

Math 111B HW2

Zih-Yu Hsieh

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Question 1 Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable on (a, b) .

Prove : if $\forall x \in (a, b)$, $f'(x) \neq 0$, then f is one-to-one on (a, b) .

Give an example showing that the converse statement is in general not true.

Pf:

Suppose $\forall x \in (a, b)$, $f'(x) \neq 0$:

(1) $f'(x)$ is strictly less than or greater than 0 on (a, b) :

We'll prove by contradiction: Suppose $f'(x)$ is neither strictly less than 0 nor strictly greater than 0 on (a, b) , then there exists $x_0, x_1 \in (a, b)$, with $f'(x_0) \leq 0$ and $f'(x_1) \geq 0$, and by the assumption that $f'(x) \neq 0$, the strict inequality $f'(x_0) < 0$ and $f'(x_1) > 0$ is applied. (This also implies $x_0 \neq x_1$, since derivatives are different at the two points).

Recall that for function $f : [a, b] \rightarrow \mathbb{R}$ be differentiable on (a, b) , if $a < c < d < b$ and $f'(c) \neq f'(d)$, for any λ strictly in between $f'(c)$ and $f'(d)$ (either $f'(c) < \lambda < f'(d)$ or $f'(c) > \lambda > f'(d)$), there exists $x \in (c, d)$ with $f'(x) = \lambda$.

Then, first suppose $x_0 < x_1$: f is differentiable on (a, b) and $f'(x_0) < 0 < f'(x_1)$ implies there exists $x \in (x_0, x_1)$ with $f'(x) = 0$, which contradicts the assumption;

then suppose $x_1 < x_0$: again, f is differentiable on (a, b) and $f'(x_1) > 0 > f'(x_0)$ implies there exists $x \in (x_1, x_0)$ with $f'(x) = 0$, which again contradicts the assumption.

So, the assumption is false, $f'(x)$ must be strictly greater than 0 or less than 0 for all $x \in (a, b)$.

(2) f is strictly increasing or decreasing on (a, b) :

Based on **(1)**, $f'(x)$ is strictly less than 0 or strictly greater than 0.

Suppose $f'(x) > 0$ for all $x \in (a, b)$, then for any $x, y \in (a, b)$ with $x < y$, by the Mean Value Theorem, there exists $c \in (x, y) \subseteq (a, b)$, such that the following is true:

$$\frac{f(y) - f(x)}{y - x} = f'(c) > 0, \quad (f(y) - f(x)) = f'(c)(y - x)$$

Since $(y - x), f'(c) > 0$ by assumption, the $(f(y) - f(x)) = f'(c)(y - x) > 0$, thus $f(y) > f(x)$, showing that f is strictly increasing.

Similarly, suppose $f'(x) < 0$ for all $x \in (a, b)$, with the same x, y above, by Mean Value Theorem, there exists $c \in (x, y) \subseteq (a, b)$, such that the following is true:

$$\frac{f(y) - f(x)}{y - x} = f'(c) < 0, \quad (f(y) - f(x)) = f'(c)(y - x)$$

Since $(y - x) > 0$ and $f'(c) < 0$, then $(f(y) - f(x)) = f'(c)(y - x) < 0$, of $f(y) < f(x)$, showing that f is strictly decreasing.

With the above condition, since f is either strictly increasing or strictly decreasing on $[a, b]$, then for all $x, y \in (a, b)$, $x \neq y$ implies $f(x) \neq f(y)$ (or else it's no longer strictly increasing or decreasing). Thus, f is in fact one-to-one on (a, b) .

Counterexample of Converse:

Let $f : [-1, 1] \rightarrow \mathbb{R}$ be $f(x) = x^3$, which $f'(x) = 3x^2$, which $f'(0) = 0$. Yet, suppose $x, y \in (-1, 1)$ has $x^3 = y^3$, then:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2) = 0$$

Which, the only real solution is $x = y$ (since if treating y as constant, $y^2 - 4y^2 = -3y^2 \leq 0$; the only time with real solution is when $y = 0$, which implies $x^3 = 0$, or $x = 0$).

So, $f(x) = x^3$ is one-to-one on the region $(-1, 1)$, but still has $f'(0) = 0$, which is a counterexample.

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Question 2 Let $f : (a, b) \rightarrow \mathbb{R}$ be a function such that:

$$\exists M > 0, \exists \alpha > 0, \forall x, y \in (a, b), |f(x) - f(y)| < M|x - y|^\alpha$$

If $\alpha \in (0, 1)$, then f is Holder of order α in (a, b) . If $\alpha = 1$, then f is Lipschitz. Prove :

- (a) If $\alpha > 1$, then f is constant.
- (b) If $\alpha \in (0, 1]$, then f is uniformly continuous on (a, b) .
- (c) Give an example such that f is Lipschitz, but not differentiable.
- (d) If f is differentiable on (a, b) and $f(x)$ is bounded on (a, b) , then f is Lipschitz.

Pf:

- (a) Suppose $\alpha > 1$, then there exists $\epsilon > 0$, such that $\alpha = 1 + \epsilon$. Which, for all $x, y \in (a, b)$ (with $x \neq y$), the following is true:

$$|f(x) - f(y)| < M|x - y|^\alpha = M|x - y|^{1+\epsilon} = M|x - y| \cdot |x - y|^\epsilon$$

$$\left| \frac{f(x) - f(y)}{x - y} \right| = \frac{|f(x) - f(y)|}{|x - y|} < M|x - y|^\epsilon$$

Which, fix arbitrary $x_0 \in (a, b)$, for all $y \in (a, b)$ with $y \neq x_0$, the following is true:

$$0 \leq \left| \frac{f(x_0) - f(y)}{x_0 - y} \right| < M|x_0 - y|^\epsilon, \quad -M|x_0 - y|^\epsilon < \frac{f(x_0) - f(y)}{x_0 - y} < M|x_0 - y|^\epsilon$$

Since $\epsilon > 0$, then $\lim_{y \rightarrow x_0} |x_0 - y|^\epsilon = 0$. Which, by Squeeze Theorem, the following is true:

$$0 = \lim_{y \rightarrow x_0} -M|x_0 - y|^\epsilon \leq \lim_{y \rightarrow x_0} \frac{f(x_0) - f(y)}{x_0 - y} \leq \lim_{y \rightarrow x_0} M|x_0 - y|^\epsilon = 0$$

Thus, $\lim_{y \rightarrow x_0} \frac{f(x_0) - f(y)}{x_0 - y} = 0$, or $f'(x_0) = 0$.

This implies that $f(x)$ is a constant function: Suppose $f(x)$ is not a constant function, then there exists $c, d \in (a, b)$ with $c < d$, such that $f(c) \neq f(d)$.

Notice that since $f'(x_0)$ exists for all $x_0 \in (a, b)$, then by Mean Value Theorem, there exists $x \in (c, d)$, such that $f'(x)(d - c) = f(d) - f(c)$.

Yet, since $f'(x) = 0$, while $f(d) - f(c) \neq 0$, $0 = f'(x)(d - c) \neq f(d) - f(c)$, which it is a contradiction.

Thus, $f(x)$ must be a constant function.

(b) Suppose $\alpha \in (0, 1]$, notice that for all $x, y \in (a, b)$, the following is true:

$$a < x < b, \quad -b < -y < -a, \quad (a - b) = -(b - a) < (x - y) < (b - a), \quad |x - y| < |b - a|$$

Which, since $\alpha > 0$, then $|x - y|^\alpha < |b - a|^\alpha$. Now, for any $\epsilon > 0$, define $\delta = \left(\frac{\epsilon}{M}\right)^{\frac{1}{\alpha}} > 0$, then for all $x, y \in (a, b)$, if $|x - y| < \delta$, the following is true:

$$|f(x) - f(y)| < M|x - y|^\alpha < M \cdot \delta^\alpha$$

(Note: the above inequality is true, since $\alpha > 0$, then $0 \leq |x - y| < |b - a|$ implies $|x - y|^\alpha < |b - a|^\alpha$). Thus, it can be rewritten as:

$$|f(x) - f(y)| < M \cdot \delta^\alpha = M \cdot \left(\left(\frac{\epsilon}{M}\right)^{\frac{1}{\alpha}}\right)^\alpha = M \cdot \frac{\epsilon}{M} = \epsilon$$

Thus, since for all $\epsilon > 0$, there exists $\delta > 0$ with $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, f is uniformly continuous.

(c) Consider the function $f : (-1, 1) \rightarrow \mathbb{R}$ by $f(x) = |x|$.

Choose $M = 1.01$ and $\alpha = 1$, then the following is true:

$$\forall x, y \in (-1, 1), \quad |f(x) - f(y)| = ||x| - |y|| \leq |x - y| = |x - y|^\alpha < 1.01|x - y|^\alpha = M|x - y|^\alpha$$

Thus, f is Lipschitz continuous.

Yet, f is not differentiable at $x = 0$: For all $x < 0$ and $y > 0$ (with $x, y \in (-1, 1)$), the following is true:

$$\begin{aligned} \frac{f(x) - f(0)}{x - 0} &= \frac{|x| - 0}{x - 0} = \frac{-x}{x} = -1 \\ \frac{f(y) - f(0)}{y - 0} &= \frac{|y| - 0}{y - 0} = \frac{y}{y} = 1 \end{aligned}$$

Which, $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$ is not defined, since the left and right limits as x approaches 0 are different.

(d) Suppose f is differentiable on (a, b) and $f'(x)$ is bounded on (a, b) , then there exists $M > 0$, with $|f'(x)| < M$ for all $x \in (a, b)$. Which, for all $x, y \in (a, b)$ with $x < y$, by the Mean Value Theorem, there exists $c \in (x, y)$, such that $f(y) - f(x) = f'(c)(y - x)$. Which, the following is true:

$$|f(y) - f(x)| = |f'(c)| \cdot |y - x| < M|y - x|$$

Thus, f is Lipschitz continuous.

Question 3 For any $a \geq 0$, define $f_a : \mathbb{R} \rightarrow \mathbb{R}$ as:

$$f_a(x) = \begin{cases} x^a \sin(\frac{1}{x}) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

- (a) For which values of a is f_a continuous at 0.
 (b) For which values of a is $f'_a(0)$ defined.
 (c) For which values of a is f'_a continuous at 0.
 (d) For which values of a is $f''_a(0)$ defined.

Pf:

- (a) **Ans:** $a > 0$. For $a = 0$, the function $f_a(x)$ is not continuous: Choose the sequence $(x_n)_{n \in \mathbb{N}}$ by $x_n = \frac{1}{(2n+1/2)\pi} > 0$, then $\lim_{n \rightarrow \infty} \frac{1}{(2n+1/2)\pi} = 0$, thus x_n converges to 0; but, consider $(f_a(x_n))_{n \in \mathbb{N}}$:

$$\forall n \in \mathbb{N}, \quad f_a(x_n) = x_n^0 \sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{1}{1/(2n+1/2)\pi}\right) = \sin((2n+1/2)\pi) = 1$$

Which, $\lim_{n \rightarrow \infty} f_a(x_n) = 1 \neq 0 = f_a(0)$, thus $f_a(x_n)$ doesn't converge to $f_a(0)$, showing it's not continuous.

Now, for all $a > 0$, for any $x > 0$, since $x^a > 0$, it satisfies the following:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1, \quad -x^a \leq f_a(x) = x^a \sin\left(\frac{1}{x}\right) \leq x^a$$

Which, take the right limit of x^a of 0, $\lim_{x \rightarrow 0^+} x^a = 0$, then by Squeeze Theorem, the following is true:

$$0 = \lim_{x \rightarrow 0^+} -x^a \leq \lim_{x \rightarrow 0^+} x^a \sin\left(\frac{1}{x}\right) \leq \lim_{x \rightarrow 0^+} x^a = 0$$

Thus, $\lim_{x \rightarrow 0^+} f_a(x) = 0$.

Also, since $\lim_{x \rightarrow 0^-} f_a(x) = 0$ (since for $x < 0$, $f_a(x) = 0$), then the left and right limits both agree with $f_a(0) = 0$, showing it's continuous at 0. Every $a > 0$ has $f_a(x)$ being continuous at 0.

- (b) **Ans:** $a > 1$. In case for $f'_a(0)$ to be defined, f_a must be continuous at 0. Thus, $a > 0$ is required.

Consider the slope $\frac{f_a(x) - f_a(0)}{x - 0}$ for all $x \neq 0$. If $x < 0$, then since $f_a(x) = 0$, then the slope is 0. Thus, the left limit of the slope $\lim_{x \rightarrow 0^-} \frac{f_a(x) - f_a(0)}{x - 0} = 0$.

Now, consider the slope from the right:

$$x > 0, \quad \frac{f_a(x) - f_a(0)}{x - 0} = \frac{x^a \sin(1/x) - 0}{x - 0} = x^{a-1} \sin\left(\frac{1}{x}\right)$$

Since the left limit is evaluated as 0, in case for $f'(0)$ to be defined, the right limit also needs to converge to 0.

First, notice that if $a \leq 1$, the right limit doesn't exist:

Consider the same sequence $x_n = \frac{1}{(2n+1/2)\pi} > 0$ used in part (a), then the following is true:

$$\forall n \in \mathbb{N}, \quad (x_n)^{a-1} \sin\left(\frac{1}{x_n}\right) = \left(\frac{1}{(2n+1/2)\pi}\right)^{a-1} \sin((2n+1/2)\pi) = \left(\frac{1}{(2n+1/2)\pi}\right)^{a-1}$$

Which, if $a = 1$ (or $a - 1 = 0$), then $(x_n)^{a-1} \sin(1/x_n) = 1$ for all $n \in \mathbb{N}$, which $\lim_{n \rightarrow \infty} \frac{f_a(x_n) - f_a(0)}{x_n - 0} = 1$, while $\lim_{n \rightarrow \infty} x_n = 0$. This shows that the right limit of the slope is not 0, which $f'_a(0)$ is not defined.

Else, if $a < 1$ (or $a - 1 < 0$), then $(x_n)^{a-1} \sin(1/x_n) = \left(\frac{1}{(2n+1/2)\pi}\right)^{a-1} = ((2n+1/2)\pi)^{1-a}$ is in fact unbounded as n increases indefinitely (since $1 - a > 0$), so again the right limit of the slope is not defined, implying $f'_a(0)$ is not defined.

So, in case for the right limit to be defined, $a > 1$. Which, since $a - 1 > 0$, then for all $x > 0$, $x^{a-1} > 0$, and $\lim_{x \rightarrow 0^+} x^{a-1} = 0$. Thus based on Squeeze Theorem:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1, \quad x > 0, \quad -x^{a-1} \leq x^{a-1} \sin\left(\frac{1}{x}\right) \leq x^{a-1}$$

$$0 = \lim_{x \rightarrow 0^+} -x^{a-1} \leq \lim_{x \rightarrow 0^+} x^{a-1} \sin\left(\frac{1}{x}\right) \leq \lim_{x \rightarrow 0^+} x^{a-1} = 0$$

So, the right limit of $x^{a-1} \sin(1/x)$ is 0 when x approaches 0, which it agrees with the initial left limit, hence for $a > 1$, $\lim_{x \rightarrow 0} \frac{f_a(x) - f_a(0)}{x - 0} = 0$, $f'_a(0) = 0$ is defined.

(c) **Ans:** $a > 2$. For f'_a to be continuous at 0, $f'_a(0)$ needs to be defined. So, $a > 1$ is required.

For $x < 0$, since $f_a(x) = 0$, then $f'_a(x) = 0$, which $\lim_{x \rightarrow 0^-} f'_a(x) = 0$.

Consider $f'_a(x)$ for $x > 0$, which by the differentiation rule, it is evaluated as:

$$f'_a(x) = ax^{a-1} \sin\left(\frac{1}{x}\right) + x^a \cos\left(\frac{1}{x}\right) \frac{-1}{x^2} = ax^{a-1} \sin\left(\frac{1}{x}\right) - x^{a-2} \cos\left(\frac{1}{x}\right)$$

In case for $f'_a(x)$ to be continuous at 0, $\lim_{x \rightarrow 0^+} f'_a(x) = 0$.

Since $x^{a-1} \sin(1/x)$ has right limit exists as x approaches 0 (since we assume $a > 1$), it suffices to find values of a which $x^{a-2} \cos(1/x)$ has right limit being 0, when x approaches 0.

For $a \leq 2$, the right limit of $x^{a-2} \cos(1/x)$ is not 0:

Consider the sequence $(x_n)_{n \in \mathbb{N}}$ by $x_n = \frac{1}{2n\pi}$, then $\lim_{n \rightarrow \infty} x_n = 0$. Which, the following is true:

$$\forall n \in \mathbb{N}, \quad (x_n)^{a-2} \cos\left(\frac{1}{x_n}\right) = \left(\frac{1}{2n\pi}\right)^{a-2} \cos\left(\frac{1}{2n\pi}\right) = (2n\pi)^{2-a}$$

Which, if $a = 2$, $2 - a = 0$, hence $(x_n)^{a-2} \cos(1/x_n) = 1$, implying $\lim_{n \rightarrow \infty} (x_n)^{a-2} \cos(1/x_n) = 1 \neq 0$. This implies that $x^{a-2} \cos(1/x)$ doesn't converge to 0 as x converges to 0.

Else, if $a < 2$, then since $(2 - a) > 0$, $(2n\pi)^{2-a}$ goes unbounded as n increases indefinitely, so again $x^{a-2} \cos(1/x)$ doesn't converge to 0 when x converges to 0.

So, for right limit of $f'_a(x)$ of $x = 0$ to be 0, $a > 2$ is required. Which, for $a > 2$, since $a - 2 > 0$, then for all $x > 0$, $x^{a-2} > 0$. Thus by Squeeze Theorem:

$$-x^{a-2} \leq x^{a-2} \cos\left(\frac{1}{x}\right) \leq x^{a-2}$$

$$0 = \lim_{x \rightarrow 0^+} -x^{a-2} \leq \lim_{x \rightarrow 0^+} x^{a-2} \cos\left(\frac{1}{x}\right) \leq \lim_{x \rightarrow 0^+} x^{a-2} = 0$$

So, the right limit of $x^{a-2} \cos(1/x)$ is 0 as x approaches 0, hence the right limit of $f'_a(x) = ax^{a-1} \sin\left(\frac{1}{x}\right) - x^{a-2} \cos\left(\frac{1}{x}\right)$ is 0 as x approaches 0. Hence, for $a > 2$, $f'_a(x)$ is continuous at 0, since the left and right limit agrees with $f'_a(0)$.

- (d) **Ans:** $a > 3$. To make sense of the second derivative, $f'_a(x)$ needs to be continuous at 0, thus $a > 2$. Since for all $x < 0$, $f'_a(x) = 0$, thus $f''_a(x) = 0$. So, the left limit $\lim_{x \rightarrow 0^-} f''_a(x) = 0$.

Then, in case for $f''_a(0)$ to be defined, the right limit must also be 0.

Now, for all $x > 0$, consider the slope $\frac{f'_a(x) - f'_a(0)}{x - 0}$:

$$\begin{aligned} \frac{f'_a(x) - f'_a(0)}{x - 0} &= \frac{ax^{a-1} \sin\left(\frac{1}{x}\right) - x^{a-2} \cos\left(\frac{1}{x}\right) - 0}{x - 0} = \frac{ax^{a-1} \sin\left(\frac{1}{x}\right) - x^{a-2} \cos\left(\frac{1}{x}\right)}{x} \\ &= ax^{a-2} \sin\left(\frac{1}{x}\right) - x^{a-3} \cos\left(\frac{1}{x}\right) \end{aligned}$$

Which, in case for $\lim_{x \rightarrow 0^+} \frac{f'_a(x) - f'_a(0)}{x - 0}$ to be defined, $a > 3$.

If $a \leq 3$, then again take the sequence $x_n = \frac{1}{2n\pi}$ used in part (c), the above limit becomes:

$$\begin{aligned} \forall n \in \mathbb{N}, \quad ax_n^{a-2} \sin\left(\frac{1}{x_n}\right) - x_n^{a-3} \cos\left(\frac{1}{x_n}\right) &= a \left(\frac{1}{2n\pi}\right)^{a-2} \sin(2n\pi) - \left(\frac{1}{2n\pi}\right)^{a-3} \cos(2n\pi) \\ &= 0 - (2n\pi)^{3-a} \end{aligned}$$

If $a = 3$, then the above expression is -1 . Thus, as n approaches ∞ , the sequence $\frac{f'_a(x_n) - f'_a(0)}{x_n - 0}$ converges to $-1 \neq 0$, hence the right limit doesn't agree with the left limit, hence $f''_a(0)$ is not defined.

Else if $a < 3$, then the above expression is not bounded, since $3 - a > 0$, so the right limit doesn't exist in \mathbb{R} , hence $f''_a(0)$ is again not defined.

For all $a > 3$, and all $x > 0$, the above terms can again be approached by Squeeze Theorem:

$$\begin{aligned} -x^{a-2} &\leq x^{a-2} \sin\left(\frac{1}{x}\right) \leq x^{a-2} \\ 0 &= \lim_{x \rightarrow 0^+} -x^{a-2} \leq \lim_{x \rightarrow 0^+} x^{a-2} \sin\left(\frac{1}{x}\right) \leq \lim_{x \rightarrow 0^+} x^{a-2} = 0 \\ -x^{a-3} &\leq x^{a-3} \cos\left(\frac{1}{x}\right) \leq x^{a-3} \\ 0 &= \lim_{x \rightarrow 0^+} -x^{a-2} \leq \lim_{x \rightarrow 0^+} x^{a-2} \cos\left(\frac{1}{x}\right) \leq \lim_{x \rightarrow 0^+} x^{a-2} = 0 \end{aligned}$$

Hence, $\lim_{x \rightarrow 0^+} \frac{f'_a(x) - f'_a(0)}{x - 0} = \lim_{x \rightarrow 0^+} ax^{a-2} \sin\left(\frac{1}{x}\right) - x^{a-3} \cos\left(\frac{1}{x}\right) = 0$, which agrees with the left limit. So, for all $a > 3$, $f''_a(0)$ is defined.

Question 4 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = e^{-\frac{1}{x^2}}$ if $x \neq 0$ and $f(0) = 0$. Show that f is infinitely differentiable and $\forall n \in \mathbb{N}, f^{(n)}(0) = 0$.

Pf:

First, we'll prove that for all $n \in \mathbb{N}$, $\lim_{x \rightarrow 0} \frac{1}{x^n} e^{-\frac{1}{x^2}} = 0$. By doing the substitution $h = \frac{1}{x}$, the expression becomes $\lim_{h \rightarrow \infty} h^n e^{-h^2}$.

For base cases $n = 0$, the limit $\lim_{h \rightarrow \infty} h^0 e^{-h^2} = \lim_{h \rightarrow \infty} e^{-h^2} = 0$ (since $e^{-h^2} = 1/e^{h^2}$, and e^{h^2} is not bounded). Same applies for another base case $n = 1$, the limit $\lim_{h \rightarrow \infty} h e^{-h^2} = \lim_{h \rightarrow \infty} \frac{h}{e^{h^2}}$. Since both h and e^{h^2} are not bounded, then apply L'hospital's Rule becomes:

$$\lim_{h \rightarrow \infty} \frac{h}{e^{h^2}} = \lim_{h \rightarrow \infty} \frac{1}{2he^{h^2}} = 0$$

The second part is true since he^{h^2} is not bounded. Which, the case is also true for $n = 1$.

Then, suppose for given $n \in \mathbb{N}$ and all integer $0 < k \leq n$, $\lim_{h \rightarrow \infty} h^k e^{-h^2} = 0$, for the case of $(n + 1)$, $\lim_{h \rightarrow \infty} h^{(n+1)} e^{-h^2} = \lim_{h \rightarrow \infty} \frac{h^{n+1}}{e^{h^2}}$, which both $h^{(n+1)}$ and e^{h^2} are not bounded in this limit. THus, apply L'hospital's Rule, the limit becomes:

$$\lim_{h \rightarrow \infty} \frac{h^{(n+1)}}{e^{h^2}} = \lim_{h \rightarrow \infty} \frac{(n+1)h^n}{2he^{h^2}} = \lim_{h \rightarrow \infty} \frac{(n+1)}{2} h^{n-1} e^{-h^2}$$

If $0 < (n + 1) < n$, then based on induction hypothesis, the above limit evalutes to be 0; if $(n - 1) = 0$, then it returns to the initial case, which again evaluates to be 0; else, if $(n - 1) < 0$, then the limit becomes $\lim_{h \rightarrow \infty} \frac{(n+1)}{2h^{1-n}e^{h^2}}$, where $(1 - n) > 0$. Thus, the denominator goes unbounded, the limit again evaluates to be 0.

So, by the Principle of Mathematical Induction, the limit $\lim_{x \rightarrow 0} \frac{1}{x^n} e^{-\frac{1}{x^2}} = \lim_{h \rightarrow \infty} h^n e^{-h^2} = 0$ for all $n \in \mathbb{N}$. And, if take finite linear combination of different powers, for any real-valued polynomial $p(h) = a_n h^n + \dots + a_0$, $p(1/x) e^{-\frac{1}{x^2}}$ also converges to 0 as x approaches 0 (since $p(1/x) e^{-\frac{1}{x^2}} = a_n (1/x^n) e^{-\frac{1}{x^2}} + \dots + a_0 e^{-\frac{1}{x^2}}$, where each individual component converges to 0 as x approaches 0).

Now, we can use induction to prove that for all $n \in \mathbb{N}$, the function $f(x) = e^{-\frac{1}{x^2}}$ has n^{th} derivative in the form $p(1/x) e^{-\frac{1}{x^2}}$ for some polynomial $p(h)$, and is infinitely differentiable at 0.

First, for base case $n = 1$, for all $x \neq 0$, $f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}$ by the differentiation rules, which let polynomial $p_1(h) = 2h^3$, then $f'(x) = p_1(1/x) e^{-\frac{1}{x^2}}$. Which, $\lim_{x \rightarrow 0} \frac{2}{x^3} e^{-\frac{1}{x^2}} = 0$, since $\lim_{x \rightarrow 0} \frac{1}{x^3} e^{-\frac{1}{x^2}} = 0$ follows from the statment proven previously.

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Question 5 *From the textbook solve exercises 2, 7 and 15 (first part) of Chapter 5.*