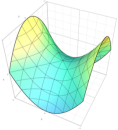

Unidade 5: Integrais triplas

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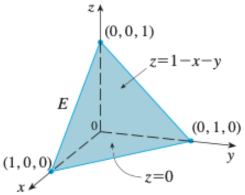
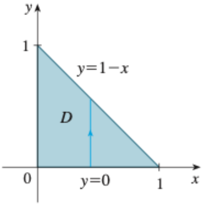
Sumário

Exemplo

Calcule a integral tripla $\iiint_E z\,dx\,dy\,dz$ *com* E *o tetraedro sólido limitado pelos plano* $x = 0, y = 0, z = 0$ *e* $x + y + z = 1$

Solução: Desenhar a região de integração E no espaço e a região plana D





Note que, $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$ e $0 \leq z \leq 1 - x - y$

Pelo teorema de Fubini:

$$\iiint_E z\,dx\,dy\,dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left(\int_0^{1-x-y} z\,dz \right) dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left(\frac{z^2}{2} \right)_0^{1-x-y} dy\,dx$$

$$= \int_0^1 \left(\int_0^{1-x} \frac{(1-x-y)^2}{2}\,dy \right) dx$$

Troca de variável: $u = 1 - x - y \; ; \; du = -dy$:

$$= \int_0^1 \left(\int_0^{1-x} \frac{(1-x-y)^2}{2}\, dy \right) dx = \int_0^1 \left(-\int_0^{1-x} \frac{u^2}{2}\, du \right) dx$$

$$= \int_0^1 \left(-\frac{1}{2} \int_0^{1-x} \frac{u^3}{3}\, du \right) dx = -\frac{1}{6} \int_0^1 (1-x-y)^3 \Big|_0^{1-x} dx$$

$$= -\frac{1}{6} \int_0^1 \Big(1 - x - (1-x)\Big)^3 - \Big(1 - x - 0\Big)^3 dx$$

$$= \frac{1}{6} \int_0^1 (1-x)^3 dx$$

Integrais triplas em regiões mais gerais

Troca de variável: $u = 1 - x \; ; \; du = -dx$:

$$= \frac{1}{6}\int_0^1 (1-x)^3 dx = -\frac{1}{6}\int_0^1 u^3 du = -\frac{1}{6}\left(\frac{u^4}{4}\right)_0^1$$

$$= -\frac{1}{24}\left((1-x)^4\right)_0^1 = -\frac{1}{24}\left((1-1)-(1-0)\right)$$

$$= \frac{1}{24}$$

Exemplo

Calcule o volume do sólido limitado por $z = x^2 = 9$, $z + y = 4$, $y = 0$ *e* $y = 4$

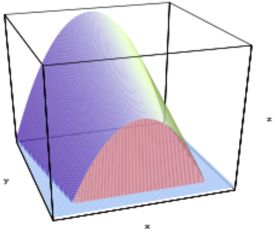
Exemplo

Calcule o volume do sólido limitado por $z = x^2 = 9$, $z + y = 4$, $y = 0$ *e* $y = 4$

O volume é dado por

$$V = \iiint_E dx\, dy\, dz$$

Note que o sólido E é limitado superiormente pelo cilindro $z = 9 - x^2$ e inferiormente pelo plano $z = 4 - y$



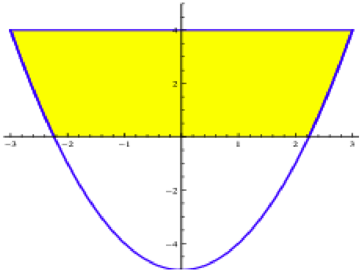
Então,

$$E = \left\{(x,y,z)\,;\,(x,y) \in D,\; 4-y \le z \le 9-x^2\right\}$$

com D a projeção de E no plano xy.

Para encontrar D vamos eliminar a variável z, ou seja,

$$z = 9 - x^2,\; z = 4 - y \Rightarrow 9 - x^2 = 4 - y \Rightarrow y = x^2 - 5$$



$$0 \leq y \leq 4 \text{ e } -\sqrt{y+5} \leq x \leq \sqrt{y+5}$$

Então,

$$E = \left\{(x, y, z)\,;\, 0 \leq y \leq 4, -\sqrt{y+5} \leq x \leq \sqrt{y+5}, 4-y \leq z \leq 9-x^2\right\}$$

e

$$V = \iiint_E dx\,dy\,dz = \int_0^4 \int_{-\sqrt{y+5}}^{\sqrt{y+5}} \int_{4-y}^{9-x^2} dz\,dx\,dy$$

$$V = \int_0^4 \int_{-\sqrt{y+5}}^{\sqrt{y+5}} \left(\int_{4-y}^{9-x^2} dz \right) dx\, dy$$

$$= \int_0^4 \int_{-\sqrt{y+5}}^{\sqrt{y+5}} z\Big|_{4-y}^{9-x^2} dx\, dy = \int_0^4 \int_{-\sqrt{y+5}}^{\sqrt{y+5}} (9-x^2)-(4-y)\, dx\, dy$$

$$= \int_0^4 \int_{-\sqrt{y+5}}^{\sqrt{y+5}} (5+y-x^2)\, dx\, dy$$

$$= \int_0^4 (5+y)x\Big|_{-\sqrt{y+5}}^{\sqrt{y+5}} dy - \int_0^4 \frac{x^3}{3}\Big|_{-\sqrt{y+5}}^{\sqrt{y+5}} dy$$

$$= 2\int_0^4 (5+y)\sqrt{y+5}\, dy - \frac{1}{3}\int_0^4 \left(\sqrt{y+5}\right)^3 - \left(-\sqrt{y+5}\right)^3 dy$$

$$V = 2\int_0^4 (y+5)\sqrt{y+5}\,dy - \frac{2}{3}\int_0^4 \left(\sqrt{y+5}\right)^3 dy$$

$$= \int_0^4 2(y+5)^{3/2} - \frac{2}{3}(y+5)^{3/2}\,dy = \frac{4}{3}\int_0^4 (y+5)^{3/2}\,dy$$

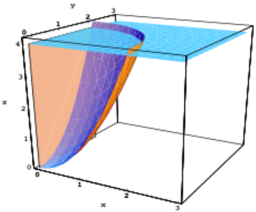
$$= \frac{4}{3}\left(\frac{(y+5)^{5/2}}{5/2}\right)_0^4 = \frac{8}{15}\left(\sqrt{9^5} - \sqrt{5^5}\right) = \frac{8}{15}\left(9^2\sqrt{9} - 5^2\sqrt{5}\right)$$

$$= \frac{8}{15}\left(3.9^2 - 25\sqrt{5}\right) = \frac{648}{5} - \frac{40}{3}\sqrt{5}$$

Exemplo

Calcule $\iiint_E x \, dx \, dy \, dz$ *com* E *limitado por* $z = x^2 + y^2$ *e* $z = 2$ *no primeiro octante.*

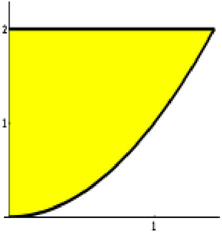
Note que o sólido E é limitado pelo paraboloide $z = x^2 + y^2$ (somente a parte do primeiro octante) e superiormente pelo plano $z = 2$.



Então,

$$E = \left\{(x,y,z);(x,z) \in D,\ 0 \leq y \leq \sqrt{z-x^2}\right\}$$

com D a projeção de E no plano xz. Para encontra D, tomamos $y = 0 \Rightarrow z = x^2$.



Assim, $0 \leq z \leq 2$ e $0 \leq x \leq \sqrt{z}$,

$$E = \left\{(x, y, z);\, 0 \leq z \leq 2,\ 0 \leq x \leq \sqrt{z}, 0 \leq y \leq \sqrt{z - x^2}\right\}$$

e

$$\iiint_E x\, dx\, dy\, dz = \int_0^2 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} x\, dy\, dx\, dz$$

$$\int_0^2 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} x\, dy\, dx\, dz = \int_0^2 \int_0^{\sqrt{z}} \left(\int_0^{\sqrt{z-x^2}} x\, dy \right) dx\, dz$$

$$= \int_0^2 \int_0^{\sqrt{z}} x \Big(y \Big)_0^{\sqrt{z-x^2}} dx\, dz = \int_0^2 \int_0^{\sqrt{z}} x\sqrt{z-x^2}\, dx\, dz$$

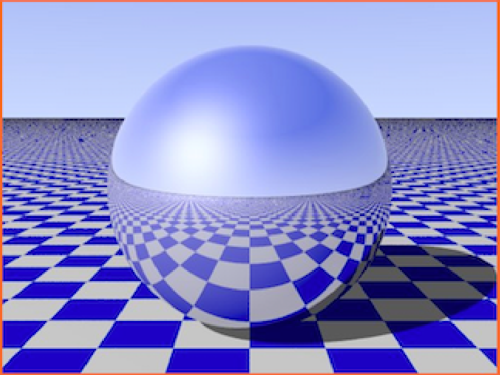
Troca de variável: $u = z - x^2,\ du = -2x\, dx$

$$\int_0^2 \int_0^{\sqrt{z}} x\sqrt{z-x^2}\,dx\,dz = -\frac{1}{2}\int_0^2 \int_0^{\sqrt{z}} u^{1/2}\,du\,dz$$

$$= -\frac{1}{2}\int_0^2 \left(\frac{u^{3/2}}{3/2}\right)_0^{\sqrt{z}} dz = -\frac{1}{2}\left(\frac{2}{3}\right)\int_0^2 \left[(z-x^2)^{3/2}\right]_0^{\sqrt{z}} dz$$

$$= \frac{1}{3}\int_0^2 z^{3/2}\,dz = \frac{1}{3}\left(\frac{z^{5/2}}{5/2}\right)_0^2 = \frac{2}{15}\sqrt{2^5}$$

$$= \frac{2}{15}\left(2^2\sqrt{2}\right) = \frac{8}{15}\sqrt{2}$$



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