# Math CS Topology HW4

Zih-Yu Hsieh

February 15, 2025

## 1

**Question 1** *Let $f : X \to Y$ be a continuous map between topological spaces, and suppose $Y$ is Hausdorff. Prove that the graph $\{(x, f(x)) \mid x \in X\}$ is a closed subset of $X \times Y$.*

**Pf:**

Let $G = \{(x, f(x)) \mid x \in X\}$ be the graph. To prove that $G$ is closed, it is equivalent to show that $X \setminus G$ is open.

For all $(x, y) \in X \setminus G$, since the element is not in $G$, then $y \neq f(x)$. Then, by the Hausdorff Property of $Y$, there exists disjoint open subsets $U, V \subseteq Y$, such that $f(x) \in U$ and $y \in V$.

Notice that because $f$ is continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are open; furthermore, since $U \cap V = \emptyset$, then $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

Now, consider the basis element $f^{-1}(U) \times V$: First, it is an open neighborhood of $(x, y)$, since $y \in V$ and $f(x) \in U$ (which implies $x \in f^{-1}(U)$); futhermore, for all $a \in f^{-1}(U)$, since $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, then $f(a) \notin V$. Hence, for all $(a, b) \in f^{-1}(U) \times V$, since $f(a) \notin V$, $(a, f(a)) \notin f^{-1}(U) \times V$, hence $(a, b) \neq (a, f(a))$ (which $b \neq f(a)$), showing that $(a, b) \notin G$.

Therefore, $(a, b) \in X \setminus G$, implying that $f^{-1}(U) \times V \subseteq G$.

So, for all $(x, y) \in X \setminus G$, there exists a basis element $B$ (Note: $B = f^{-1}(U) \times V$ in the above construction), such that $(x, y) \in B \subseteq X \setminus G$, which $X \setminus G$ is open, showing that its complement $G$ is closed.

Therefore, the graph $G = \{(x, f(x)) \mid x \in X\}$ is closed.

# 2

**Question 2** *Prove that if $X$ and $Y$ are nonempty topological spaces then $X$ is homeomorphic to a subspace of $X \times Y$.*

**Pf:**

Since $Y$ is not empty, there exists $y_0 \in Y$. Consider the following map $f : X \to X \times Y$, such that for all $x \in X$, $f(x) = (x, y_0)$, which $f(X) = X \times \{y_0\}$ (since for all $x \in X$, $f(x) = (x, y_0) \in X \times \{y_0\}$, and for all $(x, y_0) \in X \times \{y_0\}$, $f(x) = (x, y_0)$). So, we'll restrict the codomain to the set $X \times \{y_0\}$, letting $f : X \to X \times \{y_0\}$.

**$f$ is Bijective:**

First, we've verified that $f(X) = X \times \{y_0\}$, hence restricting the codomain to the image had made the map surjective.

To verify injectivity, consider $x_1, x_2 \in X$: If $f(x_1) = f(x_2)$, then $(x_1, y_0) = (x_2, y_0)$, so $x_1 = x_2$, proving that it's injective.

So, the map $f$ is bijective, and $f^{-1} : X \times \{y_0\} \to X$ satisfies $f(x, y_0) = x$.

**$f$ is Continuous:**

For all ope subset $U' \subseteq X \times \{y_0\}$, there exists open subset $U \subseteq X \times Y$, with $U \cap (X \times \{y_0\}) = U'$. Now, consider the preimage $f^{-1}(U')$: For all $x \in f^{-1}(U')$, since $f(x) = (x, y_0) \in U' \subseteq U$, there exists a basis element $A \times B$ (where $A \subseteq X$ and $B \subseteq Y$ are both open), such that $(x, y_0) \in A \times B \subseteq U$. Which:

$$A \times \{y_0\} = (A \cap X) \times (B \cap \{y_0\}) = (A \times B) \cap (X \times \{y_0\}) \subseteq U \cap (X \times \{y_0\}) = U'$$

So, $A \times \{y_0\}$ is an open subset of $X \times \{y_0\}$ under subspace topology, and $(A \times \{y_0\}) \subseteq U'$.

Now, consider all $a \in A \subseteq X$: Since $f(a) = (a, y_0) \in (A \times \{y_0\}) \subseteq U'$, then $a \in f^{-1}(U')$. Hence, $A \subseteq f^{-1}(U')$. Also, recall that $x \in A$, hence $x \in A \subseteq f^{-1}(U')$.

So, for every $x \in f^{-1}(U')$, there is an open subset $A \subseteq X$, with $x \in A \subseteq f^{-1}(U')$, showing that $f^{-1}(U') \subseteq X$ is open.

Therefore, we can conclude that $f$ is continuous, since every open subset of $X \times \{y_0\}$ the image, the preimage in $X$ is open.

**$f^{-1}$ is Continuous:**

For all open subset $U \subseteq X$, notice that for all $(x, y_0) \in X \times \{y_0\}$, $f^{-1}(x, y_0) = x \in U$ if and only if $x \in U$, hence the preimage $(f^{-1})^{-1}(U) = U \times \{y_0\}$. Which, consider $U \times Y$ an open subset of $X \times Y$, the following is true:

$$(U \times Y) \cap (X \times \{y_0\}) = (U \cap X) \times (Y \cap \{y_0\}) = U \times \{y_0\}$$

Hence, $U \times \{y_0\}$ is an intersection of $X \times \{y_0\}$ and $(U \times Y)$, proving that $U \times \{y_0\}$ is an open subset of $X \times \{y_0\}$ under subspace topology, so the preimage of $U$ under $f^{-1}$, $(f^{-1})^{-1}(U) = U \times \{y_0\}$ is open, showing that $f^{-1}$ is continuous, since all open subset of $X$ has a preimage being open.

Because $f^{-1}$ exists when restricting the codomain to $X \times \{y_0\}$, and both $f$ and $f^{-1}$ are continuous using the given topology, hence $f$ is a homeomorphism, showing that $X$ and $X \times \{y_0\}$ (as a subspace of $X \times Y$) are homeomorphic.

# 3

**Question 3** *If $X$ is a metric space, prove that the distance function $d : X \times X \to \mathbb{R}$ is continuous, where $X \times X$ has the product of the metric topologies.*

**Pf:**

For all open subset $U \subseteq \mathbb{R}$, consider the preimage $d^{-1}(U) \subseteq X \times X$:

For all $(x_1, x_2) \in d^{-1}(U)$, since $y = d(x_1, x_2) \in U$ while $U$ is open under standard topology of $\mathbb{R}$, then there exists $r > 0$, such that $(y - r, y + r) \subseteq U$.

Now, consider the basis element $\big(B_d(x_1, \frac{r}{2}) \times B_d(x_2, \frac{r}{2})\big) \subseteq X \times X$ under product topology: For all $(a, b) \in \big(B_d(x_1, \frac{r}{2}) \times B_d(x_2, \frac{r}{2})\big)$, the following is true:

$$d(a,b) \leq d(a,x_1) + d(x_1,b) \leq d(a,x_1) + d(x_1,x_2) + d(x_2,b) < \frac{r}{2} + y + \frac{r}{2} = y + r$$

$$y = d(x_1,x_2) \leq d(x_1,a) + d(a,x_2) \leq d(x_1,a) + d(a,b) + d(b,x_2) < \frac{r}{2} + d(a,b) + \frac{r}{2} = d(a,b) + r$$

$$y - r < d(a,b)$$

(Note: the above is true, since $a \in B_d(x_1, \frac{r}{2})$ and $b \in B_d(x_2, \frac{r}{2})$).

Hence, since $y - r < d(a,b) < y + r$, then $d(a,b) \in (y - r, y + r) \subseteq U$, showing that $(a,b) \in d^{-1}(U)$. And, since the choice of $(a,b) \in \big(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\big)$ is arbitrary, $\big(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\big) \subseteq f^{-1}(U)$.

So, for all $(x_1, x_2) \in d^{-1}(U) \subseteq X \times X$, there exists a basis element $\big(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\big)$, such that $(x_1, x_2) \in \big(B_d(x_1, \frac{r}{3}) \times B_d(x_2, \frac{r}{3})\big) \subseteq d^{-1}(U)$, hence $d^{-1}(U)$ is open.

Which, we can conclude that $d : X \times X \to \mathbb{R}$ is continuous under product topology of $X \times X$ (based on metric topology of $X$), and standard topology of $\mathbb{R}$.

# 4

**Question 4** *Let $x$ be a point in a metric space $X$. Prove that $\{y \in X \mid d(x,y) \leq 1\}$ is closed, but is not necessarily equal to the closure of the unit open ball $B(x,1)$. (This is contrary to Exercise 5.14b in my copy of the textbook.)*

**Pf:**

To prove that $C = \{y \in X \mid d(x,y) \leq 1\}$ is closed, it suffices to prove that $X \setminus C = \{y \in X \mid d(x,y) > 1\}$ is open.

For all $y \in X \setminus C$, $d(x,y) > 1$. Which, consider $r = d(x,y) - 1 > 0$, and the open ball $B(y,r)$: For all $z \in B(y,r)$, $d(y,z) < r = d(x,y) - 1$. Which, consider $d(x,z)$, the following is true:

$$d(x,y) \leq d(x,z) + d(y,z) < d(x,z) + d(x,y) - 1$$

$$0 < d(x,z) - 1, \quad 1 < d(x,z)$$

Hence, we can conclude that $z \in X \setminus C$, which $y \in B(y,r) \subseteq (X \setminus C)$.

Since for all points in $X \setminus C$, there exists a basis element containing the point, that is a subset of $X \setminus C$, then $X \setminus C$ is open, hence $C$ is closed.

**Closure of Open Ball and Closed Ball could be Different:**

For any nonempty set $X$ with more than one element, consider the discrete metric $d : X \times X \to \mathbb{R}$ defined as follow:

$$d(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

For all $x \in X$, the ball $B(x,1) = \{x\}$, since for all $y \in X$ with $y \neq x$, $d(x,y) = 1$, so $y \notin B(x,1)$ (since the distance is strictly smaller than 1).

Which, if we take the closed ball of distance 1 around $x$, $CB(x,1) = \{y \in X \mid d(x,y) \leq 1\} = X$ (since everything has distance at most 1 from $x$).

Yet, the closure of open ball with radius 1, is $\overline{B(x,1)} = \{x\}$, since under discrete metric, $\{x\}$ is also a closed set containing itself, hence the closure (which is the intersection of closed set containing $\{x\}$) must be $\{x\}$, because it is the smallest closed set containing itself.

Hence, $CB(x,1) \neq \overline{B(x,1)}$, showing that under extreme cases (like discrete metric), the two may not be the same.