and, when $r = \infty$, the limit of the ratio $= \frac{1}{p}$. Therefore the series will be convergent if $p > 1$.

11.

In considering the series $F_1(p^\alpha, p^\beta, p^\gamma)$, it naturally occurs that one should wish to consider the series which will be the generalization of $F(\alpha, \beta, \gamma, x)$ and the differential equation connected with it.

The series $$1 + p^{\gamma-\alpha-\beta}\frac{p^\alpha-1\,.\,p^\beta-1}{p^\gamma-1\,.\,p-1}x+\dots$$

is not the series sought, for, on differentiating the powers of x, we do not obtain coefficients which cancel with the factors $p-1$, p^2-1, p^3-1, &c., of the denominators.

It seems that the ordinary notion of differentiation is not sufficient, and that differentiation must be generalized into some operation of the following type:—

$$u = \left(\frac{p^x-1}{p-1}\right)^{(p^n-1)/(p-1)},$$

$$\frac{du}{d(x)} = \frac{p^n-1}{p-1}\left(\frac{p^x-1}{p-1}\right)^{(p^{n-1}-1)/(p-1)},$$

reducing when $p=1$ to ordinary differentiation.

On the Partition of Numbers. By G. B. Mathews, M.A.

Read May 13th, 1897. Received, in revised form, June 25th, 1897.

The problem of the partition of a given multipartite number $(m, m', m'', \dots)$ into assigned parts $(a, a', a'', \dots)$, $(b, b', b'', \dots)$, &c., is identical with that of finding integers $x, y, z, \dots t$, none of which are negative, although some of them may be zero, so that

$$\left.\begin{aligned} u &= ax+by+cz+\dots+lt = m, \\ u' &= a'x+b'y+c'z+\dots+l't = m', \\ u'' &= a''x+b''y+c''z+\dots+l''t = m'', \\ &\text{\&c.,} \end{aligned}\right\} \qquad (1)$$

the number of equations being equal to that of the elements m, m', m'', &c.

I propose to show, by very elementary considerations, that this general problem is reducible, in an indefinite number of ways, to one of simple partition. For convenience, the term "solution" will be used for a solution in which $x, y, z, \dots t$ satisfy the condition above stated.

In order to avoid the discussion of exceptional cases, it will be supposed that none of the coefficients $a, b, \dots l$; $a', b', \dots l'$; &c., is zero. No real loss of generality arises from this; because if, for instance, we have the two equations $u = m$, $u' = m'$, and zero coefficients occur in them, we may replace them by the equivalent pair

$$u+u' = m+m', \quad u+2u' = m+2m',$$

in which the coefficients are all positive, and a similar process may be applied in the general case.

Suppose, in the first place, that we have the two equations

$$\left.\begin{aligned} u &= ax+by+cz+dw = m, \\ u' &= a'x+b'y+c'z+d'w = m'; \end{aligned}\right\} \tag{2}$$

then, if λ, μ are any two positive integers, every solution of (2) is also a solution of

$$(\lambda a+\mu a')\,x+(\lambda b+\mu b')\,y+(\lambda c+\mu c')\,z+(\lambda d+\mu d')\,w = \lambda m+\mu m'. \tag{3}$$

But, conversely, if λ, μ are suitably chosen, then every solution of (3) is also a solution of (2). For suppose λ, μ are prime to each other; then every solution of (3) must give

$$u = ax+by+cz+dw = m \pm \mu\theta,$$

$$u' = a'x+b'y+c'z+d'w = m' \mp \lambda\theta,$$

where θ is some integer or zero. If we take $\lambda \overline{\overline{>}} m'$ and $\mu \overline{\overline{>}} m$, one or other of the quantities $m \mp \mu\theta$, $m' \mp \lambda\theta$ will be zero or negative, except when $\theta = 0$. But every solution reckoned makes u and u' positive;* therefore $\theta = 0$, and every solution of (3) is also a solution of (2). In other words, the problem of bipartition represented by the pair of equations (2) has been reduced to that of a simple partition, corresponding to the equation (3).

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* The value zero for u or u' is excluded by the supposition that all the coefficients of u and u' are positive.

As a particular case, if m, m' are prime to each other, we may put $\lambda = m'$, $\mu = m$, so that (3) becomes

$$(m'a+ma')\,x+(m'b+mb')\,y+(m'c+mc')\,z+(m'd+md')\,w = 2mm'.$$

For example, the solutions of

$$x+y+z+w = 9, \quad x+2y+3z+4w = 20$$

coincide with those of

$$29x+38y+47z+56w = 360.$$

The argument has not been affected by the number of unknown quantities x, y, z, &c., so that the general problem of the partition of a bipartite number has been reduced to that of a simple partition. But the process may evidently be extended so as to cover the most general case of multiple partition; thus suppose that there are three equations

$$\left.\begin{aligned} u &= ax+by+cz+dw = m, \\ u' &= a'x+b'y+c'z+d'w = m', \\ u'' &= a''x+b''y+c''z+d''w = m''; \end{aligned}\right\} \qquad (4)$$

then, by suitably choosing λ, μ, the solutions of the first two may be made to coincide with those of

$$\lambda u+\mu u' = \lambda m+\mu m',$$

and now we may again choose integers λ', μ' so that the solutions common to this and the third of the given equations coincide with those of the single equation

$$\lambda'(\lambda u+\mu u')+\mu' u'' = \lambda'(\lambda m+\mu m')+\mu' m'';$$

or, changing the notation, we may say that it is possible, in an infinite number of ways, to choose positive integers λ, μ, ν so that the solutions of (4) coincide with those of

$$(\lambda a+\mu a'+\nu a'')\,x+\ldots = \lambda m+\mu m'+\nu m''.$$

It is clear that, by a process of induction, a similar result may be established for any number of equations.

In the case of two equations with three undetermined quantities

$$ax+by+cz-m = 0, \quad a'x+b'y+c'z-m' = 0,$$

we have a geometrical interpretation; namely, that, if there are integral points on the line of intersection of the planes represented by these equations, then an indefinite number of planes

$$\lambda\,(ax+by+cz-m)+\mu\,(a'x+b'y+c'z-m') = 0$$

can be found (λ, μ being positive integers), so as to pass through no integral points in the first octant except those on their common line of intersection. A quasi-geometrical interpretation of the same kind may, of course, be given to the result in the general case.

It may be observed that the above method of finding λ, μ for the reduction of a bipartition is not the only one; nor does it necessarily give the most convenient values. Another way of proceeding is as follows:—Let the two given equations be

$$\left.\begin{aligned} ax+by+cz+dw &= m, \\ a'x+b'y+c'z+d'w &= m'; \end{aligned}\right\} \qquad (5)$$

then (i.), if there is any solution at all, one at least of the quantities

$$\frac{m'a}{a'},\ \frac{m'b}{b'},\ \frac{m'c}{c'},\ \frac{m'd}{d'}$$

must be equal to or exceed m; and (ii.), if i is any positive integer which exceeds the greatest of the four quantities, the solutions of (5) coincide with those of

$$(a+ia')\,x+(b+ib')\,y+(c+ic')\,z+(d+id')\,w = m+im'. \qquad (6)$$

To prove the first statement, suppose

$$m'a < ma', \quad m'b < mb', \quad \&\text{c.};$$

then $$m'\,(ax+by+cz+dw) < m\,(a'x+b'y+c'z+d'w),$$

if none of the quantities x, y, z, w is negative, and they are not all zero. But every solution of (5) makes

$$m'\,(ax+by+cz+dw) = m\,(a'x+b'y+c'z+d'w);$$

therefore the assumed inequalities cannot all hold if the given equations are soluble.

Every solution of (6) leads to

$$\left.\begin{aligned} ax+by+cz+dw &= m+iq, \\ a'x+b'y+c'z+d'w &= m'-q, \end{aligned}\right\} \tag{7}$$

where q is zero or some positive integer. But, since $ia' > m'a$, &c., it follows that

$$i(a'x+b'y+c'z+d'w) > m'(ax+by+cz+dw);$$

and therefore, if $$ax+by+cz+dw > i,$$

it follows, *a fortiori*, that

$$a'x+b'y+c'z+d'w > m';$$

so that the equations (7) are inconsistent, except for $q = 0$. This proves that every solution of (6) is also a solution of (5).

Thus, if, in the numerical example above given, we take $m = 9$, $m' = 20$, the process just explained gives $i \geqq 21$, and the derived equation for $i = 21$ is

$$22x+43y+64z+85w = 429,$$

while, if we take $m = 20$, $m' = 9$, we have $i \geqq 37$, and the derived equation for $i = 37$ is

$$38x+39y+40z+41w = 353.$$

The second process, like the first, may be used to replace any number of linear indeterminate equations by a single equivalent one.

It may be of interest to state that the argument upon which this paper is based occurred to me after reading the memoirs of Dedekind and Kronecker on the theory of ideal numbers. Kronecker makes extensive use of "undetermined quantities," which are, in fact, what Sylvester called *umbræ;* in Dedekind's ideals we have undetermined integers, or compound moduli, which serve the same purpose as Kronecker's umbræ.