Applications

Goran Banjac

Large-Scale Convex Optimization
ETH Zurich

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Optimal control

- consider the state-space representation of a discrete-time dynamical system

$$x_{k+1} = f(x_k, u_k)$$

- we want to drive the system to some desired state x_{ref} while respecting constraints on inputs u_k and states x_k
- we can compute the sequence of inputs u_k that solve the following optimal control problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=0}^{N-1} l_k(x_k - x_{\mathsf{ref}}, u_k) + l_N(x_N - x_{\mathsf{ref}}) \\ \text{subject to} & x_0 = x_{\mathsf{init}} \\ & x_{k+1} = f(x_k, u_k), \quad k = 0, \ldots, N-1 \\ & x_k \in \mathcal{X},\ u_k \in \mathcal{U}, \quad\ \ k = 0, \ldots, N-1 \\ & x_N \in \mathcal{X}_N \end{array}$$

- warm starting typically improves computation times considerably

Optimal spacecraft landing

https://youtu.be/2t15vP1PyoA

Optimal spacecraft landing

- we want to optimize the thrust profile of a spacecraft to carry out a landing at a target position
- the spacecraft dynamics are

$$m\ddot{p} = f - mge_3$$

  where m is its mass, $p(t) \in \mathbb{R}^3$ the position, with 0 the target landing position, $f(t) \in \mathbb{R}^3$ the thrust force
- the thrust force is constrained by $\|f(t)\|_2 \leq F^{\mathsf{max}}$
- the spacecraft must remain in a region given by the glide slope constraint

$$p_3(t) \geq \alpha \|(p_1(t), p_2(t))\|_2$$

- the fuel use rate is proportional to the thrust force magnitude, so the total fuel use is

$$\int_0^{T^{\mathsf{td}}} \gamma \, \|f(t)\|_2 \, dt$$

Time discretization

- we discretize the thrust force in time, *i.e.*, it is constant over time period of length $h > 0$, where $T^{\text{td}} = Kh$
- the spacecraft dynamics then take the following form

$$\begin{aligned} v_{k+1} &= v_k + (h/m) f_k - hge_3 \\ p_{k+1} &= p_k + (h/2)(v_k + v_{k+1}) \end{aligned}$$

- for simplicity, we will impose the glide slope constraint only at the times $t = 0, h, 2h, \ldots, Kh$
- the total fuel use is then

$$\sum_{k=1}^{K} \gamma\, h \|f_k\|_2$$

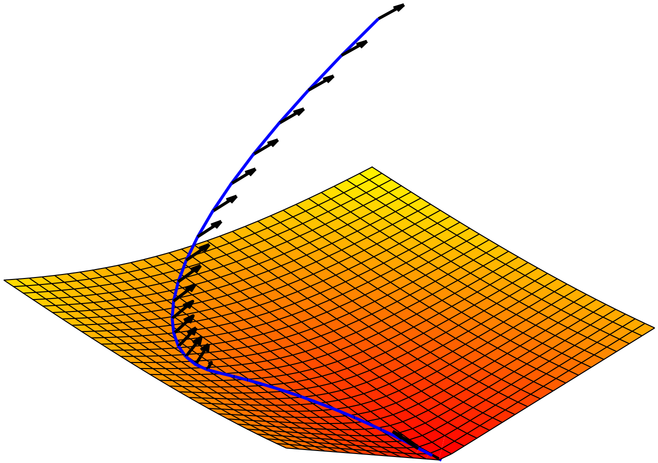
Minimum fuel descent

- we minimize fuel consumption, given the touchdown time $T^{\mathsf{td}} = Kh$

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{K} \|f_k\|_2 \\ \text{subject to} & v_{k+1} = v_k + (h/m) f_k - hge_3 \\ & p_{k+1} = p_k + (h/2)(v_k + v_{k+1}) \\ & \|f_k\|_2 \leq F^{\mathsf{max}}, \quad (p_k)_3 \geq \|((p_k)_1, (p_k)_2)\|_2 \\ & p_{K+1} = 0,\ v_{K+1} = 0,\ p_1 = p(0),\ v_1 = \dot{p}(0) \end{array}$$

- this is a convex optimization problem

Minimum fuel descent



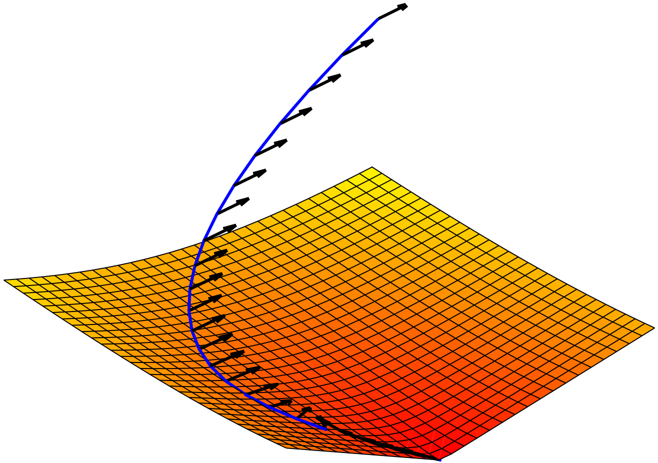
Minimum time descent

- to find the thrust profile that minimizes the touchdown time, we can solve a sequence of feasibility problems, *i.e.*, for each K we solve

$$
\begin{array}{ll}
\text{minimize} & 0 \\
\text{subject to} & v_{k+1} = v_k + (h/m) f_k - hge_3 \\
& p_{k+1} = p_k + (h/2)(v_k + v_{k+1}) \\
& \|f_k\|_2 \leq F^{\mathsf{max}}, \quad (p_k)_3 \geq \|((p_k)_1, (p_k)_2)\|_2 \\
& p_{K+1} = 0,\ v_{K+1} = 0,\ p_1 = p(0),\ v_1 = \dot{p}(0)
\end{array}
$$

- if the problem is feasible, we reduce K, otherwise we increase K

Minimum time descent



Portfolio allocation

- invest fraction w_i in asset i, $i \in 1, \ldots, n$
- *portfolio allocation vector* $w \in \mathbb{R}^n$ satisfies $\mathbf{1}^T w = 1$
- initial prices $p_i > 0$; end of period prices $p_i^+ > 0$
- asset (fractional) returns $r_i = (p_i^+ - p_i)/p_i$
- portfolio (fractional) return $R = r^T w$
- common model: r is a random variable with mean $\mathbb{E}r = \mu$ and covariance $\mathbb{E}(r-\mu)(r-\mu)^T = \Sigma$
- therefore, R is a random variable with $\mathbb{E}R = \mu^T w$ and $\mathrm{var}\, R = w^T \Sigma w$
- $\mathbb{E}R$ is (mean) *return* of portfolio
- $\mathrm{var}\, R$ is *risk* of portfolio

- two objectives: high return, low risk

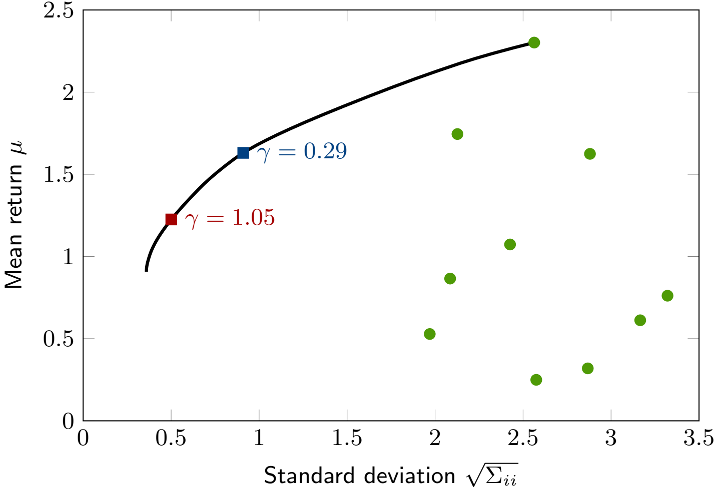
Markowitz portfolio optimization

$$\begin{array}{ll} \text{maximize} & \mu^T w - \gamma w^T \Sigma w \\ \text{subject to} & \mathbf{1}^T w = 1, \quad w \in \mathcal{W} \end{array}$$

- $\mathcal{W}$ is set of allowed portfolios; common case: $\mathcal{W} = \mathbb{R}^n_+$
- $\gamma > 0$ is the *risk aversion parameter*
- $\mu^T w - \gamma w^T \Sigma w$ is *risk-adjusted return*
- varying γ gives optimal *risk-return trade-off*
- can also fix return and minimize risk, etc.

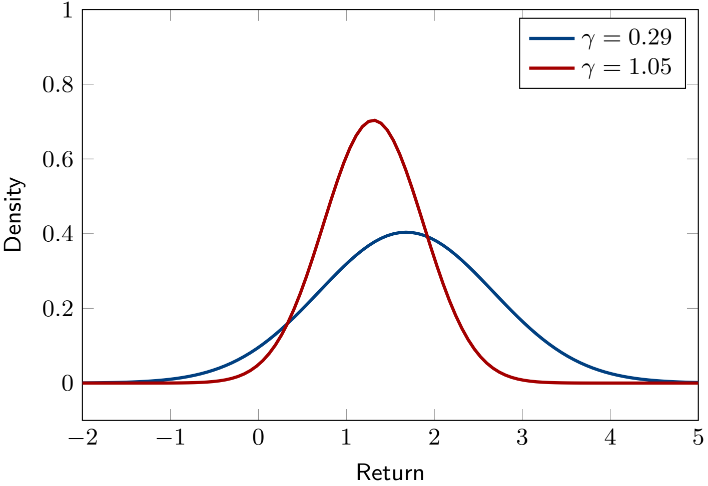
Numerical example

- optimal risk-return trade-off for 10 assets



Numerical example

- return distributions for two risk aversion values



Matrix completion

- image inpainting, recommender systems, optimal advertising