DRAFT NOTES FOR BOARD TALK Limits and colimits in double categories, by example BRYCE CLARKE

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A double category consists of 4 kinds of things:

- · objects A, B, C, D, ...
- · tight morphisms A -> B
- · loose morphisms A --- C

· cells 
$$A \xrightarrow{\rho} C$$
  
 $f \downarrow \propto \downarrow g$   
 $B \xrightarrow{\alpha} D$ 

We have both a tight composition (vertically) and a loose composition (horizontally).

The latter may only be unital & associative up to specified isomorphism (like a bicategory).

### Examples:

· For each category C, let Sq(e) be the double category whose cells are commutative squares in C.

$$A \xrightarrow{h} C$$

$$f \downarrow Q \downarrow g$$

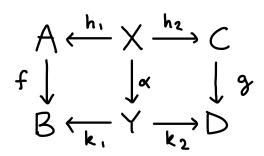
$$B \xrightarrow{k} D$$

$$gh = kf$$

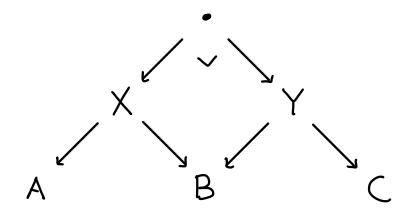
· For each 2-category K, let Q(K) be the "double category of quintets" with cells given by the 2-cells.

$$\begin{array}{ccc}
A & \xrightarrow{h} & C \\
f & \downarrow \downarrow \alpha & \downarrow g & \alpha : gh \Rightarrow kf \\
B & \xrightarrow{R} & D
\end{array}$$

· Let Span be the double category of sets, functions, and spans. A cell is given by a diagram:



Composition of loose morphisms is given by pullback.



This extends to loose composition of cells via universal property.

· Let IProf be the double category of categories, functors, and profunctors.

A P C

P: C°\* A - Set

A cell is given by a natural transformation

$$A \xrightarrow{P} C \qquad C^{\circ P} A \xrightarrow{P}$$

$$f \downarrow \qquad \alpha \qquad \downarrow g \qquad \omega \qquad g \times f \downarrow \qquad \downarrow \alpha \qquad Set$$

$$B \xrightarrow{Q} D \qquad D^{\circ P} \times B \xrightarrow{Q} \qquad \alpha_{a,c} : P(c,a) \longrightarrow Q(gc,fa)$$

Composition of loose morphisms is given by coends.

$$A \xrightarrow{P} B \xrightarrow{Q} C$$

$$P \cdot Q : C^{P} \times A \longrightarrow Set$$

$$(c,a) \longmapsto \int_{Q(c,b) \times P(b,a)}^{b \in B} Q(c,b) \times P(b,a)$$

u Ø p' = p

$$c \xrightarrow{q} b \xrightarrow{p} a \qquad \sim c \xrightarrow{q'} b' \xrightarrow{p'} a$$

if  $\exists u: b \rightarrow b'$  in B such that  $q \otimes u = q'$ 

# Terminal object (in 1D)

A terminal object is an object T such that:

- (i) for each object A there exists a unique tight morphism t:A→T.
- (ii) for each loose morphism P: A+B, there exists a unique cell

$$\begin{array}{ccc}
A & \xrightarrow{P} & B \\
t \downarrow & & \downarrow t \\
T & \xrightarrow{id_{T}} & T
\end{array}$$

An <u>initial object</u> I is defined dually.

• In Span, the singleton set is a terminal object

$$\{*\} = \{*\} = \{*\}$$

· In Span, the empty set is an initial object.

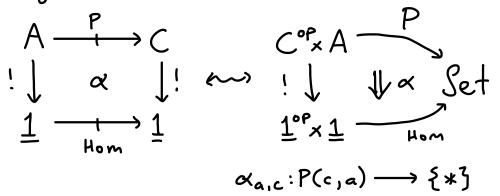
$$\emptyset = \emptyset = \emptyset$$

$$\downarrow \downarrow \qquad \downarrow \downarrow$$

$$\downarrow \downarrow \qquad \downarrow \downarrow$$

$$\Diamond \Rightarrow \Diamond$$

· In IProf, the category 1 is a terminal object.

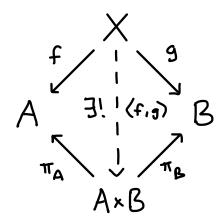


· In IProf, the empty category is an initial object.

#### Products

Given a pair of objects A and B, their product is an object  $A \times B$ together with tight morphisms  $\pi_A: A \times B \longrightarrow A$  and  $\pi_B: A \times B \longrightarrow B$ such that:

(i) For each span of tight morphisms



there exists a unique tight morphism (fig) such that above diagram commutes.

(ii) Given cells

$$\begin{array}{cccc}
X & \xrightarrow{P} & Y & X & \xrightarrow{P} & Y \\
f \downarrow & \alpha & \downarrow 9 & h \downarrow & \beta & \downarrow k \\
A & \xrightarrow{id_A} & A & B & \xrightarrow{id_B} & B
\end{array}$$

there exists a unique cell

$$\begin{array}{c}
X \xrightarrow{P} Y \\
\langle \epsilon, h \rangle \downarrow \langle \alpha, \beta \rangle \downarrow \langle 9, k \rangle \\
A \times B \xrightarrow{id_{A \times B}} A \times B
\end{array}$$

such that

$$\frac{\langle \alpha, \beta \rangle}{i d_{\pi_A}} = \alpha \qquad \frac{\langle \alpha, \beta \rangle}{i d_{\pi_B}} = \beta$$

The coproduct A+B of a pair of objects is defined dually.

· In Span, the product is just the usual product of sets. Given cells

there exists a unique cell

· In Span, the coproduct is just the usual coproduct of sets. Given cells,

$$A = A = A$$

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there exists a unique cell

$$A+B = A+B = A+B$$

$$(f,h) \downarrow \qquad \qquad \downarrow (\alpha,\beta) \qquad \downarrow (g,k)$$

$$C \leftarrow X \longrightarrow D$$

· In IProf, the product is the usual product of categories.

Given cells

 $\alpha_{a,c}: P(c,a) \longrightarrow A(fc,9a)$ 

 $\beta_{a,c}: P(c,a) \longrightarrow B(hc,ka)$ 

there exists a unique cell

(da,c, Ba,c): P(c,a) - A (fc,ga) × B (hc,ka)

. In Prof, the coproduct is the usual coproduct of categories. Given cells

$$\beta_{c,c'}: C(c,c') \longrightarrow Q(hc,kc')$$

there exists a unique cell

$$A^{\circ p} A + C^{\circ p} A + \cong (A + C)^{\circ p} (A + C) \longrightarrow A^{\circ p} A + C^{\circ p} A + C^$$

where

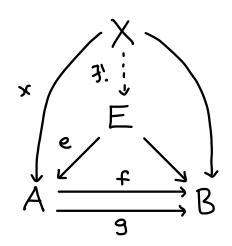
Where
$$\begin{cases}
0 & \xrightarrow{\cdot} Q(fa, kc) \quad x=a \quad y=c \\
0 & \xrightarrow{\cdot} Q(hc, ga) \quad x=c \quad y=a \\
x=a \quad y=a \quad y=a \quad y=a \quad y=c \quad y$$

#### Equalisers

Given a parallel pair of tight morphisms fig: A = B, their equaliser is an object E and a cone of tight morphisms

$$A \xrightarrow{\text{fe=ge}} B$$

(i) For each cone



there exists a unique tight morphism  $X \rightarrow E$  such that the diagram commutes.

(ii) For each cell  $\alpha$  such that

there exists a unique cell

$$\begin{array}{cccc}
X & \xrightarrow{P} & Y \\
t & & & & \downarrow s \\
E & \xrightarrow{id_E} & E
\end{array}$$
 such that  $\frac{X}{id_e} = \alpha$ 

The coequaliser is defined dually.

- · Span has all (co)equalisers.
- · IProf has all (co) equalicers

#### Tabulators

Given a loose morphism  $P:A \rightarrow B$ , its <u>tabulator</u> is an object TP together with a cell

$$\begin{array}{ccc}
 & TP \xrightarrow{id} & TP \\
 & \downarrow & \uparrow_{P} & \downarrow \uparrow_{g} \\
 & A \xrightarrow{P} & B
\end{array}$$

Such that:

there exists a unique tight morphism

$$t: X \longrightarrow TP$$
 such that 
$$\frac{id_t}{\tau_p} = \alpha.$$

(ii) Given cells a, B, X, & such that

there exists a unique cell

$$X \xrightarrow{\mathbb{Q}} Y$$

$$e_1 \downarrow \qquad \mathcal{E} \qquad \downarrow e_2$$

$$TP \xrightarrow{id} TP$$

Such that

$$\frac{\mathcal{E}}{id_{\pi_A}} = \delta \frac{\mathcal{E}}{id_{\pi_B}} = \lambda \frac{id_{e_1}}{\tau_P} = \alpha \frac{id_{e_2}}{\tau_P} = \beta$$

· In Span the tabulator is given by the cell:

The cotabulator is given by the cell:

$$A \xleftarrow{f} X \xrightarrow{g} B$$

$$\downarrow^{1}_{B} \qquad \qquad \downarrow^{1}_{B}$$

$$A+_{x}B \xrightarrow{A+_{x}B} A+_{x}B \xrightarrow{pushout of f g}$$

- ·In Prof, the tabulator of
  P: B\*\* × A ---> Set is category
  TP whose:

  of elements
  - -objects are elements  $p:b \rightarrow a$  in  $\sum P(b,a)$ .
  - morphisms are given by

$$\begin{array}{cccc}
b & \xrightarrow{P} & \text{such that} \\
B(b,b') \ni u & & & & & \\
b' & & & & & \\
b' & & & & & \\
p' & & & & & \\
\end{array}$$
such that
$$p \otimes v = u \otimes p'$$

Given a cell in Prof

$$\begin{array}{ccc}
& & & & & \\
X & & & & \\
\downarrow & & & & \\
f & & & & \\
A & & & \\
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there is a unique functor.

$$X \longrightarrow TP$$

$$x \longmapsto \alpha_{x,x} (id_x) : gx \longrightarrow fx$$

· In Prof, the cotabulator
P: Box A ---> Set is category
Coll(P) whose:

- set of objects is obj(A) + obj(B)

- homset  $Coll(P)(x,y) = \begin{cases} A(x,y) & \text{if } x,y \in A \\ B(x,y) & \text{if } x,y \in B \end{cases}$   $P(x,y) & \text{if } x \in B \\ y \in A & \text{otherwise} \end{cases}$ 

- composition is determined by composition in A, B, and actions of P.

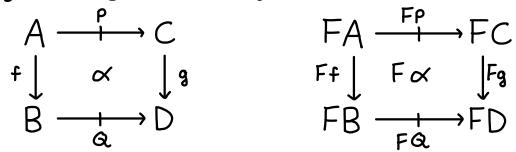
Civen a cell in Prof

$$\begin{array}{ccc}
A & \xrightarrow{P} & B \\
f \downarrow & \alpha & \downarrow g & \alpha_{b,a} : P(b,a) \longrightarrow X(gb,fa) \\
\times & \xrightarrow{id} & \times
\end{array}$$

there is a unique functor  $Coll(P) \longrightarrow X$   $p:b \rightarrow a \longmapsto \alpha_{b,a}(p)$ 

## General theory of limits

A lax double functor  $F: \mathbb{C} \longrightarrow \mathbb{D}$  is given by an assignment



$$\begin{array}{ccc}
FA & \xrightarrow{FP} & FC \\
Ff & F & & \downarrow F_{9} \\
FB & \xrightarrow{FQ} & FD
\end{array}$$

that preserves the fight direction strictly, and the loose direction up to comparison cells:

$$FA \xrightarrow{idFA} FA \qquad FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$$

$$\parallel M_A \parallel \qquad \parallel M_{fig} \parallel$$

$$FA \xrightarrow{F(id_A)} FA \qquad FA \xrightarrow{F(f \bullet g)} FC$$

A tight transformation between lax double functors  $\alpha: F \rightarrow G: C \rightarrow D$ consists of:

- (i) For each object AEC, a tight morphism  $\alpha_A: FA \rightarrow GA$
- (ii) For each look morphism P:A+B in C, a cell  $FA \xrightarrow{FP} FB$  $\alpha_A \downarrow \alpha_P \downarrow \alpha_B$

which is natural, and coherent with loose identities & composition.

A pseudo loose transformation between lax double functors  $\sigma: F \rightarrow G: C \rightarrow D$  consists of

GA -++GB

(i) For each object A∈C, a loose morphism  $\sigma_A: FA \longrightarrow CA$ 

(ii) For each tight morphism 
$$f: A \rightarrow B$$

a cell

 $FA \xrightarrow{\sigma_A} GA$ 
 $Ff \downarrow \sigma_f \downarrow Gf$ 
 $FB \xrightarrow{\sigma_B} GB$ 

(iii) For loose morphism P: A +> B

in C, an isocell

FA FP FB GB

$$\begin{array}{c|c}
 & \sigma_{P} & \parallel \\
 & FA & \longrightarrow GA & \longrightarrow GB \\
\hline
 & \sigma_{A} & GP
\end{array}$$

which is natural and satisfies 4 coherence laws.

has components  $\mathcal{O}_A$  for each object  $A \in \mathbb{C}$ 

which is coherent in both tight and loose directions.

We have a double category.

ILX (C, ID) of lax double

functors, tight transformations,

pseudo loose transformations,

and modifications for each

pair of double categories C and D.

If C is not are we have a

If C is unitary, we have a diagonal  $\Delta: \mathbb{D} \longrightarrow \mathbb{IL} \times (C, \mathbb{D})$ 

which sends each object to the constant double functor at that object.

The double category of cones over a lax double functor  $F: C \rightarrow D$  is given by the comma double category

$$\begin{array}{c} \mathbb{C}_{one}(F) \longrightarrow \mathbb{1} \\ \downarrow & \mathbb{D} & \downarrow F \\ \mathbb{D} & \longrightarrow \mathbb{L}_{\times}(\mathbb{C}, \mathbb{D}) \end{array}$$

The <u>limit</u> of the double functor  $F: C \rightarrow D$  is a terminal object in Cone(F).

### Theorem:

A double category has all limits if and only if it has products, equalisers, and tabulators.

Prop Disc: Span in Prof preserves all limits and coproducts scoequalisers.

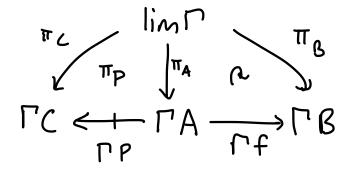
## References

Grandis & Paré (1999). Limits in double categories.

Crandis (2019). Higher Dimensional Categories. The <u>limit</u> of a double functor

II — D is an object limp

with projections

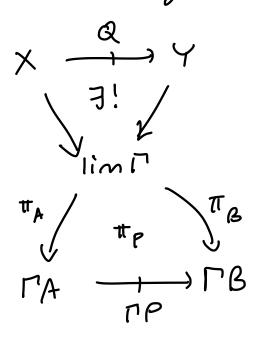


such that

- (i) for any other cone there exists a unique tight morphism

  → lim?

Factors uniquely as



dr,xi X°PxX Hom  $\chi(x'x,)$ g×f J y & Set BorxA p P(gx, fx')YorxX Q dy,x hxgl U8 Set Q(y,x) BorxB Non B(hy,gx) Y 00 × X Q Jy, x  $k \times f \int (J S) \int S dt Q(y_1 \times f)$ AOOXA HOM A(ky,fx) YOPXY Mon Byiy' hxk J JB Sel Y(y,y') Boox A P P (hy, ky')