How do they synchronize? Interpretable feature learning for coupled oscillators

Agam Goyal, Bella Wu, Binhao Chen, Bryan Xu, Hanbaek Lyu

University of Wisconsin - Madison

Background

Suppose we are given a system of coupled oscillators on an arbitrary graph along with the trajectory of the system for some time period $0 \le t \le T$. Can we predict whether the system will eventually synchronize? This is an important but analytically intractable question especially when the structure of the underlying graph is highly varied. Prior work [1] has shown that it is possible to use machine learning methods to predict coupled oscillator synchronization with surprisingly high accuracy, far exceeding the prediction accuracy we can get by classical oscillator theory literature.

What enables these models to do so? Can we use our knowledge of coupled oscillators in tandem with the features learned by the machine learning models to create an interpretable framework that makes it easy to understand this phenomenon of synchronization of coupled oscillators? This will enable us to learn what machine learning algorithms learned from a massive amount of data and use it to advance our theoretical understanding of coupled oscillators.

Models of Coupled Oscillators

We consider three well-studied models of coupled oscillators for our work, namely the **Kuramoto Model (KM)** [2, 3], the **Firefly Cellular Automata (FCA)** [4], and the **Greenberg-Hastings Model (GHM)** [5]

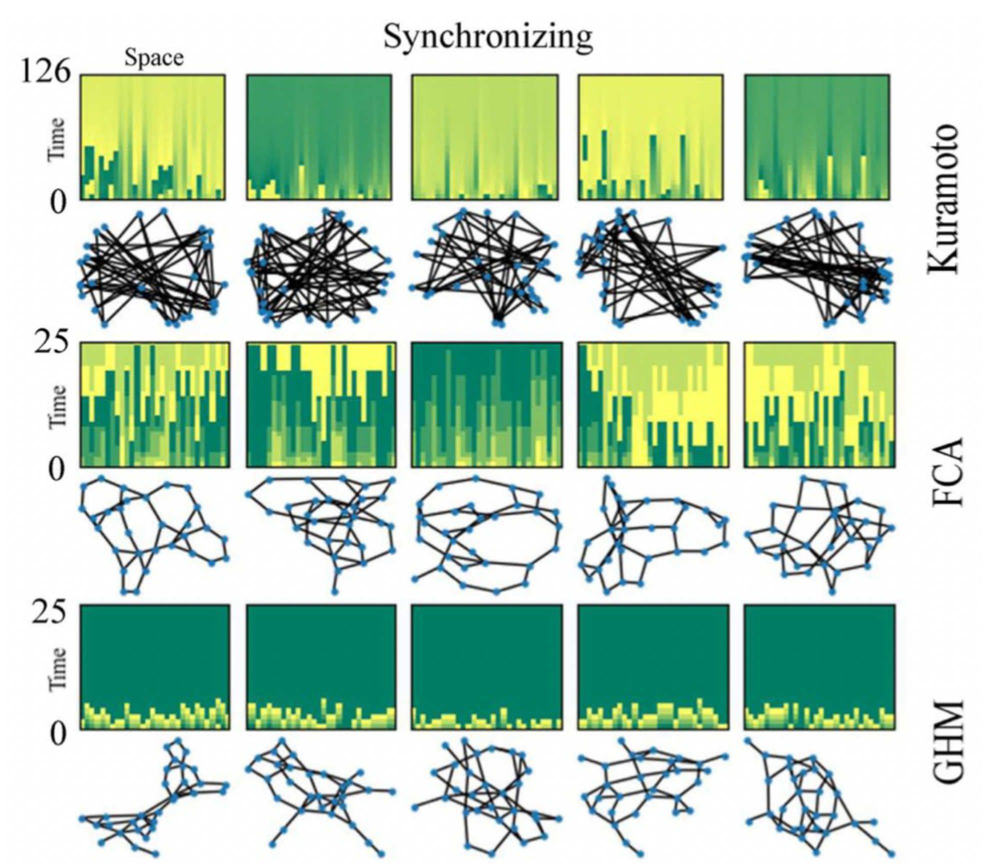

Fig. 1: Synchronizing Examples

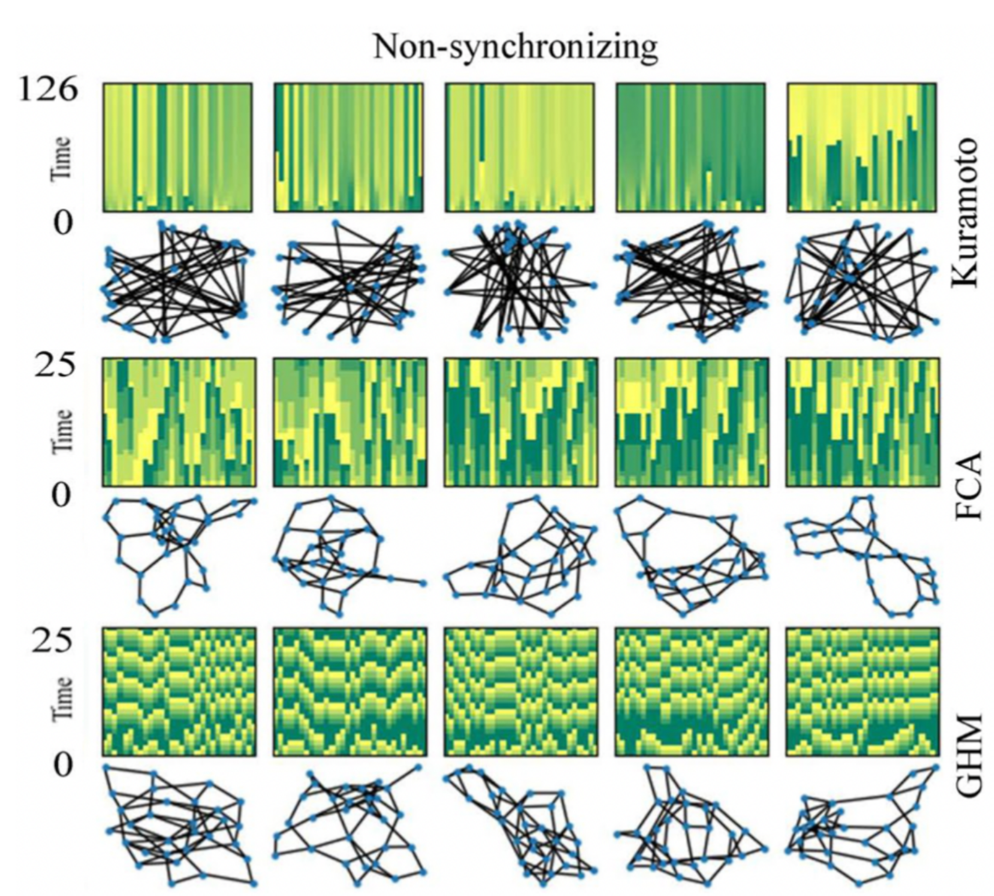

Fig. 2: Non-Synchronizing Examples

Generalized Latent Linear Model (GLLM)

Given a pair of graph and observed phase dynamics $(G, (X_t)_{0\le t<T})$, we can represent this input data by a nonnegative tensor $\mathbf{X}$ of shape $k \times k \times T$, where each slice $\mathbf{X}[:,:,t]$ represents the graph topology G decorated by the phase configuration X_t.

Our GLLM for synchronization prediction assumes that we have a 'dictionary of latent dynamics filters', which is an r−tuple where $r \in \mathbb{Z}^+$ is the 'size' of the dictionary $\mathcal{D}$.

Our modeling assumption is

$$\mathbb{1}(X_t \text{ synchronizes as } t \to \infty \,|\, \mathbf{X})) \approx Y \,|\, \mathbf{X} \sim \text{Bernoulli}(p),$$

where the predictive probability p is given by

$$p = \exp\left(\sum_{i=1}^{r} \beta_i \langle \mathbf{F}_i, \mathbf{X}\rangle\right) \Big/ \left(1 + \exp\left(\sum_{i=1}^{r} \beta_i \langle \mathbf{F}_i, \mathbf{X}\rangle\right)\right).$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r) \in \mathbb{R}^r$ is a vector of regression coefficients.

Using this general activation function, our GLLM can be concisely stated as

$$\mathbb{P}(Y = 1 \,|\, \mathbf{X}) = g(\boldsymbol{\beta}^T \, \mathtt{MAT}(\mathcal{D})^T \, \mathtt{VEC}(\mathbf{X})),$$

where $\mathtt{MAT}(\cdot)$ and $\mathtt{VEC}(\cdot)$ are the matricization and the vectorization operators. Observe that $\mathtt{MAT}(\mathcal{D})^T \, \mathtt{VEC}(X) \in \mathbb{R}^r$, is the r-vector of proximity scores.

The figure below is representative of our Generalized Latent Linear Model in its entirety.

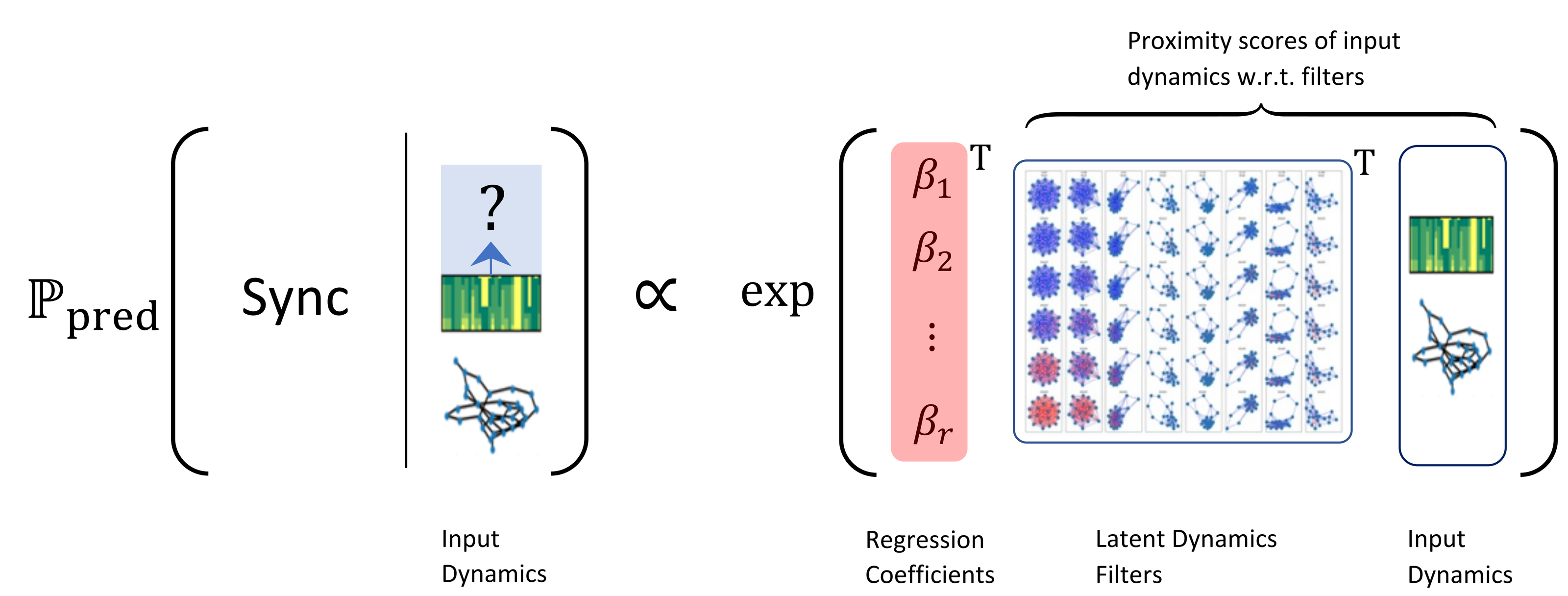

Fig. 3: Working Scheme of the Generalized Latent Linear Model (GLLM)

Model Performance

Our GLLM using both theory- and data-informed pipelines significantly outperforms the baseline predictor, and is close to the performance of the Random Forest (RF) classifier. It does so while also allowing us to interpret how the model is making its prediction. For the Caltech and NWS networks, GLLM performs almost at par with Random Forest, and for GHM it has near-perfect accuracy.

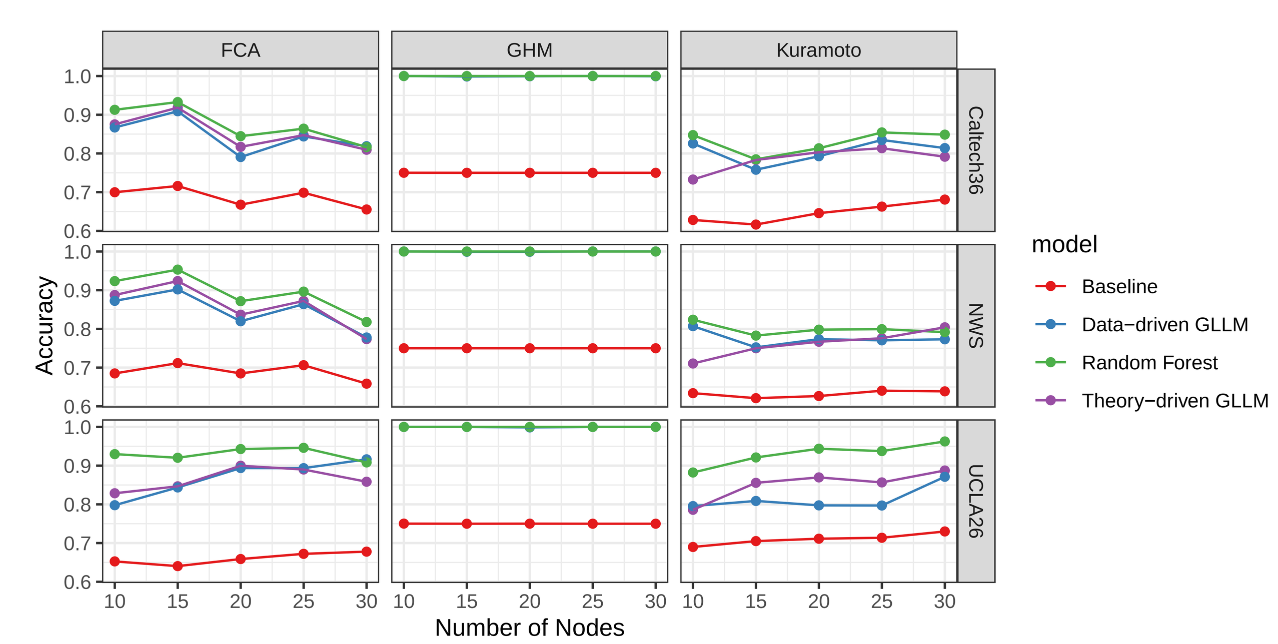

Fig. 4: Accuracy Comparison of Various Models

Dictionary of Latent Dynamics Filters

We employ motif-sampling [6], and follow that up with unsupervised feature extraction using Nonnegative Matrix Factorization (NMF) [7], or Supervised Dictionary Learning (SDL) [8] in order to learn the latent dynamic filters.

Here are the visualizations of the dictionary learned by data-informed GLLM on the dynamics generated by FCA on NWS sub-graphs.

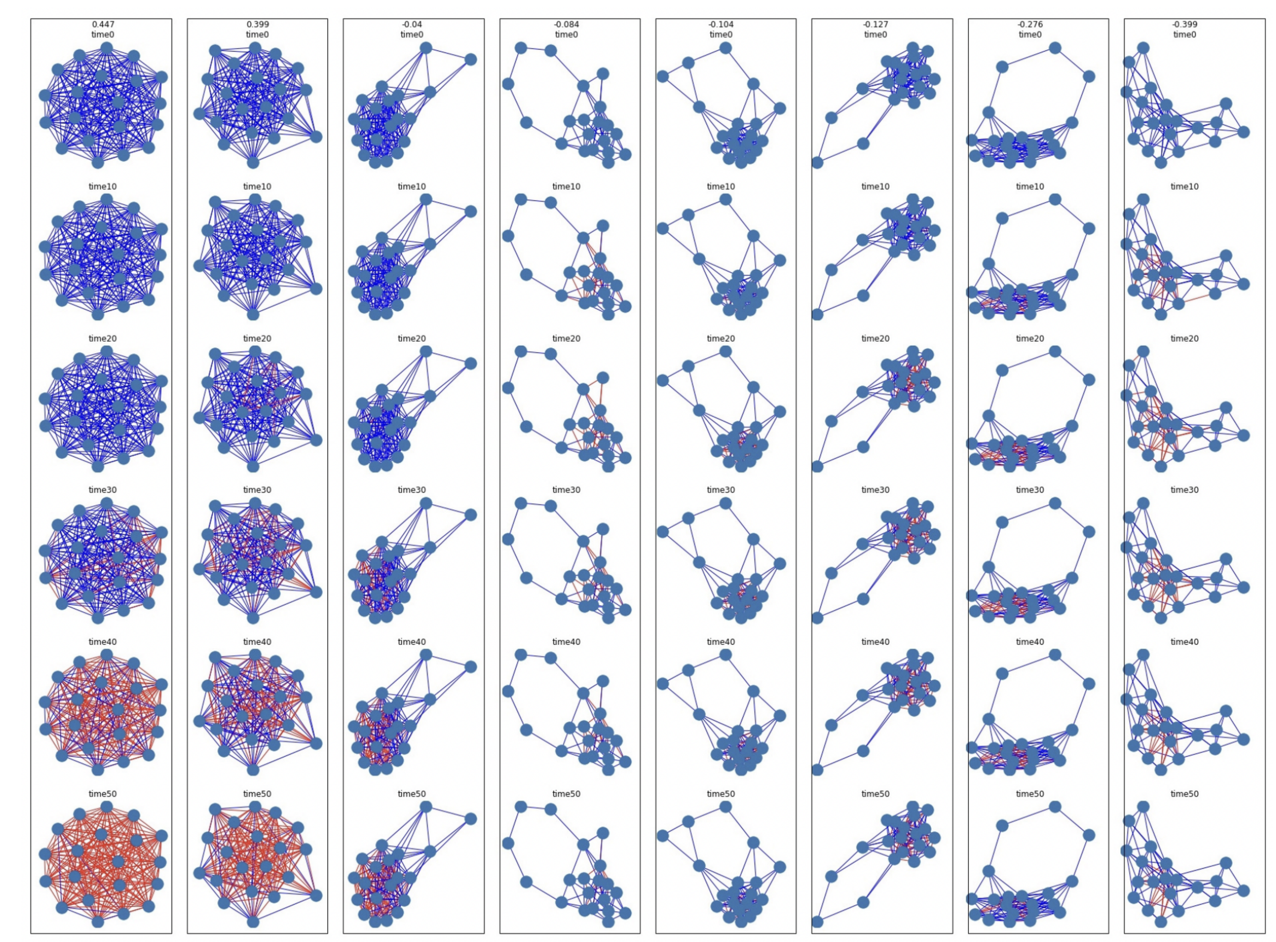

Fig. 5: Example of Dictionary Elements

Each column is a dictionary element and the number on the top of the column is the regression coefficient for that particular dictionary. Each row represents the dictionary element at different times in the phase space. If an edge is colored red, then the two vertices of the corresponding edge are synchronized. If an edge is colored blue, then the two vertices of the edge are not synchronized. The two dictionary elements with highest regression coefficients appear to be of densely connected sub-graphs, with no isolated or weakly connected nodes. In contrast, the dictionary element with the lowest regression coefficient appears to be from a sparse sub-graph.

What's more? For dictionary elements with positive regression coefficients, the networks gradually progress towards synchronization with time, and almost all nodes synchronize at the end of the training iteration.

References

[1] Hardeep Bassi et al. "Learning to predict synchronization of coupled oscillators on randomly generated graphs". In: *Scientific reports* 12.1 (2022), pp. 1–15.

[2] Yoshiki Kuramoto. "International symposium on mathematical problems in theoretical physics". In: *Lecture notes in Physics* 30 (1975), p. 420.

[3] Yoshiki Kuramoto. "Chemical turbulence". In: *Chemical oscillations, waves, and turbulence.* Springer, 1984, pp. 111–140.

[4] Hanbaek Lyu. "Synchronization of finite-state pulse-coupled oscillators". In: *Physica D: Nonlinear Phenomena* 303 (2015), pp. 28–38.

[5] James M Greenberg and Stuart P Hastings. "Spatial patterns for discrete models of diffusion in excitable media". In: *SIAM Journal on Applied Mathematics* 34.3 (1978), pp. 515–523.

[6] Hanbaek Lyu et al. "Learning low-rank latent mesoscale structures in networks". In: *Draft available: https://hanbaeklyudotcom.files.wordpress.com/2020/10/ndl-1.pdf* (2020).

[7] Daniel D Lee and H Sebastian Seung. "Learning the parts of objects by non-negative matrix factorization". In: *Nature* 401.6755 (1999), pp. 788–791.

[8] Julien Mairal et al. "Supervised dictionary learning". In: *Advances in neural information processing systems* 21 (2008).