Tutorial 6

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Group 3

Let $n \geq 0$ be an integer. Let G be the graph defined as follows: $V(G)$ is the set of all binary strings of length n having at most one block of 1s, and two vertices are adjacent if and only if they differ in exactly one position. Determine $|V(G)|$ and $|E(G)|$.

We can define the set of of $V(G)$ using a regular expression: $R = 0^*11^*0^* \smile 0^*$
R is unambiguous by definition since we only have strings that are all 0, all 1, or one block of 1s.

Then we can rewrite R as a generating series,

$$\begin{aligned}
\implies \frac{x}{(1-x)^3} + \frac{1}{1-x} &\overset{\text{NBT}}{\underset{\text{GS}}{=}} x \sum_{n \geq 0} \binom{n+3-1}{3-1} x^n + \sum_{n \geq 0} x^n \\
&= \sum_{n \geq 0} \left(\binom{n+2}{2} + 1 \right) x^{n+1} \\
&= \sum_{n \geq 1} \left(\binom{n+1}{2} + 1 \right) x^n
\end{aligned}$$

Extracting the coefficient $[x^n]$ of the generating series, we will obtain $|V(G)|$ given some $n \geq 1$ since we know $n = 0$ results in 0 edges and 0 vertices, so

$$|V(G)| = [x^n] \sum_{n \geq 1} \left(\binom{n+1}{2} + 1 \right) x^n = \binom{n+1}{2} + 1$$

Then to obtain $|E(G)|$ we begin by finding the number of vertices for each possible $deg(v)$:
- For $deg(v) = n$, there is 1 vertex;
  - where every digit is 0.
- For $deg(v) = 2$, there are 3 vertices;
  - one where every digit is 1, and two where there is only one 1 on the far left/right position.
- For $deg(v) = 3$, there are $3(n-2)$ vertices;
  - $n-2$ each for when there is more than one 1 in the left, right and center positions.

Note: this method is valid only for $n \geq 2$, so $|V(G)|$ when $n = 0$ and $n = 1$ will be listed separately.

Thus for $deg(v) = 4$, we remove the amount of vertices in $deg(v) = i$, $i \in \{n, 2, 3\}$, from the total amount of vertices,

$$\implies \binom{n+1}{2} + 1 - (1) - (3) - (3n-6) = \binom{n+1}{2} - 3n + 3 = \binom{n+1}{2} - 3(n-1)$$

Now we can use the handshaking theorem to obtain that total edges $= \frac{1}{2} \sum_{v \in V(G)} deg(v)$. Substituting in our known degree amounts, we have

$$\begin{aligned}
\frac{1}{2}\sum_{v\in V(G)} deg(v) &= \frac{(n)(1)+(2)(3)+(3)(3n-6)+(4)\left(\binom{n+1}{2}-3(n-1)\right)}{2} \\
&= \frac{n+6+9n-18+4\binom{n+1}{2}-12n+12}{2} \\
&= 2\binom{n+1}{2}-n = 2\cdot\frac{(n+1)!}{2!(n-1)!}-n \\
&= \frac{(n+1)!}{(n-1)!}-n \\
&= (n+1)n-n \\
&= n^2 = |E(G)|
\end{aligned}$$

$\therefore$ For a graph G and some integer $n \geq 2$, we have that under the required circumstances,

$$\begin{aligned}
|V(G)| &= \binom{n+1}{2}+1 \\
|E(G)| &= n^2
\end{aligned}$$

and for $n=0$ and $n=1$, we have that $|V(G)|=0$, $|E(G)|=0$ and $|V(G)|=2$, $|E(G)|=1$ respectively.