Math 118B HW 1

Zih-Yu Hsieh

January 7, 2025

1

Question 1 ***Use just the definition, prove:***

(a) $f : \mathbf{R} \to \mathbf{R},\ f(x) = x^2$***, is not uniformly continuous.***

(b) $f : (-10^6, 10^6) \to \mathbf{R},\ f(x) = x^2$***, is uniformly continuous.***

(a) Given $f : \mathbf{R} \to \mathbf{R},\ f(x) = x^2$.

Choose $\epsilon = 1$. For any $\delta > 0$, by Archimedean Property, there exists $n \in \mathbf{N}$ with $1 < n\delta$, which implies that $\frac{1}{n} < \delta$. Then, consider $(n + \frac{1}{n}), n \in \mathbf{R}$:

First, $|(n + \frac{1}{n}) - n| = |\frac{1}{n}| < \delta$. Also, the following is true:

$$|f(n + \frac{1}{n}) - f(n)| = |(n + \frac{1}{n})^2 - n^2| = |\frac{1}{n}(2n + \frac{1}{n})| = (2 + \frac{1}{n^2}) > 1 = \epsilon$$

So, for any given $\delta > 0$, there exists $x_1, x_2 \in \mathbf{R}$, with $|x_1 - x_2| < \delta$, but $|f(x_1) - f(x_2)| \geq \epsilon$, proving that $f : \mathbf{R} \to \mathbf{R},\ f(x) = x^2$, is not uniformly continuous.

(b) Given $f : (-10^6, 10^6) \to \mathbf{R},\ f(x) = x^2$.

For all $\epsilon > 0$, let $\delta = \epsilon/(2 \cdot 10^6)$. Then, for all $x, y \in (-10^6, 10^6)$, suppose $|x - y| < \delta$, then consider $|f(x) - f(y)|$:

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y| \cdot |x + y| < \delta \cdot |x + y|$$

Which, since $x, y \in (-10^6, 10^6)$, then $|x|, |y| < 10^6$. So, $|x + y|$ is limited by the following inequality:

$$|x + y| \leq |x| + |y| < 10^6 + 10^6 = 2 \cdot 10^6$$

Thus the following is true:

$$|f(x) - f(y)| < \delta \cdot |x + y| < \delta \cdot (2 \cdot 10^6)$$

$$|f(x) - f(y)| < (2 \cdot 10^6) \cdot \epsilon/(2 \cdot 10^6) = \epsilon$$

So, the above proves that $f : (-10^6, 10^6) \to \mathbf{R},\ f(x) = x^2$, is uniformly continuous.

2

Question 2 ***Given*** (X, d_X), (Y, d_Y) ***two metric spaces. Let*** $f : X \to Y$ ***be an uniformly continuous function.***

(a) Prove that if $(x_n)_{n=1}^{\infty}$ ***is a cauchy sequence in*** X***, then*** $(f(x_n))_{n=1}^{\infty}$ ***is a cauchy sequence in*** Y***.***

(b) Given an example of $g : (0, 1) \to \mathbf{R}$ ***continuous,*** $(x_n)_{n=1}^{\infty}$ ***is a cauchy sequence in*** X ***and*** $(g(x_n))_{n=1}^{\infty}$ ***is not a cauchy sequence in*** Y***.***

(c) Prove that if $f : (0, 1) \to \mathbf{R}$ ***is uniformly continuous, it can be extended continuously to*** $[0, 1]$***.***

(a) Suppose $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X. Then, consider $(f(x_n))_{n=1}^{\infty}$:

Since f is uniformly continuous, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $x, x' \in X$, $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \epsilon$.

Also, since $(x_n)_{n=1}^{\infty}$ is Cauchy, for the given $\delta > 0$ above, there exists N, such that $m, n \geq N$ implies $d_X(x_m, x_n) < \delta$.

Which, for this specific N, since $m, n \geq N$ implies $d_X(x_m, x_n) < \delta$, and by the definition of Uniform Continuity, this further implies that $d_Y(f(x_m), f(x_n)) < \epsilon$. So, for all $\epsilon > 0$, there exists such N, such that $m, n \geq N$ implies $d_Y(f(x_m), f(x_n)) < \epsilon$. This proves that $(f(x_n))_{n=1}^{\infty}$ is a Cauchy Sequence.

(b) Consider $g : (0, 1) \to \mathbf{R}$ defined as $g(x) = \frac{1}{x}$, and let $(x_n)_{n=1}^{\infty}$ be defined as $x_n = \frac{1}{2^n}$ for all $n \in \mathbf{N}$.

First, we'll prove that $(x_n)_{n=1}^{\infty}$ is Cauchy: For all $\epsilon > 0$, since $\frac{\epsilon}{2} > 0$, there exists $k \in \mathbf{N}$, such that1 $< k\frac{\epsilon}{2}$, which $\frac{1}{k} < \frac{\epsilon}{2}$. Now, choose $N = \log_2(k)$. For all $n \geq N = \log_2(k)$, $2^n \geq 2^N = k$, thus $|x_n - 0| = |\frac{1}{2^n}| = \frac{1}{2^n} \leq \frac{1}{k} < \frac{\epsilon}{2}$. Then, for all $m, n \geq N$, the following is true:

$$|x_m - x_n| = |(x_m - 0) + (0 - x_n)| \leq |x_m - 0| + |0 - x_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, for all $\epsilon > 0$, there exists N, with $m, n \geq N$ implies that $|x_m - x_n| < \epsilon$, proving that $(x_n)_{n=1}^{\infty}$ is Cauchy.

Next, we'll prove that $(g(x_n))_{n=1}^{\infty}$ is not Cauchy: For all $n \in \mathbf{N}$, since $x_n = \frac{1}{2^n}$, then $g(x_n) = \frac{1}{1/2^n} = 2^n$. Then, choose $\epsilon = 1 > 0$. For all $N \in \mathbf{R}$, by Archimedean's Property, there exists $k \in \mathbf{N}$, such that $N < k \leq 2^k$. Which, consider $g(x_k)$ and $g(x_{k+1})$:

$$|g(x_k) - g(x_{k+1})| = |2^k - 2^{k+1}| = |-2^k| = 2^k > 2 > 1 = \epsilon$$

So, for $\epsilon = 1$, for all N, there exists $m, n \geq N$, such that $|g(x_k) - g(x_{k+1})| > \epsilon$, proving that $(g(x_n))_{n=1}^{\infty}$ is not Cauchy.

(c) Suppose $f(0,1) \to \mathbf{R}$ is Uniformly Continuous.

Limit near 0 and 1:

Given arbitrary $a \in \{0, 1\}$. For all $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset (0,1)$ that converges to a, the two sequences are Cauchy. Based on the statement in **Problem 2 Part (a)**, f is uniformly continuous and the two sequences being Cauchy, implies that $(f(x_n))_{n=1}^{\infty}, (f(y_n))_{n=1}^{\infty} \subset \mathbf{R}$ are Cauchy, and due to the Completeness of $\mathbf{R}$, the two sequences converge. Thus, $\lim_{n\to\infty} f(x_n) = L_x \in \mathbf{R}$, and $\lim_{n\to\infty} f(y_n) = L_y \in \mathbf{R}$.

Now, to prove that the limit is unique, consider $|L_x - L_y|$: For any $n \in \mathbf{N}$, the following is true:

$$0 \leq |L_x - L_y| = |(L_x - f(x_n)) + (f(x_n) - f(y_n)) + (f(y_n) - L_y)|$$

$$0 \leq |L_x - L_y| \leq |L_x - f(x_n)| + |f(x_n) - f(y_n)| + |L_y - f(y_n)|$$

Which, for all $\epsilon > 0$ (which $\frac{\epsilon}{3} > 0$), based on the definition of convergence, there exists N_1, N_2, such that $n \geq N_1$ implies that $|L_x - f(x_n)| < \frac{\epsilon}{3}$, and $n \geq N_2$ implies that $|L_y - f(y_n)| < \frac{\epsilon}{3}$.

Also, based on the definition of Uniform Continuity, given $\frac{\epsilon}{3} > 0$, there exists $\delta > 0$, such that for all $x, x' \in (0,1)$, $|x - x'| < \delta$ implies $|f(x) - f(x')| < \frac{\epsilon}{3}$.

Then, since $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ both converges to a, then given $\delta > 0$ (which $\frac{\delta}{2} > 0$), there exists N_3, N_4, such that $n \geq N_3$ implies $|a - x_n| < \frac{\delta}{2}$, and $n \geq N_4$ implies $|a - y_n| < \frac{\delta}{2}$.

Now, consider $N = \max\{N_1, N_2, N_3, N_4\}$. For any $n \geq N$:

Since $n \geq N_3$ and $n \geq N_4$, then $|a - x_n| < \frac{\delta}{2}$, and $|a - y_n| < \frac{\delta}{2}$. Which:

$$|x_n - y_n| = |(x_n - a) + (a - y_n)| \leq |a - x_n| + |a - y_n| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

Because $x_n, y_n \in (0,1)$ and $|x_n - y_n| < \delta$, this implies that $|f(x_n) - f(y_n)| < \frac{\epsilon}{3}$ based on the above definition of uniform continuity.

Also, since $n \geq N_1$, it implies that $|L_x - f(x_n)| < \frac{\epsilon}{3}$, and $n \geq N_2$ implies that $|L_y - f(y_n)| < \frac{\epsilon}{3}$.

Then, recall the initial inequality, the following is true:

$$0 \leq |L_x - L_y| \leq |L_x - f(x_n)| + |f(x_n) - f(y_n)| + |L_y - f(y_n)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Thus, for all $\epsilon > 0$, $0 \leq |L_x - L_y| < \epsilon$. This implies that $|L_x - L_y| = 0$, so $L_x = L_y$. Hence, we can conclude that for all $(x_n)_{n=1}^{\infty} \subset (0,1)$ satisfying $\lim_{n\to\infty} x_n = a$ (with $a \in \{0,1\}$), $\lim_{n\to\infty} f(x_n)$ converges to a unique element in $\mathbf{R}$, regardless the choice of $(x_n)_{n=1}^{\infty}$.

So, there exists unique $L, R \in \mathbf{R}$, such that for all $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \in (0,1)$ that satisfy $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 1$, the following is true:

$$\lim_{n\to\infty} f(x_n) = L, \quad \lim_{n\to\infty} f(y_n) = R$$

Continuity of f:

Define $f : [0,1] \to \mathbf{R}$ as follow:

$$f(x) = \begin{cases} L & x = 0 \\ f(x) & x \in (0,1) \\ R & x = 1 \end{cases}$$

The extension is continuous on $(0,1)$, to prove this extension is continuous, it suffices to prove that it is continuous at $x = 0$ and $x = 1$. We'll approach this by contradiction.

Given $a \in \{0,1\}$, suppose f is not continuous at a, then there exists $\epsilon > 0$, such that for all $\delta > 0$, there exists $x \in [0,1]$, with $|x - a| < \delta$, but $|f(x) - f(a)| \geq \epsilon$. Then, consider the following process:

Step 1. Choose $\delta_1 = \frac{1}{10^1}$, there exists $x_1 \in [0,1]$, such that $|x_1 - a| < \delta_1$, but $|f(x_1) - f(a)| \geq \epsilon$.

Notice that $x_1 \neq 0$ and $x_1 \neq 1$: If $a = 0$, then $x_1 \neq 0$ (since $|f(x_1) - f(0)| \geq \epsilon > 0$, so $f(x_1) \neq f(0)$, implying that $x_1 \neq 0$); also, since $|x_1 - 0| < \delta_1 = \frac{1}{10^1} < 1 = |1 - 0|$, then $x_1 \neq 1$. Else, if $a = 1$, then $x_1 \neq 1$ (since $|f(x_1) - f(1)| \geq \epsilon > 0$, so $f(x_1) \neq f(1)$, implying that $x_1 \neq 1$); then, since $|x_1 - 1| < \delta_1 = \frac{1}{10^1} < 1 = |0 - 1|$, $x_1 \neq 0$.

So, $x_1 \in (0,1)$.

Step k. Given integer $k \geq 2$, Choose $\delta_k = \frac{1}{10^k}$, there exists $x_k \in [0,1]$, such that $|x_k - a| < \delta_k$, but $|f(x_k) - f(a)| \geq \epsilon$.

Based on similar reason, $x_k \neq 0$ and $x_k \neq 1$: If $a = 0$, then $x_k \neq 0$ (since $|f(x_k) - f(0)| \geq \epsilon > 0$, so $f(x_k) \neq f(0)$, implying that $x_k \neq 0$); also, since $|x_k - 0| < \delta_1 = \frac{1}{10^k} < 1 = |1 - 0|$, then $x_k \neq 1$. Else, if $a = 1$, then $x_k \neq 1$ (since $|f(x_k) - f(1)| \geq \epsilon > 0$, so $f(x_k) \neq f(1)$, implying that $x_k \neq 1$); then, since $|x_k - 1| < \delta_1 = \frac{1}{10^k} < 1 = |0 - 1|$, $x_k \neq 0$.

So, $x_k \in (0,1)$.

From the above process, we constructed $(x_k)_{k=1}^{\infty} \subset (0,1)$, such that the following is true: For all $\epsilon' > 0$, there exists $K \in \mathbf{N}$, such that $1 < K\epsilon$, or $\frac{1}{10^K} < \frac{1}{K} < \epsilon'$. Then, for all $k \geq K$, since $10^k \geq 10^K$, $\frac{1}{10^k} \leq \frac{1}{10^K}$. Which, the following is true:

$$|x_k - a| < \delta_k = \frac{1}{10^k} \leq \frac{1}{10^K} < \epsilon'$$

Thus, this implies that x_k converges to a.

So, $(s_k)_{k=1}^{\infty} \subset (0,1)$ is a sequence satisfying $\lim_{k\to\infty} x_k = a$ (with $a \in \{0,1\}$), while $\lim_{k\to\infty} f(x_k) \neq f(a)$ (since for all $k \in \mathbf{N}$, $|f(x_k) - f(a)| \geq \epsilon > 0$). Yet, this is a contradiction:

Recall that for all $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \in (0,1)$ that satisfy $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 1$, $\lim_{n\to\infty} f(x_n) = L$, and $\lim_{n\to\infty} f(y_n) = R$.

If $a = 0$, then $(x_k)_{k=1}^{\infty}$ satisfies $\lim_{k\to\infty} x_k = a = 0$, while $\lim_{k\to\infty} f(x_k) \neq f(a) = L$, which is a contradiction. Else if $a = 1$, then $(x_k)_{k=1}^{\infty}$ satisfies $\lim_{k\to\infty} x_k = a = 1$, while $\lim_{k\to\infty} f(x_k) \neq f(a) = R$, which is again a contradiction.

So, our initial assumption must be false, the extended f must be continuous at a. And, since $a \in \{0,1\}$ is arbitrary, then the extended f is continuous on both $x = 0$ and $x = 1$, showing that we can extend f to be continuous on $[0,1]$.

3

Question 3 ***Textbook:***

2. ***If*** $f : X \to Y$ ***is continuous, prove that*** $f(\overline{E}) \subseteq \overline{f(E)}$ ***for all*** $E \subset X$***, and show that proper inclusion is possible.***

7. ***Define*** f ***and*** g ***mapping from*** $\mathbf{R}^2$ ***to*** $\mathbf{R}$ ***by*** $f(0,0) = g(0,0) = 0$***, and*** $f(x,y) = xy^2/(x^2 + y^4)$, $g(x,y) = xy^2/(x^2 + y^6)$ ***if*** $(x,y) \neq (0,0)$***. Prove that*** f ***is bounded on*** $\mathbf{R}$***,*** g ***is unbounded on every neighborhood of*** $(0,0)$***, and*** f ***is not continuous on*** $(0,0)$***. Also, show that the restriction of all straight line in*** $\mathbf{R}^2$ ***is continuous.***

18. ***For all*** $x \in \mathbf{Q} \setminus \{0\}$***, there exists unique*** $m, n \in \mathbf{Z}$ ***with*** $n > 0$***, such that*** $x = \frac{m}{n}$***, and*** m, n ***are coprime (if*** $x = 0$***, take*** $n = 1$***). Take the following function*** $f : \mathbf{R} \to \mathbf{R}$***:***

$$f(x) = \begin{cases} 0 & x \in \mathbf{Q}^C \\ \frac{1}{n} & x = \frac{m}{n},\ \gcd(m,n) = 1 \end{cases}$$

Prove that f ***is continuous at every irrational points, while discontinuous at every rational points.***

21. ***Suppose*** $K, F \subset X$ ***are disjoint sets with*** K ***being compact and*** F ***is closed. Prove that there exists*** $\delta > 0$ ***such that if*** $p \in K$ ***and*** $q \in F$***, then*** $d(p,q) > \delta$***. And, Show that the conclusion may fail for two disjoint closed sets if neither is compact.***

23. ***Prove that every convex function is continuous, every increasing convex function of a convex function is convex, and if*** $f : (a,b) \to \mathbf{R}$ ***is convex, given*** $a < s < t < u < b$***, the following is true:***

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$$

Q2. Suppose $f: X \to Y$ is continuous, and arbitrary $E \subseteq X$.

For any $x \in \overline{E}$, there are two cases to consider:

First, if $x \in E$, then $f(x) \in f(E) \subseteq \overline{f(E)}$.

Else, if $x \in E'$ with $x \notin E$:

Suppose $f(x) \in f(E) \subseteq \overline{f(E)}$, it is already done;

for the other case, if $f(x) \notin f(E)$, by the definition of continuity, for any $\epsilon > 0$, there exists $\delta > 0$, with $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$. Also, since $x \in E'$, then for the given $\delta > 0$, there exists $a \in B_\delta(x) \setminus \{x\} \cap E$. Thus, $a \in E$ satisfies $a \in B_\delta(x)$, which implies that $f(a) \in f(B_\delta(x)) \subseteq B_\epsilon(f(x))$, and $f(a) \in f(E)$. Also, since $f(x) \notin f(E)$ by assumption, then $f(a) \neq f(x)$. So, $f(a) \in B_\epsilon(f(x)) \setminus \{f(x)\} \cap f(E)$. This proves that $f(x)$ is a limit point of $f(E)$, hence $f(x) \in \overline{f(E)}$.

So, under all cases, $x \in \overline{E}$ implies that $f(x) \in \overline{f(E)}$, thus $f(\overline{E}) \subseteq \overline{f(E)}$.

Example of Proper Inclusion:

Consider the following function $f: \mathbf{R} \to \mathbf{R}$ with $f(x) = e^x$, and the set $E = (-\infty, 0)$.

Which, $\overline{E} = (-\infty, 0]$, we have $f(\overline{E}) = (0, 1]$.

However, $f(E) = (0, 1)$, which $\overline{f(E)} = [0, 1]$, $0 \in \overline{f(E)}$ while $0 \notin f(\overline{E})$, thus $f(\overline{E}) \subsetneq \overline{f(E)}$.

Q7. f **Bounded:**

For all $x, y \in \mathbf{R}$, with $(x, y) \neq (0, 0)$ (so, $x^2 + y^4 > 0$, since at least one of the entry is nonzero), since both $x^2, y^4 \geq 0$, then the following inequality is true:

$$\sqrt{x^2y^4} \leq \frac{x^2+y^4}{2}, \quad |xy^2| \leq \frac{x^2+y^4}{2}$$

$$\frac{1}{x^2+y^4} \leq \frac{1}{2|xy^2|}$$

Which, for $f(x, y)$, there are three cases to consider:

If $xy^2 > 0$, then $f(x) = \frac{xy^2}{x^2+y^4} \leq \frac{xy^2}{2|xy^2|} \leq \frac{1}{2}$, and $-\frac{1}{2} < 0 < f(x)$ (since $xy^2, (x^2 + y^4) > 0$).

If $xy^2 = 0$, thus $f(x) = \frac{xy^2}{x^2+y^4} = 0$.

Else if $xy^2 < 0$, then $f(x) = \frac{xy^2}{x^2+y^4} \geq \frac{xy^2}{2|xy^2|} \geq -\frac{1}{2}$, and $f(x) < 0 < \frac{1}{2}$ (since $xy^2 < 0$, and $(x^2+y^4) > 0$).

Thus, in all cases, $-\frac{1}{2} \leq f(x) \leq \frac{1}{2}$ (including $f(0, 0) = 0$), which f is bounded.

g **Unbounded:**

For all $r > 0$, consider $B_r(0, 0)$: Given arbitrary $M > 0$, consider the set $D = \{y > 0 \mid y < \frac{1}{2M}$ and $y < \frac{r}{\sqrt{2}}\}$ which is not empty (since both $\frac{1}{2M}, \frac{r}{\sqrt{2}} > 0$).

Note that it is always possible to find $y \in D$ with $y^3 < \frac{r}{\sqrt{2}}$:

If $y \geq 1$, take $y' = y^{\frac{1}{3}} > 0$, then $y' = y^{\frac{1}{3}} \leq y = (y')^3$, implying that $y' \leq (y')^3 = y < \frac{r}{\sqrt{2}}$ and $y' \leq (y')^3 = y < \frac{1}{2M}$, which y' satisfies the condition; else if $y < 1$, then $y^3 < y < \frac{r}{\sqrt{2}}$ and $y^3 < y < \frac{1}{2M}$, which y satisfies the condition.

Then, choose the $y \in D$ satisfying $y, y^3 < \frac{r}{\sqrt{2}}$, consider the element $(y^3, y) \in \mathbf{R}^2$: since both $y, y^3 < \frac{r}{\sqrt{2}}$, then:

$$\|(y^3, y)\|^2 = (y^3)^2 + y^2 < (\frac{r}{\sqrt{2}})^2 + (\frac{r}{\sqrt{2}})^2 = r^2, \quad \|(y^3, y)\| < \sqrt{r^2} = r$$

Thus, $(y^3, y) \in B_r(0, 0)$. Also, consider $g(y^3, y)$:

$$g(y^3, y) = \frac{y^3 \cdot y^2}{(y^3)^2 + y^6} = \frac{y^5}{2y^6} = \frac{1}{2y}$$

$$y < \frac{1}{2M}, \quad M < \frac{1}{2y} = g(y^3, y)$$

Hence, for all $r, M > 0$, there exists element $(x, y) \in B_r(0, 0)$ with $g(x, y) > M$, proving that g is unbounded in every neighborhood of $(0, 0)$.

f **Not continuous at** $(0, 0)$**:**

Take $\epsilon = \frac{1}{4}$, for all $\delta > 0$, since $\frac{\delta}{\sqrt{2}} > 0$, then there exists $n \in \mathbf{N}$ with $1 < n\frac{\delta}{\sqrt{2}}$, thus $\frac{1}{n} < \frac{\delta}{\sqrt{2}}$. (Note: since $n \geq 1$, then $n^2 \geq n$, $\frac{1}{n^2} \leq \frac{1}{n}$, implying that $\frac{1}{n^4} \leq \frac{1}{n^2}$).

Which, consider $(\frac{1}{n^2}, \frac{1}{n})$: First, about the norm:

$$\|(\frac{1}{n^2}, \frac{1}{n})\|^2 = (\frac{1}{n^2})^2 + (\frac{1}{n})^2 \leq \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2} < 2 \cdot (\frac{\delta}{\sqrt{2}})^2 = \delta^2$$

$$\|(\frac{1}{n^2}, \frac{1}{n})\| < \sqrt{\delta^2} = \delta$$

Thus, $(\frac{1}{n^2}, \frac{1}{n}) \in B_\delta(0,0)$. However, consider $f(\frac{1}{n^2}, \frac{1}{n})$:

$$f(\frac{1}{n^2}, \frac{1}{n}) = \frac{(\frac{1}{n^2}) \cdot (\frac{1}{n})^2}{(\frac{1}{n^2})^2 + (\frac{1}{n})^4} = \frac{1/n^4}{2/n^4} = \frac{1}{2}$$

Which, $|f(\frac{1}{n^2}, \frac{1}{n}) - f(0,0)| = |\frac{1}{2} - 0| = \frac{1}{2} > \frac{1}{4} = \epsilon$.

So, the above proves that f is not continuous at $(0,0)$, since the chosen $\epsilon > 0$ satisfies for all $\delta > 0$, there exists $(x,y) \in B_\delta(0,0)$ with $|f(x,y) - f(0,0)| \geq \epsilon$.

Restriction onto Straight Line:

For all straight line in $\mathbf{R}^2$, every (x,y) on the line satisfies $ax + by = c$ for some $a,b,c \in \mathbf{R}$ (which $(a,b) \neq (0,0)$).

If $c \neq 0$, then the line isn't including $(0,0)$, thus f, g are following the given rational function with every point being well-defined, which is continuous.

Else, if $c = 0$, then the line is including $(0,0)$ (everywhere else is defined with the rational function), the goal is to prove that the function is continuous at $(0,0)$. Again, there are 2 cases to consider:

First, if $b = 0$, then $ax + 0 = 0$, which $x = 0$ (since $(a,b) \neq (0,0)$, so $a \neq 0$). Then, $f(0,y) = \frac{0y^2}{0^2+y^4} = 0$, $g(0,y) = \frac{0y^2}{0^2+y^6} = 0$, which given the domain as the straight line $ax = 0$, the function has output 0, which is a constant function (and it is continuous). The same concept applies when $a = 0$ (which changes to $y = 0$, but the functions are still constant function of 0).

Else, if $a, b \neq 0$, then $ax = -by$, which $y = \frac{-ax}{b}$. If $(x,y) \neq (0,0)$ (which $x \neq 0$, or else it implies $y = 0$, causing $(x,y) = (0,0)$), then the following is true:

$$f(x,y) = f(x, \frac{-ax}{b}) = \frac{x(\frac{-ax}{b})^2}{x^2 + (\frac{-ax}{b})^4} = \frac{(-a/b)^2 x^3}{x^2 + (-a/b)^4 x^4} = \frac{(-a/b)^2 x}{1 + (-a/b)^4 x^2}$$

$$g(x,y) = g(x, \frac{-ax}{b}) = \frac{x(\frac{-ax}{b})^2}{x^2 + (\frac{-ax}{b})^6} = \frac{(-a/b)^2 x^3}{x^2 + (-a/b)^4 x^6} = \frac{(-a/b)^2 x}{1 + (-a/b)^6 x^4}$$

Which, notice that $(-a/b)^4 x^2, (-a/b)^6 x^4 \geq 0$, then $1 + (-a/b)^4 x^2, 1 + (-a/b)^6 x^4 \geq 1$, or:

$$\frac{1}{1 + (-a/b)^4 x^2}, \frac{1}{1 + (-a/b)^6 x^4} \leq 1$$

Thus, for all $(x,y) \neq (0,0)$ on the line, the following is true:

$$|f(x,y) - f(0,0)| = \left| \frac{(-a/b)^2 x}{1 + (-a/b)^4 x^2} - 0 \right| = \frac{|(-a/b)^2 x|}{1 + (-a/b)^4 x^2} \leq |(-a/b)^2 x|$$

$$|g(x,y) - g(0,0)| = \left| \frac{(-a/b)^2 x}{1 + (-a/b)^6 x^4} - 0 \right| = \frac{|(-a/b)^2 x|}{1 + (-a/b)^6 x^4} \le |(-a/b)^2 x|$$

So, for all $\epsilon > 0$, choose $\delta = (b/a)^2 \epsilon$. Then, for all $(x,y) \in \mathbf{R}^2$ (with $(x,y) \neq (0,0)$) satisfying $\|(x,y)\| = \sqrt{x^2+y^2} < \delta$ (in other word, $(x,y) \in B_\delta(0,0)$), since $|x| = \sqrt{x^2} \le \|(x,y)\| < \delta = (b/a)^2 \epsilon$, then the following is true:

$$|f(x,y) - f(0,0)|,\ |g(x,y) - g(0,0)| \le |(-a/b)^2 x| = (a/b)^2 |x| < (a/b)^2 \cdot (b/a)^2 \epsilon = \epsilon$$

Thus, for all $\epsilon > 0$, there exists $\delta > 0$, with $(x,y) \in B_\delta(0,0)$ (restricted to the straight line), it implies $|f(x,y) - f(0,0)|,\ |g(x,y) - g(0,0)| < \epsilon$. Thus, when restricted to any straight line passing through $(0,0)$, the functions are continuous at $(0,0)$, hence continuous on the whole straight line.

Q18. Given the following function:

$$f(x) = \begin{cases} 0 & x \in \mathbf{Q}^C \\ \frac{1}{n} & x = \frac{m}{n},\ \gcd(m,n) = 1 \end{cases}$$

And assume that at $x = 0$, $n = 1$ (so $f(0) = 1$).

Discontinuity on Q:

For any $x \in \mathbf{Q}$, $f(x) = \frac{1}{n}$ for some $n \in \mathbf{N}$, then choose $\epsilon = \frac{1}{2n}$. For all $\delta > 0$, since the open ball $(x-\delta, x+\delta)$ contains some irrational number r due to denseness of $\mathbf{Q}^C$ in $\mathbf{R}$, then r satisfies $|x-r| < \delta$, and $f(r) = 0$, which $|f(r) - f(x)| = |0 - \frac{1}{n}| = \frac{1}{n} \geq \frac{1}{2n} = \epsilon$. This shows that f is not continuous on x, which f is not continuous on $\mathbf{Q}$.

Continuity on $\mathbf{Q}^C$:

For any $x \in \mathbf{Q}^C$, consider $U = \{n \in \mathbf{Z} \mid x < n\}$. Then, since U is bounded below, $k = \inf(U)$ exists; and by the well-ordering property, $k \in U$. So, $k \in \mathbf{Z}$ satisfies $(k-1) \leq x < k$ ($(k-1) < k$, with $k = \inf(U)$, so $(k-1) \notin U$, or $(k-1) \leq x$). Also, since $(k-1)$ is rational, then $x \neq (k-1)$. Thus, $(k-1) < x < k$.

Now, for all $\epsilon > 0$, by Archimedean's Property, there exists $n \in \mathbf{N}$ with $1 < n\epsilon$ (or $\frac{1}{n} < \epsilon$). First, let $D = \{(k-1), k\} \cup \{(k-1) + \frac{i}{j} \mid i, j \in \mathbf{N},\ i < j < n\}$ (which D is finite, and every element is rational).

For any $q \in (k-1, k)$ with $f(q) > \frac{1}{n}$, if $q \in \mathbf{Q}^C$, then $f(q) = 0 < \frac{1}{n}$, which violates the desired condition, so $q \in \mathbf{Q}$; then, there exists $m \in \mathbf{Z}$ and $j \in \mathbf{N}$ with $\gcd(m, j) = 1$, such that $q = \frac{m}{j}$. Now, there are some conditions:

Since $f(q) = \frac{1}{j} > \frac{1}{n}$, then $n > j$.

Also, since $(k-1) < q = \frac{m}{j} < k$, then $(k-1)j < m < kj$, which $0 < (m - (k-1)j) < j$.

So, let $i = (m - (k-1)j)$, consider $(k-1) + \frac{i}{j}$:

$$(k-1) + \frac{i}{j} = (k-1) + \frac{m - (k-1)j}{j} = (k-1) + \frac{m}{j} - \frac{(k-1)j}{j}$$

$$(k-1) + \frac{i}{j} = (k-1) + \frac{m}{j} - (k-1) = \frac{m}{j}$$

Which, $q = \frac{m}{j} = (k-1) + \frac{i}{j}$, with $0 < i < j < n$ (since $i = (m - (k-1)j)$), then $q \in D$ (since $q = (k-1) + \frac{i}{j}$, which satisfies the set axiom of $\{(k-1) + \frac{i}{j} \mid i, j \in \mathbf{N},\ i < j < n\}$).

Now, let $\delta = \min\{|x - q| \mid q \in D\}$, which $\delta > 0$ since for all $q \in D$, q is rational, which $q \neq x$, so $|q - x| > 0$. For all $a \in \mathbf{R}$ satisfying $|a - x| < \delta$, then the following is true:

$$-\delta < a - x < \delta, \quad x - \delta < a < x + \delta$$

Which, since $(k-1) \in D$, $\delta \leq |x-(k-1)| = x-(k-1)$, then $(k-1) < (x-\delta)$; also, since $k \in D$, $\delta \leq |x-k| = k-x$, then $(x+\delta) < k$. So, the following is true:

$$(k-1) < (x-\delta) < a < (x+\delta) < k$$

So, $a \in (k-1, k)$. Also, since for all $q \in (k-1, k)$ satisfying $f(q) > \frac{1}{n}$, we've proven that $q \in D$, then $a \notin D$, since for all $q \in D$, $|x-a| < \delta \leq |x-q|$ by how we define δ.

Thus, $0 \leq f(a) \leq \frac{1}{n}$ (since $a \notin D$, it can't have $f(a) > \frac{1}{n}$). Hence, $|f(x)-f(a)| = |0-f(a)| = f(a) \leq \frac{1}{n} < \epsilon$.

This proves that f is continuous at x, which since $x \in \mathbf{Q}^C$ is arbitrary, then f is continuous on all $\mathbf{Q}^C$.

Q21. Given $K, F \subset X$ that are disjoint, with K being compact and F is closed.

First, consider the function $\rho_F : X \to \mathbf{R}$ with $\rho_F(x) = \inf\{d(x,q) \mid q \in F\}$, we'll prove that ρ_F is uniformly continuous:

For all $x, y \in X$ and all $q \in F$, notice that $\rho_F(x) \leq d(x,q) \leq d(x,y) + d(y,q)$, so $\rho_F(x) \leq d(x,y) + d(y,q)$ for all $q \in F$ (or, $\rho_F(x) - d(x,y) \leq d(y,q)$ for all $q \in F$), hence $(\rho_F(x) - d(x,y))$ is a lower bound of the set $\{d(y,q) \mid q \in F\}$, or $(\rho_F(x) - d(x,y)) \leq \inf\{d(y,q) \mid q \in F\} = \rho_F(y)$. Thus, $\rho_F(x) - d(x,y) \leq \rho_F(y)$, which $(\rho_F(x) - \rho_F(y)) \leq d(x,y)$.

So, given any $\epsilon > 0$, choose $\delta = \epsilon$, then for all $x, y \in X$ (Without Loss of Generality, assume $\rho_F(x) \geq \rho_F(y)$), if $d(x,y) < \delta = \epsilon$, then:

$$|\rho_F(x) - \rho_F(y)| = (\rho_F(x) - \rho_F(y)) \leq d(x,y) < \epsilon$$

Thus, the function $\rho_F(x)$ is uniformly continuous.

Now, since ρ_F is continuous and K is compact, then $\rho_F(K) \subset \mathbf{R}$ is compact (which is closed and bounded), thus $\inf(\rho_F(K)) \in \rho_F(K)$, there exists $p_0 \in K$, with $\rho_F(p_0) = \inf(\rho_F(K))$.

Then, we'll prove by contradiction that $\inf(\rho_F(K)) > 0$: Suppose this statement is false, then $\inf(\rho_F(K)) \leq 0$; furthermore, since 0 is always the lower bound of a set of distance, then for any $x \in X$, $\rho_F(x) \geq 0$, so $\inf(\rho_F(K)) = \rho_F(p_0) \geq 0$, showing that $\inf(\rho_F(K)) = \rho_F(p_0) = 0$.

However, this implies that for any $r > 0$, since $r = r + \rho_F(p_0)$ is no longer a lower bound of the set $\{d(p_0, q) \mid q \in F\}$, then there exists $q \in F$, with $d(p_0, q) < r$ (or $q \in B_r(p_0)$). Since K and F are disjoint, then $p_0 \neq q$, which $q \in B_r(p_0) \setminus \{p_0\} \cap F$, showing that p_0 is a limit point of F. With the assumption that F is closed, then $p_0 \in F' \subseteq F$, so $p_0 \in K \cap F$; yet, this contradicts the fact that the two sets are disjoint, so the assumption is false, or $\inf(\rho_F(K)) > 0$.

Eventually, let $\delta = \frac{\inf(\rho_F(K))}{2} > 0$, then for all $p \in K$ and $q \in F$, since $\rho_F(p) \in \rho_F(K)$, then $\delta = \frac{\inf(\rho_F(K))}{2} < \inf(\rho_F(K)) \leq \rho_F(p) \leq d(p,q)$.

Example if Both Closed Sets are not Compact:

Given $K = \mathbf{N}$ and $F = \{n + \frac{1}{n} \mid n \in \mathbf{N},\ n \geq 2\}$. Both sets are not compact as they're not bounded; both are closed as there are no limit points for either of them, and the two sets are disjoint (since for all $n \in \mathbf{N}$ with $n \geq 2$, $(n + \frac{1}{n})$ is not an integer, which F contains no elements from $K = \mathbf{N}$).

However, for all $\delta > 0$, there exists $n \in \mathbf{N}$ with $1 < n\delta$ (or $\frac{1}{n} < \delta$), which, choose $(n+1) \in K$ and $((n+1) + \frac{1}{(n+1)}) \in F$ (note: $n \geq 1$, so $(n+1) \geq 2$). Then, the following is true:

$$d\left((n+1), (n+1) + \frac{1}{(n+1)}\right) = \frac{1}{n+1} < \frac{1}{n} \leq \delta$$

Which it is a counterexample of the desired property.

Q23. **Inequality of Convex Function:**

To prove the continuity of convex function, we'll use this as a tool (that's why we're proving it first). Given a convex function $f : (a, b) \to \mathbf{R}$, and given $a < s < t < u < b$. Then, let $\lambda = \frac{u-t}{u-s}$, the following is true:

$$\lambda s + (1-\lambda)u = \frac{u-t}{u-s}s + \left(1 - \frac{u-t}{u-s}\right)u = \frac{us - ts}{u-s} + \frac{(u-s)-(u-t)}{u-s}u$$
$$= \frac{us - ts}{u-s} + \frac{ut - us}{u-s} = \frac{ut - ts}{u-s} = t\frac{u-s}{u-s} = t$$

Since $s < u,\ 0 < (u - s)$, which the above quantity is defined.

Also, since $s < t < u$, which $0 < (u - t) < (u - s)$, thus $0 < \frac{u-t}{u-s} < 1$, which $\lambda = \frac{u-t}{u-s} \in (0, 1)$.

With this parametrization, $t = \lambda s + (1 - \lambda)u$ for some $\lambda \in (0, 1)$, thus:

$$f(t) = f(\lambda s + (1-\lambda)u) \leq \lambda f(s) + (1-\lambda)f(u)$$

$$f(t) - f(s) \leq (\lambda f(s) + (1-\lambda)f(u)) - f(s) = (1-\lambda)(f(u) - f(s))$$

Given that $(t - s) > 0$, the following is true:

$$\frac{f(t) - f(s)}{t - s} \leq \frac{(1-\lambda)(f(u) - f(s))}{t - s}$$

And again, by the parametrization of t, the following is true:

$$t - s = (\lambda s + (1-\lambda)u) - s = (1-\lambda)(u - s)$$

So, the inequality can be rewrite as:

$$\frac{f(t) - f(s)}{t - s} \leq \frac{(1-\lambda)(f(u) - f(s))}{t - s} = \frac{(1-\lambda)(f(u) - f(s))}{(1-\lambda)(u - s)} = \frac{f(u) - f(s)}{u - s}$$
$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s}$$

Now, based on the same reasoning:

$$f(t) \leq \lambda f(s) + (1-\lambda)f(u), \quad f(u) - f(t) \geq f(u) - (\lambda f(s) + (1-\lambda)f(u))$$

$$\lambda(f(u) - f(s)) \leq f(u) - f(t)$$

Which, since $(t < u),\ 0 < (u - t)$, so:

$$\frac{\lambda(f(u) - f(s))}{u - t} \leq \frac{f(u) - f(t)}{u - t}$$

Rewrite $(u - t)$ with the given parametrization, the following is true:

$$u - t = u - (\lambda s + (1-\lambda)u) = \lambda(u - s)$$

So, the following inequality is also true:

$$\frac{\lambda(f(u)-f(s))}{u-t} = \frac{\lambda(f(u)-f(s))}{\lambda(u-s)} = \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$$

$$\frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$$

Combining the two inequality, the following is true:

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$$

Convex Implies Continuity:

Given convex function $f : (a, b) \to \mathbf{R}$. For all $x_0 \in (a, b)$, since $a < x_0 < b$, we can choose c, d satisfying $a < c < x_0 < d < b$.

Now, consider any $y \in (c, d)$ with $y \neq x_0$, there are two cases:

First, if $y < x_0$, then $a < c < y < x_0 < b$, which from the inequality proven beforehand:

$$\frac{f(y)-f(c)}{y-c} \leq \frac{f(x_0)-f(c)}{x_0-c} \leq \frac{f(x_0)-f(y)}{x_0-y}, \quad \frac{f(x_0)-f(c)}{x_0-c} \leq \frac{f(x_0)-f(y)}{x_0-y}$$

Else, if $y > x_0$, then $a < x_0 < y < d < b$, which using the same inequality:

$$\frac{f(y)-f(x_0)}{y-x_0} \leq \frac{f(d)-f(x_0)}{d-x_0} \leq \frac{f(d)-f(y)}{d-y}, \quad \frac{f(x_0)-f(y)}{x_0-y} = \frac{f(y)-f(x_0)}{y-x_0} \leq \frac{f(d)-f(x_0)}{d-x_0}$$

So, regardless of the case, the following is true:

$$\frac{f(x_0)-f(c)}{x_0-c} \leq \frac{f(x_0)-f(y)}{x_0-y} \leq \frac{f(d)-f(x_0)}{d-x_0}$$

Which, let $M = \max\left\{\left|\frac{f(x_0)-f(c)}{x_0-c}\right|, \left|\frac{f(d)-f(x_0)}{d-x_0}\right|\right\}$, then the above inequality implies that $\left|\frac{f(x_0)-f(y)}{x_0-y}\right| \leq M < (M+1)$, which:

$$|f(x_0) - f(y)| < (M+1)|x_0 - y|$$

(Note: since M is the maximum among absolute values, $M \geq 0$, so $(M+1) > 0$).

Then, to prove that f is continuous at x_0, given any $\epsilon > 0$, let $\delta = \min\left\{(x_0-c), (d-x_0), \frac{\epsilon}{(M+1)}\right\}$ (Note: all of these are positive, since $c < x_0 < d$, and $(M+1) > 0$). Then, for all y that satisfy $|y - x_0| < \delta$, if $y = x_0$, obviously $|f(y) - f(x_0)| = 0 < \epsilon$.

Else, if $y \neq x_0$, the following is true:

$$-\delta < (y - x_0) < \delta, \quad (x_0 - \delta) < y < (x_0 + \delta)$$

Also, since $\delta \leq (d - x_0)$ and $\delta \leq (x_0 - c)$ (which $-\delta \geq -(x_0 - c)$), the following is true:

$$c = (x_0 - (x_0 - c)) \leq (x_0 - \delta) < y < (x_0 + \delta) \leq (x_0 + (d - x_0)) = d$$

So, $c < y < d$ (or $y \in (c, d)$). Now, based on the inequality proven above, we have:

$$|f(x_0) - f(y)| < (M+1)|x_0 - y| < (M+1)\delta \leq (M+1)\frac{\epsilon}{(M+1)} = \epsilon$$

Which we deduced $|f(y) - f(x_0)| < \epsilon$, showing that f is continuous at x_0. Since the choice of $x_0 \in (a, b)$ is arbitrary, this proves that f is continuous on (a, b). So, $f : (a, b) \to \mathbf{R}$ is convex implies that f is continuous on (a, b).

Composition of Increasing Convex Function and Convex Function:

Suppose $g : \mathbf{R} \to \mathbf{R}$ is an increasing convex function, and $f : \mathbf{R} \to \mathbf{R}$ is a convex function. Then, for all $x, y \in \mathbf{R}$ and $\lambda \in (0, 1)$, since f is convex, the following is true:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

Now, treat $f(x), f(y)$ as two inputs, since g is also convex, the following is true:

$$g(\lambda f(x) + (1-\lambda)f(y)) \leq \lambda g(f(x)) + (1-\lambda)g(f(y))$$

Then, since g is increasing, while $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, then the following is true:

$$g(f(\lambda x + (1-\lambda)y)) \leq g(\lambda f(x) + (1-\lambda)f(y))$$

Thus, combining the inequality we get:

$$g(f(\lambda x + (1-\lambda)y)) \leq g(\lambda f(x) + (1-\lambda)f(y)) \leq \lambda g(f(x)) + (1-\lambda)g(f(y))$$

$$g(f(\lambda x + (1-\lambda)y)) \leq \lambda g(f(x)) + (1-\lambda)g(f(y))$$

This proves that $g \circ f$ is also a convex function.

4

Question 4 ***In each case give an example of $f : \mathbf{R} \to \mathbf{R}$ continuous and:***

- ***i*** K ***compact with*** $f^{-1}(K)$ ***no compact.***
- ***ii*** A ***connected with*** $f^{-1}(A)$ ***no connected.***
- ***iii*** B ***open with*** $f(B)$ ***not open.***
- ***iv*** C ***closed with*** $f(C)$ ***not closed.***

i Given $f(x) = 0$ the constant function. Since $K = \{1\}$ is a singleton set, then K is compact.

Yet, since for all $x \in \mathbf{R},\ f(x) = 1$, so $x \in f^{-1}(K)$, or $f^{-1}(K) = \mathbf{R}$, which is not compact.

ii Given $f(x) = x^2$ and $A = \{1\}$, which it is connected.

For all $x \in \mathbf{R}$, if $f(x) = x^2 = 1$ (or $x \in f^{-1}(A)$), then $x = 1$ or $x = -1$. So, $f^{-1}(A) = \{-1, 1\}$.

However, since $C = \{-1\},\ D = \{1\}$ satisfy $C \cup D = f^{-1}(A)$, and $\overline{C} \cup D = \overline{D} \cup C = \emptyset$, so $f^{-1}(A)$ is not connected.

iii Given $f(x) = 0$ again. The open interval $B = (0, 1) \subset \mathbf{R}$ is open.

Yet since for all $x \in B$, $f(x) = 1$, then $f(B) = \{1\}$, which $f(B)$ is closed.

iv G9ven $f(x) = e^x$ and $C = (-\infty, 0] \subset \mathbf{R}$.

For all $x \in C$, since $x \leq 0$, so $f(x) = e^x \leq e^0 = 1$; also, since $f(x) = e^x > 0$, then the inequality $0 < f(x) \leq 1$ is true. Which, $f(C) = (0, 1]$, and it is not closed.