# Math 111B HW2

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## 1

**Question 1** *Let $f:(a,b)\to\mathbb{R}$ be differentiable on $(a,b)$.*
*Prove : if $\forall x\in(a,b),\ f'(x)\neq 0$, then $f$ is one-to-one on $(a,b)$.*
*Give an example showing that the converse statement is in general not true.*

**Pf:**
Suppose $\forall x\in(a,b),\ f'(x)\neq 0$:

**(1) $f'(x)$ is strictly less than or greater than $0$ on $(a,b)$:**
We'll prove by contradiction: Suppose $f'(x)$ is neither strictly less than 0 nor strictly greater than 0 on $(a,b)$, then there exists $x_0,x_1\in(a,b)$, with $f'(x_0)\leq 0$ and $f'(x_1)\geq 0$, and by the assumption that $f'(x)\neq 0$, the strict inequality $f'(x_0)<0$ and $f'(x_1)>0$ is applied. (This also implies $x_0\neq x_1$, since derivatives are different at the two points).

Recall that for function $f:[a,b]\to\mathbb{R}$ be differentiable on $(a,b)$, if $a<c<d<b$ and $f'(c)\neq f'(d)$, for any $\lambda$ strictly in between $f'(c)$ and $f'(d)$ (either $f'(c)<\lambda<f'(d)$ or $f'(c)>\lambda>f'(d)$), there exists $x\in(c,d)$ with $f'(x)=\lambda$.

Then, first suppose $x_0<x_1$: $f$ is differentiable on $(a,b)$ and $f'(x_0)<0<f'(x_1)$ implies there exists $x\in(x_0,x_1)$ with $f'(x)=0$, which contradicts the assumption;

then suppose $x_1<x_0$: again, $f$ is differentiable on $(a,b)$ and $f'(x_1)>0>f'(x_0)$ implies there exists $x\in(x_1,x_0)$ with $f'(x)=0$, which again contradicts the assumption.

So, the assumption is false, $f'(x)$ must be strictly greater than 0 or less than 0 for all $x\in(a,b)$.

**(2) $f$ is strictly increasing or decreasing on $(a,b)$:**
Based on **(1)**, $f'(x)$ is strictly less than 0 or strictly greater than 0.

Suppose $f'(x)>0$ for all $x\in(a,b)$, then for any $x,y\in(a,b)$ with $x<y$, by the Mean Value Theorem, there exists $c\in(x,y)\subseteq(a,b)$, such that the following is true:

$$\frac{f(y)-f(x)}{y-x}=f'(c)>0,\quad (f(y)-f(x))=f'(c)(y-x)$$

Since $(y-x),f'(c)>0$ by assumption, the $(f(y)-f(x))=f'(c)(y-x)>0$, thus $f(y)>f(x)$, showing that $f$ is strictly increasing.

Similarly, suppose $f'(x)<0$ for all $x\in(a,b)$, with the same $x,y$ above, by Mean Value Theorem, there exists $c\in(x,y)\subseteq(a,b)$, such that the following is true:

$$\frac{f(y)-f(x)}{y-x}=f'(c)<0,\quad (f(y)-f(x))=f'(c)(y-x)$$

Since $(y-x)>0$ and $f'(c)<0$, then $(f(y)-f(x))=f'(c)(y-x)<0$, of $f(y)<f(x)$, showing that $f$ is strictly decreasing.

With the above condition, since $f$ is either strictly increasing or strictly decreasing on $[a, b]$, then for all $x, y \in (a, b)$, $x \neq y$ implies $f(x) \neq f(y)$ (or else it's no longer strictly increasing or decreasing). Thus, $f$ is in fact one-to-one on $(a, b)$.

**Counterexample of Converse:**

Let $f : [-1, 1] \to \mathbb{R}$ be $f(x) = x^3$, which $f'(x) = 3x^2$, which $f'(0) = 0$. Yet, suppose $x, y \in (-1, 1)$ has $x^3 = y^3$, then:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2) = 0$$

Which, the only real solution is $x = y$ (since if treating $y$ as constant, $y^2 - 4y^2 = -3y^2 \leq 0$; the only time with real solution is when $y = 0$, which implies $x^3 = 0$, or $x = 0$).

So, $f(x) = x^3$ is one-to-one on the region $(-1, 1)$, but still has $f'(0) = 0$, which is a counterexample.

# 2

**Question 2** *Let $f : (a, b) \to R$ be a function such that:*

$$\exists M > 0, \exists \alpha > 0,\ \forall x, y \in (a, b),\ |f(x) - f(y)| < M|x - y|^\alpha$$

*If $\alpha \in (0, 1)$, then $f$ is Holder of order $\alpha$ in $(a, b)$. If $\alpha = 1$, then $f$ is Lipschitz. Prove :*
*(a) If $\alpha > 1$, then $f$ is constant.*
*(b) If $\alpha \in (0, 1]$, then $f$ is uniformly continuous on $(a, b)$.*
*(c) Give an example such that $f$ is Lipschitz, but not differentiable.*
*(d) If $f$ is differentiable on $(a, b)$ and $f(x)$ is bounded on (a, b), then f is Lipschitz.*

**Pf:**

(a) Suppose $\alpha > 1$, then there exists $\epsilon > 0$, such that $\alpha = 1 + \epsilon$. Which, for all $x, y \in (a, b)$ (with $x \neq y$), the following is true:

$$|f(x) - f(y)| < M|x - y|^\alpha = M|x - y|^{1+\epsilon} = M|x - y| \cdot |x - y|^\epsilon$$

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \frac{|f(x) - f(y)|}{|x - y|} < M|x - y|^\epsilon$$

Which, fix arbitrary $x_0 \in (a, b)$, for all $y \in (a, b)$ with $y \neq x_0$, the following is true:

$$0 \leq \left|\frac{f(x_0) - f(y)}{x_0 - y}\right| < M|x_0 - y|^\epsilon, \quad -M|x_0 - y|^\epsilon < \frac{f(x_0) - f(y)}{x_0 - y} < M|x_0 - y|^\epsilon$$

Since $\epsilon > 0$, then $\lim_{y \to x_0} |x_0 - y|^\epsilon = 0$. Which, by Squeeze Theorem, the following is true:

$$0 = \lim_{y \to x_0} -M|x_0 - y|^\epsilon \leq \lim_{y \to x_0} \frac{f(x_0) - f(y)}{x_0 - y} \leq \lim_{y \to x_0} M|x_0 - y|^\epsilon = 0$$

Thus, $\lim_{y \to x_0} \frac{f(x_0) - f(y)}{x_0 - y} = 0$, or $f'(x_0) = 0$.

This implies that $f(x)$ is a constant function: Suppose $f(x)$ is not a constant function, then there exists $c, d \in (a, b)$ with $c < d$, such that $f(c) \neq f(d)$.

Notice that since $f'(x_0)$ exists for all $x_0 \in (a, b)$, then by Mean Value Theorem, there exists $x \in (c, d)$, such that $f'(x)(d - c) = f(d) - f(c)$.

Yet, since $f'(x) = 0$, while $f(d) - f(c) \neq 0$, $0 = f'(x)(d - c) \neq f(d) - f(c)$, which it is a contradiction.

Thus, $f(x)$ must be a constant function.

(b) Suppose $\alpha \in (0, 1]$, notice that for all $x, y \in (a, b)$, the following is true:

$$a < x < b, \quad -b < -y < -a, \quad (a - b) = -(b - a) < (x - y) < (b - a), \quad |x - y| < |b - a|$$

Which, since $\alpha > 0$, then $|x - y|^\alpha < |b - a|^\alpha$. Now, for any $\epsilon > 0$, define $\delta = (\frac{\epsilon}{M})^{\frac{1}{\alpha}} > 0$, then for all $x, y \in (a, b)$, if $|x - y| < \delta$, the following is true:

$$|f(x) - f(y)| < M|x - y|^\alpha < M \cdot \delta^\alpha$$

(Note: the above inequality is true, since $\alpha > 0$, then $0 \leq |x - y| < |b - a|$ implies $|x - y|^\alpha < |b - a|^\alpha$). Thus, it can be rewritten as:

$$|f(x) - f(y)| < M \cdot \delta^\alpha = M \cdot \left( \left( \frac{\epsilon}{M} \right)^{\frac{1}{\alpha}} \right)^\alpha = M \cdot \frac{\epsilon}{M} = \epsilon$$

Thus, since for all $\epsilon > 0$, there exists $\delta > 0$ with $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, $f$ is uniformly continuous.

(c) Consider the function $f : (-1, 1) \to \mathbb{R}$ by $f(x) = |x|$.

Choose $M = 1.01$ and $\alpha = 1$, then the following is true:

$$\forall x, y \in (-1, 1), \quad |f(x) - f(y)| = ||x| - |y|| \leq |x - y| = |x - y|^\alpha < 1.01|x - y|^\alpha = M|x - y|^\alpha$$

Thus, $f$ is Lipschitz continuous.

Yet, $f$ is not differentiable at $x = 0$: For all $x < 0$ and $y > 0$ (with $x, y \in (-1, 1)$), the following is true:

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x| - 0}{x - 0} = \frac{-x}{x} = -1$$

$$\frac{f(y) - f(0)}{y - 0} = \frac{|y| - 0}{y - 0} = \frac{y}{y} = 1$$

Which, $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ is not defined, since the left and right limits as $x$ approaches 0 are different.

(d) Suppose $f$ is differentiable on $(a, b)$ and $f'(x)$ is bounded on $(a, b)$, then there exists $M > 0$, with $|f'(x)| < M$ for all $x \in (a, b)$. Which, for all $x, y \in (a, b)$ with $x < y$, by the Mean Value Theorem, there exists $c \in (x, y)$, such that $f(y) - f(x) = f'(c)(y - x)$. Which, the following is true:

$$|f(y) - f(x)| = |f'(c)| \cdot |y - x| < M|y - x|$$

Thus, $f$ is Lipschitz continuous.

## 3

**Question 3** *For any $a \geq 0$, define $f_a : \mathbb{R} \to \mathbb{R}$ as:*

$$f_a(x) = \begin{cases} x^a sin(\frac{1}{x}) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

*(a) For which values of a is $f_a$ continuous at 0.*
*(b) For which values of a is $f_a'(0)$ defined.*
*(c) For which values of a is $f_a'$ continuous at 0.*
*(d) For which values of a is $f_a''(0)$ defined.*

**Pf:**

(a) **Ans:** $a > 0$. For $a = 0$, the function $f_a(x)$ is not continuous: Choose the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_n = \frac{1}{(2n+1/2)\pi} > 0$, then $\lim_{n\to\infty} \frac{1}{(2n+1/2)\pi} = 0$, thus $x_n$ converges to 0; but, consider $(f_a(x_n))_{n\in\mathbb{N}}$:

$$\forall n \in \mathbb{N}, \quad f_a(x_n) = x_n^0 \sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{1}{1/(2n+1/2)\pi}\right) = \sin((2n+1/2)\pi) = 1$$

Which, $\lim_{n\to\infty} f_a(x_n) = 1 \neq 0 = f_a(0)$, thus $f_a(x_n)$ doesn't converge to $f_a(0)$, showing it's not continuous.

Now, for all $a > 0$, for any $x > 0$, since $x^a > 0$, it satisfies the following:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1, \quad -x^a \leq f_a(x) = x^a \sin\left(\frac{1}{x}\right) \leq x^a$$

Which, take the right limit of $x^a$ of 0, $\lim_{x\to 0^+} x^a = 0$, then by Squeeze Theorem, the following is true:

$$0 = \lim_{x\to 0^+} -x^a \leq \lim_{x\to 0^+} x^a \sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0^+} x^a = 0$$

Thus, $\lim_{x\to 0^+} f_a(x) = 0$.

Also, since $\lim_{x\to 0^-} f_a(x) = 0$ (since for $x < 0$, $f_a(x) = 0$), then the left and right limits both agree with $f_a(0) = 0$, showing it's continuous at 0. Every $a > 0$ has $f_a(x)$ being continuous at 0.

(b) **Ans:** $a > 1$**.** In case for $f_a'(0)$ to be defined, $f_a$ must be continuous at 0. Thus, $a > 0$ is required.

Consider the slope $\frac{f_a(x)-f_a(0)}{x-0}$ for all $x \neq 0$. If $x < 0$, then since $f_a(x) = 0$, then the slope is 0. Thus, the left limit of the slope $\lim_{x\to 0^-} \frac{f_a(x)-f_a(0)}{x-0} = 0$.

Now, consider the slope from the right:

$$x > 0, \quad \frac{f_a(x) - f_a(0)}{x - 0} = \frac{x^a \sin(1/x) - 0}{x - 0} = x^{a-1} \sin\left(\frac{1}{x}\right)$$

Since the left limit is evaluated as 0, in case for $f'(0)$ to be defined, the right limit also needs to converge to 0.

First, notice that if $a \leq 1$, the right limit doesn't exist:

Consider the same sequence $x_n = \frac{1}{(2n+1/2)\pi} > 0$ used in part (a), then the following is true:

$$\forall n \in \mathbb{N}, \quad (x_n)^{a-1} \sin\left(\frac{1}{x_n}\right) = \left(\frac{1}{(2n+1/2)\pi}\right)^{a-1} \sin((2n+1/2)\pi) = \left(\frac{1}{(2n+1/2)\pi}\right)^{a-1}$$

Which, if $a = 1$ (or $a-1 = 0$), then $(x_n)^{a-1}\sin(1/x_n) = 1$ for all $n \in \mathbb{N}$, which $\lim_{n\to\infty} \frac{f_a(x_n)-f_a(0)}{x_n-0} = 1$, while $\lim_{n\to\infty} x_n = 0$. This shows that the right limit of the slope is not 0, which $f_a'(0)$ is not defined.

Else, if $a < 1$ (or $a - 1 < 0$), then $(x_n)^{a-1}\sin(1/x_n) = \left(\frac{1}{(2n+1/2)\pi}\right)^{a-1} = ((2n+1/2)\pi)^{1-a}$ is in fact unbounded as $n$ increases indefinitely (since $1 - a > 0$), so again the right limit of the slope is not defined, implying $f_a'(0)$ is not defined.

So, in case for the right limit to be defined, $a > 1$. Which, since $a-1 > 0$, then for all $x > 0$, $x^{a-1} > 0$, and $\lim_{x\to 0^+} a^{a-1} = 0$. Thus based on Squeeze Theorem:

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1, \quad x > 0, \quad -x^{a-1} \le x^{a-1}\sin\left(\frac{1}{x}\right) \le x^{a-1}$$

$$0 = \lim_{x\to 0^+} -x^{a-1} \le \lim_{x\to 0^+} x^{a-1}\sin\left(\frac{1}{x}\right) \le \lim_{x\to 0^+} x^{a-1} = 0$$

So, the right limit of $x^{a-1}\sin(1/x)$ is 0 when $x$ approaches 0, which it agrees with the initial left limit, hence for $a > 1$, $\lim_{x\to 0} \frac{f_a(x)-f_a(0)}{x-0} = 0$, $f_a'(0) = 0$ is defined.

(c) **Ans:** $a > 2$. For $f_a'$ to be continuous at 0, $f_a'(0)$ needs to be defined. So, $a > 1$ is required.
For $x < 0$, since $f_a(x) = 0$, then $f_a'(x) = 0$, which $\lim_{x\to 0^-} f_a'(x) = 0$.
Consider $f_a'(x)$ for $x > 0$, which by the differentiation rule, it is evaluated as:

$$f_a'(x) = ax^{a-1}\sin\left(\frac{1}{x}\right) + x^a\cos\left(\frac{1}{x}\right)\frac{-1}{x^2} = ax^{a-1}\sin\left(\frac{1}{x}\right) - x^{a-2}\cos\left(\frac{1}{x}\right)$$

In case for $f_a'(x)$ to be continuous at 0, $\lim_{x\to 0^+} f_a'(x) = 0$.
Since $x^{a-1}\sin(1/x)$ has right limit exists as $x$ approaches 0 (since we assume $a > 1$), it suffices to find values of $a$ which $x^{a-2}\cos(1/x)$ has right limit being 0, when $x$ approaches 0.

For $a \le 2$, the right limit of $x^{a-2}\cos(1/x)$ is not 0:
Consider the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_n = \frac{1}{2n\pi}$, then $\lim_{n\to\infty} x_n = 0$. Which, the following is true:

$$\forall n \in \mathbb{N}, \quad (x_n)^{a-2}\cos\left(\frac{1}{x_n}\right) = \left(\frac{1}{2n\pi}\right)^{a-2}\cos\left(\frac{1}{2n\pi}\right) = (2n\pi)^{2-a}$$

Which, if $a = 2$, $2 - a = 0$, hence $(x_n)^{a-2}\cos(1/x_n) = 1$, implying $\lim_{n\to\infty}(x_n)^{a-2}\cos(1/x_n) = 1 \ne 0$. This implies that $x^{a-2}\cos(1/x)$ doesn't converge to 0 as $x$ converges to 0.
Else, if $a < 2$, then since $(2 - a) > 0$, $(2n\pi)^{2-a}$ goes unbounded as $n$ increases indefinitely, so again $x^{a-2}\cos(1/x)$ doesn't converge to 0 when $x$ converges to 0.

So, for right limit of $f_a'(x)$ of $x = 0$ to be 0, $a > 2$ is required. Which, for $a > 2$, since $a - 2 > 0$, then for all $x > 0$, $x^{a-2} > 0$. Thus by Squeeze Theorem:

$$-x^{a-2} \le x^{a-2}\cos\left(\frac{1}{x}\right) \le x^{a-2}$$

$$0 = \lim_{x\to 0^+} -x^{a-2} \le \lim_{x\to 0^+} x^{a-2}\cos\left(\frac{1}{x}\right) \le \lim_{x\to 0^+} x^{a-2} = 0$$

So, the right limit of $x^{a-2}\cos(1/x)$ is 0 as $x$ approaches 0, hence the right limit of $f_a'(x) = ax^{a-1}\sin\left(\frac{1}{x}\right) - x^{a-2}\cos\left(\frac{1}{x}\right)$ is 0 as $x$ approaches 0. Hence, for $a > 2$, $f_a'(x)$ is continuous at 0, since the left and right limit agrees with $f_a'(0)$.

(d) **Ans:** $a > 3$. To make sense of the second derivative, $f'_a(x)$ needs to be continuous at 0, thus $a > 2$. Since for all $x < 0$, $f'_a(x) = 0$, thus $f''_a(x) = 0$. So, the left limit $\lim_{x\to 0^-} f''_a(x) = 0$.

Then, in case for $f''_a(0)$ to be defined, the right limit must also be 0.

Now, for all $x > 0$, consider the slope $\frac{f'_a(x)-f'_a(0)}{x-0}$:

$$\frac{f'_a(x) - f'_a(0)}{x - 0} = \frac{ax^{a-1}\sin\left(\frac{1}{x}\right) - x^{a-2}\cos\left(\frac{1}{x}\right) - 0}{x - 0} = \frac{ax^{a-1}\sin\left(\frac{1}{x}\right) - x^{a-2}\cos\left(\frac{1}{x}\right)}{x}$$

$$= ax^{a-2}\sin\left(\frac{1}{x}\right) - x^{a-3}\cos\left(\frac{1}{x}\right)$$

Which, in case for $\lim_{x\to 0^+} \frac{f'_a(x)-f'_a(0)}{x-0}$ to be defined, $a > 3$.

If $a \leq 3$, the again take the sequence $x_n = \frac{1}{2n\pi}$ used in part (c), the above limit becomes:

$$\forall n \in \mathbb{N}, \quad ax_n^{a-2}\sin\left(\frac{1}{x_n}\right) - x^{a-3}\cos\left(\frac{1}{x_n}\right) = a\left(\frac{1}{2n\pi}\right)^{a-2}\sin(2n\pi) - \left(\frac{1}{2n\pi}\right)^{a-3}\cos(2n\pi)$$

$$= 0 - (2n\pi)^{3-a}$$

If $a = 3$, then the above expression is $-1$. Thus, as $n$ approaches $\infty$, the sequence $\frac{f'_a(x_n)-f'_a(0)}{x_n-0}$ converges to $1 \neq 0$, hence the right limit doesn't agree with the left limit, hence $f''_a(0)$ is not defined.

Else if $a < 3$, then the above expression is not bounded, since $3 - a > 0$, so the right limit doesn't exist in $\mathbb{R}$, hence $f''_a(0)$ is again not defined.

For all $a > 3$, and all $x > 0$, the above terms can again be approached by Squeeze Theorem:

$$-x^{a-2} \leq x^{a-2}\sin\left(\frac{1}{x}\right) \leq x^{a-2}$$

$$0 = \lim_{x\to 0^+} -x^{a-2} \leq \lim_{x\to 0^+} x^{a-2}\sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0^+} x^{a-2} = 0$$

$$-x^{a-3} \leq x^{a-3}\cos\left(\frac{1}{x}\right) \leq x^{a-3}$$

$$0 = \lim_{x\to 0^+} -x^{a-2} \leq \lim_{x\to 0^+} x^{a-2}\cos\left(\frac{1}{x}\right) \leq \lim_{x\to 0^+} x^{a-2} = 0$$

Hence, $\lim_{x\to 0^+} \frac{f'_a(x)-f'_a(0)}{x-0} = \lim_{x\to 0^+} ax^{a-2}\sin\left(\frac{1}{x}\right) - x^{a-3}\cos\left(\frac{1}{x}\right) = 0$, which agrees with the left limit. So, for all $a > 3$, $f''_a(0)$ is defined.

## 4

**Question 4** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined as* $f(x) = e^{-\frac{1}{x^2}}$ *if* $x \neq 0$ *and* $f(0) = 0$. *Show that* $f$ *is infinitely differentiable and* $\forall n \in \mathbb{N},\ f^{(n)}(0) = 0$.

**Pf:**

First, we'll prove that for all $n \in \mathbb{N}$, $\lim_{x\to 0} \frac{1}{x^n} e^{-\frac{1}{x^2}} = 0$. By doing the substitution $h = \frac{1}{x}$, the expression becomes $\lim_{h\to\infty} h^n e^{-h^2}$.

For base cases $n = 0$, the limit $\lim_{h\to\infty} h^0 e^{-h^2} = \lim_{n\to\infty} e^{-h^2} = 0$ (since $e^{-h^2} = 1/e^{h^2}$, and $e^{h^2}$ is not bounded). Same applies for another base case $n = 1$, the limit $\lim_{h\to\infty} he^{-h^2} = \lim_{h\to\infty} \frac{h}{e^{h^2}}$. Since both $h$ and $e^{h^2}$ are not bounded, then apply L'hopital's Rule becomes:

$$\lim_{h\to\infty} \frac{h}{e^{h^2}} = \lim_{h\to\infty} \frac{1}{2he^{h^2}} = 0$$

The second part is true since $he^{h^2}$ is not bounded. Which, the case is also true for $n = 1$.

Then, suppose for given $n \in \mathbb{N}$ and all integer $0 < k \leq n$,$\lim_{h\to\infty} h^k e^{-h^2} = 0$, for the case of $(n+1)$, $\lim_{h\to\infty} h^{(n+1)} e^{-h^2} = \lim_{h\to\infty} \frac{h^{n+1}}{e^{h^2}}$, which both $h^{(n+1)}$ and $e^{h^2}$ are not bounded in this limit. THus, apply L'hopital's Rule, the limit becomes:

$$\lim_{h\to\infty} \frac{h^{(n+1)}}{e^{h^2}} = \lim_{h\to\infty} \frac{(n+1)h^n}{2he^{h^2}} = \lim_{h\to\infty} \frac{(n+1)}{2} h^{n-1} e^{-h^2}$$

If $0 < (n+1) < n$, then based on induction hypothesis, the above limit evalutes to be 0; if $(n-1) = 0$, then it returns to the initial case, which again evaluates to be 0; else, if $(n-1) < 0$, then the limit becomes $\lim_{h\to\infty} \frac{(n+1)}{2h^{1-n} e^{h^2}}$, where $(1-n) > 0$. Thus, the denominator goes unbounded, the limit again evaluates to be 0.

So, by the Principle of Mathematical Induction, the limit $\lim_{x\to 0} \frac{1}{x^n} e^{-\frac{1}{x^2}} = \lim_{h\to\infty} h^n e^{-h^2} = 0$ for all $n \in \mathbb{N}$. And, if take finite linear combination of different powers, for any real-valued polynomial $p(h) = a_n h^n + ... + a_0$, $p(1/x)e^{-\frac{1}{x^2}}$ also converges to 0 as $x$ approaches 0 (since $p(1/x)e^{-\frac{1}{x^2}} = a_n(1/x^n)e^{-\frac{1}{x^2}} + ... + a_0 e^{-\frac{1}{x^2}}$, where each individual component converges to 0 as $x$ approaches 0).

Now, we can use induction to prove that for all $n \in \mathbb{N}$, the function $f(x) = e^{-\frac{1}{x^2}}$ has $n^{th}$ derivative in the form $p(1/x)e^{-\frac{1}{x^2}}$ for some polynomial $p(h)$, and is differentiable at 0, with $f^{(n)}(0) = 0$.

First, for base case $n = 1$, for all $x \neq 0$, $f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}$ by the differentiation rules, which let polynomial $p_1(h) = 2h^3$, then $f'(x) = p_1(1/x)e^{-\frac{1}{x^2}}$. Which, $\lim_{x\to 0} \frac{2}{x^3} e^{-\frac{1}{x^2}} = 0$, since $\lim_{x\to 0} \frac{1}{x^3} e^{-\frac{1}{x^2}} = 0$ follows from the statment proven previously.

Now, for $f'(0)$, consider $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}} - 0}{x} = \lim_{x\to 0} \frac{1}{x} e^{-\frac{1}{x^2}} = 0$$

Thus, we can conclude that $f'(0) = 0$.

Then, suppose for given $n \in \mathbb{N}$, the $n^{th}$ derivative is in the form $f^{(n)}(x) = p_n(1/x)e^{-\frac{1}{x^2}}$ for some real coefficient polynomial $p_n(h)$, and is differentiable at 0.

Which, for the $(n+1)^{th}$ derivative, since for $x \neq 0$, using differentiation rule:

$$f^{(n+1)}(x) = p_n'(1/x)\frac{-1}{x^2} e^{-\frac{1}{x^2}} + p_n(1/x)e^{-\frac{1}{x^2}} \frac{-2}{x^3} = (\frac{2}{x^3} p_n(1/x) - \frac{1}{x^2} p_n'(1/x))e^{-\frac{1}{x^2}}$$

Which, let $p_{(n+1)}(h) = 2h^3 p_n(h) - h^2 p_n'(h)$ be the polynomial, $f^{(n+1)}(x) = p_{(n+1)}(1/x)e^{-\frac{1}{x^2}}$. Which, $\lim_{x\to 0} p_{(n+1)}(1/x)e^{-\frac{1}{x^2}} = 0$ is proven initially.

Now, for $f^{(n+1)}(0)$, consider $\lim_{x\to 0} \frac{f^{(n)}(x)-f^{(n)}(0)}{x-0}$:

$$\lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x\to 0} \frac{p_n(1/x)e^{-\frac{1}{x^2}} - 0}{x} = \lim_{x\to 0} \frac{1}{x} p_n(1/x)e^{-\frac{1}{x^2}}$$

Let $p(h) = hp_n(h)$ be the polynomial, the $p(1/x)e^{-\frac{1}{x^2}} = \frac{1}{x}p_n(1/x)e^{-\frac{1}{x^2}}$, thus the above limit is evaluated as 0. Which, $f^{(n+1)}(0) = 0$.

By the principle of mathematical induction, we can conclude that for all $n \in \mathbb{N}$, the $n^{th}$ derivative is in the form $f^{(n)}(x) = p_n(1/x)e^{-\frac{1}{x^2}}$ for some polynomial $p_n(h)$, and $f^{(n)}(0) = 0$. Thus, $f(x)$ described in the problem is in fact infinitely differentiable, and $f^{(n)}(0) = 0$ for all natural number

## 5

**Question 5** *From the textbook solve exercises 2, 7 and 15 (first part) of Chapter 5.*

- **Q2:** Suppose $f'(x) > 0$ in $(a,b)$ Prove that $f$ is strictly increasing in $(a,b)$, and let $g$ be its inverse function. Prove that $g$ is differentiable, and that $g'(f(x)) = 1/f'(x)$ for $a < x < b$.

**Pf:**

First, to prove that $f$ is strictly increasing in $(a,b)$, we'll use contradiction: Suppose $f$ is not strictly increasing in $(a,b)$. Then, there exists $c, d \in (a,b)$, where $c < d$, such that $f(c) \geq f(d)$. But, by Mean Value Theorem, thre exists $x \in (c, d0)$, with $'(x) = \frac{f(d)-f(c)}{d-c}$. Which, since $f(d) \leq f(c)$, $f(d)-f(c) \leq 0$; and $c < d$ implies $\frac{1}{d-c} > 0$. Thus, $f'(x) = \frac{f(d)-f(c)}{d-c} \leq 0$, but this is a contradiction (since $x \in (a,b)$ implies $f'(x) > 0$). So, the assumption is false, $f$ must be strictly increasing in $(a,b)$.

Now, given that $g$ if the inverse of $f$, then for all $x \in (a,b)$, then $g(f(x)) = x$. Since $g$ is defined on the set $f((a,b))$, and $(a,b)$ is a connected interval while $f$ is continuous (recall that differentiability of $f$ implies continuity).

- **Q7:** Suppose $f'(x), g'(x)$ exists, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that $\lim_{t\to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$.

**Pf:**

Since $f'(x), g'(x)$ exists, within some neightborhood $(x-\epsilon, x+\epsilon)$, if $t$ is in the neighborhood, $\lim_{t\to x} \frac{f(t)-f(x)}{t-x} = f'(x)$ and $\lim_{t\to x} \frac{g(t)-g(x)}{t-x} = g'(x)$. Thus, for all $t \neq x$ within the given neighborhood, if $g(t) \neq 0$, the following is true:

$$\frac{f(t)}{g(t)} = \frac{f(t)-0}{g(t)-0} = \frac{f(t)-f(x)}{g(t)-g(x)} = \frac{f(t)-f(x)}{t-x}\frac{t-x}{g(t)-g(x)} = \frac{f(t)-f(x)}{t-x}\frac{1}{\frac{g(t)-g(x)}{t-x}}$$

Notice that since $\lim_{t\to x} \frac{f(t)-f(x)}{t-x} = f'(x)$, and $\lim_{t\to x} \frac{g(t)-g(x)}{t-x} = g'(x) \neq 0$, thus $\lim_{t\to x} 1/\left(\frac{g(t)-g(x)}{t-x}\right) = 1/g'(x)$. So, the limit is given as follow:

$$\lim_{t\to x}\frac{f(t)}{g(t)} = \lim_{t\to x}\frac{f(t)-f(x)}{t-x}\frac{1}{\frac{g(t)-g(x)}{t-x}} = \left(\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\right)\left(\lim_{t\to x}\frac{1}{\frac{g(t)-g(x)}{t-x}}\right) = f'(x)\frac{1}{g'(x)}$$

Hence, $\lim_{t\to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$.

- **Q15:** Suppose $a \in \mathbb{R}$, $f$ is a twice-differentiable real function on $(a,\infty)$, and $M_0, M_1, M_2$ are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on $(a,\infty)$. Prove that $M_1^2 \leq 4M_0M_2$.

**Pf:**

For all $x_0 \in (a,\infty)$, consider the Taylor Polynomial $P_1(x) = f(x_0) + f'(x_0)(x - x_0)$. Which, for all $h > 0$ $(2h > 0)$, since $x_0 + 2h > x_0$, so $(x_0 + 2h) \in (a,\infty)$. Thus, by Taylor's Theorem, there exists $z \in (x_0, x_0 + 2h)$, with $f(x_0 + 2h) - P_1(x_0 + 2h) = \frac{f''(z)}{2!}((x_0 + 2h) - x_0)^2$. Thus:

$$f(x_0 + 2h) - P_1(x_0 + 2h) = f(x_0 + 2h) - (f(x_0) + f'(x_0)((x_0 + 2h) - x_0))$$

$$= f(x_0 + 2h) - f(x_0) - 2hf'(x_0)$$

$$\frac{f''(z)}{2!}((x_0 + 2h) - x_0)^2 = \frac{f''(z)}{2}(2h)^2$$

So, $f(x_0 + 2h) - f(x_0) - 2hf'(x_0) = \frac{f''(z)}{2}4h^2$, thus $2hf'(x_0) = f(x_0 + 2h) - f(x_0) - f''(z)2h^2$, or $f'(x_0) = \frac{1}{2h}f(x_0 + 2h) - f(x_0)) - hf''(z)$. Hence, the following inequality is true:

$$|f'(x_0)| = \left|\frac{1}{2h}f(x_0 + 2h) - f(x_0)) - hf''(z)\right| \leq \frac{1}{2h}(|f(x_0 + 2h)| + |f(x_0)|) + h|f''(z)|$$

$$|f'(x_0)| \leq \frac{1}{2h}2M_0 + hM_2 = \frac{M_0}{h} + hM_2$$

Which, if choose $h = \sqrt{M_0/M_2}$, the following is true:

$$|f'(x_0)| \leq \frac{M_0}{\sqrt{M_0/M_2}} + \sqrt{\frac{M_0}{M_2}}M_2 = \sqrt{M_0M_2} + \sqrt{M_0M_2} = 2\sqrt{M_0M_2}$$

Thus, $2\sqrt{M_0M_2}$ is an upper bound of $|f'(x)|$ for all $x \in (a, \infty)$, hence $M_1 \leq 2\sqrt{M_0M_2}$ (since $M_1$ by definition is the least upper bound of $|f'(x)|$). So:

$$M_1^2 \leq (2\sqrt{M_0M_2})^2 = 4M_0M_2$$