

# Math 118B HW 1

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**Question 1** *Use just the definition, prove:*

(a)  $f : \mathbf{R} \rightarrow \mathbf{R}$ ,  $f(x) = x^2$ , *is not uniformly continuous.*

(b)  $f : (-10^6, 10^6) \rightarrow \mathbf{R}$ ,  $f(x) = x^2$ , *is uniformly continuous.*

(a) Given  $f : \mathbf{R} \rightarrow \mathbf{R}$ ,  $f(x) = x^2$ .

Choose  $\epsilon = 1$ . For any  $\delta > 0$ , by Archimedean Property, there exists  $n \in \mathbf{N}$  with  $1 < n\delta$ , which implies that  $\frac{1}{n} < \delta$ . Then, consider  $(n + \frac{1}{n}), n \in \mathbf{R}$ :

First,  $|(n + \frac{1}{n}) - n| = |\frac{1}{n}| < \delta$ . Also, the following is true:

$$|f(n + \frac{1}{n}) - f(n)| = |(n + \frac{1}{n})^2 - n^2| = |\frac{1}{n}(2n + \frac{1}{n})| = (2 + \frac{1}{n^2}) > 1 = \epsilon$$

So, for any given  $\delta > 0$ , there exists  $x_1, x_2 \in \mathbf{R}$ , with  $|x_1 - x_2| < \delta$ , but  $|f(x_1) - f(x_2)| \geq \epsilon$ , proving that  $f : \mathbf{R} \rightarrow \mathbf{R}$ ,  $f(x) = x^2$ , is not uniformly continuous.

(b) Given  $f : (-10^6, 10^6) \rightarrow \mathbf{R}$ ,  $f(x) = x^2$ .

For all  $\epsilon > 0$ , let  $\delta = \epsilon/(2 \cdot 10^6)$ . Then, for all  $x, y \in (-10^6, 10^6)$ , suppose  $|x - y| < \delta$ , then consider  $|f(x) - f(y)|$ :

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y| \cdot |x + y| < \delta \cdot |x + y|$$

Which, since  $x, y \in (-10^6, 10^6)$ , then  $|x|, |y| < 10^6$ . So,  $|x + y|$  is limited by the following inequality:

$$|x + y| \leq |x| + |y| < 10^6 + 10^6 = 2 \cdot 10^6$$

Thus the following is true:

$$|f(x) - f(y)| < \delta \cdot |x + y| < \delta \cdot (2 \cdot 10^6)$$

$$|f(x) - f(y)| < (2 \cdot 10^6) \cdot \epsilon/(2 \cdot 10^6) = \epsilon$$

So, the above proves that  $f : (-10^6, 10^6) \rightarrow \mathbf{R}$ ,  $f(x) = x^2$ , is uniformly continuous.

**Question 2** Given  $(X, d_X)$ ,  $(Y, d_Y)$  two metric spaces. Let  $f : X \rightarrow Y$  be an uniformly continuous function.

- (a) Prove that if  $(x_n)_{n=1}^\infty$  is a cauchy sequence in  $X$ , then  $(f(x_n))_{n=1}^\infty$  is a cauchy sequence in  $Y$ .
- (b) Given an example of  $g : (0, 1) \rightarrow \mathbf{R}$  continuous,  $(x_n)_{n=1}^\infty$  is a cauchy sequence in  $X$  and  $(g(x_n))_{n=1}^\infty$  is not a cauchy sequence in  $Y$ .
- (c) Prove that if  $f : (0, 1) \rightarrow \mathbf{R}$  is uniformly continuous, it can be extended continuously to  $[0, 1]$ .

- (a) Suppose  $(x_n)_{n=1}^\infty$  is a Cauchy sequence in  $X$ . Then, consider  $(f(x_n))_{n=1}^\infty$ :

Since  $f$  is uniformly continuous, for all  $\epsilon > 0$ , there exists  $\delta > 0$ , such that for all  $x, x' \in X$ ,  $d_X(x, x') < \delta$  implies  $d_Y(f(x), f(x')) < \epsilon$ .

Also, since  $(x_n)_{n=1}^\infty$  is Cauchy, for the given  $\delta > 0$  above, there exists  $N$ , such that  $m, n \geq N$  implies  $d_X(x_m, x_n) < \delta$ .

Which, for this specific  $N$ , since  $m, n \geq N$  implies  $d_X(x_m, x_n) < \delta$ , and by the definition of Uniform Continuity, this further implies that  $d_Y(f(x_m), f(x_n)) < \epsilon$ . So, for all  $\epsilon > 0$ , there exists such  $N$ , such that  $m, n \geq N$  implies  $d_Y(f(x_m), f(x_n)) < \epsilon$ . This proves that  $(f(x_n))_{n=1}^\infty$  is a Cauchy Sequence.

- (b) Consider  $g : (0, 1) \rightarrow \mathbf{R}$  defined as  $g(x) = \frac{1}{x}$ , and let  $(x_n)_{n=1}^\infty$  be defined as  $x_n = \frac{1}{2^n}$  for all  $n \in \mathbf{N}$ .

First, we'll prove that  $(x_n)_{n=1}^\infty$  is Cauchy: For all  $\epsilon > 0$ , since  $\frac{\epsilon}{2} > 0$ , there exists  $k \in \mathbf{N}$ , such that  $1 < k\frac{\epsilon}{2}$ , which  $\frac{1}{k} < \frac{\epsilon}{2}$ . Now, choose  $N = \log_2(k)$ . For all  $n \geq N = \log_2(k)$ ,  $2^n \geq 2^N = k$ , thus  $|x_n - 0| = |\frac{1}{2^n}| = \frac{1}{2^n} \leq \frac{1}{k} < \frac{\epsilon}{2}$ . Then, for all  $m, n \geq N$ , the following is true:

$$|x_m - x_n| = |(x_m - 0) + (0 - x_n)| \leq |x_m - 0| + |0 - x_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, for all  $\epsilon > 0$ , there exists  $N$ , with  $m, n \geq N$  implies that  $|x_m - x_n| < \epsilon$ , proving that  $(x_n)_{n=1}^\infty$  is Cauchy.

Next, we'll prove that  $(g(x_n))_{n=1}^\infty$  is not Cauchy: For all  $n \in \mathbf{N}$ , since  $x_n = \frac{1}{2^n}$ , then  $g(x_n) = \frac{1}{1/2^n} = 2^n$ . Then, choose  $\epsilon = 1 > 0$ . For all  $N \in \mathbf{R}$ , by Archimedean's Property, there exists  $k \in \mathbf{N}$ , such that  $N < k \leq 2^k$ . Which, consider  $g(x_k)$  and  $g(x_{k+1})$ :

$$|g(x_k) - g(x_{k+1})| = |2^k - 2^{k+1}| = |-2^k| = 2^k > 2 > 1 = \epsilon$$

So, for  $\epsilon = 1$ , for all  $N$ , there exists  $m, n \geq N$ , such that  $|g(x_k) - g(x_{k+1})| > \epsilon$ , proving that  $(g(x_n))_{n=1}^{\infty}$  is not Cauchy.

(c) Suppose  $f(0, 1) \rightarrow \mathbf{R}$  is Uniformly Continuous.

**Limit near 0 and 1:**

Given arbitrary  $a \in \{0, 1\}$ . For all  $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset (0, 1)$  that converges to  $a$ , the two sequences are Cauchy. Based on the statement in **Problem 2 Part (a)**,  $f$  is uniformly continuous and the two sequences being Cauchy, implies that  $(f(x_n))_{n=1}^{\infty}, (f(y_n))_{n=1}^{\infty} \subset \mathbf{R}$  are Cauchy, and due to the Completeness of  $\mathbf{R}$ , the two sequences converge. Thus,  $\lim_{n \rightarrow \infty} f(x_n) = L_x \in \mathbf{R}$ , and  $\lim_{n \rightarrow \infty} f(y_n) = L_y \in \mathbf{R}$ .

Now, to prove that the limit is unique, consider  $|L_x - L_y|$ : For any  $n \in \mathbf{N}$ , the following is true:

$$0 \leq |L_x - L_y| = |(L_x - f(x_n)) + (f(x_n) - f(y_n)) + (f(y_n) - L_y)|$$

$$0 \leq |L_x - L_y| \leq |L_x - f(x_n)| + |f(x_n) - f(y_n)| + |L_y - f(y_n)|$$

Which, for all  $\epsilon > 0$  (which  $\frac{\epsilon}{3} > 0$ ), based on the definition of convergence, there exists  $N_1, N_2$ , such that  $n \geq N_1$  implies that  $|L_x - f(x_n)| < \frac{\epsilon}{3}$ , and  $n \geq N_2$  implies that  $|L_y - f(y_n)| < \frac{\epsilon}{3}$ .

Also, based on the definition of Uniform Continuity, given  $\frac{\epsilon}{3} > 0$ , there exists  $\delta > 0$ , such that for all  $x, x' \in (0, 1)$ ,  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < \frac{\epsilon}{3}$ .

Then, since  $(x_n)_{n=1}^{\infty}$  and  $(y_n)_{n=1}^{\infty}$  both converges to  $a$ , then given  $\delta > 0$  (which  $\frac{\delta}{2} > 0$ ), there exists  $N_3, N_4$ , such that  $n \geq N_3$  implies  $|a - x_n| < \frac{\delta}{2}$ , and  $n \geq N_4$  implies  $|a - y_n| < \frac{\delta}{2}$ .

Now, consider  $N = \max\{N_1, N_2, N_3, N_4\}$ . For any  $n \geq N$ :

Since  $n \geq N_3$  and  $n \geq N_4$ , then  $|a - x_n| < \frac{\delta}{2}$ , and  $|a - y_n| < \frac{\delta}{2}$ . Which:

$$|x_n - y_n| = |(x_n - a) + (a - y_n)| \leq |x_n - a| + |a - y_n| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

Because  $x_n, y_n \in (0, 1)$  and  $|x_n - y_n| < \delta$ , this implies that  $|f(x_n) - f(y_n)| < \frac{\epsilon}{3}$  based on the above definition of uniform continuity.

Also, since  $n \geq N_1$ , it implies that  $|L_x - f(x_n)| < \frac{\epsilon}{3}$ , and  $n \geq N_2$  implies that  $|L_y - f(y_n)| < \frac{\epsilon}{3}$ .

Then, recall the initial inequality, the following is true:

$$0 \leq |L_x - L_y| \leq |L_x - f(x_n)| + |f(x_n) - f(y_n)| + |L_y - f(y_n)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Thus, for all  $\epsilon > 0$ ,  $0 \leq |L_x - L_y| < \epsilon$ . This implies that  $|L_x - L_y| = 0$ , so  $L_x = L_y$ . Hence, we can conclude that for all  $(x_n)_{n=1}^\infty \subset (0, 1)$  satisfying  $\lim_{n \rightarrow \infty} x_n = a$  (with  $a \in \{0, 1\}$ ),  $\lim_{n \rightarrow \infty} f(x_n)$  converges to a unique element in  $\mathbf{R}$ , regardless the choice of  $(x_n)_{n=1}^\infty$ .

So, there exists unique  $L, R \in \mathbf{R}$ , such that for all  $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \in (0, 1)$  that satisfy  $\lim_{n \rightarrow \infty} x_n = 0$  and  $\lim_{n \rightarrow \infty} y_n = 1$ , the following is true:

$$\lim_{n \rightarrow \infty} f(x_n) = L, \quad \lim_{n \rightarrow \infty} f(y_n) = R$$

### Continuity of $f$ :

Define  $f : [0, 1] \rightarrow \mathbf{R}$  as follow:

$$f(x) = \begin{cases} L & x = 0 \\ f(x) & x \in (0, 1) \\ R & x = 1 \end{cases}$$

The extension is continuous on  $(0, 1)$ , to prove this extension is continuous, it suffices to prove that it is continuous at  $x = 0$  and  $x = 1$ . We'll approach this by contradiction.

Given  $a \in \{0, 1\}$ , suppose  $f$  is not continuous at  $a$ , then there exists  $\epsilon > 0$ , such that for all  $\delta > 0$ , there exists  $x \in [0, 1]$ , with  $|x - a| < \delta$ , but  $|f(x) - f(a)| \geq \epsilon$ . Then, consider the following process:

Step 1. Choose  $\delta_1 = \frac{1}{10^1}$ , there exists  $x_1 \in [0, 1]$ , such that  $|x_1 - a| < \delta_1$ , but  $|f(x_1) - f(a)| \geq \epsilon$ .

Notice that  $x_1 \neq 0$  and  $x_1 \neq 1$ : If  $a = 0$ , then  $x_1 \neq 0$  (since  $|f(x_1) - f(0)| \geq \epsilon > 0$ , so  $f(x_1) \neq f(0)$ , implying that  $x_1 \neq 0$ ); also, since  $|x_1 - 0| < \delta_1 = \frac{1}{10^1} < 1 = |1 - 0|$ , then  $x_1 \neq 1$ . Else, if  $a = 1$ , then  $x_1 \neq 1$  (since  $|f(x_1) - f(1)| \geq \epsilon > 0$ , so  $f(x_1) \neq f(1)$ , implying that  $x_1 \neq 1$ ); then, since  $|x_1 - 1| < \delta_1 = \frac{1}{10^1} < 1 = |0 - 1|$ ,  $x_1 \neq 0$ .

So,  $x_1 \in (0, 1)$ .

Step k. Given integer  $k \geq 2$ , Choose  $\delta_k = \frac{1}{10^k}$ , there exists  $x_k \in [0, 1]$ , such that  $|x_k - a| < \delta_k$ , but  $|f(x_k) - f(a)| \geq \epsilon$ .

Based on similar reason,  $x_k \neq 0$  and  $x_k \neq 1$ : If  $a = 0$ , then  $x_k \neq 0$  (since  $|f(x_k) - f(0)| \geq \epsilon > 0$ , so  $f(x_k) \neq f(0)$ , implying that  $x_k \neq 0$ ); also, since  $|x_k - 0| < \delta_k = \frac{1}{10^k} < 1 = |1 - 0|$ , then  $x_k \neq 1$ . Else, if  $a = 1$ , then  $x_k \neq 1$  (since  $|f(x_k) - f(1)| \geq \epsilon > 0$ , so  $f(x_k) \neq f(1)$ , implying that  $x_k \neq 1$ ); then, since  $|x_k - 1| < \delta_k = \frac{1}{10^k} < 1 = |0 - 1|$ ,  $x_k \neq 0$ .

So,  $x_k \in (0, 1)$ .

From the above process, we constructed  $(x_k)_{k=1}^\infty \subset (0, 1)$ , such that the following is true: For all  $\epsilon' > 0$ , there exists  $K \in \mathbf{N}$ , such that  $1 < K\epsilon$ , or  $\frac{1}{10^K} < \frac{1}{K} < \epsilon'$ . Then, for all  $k \geq K$ , since  $10^k \geq 10^K$ ,  $\frac{1}{10^k} \leq \frac{1}{10^K}$ . Which, the following is true:

$$|x_k - a| < \delta_k = \frac{1}{10^k} \leq \frac{1}{10^K} < \epsilon'$$

Thus, this implies that  $x_k$  converges to  $a$ .

So,  $(s_k)_{k=1}^\infty \subset (0, 1)$  is a sequence satisfying  $\lim_{k \rightarrow \infty} x_k = a$  (with  $a \in \{0, 1\}$ ), while  $\lim_{k \rightarrow \infty} f(x_k) \neq f(a)$  (since for all  $k \in \mathbf{N}$ ,  $|f(x_k) - f(a)| \geq \epsilon > 0$ ). Yet, this is a contradiction:

Recall that for all  $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \in (0, 1)$  that satisfy  $\lim_{n \rightarrow \infty} x_n = 0$  and  $\lim_{n \rightarrow \infty} y_n = 1$ ,  $\lim_{n \rightarrow \infty} f(x_n) = L$ , and  $\lim_{n \rightarrow \infty} f(y_n) = R$ .

If  $a = 0$ , then  $(x_k)_{k=1}^\infty$  satisfies  $\lim_{k \rightarrow \infty} x_k = a = 0$ , while  $\lim_{k \rightarrow \infty} f(x_k) \neq f(a) = L$ , which is a contradiction. Else if  $a = 1$ , then  $(x_k)_{k=1}^\infty$  satisfies  $\lim_{k \rightarrow \infty} x_k = a = 1$ , while  $\lim_{k \rightarrow \infty} f(x_k) \neq f(a) = R$ , which is again a contradiction.

So, our initial assumption must be false, the extended  $f$  must be continuous at  $a$ . And, since  $a \in \{0, 1\}$  is arbitrary, then the extended  $f$  is continuous on both  $x = 0$  and  $x = 1$ , showing that we can extend  $f$  to be continuous on  $[0, 1]$ .

## Question 3 Textbook:

2. If  $f : X \rightarrow Y$  is continuous, prove that  $f(\overline{E}) \subseteq \overline{f(E)}$  for all  $E \subset X$ , and show that proper inclusion is possible.
7. Define  $f$  and  $g$  mapping from  $\mathbf{R}^2$  to  $\mathbf{R}$  by  $f(0,0) = g(0,0) = 0$ , and  $f(x,y) = xy^2/(x^2 + y^4)$ ,  $g(x,y) = xy^2/(x^2 + y^6)$  if  $(x,y) \neq (0,0)$ . Prove that  $f$  is bounded on  $\mathbf{R}$ ,  $g$  is unbounded on every neighborhood of  $(0,0)$ , and  $f$  is not continuous on  $(0,0)$ . Also, show that the restriction of all straight line in  $\mathbf{R}^2$  is continuous.
18. For all  $x \in \mathbf{Q} \setminus \{0\}$ , there exists unique  $m, n \in \mathbf{Z}$  with  $n > 0$ , such that  $x = \frac{m}{n}$ , and  $m, n$  are coprime (if  $x = 0$ , take  $n = 1$ ). Take the following function  $f : \mathbf{R} \rightarrow \mathbf{R}$ :

$$f(x) = \begin{cases} 0 & x \in \mathbf{Q}^C \\ \frac{1}{n} & x = \frac{m}{n}, \gcd(m,n) = 1 \end{cases}$$

Prove that  $f$  is continuous at every irrational points, while discontinuous at every rational points.

21. Suppose  $K, F \subset X$  are disjoint sets with  $K$  being compact and  $F$  is closed. Prove that there exists  $\delta > 0$  such that if  $p \in K$  and  $q \in F$ , then  $d(p,q) > \delta$ . And, Show that the conclusion may fail for two disjoint closed sets if neither is compact.
23. Prove that every convex function is continuous, every increasing convex function of a convex function is convex, and if  $f : (a,b) \rightarrow \mathbf{R}$  is convex, given  $a < s < t < u < b$ , the following is true:

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$$

Q2. Suppose  $f : X \rightarrow Y$  is continuous, and arbitrary  $E \subseteq X$ .

For any  $x \in \overline{E}$ , there are two cases to consider:

First, if  $x \in E$ , then  $f(x) \in f(E) \subseteq \overline{f(E)}$ .

Else, if  $x \in E'$  with  $x \notin E$ :

Suppose  $f(x) \in f(E) \subseteq \overline{f(E)}$ , it is already done;

for the other case, if  $f(x) \notin f(E)$ , by the definition of continuity, for any  $\epsilon > 0$ , there exists  $\delta > 0$ , with  $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$ . Also, since  $x \in E'$ , then for the given  $\delta > 0$ , there exists  $a \in B_\delta(x) \setminus \{x\} \cap E$ . Thus,  $a \in E$  satisfies  $a \in B_\delta(x)$ , which implies that  $f(a) \in f(B_\delta(x)) \subseteq B_\epsilon(f(x))$ , and  $f(a) \in f(E)$ . Also, since  $f(x) \notin f(E)$  by assumption, then  $f(a) \neq f(x)$ . So,  $f(a) \in B_\epsilon(f(x)) \setminus \{f(x)\} \cap f(E)$ . This proves that  $f(x)$  is a limit point of  $f(E)$ , hence  $f(x) \in \overline{f(E)}$ .

So, under all cases,  $x \in \overline{E}$  implies that  $f(x) \in \overline{f(E)}$ , thus  $f(\overline{E}) \subseteq \overline{f(E)}$ .

**Example of Proper Inclusion:**

Consider the following function  $f : \mathbf{R} \rightarrow \mathbf{R}$  with  $f(x) = e^x$ , and the set  $E = (-\infty, 0)$ .

Which,  $\overline{E} = (-\infty, 0]$ , we have  $f(\overline{E}) = (0, 1]$ .

However,  $f(E) = (0, 1)$ , which  $\overline{f(E)} = [0, 1]$ ,  $0 \in \overline{f(E)}$  while  $0 \notin f(\overline{E})$ , thus  $f(\overline{E}) \subsetneq \overline{f(E)}$ .

**Q7.  $f$  Bounded:**

For all  $x, y \in \mathbf{R}$ , with  $(x, y) \neq (0, 0)$  (so,  $x^2 + y^4 > 0$ , since at least one of the entry is nonzero), since both  $x^2, y^4 \geq 0$ , then the following inequality is true:

$$\sqrt{x^2 y^4} \leq \frac{x^2 + y^4}{2}, \quad |xy^2| \leq \frac{x^2 + y^4}{2}$$

$$\frac{1}{x^2 + y^4} \leq \frac{1}{2|xy^2|}$$

Which, for  $f(x, y)$ , there are three cases to consider:

If  $xy^2 > 0$ , then  $f(x) = \frac{xy^2}{x^2 + y^4} \leq \frac{xy^2}{2|xy^2|} \leq \frac{1}{2}$ , and  $-\frac{1}{2} < 0 < f(x)$  (since  $xy^2, (x^2 + y^4) > 0$ ).

If  $xy^2 = 0$ , thus  $f(x) = \frac{xy^2}{x^2 + y^4} = 0$ .

Else if  $xy^2 < 0$ , then  $f(x) = \frac{xy^2}{x^2 + y^4} \geq \frac{xy^2}{2|xy^2|} \geq -\frac{1}{2}$ , and  $f(x) < 0 < \frac{1}{2}$  (since  $xy^2 < 0$ , and  $(x^2 + y^4) > 0$ ).

Thus, in all cases,  $-\frac{1}{2} \leq f(x) \leq \frac{1}{2}$  (including  $f(0, 0) = 0$ ), which  $f$  is bounded.

**$g$  Unbounded:**

For all  $r > 0$ , consider  $B_r(0, 0)$ : Given arbitrary  $M > 0$ , consider the set  $D = \{y > 0 \mid y < \frac{1}{2M} \text{ and } y < \frac{r}{\sqrt{2}}\}$  which is not empty (since both  $\frac{1}{2M}, \frac{r}{\sqrt{2}} > 0$ ).

Note that it is always possible to find  $y \in D$  with  $y^3 < \frac{r}{\sqrt{2}}$ :

If  $y \geq 1$ , take  $y' = y^{\frac{1}{3}} > 0$ , then  $y' = y^{\frac{1}{3}} \leq y = (y')^3$ , implying that  $y' \leq (y')^3 = y < \frac{r}{\sqrt{2}}$  and  $y' \leq (y')^3 = y < \frac{1}{2M}$ , which  $y'$  satisfies the condition; else if  $y < 1$ , then  $y^3 < y < \frac{r}{\sqrt{2}}$  and  $y^3 < y < \frac{1}{2M}$ , which  $y$  satisfies the condition.

Then, choose the  $y \in D$  satisfying  $y, y^3 < \frac{r}{\sqrt{2}}$ , consider the element  $(y^3, y) \in \mathbf{R}^2$ : since both  $y, y^3 < \frac{r}{\sqrt{2}}$ , then:

$$\|(y^3, y)\|^2 = (y^3)^2 + y^2 < \left(\frac{r}{\sqrt{2}}\right)^2 + \left(\frac{r}{\sqrt{2}}\right)^2 = r^2, \quad \|(y^3, y)\| < \sqrt{r^2} = r$$

Thus,  $(y^3, y) \in B_r(0, 0)$ . Also, consider  $g(y^3, y)$ :

$$g(y^3, y) = \frac{y^3 \cdot y^2}{(y^3)^2 + y^6} = \frac{y^5}{2y^6} = \frac{1}{2y}$$

$$y < \frac{1}{2M}, \quad M < \frac{1}{2y} = g(y^3, y)$$

Hence, for all  $r, M > 0$ , there exists element  $(x, y) \in B_r(0, 0)$  with  $g(x, y) > M$ , proving that  $g$  is unbounded in every neighborhood of  $(0, 0)$ .

**$f$  Not continuous at  $(0, 0)$ :**

Take  $\epsilon = \frac{1}{4}$ , for all  $\delta > 0$ , since  $\frac{\delta}{\sqrt{2}} > 0$ , then there exists  $n \in \mathbf{N}$  with  $1 < n \frac{\delta}{\sqrt{2}}$ , thus  $\frac{1}{n} < \frac{\delta}{\sqrt{2}}$ . (Note: since  $n \geq 1$ , then  $n^2 \geq n$ ,  $\frac{1}{n^2} \leq \frac{1}{n}$ , implying that  $\frac{1}{n^4} \leq \frac{1}{n^2}$ ).



Which, consider  $(\frac{1}{n^2}, \frac{1}{n})$ : First, about the norm:

$$\|(\frac{1}{n^2}, \frac{1}{n})\|^2 = (\frac{1}{n^2})^2 + (\frac{1}{n})^2 \leq \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2} < 2 \cdot (\frac{\delta}{\sqrt{2}})^2 = \delta^2$$

$$\|(\frac{1}{n^2}, \frac{1}{n})\| < \sqrt{\delta^2} = \delta$$

Thus,  $(\frac{1}{n^2}, \frac{1}{n}) \in B_\delta(0, 0)$ . However, consider  $f(\frac{1}{n^2}, \frac{1}{n})$ :

$$f(\frac{1}{n^2}, \frac{1}{n}) = \frac{(\frac{1}{n^2}) \cdot (\frac{1}{n})^2}{(\frac{1}{n^2})^2 + (\frac{1}{n})^4} = \frac{1/n^4}{2/n^4} = \frac{1}{2}$$

Which,  $|f(\frac{1}{n^2}, \frac{1}{n}) - f(0, 0)| = |\frac{1}{2} - 0| = \frac{1}{2} > \frac{1}{4} = \epsilon$ .

So, the above proves that  $f$  is not continuous at  $(0, 0)$ , since the chosen  $\epsilon > 0$  satisfies for all  $\delta > 0$ , there exists  $(x, y) \in B_\delta(0, 0)$  with  $|f(x, y) - f(0, 0)| \geq \epsilon$ .

### Restriction onto Straight Line:

For all straight line in  $\mathbf{R}^2$ , every  $(x, y)$  on the line satisfies  $ax + by = c$  for some  $a, b, c \in \mathbf{R}$  (which  $(a, b) \neq (0, 0)$ ).

If  $c \neq 0$ , then the line isn't including  $(0, 0)$ , thus  $f, g$  are following the given rational function with every point being well-defined, which is continuous.

Else, if  $c = 0$ , then the line is including  $(0, 0)$  (everywhere else is defined with the rational function), the goal is to prove that the function is continuous at  $(0, 0)$ . Again, there are 2 cases to consider:

First, if  $b = 0$ , then  $ax + 0 = 0$ , which  $x = 0$  (since  $(a, b) \neq (0, 0)$ , so  $a \neq 0$ ). Then,  $f(0, y) = \frac{0y^2}{0^2 + y^4} = 0$ ,  $g(0, y) = \frac{0y^2}{0^2 + y^6} = 0$ , which given the domain as the straight line  $ax = 0$ , the function has output 0, which is a constant function (and it is continuous). The same concept applies when  $a = 0$  (which changes to  $y = 0$ , but the functions are still constant function of 0).

Else, if  $a, b \neq 0$ , then  $ax = -by$ , which  $y = \frac{-ax}{b}$ . If  $(x, y) \neq (0, 0)$  (which  $x \neq 0$ , or else it implies  $y = 0$ , causing  $(x, y) = (0, 0)$ ), then the following is true:

$$f(x, y) = f(x, \frac{-ax}{b}) = \frac{x(\frac{-ax}{b})^2}{x^2 + (\frac{-ax}{b})^4} = \frac{(-a/b)^2 x^3}{x^2 + (-a/b)^4 x^4} = \frac{(-a/b)^2 x}{1 + (-a/b)^4 x^2}$$

$$g(x, y) = g(x, \frac{-ax}{b}) = \frac{x(\frac{-ax}{b})^2}{x^2 + (\frac{-ax}{b})^6} = \frac{(-a/b)^2 x^3}{x^2 + (-a/b)^4 x^6} = \frac{(-a/b)^2 x}{1 + (-a/b)^6 x^4}$$

Which, notice that  $(-a/b)^4 x^2, (-a/b)^6 x^4 \geq 0$ , then  $1 + (-a/b)^4 x^2, 1 + (-a/b)^6 x^4 \geq 1$ , or:

$$\frac{1}{1 + (-a/b)^4 x^2}, \frac{1}{1 + (-a/b)^6 x^4} \leq 1$$

Thus, for all  $(x, y) \neq (0, 0)$  on the line, the following is true:

$$|f(x, y) - f(0, 0)| = \left| \frac{(-a/b)^2 x}{1 + (-a/b)^4 x^2} - 0 \right| = \frac{|(-a/b)^2 x|}{1 + (-a/b)^4 x^2} \leq |(-a/b)^2 x|$$

$$|g(x, y) - g(0, 0)| = \left| \frac{(-a/b)^2 x}{1 + (-a/b)^6 x^4} - 0 \right| = \frac{|(-a/b)^2 x|}{1 + (-a/b)^6 x^4} \leq |(-a/b)^2 x|$$

So, for all  $\epsilon > 0$ , choose  $\delta = (b/a)^2 \epsilon$ . Then, for all  $(x, y) \in \mathbf{R}^2$  (with  $(x, y) \neq (0, 0)$ ) satisfying  $\|(x, y)\| = \sqrt{x^2 + y^2} < \delta$  (in other word,  $(x, y) \in B_\delta(0, 0)$ ), since  $|x| = \sqrt{x^2} \leq \|(x, y)\| < \delta = (b/a)^2 \epsilon$ , then the following is true:

$$|f(x, y) - f(0, 0)|, |g(x, y) - g(0, 0)| \leq |(-a/b)^2 x| = (a/b)^2 |x| < (a/b)^2 \cdot (b/a)^2 \epsilon = \epsilon$$

Thus, for all  $\epsilon > 0$ , there exists  $\delta > 0$ , with  $(x, y) \in B_\delta(0, 0)$  (restricted to the straight line), it implies  $|f(x, y) - f(0, 0)|, |g(x, y) - g(0, 0)| < \epsilon$ . Thus, when restricted to any straight line passing through  $(0, 0)$ , the functions are continuous at  $(0, 0)$ , hence continuous on the whole straight line.

Q18. Given the following function:

$$f(x) = \begin{cases} 0 & x \in \mathbf{Q}^C \\ \frac{1}{n} & x = \frac{m}{n}, \gcd(m, n) = 1 \end{cases}$$

And assume that at  $x = 0$ ,  $n = 1$  (so  $f(0) = 1$ ).

### Discontinuity on $\mathbf{Q}$ :

For any  $x \in \mathbf{Q}$ ,  $f(x) = \frac{1}{n}$  for some  $n \in \mathbf{N}$ , then choose  $\epsilon = \frac{1}{2n}$ . For all  $\delta > 0$ , since the open ball  $(x - \delta, x + \delta)$  contains some irrational number  $r$  due to denseness of  $\mathbf{Q}^C$  in  $\mathbf{R}$ , then  $r$  satisfies  $|x - r| < \delta$ , and  $f(r) = 0$ , which  $|f(r) - f(x)| = |0 - \frac{1}{n}| = \frac{1}{n} \geq \frac{1}{2n} = \epsilon$ . This shows that  $f$  is not continuous on  $x$ , which  $f$  is not continuous on  $\mathbf{Q}$ .

### Continuity on $\mathbf{Q}^C$ :

For any  $x \in \mathbf{Q}^C$ , consider  $U = \{n \in \mathbf{Z} \mid x < n\}$ . Then, since  $U$  is bounded below,  $k = \inf(U)$  exists; and by the well-ordering property,  $k \in U$ . So,  $k \in \mathbf{Z}$  satisfies  $(k - 1) \leq x < k$  ( $(k - 1) < k$ , with  $k = \inf(U)$ , so  $(k - 1) \notin U$ , or  $(k - 1) \leq x$ ). Also, since  $(k - 1)$  is rational, then  $x \neq (k - 1)$ . Thus,  $(k - 1) < x < k$ .

Now, for all  $\epsilon > 0$ , by Archimedean's Property, there exists  $n \in \mathbf{N}$  with  $1 < n\epsilon$  (or  $\frac{1}{n} < \epsilon$ ). First, let  $D = \{(k - 1), k\} \cup \{(k - 1) + \frac{i}{j} \mid i, j \in \mathbf{N}, i < j < n\}$  (which  $D$  is finite, and every element is rational).

For any  $q \in (k - 1, k)$  with  $f(q) > \frac{1}{n}$ , if  $q \in \mathbf{Q}^C$ , then  $f(q) = 0 < \frac{1}{n}$ , which violates the desired condition, so  $q \in \mathbf{Q}$ ; then, there exists  $m \in \mathbf{Z}$  and  $j \in \mathbf{N}$  with  $\gcd(m, j) = 1$ , such that  $q = \frac{m}{j}$ . Now, there are some conditions:

Since  $f(q) = \frac{1}{j} > \frac{1}{n}$ , then  $n > j$ .

Also, since  $(k - 1) < q = \frac{m}{j} < k$ , then  $(k - 1)j < m < kj$ , which  $0 < (m - (k - 1)j) < j$ .

So, let  $i = (m - (k - 1)j)$ , consider  $(k - 1) + \frac{i}{j}$ :

$$\begin{aligned} (k - 1) + \frac{i}{j} &= (k - 1) + \frac{m - (k - 1)j}{j} = (k - 1) + \frac{m}{j} - \frac{(k - 1)j}{j} \\ &= (k - 1) + \frac{i}{j} = (k - 1) + \frac{m}{j} - (k - 1) = \frac{m}{j} \end{aligned}$$

Which,  $q = \frac{m}{j} = (k - 1) + \frac{i}{j}$ , with  $0 < i < j < n$  (since  $i = (m - (k - 1)j)$ ), then  $q \in D$  (since  $q = (k - 1) + \frac{i}{j}$ , which satisfies the set axiom of  $\{(k - 1) + \frac{i}{j} \mid i, j \in \mathbf{N}, i < j < n\}$ ).

Now, let  $\delta = \min\{|x - q| \mid q \in D\}$ , which  $\delta > 0$  since for all  $q \in D$ ,  $q$  is rational, which  $q \neq x$ , so  $|q - x| > 0$ . For all  $a \in \mathbf{R}$  satisfying  $|a - x| < \delta$ , then the following is true:

$$-\delta < a - x < \delta, \quad x - \delta < a < x + \delta$$

Which, since  $(k-1) \in D$ ,  $\delta \leq |x - (k-1)| = x - (k-1)$ , then  $(k-1) < (x - \delta)$ ; also, since  $k \in D$ ,  $\delta \leq |x - k| = k - x$ , then  $(x + \delta) < k$ . So, the following is true:

$$(k-1) < (x - \delta) < a < (x + \delta) < k$$

So,  $a \in (k-1, k)$ . Also, since for all  $q \in (k-1, k)$  satisfying  $f(q) > \frac{1}{n}$ , we've proven that  $q \in D$ , then  $a \notin D$ , since for all  $q \in D$ ,  $|x - a| < \delta \leq |x - q|$  by how we define  $\delta$ .

Thus,  $0 \leq f(a) \leq \frac{1}{n}$  (since  $a \notin D$ , it can't have  $f(a) > \frac{1}{n}$ ). Hence,  $|f(x) - f(a)| = |0 - f(a)| = f(a) \leq \frac{1}{n} < \epsilon$ .

This proves that  $f$  is continuous at  $x$ , which since  $x \in \mathbf{Q}^C$  is arbitrary, then  $f$  is continuous on all  $\mathbf{Q}^C$ .

Q21. Given  $K, F \subset X$  that are disjoint, with  $K$  being compact and  $F$  is closed.

First, consider the function  $\rho_F : X \rightarrow \mathbf{R}$  with  $\rho_F(x) = \inf\{d(x, q) \mid q \in F\}$ , we'll prove that  $\rho_F$  is uniformly continuous:

For all  $x, y \in X$  and all  $q \in F$ , notice that  $\rho_F(x) \leq d(x, q) \leq d(x, y) + d(y, q)$ , so  $\rho_F(x) \leq d(x, y) + d(y, q)$  for all  $q \in F$  (or,  $\rho_F(x) - d(x, y) \leq d(y, q)$  for all  $q \in F$ ), hence  $(\rho_F(x) - d(x, y))$  is a lower bound of the set  $\{d(y, q) \mid q \in F\}$ , or  $(\rho_F(x) - d(x, y)) \leq \inf\{d(y, q) \mid q \in F\} = \rho_F(y)$ . Thus,  $\rho_F(x) - d(x, y) \leq \rho_F(y)$ , which  $(\rho_F(x) - \rho_F(y)) \leq d(x, y)$ .

So, given any  $\epsilon > 0$ , choose  $\delta = \epsilon$ , then for all  $x, y \in X$  (Without Loss of Generality, assume  $\rho_F(x) \geq \rho_F(y)$ ), if  $d(x, y) < \delta = \epsilon$ , then:

$$|\rho_F(x) - \rho_F(y)| = (\rho_F(x) - \rho_F(y)) \leq d(x, y) < \epsilon$$

Thus, the function  $\rho_F(x)$  is uniformly continuous.

Now, since  $\rho_F$  is continuous and  $K$  is compact, then  $\rho_F(K) \subset \mathbf{R}$  is compact (which is closed and bounded), thus  $\inf(\rho_F(K)) \in \rho_F(K)$ , there exists  $p_0 \in K$ , with  $\rho_F(p_0) = \inf(\rho_F(K))$ .

Then, we'll prove by contradiction that  $\inf(\rho_F(K)) > 0$ : Suppose this statement is false, then  $\inf(\rho_F(K)) \leq 0$ ; furthermore, since 0 is always the lower bound of a set of distance, then for any  $x \in X$ ,  $\rho_F(x) \geq 0$ , so  $\inf(\rho_F(K)) = \rho_F(p_0) \geq 0$ , showing that  $\inf(\rho_F(K)) = \rho_F(p_0) = 0$ .

However, this implies that for any  $r > 0$ , since  $r = r + \rho_F(p_0)$  is no longer a lower bound of the set  $\{d(p_0, q) \mid q \in F\}$ , then there exists  $q \in F$ , with  $d(p_0, q) < r$  (or  $q \in B_r(p_0)$ ). Since  $K$  and  $F$  are disjoint, then  $p_0 \neq q$ , which  $q \in B_r(p_0) \setminus \{p_0\} \cap F$ , showing that  $p_0$  is a limit point of  $F$ . With the assumption that  $F$  is closed, then  $p_0 \in F' \subseteq F$ , so  $p_0 \in K \cap F$ ; yet, this contradicts the fact that the two sets are disjoint, so the assumption is false, or  $\inf(\rho_F(K)) > 0$ .

Eventually, let  $\delta = \frac{\inf(\rho_F(K))}{2} > 0$ , then for all  $p \in K$  and  $q \in F$ , since  $\rho_F(p) \in \rho_F(K)$ , then  $\delta = \frac{\inf(\rho_F(K))}{2} < \inf(\rho_F(K)) \leq \rho_F(p) \leq d(p, q)$ .

### Example if Both Closed Sets are not Compact:

Given  $K = \mathbf{N}$  and  $F = \{n + \frac{1}{n} \mid n \in \mathbf{N}, n \geq 2\}$ . Both sets are not compact as they're not bounded; both are closed as there are no limit points for either of them, and the two sets are disjoint (since for all  $n \in \mathbf{N}$  with  $n \geq 2$ ,  $(n + \frac{1}{n})$  is not an integer, which  $F$  contains no elements from  $K = \mathbf{N}$ ).

However, for all  $\delta > 0$ , there exists  $n \in \mathbf{N}$  with  $1 < n\delta$  (or  $\frac{1}{n} < \delta$ ), which, choose  $(n + 1) \in K$  and  $((n + 1) + \frac{1}{(n+1)}) \in F$  (note:  $n \geq 1$ , so  $(n + 1) \geq 2$ ). Then, the following is true:

$$d\left((n + 1), (n + 1) + \frac{1}{(n + 1)}\right) = \frac{1}{n + 1} < \frac{1}{n} \leq \delta$$

Which it is a counterexample of the desired property.

**Q23. Inequality of Convex Function:**

To prove the continuity of convex function, we'll use this as a tool (that's why we're proving it first).  
 Given a convex function  $f : (a, b) \rightarrow \mathbf{R}$ , and given  $a < s < t < u < b$ . Then, let  $\lambda = \frac{u-t}{u-s}$ , the following is true:

$$\begin{aligned}\lambda s + (1 - \lambda)u &= \frac{u-t}{u-s}s + \left(1 - \frac{u-t}{u-s}\right)u = \frac{us-ts}{u-s} + \frac{(u-s)-(u-t)}{u-s}u \\ &= \frac{us-ts}{u-s} + \frac{ut-us}{u-s} = \frac{ut-ts}{u-s} = t \frac{u-s}{u-s} = t\end{aligned}$$

Since  $s < u$ ,  $0 < (u-s)$ , which the above quantity is defined.

Also, since  $s < t < u$ , which  $0 < (u-t) < (u-s)$ , thus  $0 < \frac{u-t}{u-s} < 1$ , which  $\lambda = \frac{u-t}{u-s} \in (0, 1)$ .

With this parametrization,  $t = \lambda s + (1 - \lambda)u$  for some  $\lambda \in (0, 1)$ , thus:

$$f(t) = f(\lambda s + (1 - \lambda)u) \leq \lambda f(s) + (1 - \lambda)f(u)$$

$$f(t) - f(s) \leq (\lambda f(s) + (1 - \lambda)f(u)) - f(s) = (1 - \lambda)(f(u) - f(s))$$

Given that  $(t-s) > 0$ , the following is true:

$$\frac{f(t) - f(s)}{t - s} \leq \frac{(1 - \lambda)(f(u) - f(s))}{t - s}$$

And again, by the parametrization of  $t$ , the following is true:

$$t - s = (\lambda s + (1 - \lambda)u) - s = (1 - \lambda)(u - s)$$

So, the inequality can be rewrite as:

$$\begin{aligned}\frac{f(t) - f(s)}{t - s} &\leq \frac{(1 - \lambda)(f(u) - f(s))}{t - s} = \frac{(1 - \lambda)(f(u) - f(s))}{(1 - \lambda)(u - s)} = \frac{f(u) - f(s)}{u - s} \\ \frac{f(t) - f(s)}{t - s} &\leq \frac{f(u) - f(s)}{u - s}\end{aligned}$$

Now, based on the same reasoning:

$$f(t) \leq \lambda f(s) + (1 - \lambda)f(u), \quad f(u) - f(t) \geq f(u) - (\lambda f(s) + (1 - \lambda)f(u))$$

$$\lambda(f(u) - f(s)) \leq f(u) - f(t)$$

Which, since  $(t < u)$ ,  $0 < (u-t)$ , so:

$$\frac{\lambda(f(u) - f(s))}{u - t} \leq \frac{f(u) - f(t)}{u - t}$$

Rewrite  $(u - t)$  with the given parametrization, the following is true:

$$u - t = u - (\lambda s + (1 - \lambda)u) = \lambda(u - s)$$

So, the following inequality is also true:

$$\frac{\lambda(f(u) - f(s))}{u - t} = \frac{\lambda(f(u) - f(s))}{\lambda(u - s)} = \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$$

$$\frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$$

Combining the two inequality, the following is true:

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$$

### Convex Implies Continuity:

Given convex function  $f : (a, b) \rightarrow \mathbf{R}$ . For all  $x_0 \in (a, b)$ , since  $a < x_0 < b$ , we can choose  $c, d$  satisfying  $a < c < x_0 < d < b$ .

Now, consider any  $y \in (c, d)$  with  $y \neq x_0$ , there are two cases:

First, if  $y < x_0$ , then  $a < c < y < x_0 < b$ , which from the inequality proven beforehand:

$$\frac{f(y) - f(c)}{y - c} \leq \frac{f(x_0) - f(c)}{x_0 - c} \leq \frac{f(x_0) - f(y)}{x_0 - y}, \quad \frac{f(x_0) - f(c)}{x_0 - c} \leq \frac{f(x_0) - f(y)}{x_0 - y}$$

Else, if  $y > x_0$ , then  $a < x_0 < y < d < b$ , which using the same inequality:

$$\frac{f(y) - f(x_0)}{y - x_0} \leq \frac{f(d) - f(x_0)}{d - x_0} \leq \frac{f(d) - f(y)}{d - y}, \quad \frac{f(x_0) - f(y)}{x_0 - y} = \frac{f(y) - f(x_0)}{y - x_0} \leq \frac{f(d) - f(x_0)}{d - x_0}$$

So, regardless of the case, the following is true:

$$\frac{f(x_0) - f(c)}{x_0 - c} \leq \frac{f(x_0) - f(y)}{x_0 - y} \leq \frac{f(d) - f(x_0)}{d - x_0}$$

Which, let  $M = \max \left\{ \left| \frac{f(x_0) - f(c)}{x_0 - c} \right|, \left| \frac{f(d) - f(x_0)}{d - x_0} \right| \right\}$ , then the above inequality implies that  $\left| \frac{f(x_0) - f(y)}{x_0 - y} \right| \leq M < (M + 1)$ , which:

$$|f(x_0) - f(y)| < (M + 1)|x_0 - y|$$

(Note: since  $M$  is the maximum among absolute values,  $M \geq 0$ , so  $(M + 1) > 0$ ).

Then, to prove that  $f$  is continuous at  $x_0$ , given any  $\epsilon > 0$ , let  $\delta = \min \left\{ (x_0 - c), (d - x_0), \frac{\epsilon}{(M + 1)} \right\}$

(Note: all of these are positive, since  $c < x_0 < d$ , and  $(M + 1) > 0$ ). Then, for all  $y$  that satisfy  $|y - x_0| < \delta$ , if  $y = x_0$ , obviously  $|f(y) - f(x_0)| = 0 < \epsilon$ .

Else, if  $y \neq x_0$ , the following is true:

$$-\delta < (y - x_0) < \delta, \quad (x_0 - \delta) < y < (x_0 + \delta)$$

Also, since  $\delta \leq (d - x_0)$  and  $\delta \leq (x_0 - c)$  (which  $-\delta \geq -(x_0 - c)$ ), the following is true:

$$c = (x_0 - (x_0 - c)) \leq (x_0 - \delta) < y < (x_0 + \delta) \leq (x_0 + (d - x_0)) = d$$

So,  $c < y < d$  (or  $y \in (c, d)$ ). Now, based on the inequality proven above, we have:

$$|f(x_0) - f(y)| < (M + 1)|x_0 - y| < (M + 1)\delta \leq (M + 1)\frac{\epsilon}{(M + 1)} = \epsilon$$

Which we deduced  $|f(y) - f(x_0)| < \epsilon$ , showing that  $f$  is continuous at  $x_0$ . Since the choice of  $x_0 \in (a, b)$  is arbitrary, this proves that  $f$  is continuous on  $(a, b)$ . So,  $f : (a, b) \rightarrow \mathbf{R}$  is convex implies that  $f$  is continuous on  $(a, b)$ .

### **Composition of Increasing Convex Function and Convex Function:**

Suppose  $g : \mathbf{R} \rightarrow \mathbf{R}$  is an increasing convex function, and  $f : \mathbf{R} \rightarrow \mathbf{R}$  is a convex function. Then, for all  $x, y \in \mathbf{R}$  and  $\lambda \in (0, 1)$ , since  $f$  is convex, the following is true:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Now, treat  $f(x), f(y)$  as two inputs, since  $g$  is also convex, the following is true:

$$g(\lambda f(x) + (1 - \lambda)f(y)) \leq \lambda g(f(x)) + (1 - \lambda)g(f(y))$$

Then, since  $g$  is increasing, while  $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ , then the following is true:

$$g(f(\lambda x + (1 - \lambda)y)) \leq g(\lambda f(x) + (1 - \lambda)f(y))$$

Thus, combining the inequality we get:

$$g(f(\lambda x + (1 - \lambda)y)) \leq g(\lambda f(x) + (1 - \lambda)f(y)) \leq \lambda g(f(x)) + (1 - \lambda)g(f(y))$$

$$g(f(\lambda x + (1 - \lambda)y)) \leq \lambda g(f(x)) + (1 - \lambda)g(f(y))$$

This proves that  $g \circ f$  is also a convex function.



**Question 4** *In each case give an example of  $f : \mathbf{R} \rightarrow \mathbf{R}$  continuous and:*

- i  $K$  compact with  $f^{-1}(K)$  no compact.*
- ii  $A$  connected with  $f^{-1}(A)$  no connected.*
- iii  $B$  open with  $f(B)$  not open.*
- iv  $C$  closed with  $f(C)$  not closed.*

i Given  $f(x) = 0$  the constant function. Since  $K = \{1\}$  is a singleton set, then  $K$  is compact.

Yet, since for all  $x \in \mathbf{R}$ ,  $f(x) = 0$ , so  $x \in f^{-1}(K)$ , or  $f^{-1}(K) = \mathbf{R}$ , which is not compact.

ii Given  $f(x) = x^2$  and  $A = \{1\}$ , which it is connected.

For all  $x \in \mathbf{R}$ , if  $f(x) = x^2 = 1$  (or  $x \in f^{-1}(A)$ ), then  $x = 1$  or  $x = -1$ . So,  $f^{-1}(A) = \{-1, 1\}$ .

However, since  $C = \{-1\}$ ,  $D = \{1\}$  satisfy  $C \cup D = f^{-1}(A)$ , and  $\overline{C} \cup D = \overline{D} \cup C = \emptyset$ , so  $f^{-1}(A)$  is not connected.

iii Given  $f(x) = 0$  again. The open interval  $B = (0, 1) \subset \mathbf{R}$  is open.

Yet since for all  $x \in B$ ,  $f(x) = 0$ , then  $f(B) = \{0\}$ , which  $f(B)$  is closed.

iv Given  $f(x) = e^x$  and  $C = (-\infty, 0] \subset \mathbf{R}$ .

For all  $x \in C$ , since  $x \leq 0$ , so  $f(x) = e^x \leq e^0 = 1$ ; also, since  $f(x) = e^x > 0$ , then the inequality  $0 < f(x) \leq 1$  is true. Which,  $f(C) = (0, 1]$ , and it is not closed.