

# COMPANIONS & CONJOINTS ARE (CO)LIMITS

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# MOTIVATION & OVERVIEW

01

A **double category**  $ID$  consists of:

- objects  $A, B, C, \dots$
  - tight morphisms  $A \rightarrow B$
  - loose morphisms  $A \rightrightarrows B$
  - cells/squares
- $\left. \begin{array}{l} \text{objects } A, B, C, \dots \\ \text{tight morphisms } A \rightarrow B \end{array} \right\} ID_0$   
 $\left. \begin{array}{l} \text{loose morphisms } A \rightrightarrows B \\ \text{cells/squares} \end{array} \right\} ID_1$

$$\begin{array}{ccc}
 A & \xrightarrow{p} & B \\
 f \downarrow & \alpha & \downarrow g \\
 C & \xrightarrow{q} & D
 \end{array}$$

A double category is a **pseudo category object** in  $CAT$ :

$$ID_0 \begin{array}{c} \xleftarrow{\text{dom}} \\ \xrightarrow{\text{id}} \\ \xleftarrow{\text{cod}} \end{array} ID_1 \xleftarrow{\circ} ID_1 \times_{ID_0} ID_1$$

**Companions** and **conjoins** determine loose morphisms from tight morphisms.

$$f: A \rightarrow B \quad \rightsquigarrow \quad \begin{array}{l} f_*: A \rightrightarrows B \\ f^*: B \rightrightarrows A \end{array}$$

Many double categories admit all companions and conjoins; these might be called:

- fibrant double categories
- framed bicategories
- (proarrow) equipments

**GOAL:** Show companions/conjoins are (co)limits!

# ANALOGY: SPLITTING OF IDEMPOTENTS

02

The **splitting** of an idempotent  $e:A \rightarrow A$  is an object  $B$  and morphisms  $r:A \rightarrow B$  and  $s:B \rightarrow A$  such that  $rs=1_B$  and  $sr=e$ .

Consider an idempotent  $e:A \rightarrow A$  and a diagram

$$\begin{array}{ccc} & B & \\ s \swarrow & \curvearrowright & \searrow s \\ A & \xrightarrow{e} & A \end{array} \quad (1)$$

such that pasting with (1) gives a bijection:

$$\begin{array}{ccc} & X & \\ f \swarrow & \curvearrowright & \searrow f \\ A & \xrightarrow{e} & A \end{array} \quad \rightsquigarrow \quad \begin{array}{c} X \\ \downarrow g \\ B \end{array}$$

Consider an idempotent  $e:A \rightarrow A$  and a diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & A \\ r \searrow & \curvearrowright & \swarrow r \\ & B & \end{array} \quad (2)$$

such that pasting with (2) gives a bijection:

$$\begin{array}{ccc} A & \xrightarrow{e} & A \\ h \searrow & \curvearrowright & \swarrow h \\ & Y & \end{array} \quad \rightsquigarrow \quad \begin{array}{c} B \\ \downarrow k \\ Y \end{array}$$

**Exercise:** Show that an idempotent  $e:A \rightarrow A$  splits iff there is a diagram (1) with the above U.P. iff there is a diagram (2) with the above U.P.

# COMPANIONS & CONJOINTS VIA STRUCTURE

03

A **companion** of a tight morphism  $f:A \rightarrow B$  is a loose morphism  $f_*:A \rightarrowtail B$  and cells

$$\begin{array}{ccc} A & \xrightarrow{f_*} & B \\ f \downarrow & \rho & \downarrow 1_B \\ B & \xrightarrow{id_B} & B \end{array} \quad \begin{array}{ccc} A & \xrightarrow{id_A} & A \\ 1_A \downarrow & \sigma & \downarrow f \\ A & \xrightarrow{f_*} & B \end{array}$$

such that the following equations hold.

$$\begin{array}{|c|} \hline \rho \\ \hline \sigma \\ \hline \end{array} = id_f \quad \begin{array}{|c|c|} \hline \sigma & \rho \\ \hline \end{array} = 1_{f_*}$$

A **conjoint** of a tight morphism  $f:A \rightarrow B$  is a loose morphism  $f^*:B \rightarrowtail A$  and cells

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ f \downarrow & \varepsilon & \downarrow 1_A \\ B & \xrightarrow{f^*} & A \end{array} \quad \begin{array}{ccc} B & \xrightarrow{f^*} & A \\ 1_B \downarrow & \eta & \downarrow f \\ B & \xrightarrow{id_B} & B \end{array}$$

such that the following equations hold.

$$\begin{array}{|c|} \hline \varepsilon \\ \hline \eta \\ \hline \end{array} = id_f \quad \begin{array}{|c|c|} \hline \eta & \varepsilon \\ \hline \end{array} = 1_{f^*}$$

# EXAMPLES OF CONJOINTS

04

- In  $\mathcal{S}pan(\mathcal{E})$ , the conjoint of  $f:A \rightarrow B$  is the span  $B \xleftarrow{f} A \xrightarrow{1_A} A$  with cells:

$$\begin{array}{ccc} A & \xleftarrow{1_A} A & \xrightarrow{1_A} A \\ f \downarrow & \downarrow 1_A & \downarrow 1_A \\ B & \xleftarrow{f} A & \xrightarrow{1_A} A \end{array} \quad \begin{array}{ccc} B & \xleftarrow{f} A & \xrightarrow{1_A} A \\ 1_B \downarrow & \downarrow f & \downarrow f \\ B & \xleftarrow{1_B} B & \xrightarrow{1_B} B \end{array}$$

- In  $\mathcal{I}Dist$ , the conjoint of a functor  $f:A \rightarrow B$  is  $B(-, f-): B^{op} \times A \rightarrow \mathcal{S}et$  with cells:

$$\begin{array}{ccc} B^{op} \times A & \xrightarrow{B(-, f-)} & \mathcal{S}et \\ 1 \times f \downarrow & \curvearrowright & \\ B^{op} \times B & \xrightarrow{B(-, -)} & \mathcal{S}et \end{array} \quad \begin{array}{ccc} A^{op} \times A & \xrightarrow{A(-, -)} & \mathcal{S}et \\ f \times 1 \downarrow & \Downarrow & \\ B^{op} \times A & \xrightarrow{B(-, f-)} & \mathcal{S}et \end{array} \quad \begin{array}{c} A(a, a') \\ \downarrow f_{a, a'} \\ B(f_a, f_{a'}) \end{array}$$

- In  $\mathcal{I}Ring$ , the conjoint of  $f:R \rightarrow S$  is the  $(S, R)$ -bimodule  $S$  with left action  $s \triangleright s' = s s'$  and right action  $s \triangleleft r = s \cdot f(r)$  and bimodule maps  $1_S: S \rightarrow S$  to the trivial  $(S, S)$ -bimodule and  $f: R \rightarrow S$  from the trivial  $(R, R)$ -bimodule.

- In  $\mathcal{Q}(\mathcal{K})$ , the conjoint of  $r:A \rightarrow B$  is a 1-cell  $\ell: B \rightarrow A$  and 2-cells:

$$\begin{array}{ccc} A & \xrightarrow{1_A} A \\ r \downarrow & \nearrow \varepsilon & \downarrow 1_A \\ B & \xrightarrow{\ell} A \end{array} \quad \begin{array}{ccc} B & \xrightarrow{\ell} A \\ 1_B \downarrow & \nearrow \eta & \downarrow r \\ B & \xrightarrow{1_B} B \end{array}$$

# CONJOINTS VIA UNIVERSAL PROPERTY

05

Consider a tight morphism  $f:A \rightarrow B$  and a cell

$$\begin{array}{ccc} B & \xrightarrow{f^*} & A \\ 1_B \downarrow & \eta & \downarrow f \\ B & \xrightarrow{id_B} & B \end{array}$$

such that composing with  $\eta$  gives a bijection:

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ g \downarrow & \alpha & \downarrow fh \\ B & \xrightarrow{id_B} & B \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} X & \xrightarrow{p} & Y \\ g \downarrow & \bar{\alpha} & \downarrow h \\ B & \xrightarrow{f^*} & A \end{array}$$

**Exercise:** Show that  $f:A \rightarrow B$  admits a conjoint iff there exists  $\eta$  with universal property above.

Consider a tight morphism  $f:A \rightarrow B$  and a cell

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ f \downarrow & \varepsilon & \downarrow 1_A \\ B & \xrightarrow{f^*} & A \end{array}$$

such that composing with  $\varepsilon$  gives a bijection:

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ kf \downarrow & \beta & \downarrow j \\ X & \xrightarrow{p} & Y \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} B & \xrightarrow{f^*} & A \\ k \downarrow & \bar{\beta} & \downarrow j \\ X & \xrightarrow{p} & Y \end{array}$$

**Exercise:** Show that  $f:A \rightarrow B$  admits a conjoint iff there exists  $\varepsilon$  with universal property above.

# SHAPES, DIAGRAMS, & CONES

06

shape  $\leadsto$  span  $S \xleftarrow{s} J \xrightarrow{t} T$  in  $\mathbf{Cat}$

$$p \in J \iff p: A \rightrightarrows B \text{ s.t. } s(p)=A, t(p)=B$$

diagram  $\leadsto$  commutative diagram

$$\begin{array}{ccccc}
 S & \xleftarrow{s} & J & \xrightarrow{t} & T \\
 F \downarrow & & \downarrow H & & \downarrow G \\
 \mathbb{D}_0 & \xleftarrow{\text{dom}} & \mathbb{D}_1 & \xrightarrow{\text{cod}} & \mathbb{D}_0
 \end{array} \quad (*)$$

double category  $\mathbb{D}$

This determines a functorial assignment:

$$\begin{array}{ccc}
 A \xrightarrow{p} B & & FA \xrightarrow{Hp} GB \\
 f \downarrow \quad \alpha \quad \downarrow g & \in J \longmapsto & Ff \downarrow \quad H\alpha \quad \downarrow Gg \in \mathbb{D} \\
 A' \xrightarrow{p'} B' & & FA' \xrightarrow{Hp'} GB'
 \end{array}$$

cone over  $(F, H, G) \leadsto$  loose morphism  $q: X \rightrightarrows Y$  and

$$\begin{array}{ccccc}
 S & \xleftarrow{s} & J & \xrightarrow{t} & T \\
 \swarrow \varphi & \downarrow F & \swarrow \theta & \downarrow H & \swarrow \psi & \downarrow G \\
 1 & \xRightarrow{x} & \mathbb{D}_0 & \xleftarrow{\text{dom}} & \mathbb{D}_1 & \xrightarrow{\text{cod}} & \mathbb{D}_0 \\
 & & \swarrow q & & \searrow y
 \end{array}$$

$$\begin{aligned}
 \varphi \cdot s &= \text{dom} \cdot \theta \\
 \psi \cdot t &= \text{cod} \cdot \theta
 \end{aligned}$$

- a tight morphism  $\varphi_A: X \rightarrow FA$  for each  $A \in S$ ;
- a tight morphism  $\psi_B: Y \rightarrow GB$  for each  $B \in T$ ;
- for each  $p: A \rightrightarrows B$  in  $J$ , a cell  $\theta_p$  as below;

$$\begin{array}{ccc}
 X & \xrightarrow{q} & Y \\
 \varphi_A \downarrow & \theta_p & \downarrow \psi_B \\
 FA & \xrightarrow{Hp} & GB
 \end{array}$$

•  $\varphi, \psi, \theta$  are natural with respect to morphisms in  $S, T, J$ .

# LIMITS INDEXED BY SPANS OF FUNCTORS

07

A **limit** of a diagram

$$\begin{array}{ccccc}
 S & \xleftarrow{s} & J & \xrightarrow{t} & T \\
 F \downarrow & & \downarrow H & & \downarrow G \\
 \mathbb{D}_0 & \xleftarrow{\text{dom}} & \mathbb{D}_1 & \xrightarrow{\text{cod}} & \mathbb{D}_0
 \end{array}$$

in a double category  $\mathbb{D}$  is a loose morphism

$\lim: \lim F \rightarrow \lim G$ , where  $(\lim F, \varphi)$  is a limit

of  $F$  and  $(\lim G, \psi)$  is a limit of  $G$ , and a natural

$$\begin{array}{ccc}
 \lim F & \xrightarrow{\lim} & \lim G \\
 \varphi_A \downarrow & \Theta_p & \downarrow \psi_B \\
 FA & \xrightarrow{H_p} & GB
 \end{array}$$

family of cells  $\Theta_p$   
indexed by  $p \in J$ ,  
with the following U.P.:

Composing with  $\Theta_p$  gives a bijection between  
cones  $(\varphi', \Theta', \psi')$  w/ apex  $q: X \rightarrow Y$  over  $(F, H, G)$   
and cells from  $q: X \rightarrow Y$  to  $\lim: \lim F \rightarrow \lim G$ .

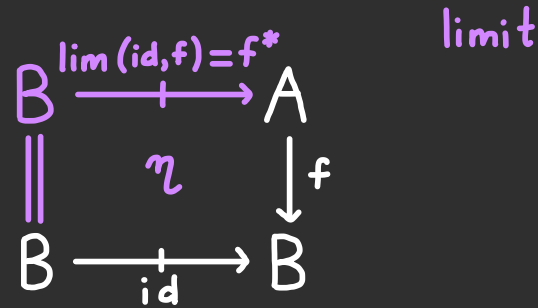
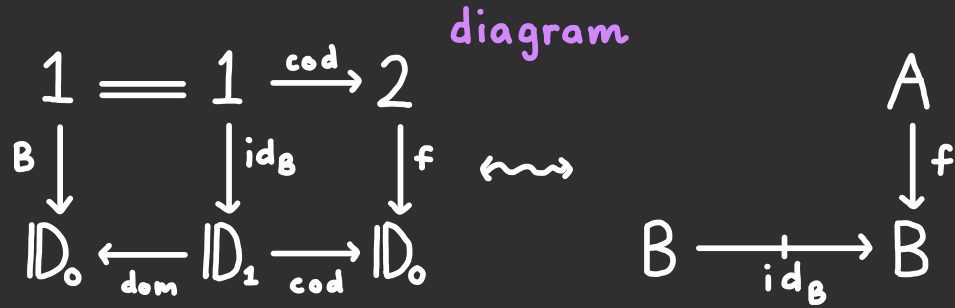
$$\begin{array}{ccc}
 X & \xrightarrow{q} & Y \\
 \varphi'_A \downarrow & \Theta'_p & \downarrow \psi'_B \\
 FA & \xrightarrow{H_p} & GB
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{ccc}
 X & \xrightarrow{q} & Y \\
 f \downarrow & \alpha & \downarrow g \\
 \lim F & \xrightarrow{\lim} & \lim G
 \end{array}$$

**Summary:** A limit in  $\mathbb{D}$  indexed by  $S \xleftarrow{s} J \xrightarrow{t} T$   
is a terminal cone over a diagram  $(F, H, G)$   
in the 2-category  $[\{ \cdot \leftarrow \cdot \rightarrow \cdot \}, \text{CAT}]$  that is  
preserved by left, right:  $[\{ \cdot \leftarrow \cdot \rightarrow \cdot \}, \text{CAT}] \rightrightarrows \text{CAT}$ .

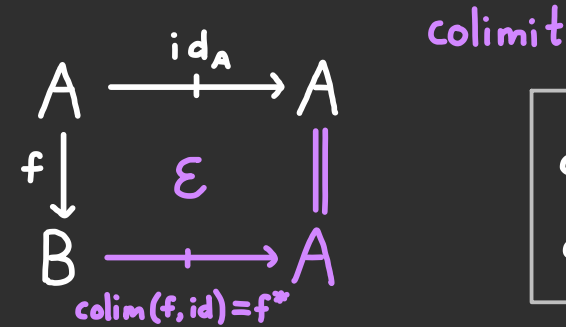
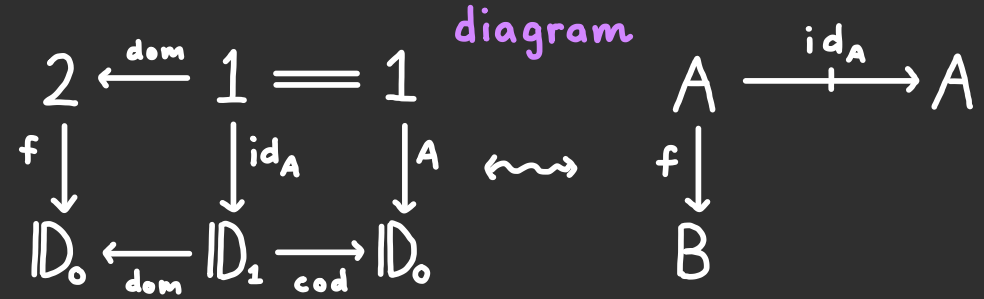
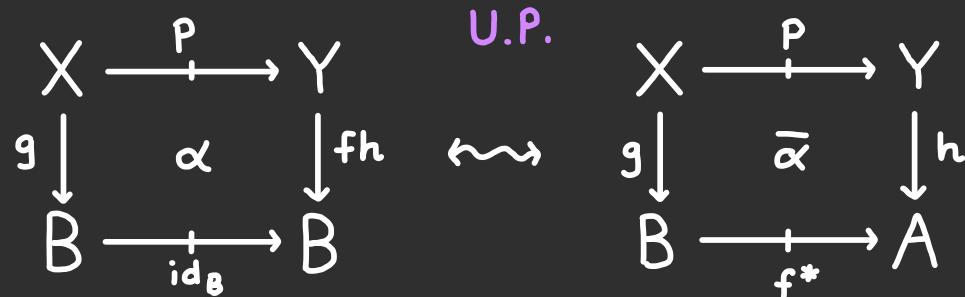


# CONJOINTS AS LIMITS & COLIMITS

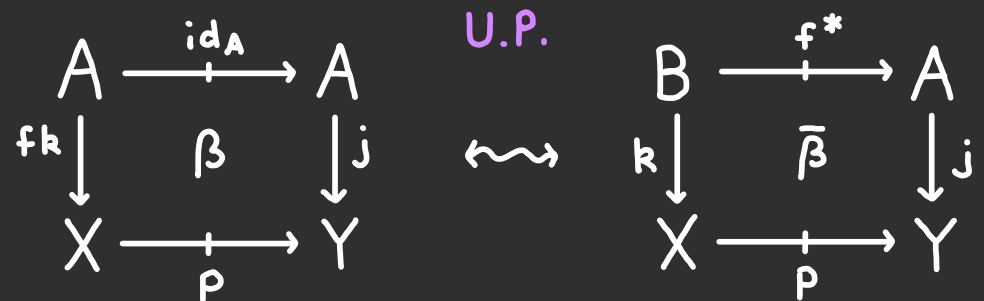
08



$$\begin{aligned}
 \lim(B: 1 \rightarrow ID_0) &= B \\
 \lim(f: 2 \rightarrow ID_0) &= A
 \end{aligned}$$



$$\begin{aligned}
 \text{colim}(A: 1 \rightarrow ID_0) &= A \\
 \text{colim}(f: 2 \rightarrow ID_0) &= B
 \end{aligned}$$



# PARALLEL LIMITS & MAIN THEOREM

09

$$\begin{array}{ccccc}
 J & \xlongequal{\quad} & J & \xlongequal{\quad} & J \\
 \downarrow & & \downarrow H & & \downarrow \\
 ID_0 & \xleftarrow{\text{dom}} & ID_1 & \xrightarrow{\text{cod}} & ID_0
 \end{array}$$

← shape for a parallel limit

• A **parallel (co)limit** is precisely a limit in  $ID_1$  that is preserved by  $\text{dom}, \text{cod}: ID_1 \rightrightarrows ID_0$ .

• E.g. a **parallel product** of  $p: A \rightrightarrows B$  and  $q: C \rightrightarrows D$  is a loose morphism  $p \times q$  and cells

$$\begin{array}{ccc}
 A \times C & \xrightarrow{p \times q} & B \times D \\
 \pi_A \downarrow & \pi_p & \downarrow \pi_B \\
 A & \xrightarrow{p} & B
 \end{array}
 \quad
 \begin{array}{ccc}
 A \times C & \xrightarrow{p \times q} & B \times D \\
 \pi_C \downarrow & \pi_q & \downarrow \pi_D \\
 C & \xrightarrow{q} & D
 \end{array}$$

with suitable U.P..

**Theorem:** A double category  $ID$  admits all (co)limits indexed by spans of functors iff  $ID$  admits:

- companions
  - conjoints
  - parallel (co)products
  - parallel (co)equalisers
- $\left. \begin{array}{l} \text{comparisons} \\ \text{conjoint restrictions} \end{array} \right\} \text{(co)restrictions}$   
 $\left. \begin{array}{l} \text{parallel (co)products} \\ \text{parallel (co)equalisers} \end{array} \right\} \text{parallel (co)limits}$

iff

- $\langle \text{dom}, \text{cod} \rangle: ID_1 \longrightarrow ID_0 \times ID_0$  is a bifibration.
- $ID_1$  admits all (co)limits and these are preserved by  $\text{dom}, \text{cod}: ID_1 \rightrightarrows ID_0$ .

# SUMMARY & FURTHER WORK

10

- We introduced (co)limits in double categories indexed by spans of functors, and showed companions and conjoiners are (co)limits.
- Companions and conjoiners are preserved by any normal (co)lax double functor — they are absolute (co)limits.
- **Conjecture:** A double category admits all absolute colimits iff it has companions, conjoiners, and (parallel) splitting of idempotents.

This talk presented a small part of a larger story on limits in double categories.

spans in CAT  $S \longleftarrow J \longrightarrow T$

⎵ generalise shapes

loose distributors between double cats.  $S \xrightarrow[\text{#}]{J} T$

⚡  
span  $S_0 \longleftarrow J \longrightarrow T_0$  with compatible left action of  $S_1 \rightrightarrows S_0$  and right action of  $T_1 \rightrightarrows T_0$