Math CS 122A HW1

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1

Question 1 ***Ahlfors Pg. 9 Problem 5: Prove Lagrange's Identity in the complex form:***

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{i \le i < j \le n} |a_i \overline{b_j} - a_j \overline{b_i}|^2$$

Pf:

2

Question 2 ***Ahlfors Pg. 11 Problem 1: Prove that:***

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$

if $|a| < 1$ ***and*** $|b| < 1$***.***

Pf:

Suppose $a, b \in \mathbb{C}$ satisfy $|a|, |b| < 1$. Then, since $|\bar{a}b| = |\bar{a}| \cdot |b| = |a| \cdot |b| < 1$, $|\bar{a}b| \neq |1| = 1$, then $\bar{a}b \neq 1$. So, $1 - \bar{a}b \neq 0$, the term $\frac{a-b}{1-\bar{a}b}$ is defined.

Now, consider the following identity:

$$\forall x, y \in \mathbb{C}, \quad |x-y|^2 = |x|^2 + |y|^2 - 2Re(x\bar{y})$$

So, the following equations are true:

$$|a-b|^2 = |a|^2 + |b|^2 - 2Re(a\bar{b})$$

$$|1-\bar{a}b|^2 = |1|^2 + |\bar{a}b|^2 - 2Re(\bar{a}b)$$

Which, $|\bar{a}b| = |\bar{a}| \cdot |b| = |a| \cdot |b|$, and $Re(\bar{a}b) = \frac{\bar{a}b + \overline{\bar{a}b}}{2} = \frac{\bar{a}b + a\bar{b}}{2} = Re(a\bar{b})$, so the equation can be simplified to:

$$|1-\bar{a}b|^2 = 1 + |a|^2 \cdot |b|^2 - 2Re(a\bar{b})$$

Then, consider the term $(1 + |a|^2 \cdot |b|^2) - (|a|^2 + |b|^2)$:

$$(1 + |a|^2 \cdot |b|^2) - (|a|^2 + |b|^2) = (1 - |b|^2) + |a|^2(|b|^2 - 1)$$

$$= (1 - |b|^2) - |a|^2(1 - |b|^2) = (1 - |a|^2)(1 - |b|^2)$$

Since both $|a|, |b| < 1$, then $|a|^2, |b|^2 < 1$, which $(1 - |a|^2), (1 - |b|^2) > 0$. Hence, we can conclude that $(1 + |a|^2 \cdot |b|^2) - (|a|^2 + |b|^2) = (1 - |a|^2)(1 - |b|^2) > 0$, which:

$$(1 + |a|^2 \cdot |b|^2) > (|a|^2 + |b|^2), \quad (1 + |a|^2 \cdot |b|^2 - 2Re(a\bar{b})) > (|a|^2 + |b|^2 - 2Re(a\bar{b}))$$

Replace the terms with the original form of absolute value, we get:

$$|1 - \bar{a}b|^2 > |a - b|^2$$

Because $(1 - \bar{a}b) \neq 0$, then $|1 - \bar{a}b|^2 > 0$. So:

$$1 > \frac{|a-b|^2}{|1-\bar{a}b|^2} = \left|\frac{a-b}{1-\bar{a}b}\right|^2, \quad \left|\frac{a-b}{1-\bar{a}b}\right| < 1$$

Which the inequality is true.

3

Question 3 ***Ahlfors Pg. 11 Problem 3: If*** $|a_i| < 1,\ \lambda_i \geq 0$ ***for all*** $i = 1, ..., n$***, and*** $\lambda_1 + ... + \lambda_n = 1$***, show that:***

$$|\lambda_1 a_1 + ... + \lambda_n a_n| < 1$$

Pf:

Given that $|a_i| < 1,\ \lambda_i \geq 0$ for all $i = 1, ..., n$, and $\lambda_1 + ... + \lambda_n = 1$. By the Triangle Inequality:

$$|\lambda_1 a_1 + ... + \lambda_n a_n| \leq |\lambda_1 a_1| + ... + |\lambda_n a_n| = \lambda_1 |a_1| + ... + \lambda_n |a_n|$$

(Note: above is true since each coefficient $\lambda_i \geq 0$). Then, let $M = \max\{|a_1|, ..., |a_n|\}$, which for all $i \in \{1, ..., n\}$, $|a_i| \leq M$; and since $|a_i| < 1$ for all index i, $M < 1$. Thus, the following is true:

$$\lambda_1 |a_1| + ... + \lambda_n |a_n| \leq \lambda_1 \cdot M + ... + \lambda_n \cdot M = M(\lambda_1 + ... + \lambda_n) = M$$

Which, combining all the inequalities above, we get:

$$|\lambda_1 a_1 + ... + \lambda_n a_n| \leq \lambda_1 |a_1| + ... + \lambda_n |a_n| \leq M < 1$$

$$|\lambda_1 a_1 + ... + \lambda_n a_n| < 1$$

Which, the given inequality is true.

4

Question 4 ***Ahlfors Pg. 16 Problem 4: If*** $w = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$ ***for some positive integer*** n***, prove that:***

$$1 + w^h + ... + w^{(n-1)h} = 0$$

For any integer h ***which is not a multiple of*** n***.***

Pf: Given that $w = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$ and $h \in \mathbb{Z}$ is not a multiple of n. Then, the term $\frac{h}{n}$ is not an integer, which consider w^h:

$$w^h = \cos\left(\frac{h \cdot 2\pi}{n}\right) + i\sin\left(\frac{h \cdot 2\pi}{n}\right) \neq 1$$

Since $\frac{h}{n}$ is not an integer, $\frac{h \cdot 2\pi}{n}$ is not an integer multiple of 2π, thus $\cos\left(\frac{h \cdot 2\pi}{n}\right) \neq 1$, or $w^h \neq 1$.

Hence, $(1 - w^h) \neq 0$, division with this number is defined. Now, consider the following:

$$1 + w^h + ... + w^{(n-1)h} = \frac{(1 - w^h)(1 + w^h + ... + w^{(n-1)h})}{(1 - w^h)} = \frac{1 - w^{nh}}{1 - w^h}$$

Which, since $w^n = \cos(\frac{n \cdot 2\pi}{n}) + i\sin(\frac{n \cdot 2\pi}{n}) = \cos(2\pi) + i\sin(2\pi) = 1$, then $w^{nh} = (w^n)^h = 1^h = 1$. Thus:

$$1 + w^h + ... + w^{(n-1)h} = \frac{1 - w^{nh}}{1 - w^h} = \frac{1 - 1}{1 - w^h} = 0$$

Which, the given equality is true.

5

Question 5 ***Ahlfors Pg. 17 Problem 5: What is the value of:***

$$1 - w^h + w^{2h} + ... + (-1)^{(n-1)}w^{(n-1)h}$$

Pf:

Using the same condition of **Question 4**, there are two cases to consider:

First, if $h = \frac{2k+1}{2}n$ for some $k \in \mathbb{Z}$, then:

$$w^h = \cos(\frac{h \cdot 2\pi}{n}) + i\sin(\frac{h \cdot 2\pi}{n})$$

$$= \cos(\frac{(2k+1)n\pi}{n}) + i\sin(\frac{(2k+1)n\pi}{n}) = \cos((2k+1)\pi) + i\sin((2k+1)\pi) = -1$$

Which the sum is as follow:

$$1 - w^h + w^{2h} + ... + (-1)^{(n-1)}w^{(n-1)h} = \sum_{j=0}^{n-1}(-1)^j(w^h)^j = \sum_{j=0}^{n-1}(-1)^j(-1)^j$$

$$1 - w^h + w^{2h} + ... + (-1)^{(n-1)}w^{(n-1)h} = \sum_{j=0}^{n-1}(1)^j = \sum_{j=0}^{n-1}1 = n$$

Under the case of $n = 2h$, the sum yields a value of n.

Else, if $h \neq \frac{2k+1}{2}n$, for all $k \in \mathbb{Z}$, then since $\frac{h \cdot 2\pi}{n} \neq \frac{(2k+1)n\pi}{n} = (2k+1)\pi$ for all $k \in \mathbb{Z}$, then $\cos(\frac{h \cdot 2\pi}{n}) \neq -1$, which $w^h = \cos(\frac{h \cdot 2\pi}{n}) + i\sin(\frac{h \cdot 2\pi}{n}) \neq -1$, or $(1 + w^h) \neq 0$.

Thus, the sum can be expressed as follow:

$$1 - w^h + w^{2h} + ... + (-1)^{(n-1)}w^{(n-1)h} = \frac{(1 + w^h)(1 - w^h + w^{2h} + ... + (-1)^{(n-1)}w^{(n-1)h})}{(1 + w^h)} = \frac{1 + w^{nh}}{1 + w^h}$$

$$1 - w^h + w^{2h} + ... + (-1)^{(n-1)}w^{(n-1)h} = \frac{2}{1 + w^h}$$

Since $w^{nh} = 1$ is proven in **Question 4**.