Some things about double categories

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Double categories

(Ehresmann) A *double category* is a category object in **Cat**

$$
\mathbb{A}: \quad
\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, shift left=2, "p_1"] \arrow[r, "\bullet" description] \arrow[r, shift right=2, "p_1"'] & \mathbf{A}_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & \mathbf{A}_0 \arrow[l, "\mathrm{id}" description]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\boxed{\mathrm{Pb}}"] & \mathbf{A}_1 \arrow[d, "d_0"] \\
\mathbf{A}_1 \arrow[r, "d_1"'] & \mathbf{A}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "\mathbf{A}_1 \times_{\mathbf{A}_0} \bullet"] \arrow[d, "\bullet \times_{\mathbf{A}_0} \mathbf{A}_1"'] \arrow[dr, phantom, "\boxed{\mathrm{Assoc}}"] & \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[d, "\bullet"] \\
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "\bullet"'] & \mathbf{A}_\bullet
\end{tikzcd}
$$

Double functor

$$
\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "F_1 \times_{F_0} F_1"'] & \mathbf{A}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "F_1"] & \mathbf{A}_0 \arrow[l] \arrow[d, "F_0"] \\
\mathbf{B}_1 \times_{\mathbf{B}_0} \mathbf{B}_1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbf{B}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbf{B}_0 \arrow[l]
\end{tikzcd}
$$

Think inside the box

$$\mathbb{A} \quad : \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\bullet}}} \mathbf{A}_1 \overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{\mathrm{id}}}} \mathbf{A}_0$$

- Objects of $\mathbf{A}_0$ are *objects* of $\mathbb{A}$
- Morphisms of $\mathbf{A}_0$ are *horizontal arrows* of $\mathbb{A}$
- Objects of $\mathbf{A}_1$ are *vertical arrows* of $\mathbb{A}$
- Morphisms of $\mathbf{A}_1$ are *double cells* of $\mathbb{A}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "w"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

A double category is a category with
two kinds of morphisms, suitably related

Opposite

A an arbitrary category

$$(\square \mathbf{A})^{co}$$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "{[v]}"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "{[w]}"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, phantom, "="] & B \\
C \arrow[u, "v"] \arrow[r, "g"'] & D \arrow[u, "w"']
\end{tikzcd}
$$

Student duality

$\mathbf{A}$ a regular category

$\mathbb{R}\mathrm{el}(\mathbf{A})$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "R"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "S", "\bullet" marking] \\
B \arrow[r, "g"'] & D
\end{tikzcd}
\qquad \text{iff} \qquad
\begin{tikzcd}
R \arrow[r, dashed, "\alpha"] \arrow[d, tail] & S \arrow[d, tail] \\
A \times B \arrow[r, "f \times g"'] & C \times D
\end{tikzcd}
$$

Proposition

There is a double functor $\square\mathbf{A}^{co} \longrightarrow \mathbb{R}\mathrm{el}(\mathbf{A})$ *which is the identity on objects and horizontal arrows, faithful on vertical arrows and full and faithful on cells*

$$
\begin{tikzcd}
A \arrow[d, "{[v]}"', "\bullet" marking] \\
B
\end{tikzcd}
\quad \longleftrightarrow \quad
B \xrightarrow{v} A
\qquad \longmapsto \qquad
\begin{tikzcd}
B \arrow[d, tail, "{\langle v, 1_B \rangle}"] \\
A \times B
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
A \arrow[d, "v^*", "\bullet" marking] \\
B
\end{tikzcd}
$$

Companions

v *companion* to f

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\psi"] & A \arrow[r, "f"] \arrow[d, "v", "\bullet"{marking}] \arrow[dr, phantom, "\chi"] & B \arrow[d, equal] \\ A \arrow[r, "f"'] & B \arrow[r, equal] & B \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}_f"] & B \arrow[d, equal] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

$$\chi\psi = \mathrm{id}_f$$

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\psi"] & A \arrow[d, "v", "\bullet"{marking}] \\ A \arrow[r, "f"'] \arrow[d, "v"', "\bullet"{marking}] \arrow[dr, phantom, "\chi"] & B \arrow[d, equal] \\ B \arrow[r, equal] & B \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "v"', "\bullet"{marking}] \arrow[dr, phantom, "1_v"] & A \arrow[d, "v", "\bullet"{marking}] \\ B \arrow[r, equal] & B \end{tikzcd}$$

$$\chi \circ \psi = 1_v$$

Proposition

(1) If f has a companion it's unique up to isomorphism: write $v = f_*$
(2) $(1_A)_* \cong \mathrm{id}_A$
(3) $(gf)_* \cong g_* f_*$

Conjoints

w is *conjoint* to f

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[r, equal] \arrow[d, "w"] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
A \arrow[r, equal] & A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}_f"] & B \arrow[d, equal] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

$$\beta\alpha = \mathrm{id}_f$$

$$
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "w"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
A \arrow[r, "f"'] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "w"] \\
A \arrow[r, equal] & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "w"'] \arrow[dr, phantom, "1_v"] & B \arrow[d, "w"] \\
A \arrow[r, equal] & A
\end{tikzcd}
$$

$$\alpha \circ \beta = 1_w$$

- Unique up to iso: write $w = f^*$
- $1_A^* \cong \mathrm{id}_A$
- $(gf)^* \cong f^* g^*$

Adjoints

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "v"'] \arrow[dr, phantom, "="] & A \arrow[r, equal] \arrow[d, "v"] & A \arrow[dd, equal] \\
B \arrow[r, equal] \arrow[dd, equal] & B \arrow[d, "w"] \arrow[r, phantom, "\epsilon"] & {} \\
{} & A \arrow[r, equal] \arrow[d, "v"] \arrow[l, phantom, "\eta"] \arrow[dr, phantom, "="] & A \arrow[d, "v"] \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "v"'] \arrow[dr, phantom, "="] & A \arrow[d, "v"] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "w"'] \arrow[dr, phantom, "="] & B \arrow[d, "w"] \\
A \arrow[r, equal] & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, equal] \arrow[dd, equal] & B \arrow[r, equal] \arrow[d, "w"] \arrow[dr, phantom, "="] & B \arrow[d, "w"] \\
{} & A \arrow[r, equal] \arrow[d, "v"] \arrow[l, phantom, "\eta"] & B \arrow[dd, equal] \\
B \arrow[r, equal] \arrow[d, "w"'] \arrow[dr, phantom, "="] & B \arrow[d, "w"] \arrow[r, phantom, "\epsilon"] & {} \\
A \arrow[r, equal] & A \arrow[r, equal] & A
\end{tikzcd}
$$

$$w \dashv v$$

Companions, conjoints, adjoints

Theorem

Any two of the following conditions imply the third:

(1) $v = f_*$

(2) $w = f^*$

(3) $v \dashv w$

Theorem

In $\mathbb{R}\mathrm{el}(\mathbf{A})$

(1) Every f *has a companion:* $f_* = (A \xrightarrow{\langle 1_A, f\rangle} A \times B)$

(2) Every f *has a conjoint:* $f^* = (A \xrightarrow{\langle f, 1_A\rangle} B \times A)$

(3) Every adjoint pair $R \dashv S$ *is of the form* $f_* \dashv f^*$

Say $\mathbb{R}\mathrm{el}(\mathbf{A})$ is *Cauchy*

Tabulators

The *tabulator* of v is a universal cell τ

$$
\begin{tikzcd}
& A \arrow[dd, "v"] \\
X \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "\xi"] & {} \\
& B
\end{tikzcd}
\quad \text{i.e.} \quad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\xi"] & A \arrow[d, "v"] \\
X \arrow[r, "g"'] & B
\end{tikzcd}
$$

$$\forall \xi \, \exists! x (\xi = \tau x)$$

$$
\begin{tikzcd}
& A \arrow[dd, "v"] \\
X \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "\xi"] & {} \\
& B
\end{tikzcd}
=
\begin{tikzcd}
& & A \arrow[dd, "v"] \\
X \arrow[r, "x"] & T(v) \arrow[ur, "t_0"] \arrow[dr, "t_1"'] \arrow[r, phantom, "\tau"] & {} \\
& & B
\end{tikzcd}
$$

$T(v)$ is *effective* if t_1 has a companion, t_0 has a conjoint and $v \cong t_{1*} \bullet t_0^*$

Tabulating relations

Proposition

$\mathbb{R}\mathrm{el}(\mathbf{A})$ *has effective tabulators*

$$
\begin{tikzcd}
 & A \arrow[d, "R", "\bullet"'] \\
R \arrow[ur, "r_0"] \arrow[dr, "r_1"'] \arrow[r, phantom, "\leq"] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
R \arrow[r, equal] \arrow[d, tail, "\Delta"'] & R \arrow[d, tail, "\langle r_0, r_1 \rangle"] \\
R \times R \arrow[r, "r_0 \times r_1"'] & A \times B
\end{tikzcd}
$$

Double functors on relations

Theorem

Double functors $\mathbb{R}\mathrm{el}(\mathbf{A}) \longrightarrow \mathbb{R}\mathrm{el}(\mathbf{B})$ *"are" functors* $\mathbf{A} \longrightarrow \mathbf{B}$ *which preserve quasi-pullbacks*

Quasi-pullback Q

$$\begin{tikzcd} Q \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{\text{Pb}}"] & A \arrow[d] \\ & B \arrow[r] & C \end{tikzcd}$$

Transformations

$\mathbf{Doub} = \mathbf{Cat}(\mathbf{Cat})$ is cartesian closed, so $\mathbf{Doub}(\mathbb{A},\mathbb{B})$ is a double category

A *horizontal transformation* $t\colon F \longrightarrow G$

- $\forall A$ a horizontal arrow $tA\colon FA \longrightarrow GA$
- $\forall v\colon A \xrightarrow{\bullet} A'$ a cell

$$\begin{tikzcd} FA \arrow[r, "tA"] \arrow[d, "Fv"', "\bullet" marking] \arrow[dr, phantom, "tv"] & GA \arrow[d, "Gv", "\bullet" marking] \\ FA' \arrow[r, "tA'"'] & GA' \end{tikzcd}$$

- Horizontally natural

$$\begin{tikzcd} FA \arrow[r, "tA"] \arrow[d, "Fv"', "\bullet" marking] \arrow[dr, phantom, "tv"] & GA \arrow[r, "Gf"] \arrow[d, "Gv", "\bullet" marking] \arrow[dr, phantom, "G\alpha"] & GC \arrow[d, "Gw", "\bullet" marking] \\ FA' \arrow[r, "tA'"'] & GA' \arrow[r, "Gg"'] & GC' \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "Fv"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FC \arrow[r, "tC"] \arrow[d, "Fw", "\bullet" marking] \arrow[dr, phantom, "tw"] & GC \arrow[d, "Gw", "\bullet" marking] \\ FA' \arrow[r, "Fg"'] & FC' \arrow[r, "tC'"'] & GC' \end{tikzcd}$$

Transformations (continued)

- Vertically functorial

$$
\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "F\,\mathrm{id}_A"'] \arrow[dr, phantom, "t(\mathrm{id}_A)"] & GA \arrow[d, "G\,\mathrm{id}_A"] \\
FA \arrow[r, "tA"'] & GA
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "\mathrm{id}_{FA}"'] \arrow[dr, phantom, "\mathrm{id}_{tA}"] & GA \arrow[d, "\mathrm{id}_{GA}"] \\
GA \arrow[r, "tA"'] & GA
\end{tikzcd}
$$

$$
\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "Fv"'] \arrow[dr, phantom, "tv"] & GA \arrow[d, "Gv"] \\
FA' \arrow[r, "tA'"'] \arrow[d, "Fv'"'] \arrow[dr, phantom, "tv'"] & GA' \arrow[d, "Gv'"] \\
FA'' \arrow[r, "tA''"'] & GA''
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "F(v'\bullet v)"'] \arrow[dr, phantom, "t(v'\bullet v)"] & GA \arrow[d, "G(v'\bullet v)"] \\
FA'' \arrow[r, "tA''"'] & GA''
\end{tikzcd}
$$

Vertical transformations and cells

A *vertical transformation* $u\colon F \longrightarrow\!\!\!\!\bullet\!\!\!\!\longrightarrow H$ is the transpose notion (switch horizontal and vertical)

- $\forall A$ a vertical arrow $uA\colon FA \longrightarrow\!\!\!\!\bullet\!\!\!\!\longrightarrow HA$
- $\forall f\colon A \longrightarrow A'$ a cell

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "uA"'] \arrow[dr, phantom, "uf"] & FA' \arrow[d, "uA'"] \\ HA \arrow[r, "Hf"'] & HA' \end{tikzcd}$$

- Vertically natural
- Horizontally functorial

A *double cell* assigns to each object A a cell νA

$$\begin{tikzcd} F \arrow[r, "t"] \arrow[d, "u"'] \arrow[dr, phantom, "\nu"] & G \arrow[d, "u'"] \\ H \arrow[r, "t'"'] & K \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "tA"] \arrow[d, "uA"'] \arrow[dr, phantom, "\nu A"] & GA \arrow[d, "u'A"] \\ HA \arrow[r, "t'A"'] & KA \end{tikzcd}$$

satisfying two conditions – horizontal and vertical naturality

Transformations for $\mathbb{R}\mathrm{el}$

$F, G \colon \mathbf{A} \longrightarrow \mathbf{B}$ quasi-pullback preserving functors
$\Phi, \Psi \colon \mathbb{R}\mathrm{el}(\mathbf{A}) \longrightarrow \mathbb{R}\mathrm{el}(\mathbf{B})$ their extensions to $\mathbb{R}\mathrm{el}$

Theorem
(1) Horizontal transformations $\Phi \longrightarrow \Psi$ are in natural bijection with natural transformations $F \longrightarrow G$
(2) Vertical transformations $\Phi \longrightarrow\!\!\!\!\!\!\bullet\;\;\; \Psi$ are in natural bijection with relations

$$V \rightarrowtail F \times G$$

in the category $QPB(\mathbf{A}, \mathbf{B})$ of quasi-pullback preserving functors and quasi-cartesian natural transformations

Question: Is $QPB(\mathbf{A}, \mathbf{B})$ a regular category?

Kleisli

$\mathbb{T} = (T, \eta, \mu)$ is a monad on $\mathbf{A}$
We get a double category $\mathbb{K}\mathrm{l}(\mathbb{T})$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "{[v]}"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "{[w]}", "\bullet" marking] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "="] & B \arrow[d, "w"] \\
TC \arrow[r, "Tg"'] & TD
\end{tikzcd}
$$

- Every horizontal arrow $f\colon A \longrightarrow B$ has a companion

$$
\begin{tikzcd}
A \arrow[d, "f_*"', "\bullet" marking] \\
B
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
A \arrow[d, "f"] \\
B \arrow[d, "\eta B"] \\
TB
\end{tikzcd}
\qquad (f_* = [\eta B \cdot f])
$$

- $f\colon A \longrightarrow B$ has a conjoint iff $T(f)$ iso

$$
\begin{tikzcd}
B \arrow[d, "f^*"', "\bullet" marking] \\
A
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
B \arrow[d, "\eta B"] \\
TB \arrow[d, "(Tf)^{-1}"] \\
TA
\end{tikzcd}
\qquad (f^* = [(Tf)^{-1} \cdot \eta B])
$$

Tabulating Kleisli

Proposition

$\mathbb{K}\mathrm{l}(T)$ *has tabulators iff* **A** *has pullbacks along* ηA*'s*

Proof.

$$
\begin{tikzcd}
 & A \arrow[d, "{[v]}", "\bullet"'] \\
X \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "\xi"] & B \\
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta X"'] \arrow[dr, phantom, "="] & A \arrow[d, "v"] \\
TX \arrow[r, "Tg"'] & TB
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dr, "v"] & \\
X \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "="] & & TB \\
 & B \arrow[ur, "\eta B"'] &
\end{tikzcd}
$$

$\square$

- The tabulators are not effective

Double functors on $\mathbb{K}\mathrm{l}$

Theorem

Double functors $\mathbb{K}\mathrm{l}(\mathbb{T}) \longrightarrow \mathbb{K}\mathrm{l}(\mathbb{S})$ correspond to monad morphisms $\mathbb{T} \longrightarrow \mathbb{S}$

Morphism of monads:

$(\Psi, \psi) \colon \mathbb{T} \longrightarrow \mathbb{S} \qquad (\mathbb{T} = (\mathbf{A}, T, \eta, \mu),\ \mathbb{S} = (\mathbf{B}, S, \kappa, \nu))$

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "\Psi"] \arrow[d, "T"'] \arrow[dr, phantom, "\overset{\psi}{\nearrow}"] & \mathbf{B} \arrow[d, "S"] \\
\mathbf{A} \arrow[r, "\Psi"'] & \mathbf{B}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Psi \arrow[dl, "\Psi\eta"'] \arrow[dr, "\kappa\Psi"] & \\
\Psi T \arrow[rr, "\psi"'] & & S\Psi
\end{tikzcd}
\qquad
\begin{tikzcd}
\Psi TT \arrow[r, "\psi T"] \arrow[dr, "\Psi\mu"'] & S\Psi T \arrow[r, "S\psi"] & SS\Psi \arrow[dl, "\nu\Psi"] \\
 & \Psi T \arrow[r, "\psi"'] & S\Psi
\end{tikzcd}$$

Transformations of monad morphisms

$(\Phi,\phi),(\Psi,\psi)\colon \mathbb{T} \longrightarrow \mathbb{S}$

A *Street 2-cell* $t\colon (\Phi,\phi) \longrightarrow (\Psi,\psi)$ is

- a natural transformation $t\colon \Phi \longrightarrow \Psi$
- satisfying

$$\begin{tikzcd}
\Phi T \arrow[r, "tT"] \arrow[d, "\phi"'] \arrow[dr, phantom, "="] & \Psi T \arrow[d, "\psi"] \\
S\Phi \arrow[r, "St"'] & S\Psi
\end{tikzcd}$$

Other transformations

A *Lack-Street 2-cell* $u\colon (\Phi,\phi) \xrightarrow{\bullet} (\Psi,\psi)$ is

- a natural transformation $u\colon \Phi \longrightarrow S\Psi$
- satisfying

$$
\begin{tikzcd}
\Phi T \arrow[rr, "\phi"] \arrow[dd, "uT"'] & & S\Phi \arrow[d, "Su"] \\
& & SS\Psi \arrow[d, "\nu\Psi"] \\
S\Psi T \arrow[r, "S\psi"'] & SS\Psi \arrow[r, "\nu\Psi"'] & S\Psi
\end{tikzcd}
$$

Double category version

Theorem

Let (Φ,ϕ) *and* (Ψ,ψ) *be monad morphisms* $\mathbb{T} \longrightarrow \mathbb{S}$ *giving rise to double functors* $\overline{\Phi}, \overline{\Psi} \colon \mathbb{K}\mathrm{l}(\mathbb{T}) \longrightarrow \mathbb{K}\mathrm{l}(\mathbb{S})$*. Then*

(1) horizontal transformations $\overline{\Phi} \longrightarrow \overline{\Psi}$ *correspond to Street* 2*-cells* $(\Phi,\phi) \longrightarrow (\Psi,\psi)$

(2) vertical transformations $\overline{\Phi} \longrightarrow\!\!\bullet\!\!\longrightarrow \overline{\Psi}$ *correspond to Lack-Street* 2*-cells* $(\Phi,\phi) \longrightarrow\!\!\bullet\!\!\longrightarrow (\Psi,\psi)$

Lax morphisms of monads

- A *lax morphism of monads* (Φ, ϕ)

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\Phi"] \arrow[d, "T"'] \arrow[dr, phantom, "\swarrow\phi"] & \mathbf{A}' \arrow[d, "T'"] \\ \mathbf{A} \arrow[r, "\Phi"'] & \mathbf{A}' \end{tikzcd}$$

satisfying

$$\begin{tikzcd} & \Phi \arrow[dl, "\eta'\Phi"'] \arrow[dr, "\Phi\eta"] & \\ T'\Phi \arrow[rr, "\phi"'] & & \Phi T \end{tikzcd}
\qquad
\begin{tikzcd} T'T'\Phi \arrow[r, "T'\phi"] \arrow[d, "\mu'\Phi"'] & T'\Phi T \arrow[r, "\phi T"] & \Phi TT \arrow[d, "\Phi\mu"] \\ T'\Phi \arrow[rr, "\phi"'] & & \Phi T \end{tikzcd}$$

Lax vs oplax

- (Ψ, ψ) was oplax

$$\begin{tikzcd}
\mathbf{EM}(\mathbb{T}) \arrow[r, "\mathbf{EM}(\Phi,\phi)"] \arrow[d] & \mathbf{EM}(\mathbb{T}') \arrow[d] \\
\mathbf{A} \arrow[r, "\Phi"'] & \mathbf{A}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Kl}(\mathbb{T}) \arrow[r, "\mathbf{Kl}(\Psi,\psi)"] & \mathbf{Kl}(\mathbb{S}) \\
\mathbf{A} \arrow[r, "\Psi"'] \arrow[u] & \mathbf{B} \arrow[u]
\end{tikzcd}$$

Lax and oplax together at last

The double category $\mathbb{M}\mathrm{onad}$

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "(\Phi{,}\phi)"] \arrow[d, "(\Psi{,}\psi)"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & \mathbb{T}' \arrow[d, "(\Psi'{,}\psi')", "\bullet" marking] \\
\mathbb{S} \arrow[r, "(\Theta{,}\theta)"'] & \mathbb{S}'
\end{tikzcd}$$

$$\Psi'\Phi \xrightarrow{\alpha} \Theta\Psi$$

$$\begin{tikzcd}
 & \Psi'\Phi T \arrow[r, "\alpha T"] & \Theta\Psi T \arrow[dr, "\Theta\psi"] & \\
\Psi' T'\Phi \arrow[ur, "\Psi'\phi"] \arrow[dr, "\psi'\Phi"'] & & & \Theta S\Psi \\
 & S'\Psi'\Phi \arrow[r, "S'\alpha"'] & S'\Theta\Psi \arrow[ur, "\theta\Psi"'] &
\end{tikzcd}$$

Fear of hexagons

$$
\begin{tikzcd}
\mathbf{A} \arrow[rr, "\Psi"] \arrow[dd, "\Phi"'] \arrow[ddrr, phantom, "\alpha \Nearrow"] & & \mathbf{B} \arrow[dr, "S"] \arrow[dd, "\Theta"] & \\
& & & \mathbf{B} \arrow[dd, "\Theta"] \\
\mathbf{A}' \arrow[rr, "\Psi'"'] \arrow[dr, "T'"'] & & \mathbf{B}' \arrow[dr, "S'"] \arrow[ur, phantom, "\theta \Nearrow"] & \\
& \mathbf{A}' \arrow[rr, "\Psi'"'] \arrow[ur, phantom, "\psi' \Nearrow"] & & \mathbf{B}'
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{A} \arrow[rr, "\psi"] \arrow[dr, "T"] \arrow[dd, "\Phi"'] & & \mathbf{B} \arrow[dr, "S"] & \\
& \mathbf{A} \arrow[rr, "\psi"] \arrow[dd, "\Phi"'] \arrow[ur, phantom, "\psi \Nearrow"] \arrow[ddrr, phantom, "\alpha \Nearrow"] & & \mathbf{B} \arrow[dd, "\Theta"] \\
\mathbf{A}' \arrow[dr, "T'"'] \arrow[ur, phantom, "\Nearrow \phi"] & & & \\
& \mathbf{A}' \arrow[rr, "\Psi'"'] & & \mathbf{B}'
\end{tikzcd}
$$

Properties of $\mathbb{M}$onad

Theorem

(1) (Φ,ϕ) *has a companion iff* ϕ *is iso*

(2) (Φ,ϕ) *has a conjoint iff* Φ *has a left adjoint*

(3) $\mathbb{M}\mathrm{nd}$ *has tabulators and they are effective*

The tabulator

The tabulator of $(\Psi,\psi)\colon (\mathbf{A},T,\eta,\mu) \mathrel{-\!\!\!\bullet\!\!\!\rightarrow} (\mathbf{B},S,\kappa,\nu)$ is given by the comma category $\Psi\downarrow\mathbf{B}$ with monad

$$\Psi\downarrow\mathbf{B} \xrightarrow{T\downarrow S} \Psi\downarrow\mathbf{B}$$

$$(A, \Psi A \xrightarrow{b} B) \quad \longmapsto \quad (TA, \Psi TA \xrightarrow{\psi A} S\Psi A \xrightarrow{Sb} SB)$$

$$\begin{tikzcd}[column sep=small]
 & & A & & & & \mathbf{A} \arrow[dd, "\Psi" description, "\bullet" marking] \\
(A, \Psi A \rightarrow B) \arrow[urr, mapsto] \arrow[drr, mapsto] & & & & \Psi\downarrow\mathbf{B} \arrow[urr, "P_1"] \arrow[drr, "P_2"'] & \Downarrow & \\
 & & B & & & & \mathbf{B}
\end{tikzcd}$$

Eilenberg-Moore for a change

A lax morphism $(\Phi,\phi)\colon \mathbb{T} \longrightarrow \mathbb{T}'$ gives an algebraic functor over Φ

$$(TA \xrightarrow{\alpha} A) \longmapsto (T'\Phi A \xrightarrow{\phi A} \Phi TA \xrightarrow{\Phi a} \Phi A)$$

$$\begin{tikzcd}
\mathbf{EM}(\mathbb{T}) \arrow[r, "\mathbf{EM}(\Phi,\phi)"] \arrow[d] & \mathbf{EM}(\mathbb{T}') \arrow[d] \\
\mathbf{A} \arrow[r, "\Phi"'] & \mathbf{A}'
\end{tikzcd}$$

But what about oplax morphisms $(\Psi,\psi)\colon \mathbb{T} \longrightarrow\!\!\bullet\!\!\longrightarrow \mathbb{S}$?

Profunctors make a cameo appearance

$$\mathbf{EM}(\Psi,\psi)\colon \mathbf{EM}(\mathbb{T}) \longrightarrow\!\!\!\!\!\!\!\bullet\;\; \mathbf{EM}(\mathbb{S})$$

$$\mathbf{EM}(\Psi,\psi)\colon \mathbf{EM}(\mathbb{T})^{op} \times \mathbf{EM}(\mathbb{S}) \longrightarrow \mathbf{Set}$$

An element of $\mathbf{EM}(\Psi,\psi)((A,a),(B,b))$ is $x\colon \Psi A \longrightarrow B$

$$\begin{tikzcd}
\Psi TA \arrow[r, "\psi A"] \arrow[d, "\Psi a"'] & S\Psi A \arrow[r, "Sx"] & SB \arrow[d, "b"] \\
\Psi A \arrow[rr, "x"'] & & B
\end{tikzcd}$$

EM extends to cells in $\mathbb{M}$onad

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "(\Phi{,}\phi)"] \arrow[d, "(\Psi{,}\psi)"'] \arrow[dr, phantom, "\alpha"] & \mathbb{T}' \arrow[d, "(\Psi'{,}\psi')"] \\
\mathbb{S} \arrow[r, "(\Theta{,}\theta)"'] & \mathbb{S}'
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{EM}(\mathbb{T}) \arrow[r, "\mathbf{EM}(\Phi{,}\phi)"] \arrow[d, "(\Psi{,}\psi)"'] \arrow[dr, phantom, "\overset{\mathbf{EM}(\alpha)}{\Longrightarrow}"] & \mathbf{EM}(\mathbb{T}') \arrow[d, "(\Psi'{,}\psi')"] \\
\mathbf{EM}(\mathbb{S}) \arrow[r, "\mathbf{EM}(\Theta{,}\theta)"'] & \mathbf{EM}(\mathbb{S}')
\end{tikzcd}$$

$$\mathbf{EM}(\alpha)\colon (\Psi A \xrightarrow{x} B) \quad \longmapsto \quad (\Psi'\Phi A \xrightarrow{\alpha A} \Theta\Psi A \xrightarrow{\Theta x} \Theta B)$$

$$\mathbf{EM}\colon \mathbb{M}\mathrm{nd} \longrightarrow \mathbb{C}\mathrm{at}$$

$\mathbb{C}\mathrm{at}$

$$\begin{tikzcd} \mathbf{A} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\overset{t}{\Rightarrow}"] & \mathbf{B} \arrow[d, "Q"] \\ \mathbf{C} \arrow[r, "G"'] & \mathbf{D} \end{tikzcd}$$

- $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ categories
- F, G functors
- P, Q profunctors

$$P\colon \mathbf{A}^{op} \times \mathbf{C} \longrightarrow \mathbf{Set}\ , Q\colon \mathbf{B}^{op} \times \mathbf{D} \longrightarrow \mathbf{Set}$$

- t natural transformation

$$t\colon P(-,=) \longrightarrow Q(F-, G=)$$

- Composition of profunctors uses coends, and is not associative on the nose

$\mathbb{C}\mathrm{at}$ is a *weak double category*
$\mathbf{EM}\colon \mathbb{M}\mathrm{nd} \longrightarrow \mathbb{C}\mathrm{at}$ is a *lax double functor*

Double categories

(Ehresmann) A *double category* is a category object in **Cat**

$$\mathbb{A}\colon \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \overset{p_1}{\underset{p_1}{\overset{\bullet}{\rightrightarrows}}} \mathbf{A}_1 \overset{d_0}{\underset{d_1}{\overset{\mathrm{id}}{\leftrightarrows}}} \mathbf{A}_0$$

$$\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\boxed{\text{Pb}}"] & \mathbf{A}_1 \arrow[d, "d_0"] \\
\mathbf{A}_1 \arrow[r, "d_1"'] & \mathbf{A}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "\mathbf{A}_1 \times_{\mathbf{A}_0} \bullet"] \arrow[d, "\bullet \times_{\mathbf{A}_0} \mathbf{A}_1"'] \arrow[dr, phantom, "\boxed{\text{Assoc}}"] & \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[d, "\bullet"] \\
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "\bullet"'] & \mathbf{A}_\bullet
\end{tikzcd}$$

Double functor

$$\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "F_1 \times_{F_0} F_1"'] & \mathbf{A}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "F_1"] & \mathbf{A}_0 \arrow[l] \arrow[d, "F_0"] \\
\mathbf{B}_1 \times_{\mathbf{B}_0} \mathbf{B}_1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbf{B}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbf{B}_0 \arrow[l]
\end{tikzcd}$$

Weak double categories

A *(weak) double category* is a weak category object in $\mathscr{C}at$

$$\mathbb{A}\colon \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \overset{p_1}{\underset{p_1}{\overset{\bullet}{\rightrightarrows}}} \mathbf{A}_1 \overset{d_0}{\underset{d_1}{\overset{\mathrm{id}}{\leftrightarrows}}} \mathbf{A}_0$$

$$\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\boxed{\mathrm{Pb}}"] & \mathbf{A}_1 \arrow[d, "d_0"] \\
\mathbf{A}_1 \arrow[r, "d_1"'] & \mathbf{A}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "\mathbf{A}_1 \times_{\mathbf{A}_0} \bullet"] \arrow[d, "\bullet \times_{\mathbf{A}_0} \mathbf{A}_1"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}\cong"] & \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[d, "\bullet"] \\
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "\bullet"'] & \mathbf{A}_\bullet
\end{tikzcd}$$

Double functor

$$\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[d, "F_1 \times_{F_0} F_1"'] & \mathbf{A}_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "F_1"] & \mathbf{A}_0 \arrow[l] \arrow[d, "F_0"] \\
\mathbf{B}_1 \times_{\mathbf{B}_0} \mathbf{B}_1 \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & \mathbf{B}_1 \arrow[r, shift left] \arrow[r, shift right] & \mathbf{B}_0 \arrow[l]
\end{tikzcd}$$

Lax double functors of weak double categories

A *(weak) double category* is a weak category object in $\mathscr{C}at$

$$\mathbb{A}\colon \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \underset{p_1}{\overset{p_1}{\underset{\bullet}{\rightrightarrows}}} \mathbf{A}_1 \underset{d_1}{\overset{d_0}{\underset{\mathrm{id}}{\leftrightarrows}}} \mathbf{A}_0$$

$$\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\boxed{\mathrm{Pb}}"] & \mathbf{A}_1 \arrow[d, "d_0"] \\
\mathbf{A}_1 \arrow[r, "d_1"'] & \mathbf{A}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "\mathbf{A}_1 \times_{\mathbf{A}_0} \bullet"] \arrow[d, "\bullet \times_{\mathbf{A}_0} \mathbf{A}_1"'] \arrow[dr, phantom, "\alpha \Swarrow \cong"] & \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[d, "\bullet"] \\
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "\bullet"'] & \mathbf{A}_\bullet
\end{tikzcd}$$

Lax double functor

$$\begin{tikzcd}
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \arrow[r, "\bullet"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\phi \Nearrow"] & \mathbf{A}_1 \arrow[d, "F_1"] \arrow[dr, phantom, "\Nwarrow \phi_0"] & \mathbf{A}_0 \arrow[l, "\mathrm{id}"'] \arrow[d, "F_0"] \\
\mathbf{B}_1 \times_{\mathbf{B}_0} \mathbf{B}_1 \arrow[r, "\bullet"'] & \mathbf{B}_1 & \mathbf{B}_0 \arrow[l, "\mathrm{id}"]
\end{tikzcd}$$

And, this is where the story begins...

Thank you!