Small-area estimation using GaussianProcess grouped IRT regression andpost-stratification

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  - … but it can take months to get estimates back!

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  - A grouped hierarchical IRT model
  - A simple GP binomial regression model
- We derive a fast expectation-maximization algorithm based on the a Pólya-Gamma data augmentation strategy
- We implement the proposed approach in an open-source package based on R and C++.

Section 1

A Gaussian Process Primer

A distribution over functions

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$$f(\mathbf{x}) \sim \mathrm{GP}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x}'))$$

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- The choice of covariance (or **kernel**) determines the types of functions that this prior can generate
  - Must be continuous, symmetric, and (preferably in optimization problems) positive definite.

Kernels and the functions they support

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- We will be working with a version of a simple squared-exponential kernel, given by

$$k(\mathbf{x}, \mathbf{x'}) = \exp\left[-\frac{1}{2}\frac{\sum_d (x_d - x'_d)^2}{\rho}\right]$$

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- For positive-definite kernels there is and equivalent representation using (an inner product of) basis functions that map inputs onto higher dimensional spaces.

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- Used to great success in many disciplines!
  - In political science, see e.g. Hainmueller and Hazlett (2014) or Hartman et al.'s KPop

Section 2

GrP: GP (IRT) regression and Post-stratification

The 2-PL (grouped) IRT

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  - One dimensional latent trait θ_i
- Gaussian prior over the vector of item parameters

$$
\boldsymbol{\beta}_j \overset{\text{iid}}{\sim} N_2(\mathbf{0}, \boldsymbol{\Lambda}_{\boldsymbol{\beta}}^{-1})
$$

GP Regression of Ideal Points

- Our main innovation: use a flexible GP regression to model ideal points as function of $\mathbf{Z}_{N \times D}$ matrix of demographic predictive features

$$\theta_i \overset{\text{indep.}}{\sim} N(f_i, 1.0)$$
$$\mathbf{f} \sim \text{GP}(\mathbf{0}, \mathbf{K}_\rho)$$

where

$$\mathbf{K}_{ij} = k(\mathbf{z}_i, \mathbf{z}_j) = \exp\left[-\frac{1}{2}\sum_{d=1}^{D}\frac{(z_{id} - z_{jd})^2}{\rho_d}\right]$$

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  - **Post-stratification**!

Pólya-Gamma Augmentation for Logit IRT

- The joint likelihood of aggregated responses $\mathbf{Y}$ is given by

$$p(\mathbf{Y} \mid \boldsymbol{\theta}, \boldsymbol{\beta}) \propto \prod_{ij} \frac{\exp(\mu_{ij})^{y_{ij}}}{[1 + \exp(\mu_{ij})]^{n_{ij}}}$$
$$= \prod_{ij} \exp \kappa_{ij}\mu_{ij} \; \mathbb{E}_{\omega_{ij}}[\exp(-\omega_{ij}\mu_{ij}^2/2)]$$

where $\kappa_{ij} = y_{ij} - n_{ij}/2$, ω_{ij} is a Pólya-Gamma (PG) random variable distributed $\mathrm{PG}(n_{ij}, 0)$, and the equality obtains from the integral identity derived by Polson & Scott (2013).

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  - Weird marginal likelihood…
  - … incredibly simple conditionals! $\rightsquigarrow$ Great for EM! (See Goplerud 2019)

A fast ECM algorithm for estimation I

- We begin by deriving the conditional expectation of the log joint posterior under the posterior of ω_{ij}:

$$
\begin{aligned}
Q(\mathbf{f}, \boldsymbol{\theta}, \beta, \boldsymbol{\rho}) = \sum_{ij} \kappa_{ij}\mu_{ij} &- \mathbb{E}_{\boldsymbol{\omega}}[\omega_{ij} \mid \boldsymbol{\theta}^{(t-1)}, \boldsymbol{\beta}^{(t-1)}]\mu_{ij}^2/2 \\
&- \frac{1}{2}\Big(\log\left[\det(\mathbf{K}_\rho)\right] + \mathbf{f}^\top \mathbf{K}_\rho^{-1}\mathbf{f})\Big) \\
&- \frac{1}{2}\sum_i(\theta_i^2 - 2\theta_i f_i) \\
&- \frac{1}{2}\sum_j \boldsymbol{\beta}_j^\top \boldsymbol{\Lambda}_\beta \boldsymbol{\beta}_j + \text{const.}
\end{aligned}
$$

A fast ECM algorithm for estimation II

- E-step: Evaluate Q-function, which requires

$$\mathbb{E}_{\boldsymbol{\omega}}[\omega_{ij} \mid \boldsymbol{\theta}^{(t-1)}, \boldsymbol{\beta}^{(t-1)}] = \frac{n_{ij}}{2\mu_{ij}^{(t-1)}} \tanh\left(\mu_{ij}^{(t-1)}/2\right)$$

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  - Extremely easy to compute!
  - Hard to sample from corresponding $\mathrm{PG}(n_{ij}, \mu_{ij})$ (though check out *Ultimate* PGS by Frühwirth-Schnatter et al.!)

A fast ECM algorithm for estimation II

- Conditional M steps optimize Q function w.r.t parameters (and hyper-parameters in Kernel).

where $\boldsymbol{\Omega}_j = \text{diag}\left(\{\omega_{ij}^{(t)}\}_{i=1}^{N}\right)$, matrix $\mathbf{X}$ has rows $\mathbf{x}_i = [\theta_i^{(t-1)}, -1]$, and $\boldsymbol{\kappa}_j = [\kappa_{1j}, \ldots, \kappa_{Nj}]^\top$

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- Conditional M steps optimize Q function w.r.t parameters (and hyper-parameters in Kernel).
- The update for the item parameters $\boldsymbol{\beta}_j$ is given by its conditional posterior mean,

$$\boldsymbol{\beta}_j^{(t)} = \left(\boldsymbol{\Lambda}_\beta + \mathbf{X}^\top \boldsymbol{\Omega}_j \mathbf{X}\right)^{-1} \mathbf{X}^\top \boldsymbol{\kappa}_j \tag{1}$$

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A fast ECM algorithm for estimation III

- The update for ideal point θ_i is again given by a conditional posterior mean...

$$\theta_i^{(t)} = \left(1.0 + \sum_j \left(\beta_{j1}^{(t)}\right)^2\right)^{-1} \left(f_i + \sum_j \beta_{j1}^{(t)}(\kappa_{ij} + \beta_{j2}^{(t)})\right)$$

A fast ECM algorithm for estimation III

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- ... and the update for $\mathbf{f}$ is given by

$$\mathbf{f}^{(t)} = \mathbf{K}_\rho^{(t-1)} \left(\mathbf{K}_\rho^{(t-1)} + \mathbf{I}_N\right)^{-1} \boldsymbol{\theta}$$

A fast ECM algorithm for estimation IV

- Finally, we take empirical Bayes route, and optimize (marginal) Q over hyper-parameters

$$\boldsymbol{\rho}^{(t)} = \arg\max_{\rho} \left[-\frac{1}{2} (\boldsymbol{\theta}^{(t)})^\top \left(\mathbf{K}_\rho + \mathbf{I}_N \right)^{-1} \boldsymbol{\theta}^{(t)} - \frac{1}{2} \log \left| (\mathbf{K}_\rho + \mathbf{I}_N) \right| \right]$$

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  - Use one pass of conjugate gradients to take single climbing step

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  - Most computationally expensive step (requires inversion of dense $\mathbf{K}_{\rho} + \mathbf{I}_N$)
  - Use one pass of conjugate gradients to take single climbing step
  - Define inverse Gamma prior to bound away from zero and away from large values.

Simulation results

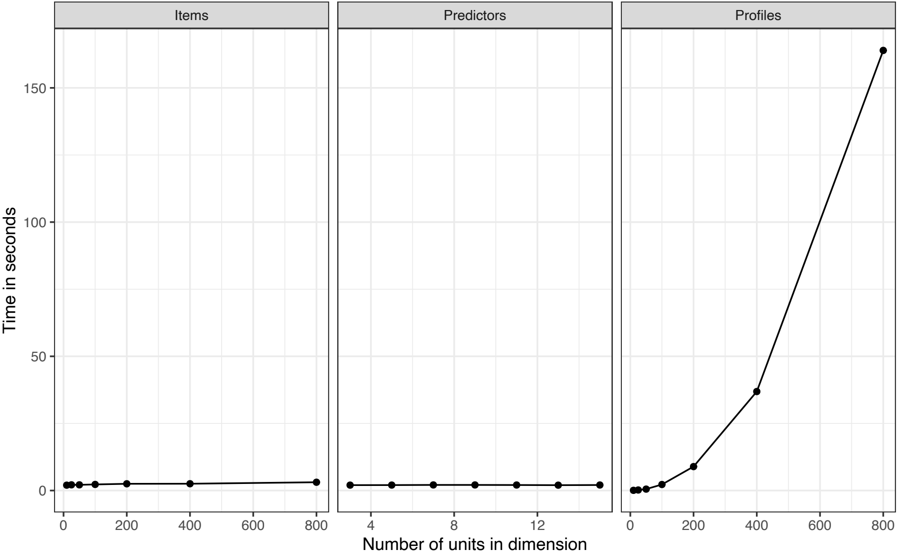

Figure 2: SimResTimes

Binomial GP Regression + Post-stratification

- For single items, we can sidestep measurement model

$$
\begin{aligned}
y_i &\overset{\text{indep.}}{\sim} \text{Binomial}(n_i, \pi_i) \\
\pi_i &= \text{logit}^{-1}(f_i) \\
\mathbf{f} &\sim N(\mathbf{0}, \mathbf{K}_\rho)
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$$

Binomial GP Regression: posterior predictive

- First, we compute the posterior predictive distribution of the latent function $f_\star$ for a hypothetical demographic profile, given by

$$p(f_\star \mid \mathbf{y}, \mathbf{X}, \mathbf{x}_\star) = \int p(f_\star \mid \mathbf{X}, \mathbf{x}_\star, \mathbf{f}) p(\mathbf{f} \mid \mathbf{y}, \mathbf{X}) \, d\mathbf{f}$$

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- Then, conditional on $f_\star$, we obtain a posterior predictive expectation of the probability someone with that profile answers the item in the affirmative, $\hat{\pi}_\star$,

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- With a logistic inverse-link, neither integral has a simple closed-form solution.

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$$p(f_\star \mid \mathbf{y}, \mathbf{X}, \mathbf{x}_\star) = \int p(f_\star \mid \mathbf{X}, \mathbf{x}_\star, \mathbf{f}) p(\mathbf{f} \mid \mathbf{y}, \mathbf{X}) \, d\mathbf{f}$$

- Then, conditional on $f_\star$, we obtain a posterior predictive expectation of the probability someone with that profile answers the item in the affirmative, $\hat{\pi}_\star$,

$$\hat{\pi}_\star = \int \text{logit}^{-1}(f_\star) p(f_\star \mid \mathbf{y}, \mathbf{X}) \, df_\star$$

- With a logistic inverse-link, neither integral has a simple closed-form solution.
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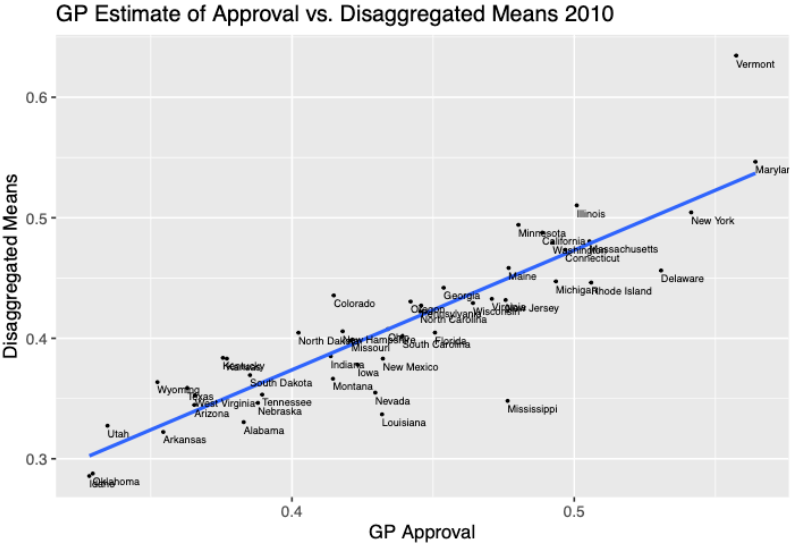
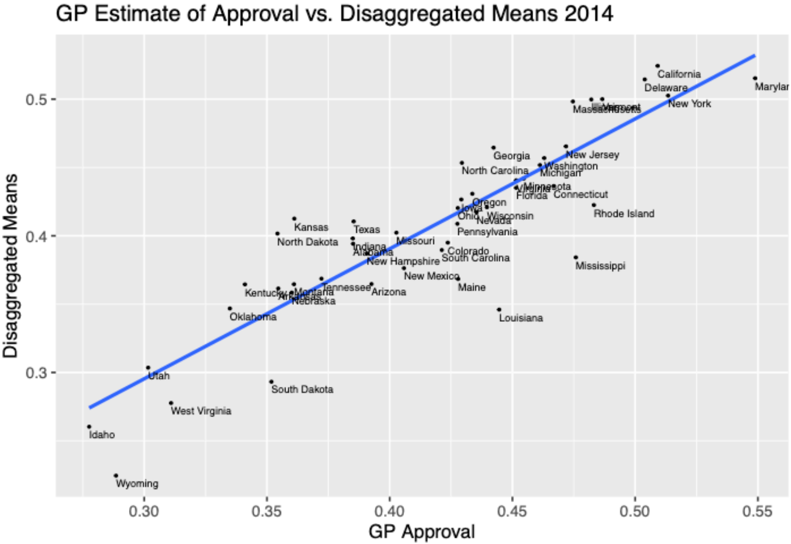
Example: Obama approval 2010



Example: Obama approval 2014



Section 3

Conclusion & in-progress

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- We are still ironing out the details of optimization of length-scale parameters