

# Math CS 122A HW8

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1

**Question 1** Ahlfors Pg. 148 Problem 2:

*Prove that the region obtained from a simply connected region by removing  $m$  points has the connectivity  $m + 1$ , and find a homology basis.*

**Pf:**

Given the open region  $\Omega$  is simply connected, which the complement  $\Omega^C$  is connected in the extended complex plane. Now, let  $z_1, \dots, z_m \in \Omega$  denote the  $m$  distinct points being removed. Then, the new region  $\Omega' = \Omega \setminus \{z_1, \dots, z_m\}$ , and  $(\Omega')^C = \Omega^C \cup \{z_1, \dots, z_m\}$ .

First, all  $i \in \{1, \dots, m\}$  has  $\{z_i\}$  being disjoint from  $\Omega^C$ : Since  $z_i \in \Omega$ , while  $\Omega$  is open, then there exists  $\epsilon_i > 0$ , with  $B_{\epsilon_i}(z_i) \subseteq \Omega$ . Hence, for all  $a \in \Omega^C$ ,  $d(z_i, a) \geq \epsilon_i$ , showing that  $z_i$  and  $\Omega^C$  are disjoint.

Then, since  $\{z_i\}$  and  $\Omega^C$  are both closed under standard topology, the two being disjoint implies the two are not connected, hence belong to different connected components.

Furthermore, all distinct  $i, j \in \{1, \dots, m\}$  have  $\{z_i\}, \{z_j\}$  being disjoint, since they're distinct points by assumption. Then, the two sets are also not connected, hence they belong to different connected components.

Then, because each set  $\Omega^C, \{z_1\}, \dots, \{z_m\}$  all belong to distinct connected component, while each set is connected ( $\Omega^C$  is connected in the extended complex plane by assumption, while a singleton is always connected under standard topology), then, there are  $m + 1$  connected components for the above collection. Hence,  $(\Omega')^C$  has  $m + 1$  connected components, showing that the new region  $\Omega'$  has connectivity  $m + 1$ .

**Homology Basis:**

Recall the above definition, each  $i \in \{1, \dots, m\}$  exists  $\epsilon_i > 0$ , with  $d(z_i, a) \geq \epsilon_i$  for all  $a \in \Omega^C$ ; also, since for  $j \neq i$ ,  $d(z_i, z_j) > 0$ , then let  $d = \min\{\epsilon_i \mid 1 \leq i \leq m\} \cup \{d(z_i, z_j) \mid 1 \leq i < j \leq m\}$ . Which,  $d > 0$ , and  $B_d(z_i)$  contains no points from other connected components (for  $a \in \Omega^C$ ,  $d(z_i, a) \geq \epsilon_i \geq d$ , showing that  $a \notin B_d(z_i)$ ; and for  $j \neq i$ ,  $d(z_i, z_j) \geq d$ , so  $z_j \notin B_d(z_i)$  also). Therefore,  $B_d(z_i) \setminus \{z_i\}$  is disjoint from  $(\Omega')^C = \Omega^C \cup \{z_1, \dots, z_m\}$ .

Then, for each index  $i$ , let cycle  $\gamma_i$  be defined as the parametrization  $z_i + \frac{d}{2}e^{i\theta}$  for  $\theta \in [0, 2\pi]$ . Then, since for all  $a \in \gamma_i$ ,  $|a - z_i| = |\frac{d}{2}e^{i\theta}| = \frac{d}{2} < d$ , then  $a \in B_d(z_i) \setminus \{z_i\}$ . showing that  $\gamma_i \subset B_d(z_i) \setminus \{z_i\}$ . Therefore,  $\gamma_i$  is disjoint from  $(\Omega')^C$ , which is contained fully in  $\Omega'$ .

Also, since  $\gamma_i \subset B_d(z_i)$  (which is simply connected), for all  $j \neq i$ , since  $z_j \notin B_d(z_i)$ , then  $n(\gamma_i, z_j) = 0$ ; the same applies for all  $a \in \Omega^C$  also. And, if do the following integration, we get:

$$n(\gamma_i, z_i) = \frac{1}{2\pi i} \int_{\gamma_i} \frac{dz}{z - z_i} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{(z_i + \frac{d}{2}e^{i\theta}) - z_i} i \frac{d}{2} e^{i\theta} d\theta = \frac{1}{2\pi i} \int_0^{2\pi} i d\theta = 1$$

Hence,  $\gamma_i$  has winding number 1 for component  $\{z_i\}$ , while winding number 0 for other components for the complement. Therefore, the collection  $\gamma_1, \dots, \gamma_m$  forms a homology basis for  $\Sigma'$ .

## 2

**Question 2** Ahlfors Pg. 148 Problem 4:

*Show that single-valued analytic branches of  $\log z$ ,  $z^\alpha$  and  $z^z$  can be defined in any simply connected region which does not contain the origin.*

**Pf:**

**Single-Valued Branch of  $\log(z)$ :**

Given that  $\Omega$  is a simply connected region that doesn't contain 0, then since  $z \neq 0$  in the region, then based on **Corollary 2** of Generalized Cauchy's Theorem in the textbook (Ahlfors Pg. 142), one can define a single-valued branch of  $\log(z)$  in this region.

**Single-Valued Branch of  $z^\alpha$  and  $z^z$ :**

From the above part, since the single-valued branch of  $\log(z)$  can be defined, then for all  $\alpha \in \mathbb{R}$ ,  $\alpha \log(z)$  and  $z \log(z)$  both have single-valued branch.

Hence,  $z^\alpha = e^{\alpha \log(z)}$  and  $z^z = e^{z \log(z)}$  are also well-defined.

**Question 3** Ahlfors Pg. 148 Problem 5:

Show that a single-valued analytic branch of  $\sqrt{1-z^2}$  can be defined in any region such that the points  $\pm 1$  are in the same component of the complement. What are the possible values of

$$\int \frac{dz}{\sqrt{1-z^2}}$$

over a closed curve in the region?

**Pf:**

Assume  $\Omega$  is the open region, where  $1, -1$  are in the same connected component of the complement. Which, for all cycle  $\gamma \subset \Omega$ , the winding number  $n(\gamma, 1) = n(\gamma, -1)$ .

**Single-Valued Branch:**

First, consider the analytic function  $\left(\frac{1}{z+1} - \frac{1}{z-1}\right)$  on  $\Omega$ : For all cycle  $\gamma \subset \Omega$ , the following integral is true:

$$\int_{\gamma} \left( \frac{1}{z+1} - \frac{1}{z-1} \right) dz = \left( \int_{\gamma} \frac{1}{z+1} dz - \int_{\gamma} \frac{1}{z-1} dz \right) = 2\pi i(n(\gamma, 1) - n(\gamma, -1)) = 0$$

Then, this implies that an antiderivative  $F(z)$  exists on  $\Omega$  (with  $F'(z) = \frac{1}{2} \left( \frac{1}{z+1} - \frac{1}{z-1} \right)$ ). Then, since the original function can be rewrite as:

$$F'(z) = \left( \frac{1}{z+1} - \frac{1}{z-1} \right) = \frac{(z-1) - (z+1)}{(z+1)(z-1)} = \frac{-2}{z^2-1} = \frac{2}{1-z^2}$$

Hence,  $F(z)$  is an antiderivative of  $\frac{2}{1-z^2}$ .

(Note: In real numbers, the antiderivative of the above equation is also written as  $\ln(x+1) - \ln(x-1) = \ln \frac{x+1}{x-1}$ , which will be a tool for guess here).

Now, consider the equation  $\frac{z+1}{z-1}e^{-F(z)}$ , and its derivative:

$$\begin{aligned} \frac{z+1}{z-1} &= 1 + \frac{2}{z-1}, \quad \frac{d}{dz} \left( \frac{z+1}{z-1} \right) = \frac{d}{dz} \left( \frac{2}{z-1} \right) = \frac{-2}{(z-1)^2} \\ \frac{d}{dz} \left( \frac{z+1}{z-1} e^{-F(z)} \right) &= \frac{-2}{(z-1)^2} e^{-F(z)} + \frac{z+1}{z-1} \left( -\frac{dF}{dz} \right) e^{-F(z)} \\ &= \frac{-2}{(z-1)^2} e^{-F(z)} + \frac{z+1}{z-1} \left( \frac{-2}{1-z^2} \right) e^{-F(z)} \\ &= \frac{-2}{(z-1)^2} e^{-F(z)} + \frac{z+1}{z-1} \left( \frac{-2}{(1-z)(1+z)} \right) e^{-F(z)} \\ &= \frac{-2}{(z-1)^2} e^{-F(z)} + \frac{2}{(z-1)^2} e^{-F(z)} = 0 \end{aligned}$$

Then, since the derivative is 0, then the function is in fact a constant over  $\Omega$ ; and, since  $\Omega$  excludes both  $\pm 1$ , then the value of  $\frac{z+1}{z-1}e^{-F(z)}$  is always nonzero. Hence,  $C = \frac{z+1}{z-1}e^{-F(z)} \neq 0$ .

Which, rewrite the function, we get:

$$C = \frac{z+1}{z-1} e^{-F(z)} = \frac{(z+1)^2}{(z^2-1)} e^{-F(z)} = -\frac{(z+1)^2}{1-z^2} e^{-F(z)}$$

Hence, we can rewrite it as the follow:

$$1 - z^2 = -\frac{1}{C}(z+1)^2 e^{-F(z)}$$

Then, define the function  $\frac{i}{\sqrt{C}}(z+1)e^{-\frac{F(z)}{2}}$ , we get:

$$\left(\frac{i}{\sqrt{C}}(z+1)e^{-\frac{F(z)}{2}}\right)^2 = -\frac{1}{C}(z+1)^2 e^{-F(z)} = 1 - z^2$$

Hence, define this branch as  $\sqrt{1-z^2} = \frac{i}{\sqrt{C}}(z+1)e^{-\frac{F(z)}{2}}$ , it is a well-defined single-valued branch.