequence $x_n = \frac{1}{(2n+1/2)\pi} > 0$ used in part (a), then the following is true:

$$\forall n \in \mathbb{N}, \quad (x_n)^{a-1} \sin\left(\frac{1}{x_n}\right) = \left(\frac{1}{(2n+1/2)\pi}\right)^{a-1} \sin((2n+1/2)\pi) = \left(\frac{1}{(2n+1/2)\pi}\right)^{a-1}$$

Which, if $a = 1$ (or $a-1 = 0$), then $(x_n)^{a-1}\sin(1/x_n) = 1$ for all $n \in \mathbb{N}$, which $\lim_{n\to\infty} \frac{f_a(x_n)-f_a(0)}{x_n - 0} = 1$, while $\lim_{n\to\infty} x_n = 0$. This shows that the right limit of the slope is not 0, which $f_a'(0)$ is not defined.

Else, if $a < 1$ (or $a - 1 < 0$), then $(x_n)^{a-1}\sin(1/x_n) = \left(\frac{1}{(2n+1/2)\pi}\right)^{a-1} = ((2n + 1/2)\pi)^{1-a}$ is in fact unbounded as n increases indefinitely (since $1 - a > 0$), so again the right limit of the slope is not defined, implying $f_a'(0)$ is not defined.

So, in case for the right limit to be defined, $a > 1$. Which, since $a-1 > 0$, then for all $x > 0$, $x^{a-1} > 0$, and $\lim_{x\to 0^+} a^{a-1} = 0$. Thus based on Squeeze Theorem:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1, \quad x > 0, \quad -x^{a-1} \leq x^{a-1}\sin\left(\frac{1}{x}\right) \leq x^{a-1}$$

$$0 = \lim_{x\to 0^+} -x^{a-1} \leq \lim_{x\to 0^+} x^{a-1}\sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0^+} x^{a-1} = 0$$

So, the right limit of $x^{a-1}\sin(1/x)$ is 0 when x approaches 0, which it agrees with the initial left limit, hence for $a > 1$, $\lim_{x\to 0} \frac{f_a(x)-f_a(0)}{x-0} = 0$, $f_a'(0) = 0$ is defined.

(c) For f_a' to be continuous at 0, $f_a'(0)$ needs to be defined. So, $a > 1$ is required.

Consider $f_a'(x)$ for $x \neq 0$, which by the differentiation rule, it is evaluated as:

$$f_a'(x) = ax^{a-1}\sin\left(\frac{1}{x}\right) + x^a\cos\left(\frac{1}{x}\right)\frac{-1}{x^2} = ax^{a-1}\sin\left(\frac{1}{x}\right) - x^{a-2}\cos\left(\frac{1}{x}\right)$$

Since

(d)

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Question 4

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Question 5