

# Virtual double categories as coloured box operads



Universiteit  
Antwerpen

Lander Hermans, 2024

fwo

# Virtual double categories as coloured box operads

(in collaboration with Wendy Lowen and Hoang Dinh Van)



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The Road Map

monoids

The Road Map

monoids

$\subseteq$

categories

The Road Map

monoids

$\subseteq$

categories

$\cap$

operads

The Road Map

# The Road Map

monoids

$\subseteq$

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multicategories =  
coloured operads

$\cap$

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$\cap$

double  
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multicategories =  
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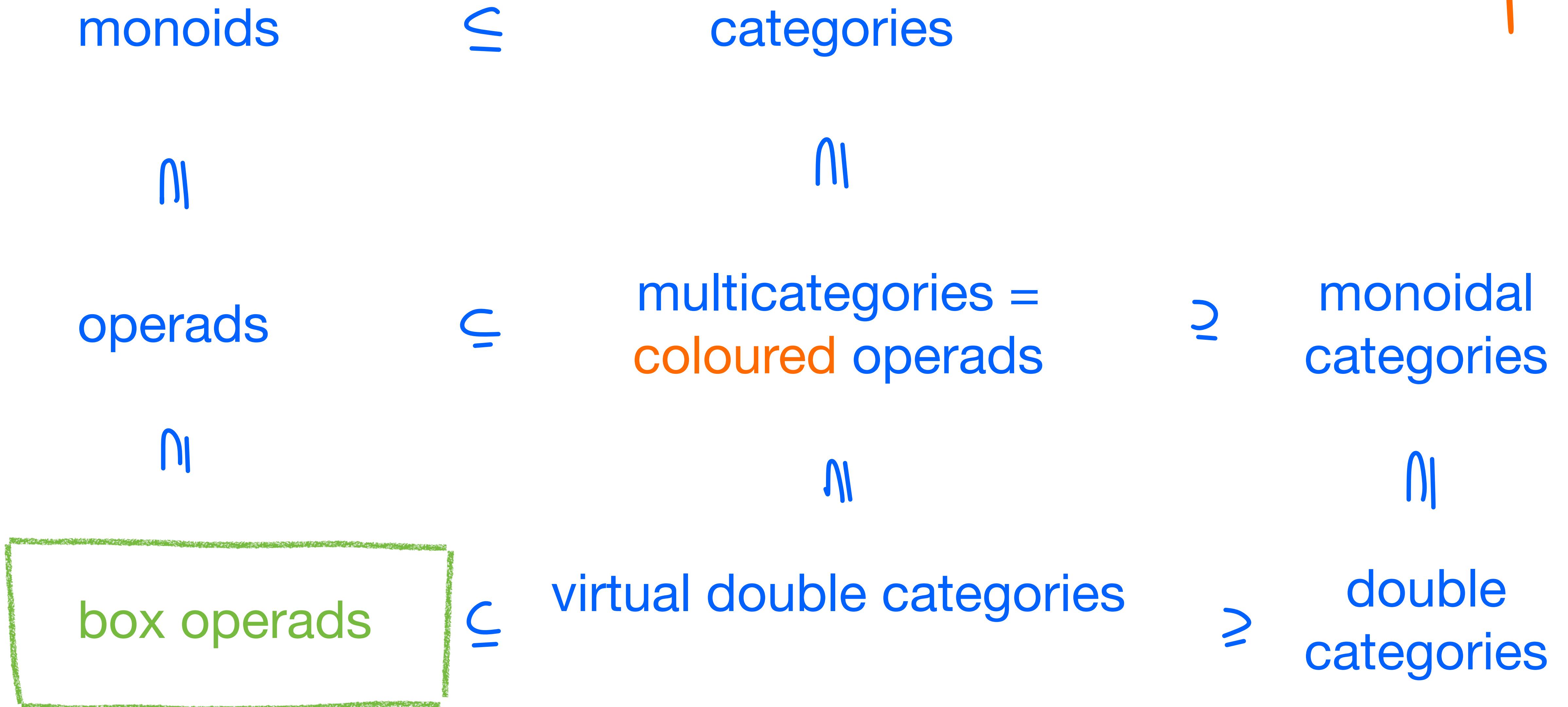
$\cap$

virtual double categories

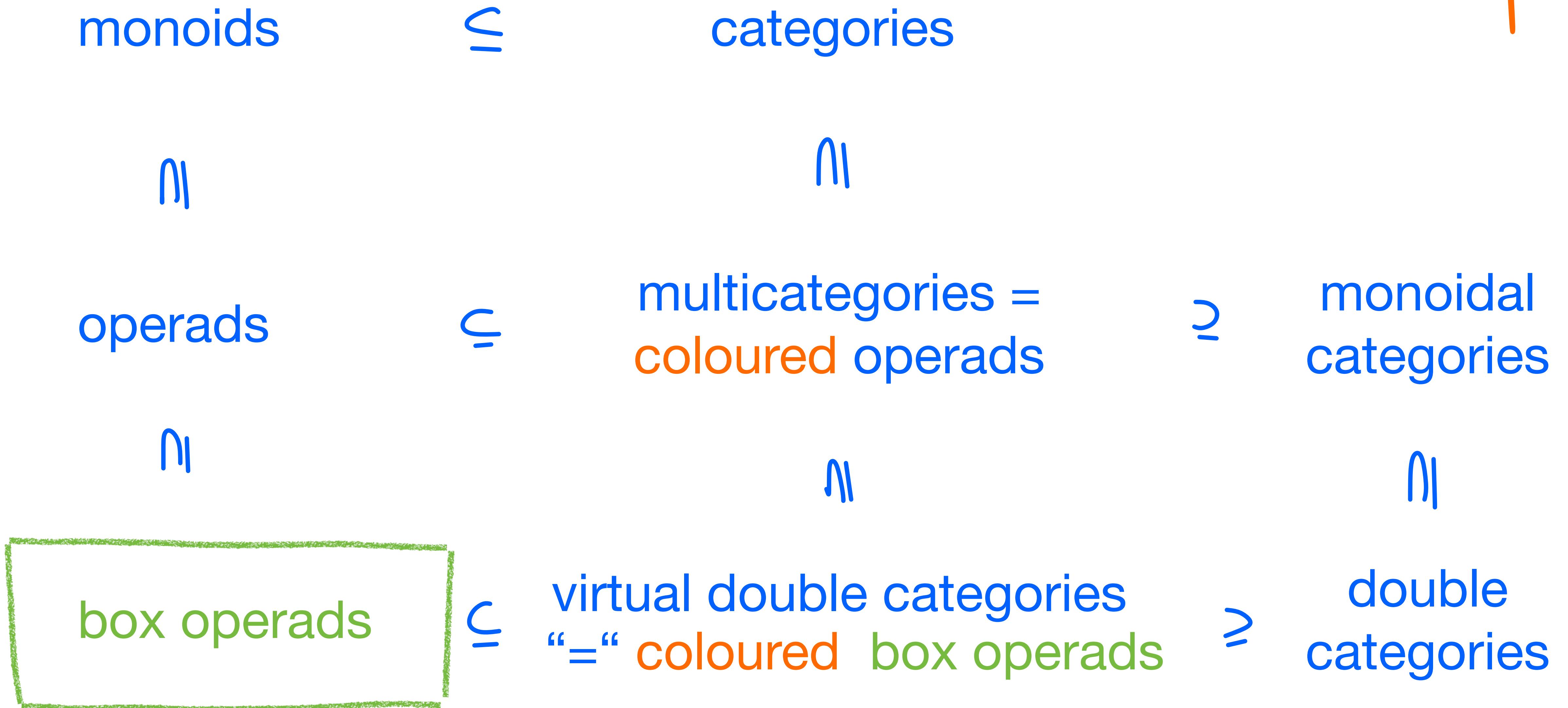
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# The Quad Map



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*A little bit of History*

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- Box operads - Dinh Van, Hermans and Lowen (2023)

Point of view

object of study

object encodes

categories

Point of view

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object of study

Gp, Ab, Set<sub>D</sub>, ...

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ring k-algbr  
 $\Delta, R, A, \dots$

Point of view

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$\Delta, R, A, \dots$

$\text{Amc}, \text{Lie}, \mathfrak{Op}, \dots$

$\text{Lax}_U, \text{colax}_U, \text{Pres}_U, \dots$

# Motivation

encode prestacks

via coloured box operads

to study

their deformation and homotopy theory

operads

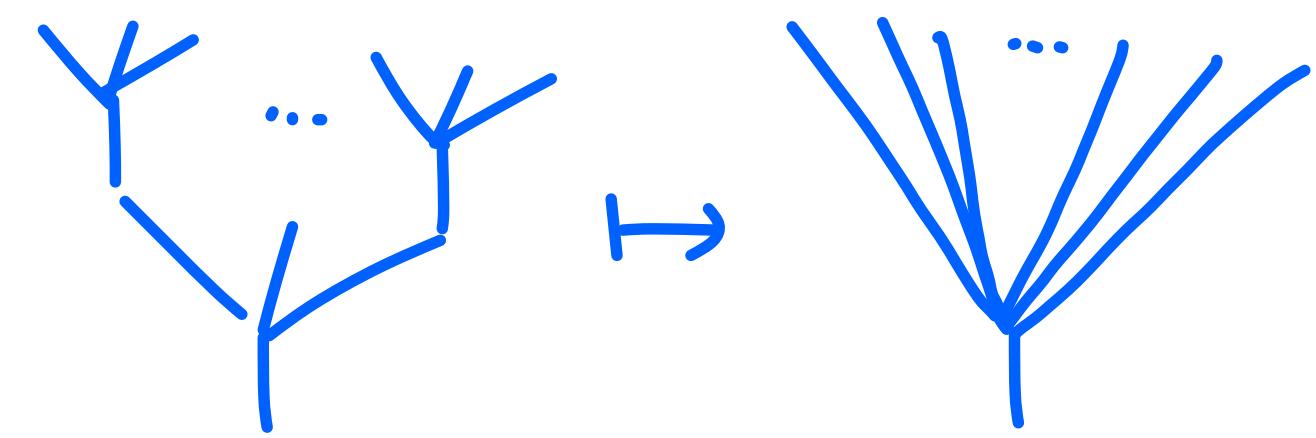
box operads

multicategories =  
coloured operads

virtual double categories “=“  
coloured box operads

monoidal  
categories

double  
categories



operads

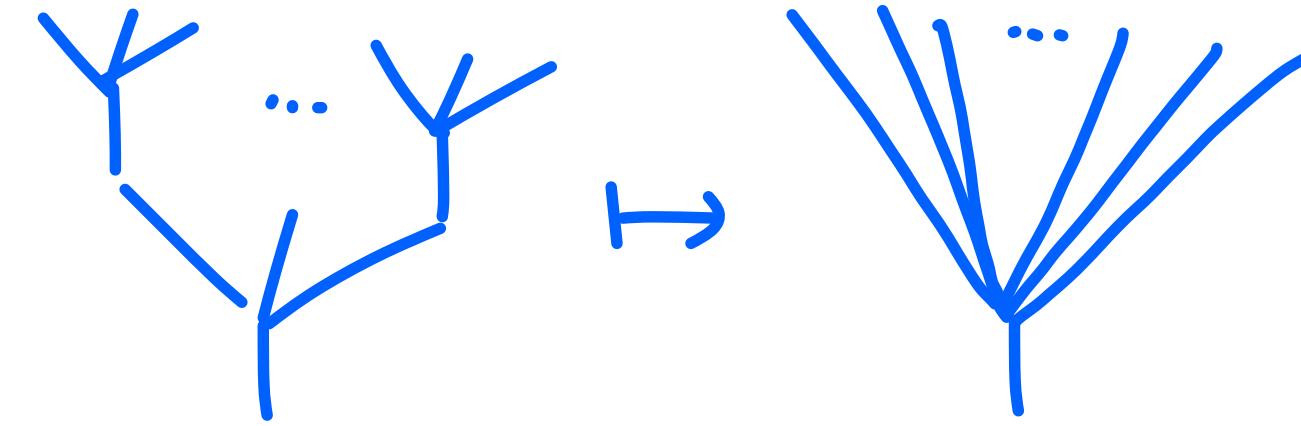
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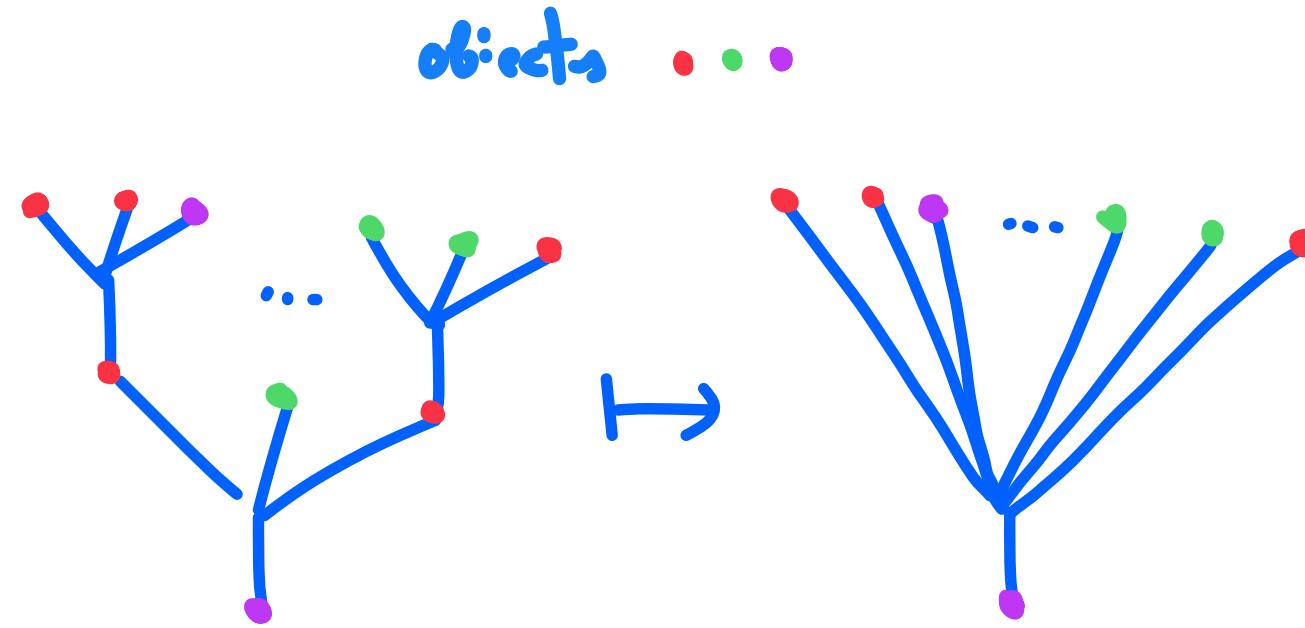
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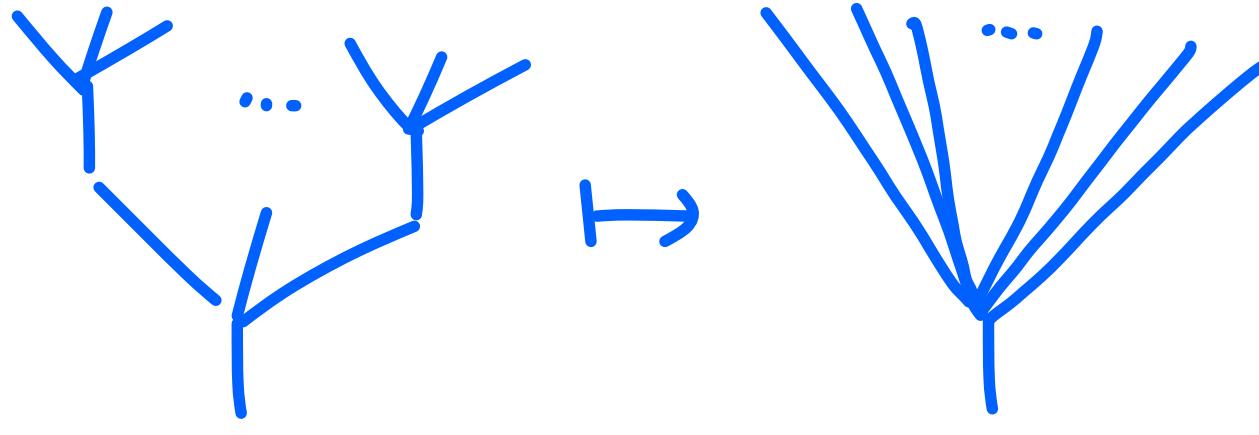


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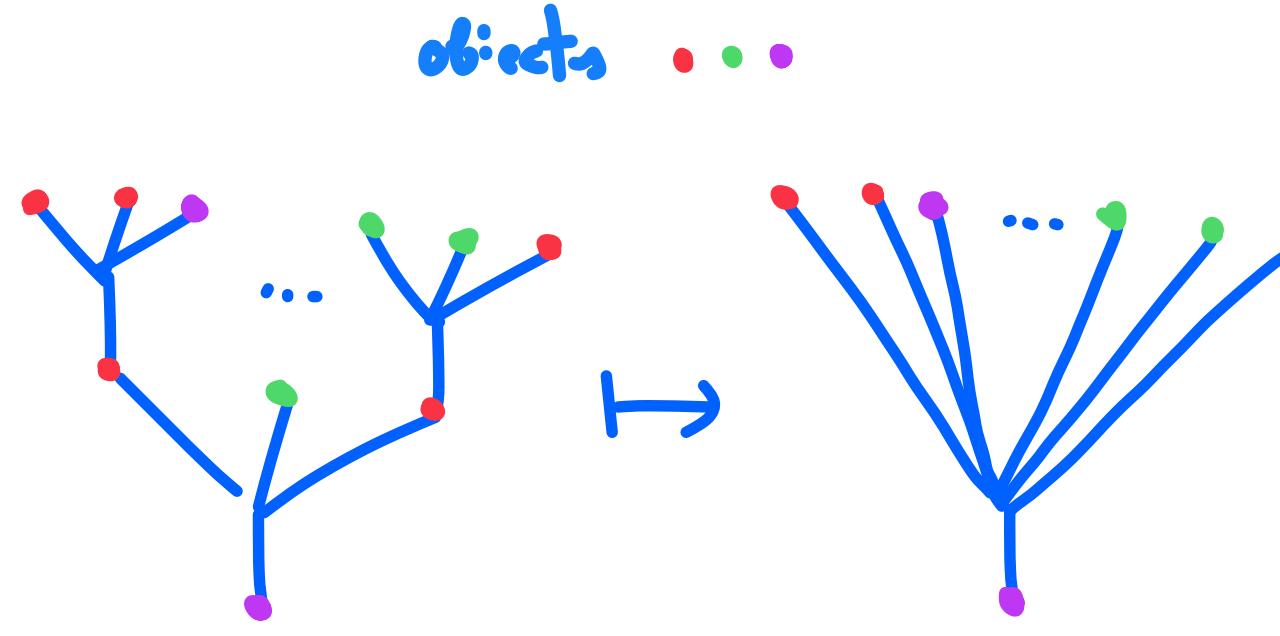
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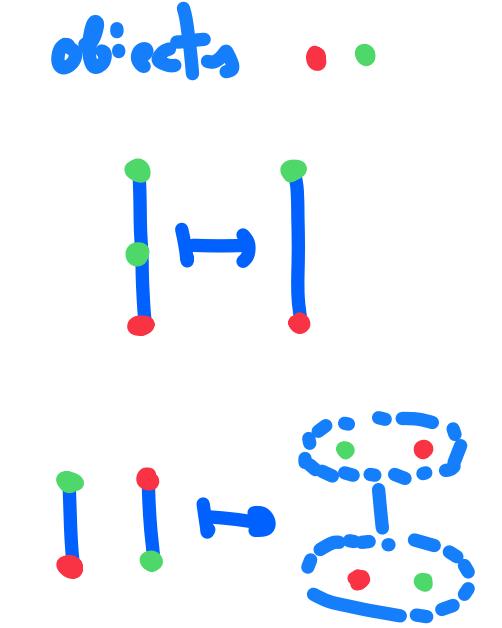
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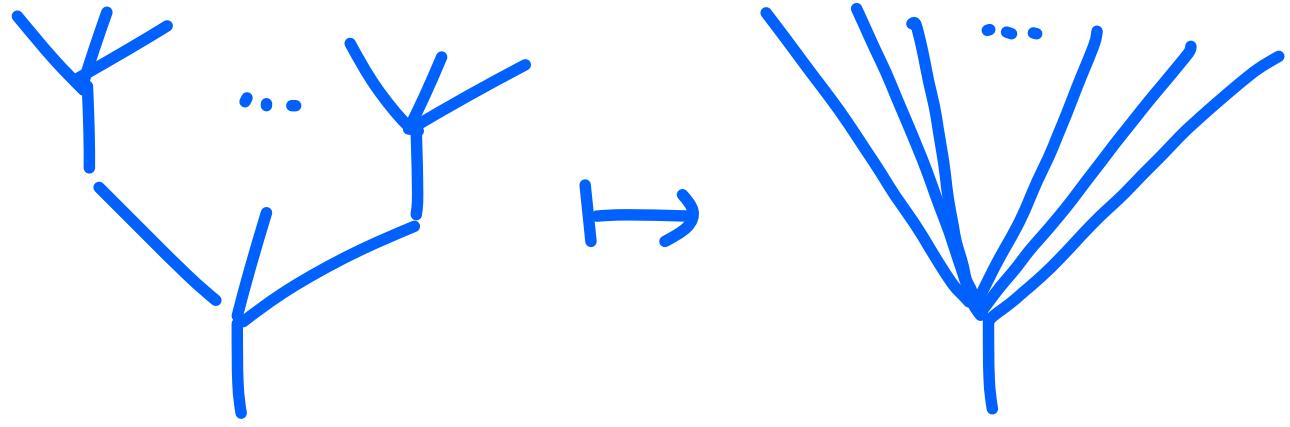
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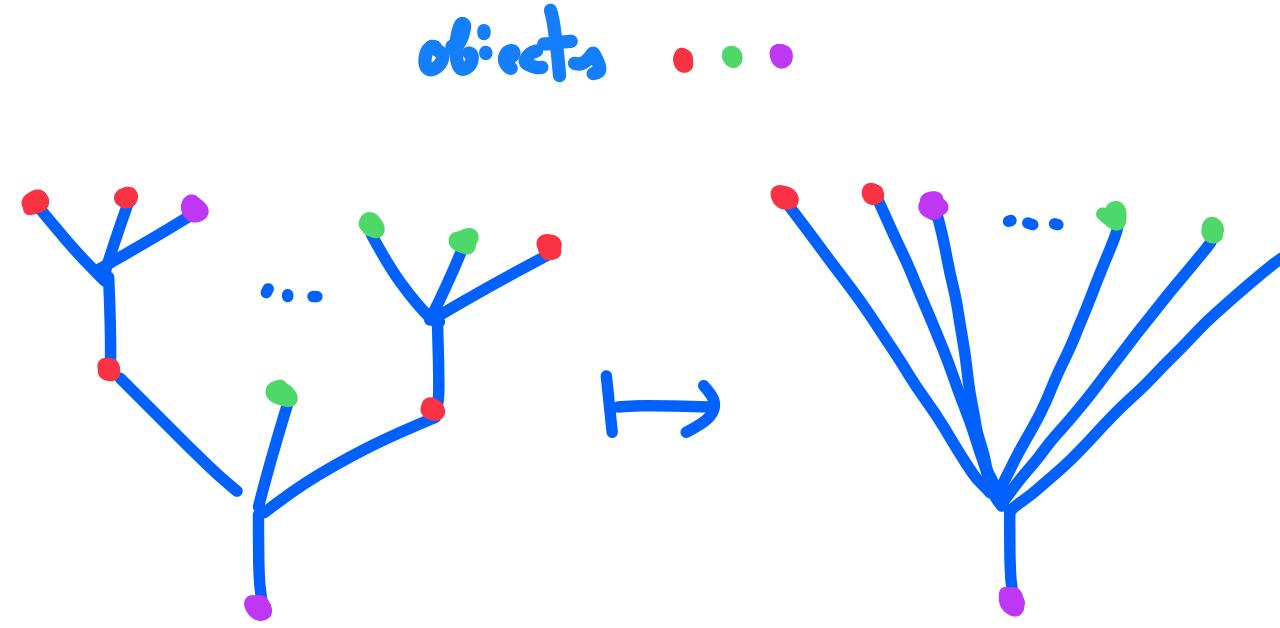
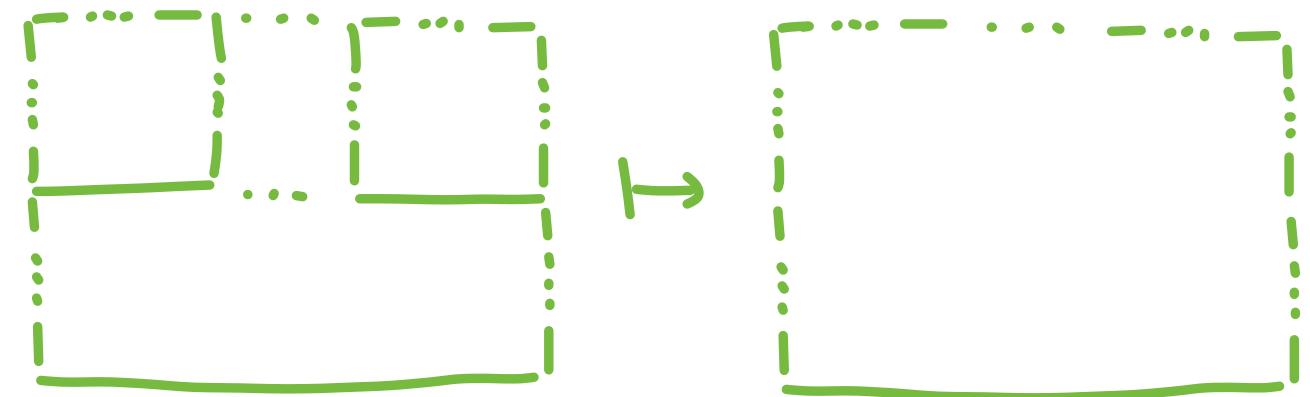
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categories

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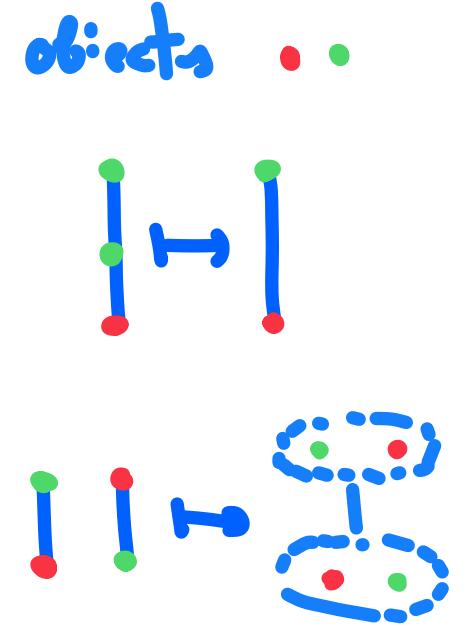
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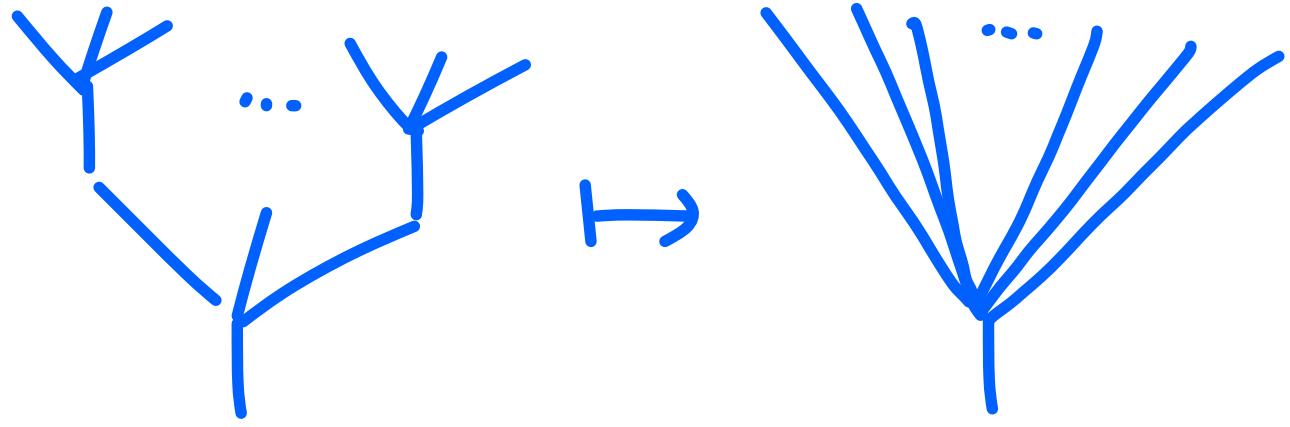
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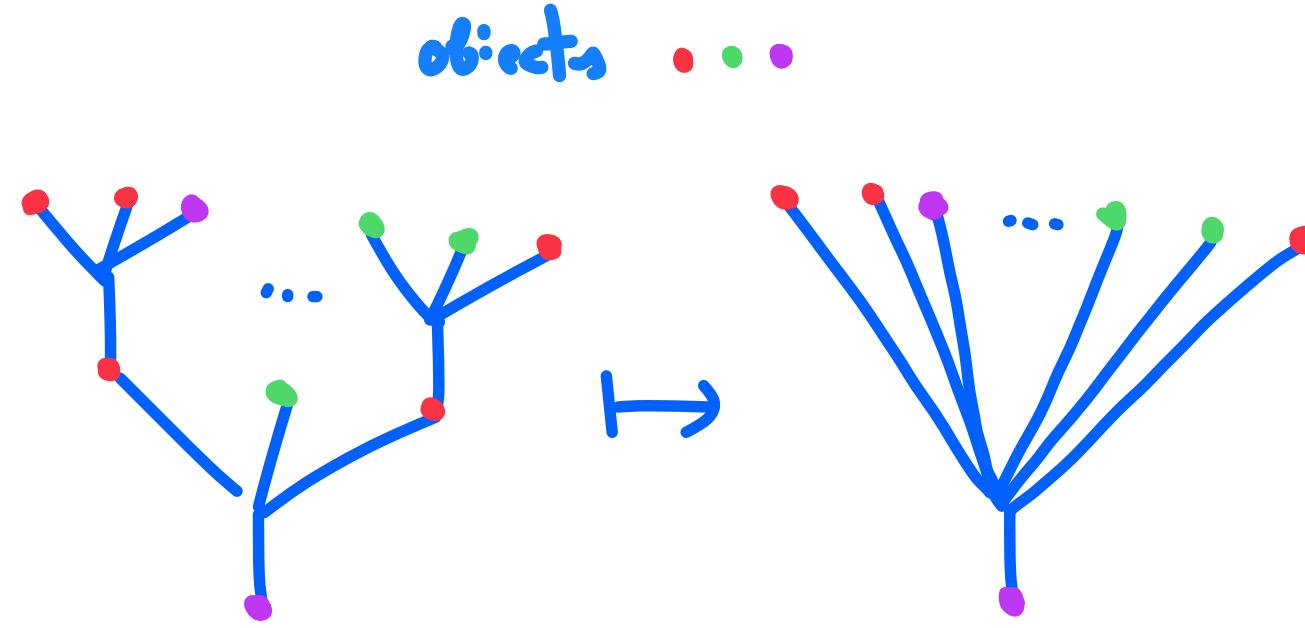
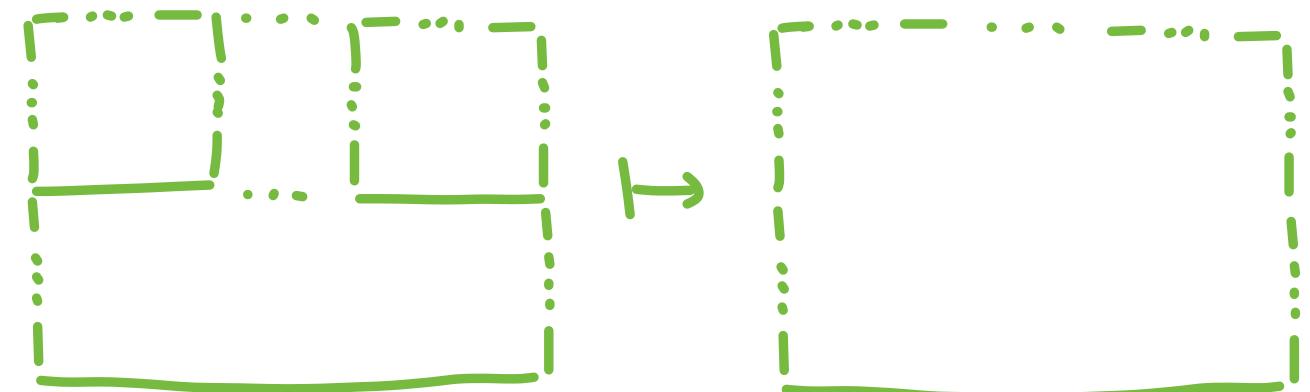
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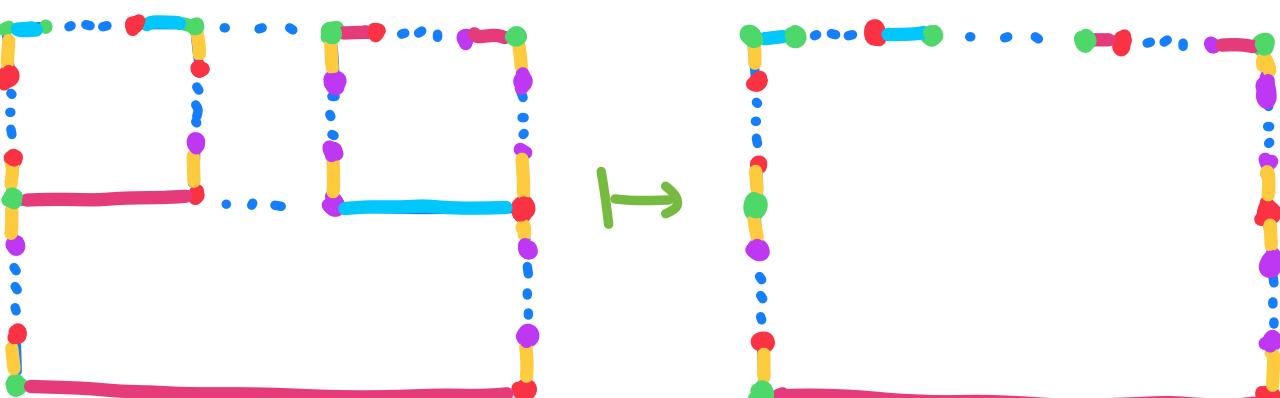
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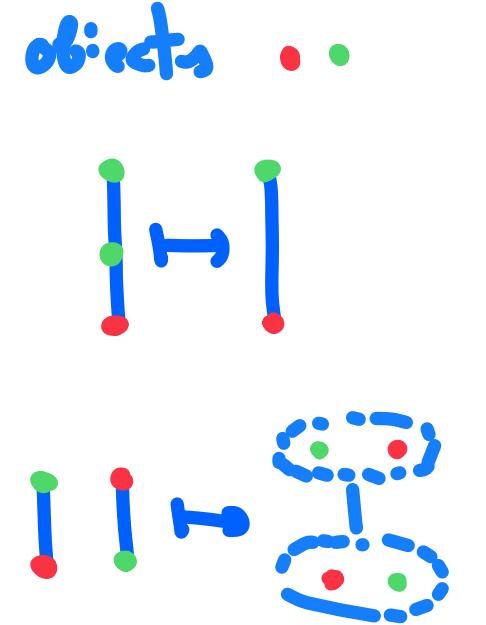


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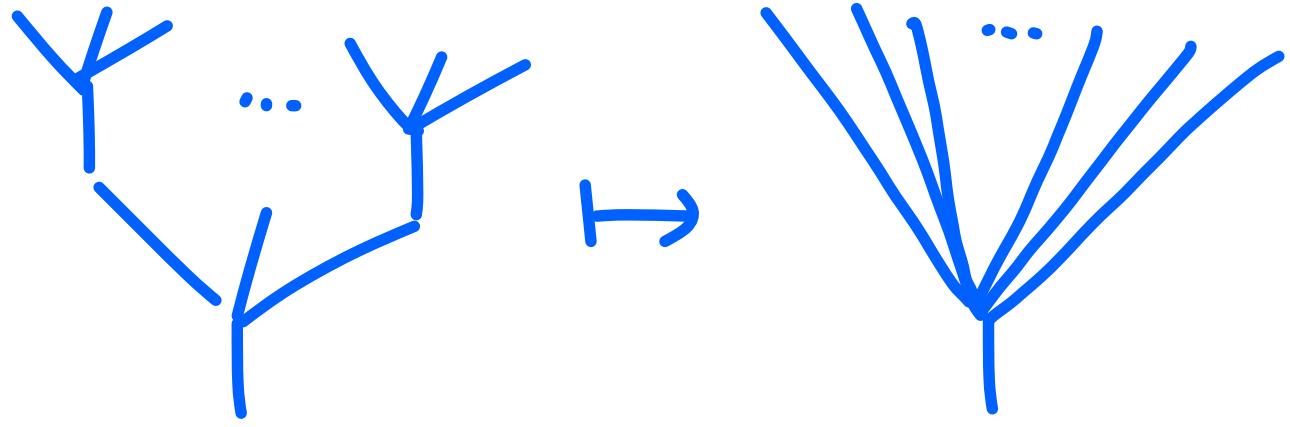


objects ...  
horizontal morphism —  
vertical morphism |



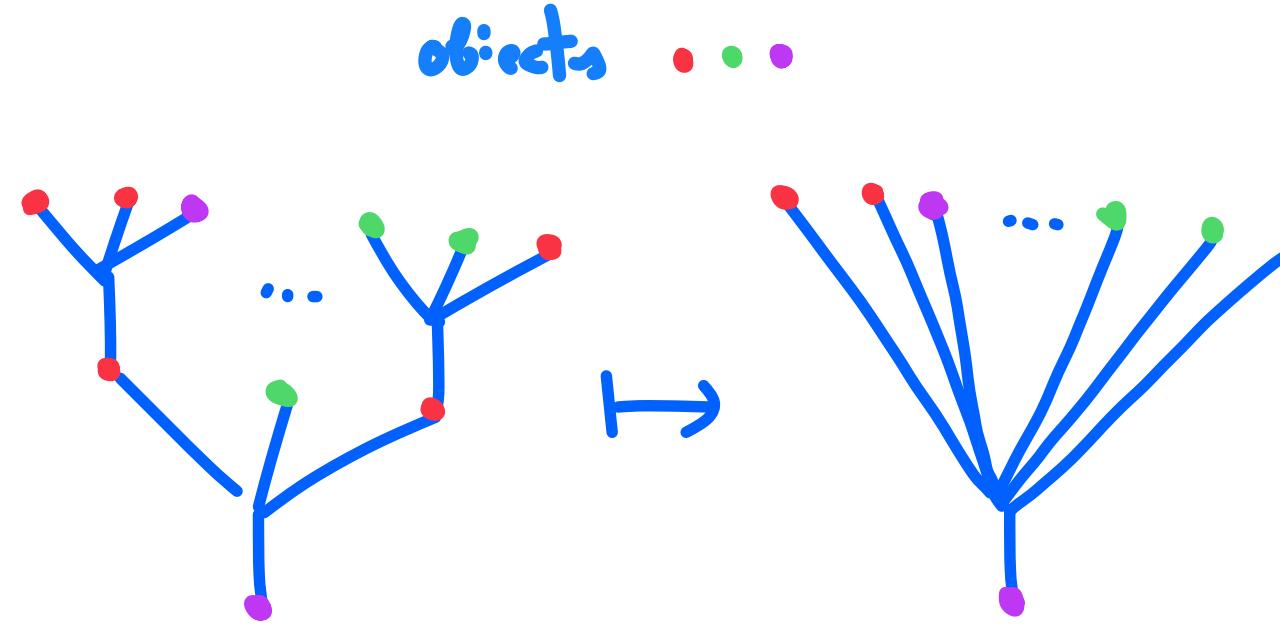
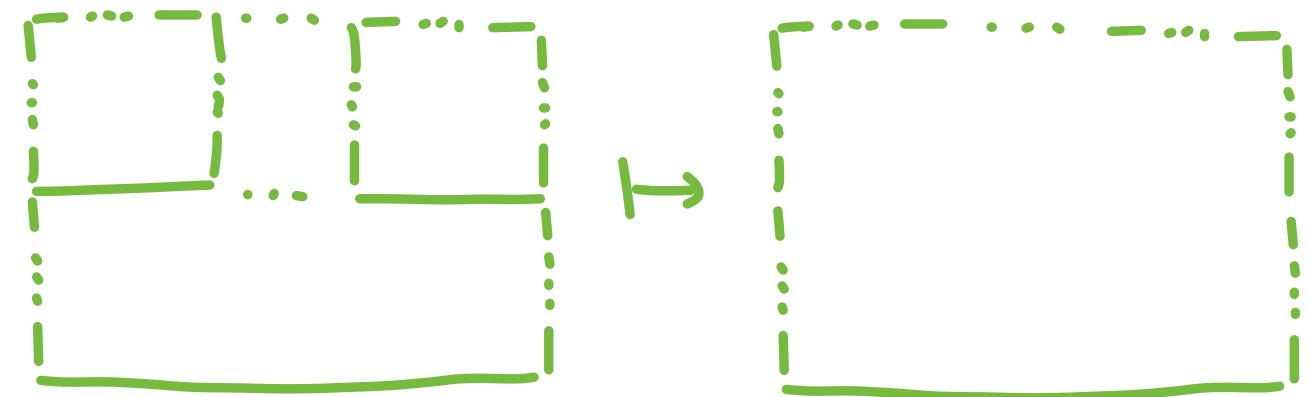
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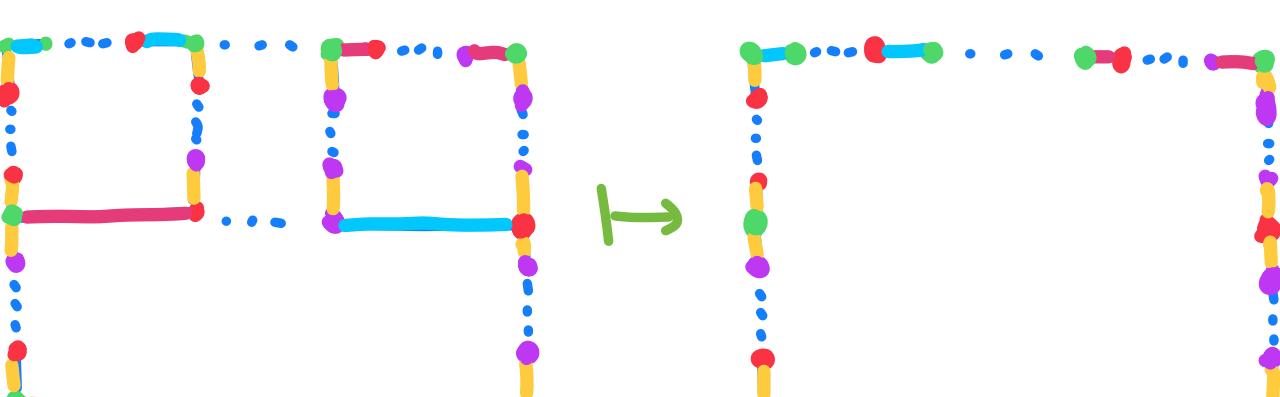
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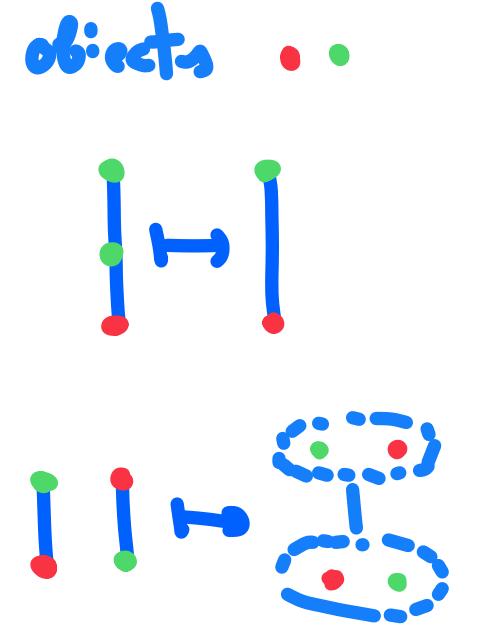


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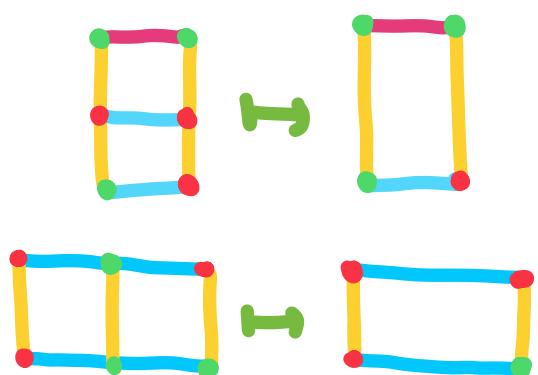


objects ...  
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Box operad  $\mathcal{B}$  enriched over  $(\mathcal{V}, \otimes, I)$

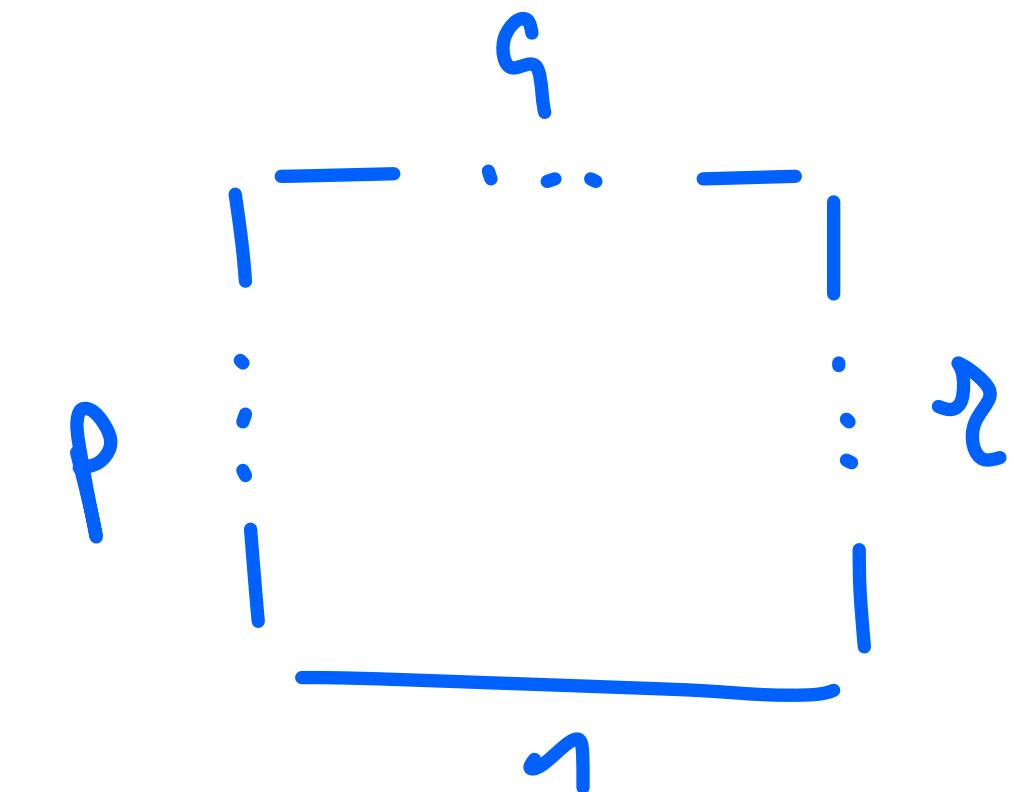
Box operad  $\beta$  enriched over  $(\mathcal{V}, \otimes, I)$  consists of

$\mathcal{V}$ -objects  $(\beta(p, q, r))_{p, q, r \geq 0}$

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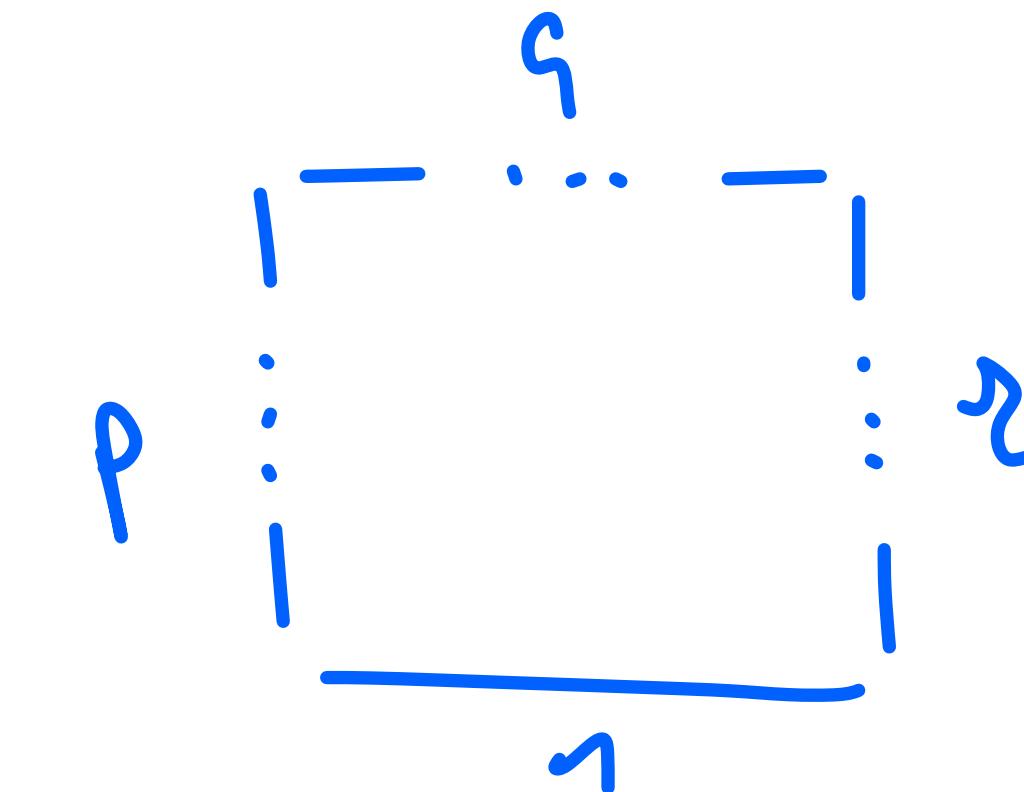
$$(\beta(p, q, r))_{p, q, r \geq 0}$$



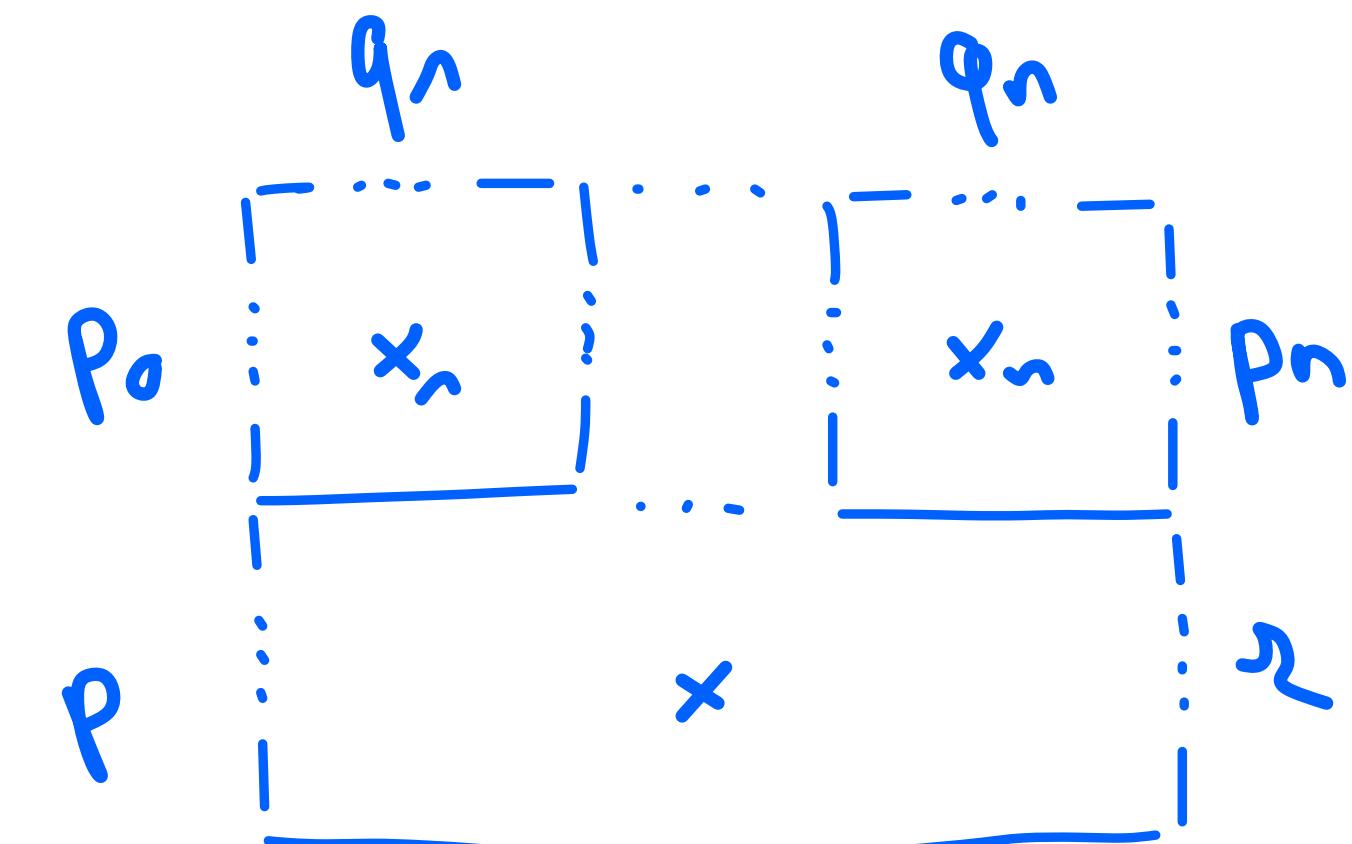
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compositions

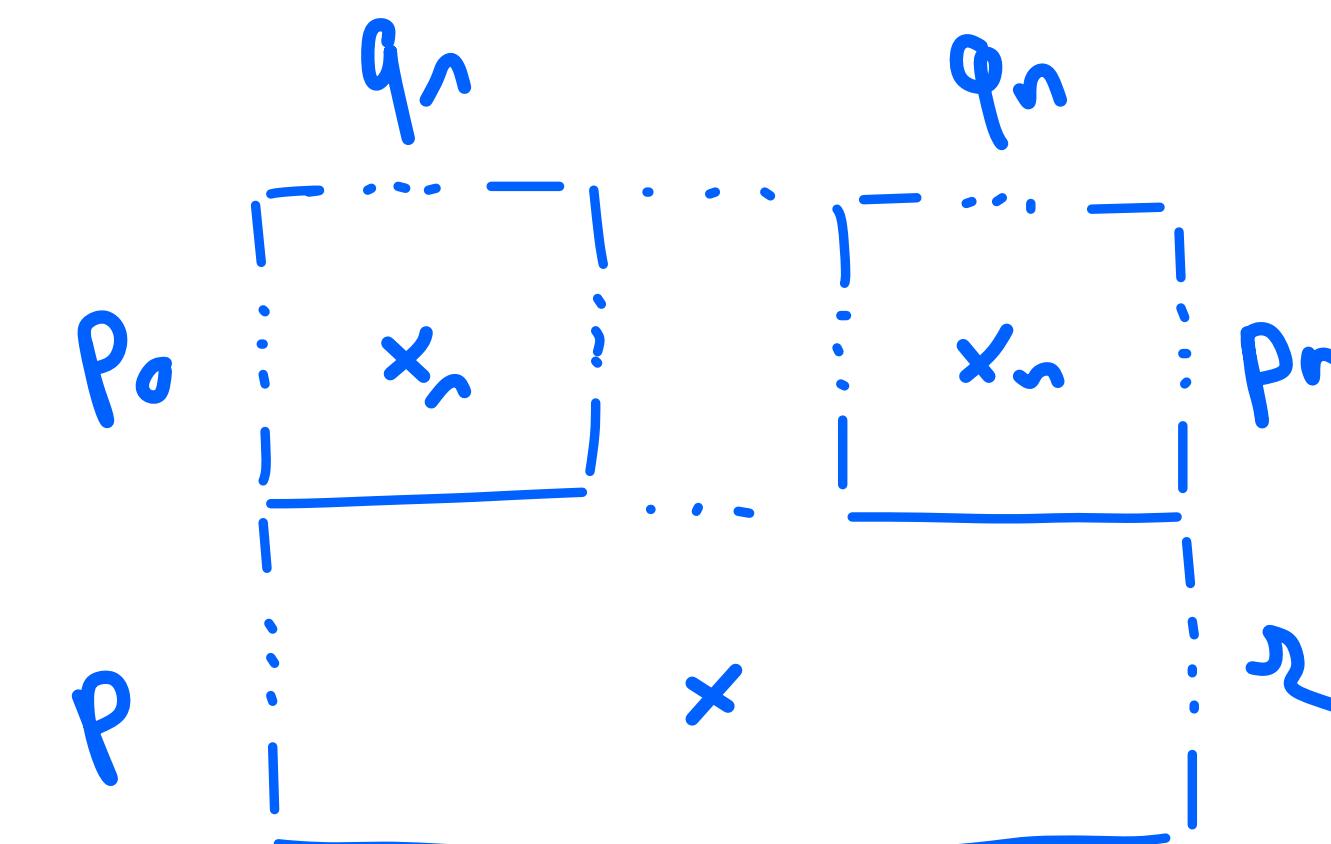


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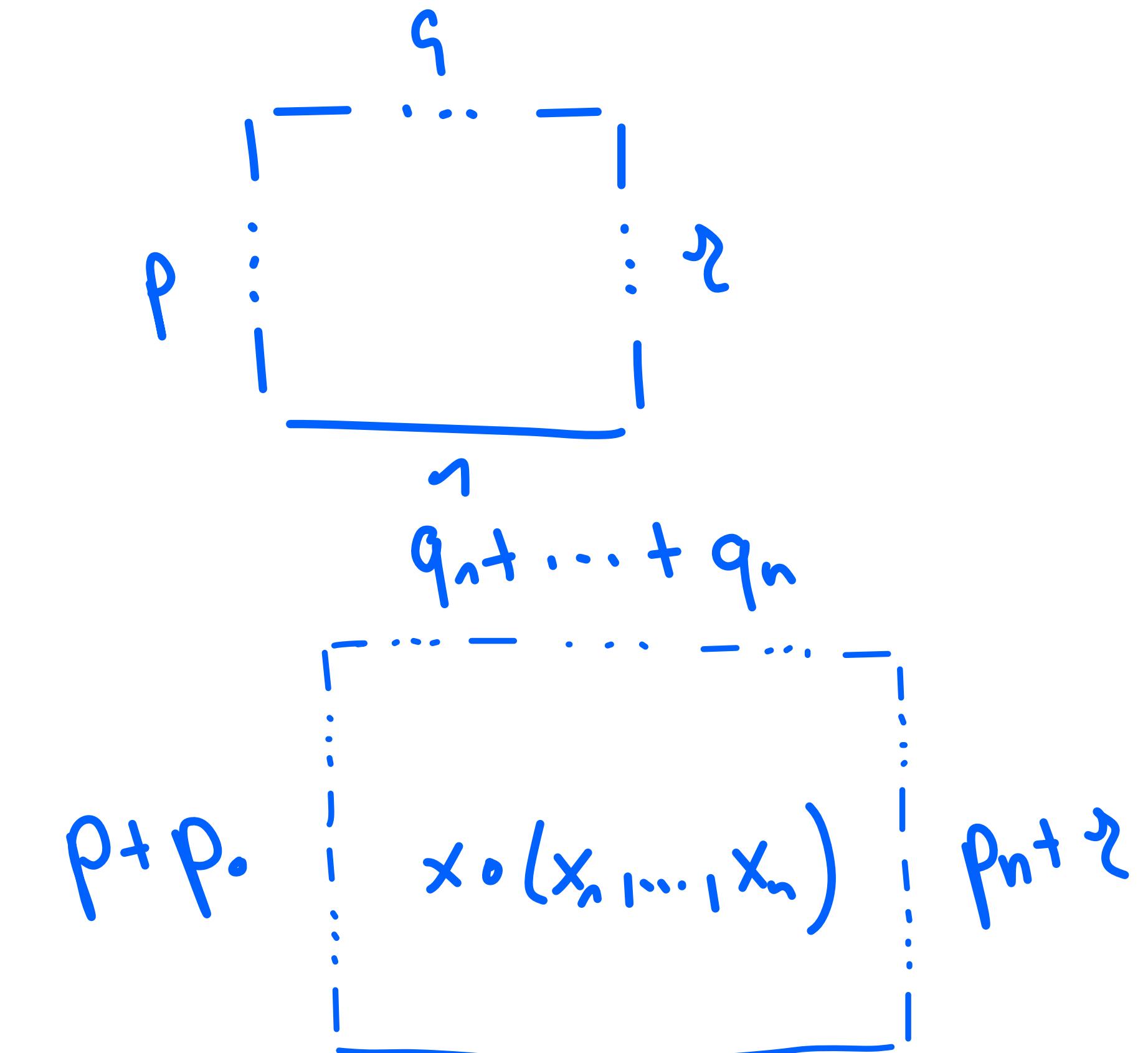
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compositions



$\mapsto$

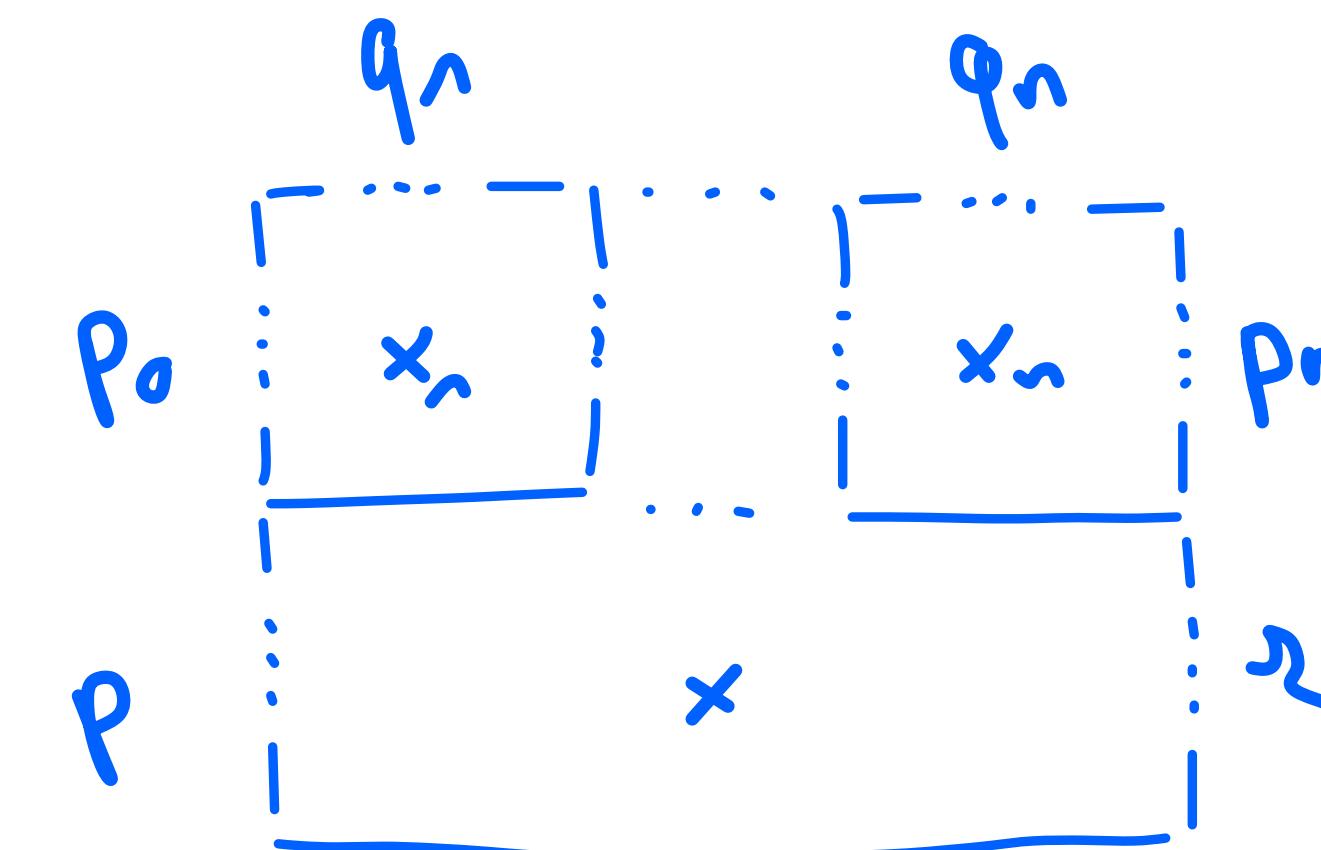


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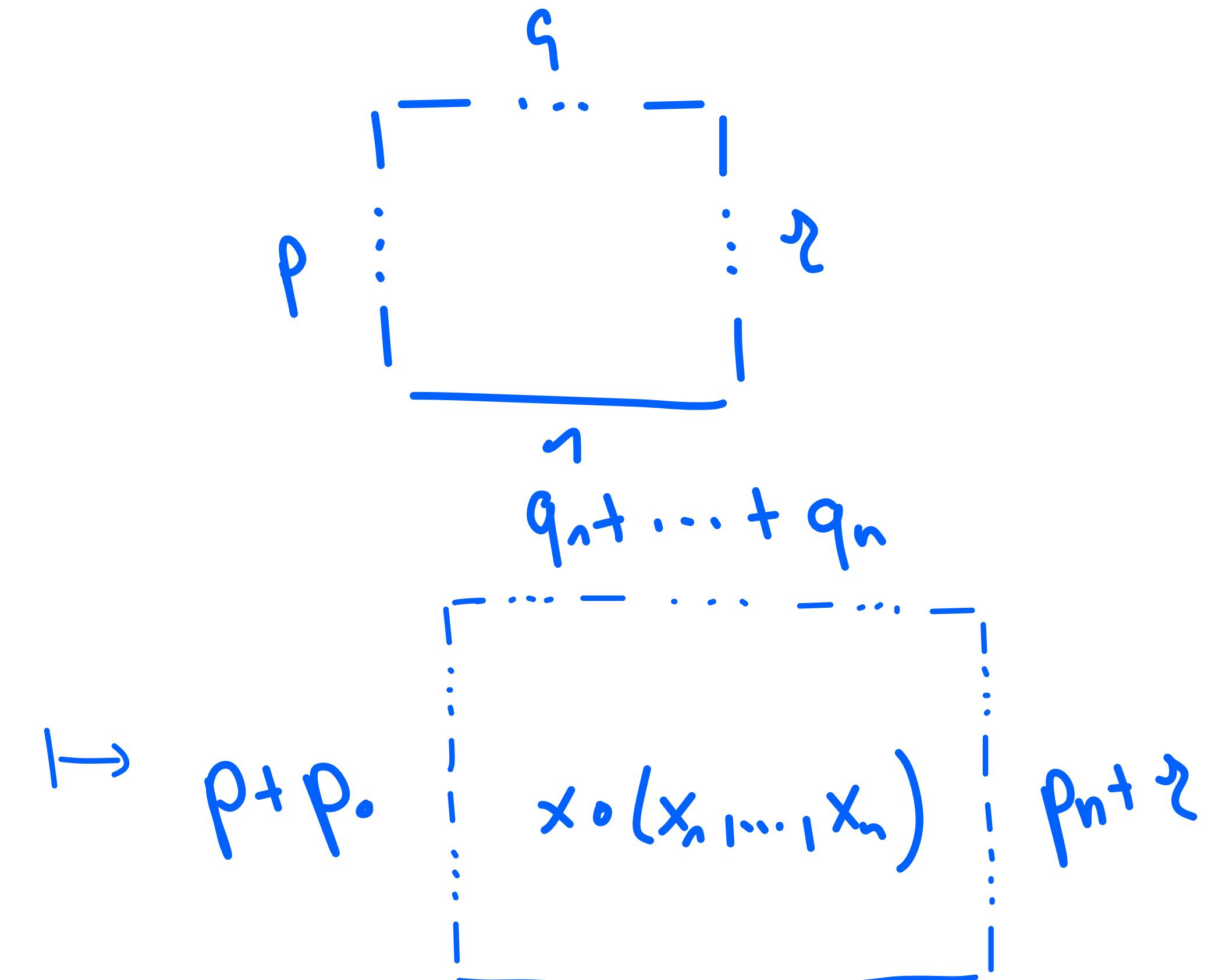
$$(\mathcal{B}(p, q, r))_{p, q, r \geq 0}$$

compositions



unit

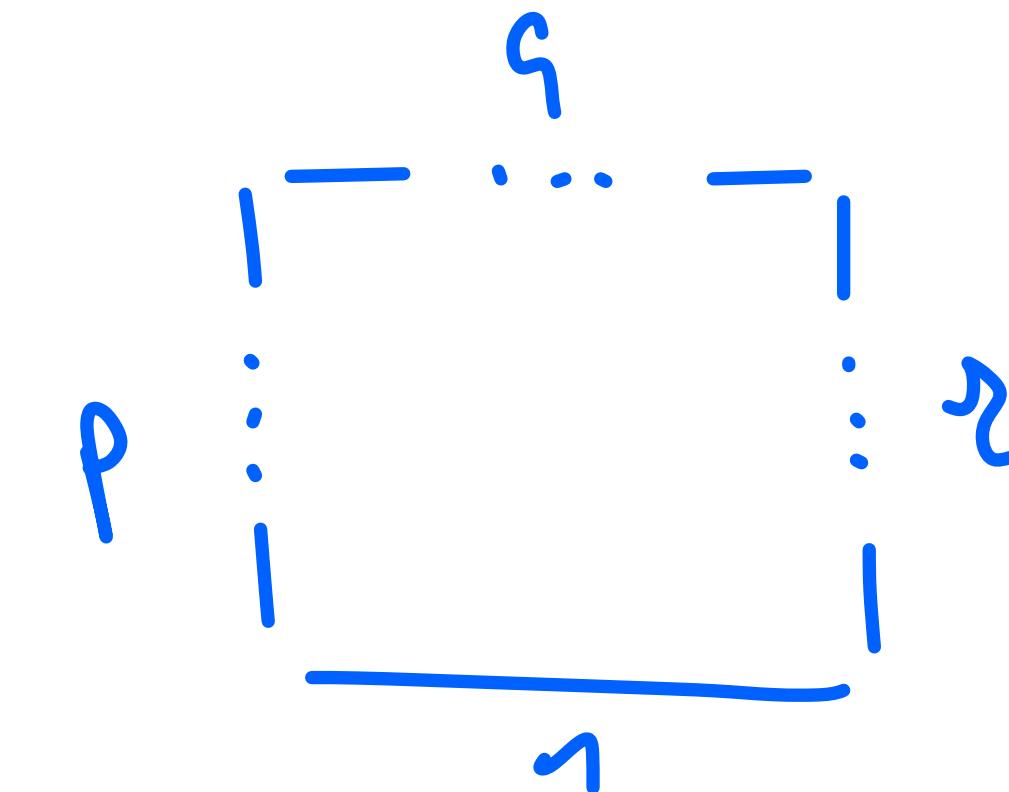
$$\circ \boxed{\eta} \circ$$



Box operad  $\mathcal{B}$  enriched over  $(\mathcal{V}, \otimes, I)$  consists of

$\mathcal{V}$ -objects

$$(\mathcal{B}(p, q, r))_{p, q, r \geq 0}$$



compositions

$$\mathcal{B}(p, n, r) \otimes \bigotimes_{i=1}^n \mathcal{B}(p_{i-1}, q_i, p_i) \xrightarrow{m} \mathcal{B}(p + p_0, \sum q_i, r + p_n)$$

unit

$$I \xrightarrow{n} \mathcal{B}(0, n, 0)$$

Box operad  $\mathcal{B}$  enriched over  $(\mathcal{V}, \otimes, I)$

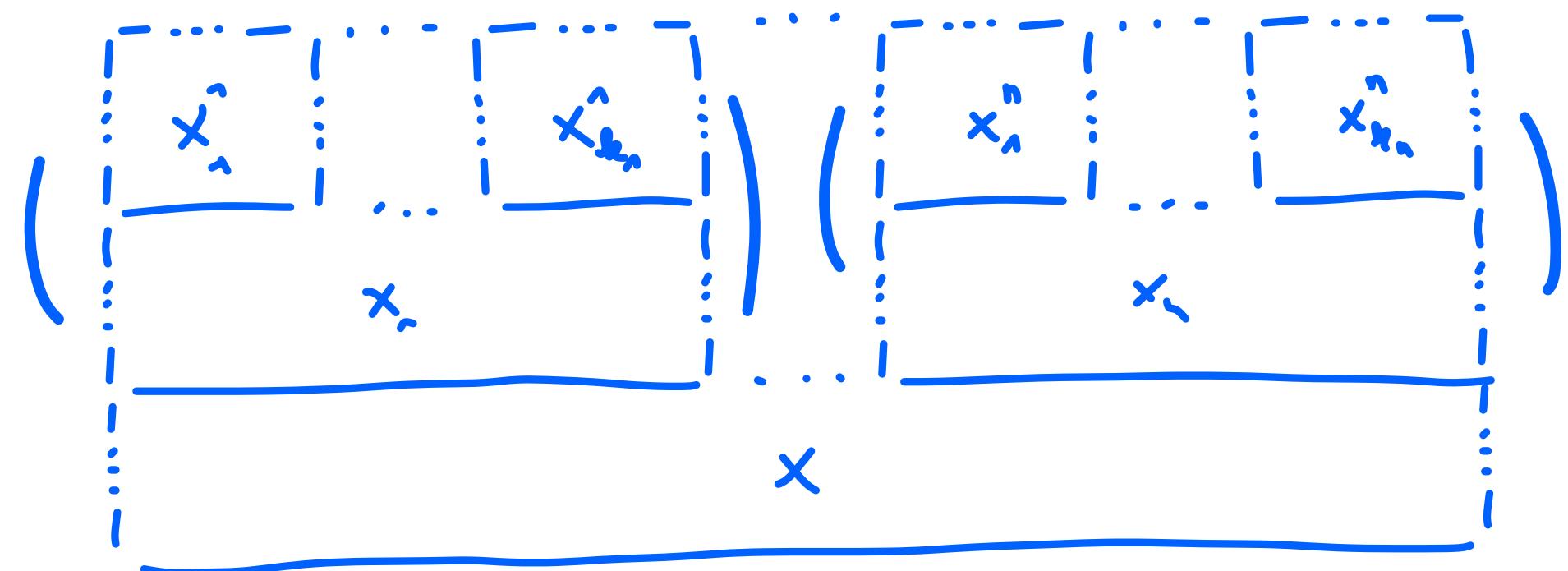
associativity

-

unit

Box operad  $\mathcal{B}$  enriched over  $(\mathcal{V}, \otimes, I)$

associativity



unit

Box operad  $\beta$  enriched over  $(\mathcal{V}, \otimes, I)$

associativity

$$\left( \left( \begin{array}{c|c} x_1 & x_2 \\ \hline x_3 & x_4 \end{array} \right) \otimes \left( \begin{array}{c|c} x_5 & x_6 \\ \hline x_7 & x_8 \end{array} \right) \right) = \left( \begin{array}{c|cc} x_1 & x_2 & x_6 \\ \hline x_3 & x_4 & x_8 \\ \hline x_5 & x_7 & x_8 \end{array} \right)$$

unit

Box operad  $\mathcal{B}$  enriched over  $(\mathcal{V}, \otimes, I)$

associativity

$$\left( \begin{array}{c|c} \text{---} & \text{---} \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ \text{---} & \text{---} \end{array} \right) \left( \begin{array}{c|c} \text{---} & \text{---} \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ \text{---} & \text{---} \end{array} \right) = \left( \begin{array}{c|c|c|c} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right)$$

unit

$$\left( \begin{array}{c|c} \text{---} & \text{---} \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ \text{---} & \text{---} \end{array} \right) = \left( \begin{array}{c|c|c|c} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right) = \left( \begin{array}{c|c} \text{---} & \text{---} \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ \text{---} & \text{---} \end{array} \right)$$

# Peculiar Degeneracies

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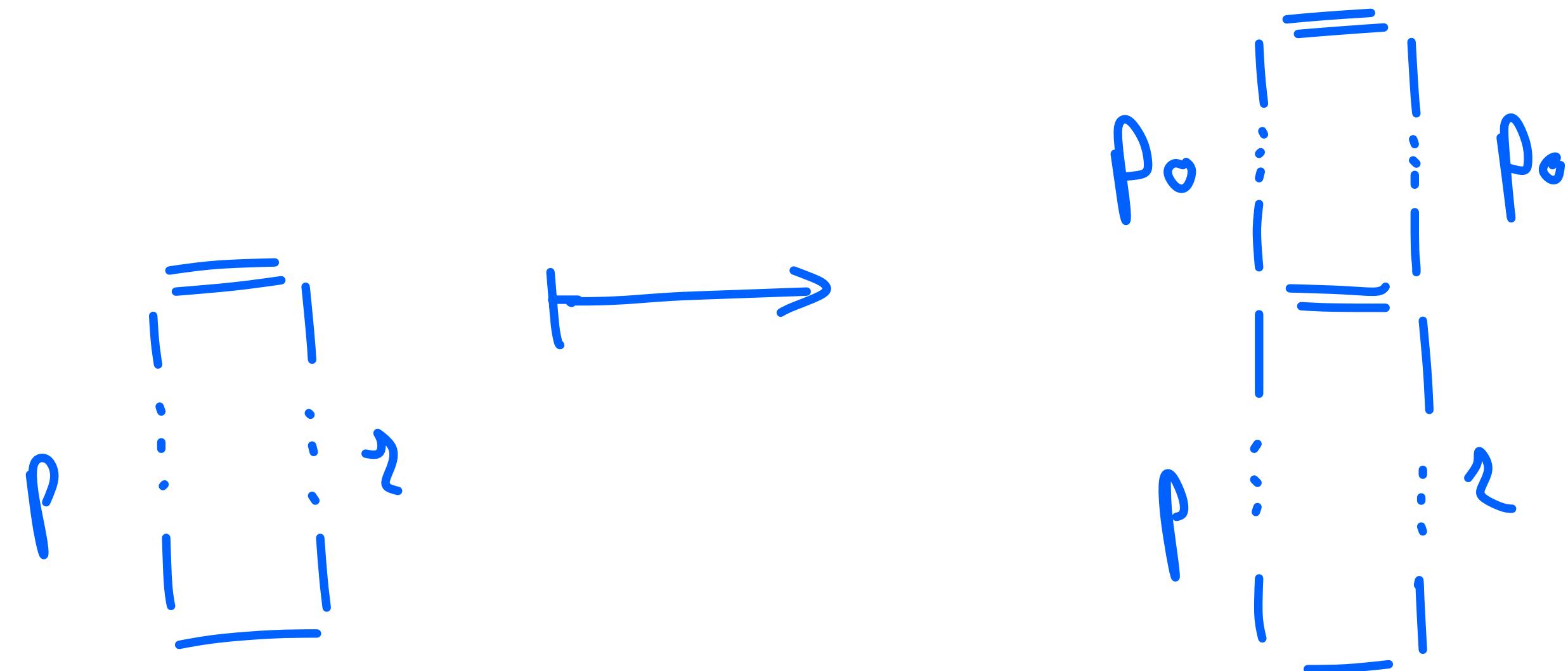
# Peculiar Degeneracies

For  $n = 0$   $B(p, 0, \gamma)$

$$\begin{matrix} & \overline{\overline{1}} \\ p & \vdots & \vdots & \gamma \\ & \underline{\underline{1}} \end{matrix}$$

# Peculiar Degeneracies

For  $n = 0$   $B(p, 0, \gamma) \xrightarrow{\mu} B(p + p_0, 0, \gamma + p_0)$



Operads = *thin* box operads

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BoxOperads ← Operads

Operads = *thin* box operads

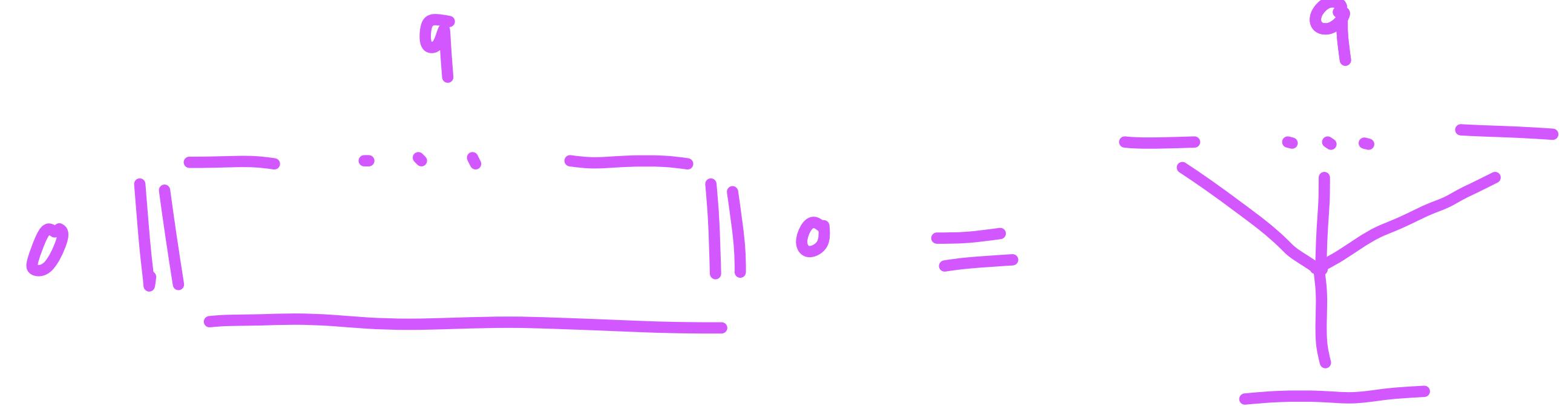
BoxOperads ← Operads

thin box

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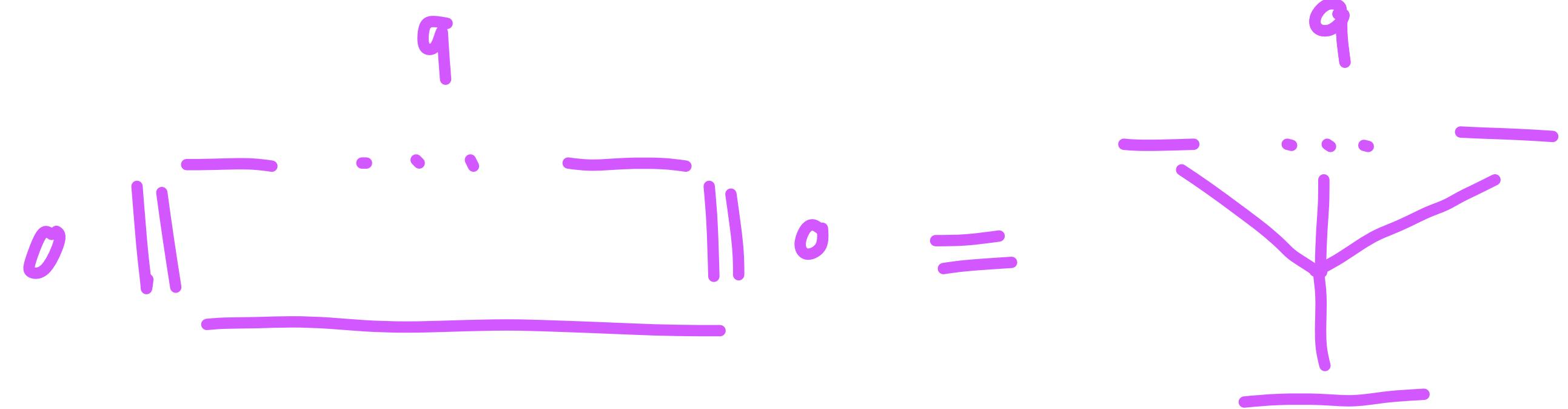
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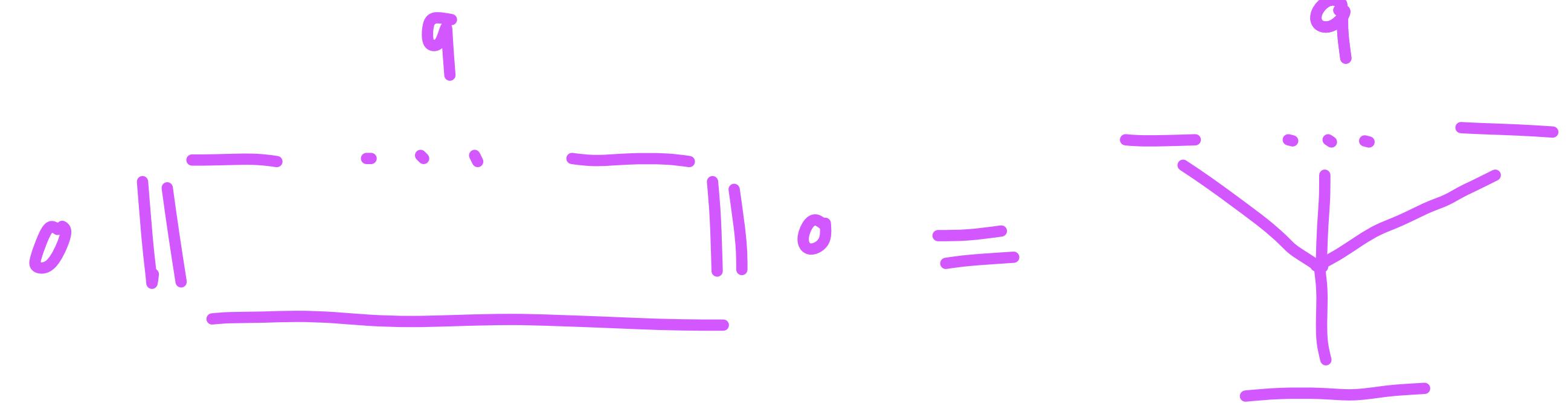


partial  
composition

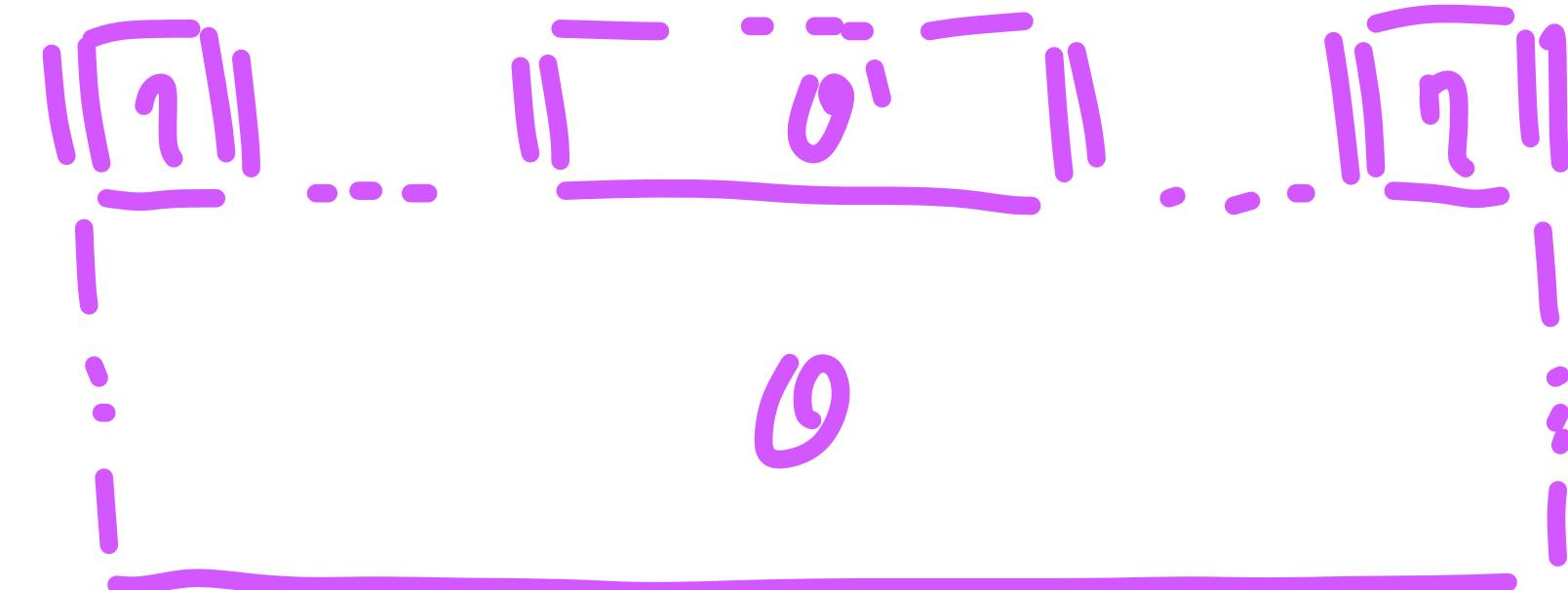
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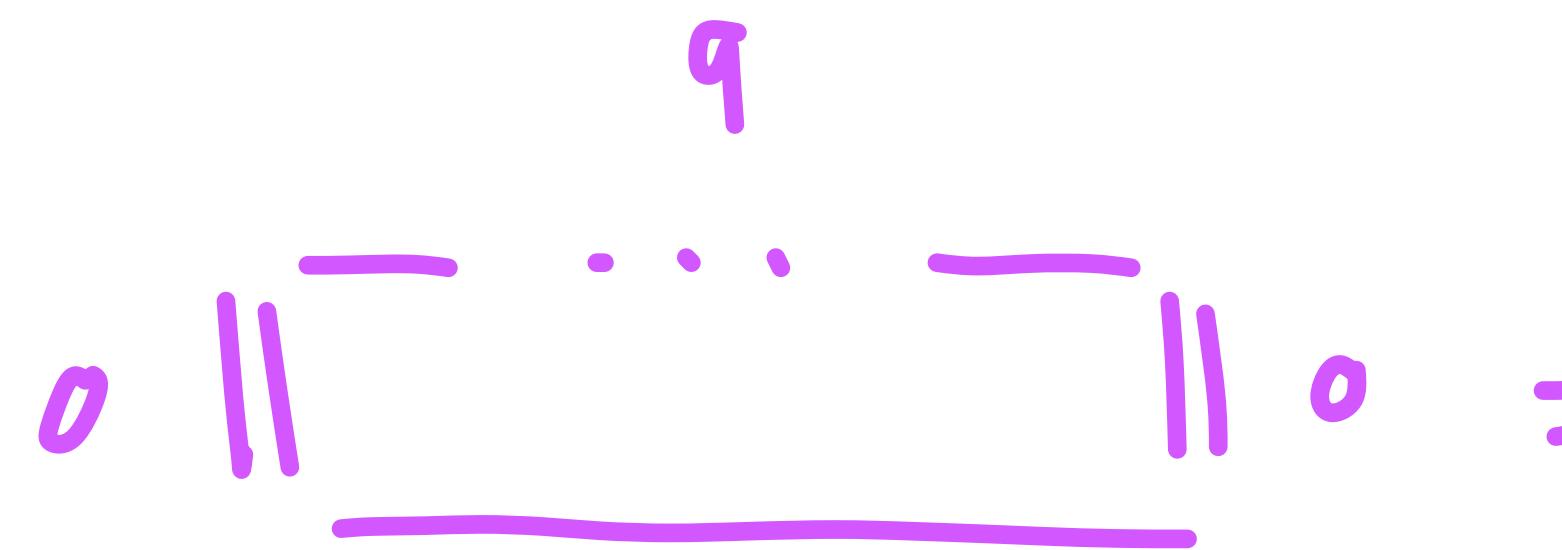
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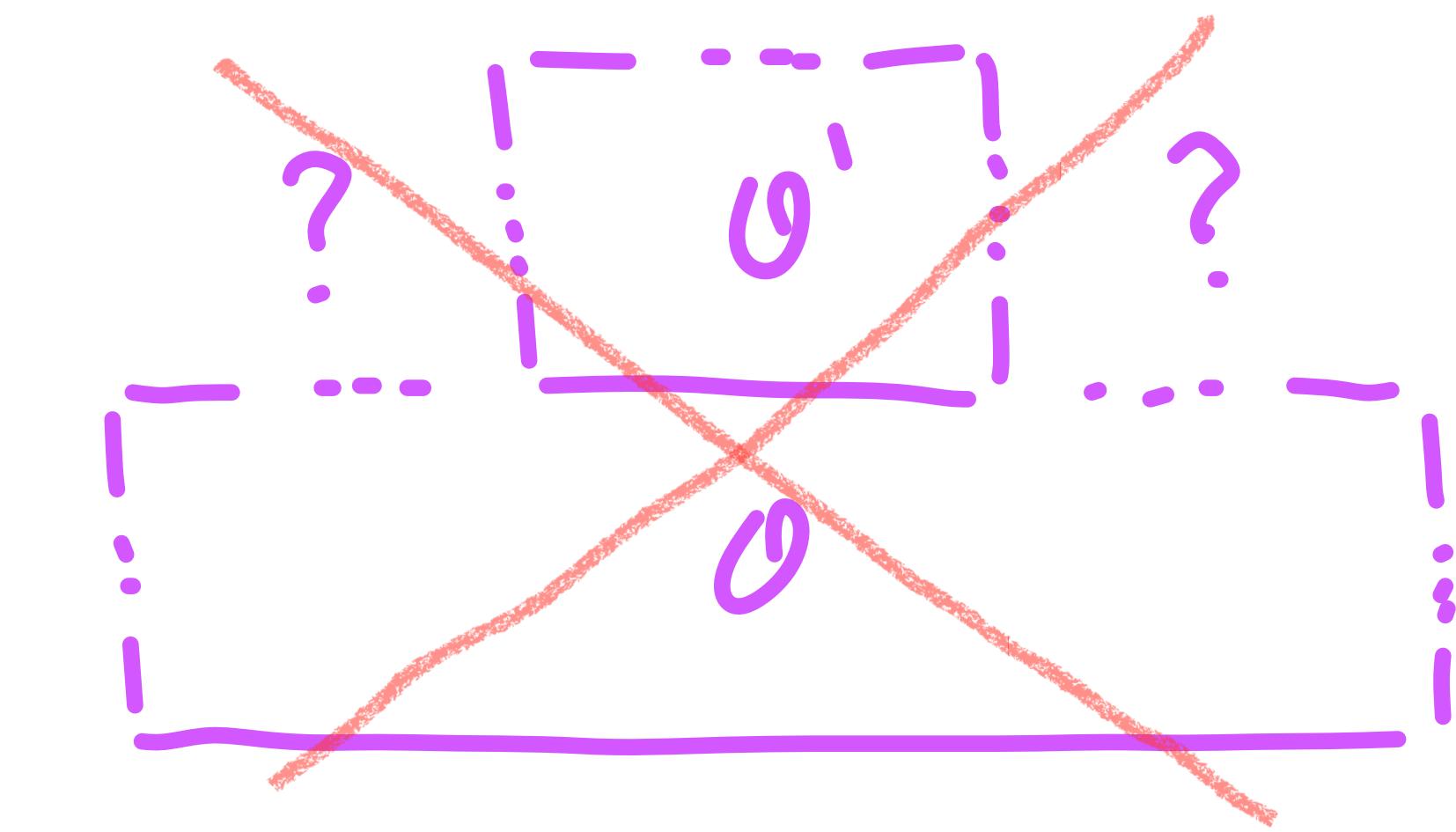
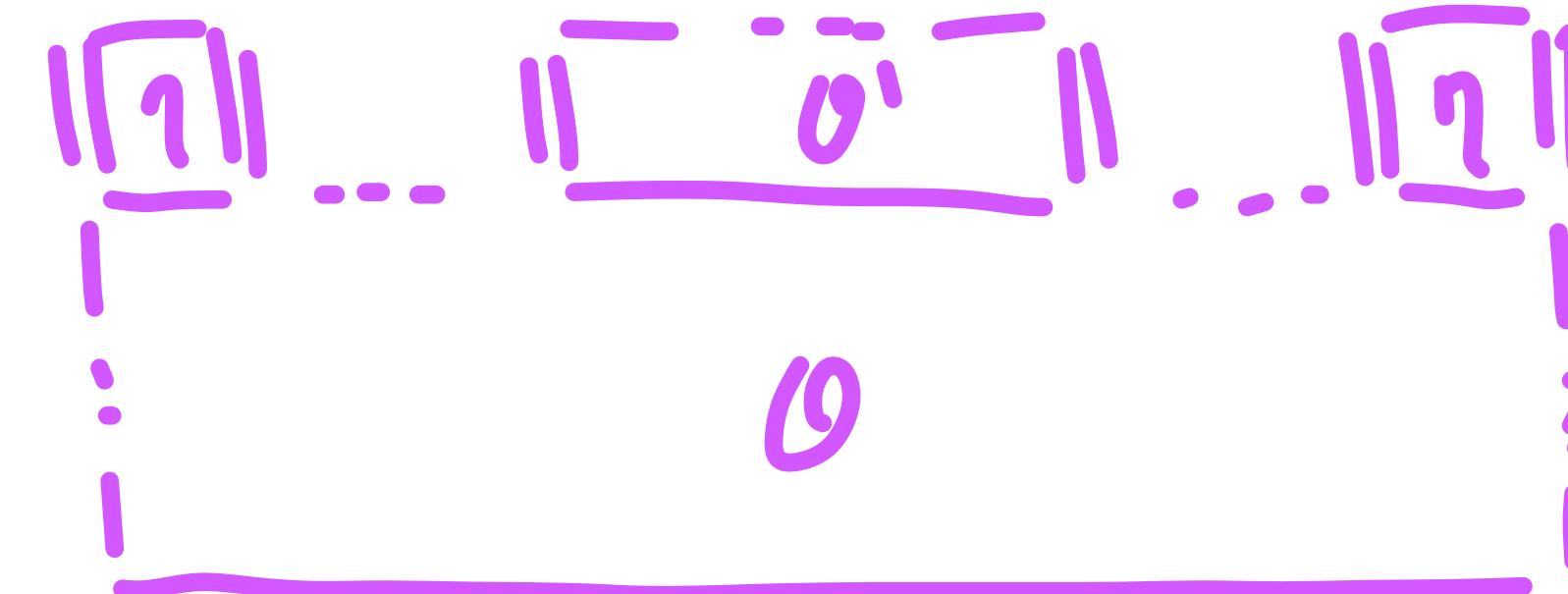
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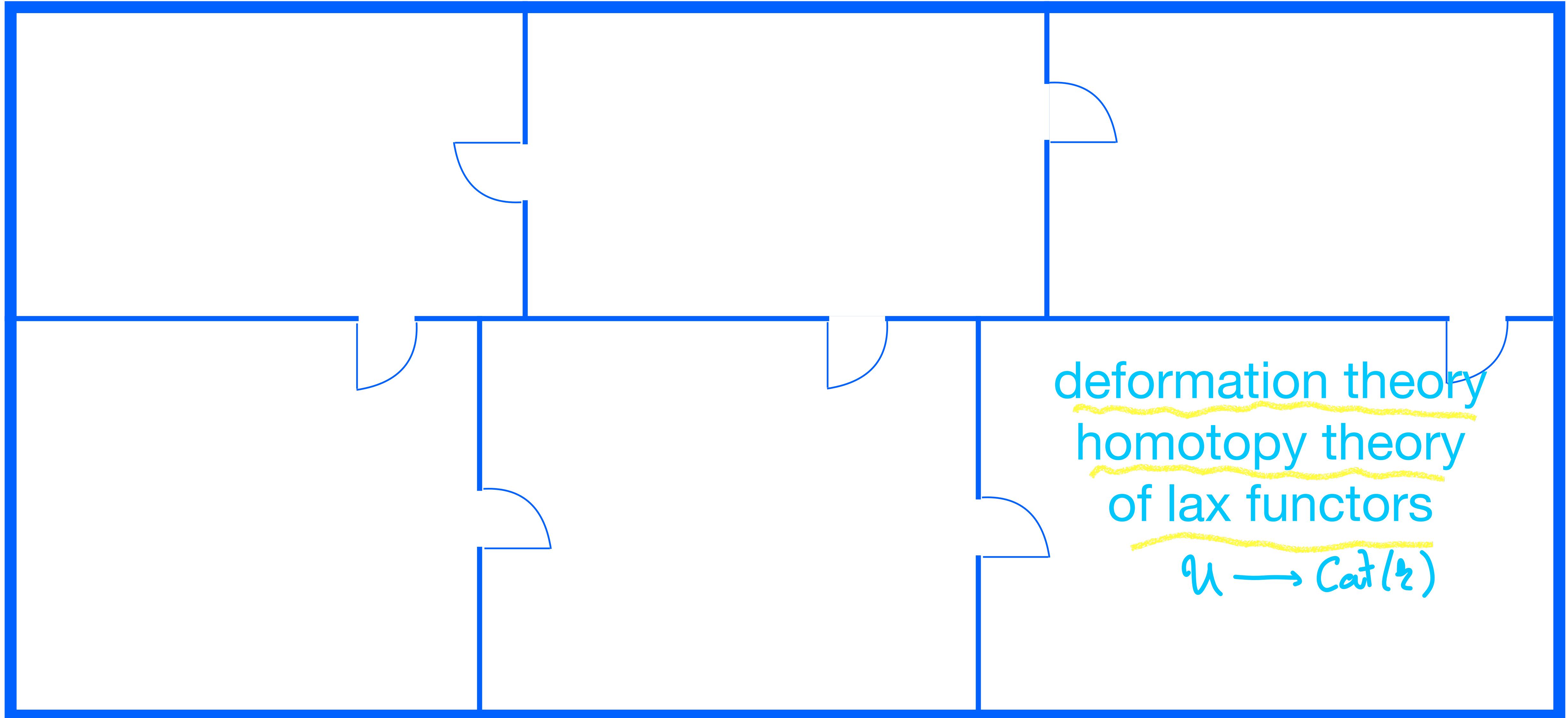


partial  
composition



Part II

# The Floorplan



# The Floorplan

box operads  
= monoids  
in a skew  
monoidal category

deformation theory  
homotopy theory  
of lax functors  
 $\mathcal{U} \rightarrow \text{Cat}(\mathbb{R})$

Box Composite

$$\square : \mathcal{V}^{\mathbb{N}^3} \times \mathcal{V}^{\mathbb{N}^3} \longrightarrow \mathcal{V}^{\mathbb{N}^3}$$

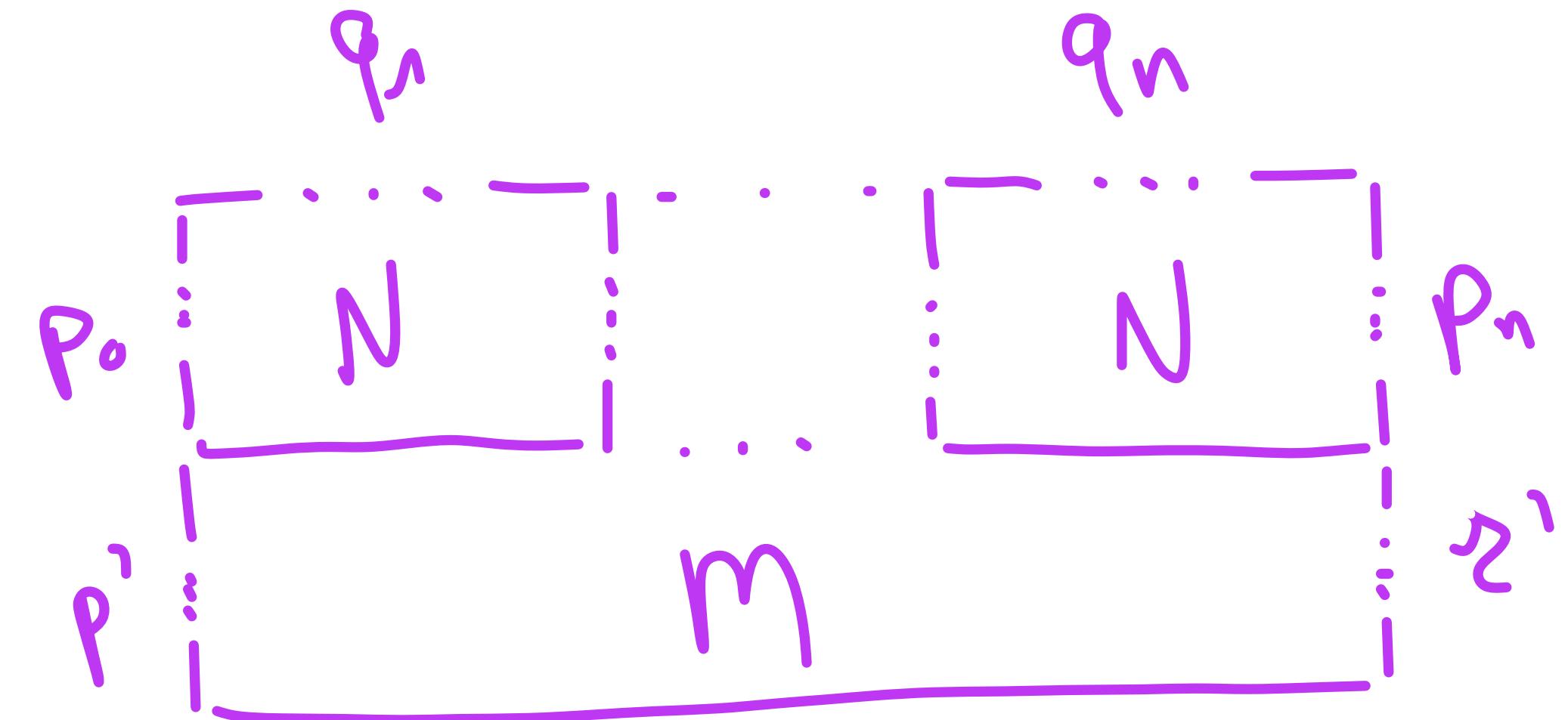
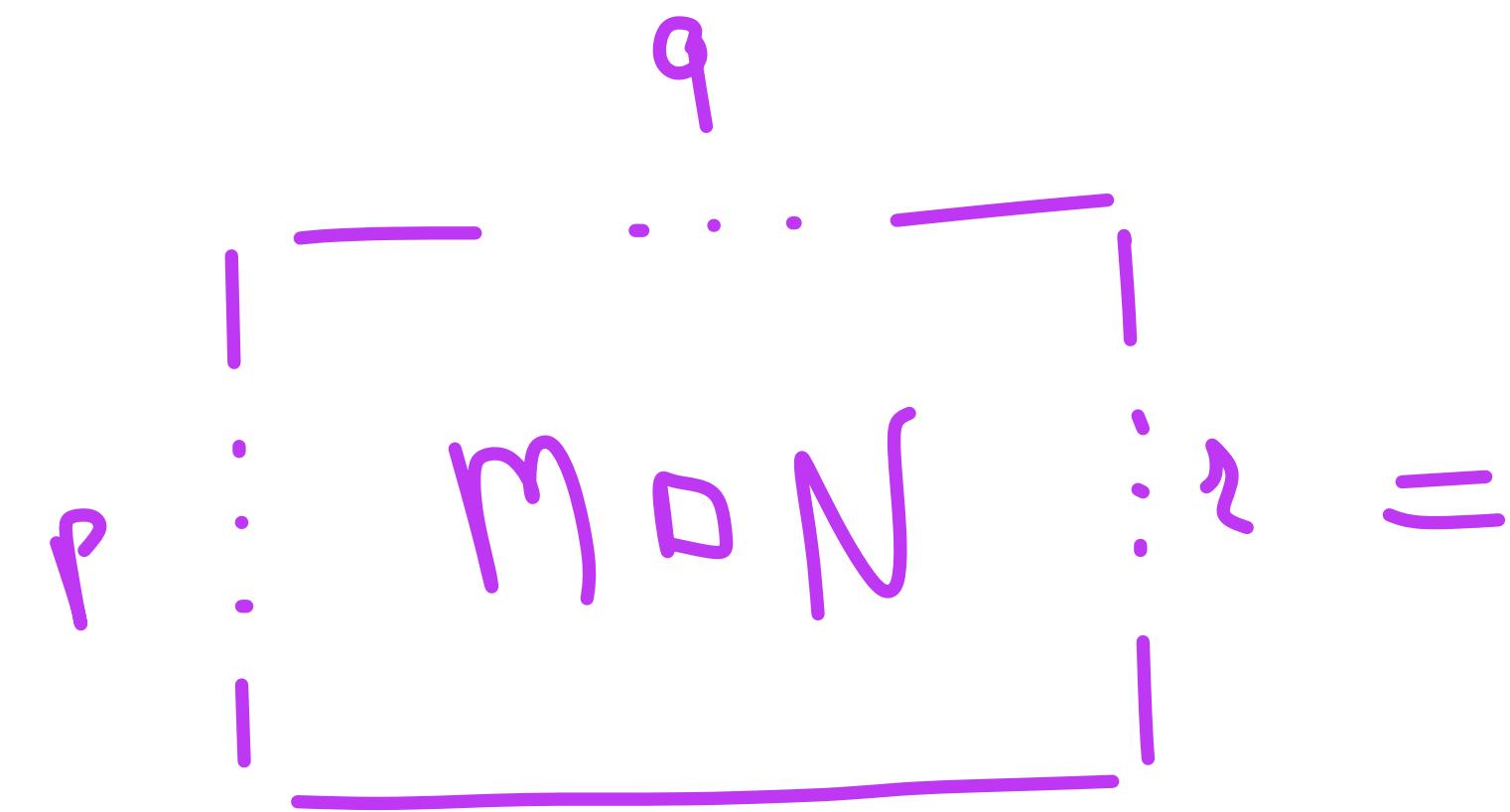
Box Composite

$$\square : \mathcal{V}^{\mathbb{N}^3} \times \mathcal{V}^{\mathbb{N}^3} \rightarrow \mathcal{V}^{\mathbb{N}^3}$$

$$p : \boxed{m \circ N} =$$

# Box Composite

$$\square : \mathcal{V}^{IN^3} \times \mathcal{V}^{IN^3} \rightarrow \mathcal{V}^{IN^3}$$



# Box Composite

$$\square : \mathcal{V}^{IN^3} \times \mathcal{V}^{IN^3} \rightarrow \mathcal{V}^{IN^3}$$

$$P : \boxed{M \bowtie N} \vdash q = \boxed{\quad}$$
$$\begin{aligned} p_0 + p' &= P \\ p_n + z' &= q \\ \sum q_i &= q \end{aligned}$$

$$P_0 \quad \boxed{\dots} \quad \boxed{N} \quad \dots \quad \boxed{N} \quad P_n$$
$$P' \quad \boxed{\dots} \quad m \quad \boxed{\dots} \quad \boxed{\dots} \quad z'$$

# Box Composite

$$\square : \mathcal{V}^{IN^3} \times \mathcal{V}^{IN^3} \rightarrow \mathcal{V}^{IN^3}$$

$$p : \boxed{m \circ N} = \boxed{\begin{array}{c} p_0 + p' = p \\ p_n + z' = z \\ \sum q_i = q \end{array}}$$

$$p_0 \quad \boxed{N} \quad \dots \quad \boxed{N} \quad p_n \\ p' \quad \quad \quad m \quad \quad \quad z'$$

Proposition  $(\mathcal{V}^{IN^3}, \square, I)$  is a skew monoidal category

# Box Composite

$$\square : \mathcal{V}^{IN^3} \times \mathcal{V}^{IN^3} \rightarrow \mathcal{V}^{IN^3}$$

$$p : \boxed{m \circ N} = \boxed{\quad} \begin{array}{l} p_0 + p' = p \\ p_n + z' = z \\ \sum q_i = q \end{array}$$

$$p_0 \boxed{\dots \overset{q_1}{\cdots} \overset{N}{\boxed{\quad}} \dots \overset{q_n}{\cdots} \overset{N}{\boxed{\quad}}} \vdash p_n \quad z'$$

Proposition  $(\mathcal{V}^{IN^3}, \square, I)$  is a skew monoidal category  
 concentrated in  $(0, 1, 0)$ .



Proposition

$$(\mathcal{V}^{\text{IN}^3}, \sqcap, \top)$$

(left normal)

is a skew monoidal category

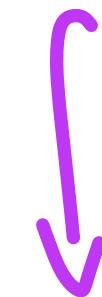
Proposition

$$(\mathcal{V}^{\text{IN}^3}, \sqcap, \mathbb{I})$$

(left normal)

is a skew monoidal category

$$(M \odot N) \square K$$



$$M \odot (N \odot K)$$

Proposition

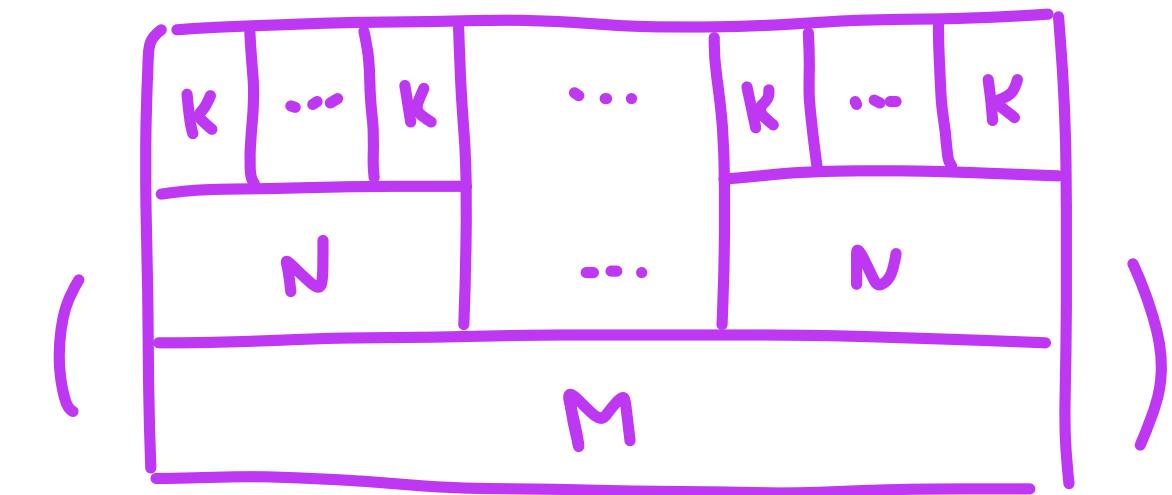
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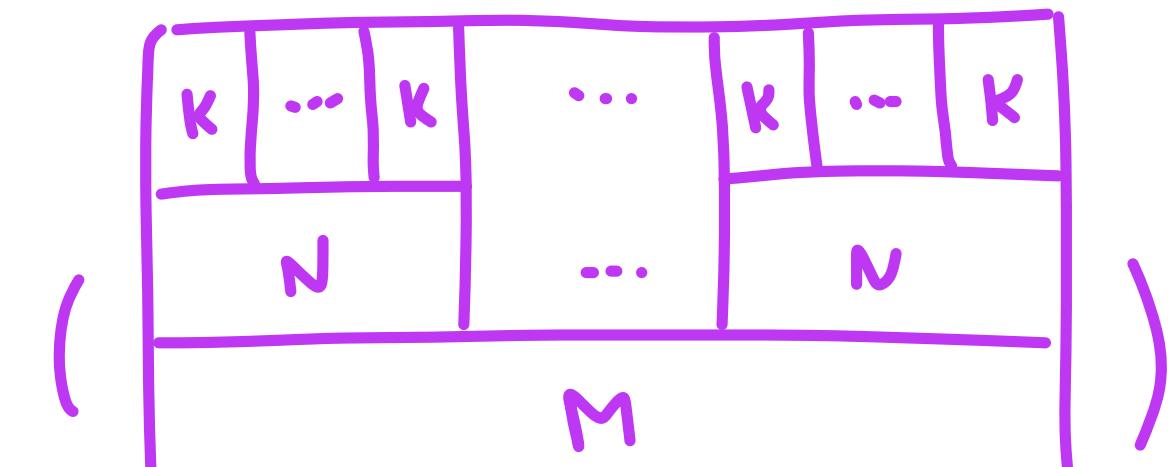
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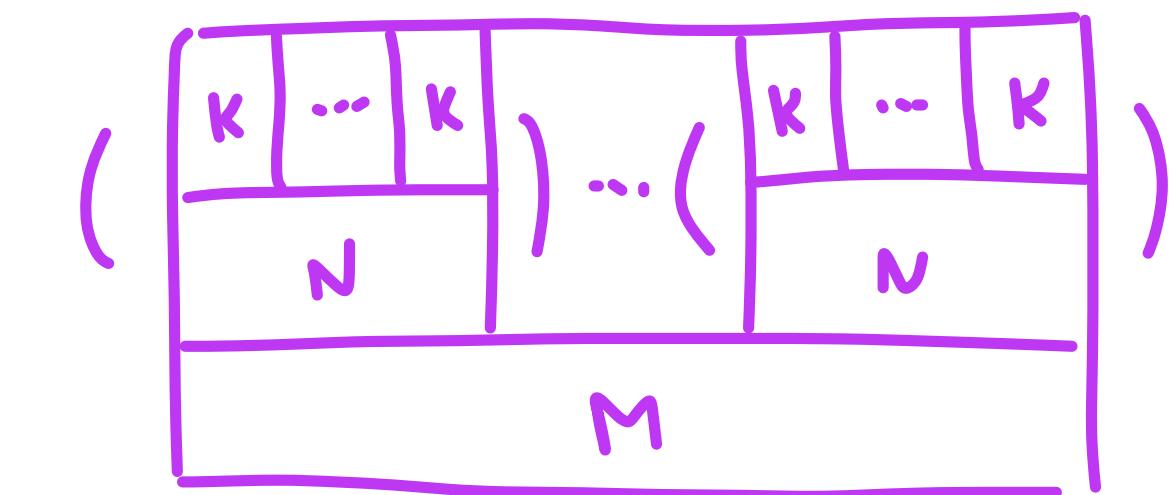
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$$M \square (N \square K)$$



IZ



Proposition

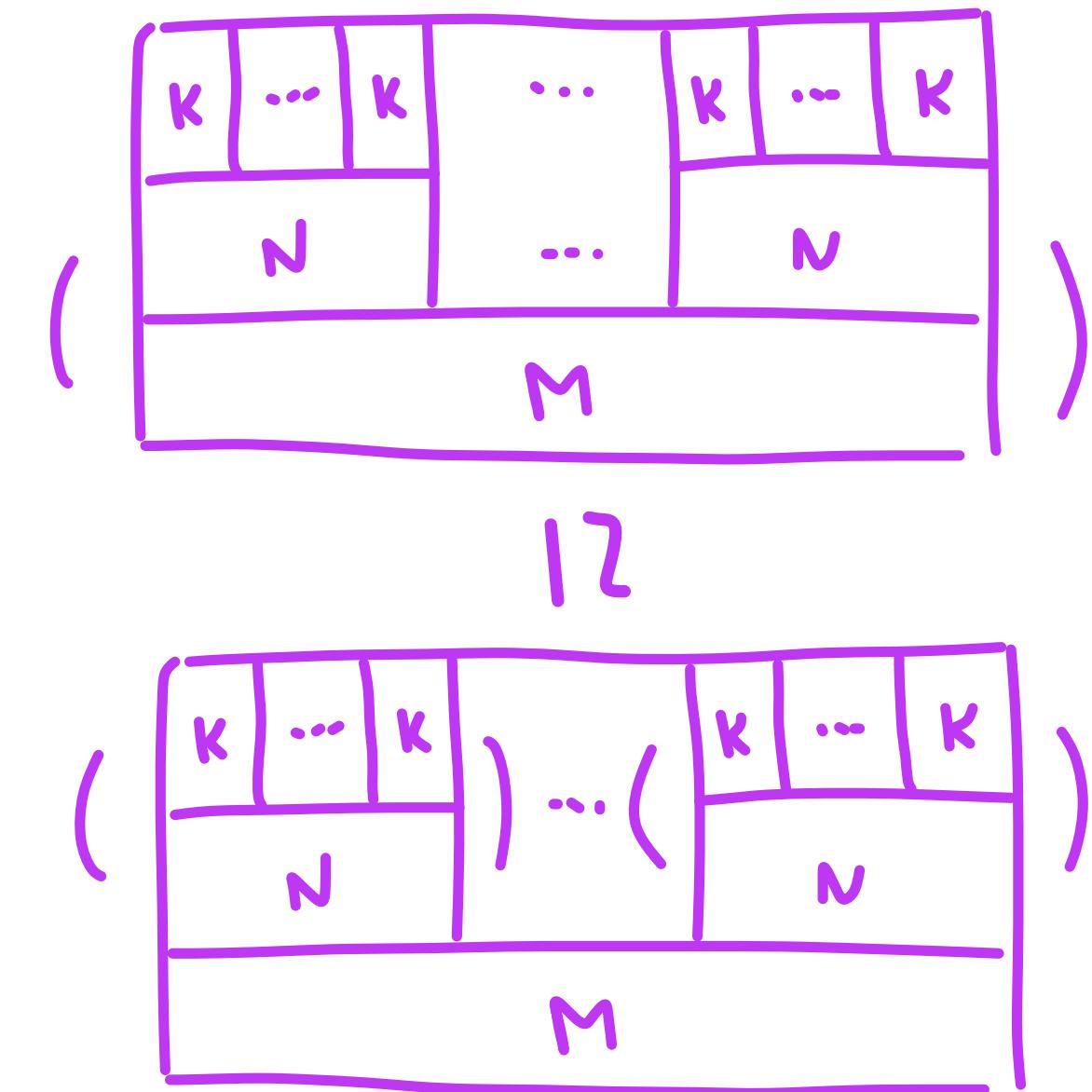
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$$(M \square N) \square K$$

$\downarrow$   ~~$\times$~~

$$M \square (N \square K)$$

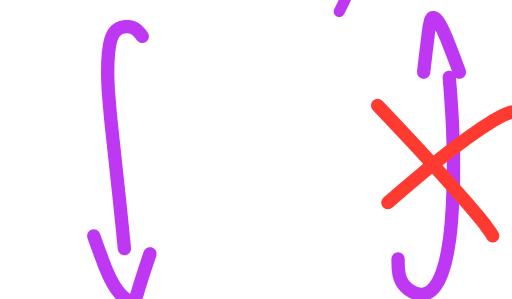


Proposition

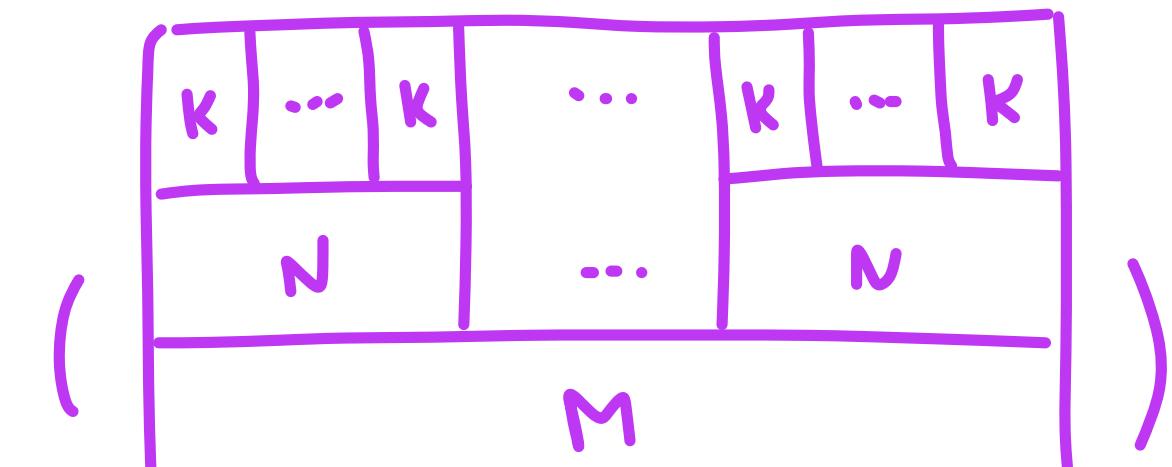
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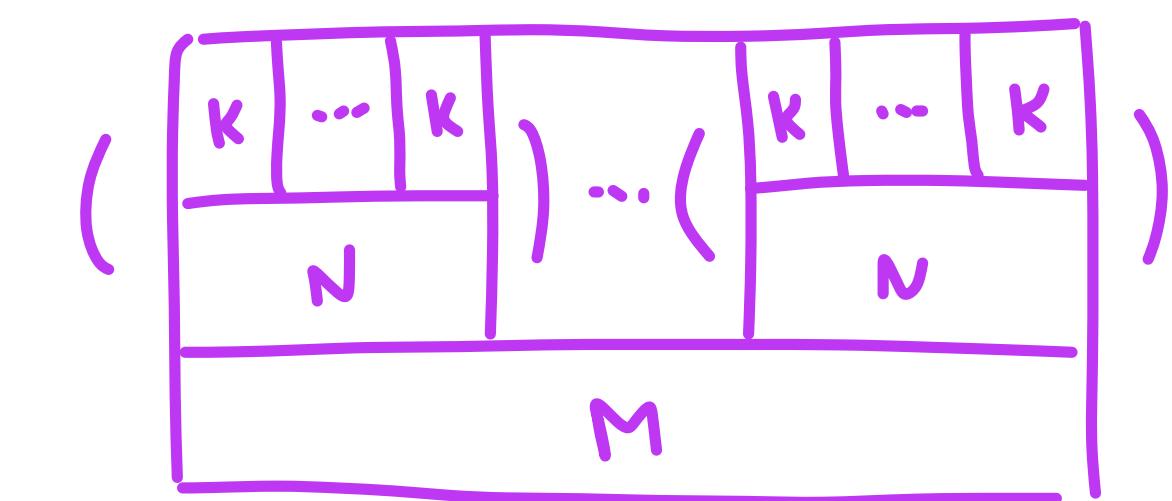
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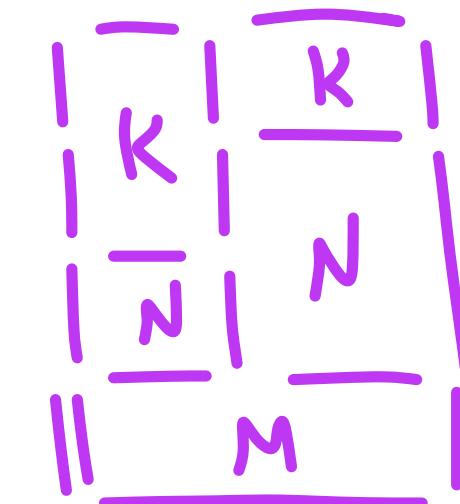
$$M \odot (N \odot K)$$



12



X

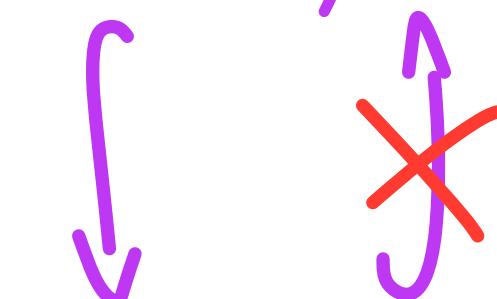


Proposition

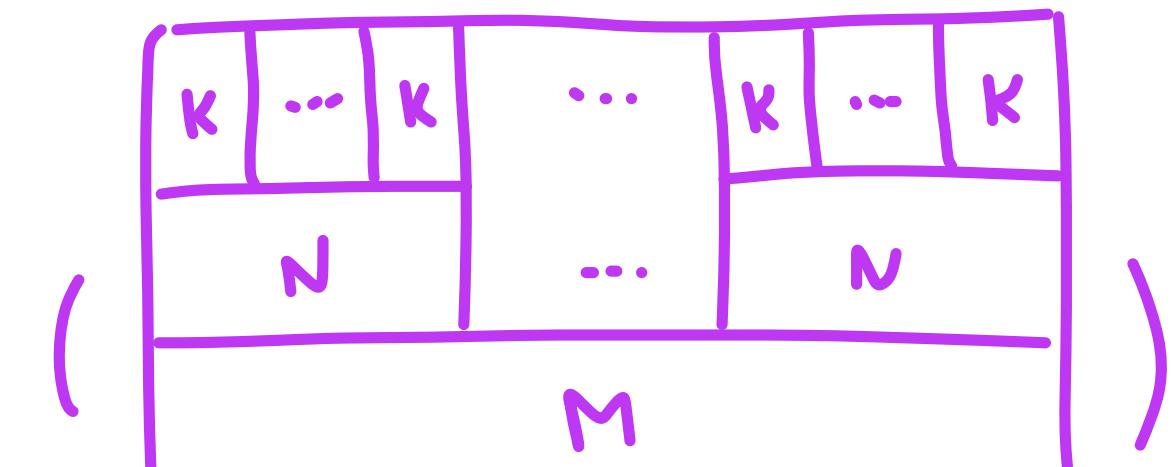
$(\mathcal{V}^{\mathbb{W}^3}, \square, \mathbf{I})$  (left normal)

is a skew monoidal category

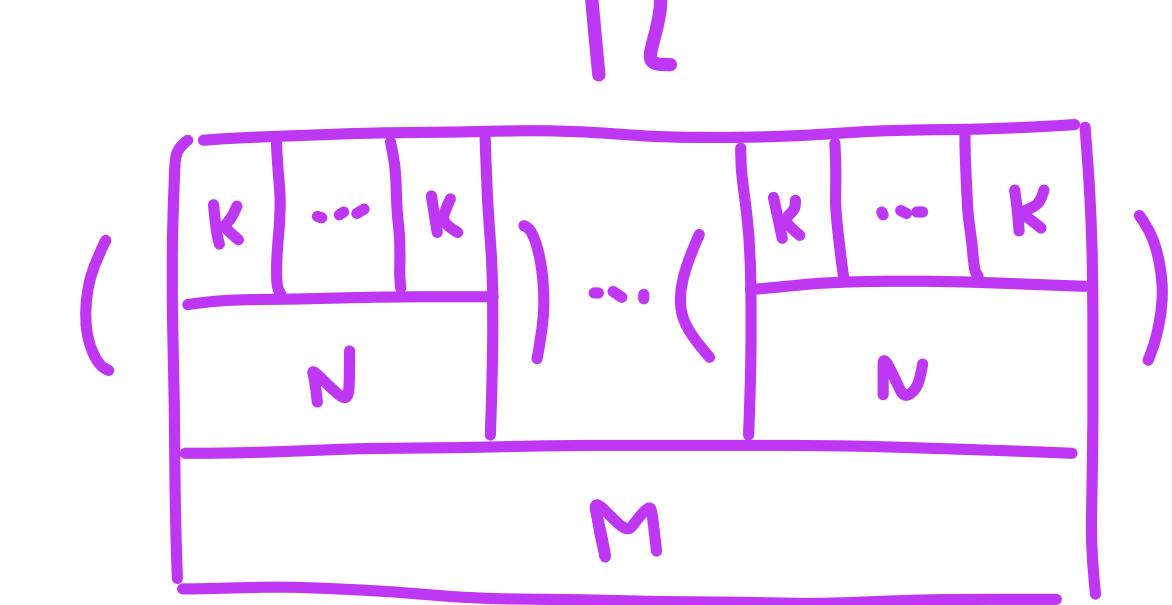
$$(M \square N) \square K$$



$$M \square (N \square K)$$



X



Proposition

box operads are monoids in

$(\mathcal{V}^{\mathbb{W}^3}, \square, \mathbf{I})$

$$\mathcal{B} \square \mathcal{B} \xrightarrow{m} \mathcal{B}$$

$$\mathbf{I} \xrightarrow{n} \mathcal{B}$$

# The Floorplan

box operads  
= monoids  
in a skew  
monoidal category

deformation theory  
homotopy theory  
of lax functors  
 $\mathcal{U} \rightarrow \text{Cat}(\mathbb{R})$

# The Floorplan

box operads  
= algebras  
over a symm.  
coloured operad

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Stackings

Stackings

a stacking  $S$  takes a sequence of boxes

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Stackings

a stacking  $S$  takes a sequence of boxes



and stacks them to form a new box

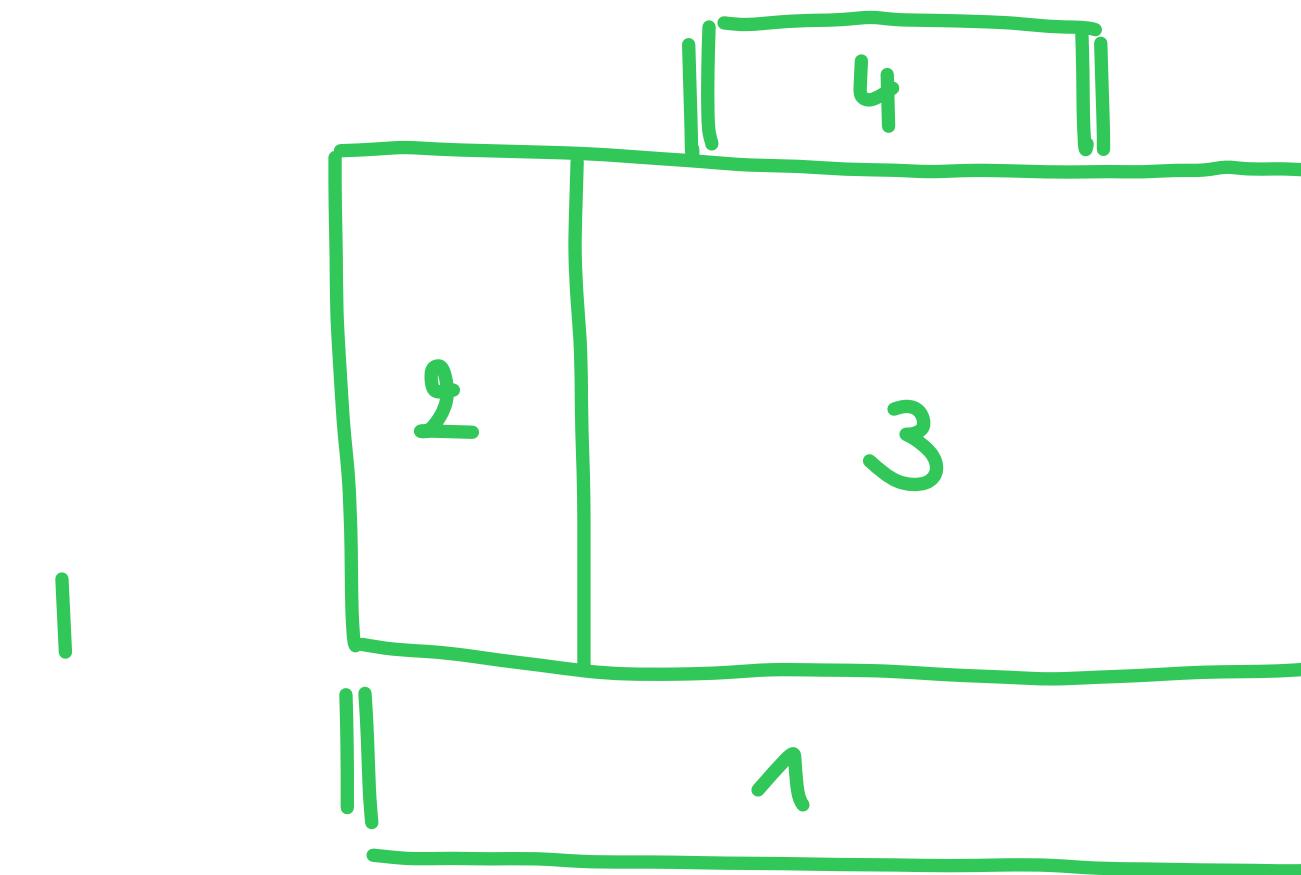
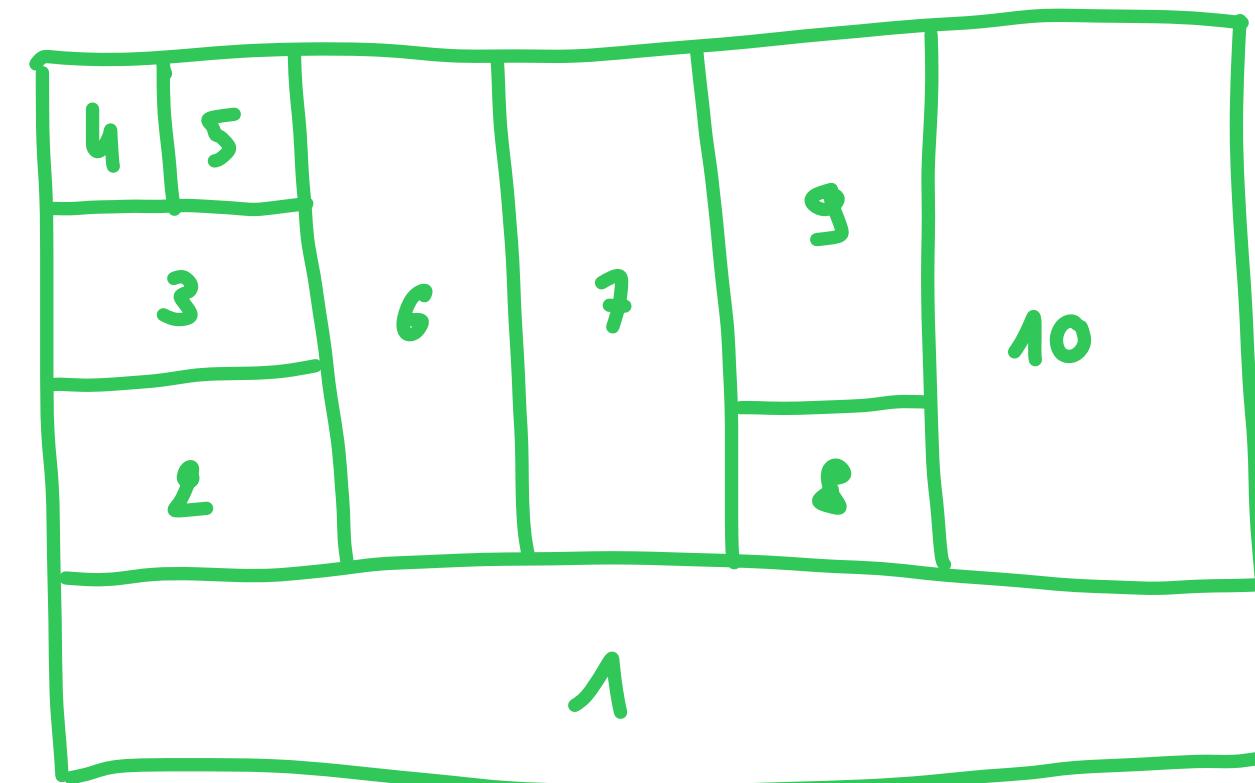
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a stacking  $S$  takes a sequence of boxes



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e.g.



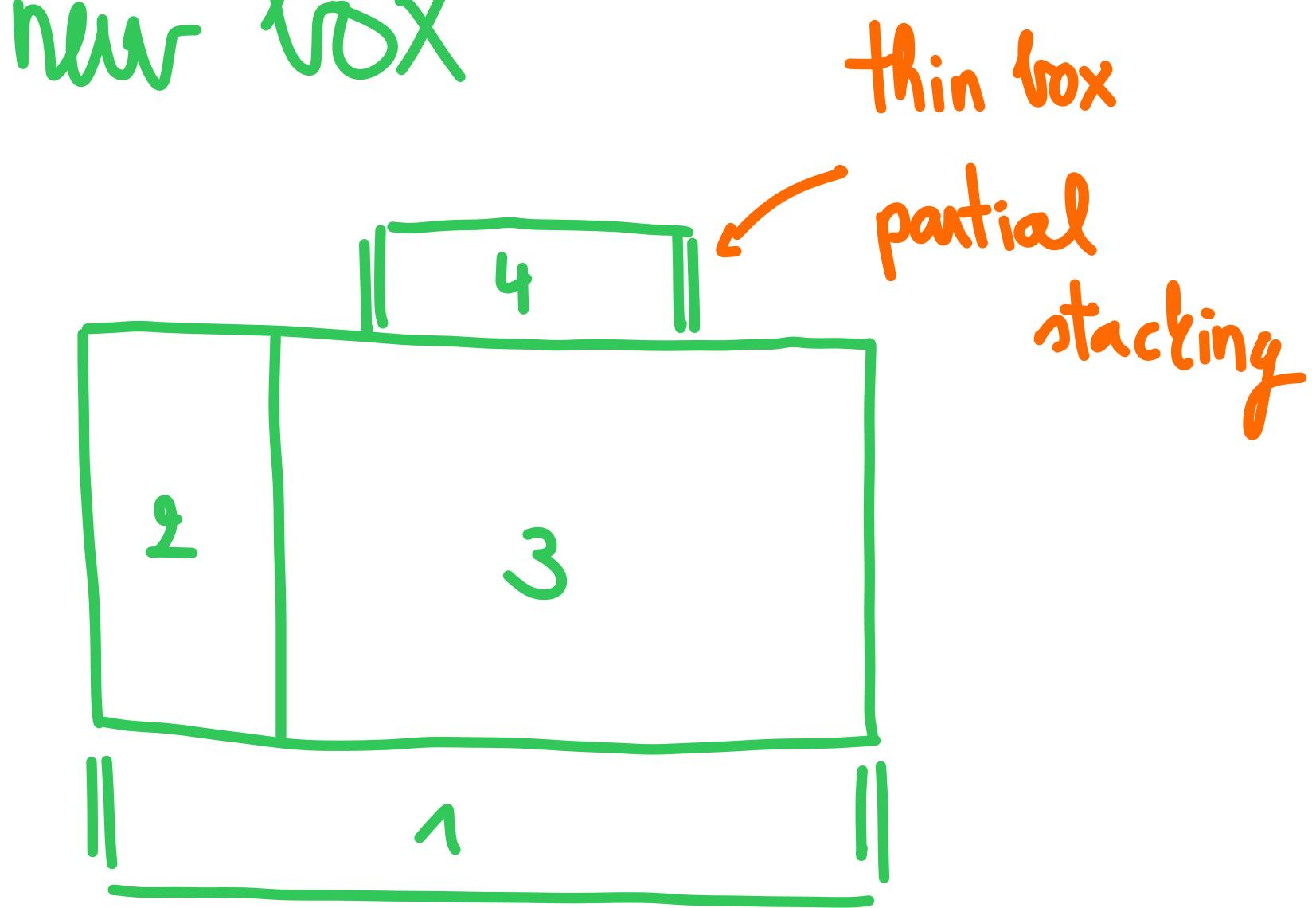
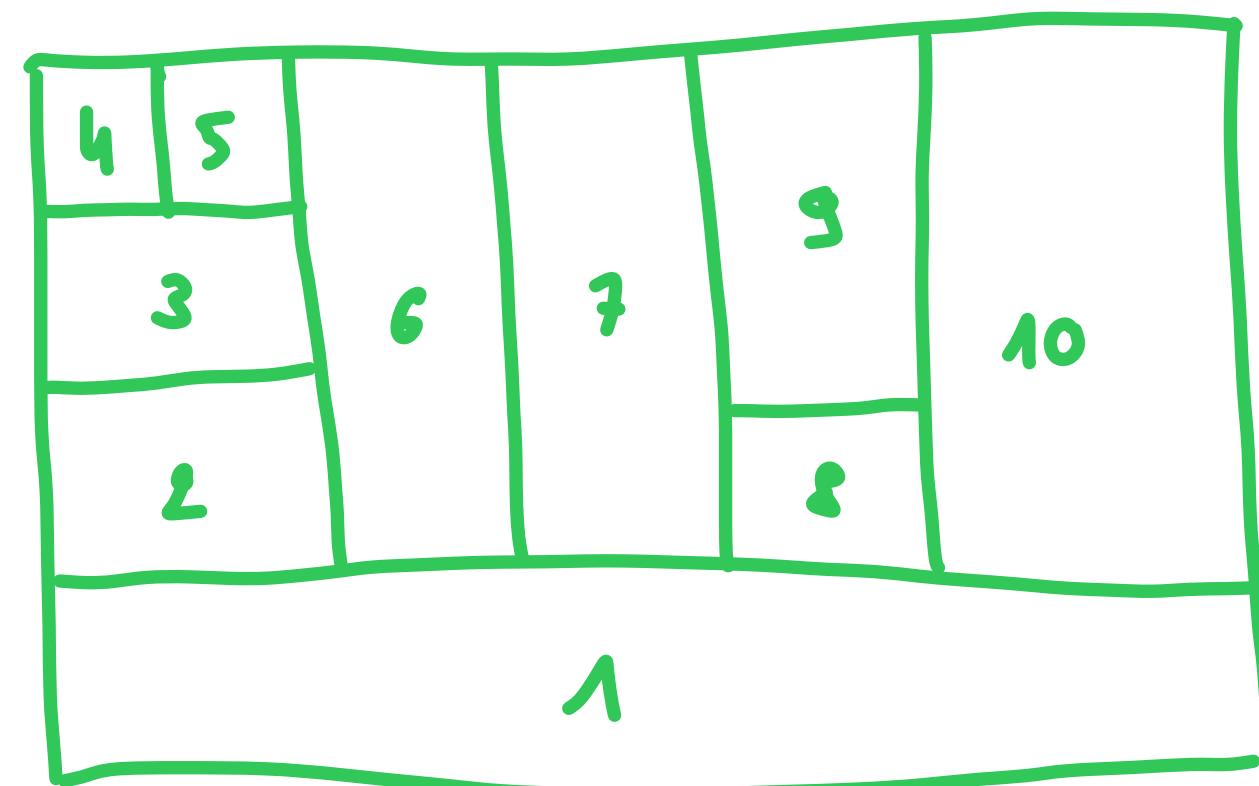
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e.g.



Coloured symmetric operad  $\square_p$

Coloured symmetric operad  $\square_p$

objects  $\mathbb{N}^3$

Coloured symmetric operad  $\square_p$

objects  $\mathbb{N}^3$

$$\square_p((p_1, q_1, \zeta_1), \dots, (p_n, q_n, \zeta_n); (p, q, \zeta)) =$$

# Coloured symmetric operad $\square_p$

objects  $\mathbb{N}^3$

$$\square_p((p_1, q_1, r_1), \dots, (p_n, q_n, r_n); (p, q, r)) = \left\{ \begin{array}{c} \text{Stacking of boxes} \\ p_1 \square^{q_1} r_1, \dots, p_n \square^{q_n} r_n \\ \text{into a box } p \square^q r \end{array} \right\}$$

Coloured symmetric operad  $\square_p$

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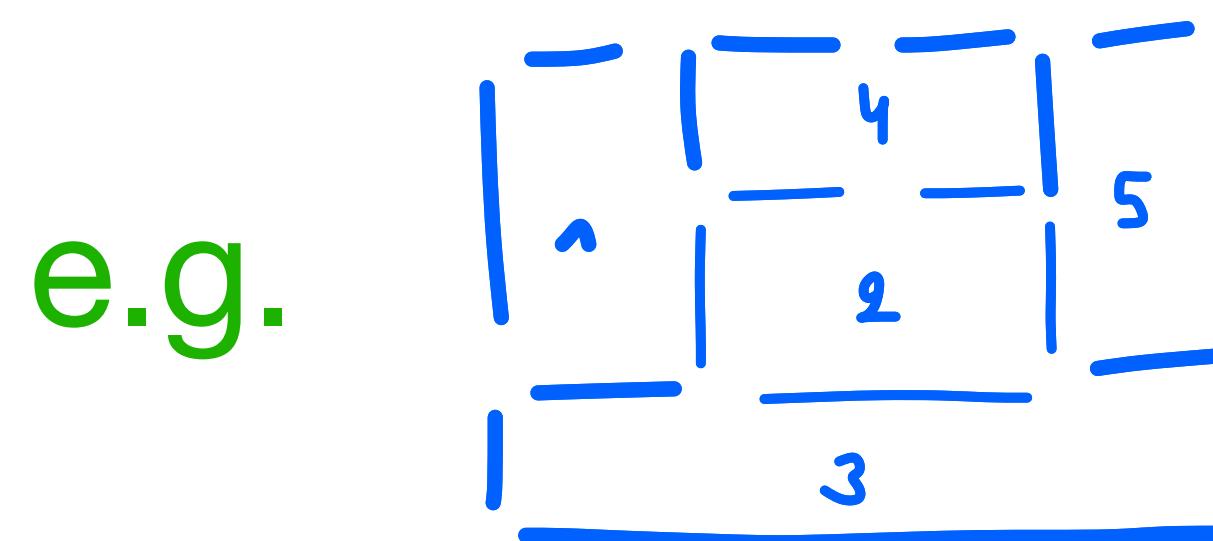
composition = substitution of a box by a stacking

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composition = substitution of a box by a stacking

e.g.

$$\circ_2 \quad \square^{(1,2,3)}_{(1,1,1)} = \square^{(1,2,3)}_{(1,1,1)}$$

# Coloured symmetric operad $\square_p$

objects  $\mathbb{N}^3$

$$\square_p((p_1, q_1, r_1), \dots, (p_n, q_n, r_n); (p, q, r)) = \left\{ \begin{array}{c} \text{Stacking of boxes} \\ p_1 \square^{q_1} r_1, \dots, p_n \square^{q_n} r_n \\ \text{into a box } p \square^q r \end{array} \right\}$$

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$$\square \circ_2 \square = \square$$

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composition = substitution of a box by a stacking

e.g.

$$\square_{(5)} \circ_2 \square_{(3)} = \square_{(7)}$$

Proposition box operads are algebras over  $\square_p$

# Higher Gerstenhaber brackets

Theorem

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Theorem we have a morphism

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$$L_\infty \longrightarrow \text{Tot}(\square_\rho)$$

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$$\ell_n \longmapsto$$

# Higher Gerstenhaber brackets

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$$L_\infty \longrightarrow \text{Tot}(\square_p)$$

$$\ell_n \mapsto \sum \text{[thin boxes]}$$

2 thin boxes

# Higher Gerstenhaber brackets

Theorem we have a morphism

$$L_\infty \longrightarrow \text{Tot}(\square_p)$$

$$\ell_n \mapsto \sum \begin{array}{c} \text{---} \\ | \end{array} + \sum \begin{array}{c} \text{---} \\ | \end{array}$$

2 thin boxes

1 thin box  
1 nonthin box

# Higher Gerstenhaber brackets

Theorem we have a morphism

$$L_\infty \longrightarrow \text{Tot}(\square_p)$$

$$l_n \mapsto \sum \begin{array}{c} \text{---} \\ | \\ \square \end{array} + \sum \begin{array}{c} \text{---} \\ | \\ \square \end{array} + \sum \begin{array}{c} \text{---} \\ | \\ \square \end{array}$$

2 thin boxes

1 thin box  
1 nonthin box

1 thin box  
 $n-1$  nonthin box  
topological condition

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box operads  
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over a symm.  
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operad  $\text{Lax}_{\mathcal{U}}$   
encoding lax functors  
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deformation theory  
homotopy theory  
of lax functors  
 $\mathcal{U} \longrightarrow \text{Cat}(\mathcal{V})$   
via minimal model  
 $\text{Lax}_{\infty}$

# The box operad Lax

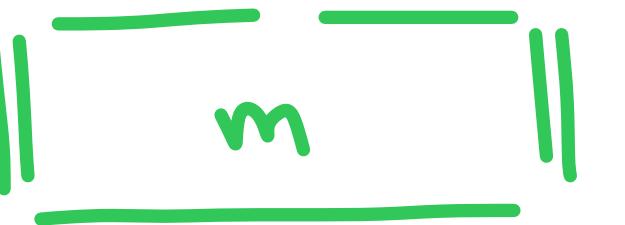
# The box operad Lax

generators

relations

# The box operad Lax

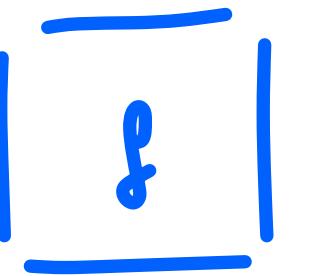
generators



relations

# The box operad Lax

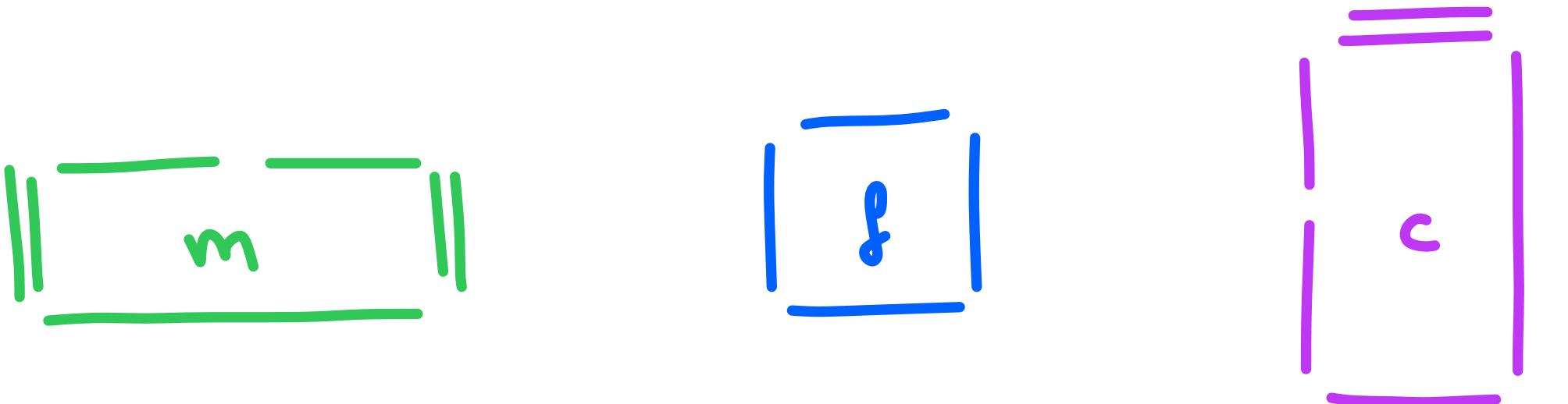
generators



relations

# The box operad Lax

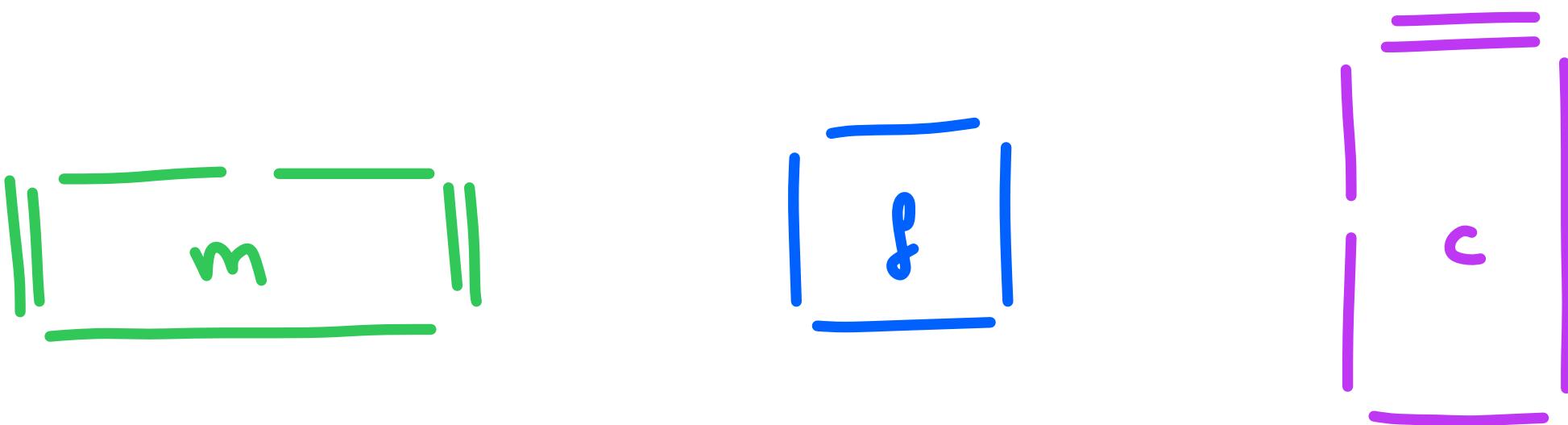
generators



relations

# The box operad Lax

generators



relations

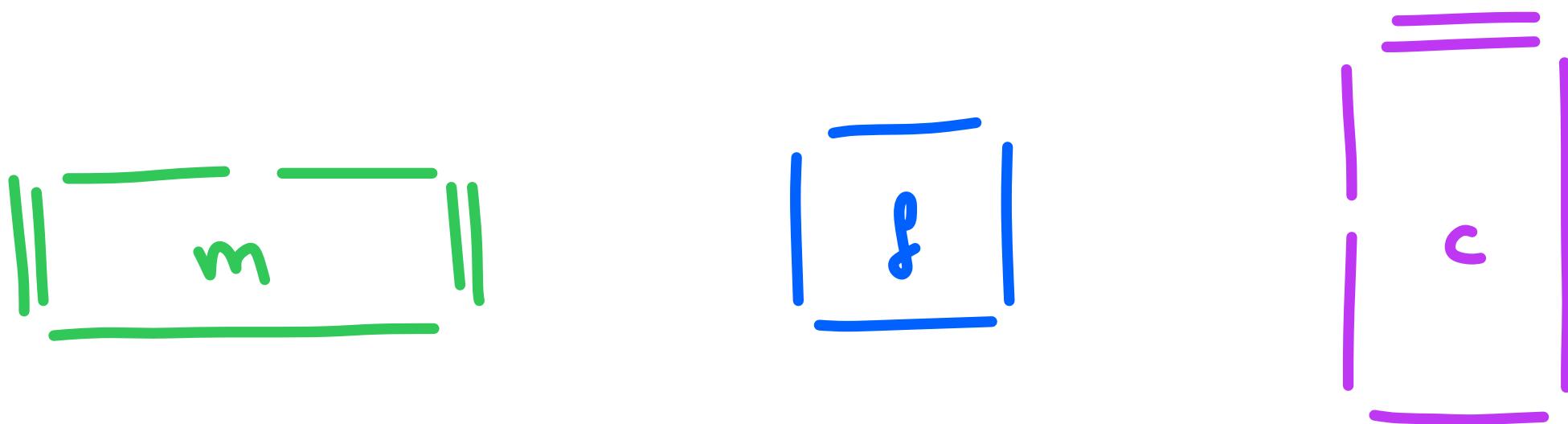
Two relations are shown involving the generator labeled 'm':

$$\begin{array}{ccc} \text{---} & = & \text{---} \\ | & & | \\ \text{---} & & \text{---} \end{array}$$

The left relation shows two 'm' boxes connected by a horizontal line. The right relation shows two 'm' boxes connected by a vertical line.

# The box operad Lax

generators



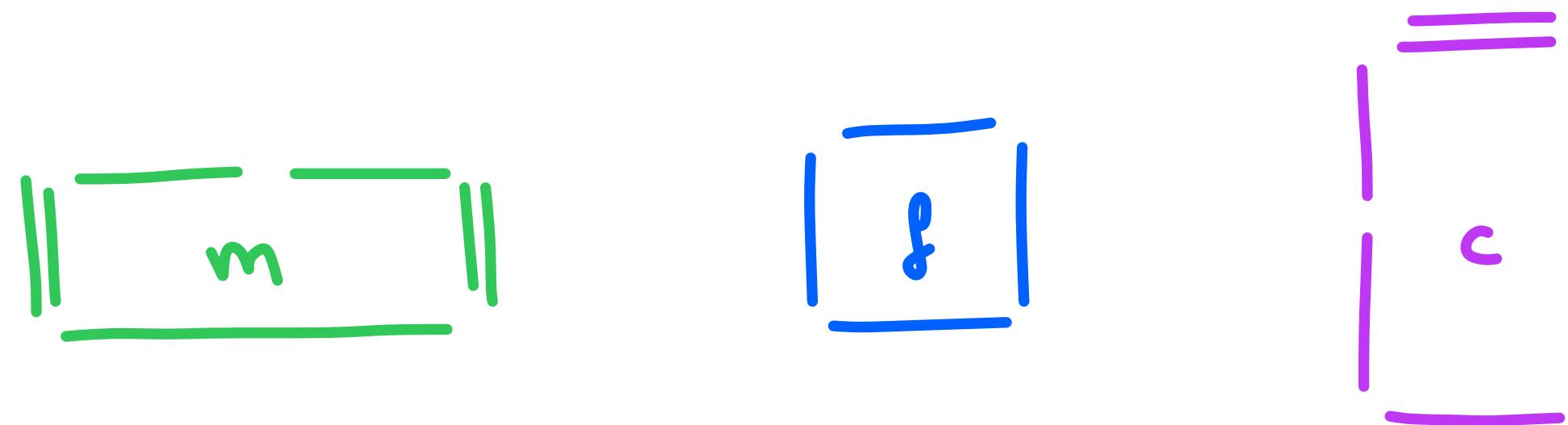
relations

Relations in the box operad Lax:

$$\begin{array}{ccc} \boxed{\begin{array}{c} \text{---} \\ \text{m} \\ \text{---} \end{array}} & = & \boxed{\begin{array}{c} \text{---} \\ \text{m} \\ \text{---} \end{array}} \\ \boxed{\begin{array}{c} \text{---} \\ \text{m} \\ \text{---} \end{array}} & & \boxed{\begin{array}{c} \text{---} \\ \text{m} \\ \text{---} \end{array}} \\ & & \boxed{\begin{array}{c} \text{---} \\ \text{g} \\ \text{---} \end{array}} \end{array}$$
$$\begin{array}{ccc} & & \boxed{\begin{array}{c} \text{---} \\ \text{m} \\ \text{---} \end{array}} \\ & = & \boxed{\begin{array}{c} \text{---} \\ \text{g} \\ \text{---} \end{array}} \end{array}$$

# The box operad Lax

generators



relations

Relations for the generator 'm':

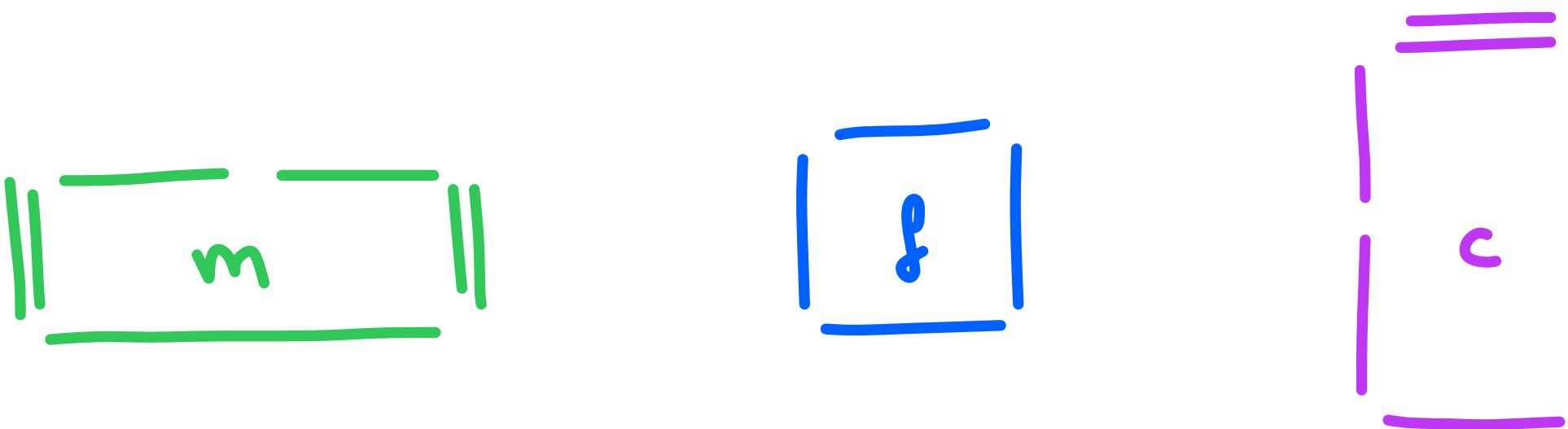
$$\boxed{\begin{array}{c} \text{---} \\ | \\ \text{m} \\ | \\ \text{---} \end{array}} = \boxed{\begin{array}{c} \text{---} \\ | \\ \text{m} \\ | \\ \text{---} \end{array}}$$
$$\boxed{\begin{array}{c} \text{---} \\ | \\ \text{g} \\ | \\ \text{---} \end{array}} = \boxed{\begin{array}{c} \text{---} \\ | \\ \text{g} \\ | \\ \text{---} \end{array}}$$

Relation for the generator 'c':

$$\boxed{\begin{array}{c} \text{---} \\ | \\ \text{g} \\ | \\ \text{---} \\ | \\ \text{c} \\ | \\ \text{---} \end{array}} = \boxed{\begin{array}{c} \text{---} \\ | \\ \text{c} \\ | \\ \text{---} \\ | \\ \text{g} \\ | \\ \text{---} \end{array}}$$

# The box operad Lax

generators



relations

Four relations are shown, each involving the three generators above:

- $m \otimes m = m$
- $\otimes \otimes = \otimes$
- $\otimes c = c \otimes$
- $c c = c$

Coherence

Theorem

# Coherence

Theorem

$$\text{Lax}(\rho, q, r) =$$

# Coherence

Theorem

$$\text{Lax}(p, q, r) = \begin{cases} \bigsqcup_{\text{Part}(p, r)} \mathcal{I} & p + q - r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

# Coherence

Theorem

$$\text{Lax}(p, q, r) = \begin{cases} \bigsqcup_{\text{Part}(p, r)} I & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

①

$$\vdash \boxed{\text{Lax}(p, q, r)} = 0$$

$\stackrel{p < r}{\dots}$

# Coherence

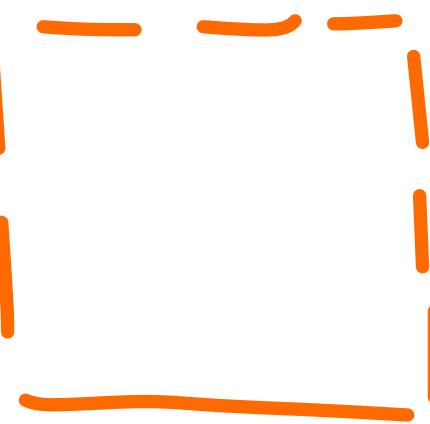
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①

$$\vdash \boxed{\text{Lax}(p, q, r)} = 0$$

e.g. there is no element



# Coherence

Theorem

$$\text{Lax}(p, q, r) = \begin{cases} \bigsqcup_{\text{Part}(p, r)} \mathcal{I} & p+q-r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

②  $\left\| \begin{bmatrix} & \cdots & & \\ & q \geq 1 & & \\ \text{Lax}(0, q, 0) & & & \end{bmatrix} \right\| = \mathcal{I} \Rightarrow \text{Lax}(0, q, 0) = \text{Anoc}(q) = m_q \mathcal{I}$

Coherence

Theorem

$$\text{Lax}(p, q, \gamma) = \begin{cases} \bigsqcup_{\text{Part}(p, \gamma)} \mathcal{I} & p + q - \gamma \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

③  $\boxed{\begin{array}{c} \vdash \cdots \\ \vdash \text{Lax}(p, q, 1) \\ \vdash \end{array} \left| \begin{array}{c} p+q \geq 2, p \geq 1 \\ \vdash \end{array} \right.} = \mathcal{I}$

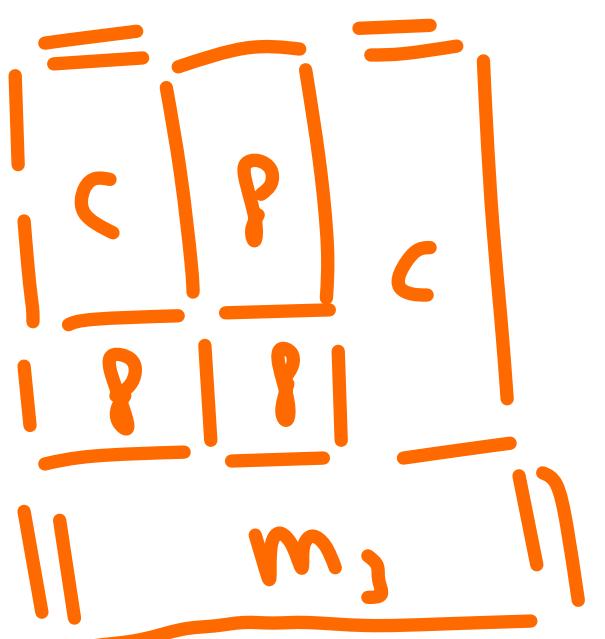
Coherence

Theorem

$$\text{Lax}(p, q, \gamma) = \begin{cases} \bigsqcup_{\text{Part}(p, \gamma)} \mathcal{I} & p+q-\gamma \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

③  $\boxed{\begin{array}{c} \vdots \\ \text{Lax}(p, q, 1) \\ \vdots \end{array} \left| \begin{array}{c} p+q \geq 2, p \geq 1 \\ \dots \end{array} \right.} = \mathcal{I}$

e.g.



Coherence

Theorem

$$\text{Lax}(p, q, \gamma) = \begin{cases} \bigsqcup_{\text{Part}(p, \gamma)} \mathcal{I} & p+q-\gamma \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

③  $\boxed{\begin{array}{c} \vdots \\ \text{Lax}(p, q, 1) \end{array}} = \mathcal{I}$

$\vdots$   $\text{p+q} \geq 2, p \geq 1$

e.g.

$$\left[ \begin{array}{|c|c|} \hline c & p \\ \hline p & c \\ \hline \end{array} \right] = \left[ \begin{array}{|c|c|} \hline c & c \\ \hline c & p \\ \hline \end{array} \right]$$

Coherence

Theorem

$$\text{Lax}(p, q, \gamma) = \begin{cases} \sqcup_{\text{Part}(p, \gamma)} \mathcal{I} & p+q-\gamma \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

③  $\boxed{\begin{array}{c} \vdash \cdots \vdash \\ \vdash \text{Lax}(p, q, 1) \\ \vdash \end{array} \left| \begin{array}{c} p+q \geq 2, p \geq 1 \end{array} \right.} = \mathcal{I}$

e.g.

$$\begin{array}{c} \overline{c} \quad \overline{p} \\ \overline{c} \quad \overline{c} \\ \overline{p} \quad \overline{p} \\ \parallel m_1 \parallel \end{array} = \begin{array}{c} \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \quad \overline{p} \\ \overline{p} \quad \overline{p} \\ \parallel m_2 \parallel \end{array} = \begin{array}{c} \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \quad \overline{p} \\ \overline{c} \quad \overline{p} \\ \parallel m_3 \parallel \end{array}$$

# Coherence

## Theorem

$$\text{Lax}(p, q, \mathcal{I}) = \begin{cases} \bigsqcup_{\text{Part}(p, \mathcal{I})} \mathcal{I} & p+q-\mathcal{I} \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

③  $\boxed{\begin{array}{c} \vdots \\ \text{Lax}(p, q, 1) \\ \vdots \end{array} \quad \begin{array}{c} \text{p+q} \geq 2, p \geq 1 \\ \dots \end{array}} = \mathcal{I}$

e.g.

$$\begin{array}{c} \boxed{\begin{array}{|c|c|} \hline c & 8 \\ \hline 8 & c \\ \hline \end{array}} \\ \parallel m_3 \parallel \end{array} = \begin{array}{c} \boxed{\begin{array}{|c|c|} \hline c & c \\ \hline 8 & c \\ \hline \end{array}} \\ \parallel m_3 \parallel \end{array} = \begin{array}{c} \boxed{\begin{array}{|c|c|} \hline c & c \\ \hline c & 8 \\ \hline \end{array}} \\ \parallel m_3 \parallel \end{array} = \begin{array}{c} \boxed{\begin{array}{|c|c|} \hline c & 8 \\ \hline 8 & c \\ \hline \end{array}} \\ \parallel m_3 \parallel \end{array}$$

Coherence

Theorem

$$\text{Lax}(p, q, \gamma) = \begin{cases} \bigsqcup_{\text{Part}(p, \gamma)} \mathcal{I} & p+q-\gamma \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

③  $\boxed{\begin{array}{c} \vdash \cdots \vdash \\ \vdash \text{Lax}(p, q, 1) \\ \vdash \end{array} \left| \begin{array}{c} p+q \geq 2, p \geq 1 \\ \vdash \end{array} \right.} = \mathcal{I}$

e.g.

$$\begin{array}{c} \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \\ \overline{m_3} \end{array} = \begin{array}{c} \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{m_3} \end{array} = \begin{array}{c} \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{m_3} \end{array} = \begin{array}{c} \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{m_3} \end{array} = \begin{array}{c} \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{c} \quad \overline{c} \quad \overline{c} \\ \overline{m_3} \end{array}$$

# Coherence

Theorem

$$\text{Lax}(p, q, r) = \begin{cases} \bigsqcup_{\text{Part}(p, r)} \mathcal{I} & p + q - r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

④

$$\boxed{\text{Lax}(3, 0, 2)} = \boxed{\begin{array}{|c|} \hline c \\ \hline s \\ \hline \end{array}} \sqcup \boxed{\begin{array}{|c|} \hline = \\ \hline c \\ \hline \end{array}}$$

The coloured box operad  $\text{Lax}_{\mathcal{U}}$

for small category  $\mathcal{U}$

generators

relations

# The coloured box operad $\text{Lax}_{\mathcal{U}}$

generators

$$\begin{array}{c} U \longrightarrow U \longrightarrow U \\ id \parallel \quad m_U \quad \parallel id \\ U \longrightarrow U \end{array}$$

$$\begin{array}{c} U \longrightarrow U \\ u \mid f^u \mid u \\ V \longrightarrow V \end{array}$$

relations

for small category  $\mathcal{U}$

$$\begin{array}{c} U = U \\ u \mid v \quad c^{u,v} \mid vu \\ r \mid w - w \end{array}$$

# The coloured box operad $\text{Lax}_{\mathcal{U}}$

for small category  $\mathcal{U}$

generators

$$\begin{array}{c} \text{id} \\ \parallel \\ \text{U} \xrightarrow{\quad m_U \quad} \text{U} \end{array} \quad \begin{array}{c} \text{id} \\ \parallel \\ \text{U} \xrightarrow{\quad m_U \quad} \text{U} \end{array}$$

$$\begin{array}{c} \text{U} \xrightarrow{\quad g^u \quad} \text{U} \\ \parallel \\ \text{V} \xrightarrow{\quad g^v \quad} \text{V} \end{array}$$

$$\begin{array}{c} \text{U} = \text{U} \\ \parallel \\ \text{U} \xrightarrow{\quad c_{\text{U}, \text{U}} \quad} \text{U} \\ \parallel \\ \text{W} = \text{W} \end{array}$$

relations

$$\begin{array}{c} \parallel \\ \boxed{\text{U} \xrightarrow{\quad m_U \quad} \text{U}} \\ \parallel \\ \text{U} \xrightarrow{\quad m_U \quad} \text{U} \end{array} = \begin{array}{c} \parallel \\ \boxed{\text{U} \xrightarrow{\quad m_U \quad} \text{U}} \\ \parallel \\ \text{U} \xrightarrow{\quad m_U \quad} \text{U} \end{array}$$

$$\begin{array}{c} \text{U} \xrightarrow{\quad g^u \quad} \text{U} \\ \parallel \\ \text{U} \xrightarrow{\quad g^v \quad} \text{U} \\ \parallel \\ \text{U} \xrightarrow{\quad g^u \quad} \text{U} \end{array} = \begin{array}{c} \parallel \\ \boxed{\text{U} \xrightarrow{\quad g^u \quad} \text{U}} \\ \parallel \\ \text{U} \xrightarrow{\quad g^v \quad} \text{U} \end{array}$$

# The coloured box operad $\text{Lax}_{\mathcal{U}}$

generators

$$\begin{array}{c} \text{id} \\ \parallel \\ \text{U} \xrightarrow{\quad m_U \quad} \text{U} \end{array} \quad \parallel \quad \text{id}$$

$$\begin{array}{c} \text{U} \xrightarrow{\quad g^u \quad} \text{U} \\ \downarrow \\ \text{V} \xrightarrow{\quad g^v \quad} \text{V} \end{array}$$

$$\begin{array}{c} \text{U} = \text{U} \\ \text{U} \xrightarrow{\quad c_{u,v} \quad} \text{vU} \\ \text{v} \xrightarrow{\quad c_{v,w} \quad} \text{w} \\ \text{w} = \text{w} \end{array}$$

relations

$$\begin{array}{c} \parallel \\ \text{U} \xrightarrow{\quad m_U \quad} \text{U} \\ \parallel \\ \text{U} \xrightarrow{\quad m_U \quad} \text{U} \end{array} = \begin{array}{c} \parallel \\ \text{U} \xrightarrow{\quad m_U \quad} \text{U} \\ \parallel \end{array}$$

$$\begin{array}{c} \text{U} \xrightarrow{\quad g^u \quad} \text{U} \\ \parallel \\ \text{U} \xrightarrow{\quad g^v \quad} \text{U} \\ \parallel \end{array} = \begin{array}{c} \parallel \\ \text{U} \xrightarrow{\quad g^u \quad} \text{U} \\ \parallel \end{array}$$

$$\begin{array}{c} \text{U} \xrightarrow{\quad g^u \quad} \text{U} \\ \text{v} \xrightarrow{\quad g^v \quad} \text{v} \\ \parallel \\ \text{U} \xrightarrow{\quad m_w \quad} \text{U} \end{array} = \begin{array}{c} \text{U} \xrightarrow{\quad c_{u,v} \quad} \text{vU} \\ \text{v} \xrightarrow{\quad c_{v,w} \quad} \text{w} \\ \parallel \\ \text{U} \xrightarrow{\quad m_w \quad} \text{U} \end{array}$$

for small category  $\mathcal{U}$

# The coloured box operad $\text{Lax}_{\mathcal{U}}$

generators

$$\begin{array}{c} \text{id} \\ \parallel \\ \text{id} \end{array} \quad \begin{array}{c} U - U - U \\ \parallel \\ m_U \end{array}$$

$$\begin{array}{c} U - U \\ u \mid g^u \mid u \\ \parallel \\ V - V \end{array}$$

$$\begin{array}{c} U = U \\ u \mid v \mid c^{u,v} \mid vu \\ r \mid w \mid w \end{array}$$

relations

$$\begin{array}{c} \parallel \\ \boxed{\begin{array}{c} m_U \\ \parallel \\ m_U \end{array}} \\ \parallel \end{array} = \begin{array}{c} \parallel \\ \boxed{\begin{array}{c} m_U \\ \parallel \\ m_U \end{array}} \\ \parallel \end{array}$$

$$\begin{array}{c} u \mid \boxed{\begin{array}{c} g^u \\ \parallel \\ m_U \end{array}} \mid g^u \mid u \\ \parallel \\ v \end{array} = \begin{array}{c} \parallel \\ \boxed{\begin{array}{c} m_U \\ \parallel \\ g^u \end{array}} \mid u \end{array}$$

$$\begin{array}{c} u \mid g^u \mid c^{u,v} \mid vu \\ v \mid g^v \mid c^{v,w} \mid vw \\ \parallel \\ m_w \end{array} = \begin{array}{c} u \mid \boxed{\begin{array}{c} c^{u,v} \\ \parallel \\ m_w \end{array}} \mid g^v \mid vw \\ \parallel \\ v \end{array}$$

$$\begin{array}{c} u \mid \boxed{\begin{array}{c} c^{u,v} \\ \parallel \\ c^{v,w} \end{array}} \mid c^{u,w} \mid uw \\ v \mid \boxed{\begin{array}{c} c^{v,w} \\ \parallel \\ g^w \end{array}} \mid gw \\ w \mid \boxed{\begin{array}{c} g^w \\ \parallel \\ m_x \end{array}} \mid mx \\ \parallel \end{array} = \begin{array}{c} \parallel \\ \boxed{\begin{array}{c} c^{u,w} \\ \parallel \\ g^w \end{array}} \mid mx \\ \parallel \\ w \end{array}$$

FOR SMALL CATEGORY  $\mathcal{U}$

Algebras over a coloured box operad  $\mathcal{B}$

# Algebras over a coloured box operad $\mathcal{B}$

↳ definition  $\text{Alg}(\mathcal{B}) := \text{Hom}(\mathcal{B}, \text{Span}(V))$

Algebras over a coloured box operad

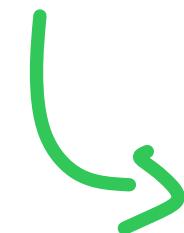
$\mathcal{B}$

virtual double  
category of

definition  $\text{Alg}(\mathcal{B}) := \text{Hom}(\mathcal{B}, \text{Span}(V))$

Algebras over a coloured box operad

$\mathcal{B}$



definition

$$\text{Alg}(\mathcal{B}) := \text{Hom}(\mathcal{B}, \text{Span}(V))$$

virtual double  
category of  
• = sets

# Algebras over a coloured box operad

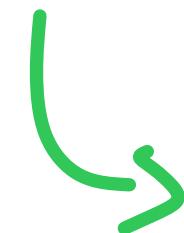
↳ definition  $\text{Alg}(\mathcal{B}) := \text{Hom}(\mathcal{B}, \text{Span}(V))$

$\mathcal{B}$

virtual double  
category of  
• = sets  
↓ = functions

Algebras over a coloured box operad

$\mathcal{B}$



definition

$$\text{Alg}(\mathcal{B}) := \text{Hom}(\mathcal{B}, \text{Span}(V))$$

virtual double  
category of

• = sets

↓ = functions

→ = spans enriched  
over  $V$

# Algebras over a coloured box operad

↳ definition  $\text{Alg}(\mathcal{B}) := \text{Hom}(\mathcal{B}, \text{Span}(\mathcal{V}))$

$\mathcal{B}$

virtual double  
category of

• = sets

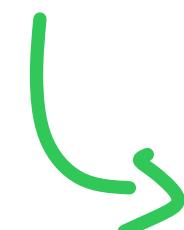
↓ = functions

→ = spans enriched  
over  $\mathcal{V}$

↓ = morphism  
of spans

Algebras over a coloured box operad

$\mathcal{B}$



definition

$$\text{Alg}(\mathcal{B}) := \text{Hom}(\mathcal{B}, \text{Span}(V))$$



$$\text{algebra over } \mathcal{B} = \text{morphism } \mathcal{B} \xrightarrow{A} \text{Span}(V)$$

virtual double  
category of

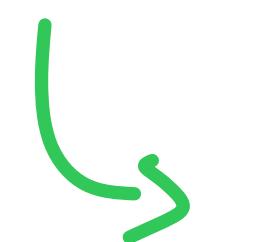
• = sets

↓ = functions

→ = spans enriched  
over  $V$

↓ = morphism  
of  $V$ -spans

Algebras over a coloured box operad  $\mathcal{B}$



definition

$$\text{Alg}(\mathcal{B}) := \text{Hom}(\mathcal{B}, \text{Span}(V))$$



$$\text{algebra over } \mathcal{B} = \text{morphism } \mathcal{B} \xrightarrow{A} \text{Span}(V)$$

virtual double  
category of

• = sets

↓ = functions

→ = spans enriched  
over  $V$

↓ = morphism  
of  $V$ -spans

Proposition

$$\text{Alg}(\text{Lax}_{\mathcal{U}}) = \text{Lax}(\mathcal{U}, \text{Cat}(V))$$

# The Floorplan

box operads  
= algebras  
over a symm.  
coloured operad

box operads  
= monoids  
in a skew  
monoidal category

a coloured box  
operad  $\text{Lax}_\omega$   
encoding lax functors

totalisation  
of a box operad  
carries a  
L-infinity structure

Koszul duality  
for  
box operads

# The Floorplan

box operads  
= algebras  
over a symm.  
coloured operad

box operads  
= monoids  
in a skew  
monoidal category

totalisation  
of a box operad  
carries a  
L-infinity structure

Koszul duality  
for  
box operads

a coloured box  
operad  $Lax_{\mathcal{U}}$   
encoding lax functors

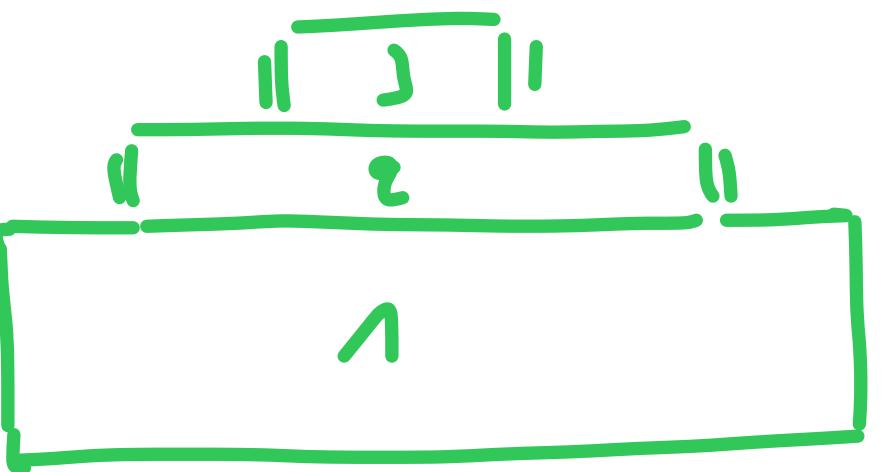
+  
deformation theory  
homotopy theory  
of lax functors  
 $\mathcal{U} \rightarrow \text{Cat}(\mathbb{R})$   
via minimal model  
 $Lax_{\infty}$

More details in

- *Box operads and higher Gerstenhaber braces* - Dinh Van, Hermans and Lowen, arXiv preprint (2023)
- *A minimal model for lax prestacks via Koszul duality for box operads* - Hermans, arXiv preprint (2023)

# Proof

# Proof



# Proof

$$\boxed{1} \stackrel{\text{def}}{=} \boxed{1} = \boxed{1}$$

The diagram shows three identical rectangular boxes, each containing the number "1". Above each box is a horizontal bar with three points: the first point is at the top left corner, the second point is in the middle of the top edge, and the third point is at the top right corner. A green equals sign connects the first box to the second, and another green equals sign connects the second box to the third.

# Proof

$$\boxed{\begin{array}{c} \overline{1} \\ \overline{2} \\ \overline{3} \end{array}} \stackrel{O_1}{\sim} \boxed{\begin{array}{c} \overline{1} \\ \overline{2} \\ \overline{3} \end{array}} = \boxed{\begin{array}{c} \overline{1} \\ \overline{2} \\ \overline{3} \end{array}} \stackrel{O_2}{=} \boxed{\begin{array}{c} \overline{1} \\ \overline{2} \\ \overline{3} \end{array}} \stackrel{O_1}{\sim} \boxed{\begin{array}{c} \overline{3} \\ \overline{2} \\ \overline{1} \end{array}}$$

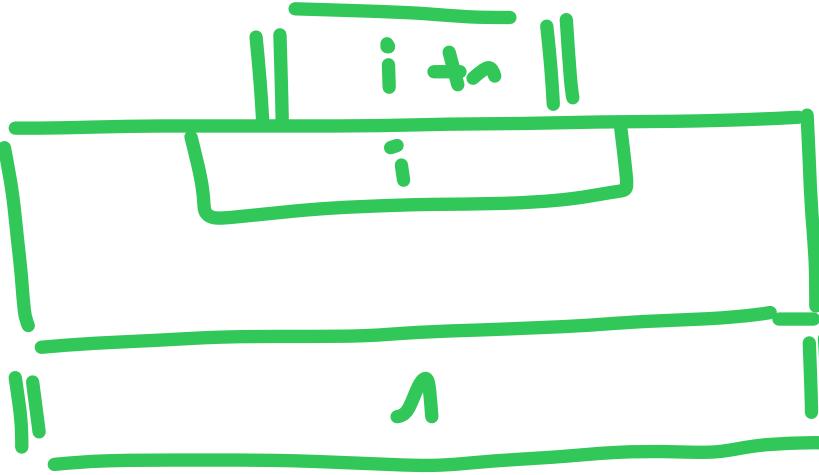
# Proof

$$\boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{O_1}{\sim} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} = \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{O_2}{=} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{O_1}{\sim} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}}$$
$$\boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{O_1}{\sim} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} = \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}}$$

# Proof

$$\boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{(2)}{=} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} = \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{(1)}{=} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{(2)}{=} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \quad (23)$$
$$\boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{(2)}{=} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} = \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{(1)}{=} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{(2)}{=} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}}$$

# Proof

$$\boxed{\begin{array}{c} \overline{i} \\ \overline{i} \\ \overline{i} \end{array}} \stackrel{?}{=} \boxed{\begin{array}{c} \overline{i} \\ \overline{i} \\ \overline{i} \end{array}} = \boxed{\begin{array}{c} \overline{i} \\ \overline{i} \\ \overline{i} \end{array}} = \boxed{\begin{array}{c} \overline{i} \\ \overline{i} \\ \overline{i} \end{array}} \stackrel{?}{=} \boxed{\begin{array}{c} \overline{i} \\ \overline{i} \\ \overline{i} \end{array}} \quad (23)$$
$$\boxed{\begin{array}{c} \overline{i} \\ \overline{i} \\ \overline{i} \end{array}} \stackrel{?}{=} \boxed{\begin{array}{c} \overline{i} \\ \overline{i} \\ \overline{i} \end{array}} = \boxed{\begin{array}{c} \overline{i} \\ \overline{i} \\ \overline{i} \end{array}} = \boxed{\begin{array}{c} \overline{i} \\ \overline{i} \\ \overline{i} \end{array}} \stackrel{?}{=} \boxed{\begin{array}{c} \overline{i} \\ \overline{i} \\ \overline{i} \end{array}}$$


# Proof

$$\begin{array}{c} \text{Diagram 1: } \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 \end{matrix}} \stackrel{?}{=} \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 \end{matrix}} = \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 \end{matrix}} = \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 \end{matrix}} \stackrel{?}{=} \boxed{\begin{matrix} 2 & 3 \\ 1 \end{matrix}} \quad (23) \\ \\ \text{Diagram 2: } \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 \end{matrix}} \stackrel{?}{=} \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 \end{matrix}} = \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 \end{matrix}} = \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 \end{matrix}} \stackrel{?}{=} \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 \end{matrix}} \quad (23) \\ \\ \text{Diagram 3: } \boxed{\begin{matrix} i & \\ 1 \end{matrix}} \stackrel{?}{=} \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 \end{matrix}} = \boxed{\begin{matrix} i & \\ 1 \end{matrix}} \end{array}$$

Diagram 1: Three boxes labeled 1, 2, 3. Box 1 contains 1. Box 2 contains 2. Box 3 contains 3. The first two boxes are swapped.

Diagram 2: Three boxes labeled 1, 2, 3. Box 1 contains 1. Box 2 contains 2. Box 3 contains 3. The first two boxes are swapped.

Diagram 3: Three boxes labeled 1, 2, 3. Box 1 contains 1. Box 2 contains 2. Box 3 contains 3. The first two boxes are swapped.

# Proof

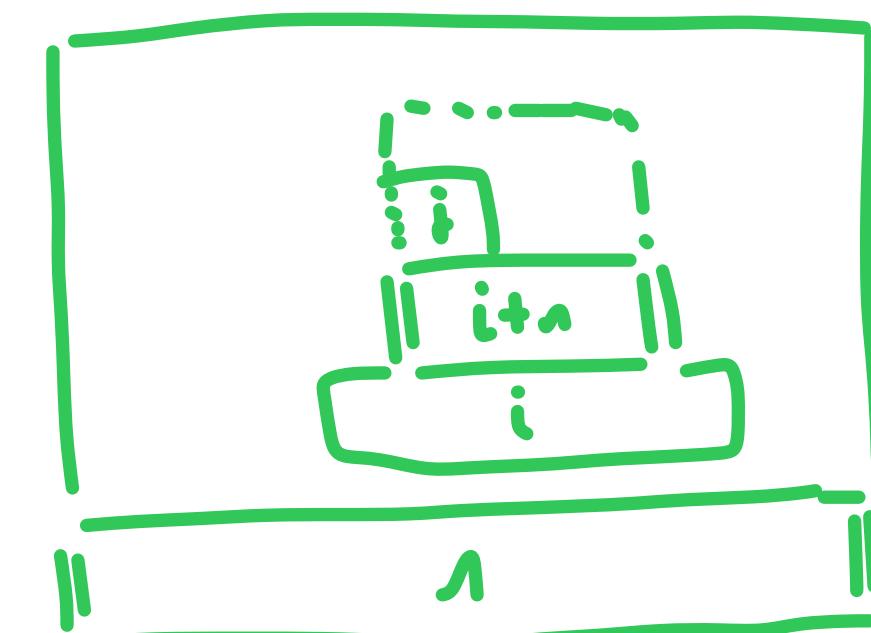
$$\begin{array}{c} \text{Diagram 1: } \boxed{\begin{matrix} 1 & 2 & 3 \\ \hline 1 \end{matrix}} \xrightarrow{o_1} \boxed{\begin{matrix} 1 & 2 & 3 \\ \hline 1 \end{matrix}} = \boxed{\begin{matrix} 1 & 2 & 3 \\ \hline 1 \end{matrix}} = \boxed{\begin{matrix} 1 & 2 \\ \hline 1 \end{matrix}} \xrightarrow{o_2} \boxed{\begin{matrix} 2 & 3 \\ \hline 1 \end{matrix}} \quad (q_3) \\ \\ \text{Diagram 2: } \boxed{\begin{matrix} 1 & 2 & 3 \\ \hline 1 \end{matrix}} \xrightarrow{o_1} \boxed{\begin{matrix} 1 & 2 & 3 \\ \hline 1 \end{matrix}} = \boxed{\begin{matrix} 1 & 2 & 3 \\ \hline 1 \end{matrix}} = \boxed{\begin{matrix} 1 & 2 \\ \hline 1 \end{matrix}} \xrightarrow{o_1} \boxed{\begin{matrix} 1 & 2 \\ \hline 1 \end{matrix}} \\ \\ \text{Diagram 3: } \boxed{\begin{matrix} i & \\ \hline 1 \end{matrix}} \xrightarrow{o_1} \boxed{\begin{matrix} 1 & 2 & 3 \\ \hline 1 \end{matrix}} = \boxed{\begin{matrix} i & i+1 \\ \hline 1 \end{matrix}} = \boxed{\begin{matrix} 1 & 2 \\ \hline 1 \end{matrix}} \xrightarrow{o_1} \boxed{\begin{matrix} i & \\ \hline 1 \end{matrix}} \end{array}$$

# Proof

$$\boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}} = \boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}} \quad (23)$$

$$\boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}} = \boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}}$$

$$\boxed{\begin{array}{c} i \\ \boxed{i} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}} = \boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \boxed{i} \\ \boxed{i} \end{array}}$$



# Proof

$$\boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} = \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{o_2}{\rightarrow} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}}$$

(23)

$$\boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} = \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{o_2}{\rightarrow} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}}$$

$$\boxed{\begin{array}{c} \text{i} \\ \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} = \boxed{\begin{array}{c} \text{i} \\ \text{i+a} \\ \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{o_2}{\rightarrow} \boxed{\begin{array}{c} \text{i} \\ \text{1} \\ \text{2} \\ \text{3} \end{array}}$$

$$\boxed{\begin{array}{c} \text{i} \\ \text{1} \\ \text{2} \\ \text{3} \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}} = \boxed{\begin{array}{c} \text{i} \\ \text{i+a} \\ \text{i} \\ \text{1} \\ \text{2} \\ \text{3} \end{array}}$$

# Proof

$$\begin{array}{c}
 \boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & & \\ \hline \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & & \\ \hline \end{array}} = \boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & \\ \hline \end{array}} = \boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & \\ \hline 1 & 2 & \\ \hline \end{array}} \stackrel{o_2}{\rightarrow} \boxed{\begin{array}{|c|c|c|} \hline 2 & 3 & \\ \hline 1 & 2 & \\ \hline \end{array}}
 \end{array} \quad (23)$$

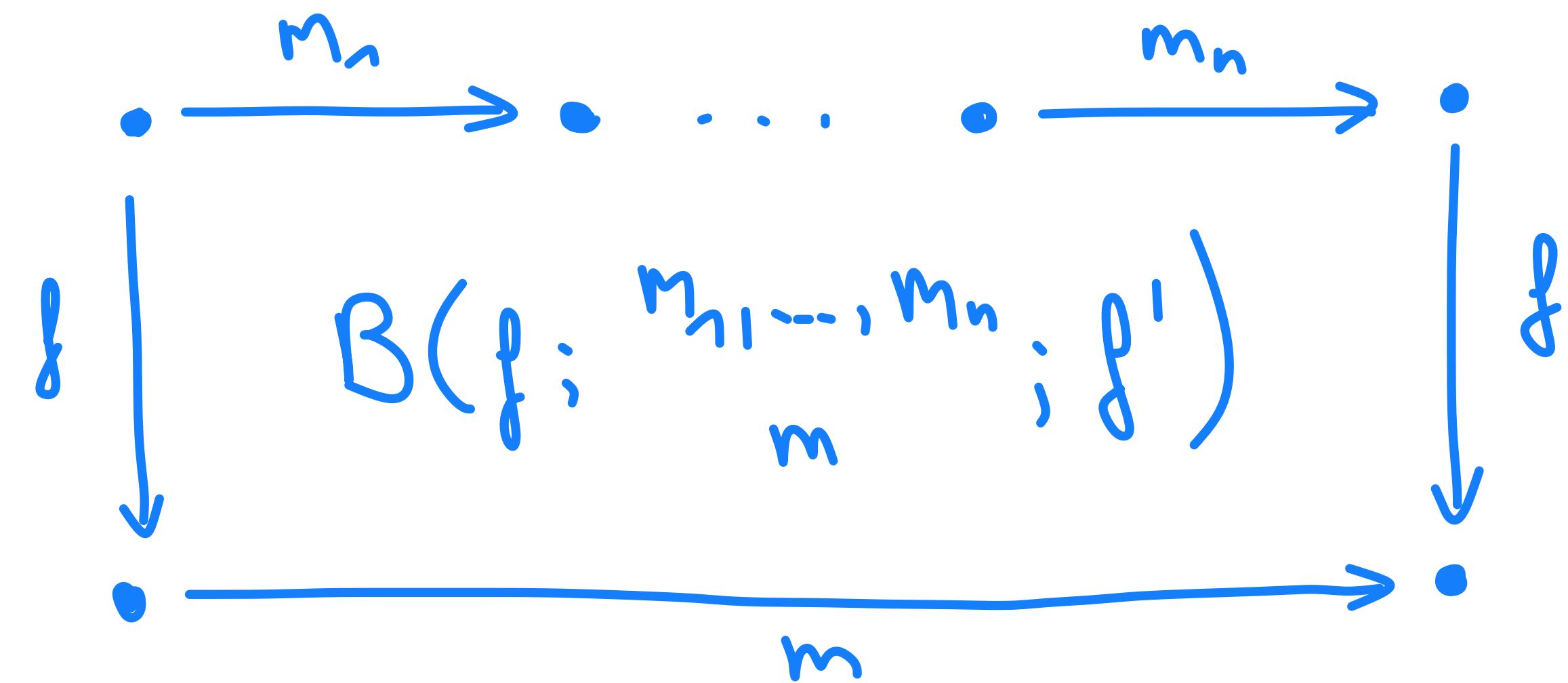
$$\begin{array}{c}
 \boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & & \\ \hline \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & & \\ \hline \end{array}} = \boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array}} = (\boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & \\ \hline 1 & 2 & \\ \hline \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & \\ \hline 1 & 2 & \\ \hline \end{array}})
 \end{array}$$

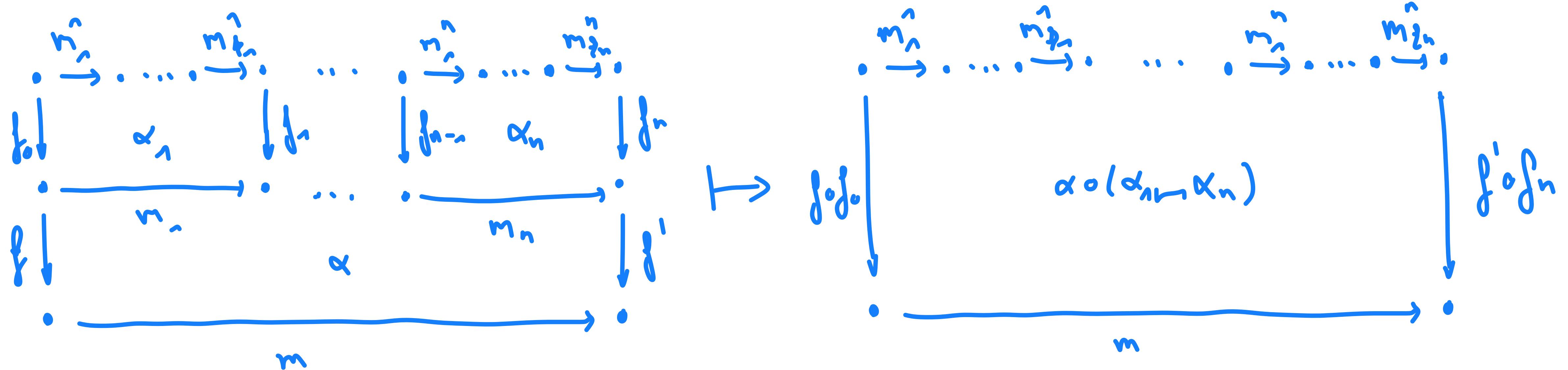
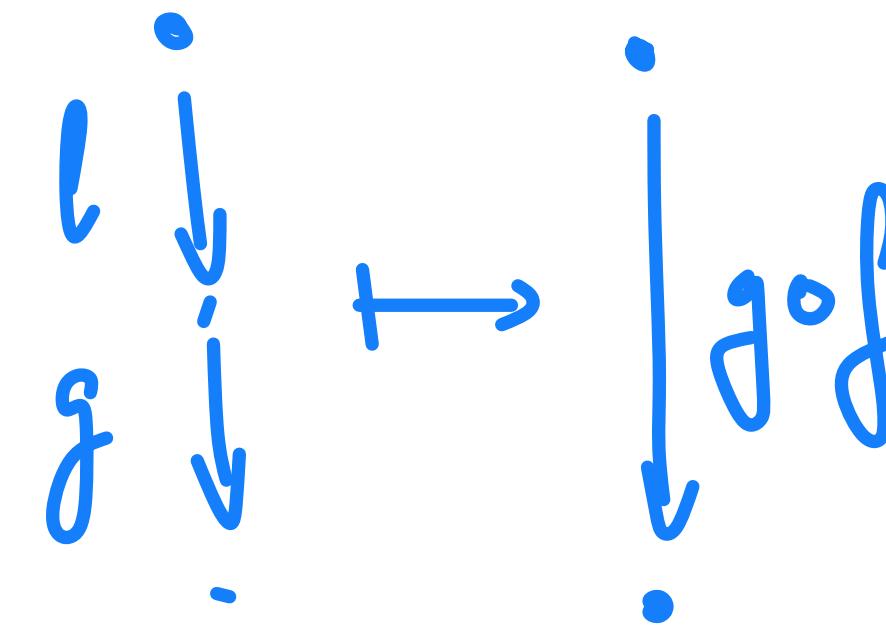
$$\begin{array}{c}
 \boxed{\begin{array}{|c|c|c|} \hline & i & \\ \hline 1 & & \\ \hline \end{array}} \stackrel{o_i}{\rightarrow} \boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & \\ \hline 1 & & \\ \hline \end{array}} = \boxed{\begin{array}{|c|c|c|} \hline & i & \\ \hline 1 & i & \\ \hline 1 & & \\ \hline \end{array}} = (\boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & \\ \hline 1 & & \\ \hline \end{array}} \stackrel{o_1}{\rightarrow} \boxed{\begin{array}{|c|c|c|} \hline & i & \\ \hline 1 & & \\ \hline \end{array}})
 \end{array}$$

$$\begin{array}{c}
 \boxed{\begin{array}{|c|c|c|} \hline & i & \\ \hline 1 & & \\ \hline \end{array}} \stackrel{o_i}{\rightarrow} \boxed{\begin{array}{|c|c|c|} \hline 1 & 2 & \\ \hline 1 & & \\ \hline \end{array}} = \boxed{\begin{array}{|c|c|c|} \hline & i & \\ \hline 1 & i & \\ \hline 1 & j & \\ \hline 1 & & \\ \hline \end{array}} = (\boxed{\begin{array}{|c|c|c|} \hline & j & \\ \hline 1 & i & \\ \hline 1 & & \\ \hline \end{array}} \stackrel{o_j}{\rightarrow} \boxed{\begin{array}{|c|c|c|} \hline & i & \\ \hline 1 & & \\ \hline \end{array}})
 \end{array}$$

Virtual double category  $\mathcal{B}$  enriched over  $(\mathcal{V}, \otimes, I)$   
 consists of

- a set of objects  $\mathcal{B}_o$
- a set of vertical arrows  $\mathcal{B}_v \rightrightarrows \mathcal{B}_o$
- a set of horizontal arrows  $\mathcal{B}_h \rightrightarrows \mathcal{B}_o$
- a collection of  $\mathcal{V}$ -objects of rectangular 2-arrows



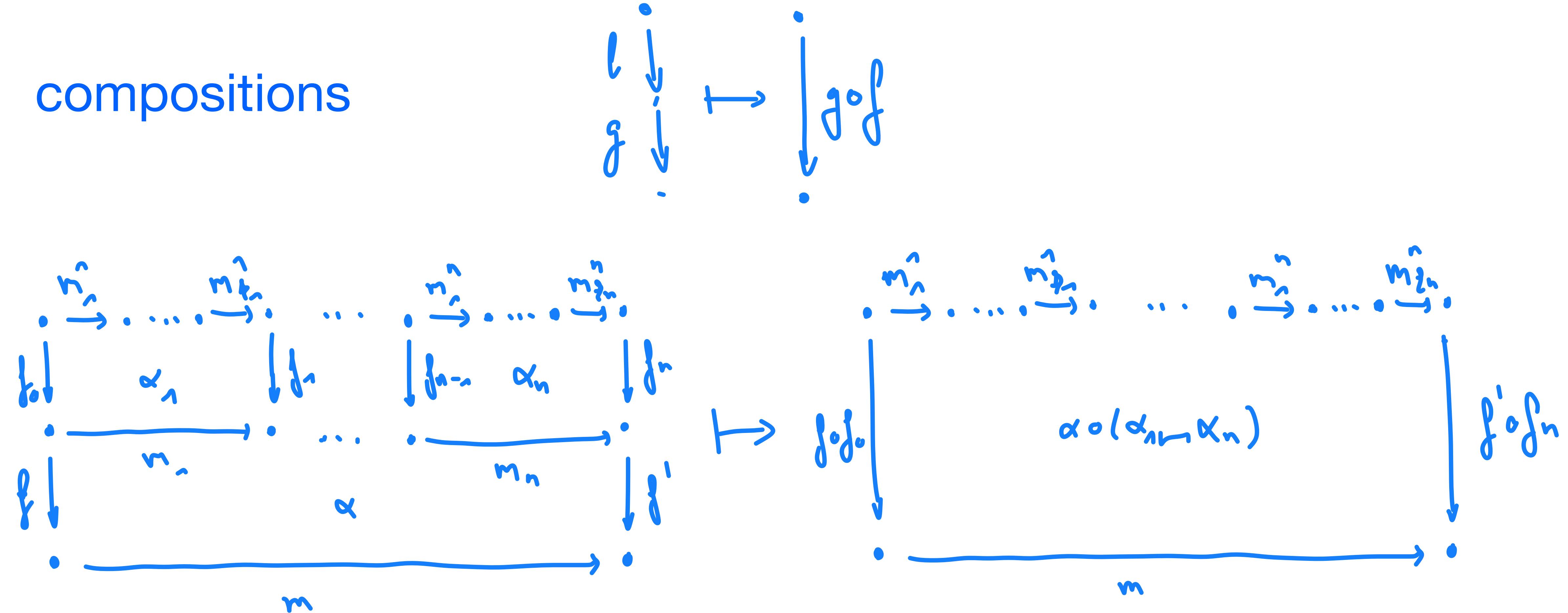


$$\begin{array}{c} a \\ \parallel id_a \\ a \end{array}$$

and

$$\begin{array}{c} id_a \\ \parallel \\ a \xrightarrow{m} b \\ \parallel id_b \\ a \xrightarrow{m} b \end{array}$$

## compositions



units

$$\begin{array}{c} a \\ \parallel id_a \\ a \end{array}$$

and

$$\begin{array}{c} a \xrightarrow{m} b \\ \parallel id_b \\ a \xrightarrow{m} b \end{array}$$