Math CS Topology Final

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Question 1 *Without looking it up, define topological space, and continuous function between topological spaces.*

Pf:

Topological Space:

Given any set X, a topology on X is a classification of what subsets are "open" in the set. More formally, a topology on X is a collection $\mathcal{T}$ collecting open subsets of X, such that the following conditions are satisfied:

(1) $\emptyset, X \in \mathcal{T}$ (empty set and the shole set are open).

(2) For arbitrary collection $\mathcal{U} \subseteq \mathcal{T}$, the union $\bigcup \mathcal{U} \in \mathcal{T}$ (union of arbitrary collection of open sets is open).

(3) For any two $U, V \in \mathcal{T}$, the intersection $U \cap V \in \mathcal{T}$ (intersection of finite open sets is open).

Which, $(X, \mathcal{T})$ (the set X along with a topology $\mathcal{T}$ on X) is called a "Topological Space".

Continuous Function:

Given X, Y two topological spaces, a function $f : X \to Y$ is continuous, if for any open sets $U \subseteq Y$, its preimage $f^{-1}(U) \subseteq X$ is open.

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Question 2 *Prove that the product of two path connected spaces is path connected.*

Pf:

Suppose X, Y are two path connected spaces (i.e. for any two points in the same space, there exists a continuous path joining the two points).

Now, consider $X \times Y$ under product topology. For any $(x_1, y_1), (x_2, y_2) \in X \times Y$, since $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, by the definition of path connected space, there exists two continuous paths $f : [0,1] \to X$ and $g : [0,1] \to Y$, with $f(0) = x_1, f(1) = x_2$, and $g(0) = y_1, g(1) = y_2$.

Then, define the function $h : [0,1] \to X \times Y$ as $h(t) = (f(t), g(t))$. Then, $h(0) = (f(0), g(0)) = (x_1, y_1)$, and $h(1) = (f(1), g(1)) = (x_2, y_2)$. So, this is potentially a function joining the two given points, it remains to show that h is continuous.

For any open set $W \subseteq X \times Y$, consider the preimage $h^{-1}(W) \subseteq [0,1]$: For any $t_0 \in h^{-1}(W)$, since $h(t_0) = (f(t_0), g(t_0)) \in W$, by the definition of openness in product topology, there exists open sets $U \subseteq X$ and $V \subseteq Y$, such that $h(t_0) = (f(t_0), g(t_0)) \in (U \times V) \subseteq W$. Then, since f, g are continuous, then $f^{-1}(U), g^{-1}(V) \subseteq [0,1]$ are open.

Then, consider $f^{-1}(U) \cap g^{-1}(V)$, which is also open in $[0,1]$: Since $f(t_0) \in U$ and $g(t_0) \in V$, then $t_0 \in f^{-1}(U)$ and $t_0 \in g^{-1}(V)$, hence $t_0 \in f^{-1}(U) \cap g^{-1}(V)$.

Now, for all $t \in f^{-1}(U) \cap g^{-1}(V)$, since $h(t) = (f(t), g(t))$ has $f(t) \in U$ and $g(t) \in V$, then $h(t) \in (U \times V) \subseteq W$, showing that $t \in h^{-1}(U \times V) \subseteq h^{-1}(W)$. So, we can conclude that $t_0 \in (f^{-1}(U) \cap g^{-1}(V)) \subseteq h^{-1}(W)$.

Since $f^{-1}(U) \cap g^{-1}(V)$ is open in $[0,1]$, therefore we can conclude that $h^{-1}(W) \subseteq [0,1]$ is in fact open.

This proves that the new function h is in fact continuous. Since it joins (x_1, y_1) and (x_2, y_2), then it is a path joining the two given points. So, because any two points in $X \times Y$ can be joined by continuous paths, we can conclude that $X \times Y$ is in fact path connected.

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Question 3 *Suppose X is a connected metric space. Prove that, for every pair of points $a, b \in X$, there exists a sequence $x_1, x_2, ..., x_n$ such that $a = x_1$, $b = x_n$, and $d(x_i, x_{i+1}) \leq 1$ for all $i = 1, ..., n-1$.*

Pf:

Since X is a connected metric space, then the only subset of X that is both open and closed (clopen subsets), is $\emptyset$ and X itself.

Based on this logic, pick any point $x \in X$, and define the set C, such that for all points $a \in C$, there exists a sequence $x_1, ..., x_n$, with $x = x_0, a = x_n$, and $d(x_i, x_{i+1}) \leq 1$ for all $i = 1, ..., (n-1)$. Which, for simplicity, assume $x \in C$ also (i.e. can have a sequence of length 1, with $x_1 = x$, connecting x to x itself).

Our first goal is to prove that $C = X$.

The set C is open:

For all points $a \in C$, there exists a sequence $x_1, ..., x_n$, such that $x_1 = x$ and $x_n = a$, while each $i = 1, ..., n-1$ has $d(x_i, x_{i+1}) \leq 1$. Then, consider the open ball $B(a, 1)$: every point $z \in B(a, 1)$ has $d(a, z) < 1$, so for point z, choose the sequence to be $x = x_1, ..., x_n = a, x_{n+1} = z$, which by the definition of set C, we know every $i = 1, ..., n-1$ has $d(x_i, x_{i+1}) \leq 1$, while for $i = n$, $d(x_n, x_{n+1}) = d(a, z) < 1$, hence this sequence joining $x = x_1$ and $z = x_{n+1}$ satisfies the condition of C, so $z \in C$. This proves that $a \in B(a, 1) \subseteq C$, or C is open under metric topology.

The set $X \setminus C$ is open:

For all $b \in X \setminus C$, none of the finite sequence $x = x_1, ..., x_n = b$ satisfies the given condition, there exists $i \in \{1, ..., n-1\}$, with $d(x_i, x_{i+1}) > 1$.

This implies that the open ball $B(b, 1) \subseteq X \setminus C$: Suppose the contrary, that $B(b, 1) \nsubseteq (X \setminus C)$, then there exists $c \in B(b, 1)$, with $c \in C$. Hence, there exists a sequence $x = x_1, ..., x_n = c$, such that all $i = 1, ..., n-1$, $d(x_i, x_{i+1}) \leq 1$; also, since $c \in B(b, 1)$, then $d(b, c) < 1$. Which, add a new point to the sequence $x_{n+1} = b$, every $i = 1, ..., n-1$ has $d(x_i, x_{i+1})$ by assumption, while for $i = n$, $d(x_n, x_{n+1}) = d(c, b) < 1$, hence the sequence $x = x_1, ..., x_n = c, x_{n+1} = b$ satisfies the given condition, implying that $b \in C$. Yet, this contradicts the assumption that $b \in X \setminus C$, hence our assumption must be false, showing that $b \in B(b, 1) \subseteq (X \setminus C)$, proving that $(X \setminus C)$ is open under metric topology.

The set $C = X$:

Since $X \setminus C$ is open, then C must be closed; yet, since C is also open, then C is clopen implies $C = \emptyset$ or $C = X$. Yet, it is clear that $x \in C$ (can assume X is nonempty, hence there exists $x \in X$), so $C \neq \emptyset$. Therefore, the only possibility is $C = X$.

The Original Statement is True:

Since $C = X$, then for all $a, b \in X = C$, there exists two sequences $x_1, ..., x_n$, and $y_1, ..., y_m$, such that $x_1 = y_1 = x$, $x_n = a$, $y_m = b$, and for all $i \in \{1, ..., n-1\}$ and $j \in \{1, ..., m-1\}$, we have $d(x_i, x_{i+1}), d(y_j, y_{j+1}) \leq 1$.

Now, define the new sequence $z_1, ..., z_{m+n}$ based on the following logic:

$$\forall i \in \{1, ..., n\}, \quad z_i = x_{(n+1)-i}, \qquad \forall i \in \{n+1, ..., n+m\}, \quad z_i = y_{i-n}$$

So, $z_1 = x_{(n+1)-1} = x_n = a$, and $z_{m+n} = y_{(m+n)-n} = y_m = b$.

Also, for all $i \in \{1, ..., n, ..., n+m-1\}$, if $i \leq (n-1)$, we have $d(z_i, z_{i+1}) = d(x_{(n+1)-i}, x_{(n+1)-(i+1)}) =\leq 1$; if $i = n$, then $d(z_n, z_{n+1}) = d(x_1, y_1) = d(x, x) = 0 \leq 1$; else, if $i \geq (n+1)$, $d(z_i, z_{i+1}) = d(y_{i-n}, y_{(i+1)-n}) \leq 1$.

Hence, the finite sequence $z_1, ..., z_{m+n}$ has $z_1 = a$, $z_{m+n} = b$, and every consequent elements have distance at most 1 apart.

This shows that for any pair $a, b \in X$, there exists a sequence $z_1, ..., z_k$ with $a = z_1$, $b = z_k$, and $d(z_i, z_{i+1}) \leq 1$ for all $i = 1, ..., k-1$, which the original statemenet is true.

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Question 4 *Prove that if X and Y are compact Hausdorff spaces then any surjective continuous function $f : X \to Y$ is a quotient map.*

Pf:

To prove if a map is a quotient map, we need all sets $U \subseteq Y$ to be open iff its preimage $f^{-1}(U) \subseteq X$ is open.

$\implies$: Suppose $U \subseteq Y$ is open, then by the definition of continuous function, its preimage $f^{-1}(U) \subseteq X$ is open.

$\impliedby$: Then, suppose for $U \subseteq Y$, its preimage $f^{-1}(U) \subseteq X$ is open. Given that X is Compact and Hausdorff, since $(X \setminus f^{-1}(U)) \subseteq X$ is closed, implies it is also a compact set.

Which, since continuous function sends compact sets to compact sets, then $f(X \setminus f^{-1}(U)) \subseteq Y$ is compact, which as Y is itself a Hausdorff space, then $f(X \setminus f^{-1}(U))$ is compact implies it is closed.

Because for all $x \in (X \setminus f^{-1}(U))$, $x \notin f^{-1}(U)$, which shows that $f(x) \notin U$, hence $f(x) \in Y \setminus U$, so $f(X \setminus f^{-1}(U)) \subseteq (Y \setminus U)$; similarly, because f is surjective, all $y \in Y \setminus U$ has preimage $f^{-1}(\{y\}) \subseteq (X \setminus f^{-1}(U))$ (since $y \notin U$, its preimage can't be in $f^{-1}(U)$), hence $y \in f(X \setminus f^{-1}(U))$, showing that $Y \setminus U \subseteq f^(X \setminus f^{-1}(U))$. This proves that $(Y \setminus U) = f(X \setminus f^{-1}(U))$.

Lastly, because $f(X \setminus f^{-1}(U)) = Y \setminus U$ is closed, then its complement U is open.

The above two statements proved that $U \subseteq Y$ is open iff $f^{-1}(U) \subseteq X$ is open, hence the map f given is in fact a quotient map.

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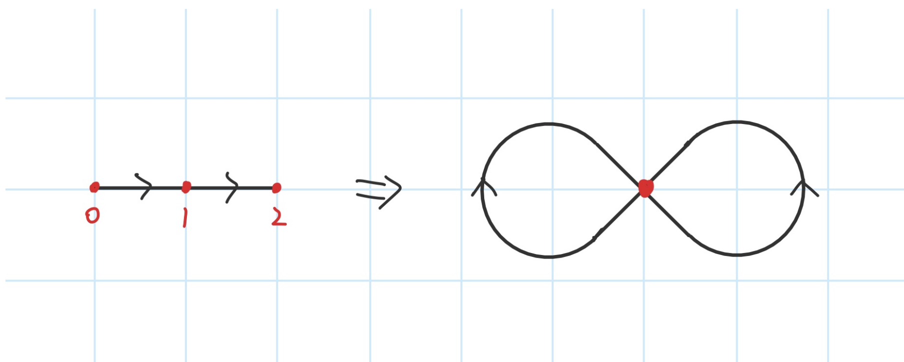
Question 5 *If A is a subspace of X, then X/A is the quotient space where A is collapsed to a point. Describe or draw (or both) the following quotient space.*

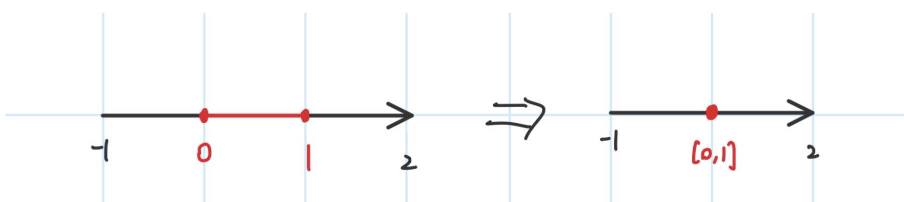
- $[0,2]/\{0,1,2\}$
- $\mathbb{R}/[0,1]$
- $\mathbb{R}/\{0,1\}$
- $\mathbb{R}^2/S^1$
- *a Möbius band / its boundary circle.*

Pf:

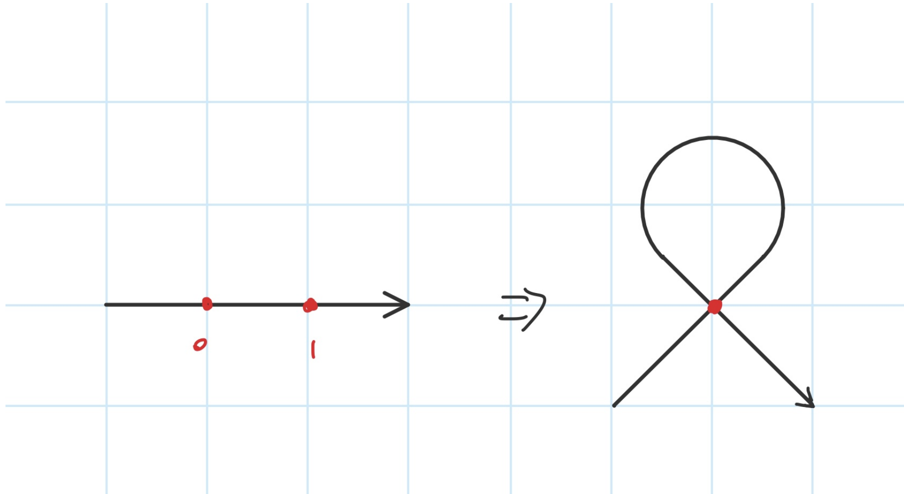
- $[0,2]/\{0,1,2\}$:



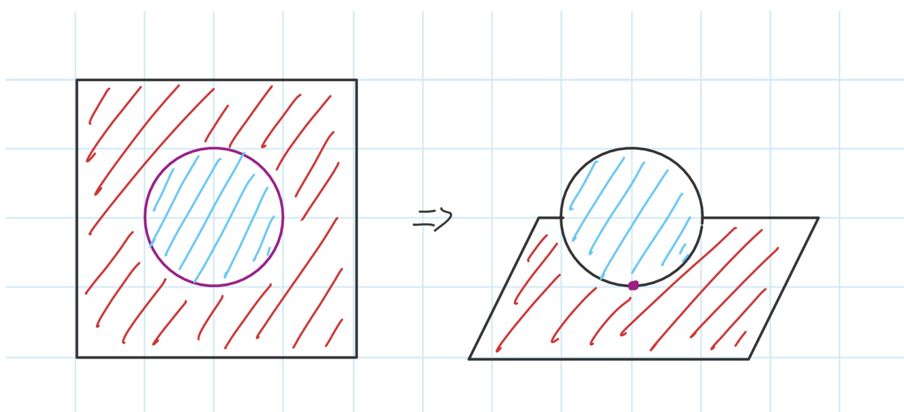
- $\mathbb{R}/[0,1]$:



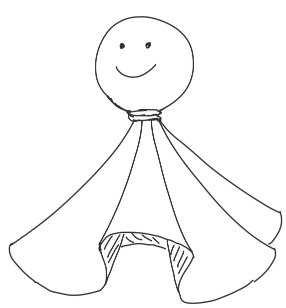
- $\mathbb{R}/\{0,1\}$:



- $\mathbb{R}^2/S^1$:



As an extra fun fact, in Japan there is a type of doll used as a charm to stop the rain, called the ”Shine, Shine Monk” (てるてる坊主, ”Teru Teru Bōzu”), and it looks a lot like $\mathbb{R}^2/S^1$:



I guess $\mathbb{R}^2/S^1$ could be a potential topological name for it.

- a Möbius band / its boundary circle: By following modification, it forms $\mathbf{RP}^2$.