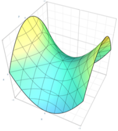

Unidade 5: Integrais múltiplas: coordenadas polares

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Sumário

Exemplo

Calcule a área da rosácea de quatro pétalas $r = \cos(2\theta)$

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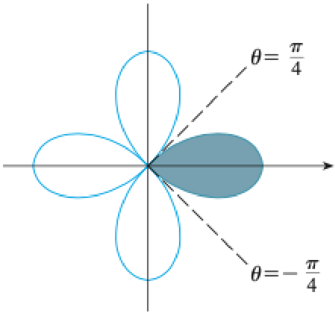


$$A(R) = 4\iint_R r\,dr\,d\theta = 4\int_{-\pi/4}^{\pi/4}\int_0^{\cos(2\theta)} r\,drd\theta$$

$$= 4\int_{-\pi/4}^{\pi/4}\left[\frac{r^2}{2}\right]_0^{\cos(2\theta)} d\theta = 2\int_{-\pi/4}^{\pi/4}\cos^2(2\theta)\,d\theta$$

Usando a fórmula $cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$ temos que

$$cos^2(2\theta) = \frac{1+\cos(4\theta)}{2},$$

e logo,

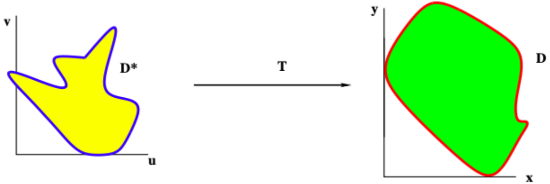
$$
\begin{aligned}
A(R) &= \int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\,d\theta = \int_{-\pi/4}^{\pi/4} \Big(1+\cos(4\theta)\Big)\,d\theta \\
&= \int_{-\pi/4}^{\pi/4} \Big(1+\cos(4\theta)\Big)\,d\theta \\
&= \theta\Big|_{-\pi/4}^{\pi/4} + \frac{1}{4}\operatorname{sen}(4\theta)\Big|_{-\pi/4}^{\pi/4} \\
&= \frac{\pi}{4} + \frac{\pi}{4} + 0 = \frac{\pi}{2}
\end{aligned}
$$

Uma mudança de coordenadas pode ser compreendida, de uma modo mais geral, como uma transformação

$$\begin{aligned} T : D^* \subset \mathbb{R}^2 &\to \mathbb{R}^2 \\ (u,v) &\mapsto T(u,v) = (x(u,v), y(u,v)) \end{aligned}$$

Também podemos representar T do seguinte modo:

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$



Note que, T transforma D^* em $D = T(D^*)$

Exemplo (1)

Considere $D^* = [0,1] \times [0,2\pi]$ *um retângulo e* $T(r,\theta) = (x(r,\theta), y(r,\theta))$ *as coordenadas polares:*

$$\begin{cases} x = r\cos(\theta) \\ y = r\,\mathrm{sen}(\theta) \end{cases}$$

Determine $D = T(D^*)$

Note que, $0 \leq r \leq 1$ e $0 \leq \theta \leq 2\pi$ e $x^2 + y^2 = r^2$.

Logo, $x^2 + y^2 \leq 1$ e $D = \left\{(x, y) \in \mathbb{R}^2 \,;\, x^2 + y^2 \leq 1\right\}$

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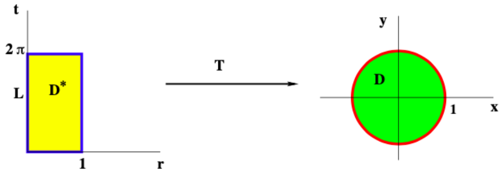


Definição

Dada uma transformação $T : D^* \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ *por*

$$\begin{cases} x &= x(u,v) \\ y &= y(u,v) \end{cases}$$

A matriz das derivadas parciais de T, *dada por*

$$J = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}$$

é chamada **matriz Jacobiana** *de* T.

Definição

O determinante da matriz jacobiana da transformação T é chamado **jaconiano** *de T e é dado por*

$$\det J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Exemplo

Calcule o jacobiano da transformação de coordenadas polares dada no exemplo (1)

Solução: T é dada por

$$\begin{cases} x &= r\cos(\theta) \\ y &= r\,\mathrm{sen}\,(\theta) \end{cases}$$

Então,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\,\mathrm{sen}\,(\theta) \\ \mathrm{sen}\,(\theta) & r\cos(\theta) \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos(\theta) & -r\,\mathrm{sen}\,(\theta) \\ \mathrm{sen}\,(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\,\mathrm{sen}\,^2(\theta)$$

Logo,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r\left(\cos^2(\theta) + \mathrm{sen}\,^2(\theta)\right) = r$$

Exemplo

Seja $D^* = [0,1] \times [0,1]$ *um quadrado e* T *dada por*

$$\begin{cases} x = u + v \\ y = u - v \end{cases}$$

Determine $D = T(D^*)$ *e calcule o jacobiano de* T.

Mudança de coordenadas
Exemplos

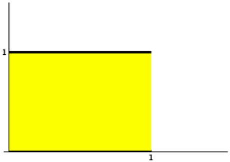
Solução: Note que, $0 \leq u \leq 1$ e $0 \leq v \leq 1$. Então,

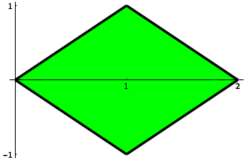
$u = 0 \Rightarrow y = -x$

$v = 0 \Rightarrow y = x$

$u = 1 \Rightarrow y = 2 - x$

$v = 1 \Rightarrow y = x - 2$


1
1


1
1
2
−1

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

Teorema

Sejam $T : D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ *uma transformação de classe* $\mathcal{C}^1$ *e* $D = T(D^*)$*. Suponha que o jabobiano de* T *é diferente de zero, então*

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

com $f(u,v) = f(x(u,v), y(u,v))$*. Em particular, a área de* D *é dada por*

$$A(D) = \iint_D dx\,dy = \iint_{D^*}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Considerando as coordenadas polares temos que

$$D^* = \{(r,\theta)\,;\, 0 \le r \le \rho, \quad \alpha \le \theta \le \beta\}$$

com $\beta - \alpha = 2\pi$ Então,

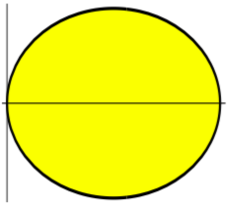
$$\begin{aligned} D = T(D^*) &= \{(x,y)\,;\, x^2 + y^2 \le \rho^2\} \\ f(r,\theta) &= f(r\cos(\theta), r\,\mathrm{sen}\,(\theta)) \\ \frac{\partial(x,y)}{\partial(r,\theta)} &= r \end{aligned}$$

Exemplo

Usando coordenadas polares, calcule a área da região limitada pelo círculo $(x-a)^2+y^2 \leq a^2$ com $a>0$ uma constante qualquer.

Mudança de coordenadas
Exemplos

Solução:



Note que, $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$

Mudança de coordenadas

Exemplos

Coordenadas polares: $x = r\cos(\theta)$ e $y = r\,\text{sen}\,(\theta)$. Então,

$(x-a)^2 + y^2 = a^2 \Rightarrow (r\cos(\theta) - a)^2 + (r\,\text{sen}\,(\theta))^2 = a^2 \Rightarrow$

$r^2\cos^2(\theta) - 2\,r\,a\,\cos(\theta) + a^2 + r^2\,\text{sen}^2(\theta) = a^2 \Rightarrow$

$r^2(\cos^2(\theta) + \text{sen}^2(\theta)) - 2\,r\,a\,\cos(\theta) = 0 \Rightarrow$

$r^2 = 2\,r\,a\,\cos(\theta) \Rightarrow r = 2\,a\,\cos(\theta)$. Logo,

$$0 \le r \le 2\,a\,\cos(\theta)$$

Portanto, $D^* = \{(r,\theta); 0 \leq r \leq 2a\cos(\theta) \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$.

Assim, lembrando que $\frac{\partial(x,y)}{\partial(r,v\theta)} = r$ temos

$$\begin{aligned} A(D) = \iint_D dx\,dy &= \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos(\theta)} r\,dr\,d\theta \end{aligned}$$

$$A(D) = \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2}\right]_0^{2a\cos(\theta)} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} 4a^2 \cos^2(\theta) d\theta$$

$$= 2a^2 \int_{-\pi/2}^{\pi/2} \cos^2(\theta) = 2a^2 \int_{-\pi/2}^{\pi/2} \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= a^2 \int_{-\pi/2}^{\pi/2} (1+\cos(2\theta))\, d\theta = a^2\theta\Big|_{-\pi/2}^{\pi/2} + \frac{a^2}{2}\operatorname{sen}(2\theta)\Big|_{-\pi/2}^{\pi/2}$$

$$= a^2\left(\frac{\pi}{2}+\frac{\pi}{2}\right) + 0 = \pi a^2$$

Exemplo

Calcule o volume do sólido situado acima do plano xy e limitado pelas superfícies $z = x^2 + y^2$ e $x^2 + y^2 - 2y = 0$.

Solução: Note que, $z = x^2 + y^2$ é um parabolóide centrado em $(,0,0,0)$ e $x^2 + y^2 - 2y = 0$ é um cilindro circular reto.

Portanto, a região de integração D é a região limitada pelo círculo $x^2 + y^2 - 2y = 0$.

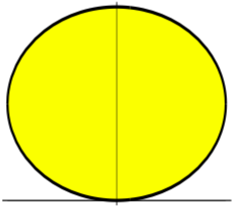
Vamos completar os quadrados para descobrir o centro e o raio deste círculo. Assim,

$x^2 + y^2 - 2y + 1 - 1 = 0 \Rightarrow x^2 + (y-1)^2 = 1 \Rightarrow$

$$\text{centro } (0,1) \text{ e raio } = 1$$

Mudança de coordenadas

Exemplos



Note que, $0 \leq \theta \leq \pi$

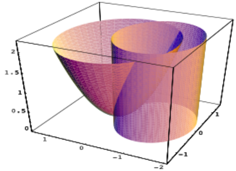
Portanto,

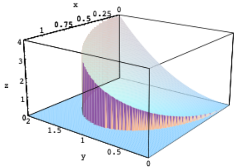
$$V = \iint_D f(x,y)\,dx\,dy = \iint_D x^2 + y^2\,dx\,dy$$

com $D = \{(x,y)\,;\, x^2 + (y-1)^2 \leq 1\}$

Mudança de coordenadas

Exemplos





Usando coordenadas polares: $x = r\cos(\theta)$ e $y = r\text{sen}\,(\theta)$ temos

$x^2 + (y-1)^2 = 1 \Rightarrow$

$(r\cos(\theta))^2 + (r\text{sen}\,(\theta))^2 - 2\,r\,\text{sen}\,(\theta) = 0 \Rightarrow$

$r^2\cos^2(\theta) + r^2\text{sen}\,^2(\theta) - r\,\text{sen}\,(\theta) = 0 \Rightarrow$

$r^2\,(\cos^2(\theta) + \text{sen}\,^2(\theta)) - r\,\text{sen}\,(\theta) = 0 \Rightarrow$

$r^2 = 2\,r\,a\,\text{sen}\,(\theta) \Rightarrow r = 2\,\text{sen}\,(\theta).$

Logo,

$$0 \leq r \leq 2\,\text{sen}\,(\theta)$$

Portanto, $D^* = \{(r,\theta); 0 \leq r \leq 2\,\text{sen}\,(\theta)\ \ 0 \leq \theta \leq \pi$. Assim, lembrando que $\dfrac{\partial(x,y)}{\partial(r,v\theta)} = r$ temos

$$V = \iint_D x^2 + y^2\,dx\,dy = \int_0^{\pi}\int_0^{2\sin(\theta)} r^2\,r\,dr\,d\theta,$$

ou seja,

$$V = \int_0^{\pi}\int_0^{2\sin(\theta)} r^3\,dr\,d\theta,$$

$$V = \int_0^{\pi} \int_0^{2\,\mathrm{sen}\,(\theta)} r^3 \, dr \, d\theta = \int_0^{\pi} \left[\frac{r^4}{4} \right]_0^{2\,\mathrm{sen}\,(\theta)} d\theta$$

$$= 4 \int_0^{\pi} \mathrm{sen}^4(\theta) \, d\theta = 4 \int_0^{\pi} \frac{(1 - \cos(2\theta))^4}{4} \, d\theta$$

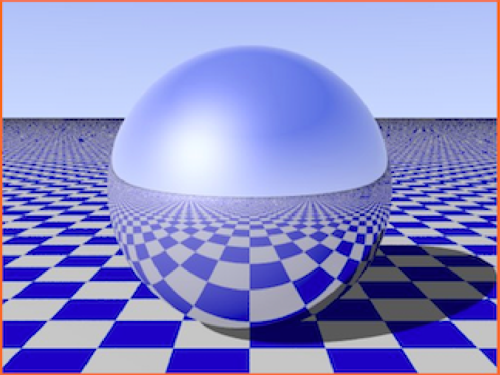
$$= \int_0^{\pi} (1 - 2\cos(2\theta) + \cos^2(2\theta)) \, d\theta$$

$$= \theta \Big|_0^{\pi} - \mathrm{sen}\,(2\theta) \Big|_0^{\pi} + \int_0^{\pi} \cos^2(2\theta) \, d\theta$$

$$= \pi + 0 + \int_0^{\pi} \frac{(1 + \cos(4\theta))}{2} \, d\theta$$

$$V = \pi + \int_0^{\pi} \frac{(1 + \cos(4\theta))}{2}\, d\theta = \pi + \left.\frac{\theta}{2}\right|_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos(4\theta)\, d\theta$$

$$= \pi + \frac{\pi}{2} + \left.\frac{1}{8}\operatorname{sen}(4\theta)\right|_0^{\pi} = \frac{3\pi}{2}$$



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