Math CS Topology HW5

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Question 1 *Prove that if X is connected and $x \in X$ then $X \times X \setminus \{(x,x)\}$ is connected*

Pf:

There are two cases to consider.

Suppose $X \times X = \{(x,x)\}$, then $X \times X \setminus \{(x,x)\} = \emptyset$ is vacuously connected. So, we can assume X is not a singleton for the rest of the proof.

Now, we'll prove the statement by contradiction.

Suppose X is connected, while $Y = X \times X \setminus \{(x,x)\}$ is disconnected, then there exists open subsets $U, V \subseteq X \times X$, such that $(U \cap Y)$ and $(V \cap Y)$ forms a separation of Y (i.e. $(U \cap Y) \cap (V \cap Y) = \emptyset$, and $(U \cap Y) \cup (V \cap Y) = Y$).

Since $(U \cap Y)$ and $(V \cap Y)$ are nonempty, there exists $(x_1, x_2) \in (U \cap Y)$, and $(x_3, x_4) \in (V \cap Y)$, and both points are not (x,x), so at least one coordinate is not x.

Then, since X is connected by assumption, while singletons are vacuously connected, then for all $y \in X$, the set $X \times \{y\}$ and $\{y\} \times X$ are connected subsets under the product topology of $X \times X$.

Notice that WLOG, we can assume that $x_2 \neq x$, and $x_3 \neq x$ in this case (so, can assume that the two points $(x_1, x_2), (x_3, x_4)$ have different coordinates that are different from x).

Under the most extreme case, we might have chosen (x_1, x) and (x_3, x) (where both points have x in the same coordinate). However, since $(\{x_1\} \times X) \subseteq Y$ is a connected subset (because $x_1 \neq x$, hence $(x,x) \notin \{x_1\} \times X$), while $(x_1, x) \in (U \cap Y)$, then $\{x_1\} \times X \subseteq (U \cap Y)$ (Note: if $(\{x_1\} \times X) \not\subseteq (U \cap Y)$, then $(\{x_1\} \times X) \cap (V \cap Y) \neq \emptyset$, showing that $(U \cap Y)$ and $(V \cap Y)$ actually forms a separation of $\{x_1\} \times X$, which contradicts the fact that $\{x_1\} \times X$ is connected).

Hence, choose another $x_2 \neq x$, we have $(x_1, x_2) \in \{x_1\} \times X \subseteq (U \cap Y)$, which (x_3, x) has $x_3 \neq x$, while (x_1, x_2) has $x_2 \neq x$.

Now, here comes the contradiction: Given the point $(x_3, x_2) \in X \times X \setminus \{(x,x)\}$.

Using the same logic as above, since $(x_3, x_4) \in (V \cap Y)$ by assumption, while $\{x_3\} \times X$ is connected, then $(\{x_3\} \times X) \subseteq (V \cap Y)$; similarly, since $(x_1, x_2) \in (U \cap Y)$ by assumption, while $X \times \{x_2\}$ is connected, then $(X \times \{x_2\}) \subseteq (U \cap Y)$.

So, $(x_3, x_2) \in (\{x_3\} \times X) \subseteq (V \cap Y)$, while also $(x_3, x_2) \in (X \times \{x_2\}) \subseteq (U \cap Y)$, showing that $(U \cap Y)$ and $(V \cap Y)$ actually has nontrivial intersection.

Yet, this contradicts the assumption that $(U \cap Y)$ and $(V \cap Y)$ forms a separation of Y. So, the initial assumption is false, $Y = X \times X \setminus \{(x,x)\}$ must be connected.

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Question 2 *Prove that every open connected subset of* $\mathbb{R}^2$ *is path connected.*

Pf:

Given an arbitrary open subset $U \subseteq \mathbb{R}^2$. If U is empty, then it is vacuously path connected, hence can assume $U \neq \emptyset$.

Now, choose any $u \in U$, consider the set $P \subseteq U$, such that every point $y \in P$, there exists a continuous path connecting the two points (So, there exists $f_y : [0,1] \to U$, with $f_y(0) = x$ and $f_y(1) = y$).

The set P is nonempty:

Since $x \in P$ (by choosing $f_x : [0,1] \to U$ with $f_x(t) = x$, it is a continuous function with $f_x(0) = f_x(1) = x$), then P is nonempty.

The set P is open:

Since U is an open subset, then there exists $\epsilon > 0$, with $B_\epsilon(x) \subseteq U$. Because every open ball is convex in vector space, then it is path connected (since for any $y, z \in B_\epsilon(x)$, choose $f : [0,1] \to U$ by $f(t) = ty + (1-t)z$, which is continuous; also, by convexity, $f([0,1]) \subseteq B_\epsilon(x)$, hence y, z are connected by paths). Therefore, $B_\epsilon(x) \subseteq P$.

Now, for all $y \in P \subseteq U$, since there exists $r > 0$ with $B_r(y) \subseteq U$, then because any open ball is path connected, then for all $z \in B_r(y)$, there exists continuous path $f : [0,1] \to U$ with $f(0) = y$ and $f(1) = z$. Then, since there exists continuous path $f_y : [0,1] \to U$ with $f_y(0) = x$ and $f_y(1) = y$ (because $y \in P$, so such path exists), define $f_z : [0,1] \to \mathbb{R}^2$ as follow:

$$f_z(t) = \begin{cases} f_y\,(2t) & t \in [0, \frac{1}{2}] \\ f\,(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Then, since $f_y(2t)$ and $f(2t-1)$ are continuous, while they agree on the point $\frac{1}{2}$ (since $f_y(2 \cdot \frac{1}{2}) = f_y(1) = y$, and $f(2 \cdot \frac{1}{2} - 1) = f(0) = y$), then by pasting lemma, f_z is a continuous function. And, because both f_y and f have the image lying in U, then f_z also has its image lie in U. Hence, since $f_z(0) = f_y(0) = x$ and $f_z(1) = f(2-1) = f(1) = z$, then f_z is a continuous path with image contained in U, that's joining x and z, showing that $z \in P$.

Therefore, $B_r(y) \subseteq P$, showing that P is in fact open. Hence, $P \subseteq U$ is open under subspace topology of U also.

The set P is closed:

It is the same as proving $U \setminus P$ is open under subspace topology of U.

For all $y \in U \setminus P$, there doesn't exist continuous path joining x and y. Also, since U is open, then there exists $r > 0$, with $B_r(y) \subseteq U$.

Notice that $B_r(y) \cap P = \emptyset$: Suppose the contrary that it is not empty, then there exists $z \in B_r(y) \cap P$. Which, there exists a path joining x and z; and since $B_r(y)$ is convex, it is also path connected. Therefore, there exists a path joining z and y. Then, using the similar method above, we can join the two paths to generate another continuous path that's starting from x and ending at y, showing that $y \in P$. Yet, this contradicts the assumption that $y \notin P$, hence the assumption is false, showing that $B_r(y) \cap P = \emptyset$.

This shows that $B_r(y) \subseteq U \setminus P$, hence $U \setminus P$ is open.

Which, because $U \setminus P$ is open in the subspace topology of U also, hence $P = U \setminus (U \setminus P)$ is closed under subspace topology.

$P = U$, and U is path connected:

Now, because P is proven to be both open and closed under subspace topology of U, while recalling that U is a connected subset of $\mathbb{R}^2$, hence the only subset of U that is both open and closed under its subspace topology, is $\emptyset$ and U itself.

Because P is nonempty, then $P = U$.

Now, for all $y, z \in P = U$, since there exists continuous paths $f_y, f_z : [0,1] \to U$, with $f_y(0) = x$ and $f_y(1) = y$, while $f_z(0) = x$ and $f_z(1) = z$. Hence, join the two paths together by $f : [0,1] \to U$ as follow:

$$f(t) = \begin{cases} f_y(1-2t) & [0, \frac{1}{2}] \\ f_z(2t-1) & [\frac{1}{2}, 1] \end{cases}$$

Because both f_y, f_z agrees at $\frac{1}{2}$ ($f_y(1-2\cdot\frac{1}{2}) = f_y(1-1) = f_y(0) = x$, and $f_z(2\cdot\frac{1}{2}-1) = f_z(1-1) = f_z(0) = x$), while both paths are continuous with image being in U, hence f is a continuous path with image in U, that is joining y and z (since $f(0) = f_y(1) = y$, and $f(1) = f_z(2-1) = f_z(1) = z$).

This shows that U is in fact path connected.

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Question 3 *Suppose X and Y are compact Hausdorff spaces. Prove that every continuous bijection $f : X \to Y$ is a homeomorphism.*

Pf:

Given that $f : X \to Y$ is a continuous bijection, while X, Y are both compact Hausdorff spaces. To prove that it is a homeomorphism, it suffices to prove that f is an open map.

(Note: Since if f is an open map, all open set $U \subseteq X$ satisfies $f(U) \subseteq Y$ is open, hence $(f^{-1})^{-1}(U) = f(U)$ is open, showing that for f^{-1}, all open subset in X has preimage in Y being open, implying that f^{-1} is continuous).

For all open set $U \subseteq X$, since $X \setminus U$ is closed, while X is compact, hence $X \setminus U$ is also compact. Then, since f is continuous, $f(X \setminus U) \subseteq Y$ is also compact. With the fact that Y is Hausdorff (which all compact subsets are closed), then $f(X \setminus U) \subseteq Y$ is open.

Now, since for all $u \in U$ (which $u \notin X \setminus U$), since f is bijective, all $v \in X \setminus U$ satisfies $f(u) \neq f(v)$, showing that $f(u) \notin f(X \setminus U)$. Hence, $f(u) \in Y \setminus f(X \setminus U)$, or $f(U) \subseteq Y \setminus f(X \setminus U)$.

Also, for all $y \in Y \setminus f(X \setminus U)$, since f is a bijection, there exists $x \in X$, with $y = f(x)$. However, since $y \notin f(X \setminus U)$, this implies that $x \notin (X \setminus U)$, showing that $x \in U$. Hence, $y = f(x) \in f(U)$, or $(Y \setminus f(X \setminus U)) \subseteq f(U)$.

The above two implication proves that $f(U) = Y \setminus f(X \setminus U)$. Since $f(X \setminus U)$ is closed in Y, then $f(U)$ must be open in Y. This proves that f is an open map. Combine with the assumption that it is a continuous bijection, then f is in fact a homeomorphism.

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Question 4 *Let X be a totally ordered set with the order topology. Prove that every nonempty compact subset of X has a maximum element.*

Pf:

Suppose a nonempty subset $K \subseteq X$ is compact. For each $\alpha \in K$, let $\Theta_\alpha = \{x \in X \mid x < \alpha\}$ (a ray in ordered topology, which is open). Then, consider the collection of open sets $\mathcal{F} = \{\Theta_\alpha \mid \alpha \in B\}$.

Notice that this can never be an open cover of K: Suppose the contrary, that $\mathcal{F}$ actually forms an open cover of K, then since K is compact, there exists $\alpha_1, ..., \alpha_n \in K$, such that $K \subseteq \bigcup_{i=1}^n \Theta_{\alpha_i}$.

Yet, since in a totally ordered set, a finite collection of elements has a maximum, hence, let $\alpha = \max\{\alpha_1, ..., \alpha_n\} \in K$, since for each $i \in \{1, ..., n\}$, $\alpha_i \leq \alpha$, then by the definition, $\alpha \notin \Theta_{\alpha_i}$ (since $\alpha < \alpha_i$ is false). Hence, $\alpha \notin \bigcup_{i=1}^n \Theta_{\alpha_i}$.

So, $\alpha \in K \subseteq \bigcup_{i=1}^n \Theta_{\alpha_i}$, while $\alpha \notin \bigcup_{i=1}^n \Theta_{\alpha_i}$, this is a contradiction. Therefore, the assumption is false, $\mathcal{F}$ cannot be an open cover of K.

Since $\mathcal{F}$ is not an open cover of K, there exists $k \in K$, with $k \notin \bigcup \mathcal{F}$. Now, we claim that k is in fact the maximum of K:

For all $\alpha \in K$, since $k \notin \bigcup \mathcal{F} = \bigcup_{\alpha' \in K} \Theta_{\alpha'}$, then $k \notin \Theta_\alpha$. Hence, $k < \alpha$ is false, showing that $\alpha \leq k$. This shows that k is actually a maximum element of K, hence all compact subset of X under order topology has a maximum element.