Distance-Weighted Kalman Consensus Filter: A Distributed State Estimation Algorithm With Dynamically Adapting Weights

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***Abstract*— During the past decades, multi-sensor state estimation has become increasingly attractive because of the significant improvement in reliability and robustness. All the estimates are supposed to reach a consensus via a distributed Kalman filtering algorithm, while a disagreement is regarded as an indicator of sensor failure. However, most distributed Kalman filters in the literature assume constant measurement noise covariance, which is not necessarily the case since measurement accuracy may change as the target-sensor distance varies. For this reason, this paper proposes a novel distance-weighted Kalman consensus filter (DW-KCF) constructed on a network of continuously moving objects. It deals with the influence of distance on measurement noises, through which the filter dynamically adapts the edge weights of the network to minimize the deviation of estimates. The stability of estimate error dynamics is rigorously proved using the Lyapunov theorem, and the functionalities of DW-KCF are verified via numerical simulations. Results demonstrate that this filter succeeds in handling the distance-dependent measurement noise without impacting either the consensus property or the accuracy of estimation.**

I. Introduction

In the past decades, Kalman filter (KF) has been statistically proved one of the most significant state estimation algorithms. It shows powerful performance in managing Gaussian uncertainties in measurement and has been extensively used in various industries[1]. As a further improvement, state estimation using multiple sensors has experienced a long history, especially in the field of robotics[2]. Scientists have developed and utilized cooperative multi-agent systems[3], which allow communication and analysis within a network of nodes. One advantage that makes distributed state estimation algorithm outstanding is the increased reliability, as the failure of one sensor will not result in complete loss of the state information. This motivation contributes to the concept of multi-hypothesis tracking[4] and decentralized Kalman filtering[5], which refers to a set of local Kalman filters communicating with all other nodes. Such a communicating network is referred to as a complete graph in graph theory[6]. In addition, the work in [7] proposes distributed Kalman filtering (DKF) algorithm such that each node only communicates with its neighbors, through which the efficiency and scalability of the communication graph is increased, especially on a relatively large network. Subsequently, this algorithm has been modified into Kalman-consensus filter (KCF), where all the sensors are designed to reach a consensus on the estimation results after communications[8]. Such a technique can be used for distributed tracking on a network of mobile sensors[9], or more practically, to accomplish the distributed control of autonomous driving[10], where each vehicle works as a mobile sensor for mutual state estimation.

However, the distributed estimation algorithms mentioned above are constructed based on a nominal Kalman filter, which assumes constant measurement noise covariance. Nevertheless, in practice, the relative measurement tends to be less and less accurate as the sensor moves away from the target, especially for time-of-flight sensors[11]. Typically, the measurement noise is proportional to the target-sensor distance[12], which is no longer negligible on a large network of mobile sensors. For this reason, we propose a distributed state estimation algorithm that computes the weighted average of the measurement from multiple sensors, with the weights being dependent on the distances between the target and the sensors. Such an algorithm is referred to as the distance-weighted Kalman consensus filter (DW-KCF). The entire algorithm iteration consists of three phases, as demonstrated in Fig. 1. Each sensor node individually estimates the states of the target node before mutually transmitting the information. Each sensor node i quantitatively determines the trustworthiness of the state estimate information from the other node j based on the distances. A sensor in the target's proximity deserves a higher weight on its estimate, since its measurement error is relatively lower than other remote sensors. Hence, for moving nodes such as mobile robots, the varying distances between nodes result in a communication graph with dynamically changing edge weights. Such graph becomes a strongly connected and weighted graph. The DW-KCF has been proved stabilizable and verified through numerical simulations.

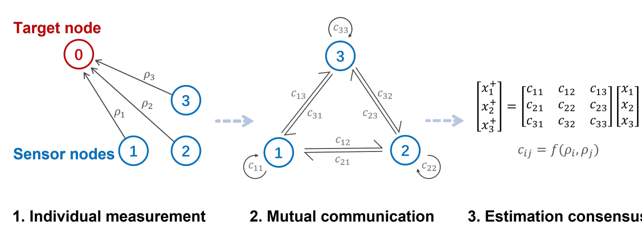


Fig. 1. At each iteration, the state of the target node is estimated by multiple sensor nodes separately. All involved sensor nodes then form a complete communication graph, with edge weights c_{ij} dependent on target-sensor distances ρ_i and ρ_j for all nodes i, j.

This paper is structured as follows. The fundamental theories are described in Section II to provide an collection of the involved mathematical preliminaries and an overview of the system model. Section III introduces the methodologies used to derive the proposed algorithm, followed by a stability analysis of the algorithm. In Section IV, a more rigorous verification is given by performing numerical simulations and discussions on the results. Eventually, the conclusion is given in Section V which summarizes the technical work.

II. Fundamentals

This section serves as a review of mathematical concepts and results to be used throughout this paper in the first part. The second part will then provide an overview of the system modeling in general cases.

A. Mathematical Preliminaries

For a given real-valued matrix A, let A^T denotes its transpose. If A is a square matrix, i.e. $A \in \mathbb{R}^{n\times n}$, let $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote its minimum and maximum eigenvalues. A is said to be positive (semi)definite if $x^T Ax > 0(\geq 0)$, for $\forall x \in \mathbb{R}^n \backslash \{0\}$. If A is a positive (semi)definite matrix, it is denoted by $A > 0(\geq 0)$. For matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$, let $A \otimes B \in \mathbb{R}^{nm\times nm}$ denote the Kronecker product between A and B. Denote $col(A_1, A_2, \ldots, A_N)$ and $diag(A_1, A_2, \ldots, A_N)$ as the block-column matrix and block-diagonal matrix consisting of matrices from A_1 to A_N respectively. Let I_n denote a $n\times n$ identity matrix, $\mathbb{1}_n$ denote a $n \times 1$ column vector of ones and $\mathbb{1}_{n\times n}$ denote a $n \times n$ square matrix of ones. Furthermore, let I denote the general identity matrix, if the dimensions of the other matrices are not specified.

Consider a discrete time-invariant system $x(k + 1) = f(x(k))$. The system is said to be Lyapunov stable if for $\forall\, \varepsilon > 0$, there exist $\delta > 0$, so that $\|x(0)\| < \delta \;\Rightarrow\; \|x(t)\| < \varepsilon$. Further, the system is said to be globally asymptotically stable if it is stable and $\lim_{t\to\infty} x(t) = 0,\ \forall x(0)$.

Consider a graph described by $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. Let $\mathcal{V} = \{1, 2, ..., N\}$ denote the set of vertices of the graph G, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denote the set of edges of the graph, $\mathcal{A}$ denote the adjacency matrix of G, with its (i, j)-th element $[a]_{ij} \neq 0$ if and only if $(i, j) \in \mathcal{E}$, otherwise, $[a]_{ij} = 0$. Additionally, let $\mathcal{L}(G)$ represent the graph Laplacian of G.

We also present some lemmas that will be used in model formulation derivation and stability analysis.

Theorem 1. *(Lyapunov Stability Theorem [13]) Let $x = 0$ be an equilibrium point for the autonomous system*

$$x(t + 1) = f(x(t))$$

where $f :\ D \to \mathbb{R}^n$ is locally Lipschitz in $D \subseteq \mathbb{R}^n$ and $0 \in D$. Let $V :\ \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

(1) $V(0) = 0$ *and* $V(x) > 0,\ \forall x \in D\backslash\{0\}$,
(2) $V(x) \to \infty$ *as* $\|x\| \to \infty$,
(3) $\delta V(x) = V(f(x)) - V(x) < 0,\ \forall x \in D\backslash\{0\}$

then $x = 0$ is globally asymptotically stable.

Lemma 1. *(The Woodbury matrix identity [14]) If A, C, U and V are conformable matrices, i.e. $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{k\times k}$, $U \in \mathbb{R}^{n\times k}$ and $V \in \mathbb{R}^{k\times n}$, then the following identity holds*

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1} VA^{-1}$$

Lemma 2. *(Matrix Calculus [15]) If $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are square matrices. Then the following differentiation rule holds:*

$$\frac{\partial}{\partial A}(A \cdot B) = B \otimes I_n$$

Lemma 3. *(Property of Kronecker Product [16]) If A and B are square matrices, then there exists a permutation matrix P, such that*

$$B \otimes A = P(A \otimes B)P^T$$

Lemma 4. *Suppose $A \in \mathbb{R}^{n\times m}$ has full rank and $X \in \mathbb{R}^{m\times m}$, then*

$$AXA^T \otimes I_n > 0(\geq 0) \;\Leftrightarrow\; X > 0(\geq 0)$$

Proof. First from basic linear algebra, one can show that

$$AXA^T > 0(\geq 0) \;\Leftrightarrow\; X > 0(\geq 0)$$

Now, applying Lemma 3, we know that $AXA^T \otimes I_n$ is congruent to $I_n \otimes AXA^T = diag(AXA^T, \ldots, AXA^T)$, which is positive (semi)definite if and only if AXA^T is positive (semi)definite. □

B. System and Estimator Models

The nominal dynamics and measurement models of the an individual discrete linear time-invariant (DLTI) system are defined as,

$$\begin{aligned} x(k + 1) &= Ax(k) + Bu(k) \\ z(k) &= Hx(k) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^m, u \in \mathbb{R}^q, z \in \mathbb{R}^p, A \in \mathbb{R}^{m\times m}, B \in \mathbb{R}^{m\times q}, H \in \mathbb{R}^{p\times m}$. Here, m denotes the number of states of the target, q denotes the number of inputs of the target, p denotes the number of measurements of each sensor. Further, we denote n as the number of sensors used to estimate the target state, and hence there are $n + 1$ agents in total. If not otherwise specified, we use i to index the sensor.

The corresponding dynamics and measurement with additive stochastic noise is defined as,

$$\begin{aligned} x(k + 1) &= Ax(k) + B\left(u(k) + w(k)\right) \\ z(k) &= Hx(k) + \mu(k) \end{aligned} \tag{2}$$

where $w(k)$ and $\mu(k)$ are actuation noise (assumed on the input only) and measurement noise respectively.

Assumption 1. *The measurement noise is assumed to be homogeneous, up to some factor depending on relative distance between sensor i and the target. Hence,*

$$\mu_i(k) = \Phi_i(\rho_i(k))v(k) \tag{3}$$

where $\rho_i(k)$ is the relative distance between sensor i and the target, $\Phi_i(\rho_i(k))$ is some function of $\rho_i(k)$, $v(k)$ is time-varying noise, but independent of distance $\rho_i(k)$.

Note. *In general, Φ_i can be any function (either linear or nonlinear) depending on ρ_i, and can be distinct for each sensor i. In this paper, we focus on Φ_i linear to ρ_i. Further, we denote $\Phi = col(\Phi_1, \Phi_2, \ldots, \Phi_n)$ as the stacked factor, and $\rho = [\rho_1, \rho_2, \ldots, \rho_n]^T$ as a vector of relative distances.*

The estimator model of the proposed DW-KCF is given in Eq.(4).

$$\hat{x}_i = \bar{x}_i + K_i(z_i - H_i\bar{x}_i) + \gamma \sum_{j\neq i, j=1}^{n} c_{ij} \cdot (\bar{x}_j - \bar{x}_i) \tag{4}$$

Here the consensus factor γ is a tuning parameter that influences the convergence rate and stability in terms of consensus, which will be discussed in Section III-C. All the other quantities are specified as follows.

- $\hat{x}_i \in \mathbb{R}^m$: Posterior update of target state by sensor i
- $\bar{x}_i \in \mathbb{R}^m$: Prior update of target state by sensor i
- $z_i \in \mathbb{R}^p$: Target state measurement from sensor i
- $K_i \in \mathbb{R}^{m\times p}$: Kalman gain matrix corresponding to sensor i
- $H_i \in \mathbb{R}^{p\times m}$: Observation matrix of sensor i
- $c_{ij} \in \mathbb{R}$: Consensus compensation factor of sensor i compared to j

Eventually, the stacked posterior update for a specific target has the following expected form,

$$\begin{aligned} \hat{x} &= \bar{x} + K(z - H\bar{x}) - \gamma\tilde{\mathcal{L}}(G)\bar{x} \\ &= \bar{x} + \epsilon^{kal} + \gamma\epsilon^{con} \end{aligned} \tag{5}$$

where $\hat{x}$ and $\bar{x}$ represent the stacked posterior and prior updates, while ϵ^{kal} and ϵ^{con} stand for Kalman compensation and consensus compensation terms respectively. To be more specific, $\hat{x} = [\hat{x}_1, \cdots, \hat{x}_n]^T$ consists of the posterior updates of all the sensors $\forall i = 1, \cdots, n$. A consensus is reached as expected if eventually $\hat{x}_1 = \hat{x}_2 = \cdots = \hat{x}_n$. The expressions of the consensus gain matrix $\tilde{\mathcal{L}}(G)$ and the Kalman gain matrix $K = diag(K_1, \cdots, K_n)$ will be respectively derived in the next sections.

III. Methodologies

In this section, we will derive the consensus gain $\mathcal{L}(G)$ and the optimal Kalman gain K, which are leveraged to compute the consensus compensation ϵ^{con} and Kalman compensation ϵ^{kal}. Nevertheless, the optimal K is approximated and replaced by the suboptimal solution, since the latter slightly delays the estimation convergence but significantly releases the computational complexity. Therefore, it is more desired to utilize the suboptimal solution, on which the stability analysis will be performed at the end of this section.

A. Consensus Compensation

The consensus compensation factor c_{ij} of node i compared to j is a function w.r.t the measured distances ρ_i and ρ_j from node i and j to the target node respectively. These two distances are then manipulated by applying the following nonlinear feature function to construct the consensus weight as follows,

$$c_{ij} = \frac{cosh(\rho_j)}{cosh(\rho_i)} = \frac{d_j}{d_i} \tag{6}$$

Note the feature function has various alternatives only if they achieve monotonous decrease with respect to ρ. The smoothness, continuity and boundedness make the hyperbolic cosine function a good candidate to generate the compensation factors. Thus, the individual consensus compensation ϵ_i^{con} becomes,

$$\begin{aligned} \epsilon_i^{con} &= \sum_{j\neq i, j=1}^{n} c_{ij} \cdot (\bar{x}_j - \bar{x}_i) \\ &= \left[c_{i1}, c_{i2}, \cdots, -\sum_{j\neq i, j=1}^{n} c_{ij}, \cdots, c_{in} \right] \cdot \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix} \\ &= \left[\frac{d_1}{d_i}, \frac{d_2}{d_i}, \cdots, -\frac{1}{d_i}\sum_{j\neq i, j=1}^{n} d_j, \cdots, \frac{d_n}{d_i} \right] \cdot \bar{x} \\ &= \left[\frac{d_1}{d_i}, \frac{d_2}{d_i}, \cdots, \frac{d_i}{d_i} - \frac{\tilde{d}}{d_i}, \cdots, \frac{d_n}{d_i} \right] \cdot \bar{x} \end{aligned} \tag{7}$$

where $\tilde{d} = \sum_{j=1}^{n} d_j$ is identical for expressions of ϵ_i^{con} $\forall i$. Hence by collecting the prior updates of all the nodes from 1 to n, the stacked consensus compensation $\epsilon^{con} = col(\epsilon_1^{con}, \cdots, \epsilon_n^{con})$ becomes,

$$\begin{aligned} \epsilon^{con} &= D^{-1} \cdot \begin{bmatrix} d_1 - \tilde{d} & d_2 & \cdots & d_n \\ d_1 & d_2 - \tilde{d} & \cdots & d_n \\ \vdots & \vdots & \ddots & \vdots \\ d_1 & d_2 & \cdots & d_n - \tilde{d} \end{bmatrix} \bar{x} \\ &= D^{-1}(\mathbb{1}_{n\times n} \cdot D - \tilde{d} \cdot I_n) \cdot \bar{x} \end{aligned} \tag{8}$$

One can then define the consensus gain matrix for the case of $m = 1$ as,

$$C = D^{-1}(\mathbb{1}_{n\times n} \cdot D - \tilde{d} \cdot I_n) \tag{9}$$

with the involved elements being,

- $d_j = cosh(\rho_j) \in \mathbb{R}$
- $D = diag(d_1, d_2, \cdots, d_n) \in \mathbb{R}^{n\times n}$

By considering the weight matrix $\mathrm{diag}\left(\{c_{ij}\}_{\{i,j\}\in\mathcal{E}}\right)$ and the incidence matrix T of the corresponding graph G, one can easily verify that

$$C = -\mathcal{L}(G) = -T \cdot \mathrm{diag}\left(\{c_{ij}\}_{\{i,j\}\in\mathcal{E}}\right) \cdot T^T \tag{10}$$

thus by definition, $\mathcal{L}(G)$ is a Laplacian matrix.

More generally, when $\bar{x}_i \in \mathbb{R}^m$ contains multiple states, i.e. $m > 1$, the complete consensus gain matrix is simply the Kronecker product of $\mathcal{L}(G)$ and the identity matrix in m-dimensional space,

$$\tilde{\mathcal{L}}(G) = \mathcal{L}(G) \otimes I_m \tag{11}$$

which concludes the derivation of the consensus gain used in Eq.(5).

B. Kalman Compensation

From the discussion above, the stacked posterior update for estimating a target node ℓ should have the following expression modified from Eq.(5):

$$\hat{x}^\ell = \bar{x}^\ell + K^\ell(z^\ell - H^\ell \bar{x}^\ell) - \gamma \tilde{\mathcal{L}}(G^\ell)\bar{x}^\ell \tag{12}$$

Here, G^ℓ denotes the graph used in state estimation of target ℓ. Hence, the posterior update of each sensor node i is denoted as $\hat{x}_i^\ell$. Following the similar procedure in [8], we address the the problem of finding optimal Kalman-like filtering by solving the following problem:

Problem 1. *(Optimal Kalman-like Consensus Filter) Find the optimal K^ℓ in the form (12) that minimizes the mean-square estimation error $\sum_{i=1}^n \mathbb{E}\left[\|\hat{x}_i^\ell - x^\ell\|^2\right]$.*

Before obtaining the optimal K^ℓ, we need to make some definitions with respect to estimation errors. Let $\eta^\ell = \hat{x}^\ell - x^\ell$ and $\bar{\eta}^\ell = \bar{x}^\ell - x^\ell$ denote the stacked estimation errors of target ℓ. Then, we define the covariance matrices M^ℓ and P^ℓ as

$$M^\ell = \mathbb{E}\left[\eta^\ell(\eta^\ell)^T\right],\ P^\ell = \mathbb{E}\left[\bar{\eta}^\ell(\bar{\eta}^\ell)^T\right] \tag{13}$$

Moreover, we assume that $w(k)$ and $v(k)$ are zero-mean white Gaussian noise with the following statistics:

$$\begin{aligned} \mathbb{E}\left[w(k)w(l)^T\right] &= Q(k)\delta_{kl} \\ \mathbb{E}\left[v(k)v(l)^T\right] &= R(k)\delta_{kl} \end{aligned} \tag{14}$$

which immediately implies that

$$\mathbb{E}\left[\mu(k)\mu(l)^T\right] = \Phi(\rho(k))R(k)(\Phi(\rho(k)))^T\delta_{kl}$$

We will then define $\mathbb{E}\left[\mu(k)\mu(l)^T\right] = \tilde{R}(k)\delta_{kl}$, where $\tilde{R}(k)$ is also some function of the relative distance $\rho(k)$. Due to nice properties of Gaussian variables, one can intuitively conclude that $\mu(k) \sim \mathcal{N}(0, \tilde{R}(k))$.

Note. *In the rest of this paper, for ease of notations, we adopt a notation that is free of the time index k and the target index ℓ.*

Theorem 2. *Consider a sensor network for the target with topology G (or $G(k)$), observing a linear time-varying process $x^+ = Ax + Bw$. Assume the stacked measurement is $z = Hx + \Phi(\rho)v$. Then the optimal DW-KCF is in the following form,*

$$\begin{aligned} \hat{x} &= \bar{x} + K(z - H\bar{x}) - \gamma\tilde{\mathcal{L}}(G)\bar{x} \\ K &= \left(P - \gamma\tilde{\mathcal{L}}(G)\right)H^T\left(\Phi(\rho)R(\Phi(\rho))^T + HPH^T\right)^{-1} \\ M &= FPF^T + \gamma^2\tilde{\mathcal{L}}(G)P\tilde{\mathcal{L}}(G)^T \\ &\quad - \gamma FP\tilde{\mathcal{L}}(G)^T - \gamma\tilde{\mathcal{L}}(G)PF^T \\ &\quad + K\Phi(\rho)R(\Phi(\rho))^T K^T \\ P^+ &= \tilde{A}M\tilde{A}^T + \tilde{B}Q\tilde{B}^T \\ \bar{x}^+ &= \tilde{A}\hat{x} \end{aligned} \tag{15}$$

where $F = I - KH$, $\tilde{A} = I_n \otimes A$ and $\tilde{B} = I_n \otimes B$.

Proof. The main task is to minimize the the estimation error $\sum_{i=1}^n \mathbb{E}\left[\|\hat{x}_i - x\|^2\right]$. Noting that

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}\left[\|\hat{x}_i - x\|^2\right] &= \sum_{i=1}^n \operatorname{tr}\left(\mathbb{E}\left[\eta_i\eta_i^T\right]\right) \\ &= \operatorname{tr}\left(\mathbb{E}\left[\eta\eta^T\right]\right) \\ &= \operatorname{tr}(M) \end{aligned}$$

which means the key in finding the optimal filter K in 12 is to compute $\operatorname{tr}(M)$ and minimize it.

In order to do so, we first examine the formula for M. Subtracting the stacked state x in both sides of 12 gives

$$\eta = F\bar{\eta} - \gamma\tilde{\mathcal{L}}(G)\bar{\eta} + K\Phi(\rho)v \tag{16}$$

Note that $M = \mathbb{E}\left[\eta\eta^T\right]$ and $P = \mathbb{E}\left[\bar{\eta}\bar{\eta}^T\right]$, we have that

$$\begin{aligned} M &= FPF^T + \gamma^2\tilde{\mathcal{L}}(G)P\tilde{\mathcal{L}}(G)^T \\ &\quad - \gamma FP\tilde{\mathcal{L}}(G)^T - \gamma\tilde{\mathcal{L}}(G)PF^T \\ &\quad + K\Phi(\rho)R(\Phi(\rho))^T K^T \end{aligned} \tag{17}$$

which means,

$$\begin{aligned} \operatorname{tr}(M) &= \operatorname{tr}(P) - \operatorname{tr}(KHP) - \operatorname{tr}\left(PH^TK^T\right) \\ &\quad + \operatorname{tr}\left(KHPH^TK^T\right) - \gamma\operatorname{tr}\left(P\tilde{\mathcal{L}}(G)^T\right) \\ &\quad + \gamma\operatorname{tr}\left(KHP\tilde{\mathcal{L}}(G)^T\right) - \gamma\operatorname{tr}\left(\tilde{\mathcal{L}}(G)P\right) \\ &\quad + \gamma\operatorname{tr}\left(\tilde{\mathcal{L}}(G)PH^TK^T\right) + \operatorname{tr}\left(K\Phi(\rho)R(\Phi(\rho))^TK^T\right) \\ &\quad + \gamma^2\operatorname{tr}\left(\tilde{\mathcal{L}}(G)P\tilde{\mathcal{L}}(G)^T\right) \\ \frac{\partial M}{\partial K} &= -2PH^T + 2K(HPH^T) \\ &\quad + 2\gamma\tilde{\mathcal{L}}(G)H^T + 2K\Phi(\rho)R(\Phi(\rho))^T \end{aligned}$$

By solving $\frac{\partial M}{\partial K} = 0$, we can find optimal filter as,

$$K^* = \left(P - \gamma\tilde{\mathcal{L}}(G)\right)H^T\left(\Phi(\rho)R(\Phi(\rho))^T + HPH^T\right)^{-1} \tag{18}$$

To clarify K^* is indeed the minimizer, we compute the second derivative of $\operatorname{tr}(M)$ with respect to K, by applying Lemma 2,

$$\frac{\partial^2 M}{\partial K^2} = 2(HPH^T)\otimes I + 2(\Phi(\rho)R(\Phi(\rho))^T)\otimes I$$

where I is identity matrix with some suitable dimension.

Meanwhile, since P and R are trivially positive semi-definite matrices, it immediately follows from Lemma 4 that $\frac{\partial^2 M}{\partial K^2}$ is positive semi-definite, thus making K^* an optimal choice. To determine the update rule for P, we utilize the system dynamic equation $x^+ = Ax + Bw$ and update equation of prior estimate $\bar{x} = \tilde{A}\hat{x}$ to obtain

$$\bar{\eta}^+ = \tilde{A}\eta - \tilde{B}w$$

which means

$$P^+ = \tilde{A}M\tilde{A}^T + \tilde{B}Q\tilde{B}^T \tag{19}$$

Combining equations (12), (17), (18), (19) with the update equation of prior estimate, we finally get the derivation of all the equations of the optimal filter. □

The main complexity in (15) lies in the computation of $\tilde{\mathcal{L}}(G)$. When tuning the consensus rate of the estimation, we tend to use a small γ to ensure stability, as we will see in Section III-C. For this reason, the magnitude of $\gamma\mathcal{L}(G)$ in the expression of K becomes negligible compared to P, while neglecting $\gamma\mathcal{L}(G)$ can significantly reduce computational complexity. Hence, we decide to use the the approximated suboptimal solution which gives the following algorithm:

$$\begin{aligned}\hat{x} &= \bar{x} + K(z - H\bar{x}) - \gamma\tilde{\mathcal{L}}(G)\bar{x}\\ K &= PH^T\left(\Phi(\rho)R(\Phi(\rho))^T + HPH^T\right)^{-1}\\ M &= FPF^T + K\Phi(\rho)R(\Phi(\rho))^T K^T\\ P^+ &= \tilde{A}M\tilde{A}^T + \tilde{B}Q\tilde{B}^T\\ \bar{x}^+ &= \tilde{A}\hat{x}\end{aligned} \tag{20}$$

C. Stability Analysis

As previously explained, the approximated suboptimal K is more desired to be implemented in the algorithm. We are then interested in the stability of the suboptimal solution, starting by specifying several lemmas as below.

Lemma 5. *Consider the suboptimal distance-weighted Kalman consensus filter in (20). Define $\tilde{R} = \Phi(\rho)R(\Phi(\rho))^T$ and $S = H^T\tilde{R}H$. Then, the following statements hold:*
1) $M = (P^{-1} + S)^{-1}$.
2) $K = MH^T\tilde{R}^{-1}$.
3) $F = MP^{-1}$.
4) $M^+ = FGF^T$, *where* $G = \tilde{A}M\tilde{A}^T + \tilde{B}Q\tilde{B}^T + PSP$.

Proof. First, we derive an alternative form for M. Substituting $F = I - KH$ into $M = FPF^T + K\tilde{R}K^T$, we have

$$\begin{aligned}M =&(I-KH)^TP(I-KH)^T + K\tilde{R}K^T\\ =&P - KHP - PH^TK^T + KHPH^TK^T + K\tilde{R}K^T\\ =&P - 2PH^T(\tilde{R}+HPH^T)^{-1}HP + PH^T\\ &(\tilde{R}+HPH^T)^{-1}(\tilde{R}+HPH^T)(\tilde{R}+HPH^T)^{-1}HP\\ =&P - PH^T(\tilde{R}+HPH^T)^{-1}HP\end{aligned}$$

Applying Lemma 1, we have

$$\begin{aligned}M =&P - PH^T(\tilde{R}+HPH^T)^{-1}HP\\ =&(P^{-1} + H\tilde{R}^{-1}H^T)^{-1}\\ =&(P^{-1}+S)^{-1}\end{aligned} \tag{21}$$

Using the results from (21), we get

$$\begin{aligned}K &= PH^T(\tilde{R}+HPH^T)^{-1}\\ &= PH^T(I+\tilde{R}^{-1}HPH^T)^{-1}\tilde{R}^{-1}\\ &= \left((I+\tilde{R}^{-1}HPH^T)^{-1}H^{-T}P^{-1}\right)^{-1}\tilde{R}^{-1}\\ &= (H^{-T}P^{-1}+\tilde{R}^{-1}H)^{-1}\tilde{R}^{-1}\\ &= (P^{-1}+H^T\tilde{R}^{-1}H)^{-1}H^T\tilde{R}^{-1}\\ &= MH^T\tilde{R}^{-1}\end{aligned} \tag{22}$$

For F: Noticing that $F = I - KH = I - MH^T\tilde{R}^{-1}H = I - MS$ and that $M(P^{-1}+S) = (P^{-1}+S)^{-1}(P^{-1}+S) = I$ we have that

$$F = I - MS = MP^{-1} \tag{23}$$

.

Finally, to compute M^+, we substitute (22) and (23) into the update Eq.(20):

$$\begin{aligned}M^+ &= FP^+F^T + K\tilde{R}K^T\\ &= F(\tilde{A}M\tilde{A}^T + \tilde{B}Q\tilde{B}^T)F^T + MH^T\tilde{R}^{-1}\tilde{R}\tilde{R}^{-1}HM^T\\ &= F(\tilde{A}M\tilde{A}^T + \tilde{B}Q\tilde{B}^T)F^T + FPH^T\tilde{R}^{-1}HPF^T\\ &= F(\tilde{A}M\tilde{A}^T + \tilde{B}Q\tilde{B}^T + PSP)F^T\\ &= FGF^T\end{aligned} \tag{24}$$

Combining (21), (22), (23) and (24) concludes the proof for this lemma. □

Lemma 6. *Suppose that $S = H^T\Phi(\rho)R(\Phi(\rho))^TH$ is a positive definite matrix for all time $k \geq 0$. Then, $\Lambda_1 = M^{-1} - \tilde{A}^TF^T(M^+)^{-1}F\tilde{A}$ is also positive definite.*

Proof. Simple calculation shows that

$$\begin{aligned}M\Lambda_1M =&M - M\tilde{A}^TF^T(M^+)^{-1}F\tilde{A}M\\ =&M - M\tilde{A}^TG^{-1}\tilde{A}M\\ =&M - M\tilde{A}^T\left[\tilde{A}M\tilde{A}^T + \left(\tilde{B}Q\tilde{B}^T + PSP\right)\right]^{-1}\tilde{A}M\\ =&\left[M^{-1} + \tilde{A}^T\left(\tilde{B}Q\tilde{B}^T + PSP\right)^{-1}\tilde{A}\right]^{-1}\end{aligned}$$

where the second equality follows from Lemma 5 and the last equality follows from Lemma 1. This means,

$$\Lambda_1 = M^{-1}\left[M^{-1} + \tilde{A}^T\left(\tilde{B}Q\tilde{B}^T + PSP\right)^{-1}\tilde{A}\right]^{-1}M^{-1}$$

Since $S > 0$, we have that $\tilde{B}Q\tilde{B}^T + PSP > 0$. Combining $M^{-1} \geq 0$ gives us $\Lambda_1 > 0$. □

Now, we are ready to presenst stability analysis of our suboptimal distance-weighted Kalman consensus filter.

Theorem 3. *(Stability of DW-KCF) Consider the suboptimal distance-weighted Kalman consensus filter in (20). Suppose that $S = H^T\Phi(\rho)R(\Phi(\rho))^TH$ and M^{-1} are positive definite matrices for all time $k \geq 0$. Then the stakced error dynamics of the suboptimal distance-weighted Kalman consensus filter is globally asymptotically stable for a will-chosen γ, which means all estimators asymptotically reach a consensus on state estimates, i.e. $\hat{x}_1(\infty) = \hat{x}_2(\infty) = \cdots = \hat{x}_n(\infty) = x(\infty)$.*

Proof. According to (16) and the update Eq.(20), the error dynamics without noise of our suboptimal distance-weighted Kalman consensus filter can be written as

$$\eta^+ = F\tilde{A}\eta - \gamma\tilde{\mathcal{L}}(G)\eta \tag{25}$$

Influenced by the choice in [8], let us define

$$V(\eta) = \eta^TM^{-1}\eta \tag{26}$$

as a candidate Lyapunov function for (25). Condition (1) and (2) in Theorem 1 are automatically satisfied due to the assumption that M^{-1} is a positive definite matrix.

Meanwhile, substituting the error dynamics (25) into the change $\delta V(\eta) = V(\eta^+) - V(\eta)$ gives,

$$\begin{aligned} \delta V(\eta) =& (\eta^+)^T (M^+)^{-1} \eta^+ - \eta^T M \eta \\ =& [F\tilde{A}\eta - \gamma\tilde{\mathcal{L}}(G)\eta]^T (M^+)^{-1} \\ & [F\tilde{A}\eta - \gamma\tilde{\mathcal{L}}(G)\eta] - \eta^T M^{-1} \eta \\ =& \eta^T \left[\tilde{A}^T F^T (M^+)^{-1} F\tilde{A} - M^{-1}\right] \eta \\ & -\gamma\eta^T \left\{ \left(\tilde{\mathcal{L}}(G)\right)^T (M^+)^{-1} F\tilde{A} \right\} \eta \\ & -\gamma\eta^T \left\{ \tilde{A}^T F^T (M^+)^{-1} \left(\tilde{\mathcal{L}}(G)\right) \right\} \eta \\ & +\gamma^2\eta^T \left\{ \left(\tilde{\mathcal{L}}(G)\right)^T (M^+)^{-1} \left(\tilde{\mathcal{L}}(G)\right) \right\} \eta \\ =& -\eta^T \left[\Lambda_1 + \gamma\Lambda_2 - \gamma^2\Lambda_3\right] \eta \end{aligned} \tag{27}$$

where

$$\begin{aligned} \Lambda_1 &= M^{-1} - \tilde{A}^T F^T (M^+)^{-1} F\tilde{A} \\ \Lambda_2 &= \left(\tilde{\mathcal{L}}(G)\right)^T (M^+)^{-1} F\tilde{A} + \tilde{A}^T F^T (M^+)^{-1} \left(\tilde{\mathcal{L}}(G)\right) \\ \Lambda_3 &= \left(\tilde{\mathcal{L}}(G)\right)^T (M^+)^{-1} \left(\tilde{\mathcal{L}}(G)\right) \end{aligned} \tag{28}$$

From Lemma 6, Λ_1 is a symmetric positive definite matrix. Λ_2 and Λ_3 are trivially symmetric matrices and thus have real eigenvalues. Indeed, one can easily verify that Λ_3 is a symmetric positive semidefinite matrix, since $M^+ = \mathbb{E}\left[(\eta^+)^T \eta^+\right]$ is positive semidefinite.

Let $\lambda_{min}(\Lambda_1)$, $\lambda_{min}(\Lambda_2)$, $\lambda_{max}(\Lambda_3)$ be the corresponding minimum or maximum eigenvalues of Λ_1, Λ_2, Λ_3. Following from the above statements, one can claim that $\lambda_{min}(\Lambda_1) > 0$ and $\lambda_{max}(\Lambda_3) \geq 0$. To be precise, we have $\lambda_{max}(\Lambda_3) > 0$ since Λ_3 could not be a zero matrix.

Therefore, simple linear algebra shows that

$$\begin{aligned} \delta V(\eta) &= -\eta^T \left(\Lambda_1 + \gamma\Lambda_2 - \gamma^2\Lambda_3\right) \eta \\ &\leq -\left[\lambda_{min}(\Lambda_1) + \gamma\lambda_{min}(\Lambda_2) - \gamma^2\lambda_{max}(\Lambda_3)\right] \|\eta\|^2 \end{aligned}$$

This means that to ensure the error dynamics to be globally asymptotically stable, we need

$$\lambda_{min}(\Lambda_1) + \gamma\lambda_{min}(\Lambda_2) - \gamma^2\lambda_{max}(\Lambda_3) > 0 \tag{29}$$

By solving (29), one can conclude that for $\gamma < \gamma^*$ where

$$\gamma^* = \frac{\lambda_{min}(\Lambda_2) + \sqrt{\lambda_{min}^2(\Lambda_2) + 4\lambda_{min}(\Lambda_1)\lambda_{max}(\Lambda_3)}}{2\lambda_{max}(\Lambda_3)} \tag{30}$$

we have $\delta V(\eta) < 0$ for all $\eta \neq 0$, which, from Theorem 1, implies that the error dynamics is globally asymptotically stable.

Since $\lambda_{min}(\Lambda_1),\ \lambda_{max}(\Lambda_3) > 0$, $\gamma^* > 0$ and thus the stability condition $\gamma < \gamma^*$ is well defined. $\square$

IV. Verification

To verify the proposed algorithm used in the DW-KCF, we have performed a numerical simulation considering a group of several identical planar robots that can mutually exchange the information of their estimated positions and relative distances. This section begins with the numerical simulation setup before discussing the simulation results in detail.

A. Simulation Setup

The simulation considers a scenario of four planar mobile robots with different motion trajectories. These robots are supposed to estimate the states of others by mutually measuring the relative distances ρ_i^ℓ and azimuth angles θ_i^ℓ, where we use ℓ and i to index the target robots and the sensor robots respectively. Specifically, a sensor robot estimates and exchanges the states of the target robot with other sensor robots for subsequent consensus filtering. The simulation starts with modeling the dynamics of all four nodes by the LTI continuous-time equations of motion in Eq.(31),

$$\dot{x}^\ell(t) = A_c x^\ell(t) + B_c u^\ell(t) + B_c w^\ell(t) \tag{31}$$

The state vector $x^\ell = [\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}]^T$ represents the Cartesian coordinates and velocities of node ℓ in 2D space. The input vector $u^\ell = [u_x^\ell, u_y^\ell]^T$ stands for the thrust forces in two directions respectively. The state matrix A_c and input matrix B_c are then given by,

$$A_c = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -1 & 0 \\ 0 & -2 & 0 & -1 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{32}$$

which resembles a controllable system that incorporates an spring-damping model to render negative eigenvalues and hence an asymptotically stable equilibrium at the origin. We then employ the MATLAB c2d function and convert the dynamics for each node into the discrete-time form given in Eq.(33):

$$x^\ell(k+1) = A_d x^\ell(k) + B_d u^\ell(k) + B_d w^\ell(k) \tag{33}$$

Specifically, the initial states and the actuation noises of four robots obey the following Gaussian distributions,

$$\begin{aligned} x^\ell(0) &\sim \mathcal{N}\left(\mathbb{E}[x^\ell(0)], P(0)\right) \\ w^\ell &\sim \mathcal{N}(0, Q) \end{aligned} \tag{34}$$

with expected initial states $\mathbb{E}[x^\ell(0)] = [\pm1, \pm1, 0, 0]^T$, and $P(0) = \mathbb{I}_4$. The actuation noise w^ℓ is assumed identical for all robots and all time step k, with a covariance matrix of $Q = 0.05\mathbb{I}_2$. Several sinusoidal signals will be used to generate the input sequence $u^\ell(k)$ to drive the system so that the distances between each two robots are continuously varying.

Besides, the observation vector z is treated in a slightly different way. For conciseness, we drop the time index k in the following descriptions. We define $z_i^\ell = [\mathbf{x}_i^\ell, \mathbf{y}_i^\ell]^T$ as the measured position of node ℓ by node i , hence the output matrix H in Eq.(3) has the following form,

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{35}$$

which ensures an observable system. Meanwhile, the measurement noises μ_i^ℓ obeys,

$$\mu_i^\ell \sim \mathcal{N}(0, \tilde{R}_i^\ell) \tag{36}$$

However, since the measurement relies on the distance and azimuth between the target and the sensor, the actual observation vector is $\bar{z}_i^\ell = [\rho_i^\ell, \theta_i^\ell]^T$. We denote the covariance matrix of $\bar{z}_i^\ell$ as $\bar{R}_i^\ell$ and expressed by,

$$\bar{R}_i^\ell(\rho_i) = \begin{bmatrix} \sigma_\rho(\rho_i^\ell) & 0 \\ 0 & \sigma_\theta(\rho_i^\ell) \end{bmatrix}^2 \tag{37}$$

by considering independent measurements of ρ_i^ℓ and θ_i^ℓ. Therefore, z_i^ℓ can be obtained by,

$$z_i^\ell = \begin{bmatrix} \mathbf{x}_i^\ell \\ \mathbf{y}_i^\ell \end{bmatrix} = \begin{bmatrix} \mathbf{x}^i + \rho_i^\ell cos(\theta_i^\ell) \\ \mathbf{y}^i + \rho_i^\ell sin(\theta_i^\ell) \end{bmatrix} \tag{38}$$

where $[\mathbf{x}^i, \mathbf{y}^i]^T$ is the estimated position of sensor i. The implemented measurement noise covariance matrix $\tilde{R}_i^\ell$ can be calculated by the covariance transformation equation[17],

$$\tilde{R}_i^\ell = \mathbf{C}(\theta_i^\ell) \cdot \bar{R}_i^\ell(\rho_i) \cdot \mathbf{C}(\theta_i^\ell)^T \tag{39}$$

with rotation matrix $\mathbf{C}(\theta_i^\ell)$ depending on the target-and-sensor azimuth angle θ_i^ℓ.

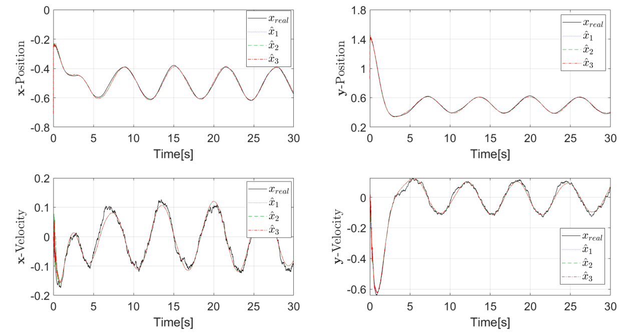
B. Results and Discussion

The proposed DW-KCF will be examined by comparing its performance against KCF as well as analyzing how γ affects the stability of consensus. For clarity, we refer to the target robot as node 0 or no index, and the sensor robots as nodes $i = 1, 2, 3$. In particular, we will illustrate the state estimation for only one target robot while the others act as mobile estimators. Since the estimation process is analogous to all robots, the scenario of estimating one target robot can be generalized to a network of multiple robots.

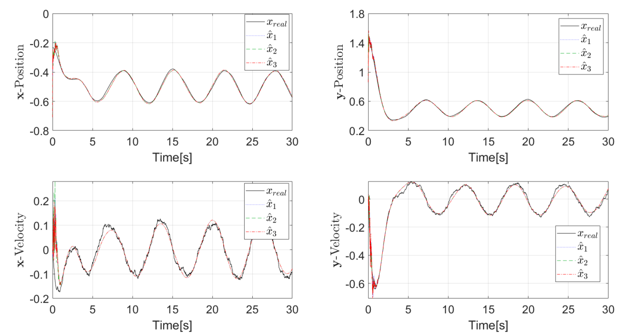
Comparison between DW-KCF and KCF

Plot (a) in Fig. 2 demonstrates the estimation response of DW-KCF when the measurement noise covariance $\tilde{R}_i$ is independent of the target-sensor distance ρ_i, while plot (b) depicts the case of distance-dependent $\tilde{R}_i$. Specifically, each plot displays the estimated states in separate subplots, with positions on the top and velocities at the bottom. In addition, each subplot illustrates the curves of the real state x_{real} of node 0 (black solid), estimated state $\hat{x}_1$ of node 0 by node 1 (blue dotted), $\hat{x}_2$ by node 2 (green dashed), and $\hat{x}_3$ by node 3 (red dash-dotted), respectively. The configuration of all the following plots will remain the same.

Both plots in Fig. 2 reveal the excellent performance of DW-KCF. Even though some deviations emerge in the first five seconds due to uncertainties in initial conditions, three estimators can quickly reach a decent consensus on the target states. It indicates that the DW-KCF satisfactorily estimates the actual state of a moving object, whether the measurement noise explicitly depends on the target-sensor distance or not.

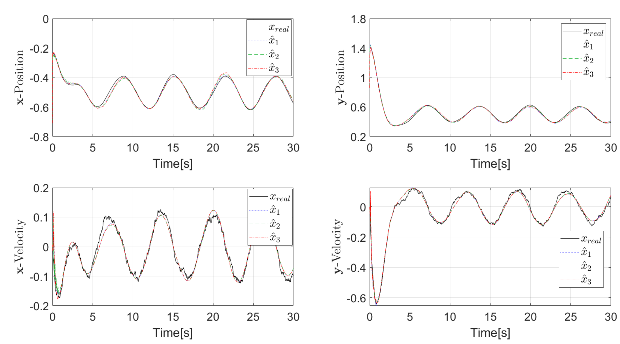


(a) DW-KCF : $\tilde{R}_i$ is independent of ρ_i

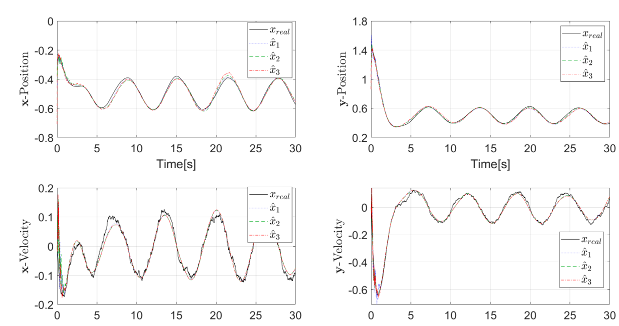


(b) DW-KCF : $\tilde{R}_i$ is dependent on ρ_i

Fig. 2. Distributed state estimates using multiple sensors by DW-KCF, with $\tilde{R}_i$ independent of or dependent on the target-sensor distance ρ_i.



(a) KCF : $\tilde{R}_i$ is independent of ρ_i



(b) KCF : $\tilde{R}_i$ is dependent on ρ_i

Fig. 3. Distributed state estimates using multiple sensors by KCF, with $\tilde{R}_i$ independent of or dependent on the target-sensor distance ρ_i.

Additionally, the deviations in velocities are more evident than those in positions. Such phenomena are attributed to the actuation noises in accelerations and the smoothing effect of integrals.

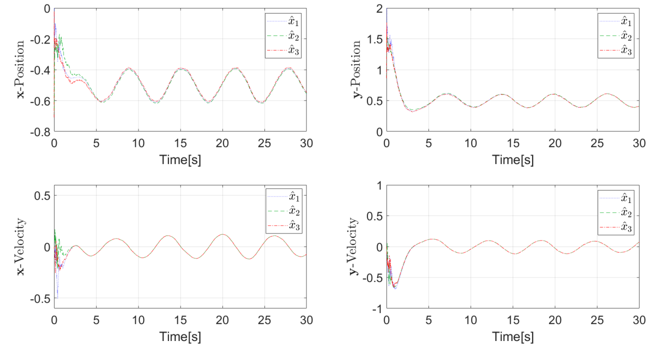
Furthermore, we are interested in the improvement of DW-KCF compared to KCF. As illustrated in Fig.3(a), KCF indeed shows relatively accurate estimation and quick consensus under distance-invariant measurement noise covariance $\tilde{R}_i$. Nevertheless, KCF can no longer guarantee decent performance when $\tilde{R}_i$ varies with the distance ρ_i. As represented in Fig.3(b), the estimates become less accurate and can hardly reach a consensus among different estimators. Therefore, we can conclude that the proposed DW-KCF is capable of handling the distance-dependent measurement noise without impacting the consensus property or accuracy of estimation. This is accomplished by virtue of considering the relation between distances and measurements, which is not taken into account by the KCF.

Stability property in terms of γ

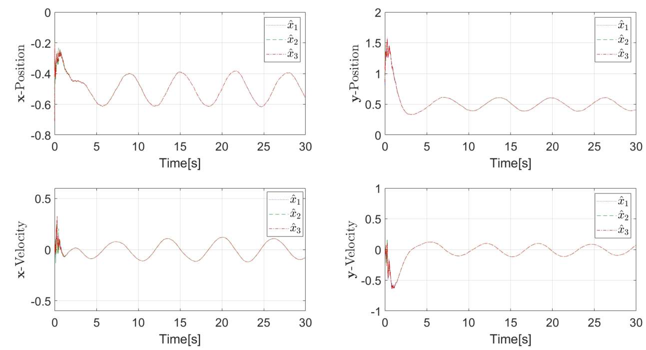
In the remaining part of this section, we will discuss how the value of γ influences the performance of DW-KCF. As analyzed in Section III-C, a relatively small γ delays the consensus while a sufficiently large γ can contribute to an unstable estimate response. Such a fact is presented in Fig.4. In (a) where $\gamma = 0$, the communication between estimators is deactivated, thus the DW-KCF is essentially a nominal KF and no consensus is guaranteed. Even though all three estimates seemingly converge to the real states, one can still notice some obvious offsets before $t = 25[s]$. By contrast, a stable consensus is quickly reached within 1 second when $\gamma = 0.3$ as displayed in (b). Finally, the estimation dynamics are expected to diverge when $\gamma < \gamma^*$ nevermore holds, as introduced in Section III-C. This can be seen in (c) where $\gamma = 0.6$ and the estimated states eventually explode as time increases. Clearly, a proper value of γ within the open interval $(0, \gamma^*)$ is desired for the sake of a decent consensus on estimates.

Variation in standard deviations

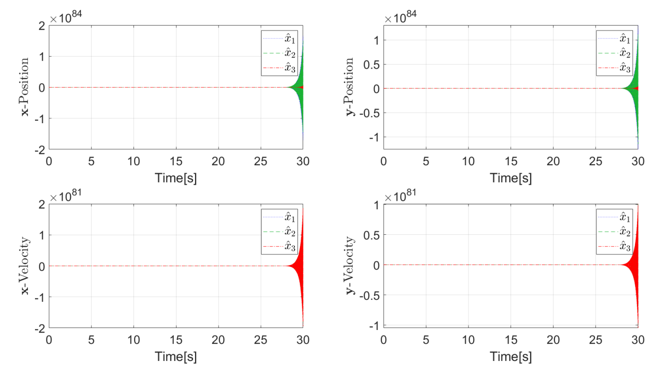
The last quantities to be analyzed are the standard deviations of the state estimates. These values are extracted from the square roots of the diagonal elements of P_m (i.e., the posterior estimation covariance matrix). As shown in Fig.5, each of the three sensors estimates four states of the target, hence giving 12 curves of standard deviations in total. These deviations decline rapidly before reaching consensus on constant values (different states have different consensus values on deviations), also known as the steady-state standard deviations. Such an response matches up with the fact that the posterior estimation covariance matrix converges to a constant matrix as time approaches infinity [18] (i.e., $P_m(k) \rightarrow P_{m,\infty}$ as $k \rightarrow \infty$).



(a) DW-KCF with $\gamma = 0$ (no consensus and slow convergence)



(b) DW-KCF with $\gamma = 0.3$ (decent consensus and quick convergence)



(c) DW-KCF with $\gamma = 0.6$ (divergence)

Fig. 4. Multi-sensor estimates by DW-KCF with various consensus factors γ. An appropriate value of γ leads to decent consensus, while relatively small or large values may result in unsatisfactory estimates.

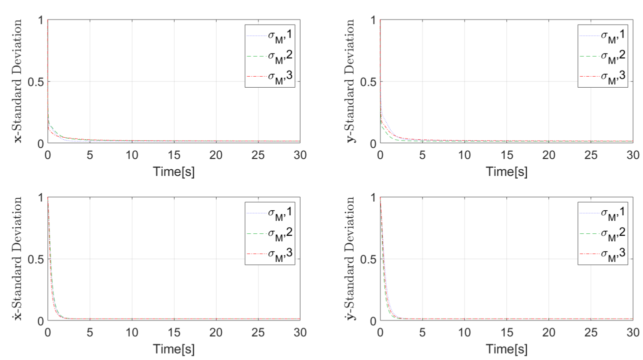


Fig. 5. Standard deviations of the state estimates decrease rapidly and converge to some constants as time goes. The curve of $\sigma_{M,i}$ represents the deviation of estimate on node 0 by node i.

V. Conclusion and Recommendation

We have proposed a novel distance-weighted Kalman consensus filter (DW-KCF), which specifically considers the impact on measurement noise by relative distances between the target and sensors. Such an impact is modeled as a non-linear function of distance. It is considered in the consensus weight c_{ij} for each communication edge between node i and node j. The entire consensus gain matrix is derived from having a similar form to a Laplacian matrix. In addition, we have acquired the update equations in stacked form for the optimal DW-KCF. Since the optimal DW-KCF is computationally consuming, we have approximated the solution with a suboptimal algorithm simplifying the Kalman gain matrix. We then employ a Lyapunov-based method to analyze the stability of the suboptimal DW-KCF. An upperbound of the tuning parameter γ in the posterior update is found to ensure asymptotic stability.

With the theoretical derivation being addressed, we have simulated the performance of our suboptimal DW-KCF under a scenario of four mobile robots. Specifically, the DW-KCF presents decent estimation accuracy and consensus property, no matter whether $\tilde{R}_i$ depends on the relative target-sensor distance ρ_i or not. The DW-KCF demonstrates improved performance in consensus compared to KCF when relative distance significantly influences the measurement noise. Additionally, the dramatically different estimation performances for different γ values prove the importance of tuning the parameter γ within the stability bound. Lastly, the standard deviations of posterior state estimates decline fast and converge to steady-state values.

However, several problems remain open regarding our work. First, the relative distance is regarded as the main factor influencing the measurement noise covariance matrix R_i and is modeled as a non-linear function in our simulation. Nonetheless, in pragmatic cases, other ambient factors, such as temperature and pressure, shall also be considered; hence, practical experiments are necessary to incorporate these factors into the covariance R_i model. Second, we apply a feature function to manipulate the distances into the consensus compensation factors, which gives a satisfactory consensus on estimates, yet is not necessarily the optimal choice. Therefore, we recommend applying different feature functions and comparing the performances. A good candidate of the feature function should be smooth, bounded, and monotonously decrease as its independent variable moves away from the origin. Last but not least, an event-triggered communication schedule is worth investigating, especially in the case of slow dynamics. For example, in [19], a prediction-based switching observer scheme is leveraged, and the error between the estimated state and the predicted state gives a metric for event-triggering. The research in [20] uses the squared difference between the current and the latest triggered observation as a rule for triggering transmission of sensor observations to estimators. As a result, communication resources are saved to achieve an efficient information transmission.

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