# Math 118B HW4

Zih-Yu Hsieh

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## 1

**Question 1** *Let $g : [0,1] \to \mathbb{R}$ be a continuous function with $g(1) = 0$. Show that the sequence $f_n : [0,1] \to \mathbb{R}$ defined as $f_n(x) = x^n g(x)$ converges uniformly to zero.*

**Pf:**

Intuitively, we'll break this into two parts: A region containing 1 (where $x^n$ is not converging to 0), and the other region by some choice of $\delta > 0$.

Before starting, fix an arbitrary $\epsilon > 0$ for all purposes.

**Behaviors around $x = 1$:**

Given that $g$ is continuous at 1, then for the chosen $\epsilon > 0$, there exists $\delta > 0$ (for simplicity, choose $\delta < 1$), such that for all $x \in [0,1]$, $|x - 1| < \delta$ implies $|g(x) - g(1)| = |g(x)| < \epsilon$.

Which, because $x \in [0,1]$, then for all $n \in \mathbb{N}$, $|x^n| \leq 1$, showing that $|f_n(x)| = |x^n g(x)| \leq |g(x)| < \epsilon$.

So, for all $n \in \mathbb{N}$ and all $x \in (1-\delta, 1] = [0,1] \cap B_\delta(1)$, $|f_n(x)| < \epsilon$.

**Behaviors for the rest of the regions:**

Since for all $x \in (1-\delta, 1]$ is well-behaved, the rest to consider is all $x \in [0, 1-\delta]$.

First, since $g$ is continuous on $[0,1]$ a compact set, then $g([0,1]) \subseteq \mathbb{R}$ is also compact, showing that $g$ is bounded. Hence, there exists $M > 0$, such that all $x \in [0,1]$ satisfies $|g(x)| \leq M$.

Then, from the previous construction, $0 < \delta < 1$, hence $0 < (1-\delta) < 1$, showing that $\lim_{n\to\infty}(1-\delta)^n = 0$. Therefore, since $\frac{\epsilon}{M} > 0$, there exists $N$, such that $n \geq N$ implies $|(1-\delta)^n| = (1-\delta)^n < \frac{\epsilon}{M}$.

Now, notice that for all $x \in [0, 1-\delta]$, since $0 \leq x \leq (1-\delta)$, then for all $n \in \mathbb{N}$, $0 \leq x^n \leq (1-\delta)^n$.

Hence, for all $n \geq N$ and all $x \in [0, 1-\delta]$, we can conclude the following:

$$|f_n(x)| = |x^n g(x)| = |x^n| \cdot |g(x)| \leq x^n \cdot M \leq (1-\delta)^n \cdot M < \frac{\epsilon}{M} \cdot M = \epsilon$$

Now, for the $N$ constructed in the second part, for any $n \geq N$, for all $x \in [0,1]$, there are two cases:

First, if $x \in (1-\delta, 1]$, then from the first part, $|f_n(x)| < \epsilon$.

Else, if $x \in [0, 1-\delta]$, then from the second part, since $n \geq N$, we have $|f_n(x)| < \epsilon$ again.

Hence, $\epsilon$ is an upper bound of the set $\{|f_n(x)|\ x \in [0,1]\}$, showing that $\sup_{x\in[0,1]} |f_n(x)| = \|f_n\|_\infty \leq \epsilon$.

So, for all $\epsilon > 0$, there exists $N$, with $n \geq N$ implies $\|f_n\|_\infty \leq \epsilon$, showing that $f_n$ converges to 0 uniformly.

## 2

**Question 2** *Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined as $f_n(x) = 1/(1+n^2x^2)$, $n \in \mathbb{Z}^+$.*

*(a) For what values of $x$ does the series $\sum f_n$ converge pointwise?*

*(b) For what values of $x$ does the series $\sum f_n$ converge uniformly?*

**Pf:**

(a) For $x = 0$, since for all $n \in \mathbb{N}$, $f_n(0) = 1/(1+n^2 \cdot 0^2) = 1$, then the series $\sum_{n=1}^{\infty} f_n(0)$ diverges.

For $x \neq 0$, recall that $\sum_{n=1}^{\infty} \frac{1}{n^2x^2}$ converges (since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges). Now, since for all $n \in \mathbb{N}$, $0 < n^2x^2 < (1+n^2x^2)$, then $0 < \frac{1}{1+n^2x^2} < \frac{1}{n^2x^2}$, hence for all $N \in \mathbb{N}$, we can conclude the following:

$$0 < \sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} \frac{1}{1+n^2x^2} < \sum_{n=1}^{N} \frac{1}{n^2x^2}$$

Then, since the series $\sum_{n=1}^{\infty} \frac{1}{n^2x^2}$ converges, while every term $f_n(x) > 0$ (since $n, x > 0$), then the above partial sum is bounded by the partial sum of $\sum_{n=1}^{\infty} \frac{1}{n^2x^2}$, implies that $\sum_{n=1}^{\infty} f_n(x)$ converges.

Hence, all $x \in \mathbb{R} \setminus \{0\}$ has $\sum_{n=1}^{\infty} f_n(x)$ converges.

(b) Even though the series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise on $\mathbb{R}\setminus\{0\}$, we can prove that it doesn't converge uniformly: Notice that if $\sum f_n$ converges to some function $F$ uniformly, then the partial sum of the functions $f_n$ is a Cauchy Sequence based on the norm $\|\cdot\|_\infty$. This also implies that $\lim_{n\to\infty} \|f_n\|_\infty = 0$.

Yet, on the set $\mathbb{R}\setminus\{0\}$, for all $n > 1$, $\|f_n\|_\infty = 1$: It is clear that for all $x \in \mathbb{R}\setminus\{0\}$, $|f_n(x)| = |\frac{1}{1+n^2x^2}| = \frac{1}{1+n^2x^2} \leq 1$. However, choose $x = \frac{1}{n^{k+2}}$ for positive integer $k$, we get:

$$f_n(1/n^{k+2}) = \frac{1}{1+n^2 \cdot (1/n^{k+2})} = \frac{1}{1+1/n^k} = \frac{n^k}{n^k+1} = 1 - \frac{1}{n^k+1}$$

Hence, since $\lim_{k\to\infty} \frac{1}{n^k+1} = 0$ (since $n > 1$), then for all $\epsilon > 0$, there exists $K$, with $k \geq K$ implies $\frac{1}{n^k+1} < \epsilon$. So, $1 - \epsilon < 1 - \frac{1}{n^k+1}$, showing that $1-\epsilon$ is no longer a supremum of the set $\{|f_n(x)| \mid x \in \mathbb{R}\setminus\{0\}\}$. Hence, on the set $\mathbb{R}\setminus\{0\}$, $\|f_n\|_\infty = 1$ (for $n > 1$). Because $\lim_{n\to\infty} \|f_n\|_\infty \neq 0$, then the series of function $\sum f_n$ doesn't converge uniformly.

However, for all $r > 0$, the series $\sum_{n=1}^{\infty} f_n(x)$ would converge uniformly on the region $(-\infty, -r] \cup [r, \infty)$: Recall that the Weierstrass's Theorem (or Weierstrass M-Test) states that given a sequence of functions $f_n : U \to \mathbb{R}$ (where $U \subseteq \mathbb{R}$), let $M_n = \sup_{x\in U} |f(x)|$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} M_n$ converges implies $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $U$.

For all $x \in (-\infty, -r] \cup [r, \infty)$, since $0 < r^2 \leq x^2$, then for all $n \in \mathbb{N}$, $0 < (1+r^2n^2) \leq (1+n^2x^2)$, showing that $0 < \frac{1}{1+n^2x^2} \leq \frac{1}{1+n^2r^2}$. Hence, we can conclude that $0 < f_n(x) \leq f_n(r)$, while $r \in (-\infty, -r] \cup [r, \infty)$, showing that $f_n(r) = \sup |f_n(x)| = \max |f_n(x)|$ on the given region $(-\infty, -r] \cup [r, \infty)$.

Then, since $\sum_{n=1}^{\infty} f_n(r)$ converges (since $r \neq 0$, which it satisfies the condition in **Part (a)**), then by Weierstrass's Theorem, since $\sum_{n=1}^{\infty} \sup |f_n(x)|$ converges for the region $(-\infty, -r] \cup [r, \infty)$, then the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

**Hence, for any $r > 0$, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $(-\infty, -r] \cup [r, \infty)$ and all of its subset.**

## 3

**Question 3** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define $f_n(x) = f(x + 1/n)$, $n \in \mathbb{Z}^+$.*

*(a) Does the sequence $\{f_n\}$ converge uniformly to $f$ in $\mathbb{R}$?*

*(b) Does the sequence $\{f_n\}$ converge uinformly to $f$ in any $K \subset \mathbb{R}$ compact?*

**Pf:**

(a) Regardless of the continuous function $f$, since for all $x \in \mathbb{R}$, the sequence $(x + 1/n)$ for $n \in \mathbb{Z}^+$ satisfies $\lim_{n\to\infty}(x + 1/n) = x$, then $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f(x + 1/n) = f(x)$ (since $f$ is continuous, which the limit of the function is the function of the limit). So, we can conclude that $f_n(x)$ converges pointwise onto $f$ for all $x \in \mathbb{R}$.

However, it's not true that $f_n$ would converge uniformly to $f$ in $\mathbb{R}$, and here is a counterexample: Take $f(x) = x^2$ a continuous function, which for all $n \in \mathbb{N}$, all $x \in \mathbb{R}$ satisfies:

$$|f_n(x) - f(x)| = \left|f\left(x + \frac{1}{n}\right) - f(x)\right| = \left|\left(x + \frac{1}{n}\right)^2 - x^2\right| = \left|\frac{1}{n}\left(2x + \frac{1}{n}\right)\right|$$

Hence, for $x > 0$, $|f_n(x) - f(x)| = \frac{2x}{n} + \frac{1}{n^2}$. Then, for all $M > 0$, choose $x = nM > 0$, we have $|f_n(x) - f(x)| = \frac{2nM}{n} + \frac{1}{n^2} > 2M > M$, showing that the collection $\{|f_n(x) - f(0)| \mid x \in \mathbb{R}\}$ is not bounded, which the supremum doesn't exists in $\mathbb{R}$. Therefore, the norm $\|f_n - f\|_\infty$ is not even defined, which is not valid to talk about uniform convergence of $f_n$.

Hence, even though $f_n$ converges to $f$ pointwise on $\mathbb{R}$, it's not guaranteed that $f_n$ converges to $f$ uniformly on $\mathbb{R}$.

(b) Given that $K \subset \mathbb{R}$ is compact, then there exists $m, M \in K$, which $m = \min(K)$ and $M = \max(K)$. Now, consider the set $[m, M + 1] \subset \mathbb{R}$: it is closed and bounded under standard topology, which is compact, hence the continuous function $f$ is uniformly continuous on $[m, M + 1]$.

Also, for all $x \in K$ and all $n \in \mathbb{N}$, since $m \leq x \leq M$, and $0 < \frac{1}{n} \leq 1$, then $m \leq x + \frac{1}{n} \leq M + 1$, hence $x, (x + 1/n) \in [m, M + 1]$ (which also $K \subseteq [m, M + 1]$).

Now, since $f$ is uniformly continuous on $[m, M+1]$, then for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $x, y \in [m, M + 1]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Then, for the given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$ based on Archimedean's Property, which for all $n \geq N$ (with $\frac{1}{n} \leq \frac{1}{N}$), the following is true:

$$\forall x \in K \subseteq [m, M + 1], \quad \left(x + \frac{1}{n}\right) \in [m, M + 1], \quad \left|\left(x + \frac{1}{n}\right) - x\right| = \frac{1}{n} \leq \frac{1}{N} < \delta$$

$$\left|\left(x + \frac{1}{n}\right) - x\right| < \delta \implies |f_n(x) - f(x)| = \left|f\left(x + \frac{1}{n}\right) - f(x)\right| < \epsilon$$

Hence, for all $\epsilon > 0$, there exists $N$, with $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in K$, showing that for all $n \geq N$, since $\epsilon$ is an upper bound for the set $\{|f_n(x) - f(x)| \mid x \in K\}$, then $\sup_{x\in K} |f_n(x) - f(x)| = \|f_n - f\|_\infty \leq \epsilon$ on $K$.

Hence, $f_n$ converges to $f$ uniformly on $K$, given that $K$ is compact.

# 4

**Question 4** *Let $f_n : [-1, 1] \to \mathbb{R}$ be defined as $f_n(x) = xe^{-nx^2}$, $n \in \mathbb{Z}^+$.*

*(a) Find the point-wise limit of the sequence $\{f_n\}$.*

*(b) Is the convergence uniform?*

*(c) Is $f$ differentiable? If so, find:*

$$f'(0), \quad \lim_{n\to\infty} f_n'(0)$$

**Pf:**

(a) First, if $x = 0$, then $f_n(0) = 0 \cdot e^{-n\cdot 0^2} = 0$, so $\lim_{n\to\infty} f_n(0) = 0$.

Else, if $x \neq 0$, since $x \in [-1, 1]$, then $|x| \leq 1$; hence, for all $n \in \mathbb{N}$, $|f_n(x)| = |xe^{-nx^2}| \leq e^{-nx^2} = (e^{x^2})^{-n}$ (while $x^2 > 0$, hence $e^{x^2} > 1$).

This implies $\lim_{n\to\infty}(e^{x^2})^{-n} = 0$. So, for all $\epsilon > 0$, there exists $N$, with $n \geq N$ implies $|(e^{x^2})^{-n}| = (e^{x^2})^{-n} < \epsilon$. Hence, for $n \geq N$, we have $|f_n(x)| \leq (e^{x^2})^{-n} < \epsilon$, showing that $\lim_{n\to\infty} f_n(x) = 0$.

We can conclude that for all $x \in [-1, 1]$, $\lim_{n\to\infty} f_n(x) = 0$, which $f_n(x)$ converges pointwise to $f(x) = 0$.

(b) Our claim is that the above convergence is in fact a uniform convergence.

We'll again break it into two parts: region containing 0, and the region not containing 0. Similarly, we'll choose an arbitrary $\epsilon > 0$ for all purposes (and for simplicity, let $\epsilon < 1$).

**Behavior about 0:**

Notice that since for all $x \in [-1, 1]$, since for all $n \in \mathbb{N}$, $-nx^2 \leq 0$, then $e^{-nx^2} \leq 1$. Hence, we have $|f_n(x)| = |xe^{-nx^2}| \leq |x|$.

Hence, for all $x \in [-1, 1]$ satisfying $|x| < \epsilon$ (or $x \in (-\epsilon, \epsilon)$), we have the following:

$$\forall n \in \mathbb{N}, \quad |f_n(x)| \leq |x| < \epsilon$$

**Behavior for the Remaining Region:**

Now, for all $x \in [-1, -\epsilon] \cup [\epsilon, 1]$ (the remaining region $[-1, 1] \setminus (-\epsilon, \epsilon)$), since $|x| \geq \epsilon$, then $x^2 \geq \epsilon^2$. Hence, for all $n \in \mathbb{N}$, we have $-nx^2 \leq -n\epsilon^2$, or $e^{-nx^2} \leq e^{-n\epsilon^2}$.

Since $|x| \leq 1$, we have $|f_n(x)| = |xe^{-nx^2}| \leq e^{-nx^2} \leq e^{-n\epsilon^2}$. Then, let $N = \frac{-\ln(\epsilon)}{\epsilon^2}$. For all positive integer $n > N = \frac{-\ln(\epsilon)}{\epsilon^2}$, the following is true:

$$n\epsilon^2 > -\ln(\epsilon), \quad -n\epsilon^2 < \ln(\epsilon), \quad e^{-n\epsilon^2} < \epsilon$$

Hence, for all $n > N$, every $x \in [-1, -\epsilon] \cup [\epsilon, 1]$ satisfies $|f_n(x)| \leq e^{-n\epsilon^2} < \epsilon$.

So, given arbitrary $\epsilon > 0$, using the $N$ proposed in the second part, for all $n \geq N$, there are two cases:

First, if $x \in (-\epsilon, \epsilon)$, then by the first part, we get $|f_n(x)| < \epsilon$.

Else, if $x \in [-1, -\epsilon] \cup [\epsilon, 1]$, then by the second part, $|f_n(x)| < \epsilon$ again.

Hence, $\epsilon$ is an upper bound of the collection $\{|f_n(x)| \mid x \in [-1, 1]\}$, showing that $\sup_{x\in[-1,1]} |f_n(x)| = \|f_n\|_\infty \leq \epsilon$.

Since for all $\epsilon > 0$, there exists $N$, with $n \geq N$ implies $\|f_n\|_\infty \leq \epsilon$, then $f_n(x)$ converges to $f(x) = 0$ uniformly on $[-1, 1]$.

(c) Since $f(x) = 0$, it is differentiable. And, $f'(0) = 0$.

Then, with $f_n(x) = xe^{-nx^2}$, its derivative $f_n'(x) = e^{-nx^2} + xe^{-nx^2} \cdot (-2nx) = e^{-nx^2}(1 - 2nx^2)$. Which, $f_n'(0) = e^{-n \cdot 0^2}(1 - 2n \cdot 0^2) = e^0 \cdot 1 = 1$. Hence, $\lim_{n\to\infty} f_n'(0) = 1$, so $1 = \lim_{n\to\infty} f_n'(0) \neq f'(0) = 0$.

# 5

**Question 5** *Let $f_n : [-1, 1] \to \mathbb{R}$ be a sequence of functions uniformly bounded, i.e.*

$$\exists M > 0 \quad s.t. \sup_{x\in[-1,1],\ n\in\mathbb{N}} |f_n(x)| \leq M$$

*Define:*

$$F(x) = \sup_{n\in\mathbb{N}} f_n(x)$$

*Give an example showing that in general $F$ is not a continuous function.*
*Assuming that the $f_n$s are differentiable and $f_n'$ are uniformly bounded, i.e.*

$$\exists K > 0 \quad s.t. \sup_{x\in[-1,1],\ n\in\mathbb{N}} |f_n'(x)| \leq K$$

*Prove that $F$ is continuous.*

**Pf:**
**Example of Not continuous $F$:**
For all $n \in \mathbb{N}$, define $f_n : [-1, 1] \to \mathbb{R}$ as follow:

$$f_n(x) = \begin{cases} \frac{1}{n} & x \in \mathbb{Q} \cap [-1, 1] \\ 0 & x \in \mathbb{Q}^C \cap [-1, 1] \end{cases}$$

Which, for all $x \in [-1, 1]$ and all $n \in \mathbb{N}$, if $x \in \mathbb{Q}$, then $|f_n(x)| = |\frac{1}{n}| \leq 1$; similarly, if $x \in \mathbb{Q}^C$, then $|f_n(x)| = |0| \leq 1$. Hence, the sequence $f_n$ is uniformly bounded.

Yet, if consider $F(x) = \sup_{n\in\mathbb{N}} f_n(x)$, these are the cases:
First, if $x \in \mathbb{Q}$, then for all $n \in \mathbb{N}$, we have $f_n(x) = \frac{1}{n}$. Hence, $F(x) = \sup_{n\in\mathbb{N}}\{\frac{1}{n}\} = 1$.
Else, if $x \in \mathbb{Q}^C$, then for all $n \in \mathbb{N}$, we have $f_n(x) = 0$. Hence, $F(x) = 0$.

So, $F(x)$ is in fact the indicator function:

$$F(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [-1,1] \\ 0 & x \in \mathbb{Q}^C \cap [-1,1] \end{cases}$$

Which is continuous nowhere on $[-1,1]$.

**$f_n'$ are uniformly bounded implies $F$ is continuous:**
Given that $f_n'$ are uniformly bounded:

$$\exists K > 0, \qquad \sup_{x \in [-1,1],\ n \in \mathbb{N}} |f_n'(x)| \leq K$$

Then, for all distinct $x, y \in [-1,1]$ (WLOG, assume $x < y$) and all $n \in \mathbb{N}$, since $f_n$ is differentiable, then by Mean Value Theorem, there exists $c \in (x,y)$, such that the following is true:

$$\left|\frac{f_n(x) - f_n(y)}{x - y}\right| = |f_n'(c)| \leq K, \quad |f_n(x) - f_n(y)| \leq K|x - y|$$

This proves that all $f_n$ are Lipschitz Continuous.

Now, for all $x_0 \in [-1,1]$ and all $\epsilon > 0$, consider $F(x_0)$ and $\delta = \frac{\epsilon}{2K} > 0$. Which, the following statements are true:

- First, since $\frac{\epsilon}{2} > 0$, and $F(x_0) = \sup_{n \in \mathbb{N}} f_n(x_0)$, then since $F(x_0) - \frac{\epsilon}{2}$ is no longer an upper bound, then there exists $n \in \mathbb{N}$, such that $F(x_0) - \frac{\epsilon}{2} < f_n(x_0) \leq F(x_0)$.

  Which, for all $x \in [-1,1]$ satisfying $|x - x_0| < \delta = \frac{\epsilon}{2K}$, by Lipschitz Continuity proven before: $|f_n(x) - f_n(x_0)| \leq K|x - x_0| < K \cdot \frac{\epsilon}{2K} = \frac{\epsilon}{2}$. Hence, the following is true:

$$-\frac{\epsilon}{2} < f_n(x) - f_n(x_0) < \frac{\epsilon}{2}, \quad f_n(x_0) - \frac{\epsilon}{2} < f_n(x)$$

  Hence, since $f_n(x) \leq F(x)$ by definition, then $f_n(x_0) - \frac{\epsilon}{2} < f_n(x) \leq F(x)$. Combining with the previous inequality, we get:

$$F(x_0) - \frac{\epsilon}{2} < f_n(x_0), \quad F(x_0) - \epsilon < f_n(x_0) - \frac{\epsilon}{2} < F(x)$$

- Then, based on the same $x$ chosen above (with $|x - x_0| < \delta = \frac{\epsilon}{2K}$), we'll prove that $F(x) < F(x_0) + \epsilon$: Suppose the contrary, that $F(x) \geq F(x_0) + \epsilon$, then $F(x) - \frac{\epsilon}{2} \geq F(x_0) + \frac{\epsilon}{2}$. Since $F(x) = \sup_{n \in \mathbb{N}} f_n(x)$, then because $F(x) - \frac{\epsilon}{2}$ is no longer an upper bound of the set, there exists $m \in \mathbb{N}$, such that $F(x) - \frac{\epsilon}{2} < f_m(x) \leq F(x)$.

  Which, based on the Lipschitz Continuity, we can conclude the following:

$$|f_m(x_0) - f_m(x)| < K|x_0 - x| < K \cdot \frac{\epsilon}{2K} = \frac{\epsilon}{2}$$

$$-\frac{\epsilon}{2} < f_m(x_0) - f_m(x) < \frac{\epsilon}{2}, \quad f_m(x) - \frac{\epsilon}{2} < f_m(x_0)$$

  Then, the following inequalities are true:

$$F(x) - \frac{\epsilon}{2} < f_m(x), \quad F(x) - \epsilon < f_m(x) - \frac{\epsilon}{2}, \quad f_m(x_0) \leq F(x_0)$$

$$F(x) - \epsilon < f_m(x) - \frac{\epsilon}{2} < f_m(x_0) \le F(x_0), \quad F(x) < F(x_0) + \epsilon$$

However, recall that $F(x) \ge F(x_0) + \epsilon$ is our initial assumption, which contradicts with the aobve inequality.

So, our assumption must be false, we must have $F(x) < F(x_0) + \epsilon$.

Combining both inequality, we can conclude that $F(x_0) - \epsilon < F(x) < F(x_0) + \epsilon$, showing that $|F(x) - F(x_0)| < \epsilon$.

Hence, for all $x \in [-1, 1]$, $\epsilon > 0$, there exists $\delta > 0$, with $|x - x_0| < \delta$ implies $|F(x) - F(x_0)| < \epsilon$, showing that $F(x)$ is continuous on $[-1, 1]$.

# 6

**Question 6** *Find the value of* $\sum_{k=1}^{\infty} k^2/3^k$.

**Pf:**

**The Series Absolutely Converges:**

Let $a_k = \frac{k^2}{3^k}$ for all $k \in \mathbb{N}$, then the following is true:

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{n\to\infty} \left| \frac{(k+1)^2}{3^{k+1}} \cdot \frac{3^k}{k^2} \right| = \lim_{n\to\infty} \frac{1}{3} \left| \frac{(k+1)^2}{k^2} \right| = \frac{1}{3}$$

Then, since $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \frac{1}{3} < 1$, then by Ratio Test, the series absolutely converges.

**Value of the Limit:**

Recall that for all $k \in \mathbb{N}$, the sum $\sum_{n=1}^{k} (2n-1) = k^2$. Hence, the partial sum of the given series can also be rewrite as the following:

$$\forall N \in \mathbb{N}, \quad \sum_{k=1}^{N} \frac{k^2}{3^k} = \sum_{k=1}^{N} \left( \sum_{n=1}^{k} \frac{2n-1}{3^k} \right)$$

Which, interchanging the summation, we get the following:

$$\sum_{k=1}^{N} \left( \sum_{n=1}^{k} \frac{2n-1}{3^k} \right) = \sum_{n=1}^{N} \left( \sum_{k=n}^{N} \frac{(2n-1)}{3^k} \right) = \sum_{n=1}^{N} (2n-1) \left( \sum_{k=n}^{N} \frac{1}{3^k} \right)$$

For the second partial sum, since $n \ge 1$, it satisfies the following equation:

$$\sum_{k=n}^{N} \frac{1}{3^k} = \sum_{k=0}^{N} \frac{1}{3^k} - \sum_{k=0}^{n-1} \frac{1}{3^k} = \frac{1 - 1/3^{N+1}}{1 - 1/3} - \frac{1 - 1/3^n}{1 - 1/3} = \frac{1/3^n - 1/3^{N+1}}{2/3} = \frac{3}{2} \left( \frac{1}{3^n} - \frac{1}{3^{N+1}} \right) \tag{1}$$

Plug back into the equation, we get:

$$\sum_{n=1}^{N} (2n-1) \left( \sum_{k=n}^{N} \frac{1}{3^k} \right) = \sum_{n=1}^{N} (2n-1) \cdot \frac{3}{2} \left( \frac{1}{3^n} - \frac{1}{3^{N+1}} \right) = \sum_{n=1}^{N} \left( \frac{(2n-1)}{2 \cdot 3^{n-1}} - \frac{(2n-1)}{2 \cdot 3^N} \right)$$

$$= \sum_{n=1}^{N} \frac{(2n-1)}{2 \cdot 3^{n-1}} - \sum_{n=1}^{N} \frac{(2n-1)}{2 \cdot 3^N} = \sum_{n=1}^{N} \frac{2n}{2 \cdot 3^{n-1}} - \sum_{n=1}^{N} \frac{1}{2 \cdot 3^{n-1}} - \frac{N^2}{2 \cdot 3^N}$$

$$= \sum_{n=1}^{N} \frac{n}{3^{n-1}} - \sum_{n=0}^{N-1} \frac{1}{2 \cdot 3^n} - \frac{N^2}{2 \cdot 3^N} = \sum_{n=1}^{N} \frac{n}{3^{n-1}} - \frac{1}{2} \cdot \frac{1 - 1/3^N}{1 - 1/3} - \frac{N^2}{2 \cdot 3^N}$$

$$= \sum_{n=1}^{N} \frac{n}{3^{n-1}} - \frac{3}{4}\left(1 - \frac{1}{3^N}\right) - \frac{N^2}{2 \cdot 3^N} \qquad (2)$$

Now, for the first summation of the term in (2), it can be rewrite as:

$$\sum_{n=1}^{N} \frac{n}{3^{n-1}} = \sum_{n=1}^{N} \left( \sum_{l=1}^{n} \frac{3}{3^n} \right) = \sum_{l=1}^{N} \left( \sum_{n=l}^{N} 3 \cdot \frac{1}{3^n} \right)$$

Which, based on the equation derived in (1), we get:

$$\sum_{l=1}^{N} \left( \sum_{n=l}^{N} 3 \cdot \frac{1}{3^n} \right) = \sum_{l=1}^{N} \left( 3 \cdot \frac{3}{2} \left( \frac{1}{3^l} - \frac{1}{3^{N+1}} \right) \right) = \frac{9}{2} \left( \sum_{l=1}^{N} \frac{1}{3^l} - \sum_{l=1}^{N} \frac{1}{3^{N+1}} \right)$$

$$= \frac{9}{2} \left( \frac{3}{2} \left( \frac{1}{3} - \frac{1}{3^{N+1}} \right) - \frac{N}{3^{N+1}} \right) = \frac{9}{4} - \frac{9}{4} \cdot \frac{1}{3^N} - \frac{9}{2} \cdot \frac{N}{3^{N+1}} \qquad (3)$$

Plug (3) back into (2), we get:

$$\sum_{n=1}^{N} \frac{n}{3^{n-1}} - \frac{3}{4}\left(1 - \frac{1}{3^N}\right) - \frac{N^2}{2 \cdot 3^N} = \left( \frac{9}{4} - \frac{9}{4} \cdot \frac{1}{3^N} - \frac{9}{2} \cdot \frac{N}{3^{N+1}} \right) - \frac{3}{4}\left(1 - \frac{1}{3^N}\right) - \frac{N^2}{2 \cdot 3^N}$$

$$= \left( \frac{9}{4} - \frac{3}{4} \right) - \left( \frac{9}{4} - \frac{3}{4} \right) \frac{1}{3^N} - \frac{3}{2} \cdot \frac{N}{3^N} - \frac{N^2}{2 \cdot 3^N}$$

$$= \frac{3}{2} - \frac{3}{2} \cdot \frac{1}{3^N} - \frac{3}{2} \cdot \frac{N}{3^N} - \frac{N^2}{2 \cdot 3^N}$$

So, connect back to the initial expression, the $N^{th}$ partial sum of the series is given by:

$$\sum_{k=1}^{N} \frac{k^2}{3^k} = \frac{3}{2} - \frac{3}{2} \cdot \frac{1}{3^N} - \frac{3}{2} \cdot \frac{N}{3^N} - \frac{N^2}{2 \cdot 3^N}$$

Now, recall that $\lim_{N\to\infty} \frac{1}{3^N} = 0$, $\lim_{N\to\infty} \frac{N}{3^N} = 0$, and $\lim_{N\to\infty} \frac{N^2}{3^N} = 0$. Hence, $\lim_{N\to\infty} \sum_{k=1}^{N} \frac{k^2}{3^k}$ is given by:

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{k^2}{3^k} = \lim_{N\to\infty} \left( \frac{3}{2} - \frac{3}{2} \cdot \frac{1}{3^N} - \frac{3}{2} \cdot \frac{N}{3^N} - \frac{N^2}{2 \cdot 3^N} \right) = \frac{3}{2}$$

Hence, we can conclude the following:

$$\sum_{k=1}^{\infty} \frac{k^2}{3^k} = \frac{3}{2}$$