Q9 steps

Ankur Garg

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We know Y is a guassian:

$$Y \sim N(X\beta, \Sigma)$$

We can compute the mean and variance of the residual:

$$\begin{aligned}
&E = (I - X(X'WX)^{-1}X'W)XB = XB - XB = 0 \\
&V = Q\Sigma Q^T \\
&R \sim M(0, Q\Sigma Q^T)
\end{aligned}$$

We can simplify $Q\Sigma Q = Q\Sigma$ (what I was missing in my explanation live) using the the fact $\Sigma = W^{-1}$:

$$\begin{aligned}
Q\Sigma Q^T &= (I - X(X'WX)^{-1}X'W)\Sigma Q^T \\
&= (\Sigma - X(X'WX)^{-1}X')Q^T \\
&= (\Sigma - X(X'WX)^{-1}X')(I - X(X'WX)^{-1}X'W)^T \\
&= \Sigma - \Sigma(X(X'WX)^{-1}X'W)^T - X(X'WX)^{-1}X' + X(X'WX)^{-1}X')(X(X'WX)^{-1}X'W)^T \\
&= \Sigma - X(X'WX)^{-1}X' \\
&= Q\Sigma
\end{aligned}$$

This is mostly just expanding and canceling the linear algebra. There is probably a simpler proof of this possible, and the unsimplified form is a valid solution

$$\begin{aligned}
E[R'\Delta R] &= tr(\Delta Q\Sigma Q^T) = tr(\Delta Q\Sigma) \\
Var[R'\Delta R] &= 2tr((\Delta Q\Sigma Q^T)^2) = 2tr((\Delta Q\Sigma)^2)
\end{aligned}$$

This last step follows from simply plugging into the trace properties in the hint(see solution)