Sea Ice Dynamics

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1 Introduction

Sea ice dynamics represent a complex system, with floe motion attributed to both underlying ocean flow fields and mutual interactions. These dynamics have significant implications for polar and global climates. Chen et al. (2021) introduced a Lagrangian data assimilation framework based on a discrete element model to describe floe dynamics, considering the impact of contact force and ocean flow field on sea ice floe trajectories. In this report, further adaptations will be explored, such as conducting data assimilation while excluding contact forces, adding dimensionality to the system, adding quadratic drag to the model, and implementing an Expectation-Maximization (EM) parameter estimation for floe thickness.

2 Ocean Field - Shallow Water Equation

One important aspect that governs the motion of ice floes is the underlying ocean flow. This is a variable that cannot be directly observed from satellite data, and thus need to be recovered through data assimilation. In our experiment, the reference ocean flow field is followed by the shallow water equation represented by a set of hyperbolic partial differential equations.

$$\frac{\partial \mathbf{u}}{\partial t} + \epsilon^{-1}\mathbf{u}^{\perp} = -\epsilon^{-1}\nabla\eta \tag{1}$$

$$\frac{\partial \eta}{\partial t} + \epsilon^{-1}\delta\nabla\cdot\mathbf{u} = 0 \tag{2}$$

In this setup, $\mathbf{u}$ embeds a two-dimensional velocity field and η is the height function. The two non-dimensional numbers are $\epsilon = R_o$, Rossby number, and $\delta = \frac{R_o^2}{Fr^2}$, where Fr is the Froude number. Higher Rossby number typically means more dominant inertial forces compared to Coriolis force induced by earth rotation. The solution is typically given as the following expression. B is geostrophically balanced mode and $\pm$ is gravity modes.

$$\begin{pmatrix}\mathbf{u}_o(\mathbf{x},t)\\ \eta(\mathbf{x},t)\end{pmatrix} = \sum_{\mathbf{k}\in\mathbb{Z}^2,\zeta\in(B,\pm)} \hat{\mathbf{u}}_{\mathbf{k},\zeta}(t)\cdot e^{i\mathbf{x}\cdot\mathbf{k}}\mathbf{p}_{\mathbf{k},\zeta} \tag{3}$$

The Fourier coefficients then updates following a linear stochastic model. The $\phi_{\mathbf{k},\zeta}$ is the phase speed of different modes correspondingly.

$$d\hat{u}_{\mathbf{k},\zeta} = ((-d_{\mathbf{k},\zeta} + i\phi_{\mathbf{k},\zeta})\hat{u}_{\mathbf{k},\zeta} + f_{\mathbf{k},\zeta})dt + \sigma_{\mathbf{k},\zeta}dW_{\mathbf{k},\zeta} \tag{4}$$

$$\phi_{\mathbf{k},B} = 0 \tag{5}$$

$$\phi_{\mathbf{k},\pm} = \pm\epsilon^{-1}\sqrt{|\mathbf{k}|^2+1} \tag{6}$$

For the convenience of determining the matrix expression, the ocean velocity evolution equations for all three modes will be explicitly listed below. In our case, the $\mathbf{k}^2$ takes value of $k = -2, -1, 0, 1, 2$ in each dimensions.

$$d\hat{u}_{\mathbf{k},B} = ((-d_{\mathbf{k},B} + i\phi_{\mathbf{k},B})\hat{u}_{\mathbf{k},B} + f_{\mathbf{k},B})dt + \sigma_{\mathbf{k},B}dW_{\mathbf{k},B} \tag{7}$$

$$d\hat{u}_{\mathbf{k},+} = ((-d_{\mathbf{k},+} + i\phi_{\mathbf{k},+})\hat{u}_{\mathbf{k},+} + f_{\mathbf{k},+})dt + \sigma_{\mathbf{k},+}dW_{\mathbf{k},+} \tag{8}$$

$$d\hat{u}_{\mathbf{k},-} = ((-d_{\mathbf{k},-} + i\phi_{\mathbf{k},-})\hat{u}_{\mathbf{k},-} + f_{\mathbf{k},-})dt + \sigma_{\mathbf{k},-}dW_{\mathbf{k},-} \tag{9}$$

This system can be combined as vector representation. And after the Fourier coefficient is reconstructed through iterations, the physical velocity field can be obtained through the inverse Fourier transform.

$$d\hat{\mathbf{u}}_o = (\mathbf{L_u}\hat{\mathbf{u}}_o + \mathbf{F_u})dt + \mathbf{N_u}d\mathbf{W_u} \tag{10}$$

As the model structure has been explicitly written out, we have the exact expression for the $\mathbf{L_u}$ matrix, forcing vector $\mathbf{F}_u$ and the noise matrix $\mathbf{N_u}$. There are no parts need to be changed for the shallow water equation model code if quadratic drag force is being implemented instead of linear counterpart since the ocean flow field does not depend on that factor. The GB has 24 modes, with 2 more damping terms, and gravity with 25 modes in each, making the dimension of 76 in total.

$$\mathbf{L_u} = \begin{bmatrix} l_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & l_2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & l_3 & \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & l_4 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \ddots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & l_5 & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & l_6 & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \ddots \end{bmatrix} \in \mathbf{C}^{76\times 76} \tag{11}$$

$$l_1 = -d_{\mathbf{k}_1,B} + i\phi_{\mathbf{k}_1,B} \tag{12}$$
$$l_2 = -d_{\mathbf{k}_2,B} + i\phi_{\mathbf{k}_2,B} \tag{13}$$
$$l_3 = -d_{\mathbf{k}_1,+} + i\phi_{\mathbf{k}_1,+} \tag{14}$$
$$l_4 = -d_{\mathbf{k}_2,+} + i\phi_{\mathbf{k}_2,+} \tag{15}$$
$$l_5 = -d_{\mathbf{k}_1,-} + i\phi_{\mathbf{k}_1,-} \tag{16}$$
$$l_6 = -d_{\mathbf{k}_2,-} + i\phi_{\mathbf{k}_2,-} \tag{17}$$

$$\mathbf{F}_u = \begin{bmatrix} f_{\mathbf{k}_1,B} \\ \vdots \\ f_{\mathbf{k}_1,+} \\ \vdots \\ f_{\mathbf{k}_1,-} \\ \vdots \end{bmatrix} \in \mathbf{C}^{76\times 1} \tag{18}$$

3 Physical System

In the realm of sea ice dynamics, both observable and unobservable variables exist, with the latter remaining inaccessible through available data sources. The goal of data assimilation is to derive these unobserved variables utilizing existing datasets. When it comes to sea ice dynamics, satellites can capture state variables like translational and angular displacements. However, recovering the driving velocities behind these state changes relies on data assimilation performed over the floe model, which outlines their mechanics. The floe dynamics model primarily involves contact forces, quadratic ocean drag forces, and random noises, culminating in the following comprehensive system.

$$d\mathbf{x}^l = \mathbf{v}^l \cdot dt + \sigma^l_{\mathbf{x}} \cdot d\mathbf{W}^l_{\mathbf{x}} \tag{19}$$
$$d\mathbf{\Omega}^l = \boldsymbol{\omega}^l \cdot dt + \sigma^l_{\mathbf{\Omega}} \cdot d\mathbf{W}^l_{\mathbf{\Omega}} \tag{20}$$
$$d\mathbf{v}^l = \frac{1}{m^l}(\sum_j (f^{lj}_n + f^{lj}_t) + \tilde{\alpha}^l(\mathbf{G}(\mathbf{x}^l)\hat{\mathbf{u}}_0 - \mathbf{v}^l)|\mathbf{G}(\mathbf{x}^l)\hat{\mathbf{u}}_0 - \mathbf{v}^l|) \cdot dt + \sigma^l_{\mathbf{v}} \cdot d\mathbf{W}^l_{\mathbf{v}} \tag{21}$$
$$d\boldsymbol{\omega}^l = \frac{1}{I^l}(\sum_j (r^l \cdot \mathbf{n}^{lj} \times \mathbf{f}^{lj}_{\mathbf{t}}) \cdot \hat{\mathbf{z}} + \tilde{\beta}^l(\nabla \times \hat{\mathbf{u}}_0/2 - \boldsymbol{\omega}^l \cdot \hat{\mathbf{z}})|\nabla \times \hat{\mathbf{u}}_0/2 - \boldsymbol{\omega}^l \cdot \hat{\mathbf{z}}|) \cdot dt + \sigma^l_{\boldsymbol{\omega}} \cdot d\mathbf{W}^l_{\boldsymbol{\omega}} \tag{22}$$

Though a bit complicated at first inspection, the main variables involved are observed sequences $\mathbf{x}^l$ and $\mathbf{\Omega}^l$, namely, translational and angular displacements, along with their unobserved velocity counterpart $\mathbf{v}^l$ and $\boldsymbol{\omega}^l$, and ocean flow field $\hat{\mathbf{u}}_0$. Henceforth, $\mathbf{X}$ will denote the two observed variables, and $\mathbf{Y}$ will include the three unobserved velocity components for all floes. However, it is convenient to perform a dimensional analysis here to settle the physical representation of α^l. Since the quadratic term takes the dimension of v^2, the coefficient $\frac{\alpha^l}{m^l}$ is then required to take the dimension of $\frac{1}{L}$. Since the mass m^l takes the form of $m^l = \rho\pi r_l^2 h^l$, α^l must have the following form, where d_o is the non-dimensional ocean drag coefficient, an inherent property of the media.

$$\alpha^l = d_o \rho_{ocn} \pi r_l^2 \tag{23}$$

Covington et al. (2022) have the similar model in their work and thus will be served as a reference (p.15).

$$\mathbf{F}_{ocn} = \rho_{ocn} C_{ocn} |\mathbf{V}_{ocn} - \mathbf{V}_{ice}|(\mathbf{V}_{ocn} - \mathbf{V}_{ice}) \tag{24}$$
$$\alpha^l = \iint_A \rho_{ocn} C_{ocn}\, dA = d_o \rho_{ocn} \pi r_l^2 \tag{25}$$

Here the force is respect to a point, and it needs to be integrated with respect to area A. It is then obvious that d_o is equivalent to C_{ocn} in Covington's work.

4 Combined System

In this section, I will combine the shallow water equation along with the floe model together for the convenience of data assimilation.

$$d\mathbf{x}^l = \mathbf{v}^l \cdot dt + \sigma^l_{\mathbf{x}} \cdot d\mathbf{W}^l_{\mathbf{x}} \tag{26}$$
$$d\mathbf{\Omega}^l = \boldsymbol{\omega}^l \cdot dt + \sigma^l_{\mathbf{\Omega}} \cdot d\mathbf{W}^l_{\mathbf{\Omega}} \tag{27}$$
$$d\mathbf{v}^l = \frac{1}{m^l}(\sum_j (f^{lj}_n + f^{lj}_t) + \tilde{\alpha}^l(\mathbf{G}(\mathbf{x}^l)\hat{\mathbf{u}}_0 - \mathbf{v}^l)|\mathbf{G}(\mathbf{x}^l)\hat{\mathbf{u}}_0 - \mathbf{v}^l|) \cdot dt + \sigma^l_{\mathbf{v}} \cdot d\mathbf{W}^l_{\mathbf{v}} \tag{28}$$
$$d\boldsymbol{\omega}^l = \frac{1}{I^l}(\sum_j (r^l \cdot \mathbf{n}^{lj} \times \mathbf{f}^{lj}_{\mathbf{t}}) \cdot \hat{\mathbf{z}} + \tilde{\beta}^l(\nabla \times \hat{\mathbf{u}}_0/2 - \boldsymbol{\omega}^l \cdot \hat{\mathbf{z}})|\nabla \times \hat{\mathbf{u}}_0/2 - \boldsymbol{\omega}^l \cdot \hat{\mathbf{z}}|) \cdot dt + \sigma^l_{\boldsymbol{\omega}} \cdot d\mathbf{W}^l_{\boldsymbol{\omega}} \tag{29}$$
$$d\hat{\mathbf{u}}_o = (\mathbf{F}_{\mathbf{u}} + \mathbf{L}_{\mathbf{u}}\hat{\mathbf{u}}_o)dt + \mathbf{N}_{\mathbf{u}} d\mathbf{W}_{\mathbf{u}} \tag{30}$$

The previous part presents the comprehensive physical system of the floe model. In order to implement it, a discretized version is necessary, and this will be explicitly elaborated in this early section as a reference for modifications in later experiments. By adopting the $\mathbf{X}$ and $\mathbf{Y}$ notation, the system of equations can be reduced to this general formulation.

$$d\mathbf{X}(t) = \mathbf{A}\mathbf{Y}(t)dt + \mathbf{B}d\mathbf{W}_{\mathbf{x}}(t) \tag{31}$$
$$d\mathbf{Y}(t) = [\mathbf{a}(\mathbf{X}(t)) + \mathbf{F}(\mathbf{X}(t), \mathbf{Y}(t))]dt + \mathbf{b}d\mathbf{W}_{\mathbf{Y}}(t) \tag{32}$$

Most of the vectors and matrices in this expression have simple structures, with the exception of $\mathbf{a}(\mathbf{X}(t))$ and $\mathbf{F}(\mathbf{X}(t), \mathbf{Y}(t))$, which include contact forces, quadratic drag forces, and torque imposed on the floes respectively. With this compact expression in hand, the discretization can be carried out effectively using Euler-Maruyama Scheme.

$$\mathbf{X}^{j+1} = \mathbf{X}^j + \mathbf{A}^j\mathbf{Y}^j\Delta t + \mathbf{B}\sqrt{\Delta t}\epsilon^j_{\mathbf{X}} \tag{33}$$
$$\mathbf{Y}^{j+1} = \mathbf{Y}^j + [\mathbf{a}(\mathbf{X}^j) + \mathbf{F}(\mathbf{X}^j, \mathbf{Y}^j)]\Delta t + \mathbf{b}\sqrt{\Delta t}\epsilon^j_{\mathbf{Y}} \tag{34}$$

Due to the nonlinear nature of $\mathbf{F}(\mathbf{X}^j, \mathbf{Y}^j)$, a conditional linearization approximation analogous to Taylor expansion is carried out to simplify the term.

$$\mathbf{F}(\mathbf{X}^j, \mathbf{Y}^j) = \mathbf{F}(\mathbf{X}^j, \boldsymbol{\mu}^j_f) + \mathbf{J}_{\mathbf{Y}}(\mathbf{X}^j, \boldsymbol{\mu}^j_f)(\mathbf{Y}^j - \boldsymbol{\mu}^j_f) + h.o.t \tag{35}$$

Then the $\mathbf{Y}$ term can be rewritten as the following.

$$\mathbf{Y}^{j+1} = \mathbf{Y}^j + [\mathbf{a}(\mathbf{X}^j) + \mathbf{F}(\mathbf{X}^j, \boldsymbol{\mu}^j_f) + \mathbf{J}_{\mathbf{Y}}(\mathbf{X}^j, \boldsymbol{\mu}^j_f)(\mathbf{Y}^j - \boldsymbol{\mu}^j_f)]\Delta t + \mathbf{b}\sqrt{\Delta t}\epsilon^j_{\mathbf{Y}} \tag{36}$$
$$= \mathbf{Y}^j + [\mathbf{a}(\mathbf{X}^j) + \mathbf{F}(\mathbf{X}^j, \boldsymbol{\mu}^j_f) - \mathbf{J}_{\mathbf{Y}}(\mathbf{X}^j, \boldsymbol{\mu}^j_f)\boldsymbol{\mu}^j_f + \mathbf{J}_{\mathbf{Y}}(\mathbf{X}^j, \boldsymbol{\mu}^j_f)\mathbf{Y}^j]\Delta t + \mathbf{b}\sqrt{\Delta t}\epsilon^j_{\mathbf{Y}} \tag{37}$$

Notice that the system dealt here after the linearization approximation matches exactly the conditional Gaussian model (Chen, 2020).

$$\mathbf{X}^{j+1} = \mathbf{X}^j + [\mathbf{A}^j_0 + \mathbf{A}^j_1\mathbf{Y}^j]\Delta t + \mathbf{B}\sqrt{\Delta t}\epsilon^j_{\mathbf{X}} \tag{38}$$
$$\mathbf{Y}^{j+1} = \mathbf{Y}^j + [\mathbf{a}^j_0 + \mathbf{a}^j_1\mathbf{Y}^j]\Delta t + \mathbf{b}\sqrt{\Delta t}\epsilon^j_{\mathbf{Y}} \tag{39}$$

Where the coefficients can be compared and the following expressions are derived.

$$\mathbf{A}_0 = 0 \tag{40}$$
$$\mathbf{A}_1 = \mathbf{A} \tag{41}$$
$$\mathbf{a}^j_0 = \mathbf{a}^j + \mathbf{F}(\mathbf{X}^j, \boldsymbol{\mu}^j_{\mathbf{f}}) - \mathbf{J}_{\mathbf{Y}}(\mathbf{X}^j, \boldsymbol{\mu}^j_{\mathbf{f}})\boldsymbol{\mu}^j_f \tag{42}$$
$$\mathbf{a}^j_1 = \mathbf{J}_{\mathbf{Y}}(\mathbf{X}^j, \boldsymbol{\mu}^j_{\mathbf{f}}) \tag{43}$$

In the preliminary code, the general nonlinear optimal filter has the following form.

$$\boldsymbol{\mu}^{j+1}_f = \boldsymbol{\mu}^j_f + (\mathbf{a}_0 + \mathbf{a}_1\boldsymbol{\mu}^j_f)\cdot\Delta t + (\mathbf{R}^j_f\mathbf{A}^\dagger)(\mathbf{B}\mathbf{B}^\dagger)^{-1}(\mathbf{X}^{j+1} - \mathbf{X}^j - \mathbf{A}\boldsymbol{\mu}^j_f\cdot\Delta t) \tag{44}$$
$$\mathbf{R}^{j+1}_f = \mathbf{R}^j_f + (\mathbf{a}_1\mathbf{R}^j_f + \mathbf{R}^j_f\mathbf{a}^\dagger_1 + \mathbf{b}\mathbf{b}^\dagger - (\mathbf{R}^j_f\mathbf{A}^\dagger)(\mathbf{B}\mathbf{B}^\dagger)^{-1}(\mathbf{A}\mathbf{R}^j_f))\cdot\Delta t \tag{45}$$

In the following sections, multiple experiments will be carried out. Due to large reconstruction error that might be caused by the short period contact force in the data assimilation, the contact force will be removed in the data assimilation part. Furthermore, the experiment will be dimensionalized to real setup. Finally, an Expectation-Maximization parameter estimation scheme is required to estimate the parameters.

5 Elimination of Contact Forces

The collision between ice floes introduces significant errors in the model due to its short observation time. To address this issue, contact force is eliminated from the code. In this section, we discuss the removal of contact forces in both the ice floe model and the associated data assimilation step. For the model, normal and tangential forces are eliminated, as well as the v_{cx} and v_{cy} components. As a result, the ice floes eventually overlap with each other. In terms of data assimilation, we only need to focus on eliminating the contact force part in unobserved variables since all observed variables are contact-free.

While performing data assimilation, contact force only takes place in the unobserved variables' equations. After the removal of contact force, $\mathbf{a}_0$ is vastly simplified since the $\mathbf{a}$ is totally eliminated. One point that requires extra caution is that the μ^j_f should be selected for the corresponding part of $\mathbf{F}$. For example, for the $\mathbf{F}$ part that is associated with the x-velocity of the floes, the columns of Jacobian and rows of μ^j_f should also be partially selected such that they match with the physical quantity of interest.

$$\mathbf{A}_0 = [0] \tag{46}$$
$$\mathbf{A}_1 = \mathbf{A} \tag{47}$$
$$\mathbf{a}_0^j = \mathbf{a}^j + \mathbf{F}(\mathbf{X}^j, \boldsymbol{\mu}_\mathbf{f}^j) - \mathbf{J}_\mathbf{Y}(\mathbf{X}^j, \boldsymbol{\mu}_\mathbf{f}^j)\boldsymbol{\mu}_f^j \tag{48}$$
$$= \mathbf{F}(\mathbf{X}^j, \boldsymbol{\mu}_\mathbf{f}^j) - \mathbf{J}_\mathbf{Y}(\mathbf{X}^j, \boldsymbol{\mu}_\mathbf{f}^j)\boldsymbol{\mu}_f^j \tag{49}$$
$$\mathbf{a}_1^j = \mathbf{J}_\mathbf{Y}(\mathbf{X}^j, \boldsymbol{\mu}_\mathbf{f}^j) \tag{50}$$

Now the system structure is clearly presented, the explicit expression of $\mathbf{A}, \mathbf{a}_0, \mathbf{a}_1$ will be written out. The reduced physical system without contact force is as following.

$$d\mathbf{v_x} = \frac{\boldsymbol{\alpha}}{\mathbf{m}}(e^{i\mathbf{x}\cdot\mathbf{k}}\hat{\mathbf{u}}_{ox} - \mathbf{v_x})|e^{i\mathbf{x}\cdot\mathbf{k}}\hat{\mathbf{u}}_{ox} - \mathbf{v_x}|dt + \boldsymbol{\sigma}_{vx}\cdot d\mathbf{W}_{vx} \tag{51}$$
$$= \mathbf{F}(\mathbf{X}, \mathbf{Y})_2 dt + \boldsymbol{\sigma}_{vx}\cdot d\mathbf{W}_{vx} \tag{52}$$
$$= (\mathbf{F}(\mathbf{X}, \boldsymbol{\mu}_f^j)_2 - [J_\mathbf{Y}(\mathbf{X}, \boldsymbol{\mu}_f^j)\boldsymbol{\mu}_f^j]_2 + [J_\mathbf{Y}(\mathbf{X}, \boldsymbol{\mu}_f^j)\mathbf{Y}]_2)dt + \boldsymbol{\sigma}_{vx}\cdot d\mathbf{W}_{vx} \tag{53}$$
$$d\mathbf{v_y} = \frac{\boldsymbol{\alpha}}{\mathbf{m}}(e^{i\mathbf{x}\cdot\mathbf{k}}\hat{\mathbf{u}}_{oy} - \mathbf{v_y})|e^{i\mathbf{x}\cdot\mathbf{k}}\hat{\mathbf{u}}_{oy} - \mathbf{v}_\mathbf{y}^l|dt + \boldsymbol{\sigma}_{vy}\cdot d\mathbf{W}_{vy} \tag{54}$$
$$= \mathbf{F}(\mathbf{X}, \mathbf{Y})_3 dt + \boldsymbol{\sigma}_{vy}\cdot d\mathbf{W}_{vy} \tag{55}$$
$$= (\mathbf{F}(\mathbf{X}, \boldsymbol{\mu}_f^j)_3 - [J_\mathbf{Y}(\mathbf{X}, \boldsymbol{\mu}_f^j)\boldsymbol{\mu}_f^j]_3 + [J_\mathbf{Y}(\mathbf{X}, \boldsymbol{\mu}_f^j)\mathbf{Y}]_3)dt + \boldsymbol{\sigma}_{vy}\cdot d\mathbf{W}_{vy} \tag{56}$$
$$d\boldsymbol{\omega} = \frac{\boldsymbol{\beta}}{\mathbf{I}}(\nabla\times(e^{i\mathbf{x}\cdot\mathbf{k}}\cdot\hat{\mathbf{u}}_0)/2 - \boldsymbol{\omega})|\nabla\times(e^{i\mathbf{x}\cdot\mathbf{k}}\cdot\hat{\mathbf{u}}_0)/2 - \boldsymbol{\omega}^l|dt + \boldsymbol{\sigma}_{\boldsymbol{\omega}}\cdot d\mathbf{W}_{\boldsymbol{\omega}} \tag{57}$$
$$= \mathbf{F}(\mathbf{X}, \mathbf{Y})_1 dt + \boldsymbol{\sigma}_{\boldsymbol{\omega}}^l\cdot d\mathbf{W}_{\boldsymbol{\omega}} \tag{58}$$
$$= (\mathbf{F}(\mathbf{X}, \boldsymbol{\mu}_f^j)_1 - [J_\mathbf{Y}(\mathbf{X}, \boldsymbol{\mu}_f^j)\boldsymbol{\mu}_f^j]_1 + [J_\mathbf{Y}(\mathbf{X}, \boldsymbol{\mu}_f^j)\mathbf{Y}]_1)dt + \boldsymbol{\sigma}_{\boldsymbol{\omega}}\cdot d\mathbf{W}_{\boldsymbol{\omega}} \tag{59}$$
$$d\hat{\mathbf{u}}_o = \mathbf{L_u}\hat{\mathbf{u}}_o dt + \mathbf{N_u}d\mathbf{W_u} \tag{60}$$
$$= \mathbf{F}(\mathbf{X}, \mathbf{Y})_4 dt + \mathbf{N_u}d\mathbf{W_u} \tag{61}$$

$$dx^l = v_x^l\cdot dt + \sigma_x^l\cdot dW_x^l \tag{62}$$
$$dy^l = v_y^l\cdot dt + \sigma_y^l\cdot dW_y^l \tag{63}$$
$$d\Omega^l = \omega^l\cdot dt + \sigma_\Omega^l\cdot dW_\Omega^l \tag{64}$$

Comparing this system to the conditional Gaussian model gives the following parameters. L will be the number of floes and M the number of modes for flow field, and n being the number of observations.

$$\mathbf{A} = \mathbf{I}^{nL\times(nL+M)} \tag{65}$$
$$= \mathbf{I}^{108\times 184} \tag{66}$$

The most important part will be the expressions for $\mathbf{F}$ given $\boldsymbol{\mu}_f^j$ and and the Jacobian matrix. Notice that the $\nabla\times$ curl for 2-D vector in the rotation part will result in a scalar value.

$$\mathbf{F}(\mathbf{X}, \boldsymbol{\mu}_f^j)_2 = \frac{\alpha^l}{m^l}(e^{i\mathbf{x}\cdot\mathbf{k}}\hat{\mathbf{u}}_{oxf}^j - v_{xf}^{lj})|e^{i\mathbf{x}\cdot\mathbf{k}}\hat{\mathbf{u}}_{oxf}^j - v_{xf}^{lj}| \tag{67}$$
$$\mathbf{F}(\mathbf{X}, \boldsymbol{\mu}_f^j)_3 = \frac{\alpha^l}{m^l}(e^{i\mathbf{x}\cdot\mathbf{k}}\hat{\mathbf{u}}_{oyf}^j - v_{yf}^{lj})|e^{i\mathbf{x}\cdot\mathbf{k}}\hat{\mathbf{u}}_{oyf}^j - v_{yf}^{lj}| \tag{68}$$
$$\mathbf{F}(\mathbf{X}, \boldsymbol{\mu}_f^j)_1 = \frac{\beta^l}{I^l}(\nabla\times(e^{i\mathbf{x}\cdot\mathbf{k}}\cdot\hat{\mathbf{u}}_{of}^j)/2 - \omega_f^{lj})|\nabla\times(e^{i\mathbf{x}\cdot\mathbf{k}}\cdot\hat{\mathbf{u}}_{of}^j)/2 - \omega_f^{lj}| \tag{69}$$
$$e^{i\mathbf{x}\cdot\mathbf{k}} \in \mathbb{R}^{36\times 76}, \hat{\mathbf{u}}_{of}^j \in \mathbb{C}^{76\times 2}, \hat{\mathbf{u}}_{oxf}^j \in \mathbb{C}^{76\times 1}, \hat{\mathbf{u}}_{oyf}^j \in \mathbb{C}^{76\times 1} \tag{70}$$

The derivation of the Jacobian matrix will be a little bit more involving. The dimensions illustrated above for $\mathbf{F}$ are all for 1 floes only. However, in the derivation of the Jacobian matrix, add the number of floes to the dimension will be a more explicit approach. An expected Jacobian matrix with respect to unobserved variables will have the following shape $\mathbf{J_Y} \in \mathbb{C}^{108\times184}$, where the 76 modes in the row is ignored since no need for calculating Jacobian for the shallow water equation due to its well-established linear structure. The first row of the Jacobian will serve as an illustrative purpose. Notice that all the $\mathbf{JR}_{1j}$ concatenates horizontally.

$$\mathbf{JR}_{11} = \begin{bmatrix} \frac{\partial F_2(X^{1j}, \boldsymbol{\mu}_f^{1j})}{\partial v_{x1}} & 0 & \cdots & 0 \end{bmatrix} \tag{71}$$

$$\mathbf{JR}_{12} = \begin{bmatrix} \frac{\partial F_2(X^{1j}, \boldsymbol{\mu}_f^{1j})}{\partial v_{y1}} & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \tag{72}$$

$$\mathbf{JR}_{13} = \begin{bmatrix} \frac{\partial F_2(X^{1j}, \boldsymbol{\mu}_f^{1j})}{\partial \omega_1} & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \tag{73}$$

$$\mathbf{JR}_{14} = \begin{bmatrix} \frac{\partial F_2(X^{1j}, \boldsymbol{\mu}_f^{1j})}{\partial u_1} & \frac{\partial F_2(X^{1j}, \boldsymbol{\mu}_f^{1j})}{\partial u_2} & \cdots & \frac{\partial F_2(X^{1j}, \boldsymbol{\mu}_f^{1j})}{\partial u_{76}} \end{bmatrix} \tag{74}$$

$$\mathbf{J_{Y1}} = [\mathbf{JR}_{11}, \mathbf{JR}_{12}, \mathbf{JR}_{13}, \mathbf{JR}_{14}] \tag{75}$$

The $\mathbf{JR}_{14}$ part needs special treatment since it is the derivative with respect to the 76 fourier modes of the shallow water equation. However, the expression $e^{i\mathbf{x}\cdot\mathbf{k}} \cdot \hat{\mathbf{u}}^j_{oxf}$ is a linear combination of fourier modes with corresponding fourier coefficients. Thus the partial derivative of the summation will only spit out a single coefficient at a time since all the modes are mutually independent. The Jacobian section for the y-direction velocity has separate fourier basis. Another point worth noticing is that the derivatives discussed above involves absolute value function, therefore the calculation should be treated carefully. Here a general form of this derivative will be illustrated.

$$\begin{aligned}
\frac{\partial (a-b)|a-b|}{\partial a} &= |a-b| + (a-b)\frac{\partial |a-b|}{\partial a} && (76)\\
&= |a-b| + (a-b)\frac{\partial |a-b|}{\partial (a-b)} \cdot \frac{\partial (a-b)}{\partial a} && (77)\\
&= |a-b| + \frac{(a-b)^2}{|a-b|} && (78)\\
&= 2|a-b| && (79)\\
\frac{\partial (a-b)|a-b|}{\partial b} &= -|a-b| + (a-b)\frac{\partial |a-b|}{\partial b} && (80)\\
&= -|a-b| + (a-b)\frac{\partial |a-b|}{\partial (a-b)} \cdot \frac{\partial (a-b)}{\partial b} && (81)\\
&= -|a-b| - \frac{(a-b)^2}{|a-b|} && (82)\\
&= -2|a-b| && (83)
\end{aligned}$$

5.1 Result Demonstration

The data assimilation technique has been elaborated above. Further experiments are carried out with full observation, partial observation for all floes, and reduced floe numbers. The goal is to examine the significance of angular position observation and number of floes being considered in the observation as less data is input is beneficial. The default number of floes is $n = 36$, and the full observations involve x-position, y-position, and angular position relative to the initial values.

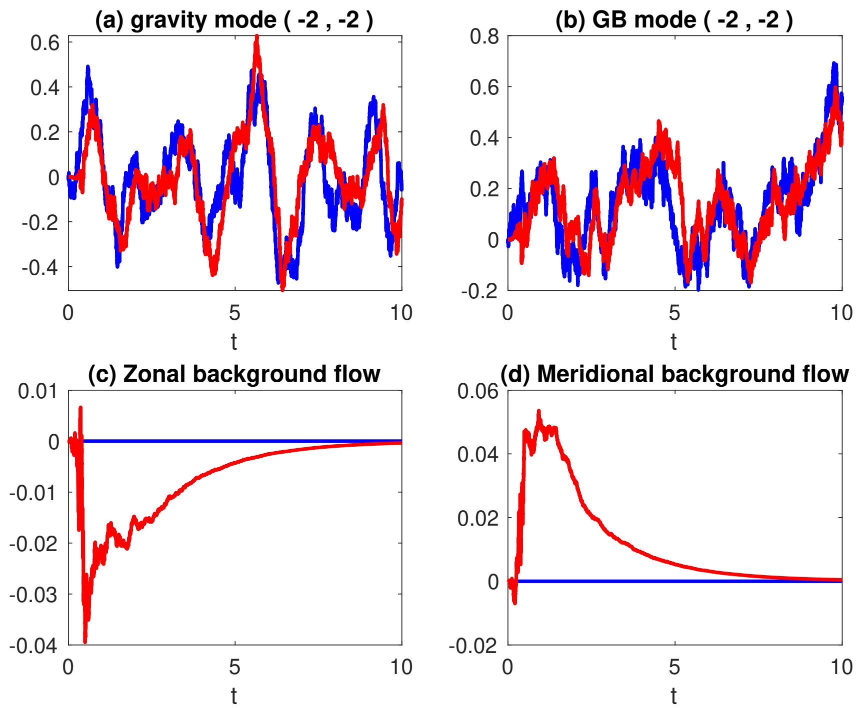

Figure 1: Ocean flow field recovery time series with full observation for $n = 36$ floes

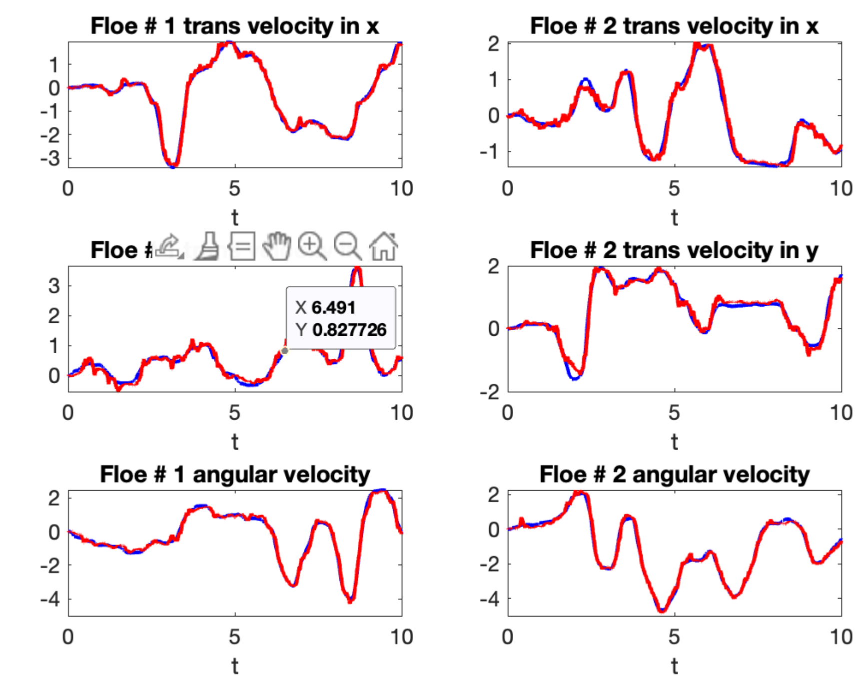

Figure 2: Velocity recovery time series with full observation for $n = 36$ floes

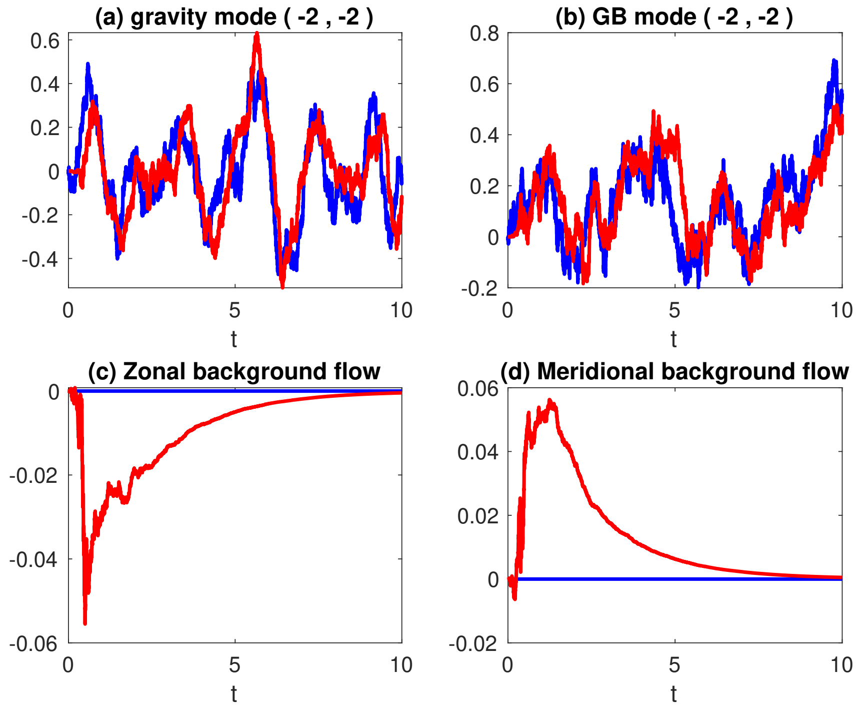

Figure 3: Ocean flow field recovery time series with full observation for $n = 36$ floes

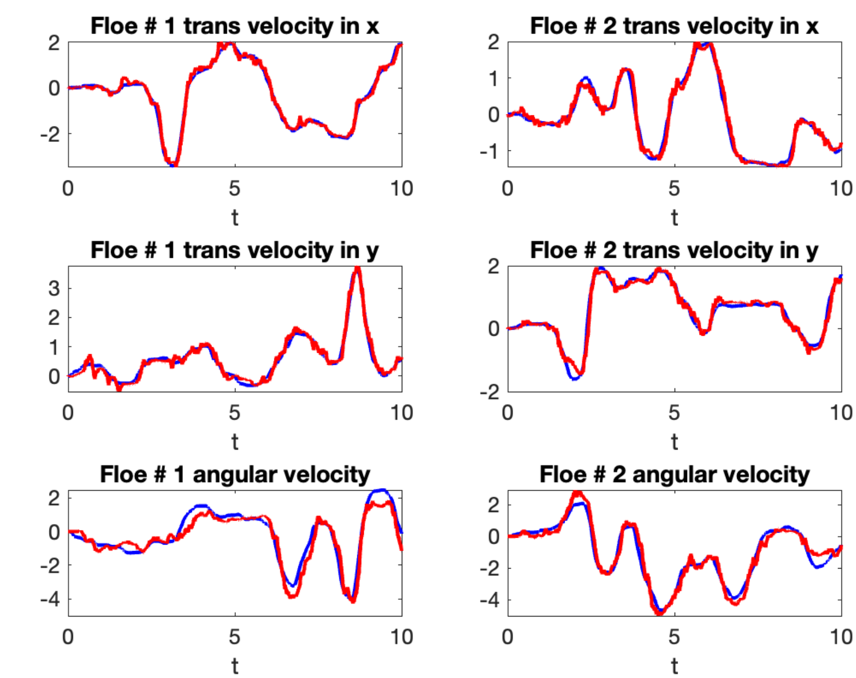

Figure 4: Velocity recovery time series with partial observation for $n = 36$ floes

6 Dimensionalization

While the currently presented experiments have been conducted in a non-dimensional context, it is essential to translate these findings into real-world scenarios by dimensionalizing the setup. This can be achieved by introducing real-world parameters such as length, mass, and time scales, as well as the actual values of the dimensionless numbers. By implementing these parameters, the model becomes more representative of the actual behavior of sea ice floes and their corresponding dynamics. Generally, there are two ways of dimensionalizing the system, one being dimensionalizing at the end of conversion, and the other dimensionalizing each variable during the process. However, here a mixed methodology is being applied here, where only the final ocean velocity field is dimensionalized since this will be a fixed quantity. For the floe model, we will have certain parameters to be also dimensionalized.

6.1 Generating Initial Floes

While generating the initial floes for the process, the radius, position, and thickness of the floes need to be dimensionalized. For the lateral length scale, all the parameters should be stretched by a factor of $\frac{50}{2\pi}$, and that includes the domain and the floe radius. For the domain, the 2π by 2π region is expanded to $50km$ by $50km$ domain. The radii for 36 floes are generated following a power law distribution $p(r) = \frac{ak^a}{r^{a+1}}$ with $k = 1.5, a = 1$ ranging from $\frac{2\pi}{25}$ to $\frac{2\pi}{10}$. The non-dimensional vector of radii generated is scaled up by $\frac{50}{2\pi}$.

$$radius_D = radius_{ND} \cdot \frac{50}{2\pi} \tag{84}$$

Empirically, the thickness of the floes should be converted to km from m and no scaling is needed.

$$thickness_D = thickness_{ND} \cdot \frac{1}{1000} \tag{85}$$

6.2 Shallow Water Equation

Besides the dimensional variables inherited from the initial floe generation, no further modification is needed for generating the underlying ocean flow field other than changing the period of the fourier basis from 2π to $50km$ and scale the final velocity field in the physical space by $\frac{50}{2\pi}$.

6.3 Floe Model

Floe model is the relatively complicated one to dimensionalize compared to the previous two. First of all, the ocean and ice density are all assumed to be $10^{12} kg/km^3$, where the non-dimensional value is 1. Then the random noise in the tracer equation, namely, position update equation should be scaled up.

$$\sigma_x^D = \frac{50}{2\pi}\sigma_x^{ND} \tag{86}$$

While standard dimensionalization of floe velocity and fourier basis function need to be scaled up by a factor of $\frac{50}{2\pi}$, the important thing is to balance the two non-dimensional coefficients $\frac{\alpha^l}{m}$ and $\frac{\beta^l}{I}$. Below a comparison for the non-dimensional coefficients and the dimensionalized ones is given in terms of their final expression, where i subscript stands for non-dimensional initial system, and f stands for the final system with dimensionalized variables. Since the value of ocean and ice density are taken to be the same in this case, they will all be denoted by the same ρ_i and ρ_f.

$$d_o = 1, \rho_i = 1, h_i = 1, \rho_f = 10^{12}, h_f = \frac{h_i}{10^3} \tag{87}$$

$$\frac{\alpha_i^l}{m_i} = \frac{d_o \rho_i \pi r_i^2}{\rho_i \pi r_i^2 h_i} = \frac{d_o}{h_i} = 1 \tag{88}$$

$$\frac{\alpha_f^l}{m_f} = \frac{d_o \rho_f \pi r_f^2}{\rho_f \pi r_f^2 h_f} = \frac{d_o}{h_f} = 10^3 \tag{89}$$

$$\tag{90}$$

Since the ratio is not dimensionless but carries $\frac{1}{L}$, it needs to be dimensionalized properly with a factor of $\frac{2\pi}{50}$ to compensate the height on the denominator. One thing worth noticing is that thickness was initially in meters and it has to be converted to kilometers with an additional factor of 10^{-3}, making it $\alpha_f^l = d_o\rho_f\pi r_f^2 \cdot 10^{-3}$. Similar analysis is applied to the angular velocity case.

$$\frac{\beta_i^l}{m_i} = \frac{d_o\rho_f\pi r_f^2 r_f^2}{\rho_i\pi r_i^2 h_i r_i^2} = \frac{d_o}{h_i} = 1 \tag{91}$$

$$\frac{\beta_i^l}{m_i} = \frac{d_o\rho_f\pi r_f^2 r_f^2}{\rho_f\pi r_f^2 h_f r_f^2} = \frac{d_o}{h_f} = 10^3 \tag{92}$$

$$\tag{93}$$

Everything left in the floe model and data assimilation part is simply changing every 2π concerned with the region size to $50km$ and a purely scaled-up version of the dynamics is achieved.

6.4 Result

After the dimensionalization process discussed above, the resulting system is the exact same amplification as the original non-dimensional system. The most important thing to check is that the first and final positions of the floes are the same.

1. discuss the difference between Fm and the upper expression, there is a compensation of Lu times ocean flow field 2. why is the parameters θ in the EM step $\frac{1}{h^l}$ instead of h^l

7 EM Parameter Estimation

Within a dynamical system, there are always important parameters that powers the underlying dynamics running besides observed and unobserved variables. Sea ice thickness is such an essential quantity for sea ice features and behaviors. On the other hand, the sea ice thickness is usually hard to observe through direct satellite images. Therefore, a method is developed to obtain the estimation of the parameter through observed and recovered physical quantities. The process involves rewriting the physical system into another form that treats all other variables as prefactors of the parameters. Therefore, a review of the initial system is essential here.

$$\mathbf{x}^{j+1} = \mathbf{x}^j + 0\cdot\Delta t + \mathbf{v}_{\mathbf{x}}^j \cdot \Delta t + \mathbf{B}_{\mathbf{x}}^j\sqrt{\Delta t}\boldsymbol{\epsilon}_{\mathbf{x}}^j \tag{94}$$

$$\mathbf{y}^{j+1} = \mathbf{y}^j + 0\cdot\Delta t + \mathbf{v}_{\mathbf{y}}^j \cdot \Delta t + \mathbf{B}_{\mathbf{y}}^j\sqrt{\Delta t}\boldsymbol{\epsilon}_{\mathbf{y}}^j \tag{95}$$

$$\boldsymbol{\Omega}^{j+1} = \boldsymbol{\Omega}^j + 0\cdot\Delta t + \boldsymbol{\omega}^j \cdot \Delta t + \mathbf{B}_{\boldsymbol{\omega}}^j\sqrt{\Delta t}\epsilon_{\omega}^j \tag{96}$$

$$\mathbf{u}^{j+1} = \mathbf{u}^j + 0\cdot\Delta t + \frac{\boldsymbol{\alpha}}{\mathbf{m}}(e^{i\mathbf{x}\cdot\mathbf{k}}\cdot\mathbf{u}_{of}^j - \mathbf{u}_f^j)|e^{i\mathbf{x}\cdot\mathbf{k}}\cdot\mathbf{u}_{of}^j - \mathbf{u}_f^j|\cdot\Delta t + \mathbf{b}_{\mathbf{u}}^j\cdot\sqrt{\Delta t}\boldsymbol{\epsilon}_u^j \tag{97}$$

$$\mathbf{v}^{j+1} = \mathbf{v}^j + 0\cdot\Delta t + \frac{\boldsymbol{\alpha}}{\mathbf{m}}(e^{i\mathbf{x}\cdot\mathbf{k}}\cdot\mathbf{v}_{of}^j - \mathbf{v}_f^j)|e^{i\mathbf{x}\cdot\mathbf{k}}\cdot\mathbf{v}_{of}^j - \mathbf{v}_f^j|\cdot\Delta t + \mathbf{b}_{\mathbf{v}}^j\cdot\sqrt{\Delta t}\boldsymbol{\epsilon}_v^j \tag{98}$$

$$\boldsymbol{\omega}^{j+1} = \boldsymbol{\omega}^j + 0\cdot\Delta t + \frac{\boldsymbol{\beta}}{\boldsymbol{I}}(\nabla\times(e^{i\mathbf{x}\cdot\mathbf{k}}\cdot\mathbf{uv}^j)/2 - \boldsymbol{\omega}^j)|\nabla\times(e^{i\mathbf{x}\cdot\mathbf{k}}\cdot\mathbf{uv}^j)/2 - \boldsymbol{\omega}^j|\cdot\Delta t + \mathbf{b}_{\boldsymbol{\omega}}^j\cdot\sqrt{\Delta t}\boldsymbol{\epsilon}_{\omega}^j \tag{99}$$

$$\mathbf{uv}^{j+1} = \mathbf{uv}^j + \mathbf{F}_{\mathbf{u}}\cdot\Delta t + \mathbf{L}_{\mathbf{u}}\mathbf{uv}^j\cdot\Delta t + \boldsymbol{\Sigma}_{\mathbf{u}}\cdot\sqrt{\Delta t}\boldsymbol{\epsilon}_{uv}^j \tag{100}$$

This system can be further simplified by using the following notation.

$$\mathbf{Z}^{j+1} - \mathbf{Z}^j - \mathbf{C}^j\Delta t = \mathbf{M}^j\boldsymbol{\theta}\Delta t + \sqrt{\mathbf{R}\Delta t}\boldsymbol{\epsilon} \tag{101}$$

Each element in the expression can be expanded.

$$\mathbf{M}^j = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ F_{21}^j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & F_{2e}^j \\ F_{31}^j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & F_{3e}^j \\ F_{11}^j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & F_{1e}^j \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \mathbf{Z}^{j+1} = \begin{bmatrix} \mathbf{x}^{j+1} \\ \mathbf{y}^{j+1} \\ \boldsymbol{\Omega}^{j+1} \\ \mathbf{u}^{j+1} \\ \mathbf{v}^{j+1} \\ \boldsymbol{\omega}^{j+1} \\ \mathbf{uv}^{j+1} \end{bmatrix}, \mathbf{Z}^j = \begin{bmatrix} \mathbf{x}^{j} \\ \mathbf{y}^{j} \\ \boldsymbol{\Omega}^{j} \\ \mathbf{u}^{j} \\ \mathbf{v}^{j} \\ \boldsymbol{\omega}^{j} \\ \mathbf{uv}^{j} \end{bmatrix} \tag{102}$$

$$\mathbf{C}^j = \begin{bmatrix} \mathbf{u}^j \\ \mathbf{v}_{\mathbf{y}}^j \\ \boldsymbol{\omega}^j \\ 0 \\ \vdots \\ 0 \\ \mathbf{F_u} + \mathbf{L_u}\mathbf{uv}^j \end{bmatrix}, \boldsymbol{\theta}^j = \begin{bmatrix} d_o/h^1 \\ d_o/h^2 \\ \vdots \\ d_o/h^L \end{bmatrix} \tag{103}$$

$$F_{21} = (e^{i\mathbf{x}_{1,:}\mathbf{k}_{:,1}}\mathbf{u}_{of}^j - u_{1f}^j)|e^{i\mathbf{x}_{1,:}\mathbf{k}_{:,1}}\mathbf{u}_{of}^j - u_{1f}^j| \tag{104}$$

$$F_{2e} = (e^{i\mathbf{x}_{end,:}\mathbf{k}_{:,end}}\mathbf{u}_{of}^j - u_{Lf}^j)|e^{i\mathbf{x}_{end,:}\mathbf{k}_{:,end}}\mathbf{u}_{of}^j - u_{Lf}^j| \tag{105}$$

$$F_{31} = (e^{i\mathbf{x}_{1,:}\mathbf{k}_{:,1}}\mathbf{v}_{of}^j - v_{1f}^j)|e^{i\mathbf{x}_{1,:}\mathbf{k}_{:,1}}\mathbf{v}_{of}^j - v_{1f}^j| \tag{106}$$

$$F_{3e} = (e^{i\mathbf{x}_{end,:}\mathbf{k}_{:,end}}\mathbf{v}_{of}^j - v_{Lf}^j)|e^{i\mathbf{x}_{end,:}\mathbf{k}_{:,end}}\mathbf{v}_{of}^j - v_{Lf}^j| \tag{107}$$

$$F_{11} = (\nabla \times (e^{i\mathbf{x}_{1,:}\mathbf{k}_{:,1}}\mathbf{uv}_{of}^j) - \omega_{1f}^j)|\nabla \times e^{i\mathbf{x}_{1,:}\mathbf{k}_{:,1}}\mathbf{uv}_{of}^j - \omega_{1f}^j| \tag{108}$$

$$F_{1e} = (\nabla \times (e^{i\mathbf{x}_{end,:}\mathbf{k}_{:,end}}\mathbf{uv}_{of}^j)/2 - \omega_{Lf}^j)|\nabla \times (e^{i\mathbf{x}_{end,:}\mathbf{k}_{:,end}}\mathbf{uv}_{of}^j)/2 - \omega_{Lf}^j| \tag{109}$$

$$\mathbf{R} = \begin{bmatrix} (\mathbf{B}_x)^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{B}_y)^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{B}_\omega)^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{b_u})^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{b_v})^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{b}_{\boldsymbol{\omega}})^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\boldsymbol{\Sigma}_\mathbf{u})^2 \end{bmatrix} \tag{110}$$

In the above expression, the bold $\mathbf{0}$ are block zero matrices that matches the dimension correspondingly to the noise coefficients.

$$\mathbf{B}_x^j = \begin{bmatrix} B_{1x}^j & 0 & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & B_{Lx}^j \end{bmatrix} \tag{111}$$

$$\mathbf{B}_y^j = \begin{bmatrix} B_{1y}^j & 0 & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & B_{Ly}^j \end{bmatrix} \tag{112}$$

$$\mathbf{B}_\omega^j = \begin{bmatrix} B_{1\omega}^j & 0 & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & B_{L\omega}^j \end{bmatrix} \tag{113}$$

$$\mathbf{b_u} = \mathbf{\Sigma_u} \tag{114}$$

Now the parameter vector $\boldsymbol{\theta}$ can be calculated using the following analytical formula.

$$\boldsymbol{\theta} = (\sum_i < (\mathbf{M}^i)^* \mathbf{R}^{-1} \mathbf{M}^i > \Delta t)^{-1} (\sum_i < (\mathbf{M}^i)^* \mathbf{R}^{-1} (\mathbf{Z}^{i+1} - \mathbf{Z}^i - \mathbf{C}^i \Delta t) >) \tag{115}$$

Rewriting this into a more convenient form for implementation, we have the following $\mathbf{A}x = \mathbf{b}$ system.

$$(\sum_i < (\mathbf{M}^i)^* \mathbf{R}^{-1} \mathbf{M}^i > \Delta t) \cdot \boldsymbol{\theta} = \sum_i < (\mathbf{M}^i)^* \mathbf{R}^{-1} (\mathbf{Z}^{i+1} - \mathbf{Z}^i - \mathbf{C}^i \Delta t) > \tag{116}$$