

# Math 118B HW5

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## 1 (Part b not done)

### Question 1

(a) Show that there exists a sequence of polynomials  $q_m : [0, 1] \rightarrow \mathbb{R}$  such that for each  $x \in [0, 1]$

$$\lim_{m \rightarrow \infty} q_m(x) = 0$$

(pointwise convergence) but it does not converge uniformly.

(b) Prove that if a sequence of polynomial  $p_m : [0, 1] \rightarrow \mathbb{R}$  converges pointwise to 0 and for all  $m \in \mathbb{N}$  one has that  $\deg(p_m) \leq 100$ , then the  $p_m$  converges uniformly to 0.

**Pf:**

#### (a) Continuous Functions Converging to 0 Pointwise, but not Uniformly:

We'll first construct a sequence of continuous functions converging to 0 pointwise, but not uniformly. For each  $n \in \mathbb{N}$ , let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be defined as:

$$f_n(x) = \begin{cases} 4nx - 2 & x \in [\frac{2}{4n}, \frac{3}{4n}] \\ -4nx + 4 & x \in (\frac{3}{4n}, \frac{4}{4n}] \\ 0 & x \notin [\frac{2}{4n}, \frac{4}{4n}] \end{cases}$$

This is a continuous function for all  $n \in \mathbb{N}$ , since the limit at  $\frac{3}{4n}$ ,  $\frac{2}{4n}$ , and  $\frac{4}{4n}$  all agrees with the function  $f_n$ 's actual values.

However, since at  $x = \frac{3}{4n} \in [0, 1]$ ,  $f_n(x) = 4n \cdot \frac{3}{4n} - 2 = 3 - 2 = 1$ , then  $\|f_n\|_\infty = \sup_{x \in [0, 1]} |f_n(x)| \geq 1$ , showing that  $f_n$  doesn't converge to 0 uniformly (since the norm  $\|\cdot\|_\infty$  is at least 1 for all  $n \in \mathbb{N}$ ).

#### Sequence of Polynomials:

Now, since  $f_n$  is continuous on  $[0, 1]$ , by Stone-Weierstrass Theorem, there exists a sequence of polynomials  $\{q_{n,k}\}_{k \in \mathbb{N}}$  that converges to  $f_n$  uniformly.

For all  $n \in \mathbb{N}$ , since  $\frac{1}{n} > 0$ , by the uniform convergence of  $\{q_{n,k}\}_{k \in \mathbb{N}}$  onto  $f_n$ , there exists  $N_n$ , such that  $k_n \geq N_n$  implies  $\|f_n - q_{n,k_n}\|_\infty < \frac{1}{n}$  (for simplicity, fix  $k_n$  to be the smallest integer with  $k_n \geq N_n$ ). For the rest of the proof of **Part (a)**, consider the sequence of polynomials  $\{q_{n,k_n}\}_{n \in \mathbb{N}}$ .

### The Sequence Pointwise Converges to 0:

For all  $x \in [0, 1]$ , there are two cases to consider:

- First, if  $x = 0$ , for all  $n \in \mathbb{N}$ , we have  $f_n(0) = 0$ . Then, for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$ , with  $\frac{1}{N} < \epsilon$  based on Archimedean's Property. For all  $n \geq N$  (which  $\frac{1}{n} \leq \frac{1}{N} < \epsilon$ ), the previous choice of polynomials satisfy:

$$|q_{n,k_n}(0)| = |q_{n,k_n}(0) - f_n(0)| \leq \|q_{n,k_n} - f_n\|_\infty < \frac{1}{n} < \epsilon$$

Hence, this states that  $\lim_{n \rightarrow \infty} q_{n,k_n}(0) = 0$ .

- Else if  $x \neq 0$  (which  $x > 0$  since  $x \in [0, 1]$ ), there exists  $N \in \mathbb{N}$ , such that  $\frac{1}{N} < x$  based on Archimedean's Property. Then, for all  $n \geq N$ , since  $\frac{4}{4n} = \frac{1}{n} \leq \frac{1}{N} < x$ ,  $f_n(x) = 0$  (since  $x \notin [\frac{2}{4n}, \frac{4}{4n}]$ ).

Again, for all  $\epsilon > 0$ , there exists  $M \in \mathbb{N}$ , with  $\frac{1}{M} < \epsilon$  again based on Archimedean's Property. Choose  $K = \max\{M, N\}$ , for all  $n \geq K$  (which  $n \geq N$ , showing that  $f_n(x) = 0$ ; and  $n \geq M$ , showing that  $\frac{1}{n} \leq \frac{1}{M} < \epsilon$ ), the previous choice of polynomials satisfy:

$$|q_{n,k_n}(x)| = |q_{n,k_n}(x) - f_n(x)| \leq \|q_{n,k_n} - f_n\|_\infty < \frac{1}{n} < \epsilon$$

Hence, this states that  $\lim_{n \rightarrow \infty} q_{n,k_n}(x) = 0$ .

So, regardless of the case,  $\lim_{n \rightarrow \infty} q_{n,k_n}(x) = 0$ , showing that  $\{q_{n,k_n}\}_{n \in \mathbb{N}}$  converges pointwise to 0.

### The Convergence is not Uniform:

Recall that for all  $n \in \mathbb{N}$ ,  $\|f_n\|_\infty \geq 1$ , and  $\|f_n - q_{n,k_n}\|_\infty < \frac{1}{n}$ . Hence, for  $n \geq 2$  (which  $\frac{1}{n} \leq \frac{1}{2}$ ), the following inequality is true:

$$\|q_{n,k_n}\|_\infty = \|(q_{n,k_n} - f_n) - (-f_n)\|_\infty \geq \left| \|q_{n,k_n} - f_n\|_\infty - \|-f_n\|_\infty \right| = \|f_n\|_\infty - \|q_{n,k_n} - f_n\|_\infty$$

$$\|q_{n,k_n}\|_\infty \geq \|f_n\|_\infty - \|q_{n,k_n} - f_n\|_\infty \geq 1 - \|q_{n,k_n} - f_n\|_\infty > 1 - \frac{1}{n} \geq 1 - \frac{1}{2} = \frac{1}{2}$$

So, since  $\| \cdot \|_\infty \geq \frac{1}{2}$  for all  $n \geq 2$ , the  $\lim_{n \rightarrow \infty} \|q_{n,k_n}\|_\infty \neq 0$ , showing that  $\{q_{n,k_n}\}_{n \in \mathbb{N}}$  doesn't converge to 0 uniformly.

In Conclusion,  $\{q_{n,k_n}\}_{n \in \mathbb{N}}$  constructed above, is a sequence of polynomial that converges pointwise to 0, yet it doesn't converge uniformly to 0. Which, it is a desired sequence for the question.

- (b) Let  $\mathcal{P}_{100}([0, 1])$  be the real vector space of polynomial defined on  $[0, 1]$  with degree at most 100 (which  $\dim(\mathcal{P}_{100}([0, 1])) = 101$ ). For this part, the sequence  $\{p_m\}_{m \in \mathbb{N}} \subset \mathcal{P}_{100}([0, 1])$ , and they converges pointwise to 0.

Now, choose distinct points  $x_1, x_2, \dots, x_{101} \in [0, 1]$ , and define the map  $T : \mathcal{P}_{100}([0, 1]) \rightarrow \mathbb{R}^{101}$  by:

$$T(p) = (p(x_1), p(x_2), \dots, p(x_{101}))$$

**The map  $T$  is a Linear Map:**

For the zero function  $0 \in \mathcal{P}_{100}([0, 1])$ , it is clear that  $T(0) = (0, 0, \dots, 0) \in \mathbb{R}^{101}$ .

Then, for all  $p, q \in \mathcal{P}_{100}([0, 1])$ :

$$\begin{aligned} T(p+q) &= ((p+q)(x_1), (p+q)(x_2), \dots, (p+q)(x_{101})) = (p(x_1)+q(x_1), p(x_2)+q(x_2), \dots, p(x_{101})+q(x_{101})) \\ &= (p(x_1), p(x_2), \dots, p(x_{101})) + (q(x_1), q(x_2), \dots, q(x_{101})) = T(p) + T(q) \end{aligned}$$

Also, for all  $\lambda \in \mathbb{R}$  and  $p \in \mathcal{P}_{100}([0, 1])$ :

$$\begin{aligned} T(\lambda p) &= ((\lambda p)(x_1), (\lambda p)(x_2), \dots, (\lambda p)(x_{101})) = (\lambda \cdot p(x_1), \lambda \cdot p(x_2), \dots, \lambda \cdot p(x_{101})) \\ &= \lambda(p(x_1), p(x_2), \dots, p(x_{101})) = \lambda T(p) \end{aligned}$$

Hence, with the above three criteria,  $T$  is a linear map from  $\mathcal{P}_{100}([0, 1]) \rightarrow \mathbb{R}^{101}$ .

**The map  $T$  is Bijective:**

Since both  $\mathcal{P}_{100}([0, 1])$  and  $\mathbb{R}^{101}$  have dimension 101, then showing  $T$  is bijective is equivalent to showing  $T$  is injective.

Suppose  $p \in \ker(T)$  (or  $T(p) = (0, 0, \dots, 0) \in \mathbb{R}^{101}$ ), since for all  $i \in \{1, 2, \dots, 101\}$ , it has  $p(x_i) = 0$ , then  $p$  has at least 101 distinct zeroes. However, since  $p \in \mathcal{P}_{100}([0, 1])$ , then its degree is at most 100. By Fundamental Theorem of Algebra, if  $p \neq 0$ , it has at most 100 distinct roots. Hence,  $p = 0$  is required.

So,  $\ker(T) = \{0\}$ , showing that  $T$  is injective, hence bijective. So,  $T^{-1}$  exists.

**The map  $T$  is Continuous:**

Let the usual dot product define the norm  $\|\cdot\|_2$  of  $\mathbb{R}^{101}$ , and let  $\|\cdot\|_\infty$  be the norm of  $\mathcal{P}_{100}([0, 1])$ .

For all  $\epsilon > 0$ , let  $\delta = \frac{\epsilon}{\sqrt{101}} > 0$ , for all  $p, q \in \mathcal{P}_{100}([0, 1])$ , if  $\|p - q\|_\infty < \delta = \frac{\epsilon}{\sqrt{101}}$ , then the output satisfies:

$$\begin{aligned} T(p) - T(q) &= T(p - q) = ((p - q)(x_1), (p - q)(x_2), \dots, (p - q)(x_{101})) \\ \|T(p) - T(q)\|_2 &= \sqrt{\sum_{i=1}^{101} |(p - q)(x_i)|^2} \leq \sqrt{\sum_{i=1}^{101} \|p - q\|_\infty^2} < \sqrt{\sum_{i=1}^{101} \left(\frac{\epsilon}{\sqrt{101}}\right)^2} \\ \|T(p) - T(q)\|_2 &< \sqrt{\sum_{i=1}^{101} \frac{\epsilon^2}{101}} = \sqrt{101 \cdot \frac{\epsilon^2}{101}} = \sqrt{\epsilon^2} = |\epsilon| = \epsilon \end{aligned}$$

Hence,  $\|p - q\|_\infty < \delta$  implies  $\|T(p) - T(q)\|_2 < \epsilon$ , showing that  $T$  is in fact uniformly continuous.

**The map  $T^{-1}$  is also Continuous:**

The goal is to prove that for all  $v_0 \in \mathbb{R}^{101}$  and all  $\epsilon > 0$ , there exists  $\delta > 0$ , such that  $\|v_0 - v\|_2 < \delta$  implies  $\|T^{-1}(v_0) - T^{-1}(v)\|_\infty < \epsilon$ .

From the previous section, we know given  $\epsilon' = \sqrt{101}\epsilon > 0$ , with  $\delta = \frac{\epsilon'}{\sqrt{101}} = \epsilon > 0$ , any  $v \in \mathbb{R}^{101}$  with  $\|T^{-1}(v_0) - T^{-1}(v)\|_\infty < \delta = \epsilon$  implies  $\|TT^{-1}(v_0) - TT^{-1}(v)\|_2 = \|v_0 - v\|_2 < \epsilon' = \sqrt{101}\epsilon$ . Hence, we need the range to be narrower than  $\sqrt{101}\epsilon$ .

**The Sequence of Polynomial Converges Uniformly to 0:**

Recall that since  $\{p_m\}_{m \in \mathbb{N}}$  converges pointwise to 0, then for all  $i \in \{1, \dots, 101\}$ ,  $\lim_{m \rightarrow \infty} p_m(x_i) = 0$ .

Also, from the previous section, since  $T^{-1}$  is continuous (possibly on a restricted domain), for all  $\epsilon > 0$ , there exists  $\delta > 0$ , such that for all  $u \in \mathbb{R}^{101}$ ,  $\|u\|_2 < \delta$  implies  $\|T^{-1}(u)\|_\infty < \epsilon$ .

Using the pointwise convergence, for the given  $\delta > 0$  (which  $\frac{\delta}{\sqrt{101}} > 0$ ), each  $i \in \{1, \dots, 101\}$  has a corresponding  $M_i$ , such that  $m \geq M_i$  implies  $|p_m(x_i)| < \frac{\delta}{\sqrt{101}}$ .

Then, let  $M = \max_{i \in \{1, \dots, 101\}} \{M_i\}$ , for all  $m \geq M$  (which  $m \geq M_i$  for all  $i \in \{1, \dots, 101\}$ ), the following is true:

$$\|T(p_m) - T(0)\|_2 = \|T(p_m)\|_2 = \sqrt{\sum_{i=1}^{101} |p_m(x_i)|^2} < \sqrt{\sum_{i=1}^{101} \left(\frac{\delta}{\sqrt{101}}\right)^2} = \sqrt{101 \cdot \frac{\delta^2}{101}} = \sqrt{\delta^2} = |\delta| = \delta$$

Hence, by the continuity of  $T^{-1}$ ,  $T^{-1}(T(p_m)) = p_m$  satisfies  $\|T^{-1}(T(p_m))\|_\infty < \epsilon$ , or  $\|p_m\|_\infty < \epsilon$ .

Therefore, this concludes that the sequence of polynomials  $p_m$  converges to 0 uniformly.

**2**

**Question 2** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a function such that  $f', f'', f^{(3)}$  are defined and continuous in  $[0, 1]$ . Prove that for any  $\epsilon > 0$  there exists a polynomial  $P$  such that

$$\sum_{j=0}^3 \|f^{(j)} - P^{(j)}\|_\infty = \sum_{j=0}^3 \sup_{x \in [0, 1]} |(f^{(j)} - P^{(j)})(x)| < \epsilon$$

**Pf:**

Before starting the prove, recall that the antiderivatives of a polynomial  $p : [0, 1] \rightarrow \mathbb{R}$  is a collection of polynomials  $\{P(x) + C \mid C \in \mathbb{R}\}$ , where  $P : [0, 1] \rightarrow \mathbb{R}$  is a polynomial satisfying  $P' = p$ .

When taking the antiderivative of any polynomial in the following steps, we'll explicitly state the initial condition to prevent ambiguity about the constant coefficients of the antiderivative.

**Generalized Statement:**

We'll prove a more general version recursively: For all  $n \in \mathbb{N}$ , let  $f : [0, 1] \rightarrow \mathbb{R}$  be a function such that  $f', \dots, f^{(n)}$  are all defined and continuous on  $[0, 1]$ , then there exists a sequence of polynomials  $\{P_m\}_{m \in \mathbb{N}}$ , such that for all  $j \in \{0, 1, \dots, n\}$ ,  $P_m^{(j)}$  converges to  $f^{(j)}$  uniformly.

For base case, since  $f^{(n)}$  is defined and continuous on  $[0, 1]$ , by Stone-Weierstrass Theorem, there exists a sequence of polynomials  $\{p_{n,m}\}$  converging to  $f^{(n)}$  uniformly.

Then as **Step (1)**, for all  $m \in \mathbb{N}$ , let polynomial  $p_{(n-1),m} : [0, 1] \rightarrow \mathbb{R}$  be an antiderivative of  $p_{n,m}$  ( $p'_{(n-1),m} = p_{n,m}$ ) such that  $p_{(n-1),m}(0) = f^{(n-1)}(0)$ .

Which, since the sequence of polynomials  $\{p_{(n-1),m}\}_{m \in \mathbb{N}}$  satisfies:  $p'_{(n-1),m} = p_{n,m}$  converges to  $(f^{(n-1)})' = f^{(n)}$  uniformly, and  $\lim_{m \rightarrow \infty} p_{(n-1),m}(0) = f^{(n-1)}(0)$ . Then, the sequence  $p_{(n-1),m}$  converges to  $f^{(n-1)}$  uniformly.

Now, for given  $k \in \{1, \dots, n-1\}$ , at **Step (k)** we constructed a sequence of  $k^{th}$  antiderivative of the sequence of polynomials  $\{p_{n,m}\}_{m \in \mathbb{N}}$  (denoted as  $\{p_{(n-k),m}\}_{m \in \mathbb{N}}$ ), such that  $p_{(n-k),m}$  converges to  $f^{(n-k)}$  uniformly:

At **Step (k+1)**, for each  $m \in \mathbb{N}$ , let polynomial  $p_{(n-(k+1)),m} : [0, 1] \rightarrow \mathbb{R}$  be an antiderivative of  $p_{(n-k),m}$  (which  $p'_{(n-(k+1)),m} = p'_{(n-k),m}$ ) such that  $p_{(n-(k+1)),m}(0) = f^{(n-(k+1))}(0)$ .

Which, since the new sequence of polynomials  $\{p_{(n-(k+1)),m}\}_{m \in \mathbb{N}}$  satisfies:  $p'_{(n-(k+1)),m} = p_{(n-k),m}$  converges to  $(f^{(n-(k+1))})' = f^{(n-k)}$ , and  $\lim_{m \rightarrow \infty} p_{(n-(k+1)),m}(0) = f^{(n-(k+1))}(0)$ . Then, the sequence  $p_{(n-(k+1)),m}$  converges to  $f^{(n-(k+1))}$  uniformly.

From the above process, since for all  $k \in \{1, \dots, n\}$ , we can find a sequence of  $k^{th}$  antiderivative of polynomials  $\{p_{n,m}\}_{m \in \mathbb{N}}$ , denoted as  $\{p_{(n-k),m}\}_{m \in \mathbb{N}}$ , that converges to  $f^{(n-k)}$  uniformly.

Then, the sequence  $\{p_{0,m}\}_{m \in \mathbb{N}}$  is a sequence of polynomial that converges to  $f^{(0)} = f$  uniformly. Which, for  $j \in \{1, \dots, n\}$ , the sequence of  $j^{th}$  derivative  $\{p_{j,m}\}_{m \in \mathbb{N}}$  converges uniformly to the  $j^{th}$  derivative of  $f$ , namely  $f^{(j)}$ . (Note: Recall that for all  $j \in \{1, \dots, n\}$  and all  $m \in \mathbb{N}$ ,  $p_{(j-1),m}$  is defined as an antiderivative of  $p_{j,m}$ ).

Hence, the sequence of polynomials  $\{p_{0,m}\}_{m \in \mathbb{N}}$  has its  $j^{th}$  derivative converges to  $f^{(j)}$  uniformly for all given  $f^{(j)}$ , satisfying the desired condition stated initially.

### The Original Problem:

From the above Generalized Statement, given  $f : [0, 1] \rightarrow \mathbb{R}$  such that  $f', f'', f^{(3)}$  that are all defined and continuous on  $[0, 1]$ , there exists a sequence of polynomials  $\{P_m\}_{m \in \mathbb{N}}$ , such that for  $j \in \{0, 1, 2, 3\}$ , its  $j^{th}$  derivative  $P_m^{(j)}$  converges to  $f^{(j)}$  uniformly.

Hence, given arbitrary  $\epsilon > 0$  (which  $\frac{\epsilon}{4} > 0$ ), for each  $j \in \{0, 1, 2, 3\}$ , there is a corresponding  $N_j$ , such that the following is true:

$$\forall m \in \mathbb{N}, \quad m \geq N_j \implies \|f^{(j)} - P_m^{(j)}\|_{\infty} < \frac{\epsilon}{4}$$

Then, choose  $N = \max_{j \in \{0, 1, 2, 3\}} N_j$ , for any index  $m \geq N$ , since  $m \geq N_j$  for all  $j \in \{0, 1, 2, 3\}$ , the above statement guarantees  $\|f^{(j)} - P_m^{(j)}\|_{\infty} < \frac{\epsilon}{4}$  for each  $j$ . Hence, the following inequality is true:

$$\sum_{j=0}^3 \|f^{(j)} - P_m^{(j)}\|_{\infty} < \sum_{j=0}^3 \frac{\epsilon}{4} = \epsilon$$

Therefore, for every  $\epsilon > 0$ , we can find a corresponding polynomial  $P$ , such that  $\sum_{j=0}^3 \|f^{(j)} - P^{(j)}\|_{\infty} < \epsilon$ .

### 3

**Question 3** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function such that

$$\int_0^1 f(x)x^j dx = 0, \quad j = 0, 1, 2, \dots$$

Prove that  $f(x) = 0, \forall x \in [0, 1]$ .

**Pf:**

Since  $f(x)$  is continuous on  $[0, 1]$  a bounded closed interval, by Stone-Weierstrass Theorem, there exists a sequence of polynomial  $\{p_n\}_{n \in \mathbb{N}}$ , such that  $p_n$  converges to  $f$  uniformly.

Now, notice that for all polynomial  $p(x) = a_0 + a_1x + \dots + a_mx^m$  (where  $a_0, a_1, \dots, a_m \in \mathbb{R}$ ), the following integral is true based on the Linearity of Riemann Integrable functions:

$$\int_0^1 f(x)p(x)dx = \int_0^1 \sum_{k=0}^m a_k x^k dx = \sum_{k=0}^m a_k \int_0^1 f(x)x^k dx = 0$$

Hence, for all  $n \in \mathbb{N}$ , we have  $\int_0^1 f(x)p_n(x)dx = 0$ .

**$fp_n$  Converges Uniformly to  $f^2$ :**

Because  $f$  is continuous on  $[0, 1]$  a compact set, hence  $f$  is bounded, there exists  $M > 0$ , such that all  $x \in [0, 1]$  satisfies  $|f(x)| < M$ .

Also, since  $p_n$  converges to  $f$  uniformly, for all  $\epsilon > 0$  (which  $\frac{\epsilon}{M} > 0$ ), there exists  $N$ , such that  $n \geq N$  implies  $\|f - p_n\|_\infty < \frac{\epsilon}{M}$ .

Hence, for all  $n \geq N$ , every  $x \in [0, 1]$  satisfies the following:

$$|f(x)p_n(x) - (f(x))^2| = |f(x)| \cdot |p_n(x) - f(x)| < M \cdot |p_n(x) - f(x)| \leq M \cdot \|f - p_n\|_\infty < M \cdot \frac{\epsilon}{M} < \epsilon$$

Hence,  $\epsilon$  is an upper bound of the set  $\{|f(x)p_n(x) - (f(x))^2| \mid x \in [0, 1]\}$ , showing that  $\|fp_n - f^2\|_\infty = \sup_{x \in [0, 1]} |f(x)p_n(x) - (f(x))^2| \leq \epsilon$ . Based on the above statement, we can conclude that  $fp_n$  converges uniformly to  $f^2$ .

**Integral of  $fp_n$  converges to Integral of  $f^2$ :**

For all  $n \in \mathbb{N}$ , we have  $fp_n$  being continuous on  $[0, 1]$  (since both  $f$  and  $p_n$  are continuous on  $[0, 1]$ ), and  $fp_n$  converges to  $f^2$  uniformly, hence the following is true:

$$\lim_{n \rightarrow \infty} \int_0^1 f(x)p_n(x)dx = \int_0^1 \lim_{n \rightarrow \infty} f(x)p_n(x)dx = \int_0^1 (f(x))^2 dx$$

Since  $\int_0^1 f(x)p_n(x)dx = 0$ , then the limit above is 0, hence  $\int_0^1 (f(x))^2 dx = 0$ .

**Integral of  $f^2$  is 0 implies  $f = 0$ :**

Since  $f$  is continuous on  $[0, 1]$ , so does  $f^2$ ; then, since for all  $x \in [0, 1]$ ,  $(f(x))^2 \geq 0$ , together with the statement  $\int_0^1 (f(x))^2 dx = 0$ , this implies that  $(f(x))^2 = 0$  for all  $x \in [0, 1]$ .

Therefore,  $f(x) = 0$  for all  $x \in [0, 1]$ .