Latex Template

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Question 1 *Let A^* denote the set of limit points of A. Prove this satisfies:*

- $\emptyset^* = \emptyset$
- $x \notin \{x\}^*$
- $A^{**} \subseteq A \cup A^*$
- $(A \cup B)^* = A^* \cup B^*$

Pf:
All the below proofs are based on a nonempty topological space X.

1. To prove that $\emptyset^* = \emptyset$, we'll use contradiction: Suppose there exists $x \in X$ with $x \in \emptyset^*$, with $x \in \emptyset^*$, then by definition, every open neighborhood U of x, the intersection $\emptyset \cap (U \setminus \{x\}) \neq \emptyset$.

   However, since every set A satisfies $\emptyset \cap A = \emptyset$, the above condition is a contradiction. Therefore, there's no such $x \in X$ satisfying $x \in \emptyset^*$, thus $\emptyset^* = \emptyset$.

2. To prove that $x \notin \{x\}^*$, consider any arbitrary open neighborhood U of x: Since $x \notin U \setminus \{x\}$, then $\{x\} \cap (U \setminus \{x\}) = \emptyset$. Thus, x is not a limit point of $\{x\}$, or $x \notin \{x\}^*$.

3. To prove that $A^{**} \subseteq A \cup A^*$, consider any $x \in A^{**}$:

   If $x \in A$, then $x \in A \cup A^*$.

   Else, if $x \notin A$, by definition, for every open neightborhood U of x, there exists $y \in A^* \cap (U \setminus \{x\})$, which $y \in A^*$ and $y \in U$, thus U is an open neighborhood of y.

   Then, since y is a limit point of A, then there exists $a \in A \cap (U \setminus \{y\})$, which $a \in A$ and $a \in U$.

   Yet, since $x \notin A$, so $a \neq x$, thus $a \in U \setminus \{x\}$, proving that $A \cap (U \setminus \{x\}) \neq \emptyset$.

   Since every open neighborhood of x satisfies $A \cap (U \setminus \{x\}) \neq \emptyset$, then x is a limit point of A, thus $x \in A^* \subseteq A \cup A^*$.

   So, regardless of the case, $x \in A^{**}$ implies $x \in A \cup A^*$, thus $A^{**} \subseteq A \cup A^*$.

4. To prove that $(A \cup B)^* = A^* \cup B^*$, consider the following:

   First, $A^* \cup B^* \subseteq (A \cup B)^*$: Since $A, B \subseteq (A \cup B)$, then if $x \in A^*$, every open neighborhood U of x satisfies $A \cap (U \setminus \{x\}) \neq \emptyset$, thus $(A \cup B) \cap (U \setminus \{x\}) \neq \emptyset$, showing that $x \in (A \cup B)^*$, or $A^* \subseteq (A \cup B)^*$. Applying the same logic on B^*, we'll get $B^* \subseteq (A \cup B)^*$, hence $(A^* \cup B^*) \subseteq (A \cup B)^*$.

Now, to prove that $(A \cup B)^* \subseteq (A^* \cup B^*)$, we'll approach by contradiction:

Suppose $(A \cup B)^* \nsubseteq (A^* \cup B^*)$, there exists $x \in (A \cup B)^*$, while $x \notin (A^* \cup B^*)$.

Then, since $x \notin A^*$, there exists open neighborhood U_1 of x, with $A \cap (U_1 \setminus \{x\}) = \emptyset$; similarly, since $x \notin B^*$, there exists open neighborhood U_2 of x, with $B \cap (U_2 \setminus \{x\}) = \emptyset$.

Now, consider $U = U_1 \cap U_2$: It is an open set, and since $x \in U_1$ and $x \in U_2$, then $x \in (U_1 \cap U_2) = U$, thus U is an open neighborhood of x.

However, since $U \subseteq U_1$, then $A \cap (U \setminus \{x\}) = \emptyset$; similarly, $U \subseteq U_2$ implies $B \cap (U \setminus \{x\}) = \emptyset$.

So, $(A \cup B) \cap (U \setminus \{x\}) = (A \cap (U \setminus \{x\})) \cup (B \cap (U \setminus \{x\})) = \emptyset$. Yet, if $x \in (A \cup B)^*$, then every open neighborhood of x should have nonempty intersection with $(A \cup B)$, while not including x.

So, this is a contradiction. Hence, $(A \cup B)^* \subseteq (A^* \cup B^*)$.

With the above two statements, $(A \cup B)^* = A^* \cup B^*$.

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Question 2 *Prove that the boundary operation satisfies:*

- $\partial A = \partial(X \setminus A)$
- $\partial\partial A \subseteq \partial A$
- $\partial(A \cup B) \subseteq \partial A \cup \partial B$
- $A \subseteq B \implies \partial A \subseteq (B \cup \partial B)$

Pf:

1. Given any set $A \subseteq X$, since $\partial A = \overline{A} \cap \overline{X \setminus A}$ and $\partial(X \setminus A) = \overline{X \setminus A} \cap \overline{X \setminus (X \setminus A)} = \overline{X \setminus A} \cap \overline{A}$, thus $\partial A = \partial(X \setminus A)$.

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Question 3

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Question 4