Math 118B HW5

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1 (Part b not done)

Question 1

(a) Show that there exists a sequence of polynomials $q_m : [0,1] \to \mathbb{R}$ *such that for each* $x \in [0,1]$

$$\lim_{m\to\infty} q_m(x) = 0$$

(pointwise convergence) but it does not converge uniformly.

(b) Prove that if a sequence of polynomial $p_m : [0,1] \to \mathbb{R}$ *converges pointwise to* 0 *and for all* $m \in \mathbb{N}$ *one has that* $\deg(p_m) \leq 100$*, then the* p_m *converges uniformly to* 0*.*

Pf:

(a) **Continuous Functions Converging to** 0 **Pointwise, but not Uniformly:**

We'll first construct a sequence of continuous functions converging to 0 pointwise, but not uniformly. For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined as:

$$f_n(x) = \begin{cases} 4nx - 2 & x \in [\frac{2}{4n}, \frac{3}{4n}] \\ -4nx + 4 & x \in (\frac{3}{4n}, \frac{4}{4n}] \\ 0 & x \notin [\frac{2}{4n}, \frac{4}{4n}] \end{cases}$$

This is a continuous function for all $n \in \mathbb{N}$, since the limit at $\frac{3}{4n}$, $\frac{2}{4n}$, and $\frac{4}{4n}$ all agrees with the function f_n's actual values.

However, since at $x = \frac{3}{4n} \in [0,1]$, $f_n(x) = 4n \cdot \frac{3}{4n} - 2 = 3 - 2 = 1$, then $\|f_n\|_\infty = \sup_{x\in[0,1]} |f_n(x)| \geq 1$, showing that f_n doesn't converge to 0 uniformly (since the norm $\|\cdot\|_\infty$ is at least 1 for all $n \in \mathbb{N}$).

Sequence of Polynomials:

Now, since f_n is continuous on $[0,1]$, by Stone-Weierstrass Theorem, there exists a sequence of polynomials $\{q_{n,k}\}_{k\in\mathbb{N}}$ that converges to f_n uniformly.

For all $n \in \mathbb{N}$, since $\frac{1}{n} > 0$, by the uniform convergence of $\{q_{n,k}\}_{k\in\mathbb{N}}$ onto f_n, there exists N_n, such that $k_n \geq N_n$ implies $\|f_n - q_{n,k_n}\|_\infty < \frac{1}{n}$ (for simplicity, fix k_n to be the smallest integer with $k_n \geq N_n$). For the rest of the proof of **Part (a)**, consider the sequence of polynomials $\{q_{n,k_n}\}_{n\in\mathbb{N}}$.

The Sequence Pointwise Converges to 0:

For all $x \in [0,1]$, there are two cases to consider:

- First, if $x = 0$, for all $n \in \mathbb{N}$, we have $f_n(0) = 0$. Then, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$, with $\frac{1}{N} < \epsilon$ based on Archimedean's Property. For all $n \geq N$ (which $\frac{1}{n} \leq \frac{1}{N} < \epsilon$), the previous choice of polynomials satisfy:

$$|q_{n,k_n}(0)| = |q_{n,k_n}(0) - f_n(0)| \leq \|q_{n,k_n} - f_n\|_\infty < \frac{1}{n} < \epsilon$$

  Hence, this states that $\lim_{n\to\infty} q_{n,k_n}(0) = 0$.

- Else if $x \neq 0$ (which $x > 0$ since $x \in [0,1]$), there exists $N \in \mathbb{N}$, such that $\frac{1}{N} < x$ based on Archimedean's Property. Then, for all $n \geq N$, since $\frac{4}{4n} = \frac{1}{n} \leq \frac{1}{N} < x$, $f_n(x) = 0$ (since $x \notin [\frac{2}{4n}, \frac{4}{4n}]$).
  Again, for all $\epsilon > 0$, there exists $M \in \mathbb{N}$, with $\frac{1}{M} < \epsilon$ again based on Archimedean's Property. Choose $K = \max\{M, N\}$, for all $n \geq K$ (which $n \geq N$, showing that $f_n(x) = 0$; and $n \geq M$, showing that $\frac{1}{n} \leq \frac{1}{M} < \epsilon$), the previous choice of polynomials satisfy:

$$|q_{n,k_n}(x)| = |q_{n,k_n}(x) - f_n(x)| \leq \|q_{n,k_n} - f_n\|_\infty < \frac{1}{n} < \epsilon$$

  Hence, this states that $\lim_{n\to\infty} q_{n,k_n}(x) = 0$.

So, regardless of the case, $\lim_{n\to\infty} q_{n,k_n}(x) = 0$, showing that $\{q_{n,k_n}\}_{n\in\mathbb{N}}$ converges pointwise to 0.

The Convergence is not Uniform:

Recall that for all $n \in \mathbb{N}$, $\|f_n\|_\infty \geq 1$, and $\|f_n - q_{n,k_n}\|_\infty < \frac{1}{n}$. Hence, for $n \geq 2$ (which $\frac{1}{n} \leq \frac{1}{2}$), the following inequality is true:

$$\|q_{n,k_n}\|_\infty = \|(q_{n,k_n} - f_n) - (-f_n)\|_\infty \geq \Big| \|q_{n,k_n} - f_n\|_\infty - \|-f_n\|_\infty \Big| = \|f_n\|_\infty - \|q_{n,k_n} - f_n\|_\infty$$

$$\|q_{n,k_n}\|_\infty \geq \|f_n\|_\infty - \|q_{n,k_n} - f_n\|_\infty \geq 1 - \|q_{n,k_n} - f_n\|_\infty > 1 - \frac{1}{n} \geq 1 - \frac{1}{2} = \frac{1}{2}$$

So, since $\|_\infty \geq \frac{1}{2}$ for all $n \geq 2$, the $\lim_{n\to\infty} \|q_{n,k_n}\|_\infty \neq 0$, showing that $\{q_{n,k_n}\}_{n\in\mathbb{N}}$ doesn't converge to 0 uniformly.

In Conclusion, $\{q_{n,k_n}\}_{n\in\mathbb{N}}$ constructed above, is a sequence of polynomial that converges pointwise to 0, yet it doesn't converge uniformly to 0. Which, it is a desired sequence for the question.

(b) Let $\mathcal{P}_{100}([0,1])$ be the real vector space of polynomial defined on $[0,1]$ with degree at most 100 (which $\dim(\mathcal{P}_{100}([0,1])) = 101$). For this part, the sequence $\{p_m\}_{m\in\mathbb{N}} \subset \mathcal{P}_{100}([0,1])$, and they converges pointwise to 0.

Now, choose distinct points $x_1, x_2, ..., x_{101} \in [0,1]$, and define the map $T : \mathcal{P}_{100}([0,1]) \to \mathbb{R}^{101}$ by:

$$T(p) = (p(x_1), p(x_2), ..., p(x_{101}))$$

The map T is a Linear Map:

For the zero function $0 \in \mathcal{P}_{100}([0,1])$, it is clear that $T(0) = (0, 0, ..., 0) \in \mathbb{R}^{101}$.

Then, for all $p, q \in \mathcal{P}_{100}([0,1])$:

$$T(p+q) = ((p+q)(x_1), (p+q)(x_2), ..., (p+q)(x_{101})) = (p(x_1)+q(x_1), p(x_2)+q(x_2), ..., p(x_{101})+q(x_{101}))$$

$$= (p(x_1), p(x_2), ..., p(x_{101})) + (q(x_1), q(x_2), ..., q(x_{101})) = T(p) + T(q)$$

Also, for all $\lambda \in \mathbb{R}$ and $p \in \mathcal{P}_{100}([0,1])$:

$$T(\lambda p) = ((\lambda p)(x_1), (\lambda p)(x_2), ..., (\lambda p)(x_{101})) = (\lambda \cdot p(x_1), \lambda \cdot p(x_2), ..., \lambda \cdot p(x_{101}))$$

$$= \lambda(p(x_1), p(x_2), ..., p(x_{101})) = \lambda T(p)$$

Hence, with the above three criteria, T is a linear map from $\mathcal{P}_{100}([0,1]) \to \mathbb{R}^{101}$.

The map T is Bijective:

Since both $\mathcal{P}_{100}([0,1])$ and $\mathbb{R}^{101}$ have dimension 101, then showing T is bijective is equivalent to showing T is injective.

Suppose $p \in \ker(T)$ (or $T(p) = (0, 0, ..., 0) \in \mathbb{R}^{101}$), since for all $i \in \{1, 2, ..., 101\}$, it has $p(x_i) = 0$, then p has at least 101 distinct zeroes. However, since $p \in \mathcal{P}_{100}([0,1])$, then its degree is at most 100. By Fundamental Theorem of Algebra, if $p \neq 0$, it has at most 100 distinct roots. Hence, $p = 0$ is required.

So, $\ker(T) = \{0\}$, showing that T is injective, hence bijective. So, T^{-1} exists.

The map T is Continuous:

Let the usual dot product define the norm $\|\cdot\|_2$ of $\mathbb{R}^{101}$, and let $\|\cdot\|_\infty$ be the norm of $\mathcal{P}_{100}([0,1])$.

For all $\epsilon > 0$, let $\delta = \frac{\epsilon}{\sqrt{101}} > 0$, for all $p, q \in \mathcal{P}_{100}([0,1])$, if $\|p - q\|_\infty < \delta = \frac{\epsilon}{\sqrt{101}}$, then the output satisfies:

$$T(p) - T(q) = T(p - q) = ((p-q)(x_1), (p-q)(x_2), ..., (p-q)(x_{101}))$$

$$\|T(p) - T(q)\|_2 = \sqrt{\sum_{i=1}^{101} |(p-q)(x_i)|^2} \leq \sqrt{\sum_{i=1}^{101} \|p-q\|_\infty^2} < \sqrt{\sum_{i=1}^{101} \left(\frac{\epsilon}{\sqrt{101}}\right)^2}$$

$$\|T(p) - T(q)\|_2 < \sqrt{\sum_{i=1}^{101} \frac{\epsilon^2}{101}} = \sqrt{101 \cdot \frac{\epsilon^2}{101}} = \sqrt{\epsilon^2} = |\epsilon| = \epsilon$$

Hence, $\|p - q\|_\infty < \delta$ implies $\|T(p) - T(q)\|_2 < \epsilon$, showing that T is in fact uniformly continuous.

The map T^{-1} is Continuous:

For this section, there are several arguments needed to be done before heading to conclusion. We'll define inner product $\langle \cdot, \cdot \rangle_{int} : \mathcal{P}_{100}([0,1]) \times \mathcal{P}_{100}([0,1]) \to \mathbb{R}$ by $\langle f, g \rangle = \int_0^1 fg dx$, and the norm $\|f\|_{int} = \langle f, f \rangle^{\frac{1}{2}}$.

- **With Respect to $\|\cdot\|_{int}$, T^{-1} is Continuous:**

  Recall that $T^{-1} : \mathbb{R}^{101} \to \mathcal{P}_{100}([0,1])$ is a linear map. Let $\mathbb{R}^{101}$ uses regular dot product as inner product, and $\mathcal{P}_{100}([0,1])$ uses $\langle \cdot, \cdot \rangle_{int}$ as inner product,and the norm for each space correspond to the given inner product.

  Then, by Singular Value Decomposition, because the map is bijective, there exists orthonormal basis $e_1, ..., e_{101} \in \mathbb{R}^{101}$, $f_1, ..., f_{101} \in \mathcal{P}_{100}([0,1])$, and positive real numbers $s_1, ..., s_{101} \in \mathbb{R}$, such that the following is true:

  $$\forall v \in \mathbb{R}^{101}, \quad T^{-1}(v) = \sum_{j=1}^{101} s_j (v \cdot e_j) f_j$$

  Hence, let $s = \max\{s_1, ..., s_{101}\} > 0$, for all $v \in \mathbb{R}^{101}$, the below inequality is satisfied:

  $$\|T^{-1}(v)\|_{int}^2 = \left\| \sum_{j=1}^{101} s_j (v \cdot e_j) f_j \right\|_{int}^2 = \sum_{j=1}^{101} \|s_j (v \cdot e_j) f_j\|_{int}^2 = \sum_{j=1}^{101} s_j^2 (v \cdot e_j)^2$$

  $$\|T^{-1}(v)\|_{int}^2 = \sum_{j=1}^{101} s_j^2 (v \cdot e_j)^2 \leq \sum_{j=1}^{101} s^2 (v \cdot e_j)^2 = s^2 \|v\|^2$$

  $$\|T^{-1}(v)\|_{int} \leq s\|v\|$$

  (Note: $v \cdot e_j$ denotes the dot product. The above is true based on Pythagorean Theorem, since the list $f_1, ..., f_{101}$ is orthonormal).

  Hence, for all $\epsilon > 0$, let $\delta = \frac{\epsilon}{s} > 0$, for all $u, v \in \mathbb{R}^{101}$ satisfying $\|u - v\| < \delta = \frac{\epsilon}{s}$, the following is true:

  $$\|T^{-1}(u) - T^{-1}(v)\|_{int} = \|T^{-1}(u - v)\|_{int} \leq s\|u - v\| < s \cdot \frac{\epsilon}{s} = \epsilon$$

  Hence, based on the norm from the inner product $\langle \cdot, \cdot \rangle_{int}$ for $\mathcal{P}_{101}([0,1])$, the linear map T^{-1} is continuous.

- **The Norms are Equivalent:**

  Given $\|\cdot\|_\infty$ and $\|\cdot\|_{int}$ the two norms, consider the set $B = \{p \in \mathcal{P}_{100}([0,1]) \mid \|p\|_{int} = 1\}$ (set of polynomials with norm 1 under $\|\cdot\|_{int}$). Because it is a compact set (unit sphere) under the topology generated by $\|\cdot\|_{int}$, and the norm function is always continuous:

  Let $p_1, ..., p_{101} \in \mathcal{P}_{100}([0,1])$ be an orthonormal basis with respect to $\langle \cdot, \cdot \rangle_{int}$. Which, $D = \max_{i \in \{1, ..., 101\}} \{\|p_i\|_\infty\} > 0$ exists. Now, for all $\epsilon > 0$, let $\delta = \frac{\epsilon}{101D} > 0$.

  Suppose $p, q \in \mathcal{P}_{100}([0,1])$ satisfy $\|p - q\|_{int} < \delta = \frac{\epsilon}{101D}$.

  Notice that $(p - q) = \sum_{i=1}^{101} a_i p_i$, where $\|p - q\|_{int}^2 = \sum_{i=1}^{101} |a_i|^2$. Then, the following is true:

  $$\Big| \|p\|_\infty - \|q\|_\infty \Big| \leq \|p - q\|_\infty = \left\| \sum_{i=1}^{101} a_i p_i \right\|_\infty \leq \sum_{i=1}^{101} |a_i| \cdot \|p_i\|_\infty \leq \sum_{i=1}^{101} |a_i| D$$

$$\left| \|p\|_\infty - \|q\|_\infty \right| \leq \sum_{i=1}^{101} |a_i| D = D \sum_{i=1}^{101} \sqrt{a_i^2} \leq D \sum_{i=1}^{101} \sqrt{\sum_{j=1}^{101} a_j^2} = D \sum_{i=1}^{101} \sqrt{\|p-q\|_{int}^2} = 101D\|p-q\|_{int}$$

$$\left| \|p\|_\infty - \|q\|_\infty \right| \leq 101D\|p-q\|_{int} < 101D \cdot \frac{\epsilon}{101D} = \epsilon$$

Hence, under the topology generated by $\|\cdot\|_{int}$, $\|p-q\|_{int} < \delta$ implies $\left| \|p\|_\infty - \|q\|_\infty \right| < \epsilon$, showing that the norm function $\|\cdot\|_\infty$ is continuous.

Then, since B is compact and $\|\cdot\|_\infty$ is continuous, the set $\|B\|_\infty \subset \mathbb{R}$ is compact, there exists a minimum m and maximum M of the set $\|B\|_\infty$, which there exists $p, q \in B$, with $\|p\|_\infty = m$ and $\|q\|_\infty = M$.
Notice that since $p, q \in B$, then $p, q \neq 0$ (because they have nonzero norm for one of the norms), hence $m = \|p\|_\infty, M = \|q\|_\infty > 0$.
Now, for all $p \in \mathcal{P}_{100}([0,1])$ with $p \neq 0$, notice that the following is true:

$$\left\| \frac{p}{\|p\|_{int}} \right\|_i nt = 1, \quad \frac{p}{\|p\|_{int}} \in B, \quad m \leq \left\| \frac{p}{\|p\|_{int}} \right\|_\infty \leq M$$

$$m\|p\|_{int} \leq \|p\|_\infty \leq M\|p\|_{int}$$

And, the above inequality is true for 0 regardless. So, we can claim that the two norms are in fact equivalent.

- **WIth Respect to $\|\cdot\|_\infty$, T^{-1} is Continuous:**
  Because the two norms are equivalent, then since T^{-1} is continuous with respect to the norm $\|\cdot\|_{int}$, it is also continuous with respect to the norm $\|\cdot\|_\infty$, which is equivalent to the previous one. Hence, we reach the desired result.

The Sequence of Polynomial Converges Uniformly to 0:

Recall that since $\{p_m\}_{m\in\mathbb{N}}$ converges pointwise to 0, then for all $i \in \{1, ..., 101\}$, $\lim_{m\to\infty} p_m(x_i) = 0$.
Also, from the previous section, since T^{-1} is continuous (possibly on a restricted domain), for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $u \in \mathbb{R}^{101}$, $\|u\|_2 < \delta$ implies $\|T^{-1}(u)\|_\infty < \epsilon$.

Using the pointwise convergence, for the given $\delta > 0$ (which $\frac{\delta}{\sqrt{101}} > 0$), each $i \in \{1, ..., 101\}$ has a corresponding M_i, such that $m \geq M_i$ implies $|p_m(x_i)| < \frac{\delta}{\sqrt{101}}$.
Then, let $M = \max_{i\in\{1,...,101\}}\{M_i\}$, for all $m \geq M$ (which $m \geq M_i$ for all $i \in \{1, ..., 101\}$), the following is true:

$$\|T(p_m) - T(0)\|_2 = \|T(p_m)\|_2 = \sqrt{\sum_{i=1}^{101} |p_m(x_i)|^2} < \sqrt{\sum_{i=1}^{101} \left(\frac{\delta}{\sqrt{101}}\right)^2} = \sqrt{101 \cdot \frac{\delta^2}{101}} = \sqrt{\delta^2} = |\delta| = \delta$$

Hence, by the continuity of T^{-1}, $T^{-1}(T(p_m)) = p_m$ satisfies $\|T^{-1}(T(p_m))\|_\infty < \epsilon$, or $\|p_m\|_\infty < \epsilon$.
Therefore, this concludes that the sequence of polynomials p_m converges to 0 uniformly.

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Question 2 *Let $f : [0,1] \to \mathbb{R}$ be a function such that $f', f'', f^{(3)}$ are defined and continuous in* $[0,1]$. *Prove that for any $\epsilon > 0$ there exists a polynomial P such that*

$$\sum_{j=0}^{3} \|f^{(j)} - P^{(j)}\|_\infty = \sum_{j=0}^{3} \sup_{x\in[0,1]} |(f^{(j)} - P^{(j)})(x)| < \epsilon$$

Pf:

Before starting the prove, recall that the antiderivatives of a polynomial $p : [0,1] \to \mathbb{R}$ is a collection of polynomials $\{P(x) + C \mid C \in \mathbb{R}\}$, where $P : [0,1] \to \mathbb{R}$ is a polynomial satisfying $P' = p$.

When taking the antiderivative of any polynomial in the following steps, we'll explicitly state the initial condition to prevent ambiguity about the constant coefficients of the antiderivative.

Generalized Statement:

We'll prove a more general version recursively: For all $n \in \mathbb{N}$, let $f : [0,1] \to \mathbb{R}$ be a function such that $f', ..., f^{(n)}$ are all defined and continuous on $[0,1]$, then there exists a sequence of polynomials $\{P_m\}_{m\in\mathbb{N}}$, such that for all $j \in \{0,1,...,n\}$, $P_m^{(j)}$ converges to $f^{(j)}$ uniformly.

For base case, since $f^{(n)}$ is defined and continuous on $[0,1]$, by Stone-Weierstrass Theorem, there exists a sequence of polynomials $\{p_{n,m}\}$ converging to $f^{(n)}$ uniformly.

Then as **Step (1)**, for all $m \in \mathbb{N}$, let polynomial $p_{(n-1),m} : [0,1] \to \mathbb{R}$ be an antiderivative of $p_{n,m}$ ($p'_{(n-1),m} = p_{n,m}$) such that $p_{(n-1),m}(0) = f^{(n-1)}(0)$.

Which, since the sequence of polynomials $\{p_{(n-1),m}\}_{m\in\mathbb{N}}$ satisfies: $p'_{(n-1),m} = p_{n,m}$ converges to $(f^{(n-1)})' = f^{(n)}$ uniformly, and $\lim_{m\to\infty} p_{(n-1),m}(0) = f^{(n-1)}(0)$. Then, the sequence $p_{(n-1),m}$ converges to $f^{(n-1)}$ uniformly.

Now, for given $k \in \{1,...,n-1\}$, at **Step (k)** we constructed a sequence of k^{th} antiderivative of the sequence of polynomials $\{p_{n,m}\}_{m\in\mathbb{N}}$ (denoted as $\{p_{(n-k),m}\}_{m\to\mathbb{N}}$), such that $p_{(n-k),m}$ converges to $f^{(n-k)}$ uniformly:

At **Step (k+1)**, for each $m \in \mathbb{N}$, let polynomial $p_{(n-(k+1)),m} : [0,1] \to \mathbb{R}$ be an antiderivative of $p_{(n-k),m}$ (which $p'_{(n-(k+1)),m} = p'_{(n-k),m}$) such that $p_{(n-(k+1)),m}(0) = f^{(n-(k+1))}(0)$.

Which, since the new sequence of polynomials $\{p_{(n-(k+1)),m}\}_{m\in\mathbb{N}}$ satisfies: $p'_{(n-(k+1)),m} = p_{(n-k),m}$ converges to $(f^{(n-(k+1))})' = f^{(n-k)}$, and $\lim_{m\to\infty} p_{(n-(k+1)),m}(0) = f^{(n-(k+1))}(0)$. Then, the sequence $p_{(n-(k+1)),m}$ converges to $f^{(n-(k+1))}$ uniformly.

From the above process, since for all $k \in \{1,...,n\}$, we can find a sequence of k^{th} antiderivative of polynomials $\{p_{n,m}\}_{m\in\mathbb{N}}$, denoted as $\{p_{(n-k),m}\}_{m\in\mathbb{N}}$, that converges to $f^{(n-k)}$ uniformly.

Then, the sequence $\{p_{0,m}\}_{m\in\mathbb{N}}$ is a sequence of polynomial that converges to $f^{(0)} = f$ uniformly. Which, for $j \in \{1,...,n\}$, the sequence of j^{th} derivative $\{p_{j,m}\}_{m\in\mathbb{N}}$ converges uniformly to the j^{th} derivative of f, namely $f^{(j)}$. (Note: Recall that for all $j \in \{1,...,n\}$ and all $m \in \mathbb{N}$, $p_{(j-1),m}$ is defined as an antiderivative of $p_{j,m}$).

Hence, the sequence of polynomials $\{p_{0,m}\}_{m\in\mathbb{N}}$ has its j^{th} derivative converges to $f^{(j)}$ uniformly for all given $f^{(j)}$, satisfying the desired condition stated initially.

The Original Problem:

From the above Generalized Statement, given $f : [0,1] \to \mathbb{R}$ such that $f', f'', f^{(3)}$ that are all defined and continuous on $[0,1]$, there exists a sequence of polynomials $\{P_m\}_{m\in\mathbb{N}}$, such that for $j \in \{0,1,2,3\}$, its j^{th} derivative $P_m^{(j)}$ converges to $f^{(j)}$ uniformly.

Hence, given arbitrary $\epsilon > 0$ (which $\frac{\epsilon}{4} > 0$), for each $j \in \{0,1,2,3\}$, there is a corresponding N_j, such that the following is true:

$$\forall m \in \mathbb{N}, \quad m \geq N_j \implies \|f^{(j)} - P_m^{(j)}\|_\infty < \frac{\epsilon}{4}$$

Then, choose $N = \max_{j\in\{0,1,2,3\}} N_j$, for any index $m \geq N$, since $m \geq N_j$ for all $j \in \{0,1,2,3\}$, the above statement guarantees $\|f^{(j)} - P_m^{(j)}\|_\infty < \frac{\epsilon}{4}$ for each j. Hence, the following inequality is true:

$$\sum_{j=0}^{3} \|f^{(j)} - P_m^{(j)}\|_\infty < \sum_{j=0}^{3} \frac{\epsilon}{4} = \epsilon$$

Therefore, for every $\epsilon > 0$, we can find a corresponding polynomial P, such that $\sum_{j=0}^{3} \|f^{(j)} - P^{(j)}\|_\infty < \epsilon$.

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Question 3 *Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that*

$$\int_0^1 f(x)x^j dx = 0, \qquad j = 0, 1, 2, ...$$

Prove that $f(x) = 0$, $\forall x \in [0,1]$.

Pf:

Since $f(x)$ is continuous on $[0,1]$ a bounded closed interval, by Stone-Weierstrass Theorem, there exists a sequence of polynomial $\{p_n\}_{n\in\mathbb{N}}$, such that p_n converges to f uniformly.

Now, notice that for all polynomial $p(x) = a_0 + a_1x + ... + a_mx^m$ (where $a_0, a_1, ..., a_m \in \mathbb{R}$), the following integral is true based on the Linearity of Riemann Integrable functions:

$$\int_0^1 f(x)p(x)dx = \int_0^1 \sum_{k=0}^m a_kx^k dx = \sum_{k=0}^m a_k \int_0^1 f(x)x^k dx = 0$$

Hence, for all $n \in \mathbb{N}$, we have $\int_0^1 f(x)p_n(x)dx = 0$.

fp_n Converges Uniformly to f^2:

Because f is continuous on $[0,1]$ a compact set, hence f is bounded, there exists $M > 0$, such that all $x \in [0,1]$ satisfies $|f(x)| < M$.

Also, since p_n converges to f uniformly, for all $\epsilon > 0$ (which $\frac{\epsilon}{M} > 0$), there exists N, such that $n \geq N$ implies $\|f - p_n\|_\infty < \frac{\epsilon}{M}$.

Hence, for all $n \geq N$, every $x \in [0,1]$ satisfies the following:

$$|f(x)p_n(x) - (f(x))^2| = |f(x)| \cdot |p_n(x) - f(x)| < M \cdot |p_n(x) - f(x)| \leq M \cdot \|f - p_n\|_\infty < M \cdot \frac{\epsilon}{M} < \epsilon$$

Hence, ϵ is an upper bound of the set $\{|f(x)p_n(x) - (f(x))^2| \mid x \in [0,1]\}$, showing that $\|fp_n - f^2\|_\infty = \sup_{x\in[0,1]} |f(x)p_n(x) - (f(x))^2| \leq \epsilon$. Based on the above statement, we can conclude that fp_n converges uniformly to f^2.

Integral of fp_n converges to Integral of f^2:

For all $n \in \mathbb{N}$, we have fp_n being continuous on $[0,1]$ (since both f and p_n are continuous on $[0,1]$), and fp_n converges to f^2 uniformly, hence the following is true:

$$\lim_{n\to\infty} \int_0^1 f(x)p_n(x)dx = \int_0^1 \lim_{n\to\infty} f(x)p_n(x)dx = \int_0^1 (f(x))^2 dx$$

Since $\int_0^1 f(x)p_n(x)dx = 0$, then the limit above is 0, hence $\int_0^1 (f(x))^2 dx = 0$.

Integral of f^2 is 0 implies $f = 0$:

Since f is continuous on $[0,1]$, so does f^2; then, since for all $x \in [0,1]$, $(f(x))^2 \geq 0$, together with the statement $\int_0^1 (f(x))^2 dx = 0$, this implies that $(f(x))^2 = 0$ for all $x \in [0,1]$.

Therefore, $f(x) = 0$ for all $x \in [0,1]$.