Math CS 122A HW7

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Question 1 *Apply the representation* $f(z) = w_0 + \zeta(z)^n$ *to* $\cos(z)$ *with* $z_0 = 0$. *Determine* $\zeta(z)$ *explicitly.*

Pf:

Given $z_0 = 0$, which $\cos(z_0) = \cos(0) = 1$. So, since $\cos(z) - 1$ has zero ath $z = 0$, and it is not identically 0, there exists some order $n \in \mathbb{N}$ and analytic function $g(z)$ such that $g(0) \neq 0$, with $\cos(z) - 1 = (z-0)^n g(z) = z^n g(z)$.

Now, consider the derivatives of $\cos(z) - 1$, and their evaluation at $z_0 = 0$:

$$\frac{d}{dz}(\cos(z) - 1) = -\sin(z), \quad -\sin(0) = 0$$

$$\frac{d}{dz}(-\sin(z)) = -\cos(z), \quad -\cos(0) = 1 \neq 0$$

Notice that this implies the order of the zero is 2, hence $\cos(z) - 1 = z^2 g(z)$, where the goal is to find the analytic branch $\zeta(z)$ such that $\cos(z) - 1 = \zeta(z)^2$.

Now, notice that $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$, hence:

$$\cos(z) - 1 = \frac{e^{iz} + e^{-iz}}{2} - 1 = \frac{e^{iz} + e^{-zi} - 2}{2} = \frac{(e^{\frac{iz}{2}} - e^{\frac{-iz}{2}})^2}{2}$$

So, define the branch $\zeta(z) = \frac{e^{\frac{iz}{2}} - e^{\frac{-iz}{2}}}{\sqrt{2}}$ would satisfy $\zeta(z)^2 = \cos(z) - 1$, its negative representation is also fitting the desired condition.

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Question 2 *Show by use of (36), or directly, that* $|f(z)| \leq 1$ *for* $|z| \leq 1$ *implies*

$$\frac{|f'(z)|}{(1 - |f(z)|^2)} \leq \frac{1}{1 - |z|^2}$$

Pf:

In the textbook, given an analytic function f, with $w_0 = f(z_0)$ with $|z_0| < R$ and $|w_0| < M$ for some $R, M > 0$, then for $|z| < R$, we have the following inequality:

$$\left|\frac{M(f(z) - w_0)}{M^2 - \bar{w_0} f(z)}\right| \leq \left|\frac{R(z - z_0)}{R^2 - \bar{z_0} z}\right|$$

In the problem, the statement has $R = M = 1$, hence fixing any z_0 with $|z_0| < 1$, for all $z \neq z_0$ with $|z| < 1$, it satisfies the following:

$$\left|\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)}\right| \leq \left|\frac{z - z_0}{1 - \bar{z_0} z}\right|$$

(Note: the original equation has $w_0 = f(z_0)$).

Which, since $z \neq z_0$, $|z - z_0| \neq 0$. Modify the equation, we get:

$$\left|\frac{f(z) - f(z_0)}{z - z_0}\right| \cdot \left|\frac{1}{1 - \overline{f(z_0)} f(z)}\right| \leq \frac{1}{|1 - \bar{z_0} z|}$$

Now, notice that $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$, $\lim_{z \to z_0} (1 - \overline{f(z_0)} f(z)) = (1 - \overline{f(z_0)} f(z_0)) = (1 - |f(z_0)|^2)$, and $\lim_{n \to \infty} (1 - \bar{z_0} z) = (1 - \bar{z_0} z_0) = (1 - |z_0|^2)$.

Which, for $|z_0| < 1$, then $(1 - |z_0|^2) > 0$; similarly, by Maximal Principle, since $|z_0| < 1$ z_0 is not on the boundary of the unit disk $\mathbb{D}$. Hence, $|f(z_0)|$ cannot be the maximum, showing that $|f(z_0)| < 1$ (since if $|f(z_0)| = 1$, because it is not the maximum, there exists $z_1 \in \mathbb{D}$, with $1 = |f(z_0)| < |f(z_1)|$, which contradicts the fact that $|f(z_1)| \leq 1$ when $|z_1| \leq 1$). Hence, $(1 - |f(z_0)|^2) > 0$.

So, the above inequality can be reduce to the following:

$$\lim_{z \to z_0} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| \cdot \left|\frac{1}{1 - \overline{f(z_0)} f(z)}\right| \leq \lim_{z \to z_0} \frac{1}{|1 - \bar{z_0} z|}$$

$$|f'(z_0)| \cdot \frac{1}{|1 - |f(z_0)|^2|} \leq \frac{1}{|1 - |z_0|^2|}$$

Hence, for $|z| < 1$, we can conclude the following:

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

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Question 3 *Prove that the arc of smallest noneuclidean length that joins two given points in the unit disk is a circular arc which is orthogonal to the unit circle. (Make use of a linear transformation that carries one end point to the origin, the other to a point on the positive real axis.) The shortest noneuclidean length is called the noneuclidean distance between the end points. Derive a formula for the noneuclidean distance between z_1 and z_2. Answer:*

$$\frac{1}{2}\log\frac{1+\left|\frac{z_1-z_2}{1-\bar{z_1}z_2}\right|}{1-\left|\frac{z_1-z_2}{1-\bar{z_1}z_2}\right|}$$

Pf:

For any distinct $z_1, z_2 \in \mathbb{D}$ (which $|z_1|, |z_2| < 1$),consider the map $S(z) = \frac{z_1-z}{1-\bar{z_1}z}$: Recall that in **HW 1 Question 4 Part (b)**, we've proven that given $|w| < 1$, the map $T(z) = \frac{w-z}{1-\bar{w}z}$ is in fact a bijection of the unit disk $\mathbb{D}$, hence the map $S(z)$ here is also a bijection of the unit disk, and specifically $S(z_1) = 0$.

Now, since S is a linear transformation, the noneuclidean distance is preserved (based on a statement given in the previous problem in the textbook); and, by the Maximum Principle, since the maximum of the function's modulus could only appear at the boundary, hence since $|z_2| < 1$ (not on the boundary of $\mathbb{D}$), then $|f(z_2)| < 1$ (since S is a bijection on $\mathbb{D}$, hence the maximum is given by $\max |S(z)| = 1$).

So, we can conclude that the Noneuclidean distance is achieved by some path γ connecting the origin and the point $S(z_2) \in \mathbb{D}$ (because the noneuclidean distance of a path is invariant under linear transformation, therefore it is sufficient to find such path after the transformation).

The Path γ is a Straight Line:

WLOG, let $\gamma : [0, 1] \to \mathbb{D}$ be a differentiable path, such that $\gamma(0) = 0$, and $\gamma(1) = S(z_2)$. Which, at every input, $\gamma(t) = r(t)e^{i\theta(t)}$ for some real-valued differentiable function $r(t)$ and $\theta(t)$ (Which, one can assume that $1 > r(t) \geq 0$ for all $t \in [0, 1]$ to fit γ in the unit disk $\mathbb{D}$). Also, it satisfies $r(0) = 0$, and $r(1) = |S(z_2)|$.

Then, we get the following:

$$|\gamma(t)| = r(t), \quad \gamma'(t) = r'(t)e^{i\theta(t)} + i\theta'(t)r(t)e^{i\theta(t)} = (r'(t) + i\theta'(t)r(t))e^{i\theta(t)}$$

$$|\gamma'(t)| = |r'(t) + i\theta'(t)r(t)| = \sqrt{(r'(t))^2 + (\theta'(t)r(t))^2}$$

Which, consider the noneuclidean distance, it is given as follow:

$$\int_\gamma \frac{|dz|}{1-|z|^2} = \int_0^1 \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}dt = \int_0^1 \frac{\sqrt{(r'(t))^2 + (\theta'(t)r(t))^2}}{1-(r(t))^2}dt$$

And, since $\sqrt{(r'(t))^2 + (\theta'(t)r(t))^2} \geq \sqrt{(r'(t))^2} = |r'(t)| \geq r'(t)$, the above integral satisfy the inequality:

$$\int_0^1 \frac{\sqrt{(r'(t))^2 + (\theta'(t)r(t))^2}}{1-(r(t))^2}dt \geq \int_0^1 \frac{r'(t)}{1-(r(t))^2}dt$$

So, doing the substitution $u = r(t)$, $du = r'(t)dt$ (which $t = 0$ satisfies $u = r(0) = 0$, and $t = 1$ satisfies $u = r(1) = |S(z_2)|$), we get the following:

$$\int_\gamma \frac{|dz|}{1-|z|^2} \geq \int_0^{|S(z_2)|} \frac{1}{1-u^2}du = \frac{1}{2}\int_0^{|S(z_2)|} \left(\frac{1}{1-u} + \frac{1}{1+u}\right)du$$

$$\int_\gamma \frac{|dz|}{1-|z|^2} \geq \frac{1}{2}\left(-\ln|1-u| + \ln|1+u|\right)\Big|_0^{|S(z_2)|} = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|\Big|_0^{|S(z_2)|} = \frac{1}{2}\ln\left|\frac{1+|S(z_2)|}{1-|S(z_2)|}\right|$$

With $|S(z_2)| = \left|\frac{z_1-z_2}{1-\bar{z_1}z_2}\right| < 1$, then $\frac{1+|S(z_2)|}{1-|S(z_2)|} > 0$, the above inequality becomes:

$$\int_\gamma \frac{|dz|}{1-|z|^2} \geq \frac{1}{2}\ln\frac{1+\left|\frac{z_1-z_2}{1-\bar{z_1}z_2}\right|}{1-\left|\frac{z_1-z_2}{1-\bar{z_1}z_2}\right|}$$

Notice that for the straight path $\gamma(t) = S(z_2)t$ ($\gamma'(t) = S(z_2)$), the path integral produces the above value, so we can claim that the shortest distance is given as the above value, and the path is given by a straight line joining 0 and $S(z_2)$.

Eventually, we can claim that the smallest noneuclidean distance between two points, is given as:

$$\frac{1}{2}\ln\frac{1+\left|\frac{z_1-z_2}{1-\bar{z_1}z_2}\right|}{1-\left|\frac{z_1-z_2}{1-\bar{z_1}z_2}\right|}$$

Also, because this minimum is achieved by having a straight path γ going through 0 and $S(z_2)$ (which is a straight line through the center, and it is orthogonal to the unit circle). Hence, the preimage $S^{-1}(\gamma)$ must be a circular arc that's orthogonal to the preimage of the unit circle, which is the unit circle itself (since the Mobius Transoformation S with $|z_1| < 1$ maps the boundary onto the boundary).

With $S(z_1) = 0$ and $S(z_2) = S(z_2)$, z_1 and z_2 are both on the preimage $S^{-1}(\gamma)$, so the shortest noneuclidean path joining the two arbitrary points in $\mathbb{D}$, is a circular arce orthogonal to the unit circle.