THE GROTHENDIECK CONSTRUCTION FOR DELTA LENSES

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WHAT IS A LENS?

- Lenses are an abstraction of product projections (in Set / Cat)

$$\begin{array}{ccc} B \times A & (b,a) \xrightarrow{(u,1_a)} (b',a) \\ \big\downarrow & \\ B & b \xrightarrow{u} b' \end{array}$$

"a lens focuses on a view of a system"

- Has forwards/backwards components (lifting: backwards)
- Model bidirectional transformations

- Lenses are an abstraction of coproducts (add together systems).

$$B \times A \simeq \sum_{b \in B} A \longrightarrow B$$

"lifting is reindexing"

- How do we adapt this to delta lenses?
  - Index by a category ...
  - ... a collection of objects (sets?)
  - Reindex along what kind of morphism?
  - How strict is reindexing?

GROTHENDIECK CONSTRUCTION(S)

Fibred vs. indexed perspectives:

Discrete opfibrations

$$\mathrm{DOpf}(B) \simeq [B, \mathrm{Set}]$$

Split opfibrations

$$\mathrm{SOpf}(B) \simeq [B, \mathrm{Cat}]$$

Functors

$$\mathrm{Cat}/B \simeq [\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{pan}]_{lax}$$

Many variations of interest in ACT:

- Monoidal Grothendieck Construction – Moeller & Vasilakopoulou
- Double categories of Open Dynamical Systems – Myers
- Structured and decorated cospans from the viewpoint of double category theory – Patterson
- Double fibrations – Cruttwell, Lambert, Pronk, & Szyld

OVERVIEW OF THE TALK

MAIN RESULT: Grothendieck construction for delta lenses

$$\boxed{\mathcal{L}\mathrm{ens}(B) \simeq [\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{Mult}]_{\mathrm{lax}}}$$

$$\begin{tikzcd} A \arrow[d, "{(f,\varphi)}"'] \\ B \end{tikzcd} \overset{①}{\simeq} \begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f\varphi"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd} \overset{②}{\simeq} \begin{tikzcd} & \mathbb{L}\mathrm{o}(B) \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\ \mathbb{S}\mathrm{q}(\mathrm{Set}) \arrow[rr] \arrow[rr, phantom, shift left=4, "\overset{\varphi}{\Rightarrow}"] & & \mathbb{S}\mathrm{pan} \end{tikzcd} \overset{③}{\simeq} \begin{tikzcd} \mathbb{L}\mathrm{o}(B) \arrow[d, "F"] \\ \mathbb{S}\mathrm{Mult} \end{tikzcd}$$

④ Examples & concluding remarks

Part 1

$$
\begin{tikzcd}
A \arrow[d, "{(f,\varphi)}"'] \\
B
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "f\varphi"'] & & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}
$$

"Delta lenses are equivalent to certain commutative diagrams in $\mathcal{C}at$"

DELTA LENSES

A **delta lens** (f,φ) is a functor $f\colon A \to B$ equipped with a lifting operation φ

$$
\begin{tikzcd}
A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"] & a' \\
B & fa \arrow[r, "u"] & b
\end{tikzcd}
$$

that satisfies three axioms:

1. $f\varphi(a,u) = u$
2. $\varphi(a, 1_{fa}) = 1_a$
3. $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a,u)$

For a category B, let $\mathcal{L}ens(B)$ be the category whose:

- objects are delta lenses into B;
- morphisms are functors

$$
\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "(f,\varphi)"'] & & C \arrow[dl, "(g,\psi)"] \\
& B &
\end{tikzcd}
$$

such that $gh = f$ and $\psi(ha,u) = h\varphi(a,u)$.

"functors which preserve chosen lifts"

BASIC EXAMPLES

- State-based lenses are delta lenses between codiscrete categories.

  $f : A \to B \qquad p : A \times B \to A$

- Discrete opfibrations are delta lenses such that $\varphi(a, fw) = w$.

- Split opfibrations are delta lenses such that the chosen lifts $\varphi(a, u)$ are opcartesian.

- Bijective-on-objects functors with a chosen section.

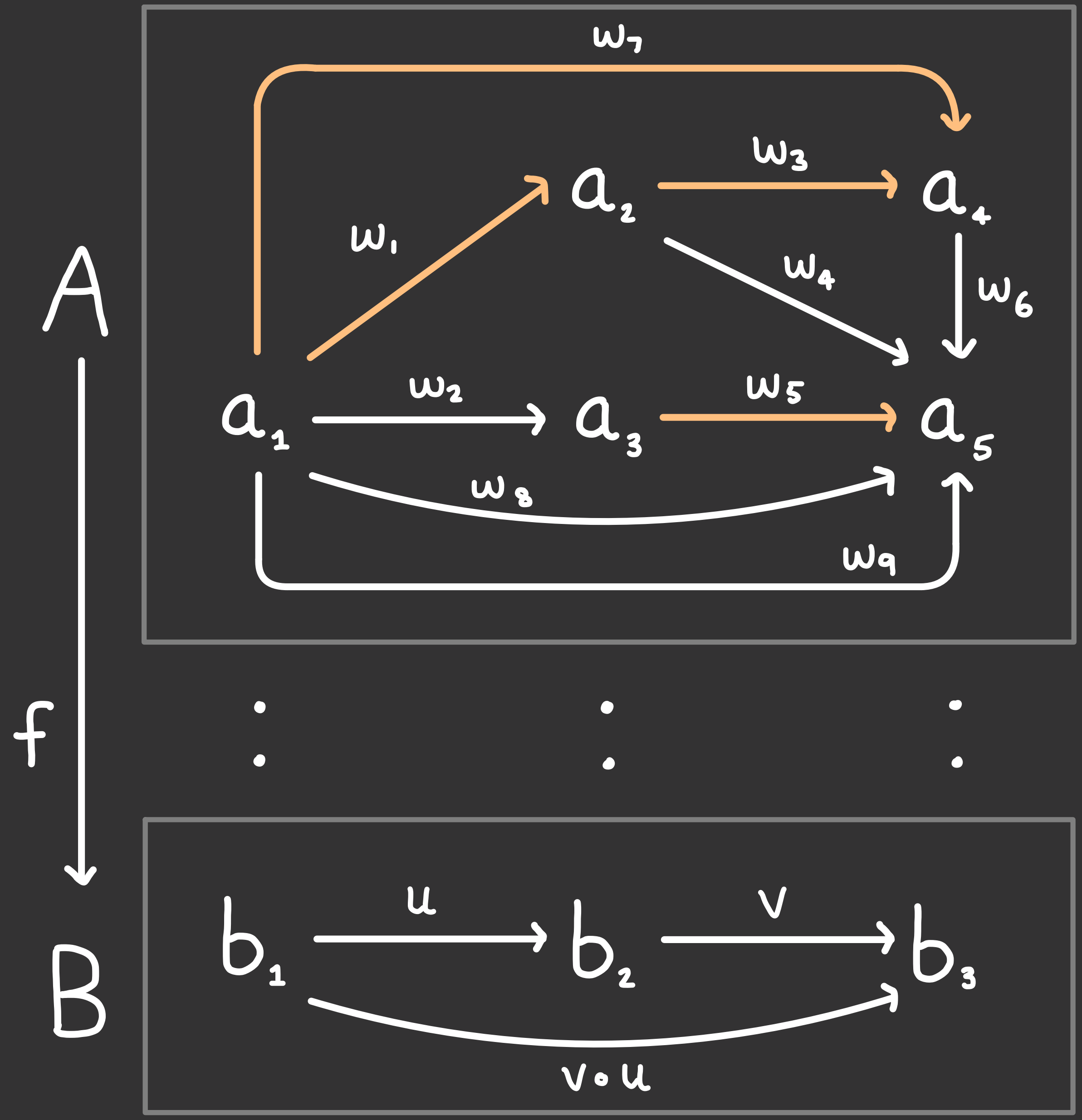
- Each functor induces a free delta lens via a monadic adjunction:

$$\mathcal{L}ens(B) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} Cat/B$$

- Each retrofunctor (i.e. cofunctor) induces a cofree delta lens:

$$\mathcal{L}ens(B) \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} Cat^{\#}(B)$$

RUNNING EXAMPLE



We require that:

$w_4 = w_6 \circ w_3 \qquad w_7 = w_3 \circ w_1 \qquad w_8 = w_5 \circ w_2$

Functor $f : A \longrightarrow B$ with

$fa_1 = b_1 \qquad fa_2 = fa_3 = b_2 \qquad fa_4 = fa_5 = b_3$

Lifting operation φ with:

$\varphi(a_1, u) = w_1 \qquad \varphi(a_2, v) = w_3$

$\varphi(a_3, v) = w_5 \qquad \varphi(a_1, v \circ u) = w_7$

DIAGRAMMATIC DELTA LENSES

A diagrammatic delta lens is a commutative diagram in $\mathcal{C}at$

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f\varphi"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

such that φ is bijective-on-objects and $f\varphi$ is a discrete opfibration.

These are objects in $\mathcal{D}ia\mathcal{L}ens(B)$.

$$\begin{tikzcd} X \arrow[r, "\bar{h}"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ A \arrow[r, "h"] \arrow[dr, "f"'] & C \arrow[d, "g"] \\ & B \end{tikzcd}$$

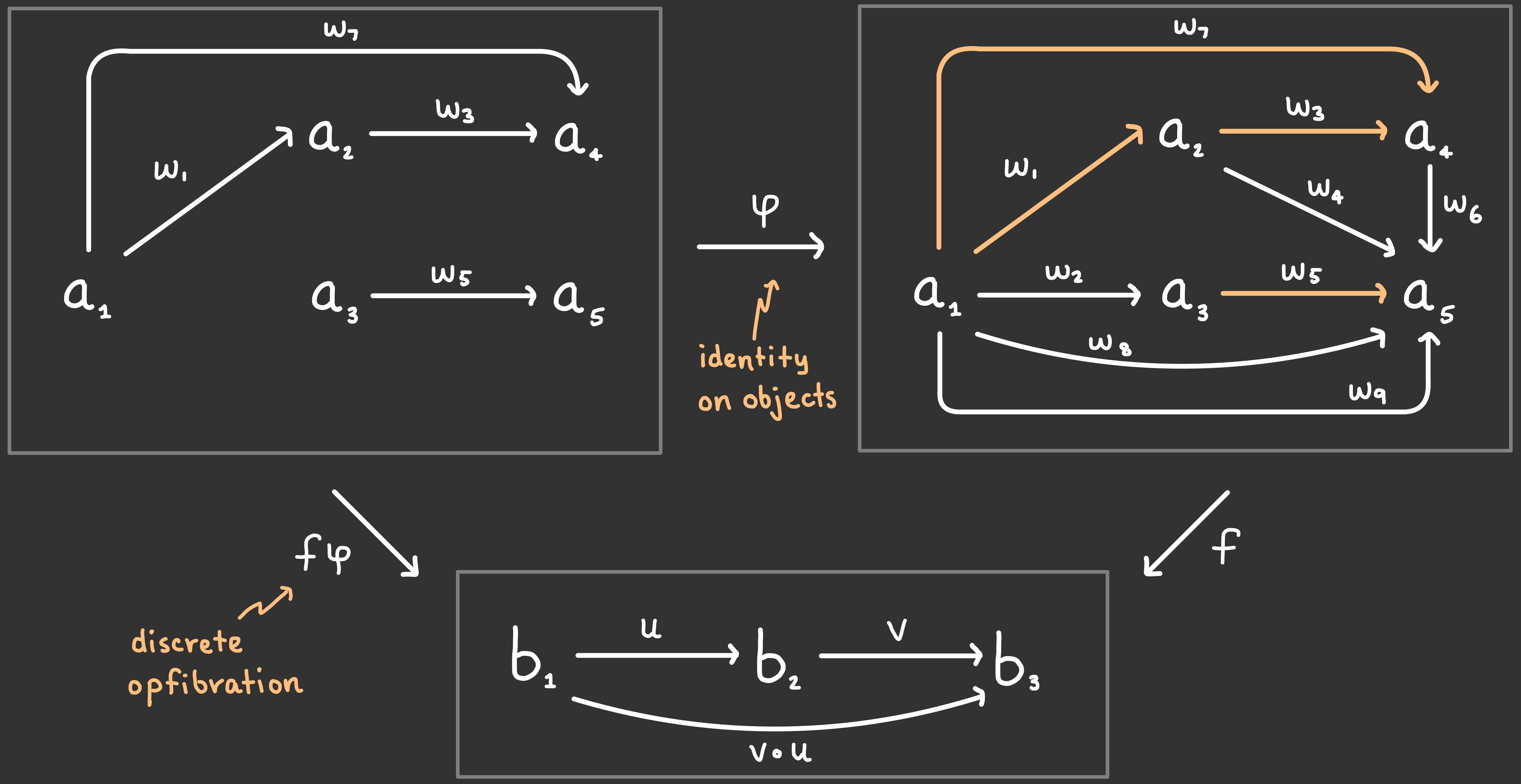
unique

Morphisms in $\mathcal{D}ia\mathcal{L}ens(B)$ are pairs* $(h, \bar{h})$ such that the diagram commutes.

$$\boxed{\mathcal{L}ens(B) \simeq \mathcal{D}ia\mathcal{L}ens(B)}$$

IDEA: Each delta lens $(f, \varphi) : A \longrightarrow B$ admits a wide subcategory of chosen lifts $\Lambda(f, \varphi) \rightarrowtail A$.

RUNNING EXAMPLE 2



Part 2

$$
\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "f\varphi"'] & & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}
\qquad \simeq \qquad
\begin{tikzcd}
& \mathbb{L}\mathrm{o}(B) \arrow[dl, "F_1"'] \arrow[dr, "F_2"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & \\
\mathbb{S}\mathrm{q}(\mathbf{Set}) \arrow[rr] & {} & \mathbb{S}\mathrm{pan}
\end{tikzcd}
$$

"Delta lenses are equivalent to certain transformations between certain lax double functors into $\mathbb{S}\mathrm{pan}$"

DOUBLE CATEGORIES

A double category $\mathbb{D}$ consists of:

- objects $A, B, C, D, \ldots$
- tight morphisms $\bullet \longrightarrow \bullet$
- loose morphisms $\bullet \overset{\shortmid}{\longrightarrow} \bullet$
- cells

$$\begin{tikzcd} A \arrow[r, "h", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "k"', "\shortmid"{marking}] & D \end{tikzcd}$$

that compose horizontally & vertically.

$\mathbb{S}\mathrm{pan}$ - sets, functions, spans

$$\begin{tikzcd} A \arrow[d, "f"'] & X \arrow[l, "h_1"'] \arrow[r, "h_2"] \arrow[d, "\alpha"] & C \arrow[d, "g"] \\ B & Y \arrow[l, "k_1"] \arrow[r, "k_2"'] & D \end{tikzcd}$$

For each category $\mathcal{C}$, we have $\mathbb{S}\mathrm{q}(\mathcal{C})$ whose cells are commuting squares in $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & C \arrow[d, equal] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

$\mathbb{L}\mathrm{o}(\mathcal{C})$ - restrict $\mathbb{S}\mathrm{q}(\mathcal{C})$ to identity tight mor.

LAX DOUBLE FUNCTORS & TIGHT TRANSFORMATIONS

A **lax double functor** $F : \mathbb{A} \to \mathbb{B}$ is given by

$$\begin{tikzcd} A \arrow[r, "u", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "f'"] \\ B \arrow[r, "v"', "|"] & B' \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} FA \arrow[r, "Fu", "|"] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FA' \arrow[d, "Ff'"] \\ FB \arrow[r, "Fv"', "|"] & FB' \end{tikzcd}$$

preserving tight direction strictly &

loose direction up to specified comparison cells:

$$\begin{tikzcd} FA \arrow[r, "\mathrm{id}_{FA}", "|"] \arrow[d, equal] \arrow[dr, phantom, "\eta_A"] & FA \arrow[d, equal] \\ FA \arrow[r, "F(\mathrm{id}_A)"', "|"] & FA \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "Fu", "|"] \arrow[d, equal] & FA' \arrow[r, "Fv", "|"] \arrow[d, phantom, "\mu_{u,v}"] & FA'' \arrow[d, equal] \\ FA \arrow[rr, "F(u \cdot v)"', "|"] & & FA'' \end{tikzcd}$$

A **tight transformation** $\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathbb{B} \end{tikzcd}$ is

$$A \overset{u}{\longrightarrow} A' \quad \rightsquigarrow \quad \begin{tikzcd} FA \arrow[r, "Fu", "|"] \arrow[d, "\tau_A"'] \arrow[dr, phantom, "\tau_u"] & FA' \arrow[d, "\tau_{A'}"] \\ GA \arrow[r, "Gu"', "|"] & GA' \end{tikzcd}$$

satisfying naturality & coherence conditions.

Globular if τ_A is identity for each object A.

Obtain a category $[\mathbb{A}, \mathbb{B}]_{lax}$ of lax double functors and tight transformations.

TWO FUNDAMENTAL RESULTS

$$\mathrm{DOpf}(B) \simeq [\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{q}(\mathrm{Set})]$$

$$\mathrm{Cat}/B \simeq [\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{pan}]_{\mathrm{lax}}$$

Thus each globular transformation

$$\begin{tikzcd} & \mathbb{L}\mathrm{o}(B) \arrow[dl, "F_1"'] \arrow[dr, "F_2"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & \\ \mathbb{S}\mathrm{q}(\mathrm{Set}) \arrow[rr, "\text{inclusion}"'] & & \mathbb{S}\mathrm{pan} \end{tikzcd}$$

is equivalent to a diagrammatic delta lens!

GlobCone$(B, \mathbb{S}\mathrm{q}(\mathrm{Set}) \to \mathbb{S}\mathrm{pan})$ has morphisms:

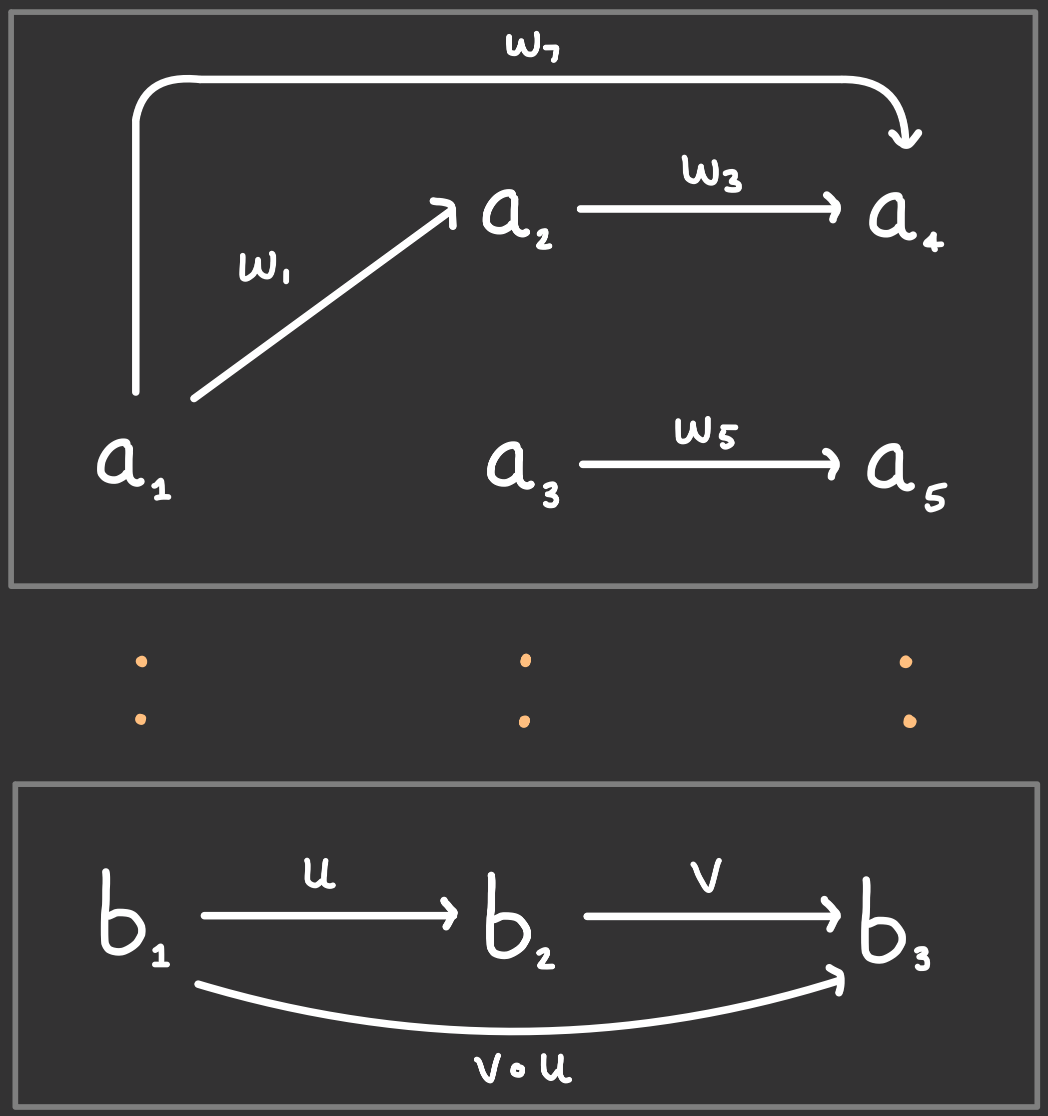
$$\begin{tikzcd} & \mathbb{L}\mathrm{o}(B) \arrow[dl, bend right=60, "F_1"'] \arrow[dl] \arrow[dr, "G_2"] \arrow[d, phantom, "\overset{\psi}{\Rightarrow}"] & \\ \mathbb{S}\mathrm{q}(\mathrm{Set}) \arrow[rr] & & \mathbb{S}\mathrm{pan} \end{tikzcd} = \begin{tikzcd} & \mathbb{L}\mathrm{o}(B) \arrow[dl, "F_1"'] \arrow[dr] \arrow[dr, bend left=60, "G_2"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & \\ \mathbb{S}\mathrm{q}(\mathrm{Set}) \arrow[rr] & & \mathbb{S}\mathrm{pan} \end{tikzcd}$$

We obtain an equivalence:

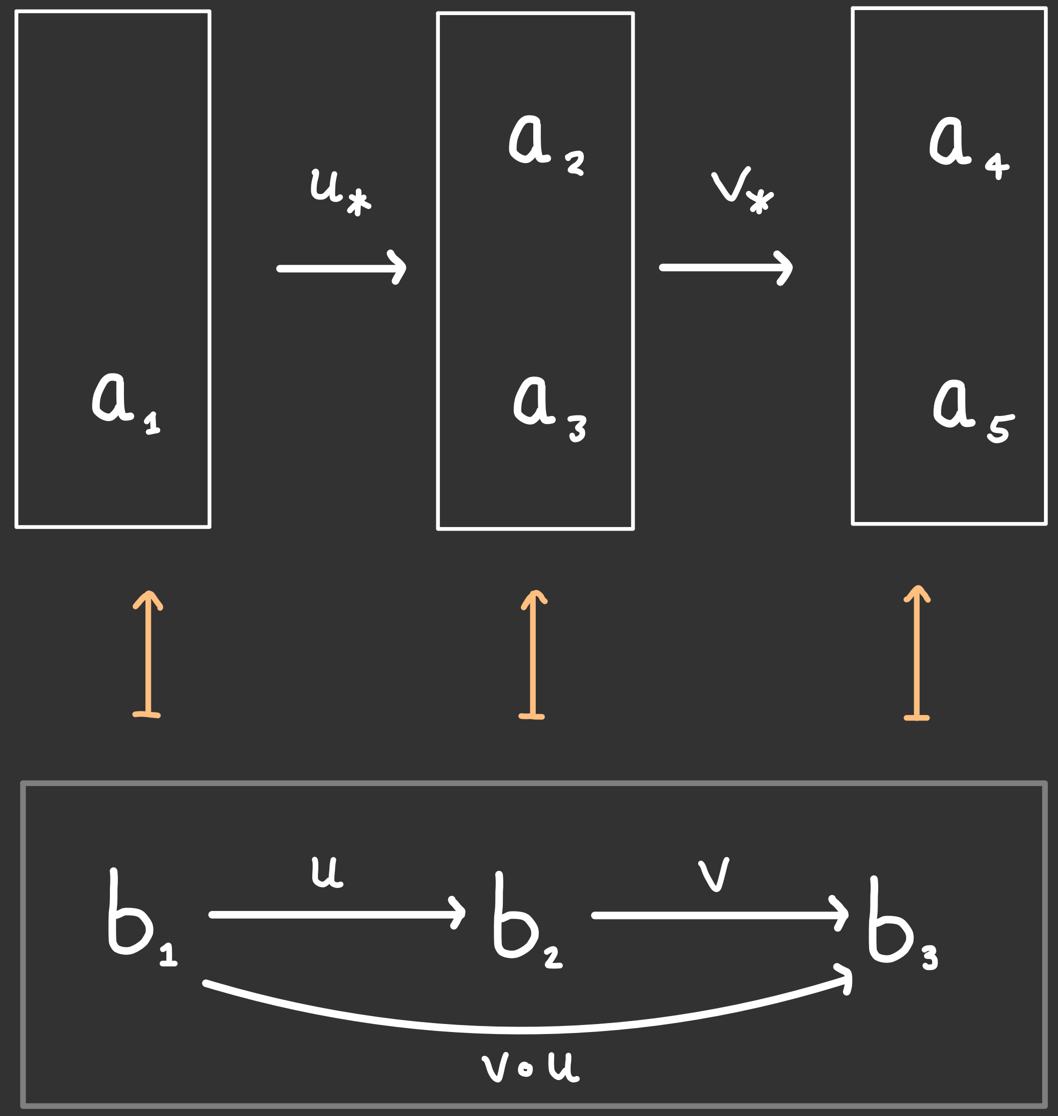
$$\boxed{\begin{gathered}\mathrm{DiaLens}(B) \simeq \\ \mathrm{GlobCone}(B, \mathbb{S}\mathrm{q}(\mathrm{Set}) \to \mathbb{S}\mathrm{pan})\end{gathered}}$$

RUNNING EXAMPLE 3.1

discrete opfibration $f': X \to B$

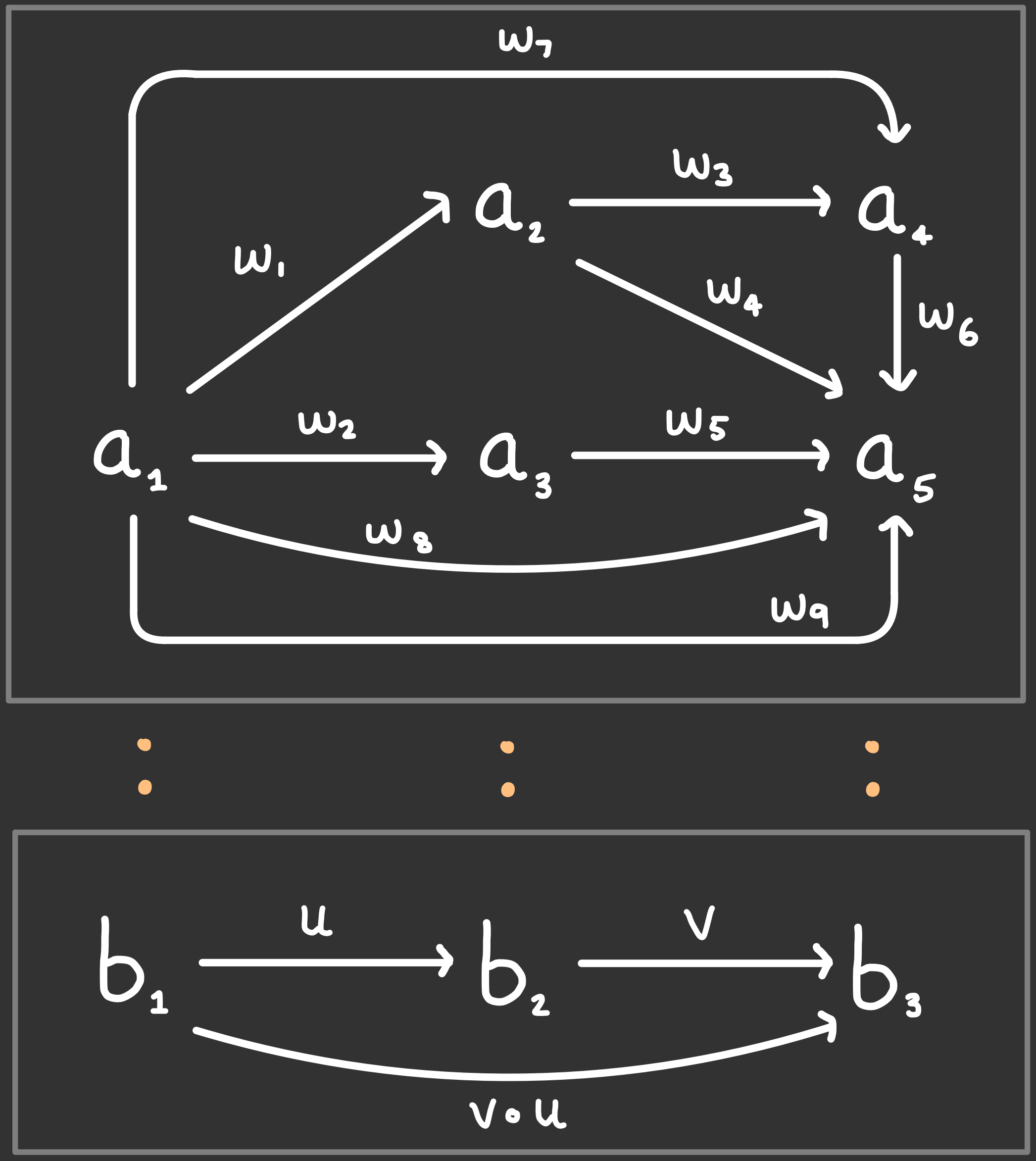


double functor $\mathbb{L}\mathrm{o}(B) \longrightarrow \mathbb{S}\mathrm{q}(\mathrm{Set})$

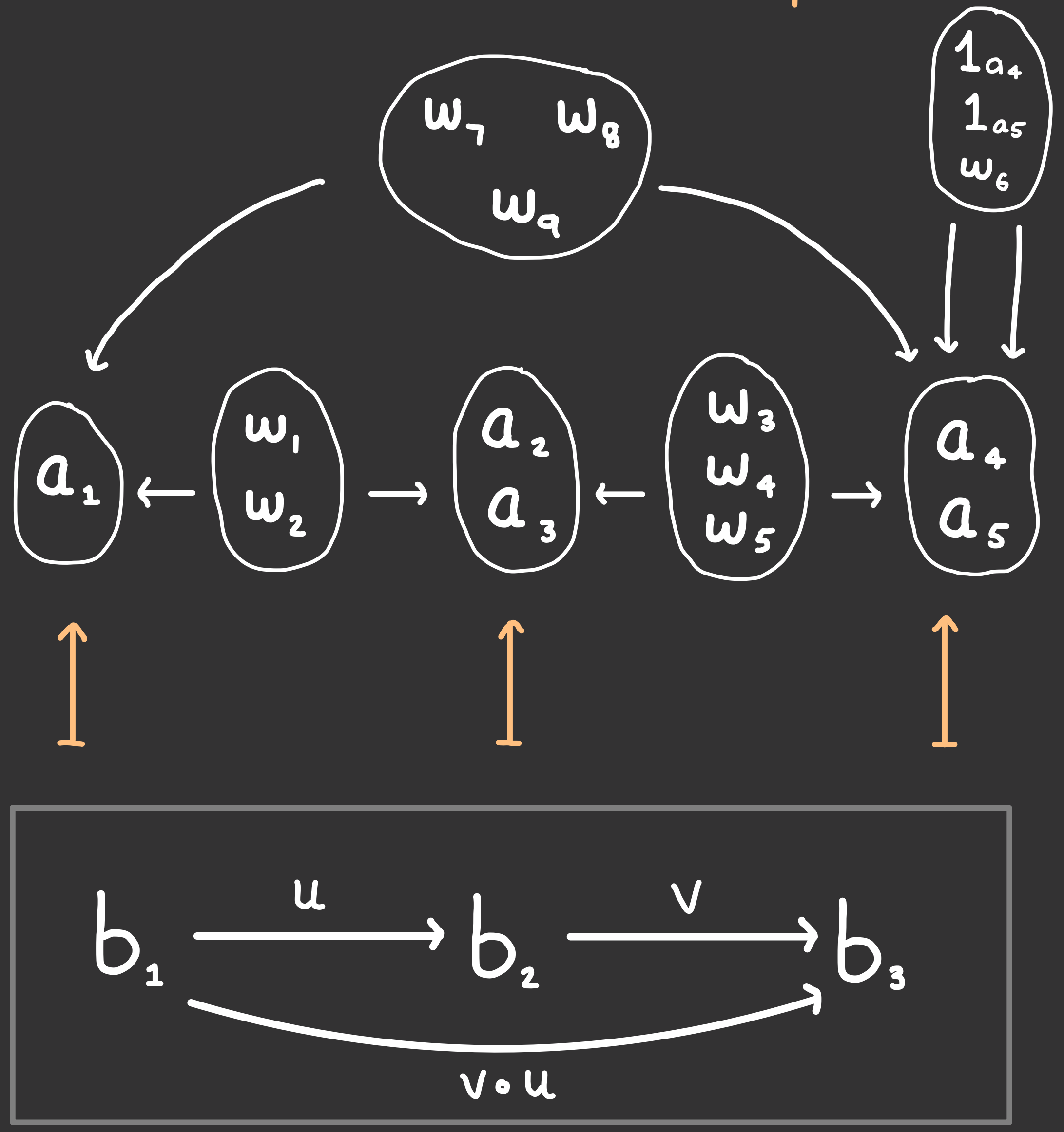


RUNNING EXAMPLE 3 2

functor $f : A \to B$



double functor $\mathbb{L}\mathrm{o}(\mathcal{B}) \longrightarrow \mathbb{S}\mathrm{pan}$



Part 3

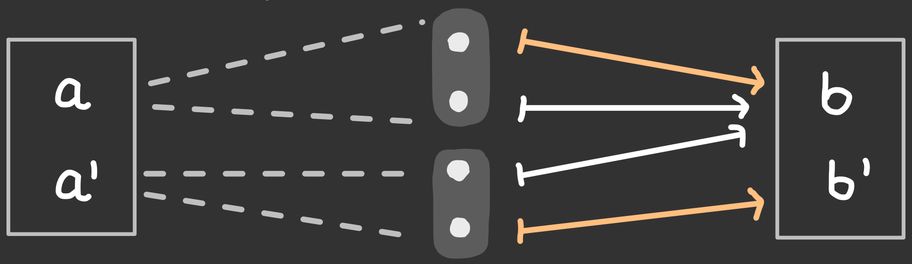
$$\begin{tikzcd}[column sep=small] & \mathbb{L}\mathrm{o}(B) \arrow[dl, "F_1"'] \arrow[dr, "F_2"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & \\ \mathbb{S}\mathrm{q}(\mathrm{Set}) \arrow[rr] & & \mathbb{S}\mathrm{pan} \end{tikzcd} \quad \simeq \quad \begin{tikzcd} \mathbb{L}\mathrm{o}(B) \arrow[d, "F"] \\ \mathbb{S}\mathrm{Mult} \end{tikzcd}$$

"Delta lenses are equivalent to lax double functors into $\mathbb{S}\mathrm{Mult}$"

SPLIT MULTI-VALUED FUNCTIONS

A **split multi-valued function** is a span whose source leg has a chosen section.

$$\begin{tikzcd} A \arrow[r, shift left, tail, "s"] & X \arrow[l, shift left, two heads, "r"] \arrow[r, "f"] & B \end{tikzcd}$$



Let **$\mathbb{M}\mathrm{ult}$** be the double category of sets, functions, and split multi-valued functions.

UNIVERSAL PROPERTY

globular transformation

$$\begin{tikzcd} & \mathbb{A} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\overset{\psi}{\Rightarrow}"] & \\ \mathbb{S}\mathrm{q}(\mathrm{Set}) \arrow[rr] & & \mathbb{S}\mathrm{pan} \end{tikzcd} = \begin{tikzcd} & \mathbb{A} \arrow[dl, bend right] \arrow[dr, bend left] \arrow[d, dashed, "\exists!"] & \\ \mathbb{S}\mathrm{q}(\mathrm{Set}) \arrow[rr] & \$\mathrm{Mult} \arrow[l] \arrow[r] \arrow[d, phantom, "\Rightarrow"] & \mathbb{S}\mathrm{pan} \\ & & \end{tikzcd}$$

Component of globular transformation:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, "fs"] \arrow[d, "s"] & B \arrow[d, equal] \\ A & X \arrow[l, two heads, "r"] \arrow[r, "f"'] & B \end{tikzcd}$$

MAIN THEOREM

For each category B, there are equivalences of categories:

$$\boxed{\begin{aligned} & \mathcal{L}\mathrm{ens}(B) \\ \simeq\ & \mathrm{Dia}\mathcal{L}\mathrm{ens}(B) \\ \simeq\ & \mathrm{GlobCone}(B, \mathbb{S}\mathrm{q}(\mathrm{Set}) \to \mathbb{S}\mathrm{pan}) \\ \simeq\ & [\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{Mult}]_{\mathrm{lax}} \end{aligned}}$$

GROTHENDIECK CONSTRUCTION:

Lax double functor $\mathbb{L}\mathrm{o}(B) \xrightarrow{F} \mathbb{S}\mathrm{Mult}$

where each $u : b \to b'$ sent to:

$$\begin{tikzcd} F(b) \arrow[r, tail, shift left, "\varphi_u"] & F[u] \arrow[l, two heads, shift left, "s_u"] \arrow[r, "t_u"] & F(b') \end{tikzcd}$$

Obtain delta lens $\int F \xrightarrow{\pi} B$ with:

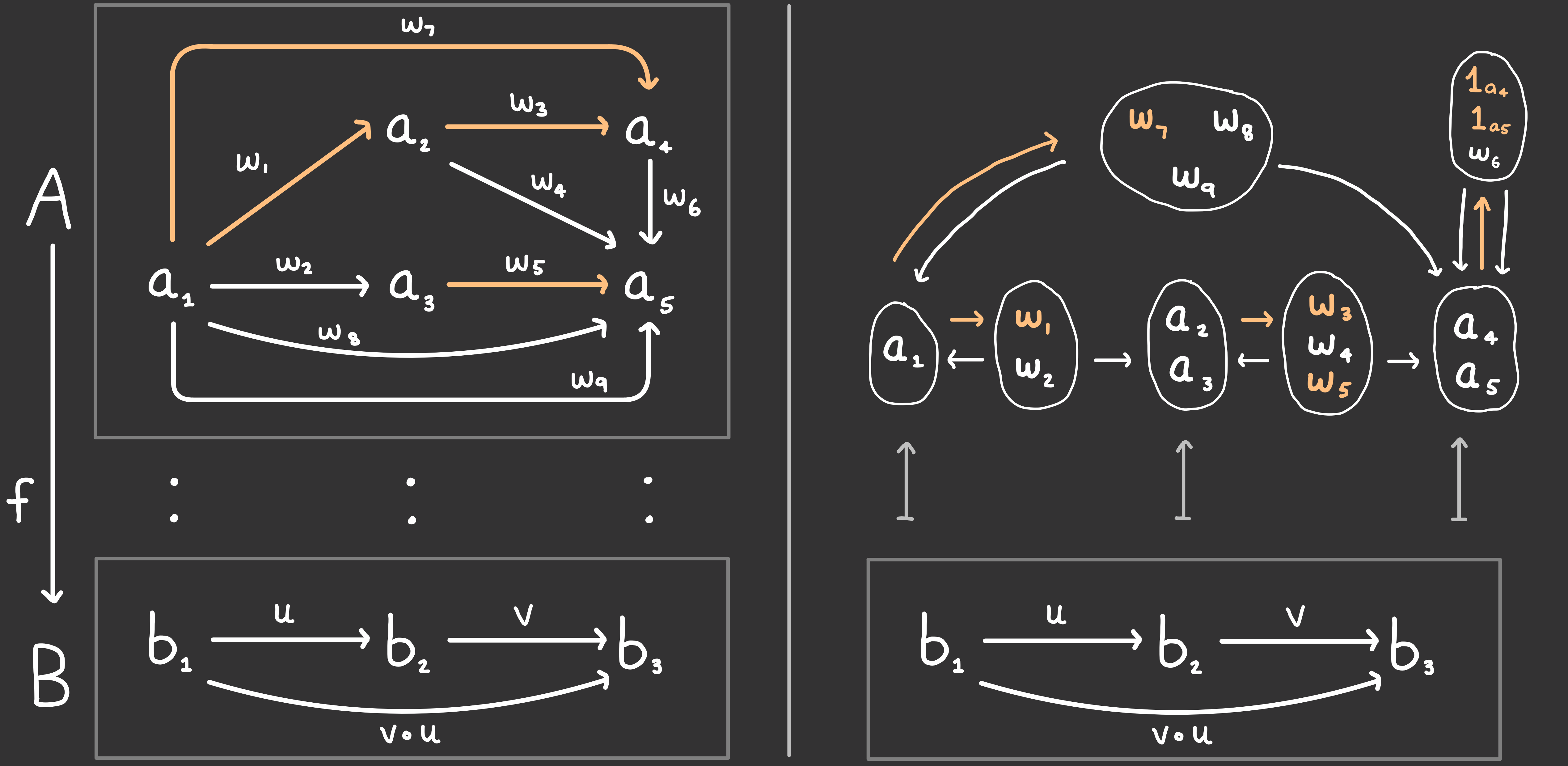
- $(b \in B, x \in F(b))$ ← objects

morphisms

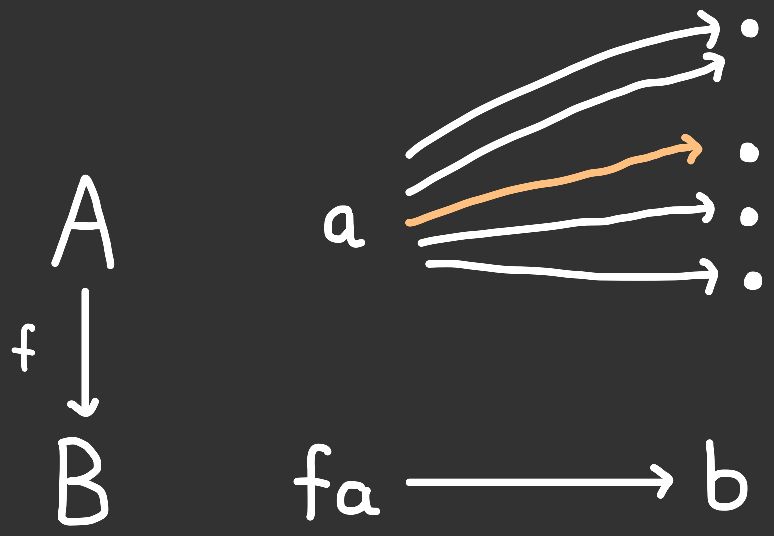
- $(u : b \to b', w \in F[u]) : (b, s_u(w)) \longrightarrow (b', t_u(w))$
- $(b \in B, x \in F(b), u : b \to b')$ chosen lifts

$(u : b \to b', \varphi_u(x)) : (b, x) \longrightarrow (b', t_u \varphi_u(x))$

RUNNING EXAMPLE 4



Part 4



"Delta lenses can be studied via the double category $\mathbb{S}\mathrm{Mult}$"

EXAMPLES & MONOIDAL PRODUCTS

Restrict $F : \mathbb{L}\mathrm{o}(B) \longrightarrow \mathbb{S}\mathrm{Mult}$ to study different classes of delta lenses:

discrete opfibration

$$F(b) = F(b) \longrightarrow F(b')$$

fully faithful

$$F(b) \rightleftarrows F(b) \times F(b') \longrightarrow F(b')$$

bijective-on-objects

$$1 \rightleftarrows F[u] \longrightarrow 1$$

discrete fibration*

$$F(b) \rightleftarrows F(b') = F(b')$$

retrofunctors

$$\begin{tikzcd}
\mathbb{L}\mathrm{o}(B) \arrow[r] \arrow[d, "\text{codiscrete}"'] \arrow[dr, phantom, "\text{globular} \Downarrow"] & \mathbb{S}\mathrm{q}(\mathrm{Set}) \arrow[d] \\
\mathbb{L}\mathrm{o}(B_\infty) \arrow[r] & \mathbb{S}\mathrm{pan}
\end{tikzcd}$$

Use monoidal products on $\mathcal{C}at$ and $\mathbb{S}\mathrm{Mult}$ to induce those on $\mathcal{L}ens(B)$:

$$\mathbb{L}\mathrm{o}(B) \xrightarrow{\langle F, G \rangle} \mathbb{S}\mathrm{Mult} \times \mathbb{S}\mathrm{Mult} \xrightarrow{\times} \mathbb{S}\mathrm{Mult}$$

$$\mathbb{L}\mathrm{o}(B + C) \xrightarrow{[F, G]} \mathbb{S}\mathrm{Mult} \times \mathbb{S}\mathrm{Mult} \xrightarrow{+} \mathbb{S}\mathrm{Mult}$$

Classical Grothendieck construction:

$$\mathsf{SOpf}(\mathcal{B}) \simeq [\mathcal{B}, \mathcal{C}at]$$

But this is full subcategory of $\mathcal{L}ens(\mathcal{B})$!

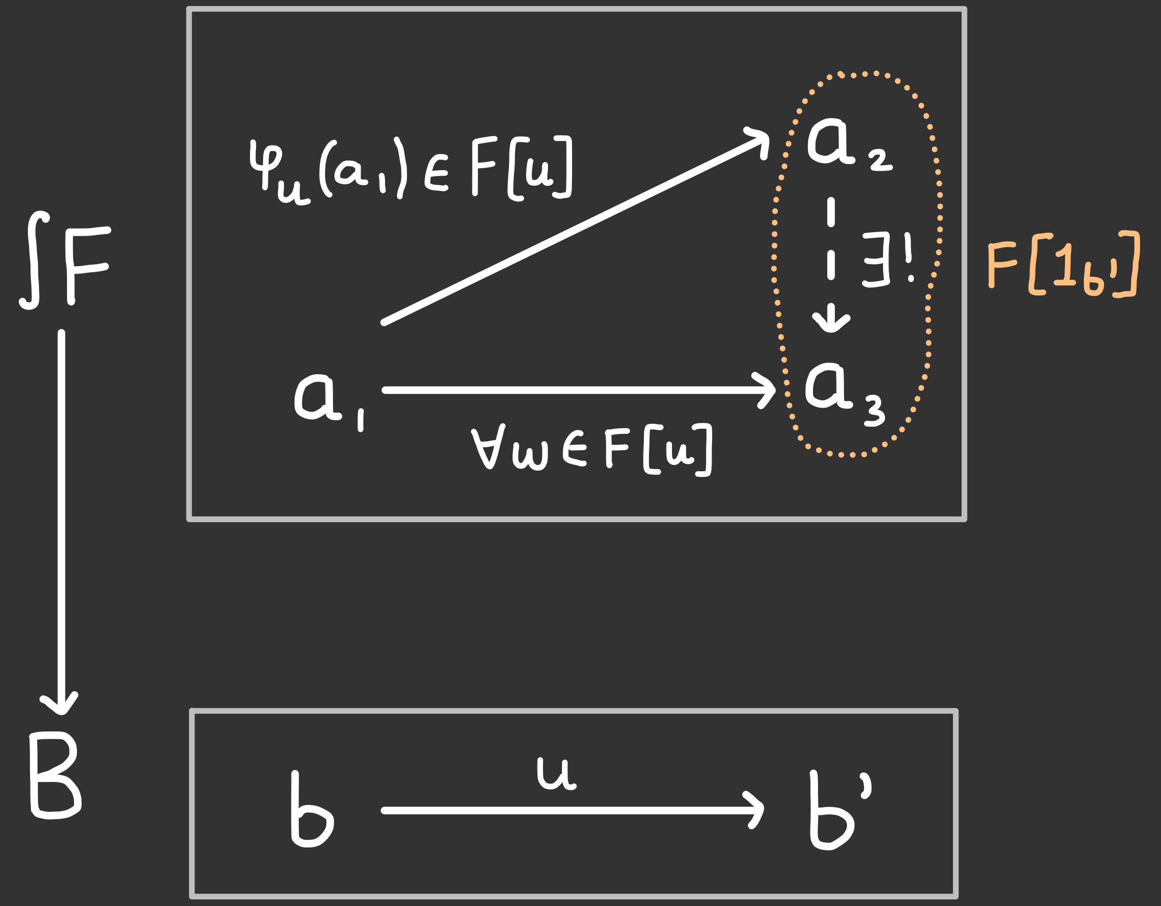
What is the image?

Split opfibration $\simeq F : \mathbb{L}o(\mathcal{B}) \longrightarrow \mathbb{S}\mathrm{Mult}$

such that the function

$$F(b) \times_{F(b')} F[1_{b'}] \xrightarrow{\varphi_u \times id} F[u] \times_{F(b')} F[1_{b'}] \xrightarrow{\mu} F[u]$$

is <u>invertible</u> for each $u : b \to b'$ in $\mathcal{B}$.



SUMMARY & FUTURE WORK

- Introduced split multi-valued functions

$$
\begin{tikzcd}
A \arrow[r, tail, shift left, "s"] & X \arrow[l, two heads, shift left, "r"] \arrow[r, "f"] & B
\end{tikzcd}
$$

- Showed that a delta lens over B is equivalent to an indexed collection of sets, with lax reindexing along split multi-valued functions.

$$\boxed{\mathcal{L}\mathrm{ens}(B) \simeq [\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{Mult}]_{\mathrm{lax}}}$$

- Characterisation as oplax colimit?
- Link with type theory / displayed cats.?
- Explicit description of left Kan lift describing free delta lens?

$$
\begin{tikzcd}
 & \mathbb{S}\mathrm{Mult} \arrow[d] \\
\mathbb{L}\mathrm{o}(B) \arrow[ur, dashed] \arrow[r] \arrow[ur, phantom, shift right=3, "\Uparrow"] & \mathbb{S}\mathrm{pan}
\end{tikzcd}
$$

- What about $F : \mathbb{B} \longrightarrow \mathbb{S}\mathrm{Mult}$?

BONUS: FURTHER IDEAS

- $\mathbb{L}\mathrm{ens}$ is a double category.

$$\mathcal{L}\mathrm{ens} \xrightarrow{\mathrm{cod}} \mathcal{C}\mathrm{at}$$ ← bifibration

$$\mathrm{cod}^{-1}\{B\} = \mathcal{L}\mathrm{ens}(B) \simeq [\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{Mult}]$$

- Can we easily enumerate finite examples of delta lenses?

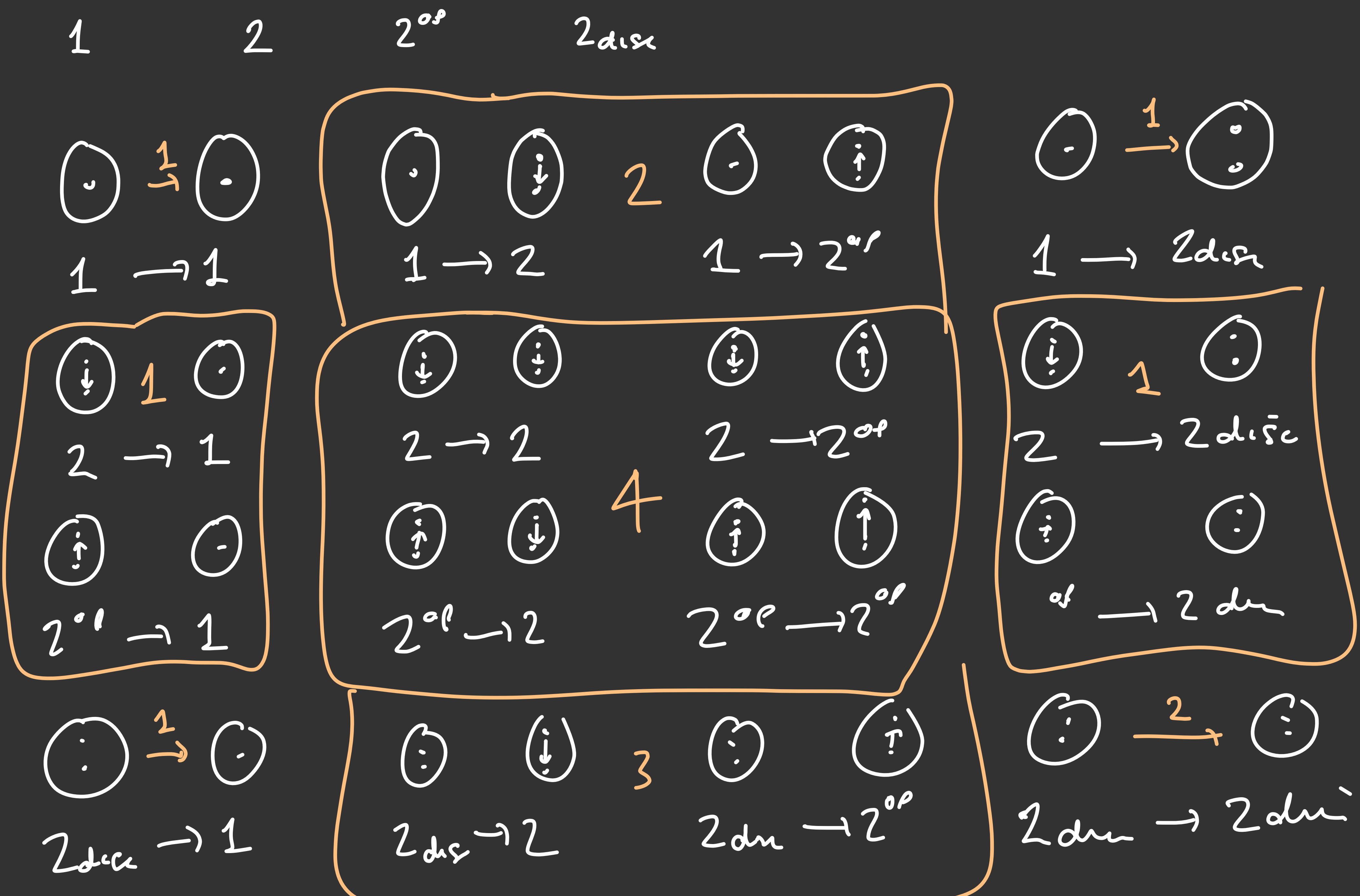
- Are split multi-valued functions a kind of decorated span?

- What is the sense in which $\mathbb{S}\mathrm{Mult}$ is a limit in the Dbl-enriched category of double categories and lax double functors?

- Is $\mathbb{S}\mathrm{Mult}$ a Kleisli double category?

- Link with double opfibrations & internal lenses in $\mathcal{C}\mathrm{at}$?

- Link with A.W.F.S.?



1
2
2^op
2disc
1
1 → 1
2
1 → 2
1 → 2^op
1
1 → 2disc
1
2 → 1
2^op → 1
4
2 → 2
2 → 2^op
2^op → 2
2^op → 2^op
1
2 → 2disc
op → 2disc
1
2disc → 1
3
2disc → 2
2disc → 2^op
2
2disc → 2disc