

# DOUBLE-CATEGORICAL COMPOSITIONAL REWRITING THEORY

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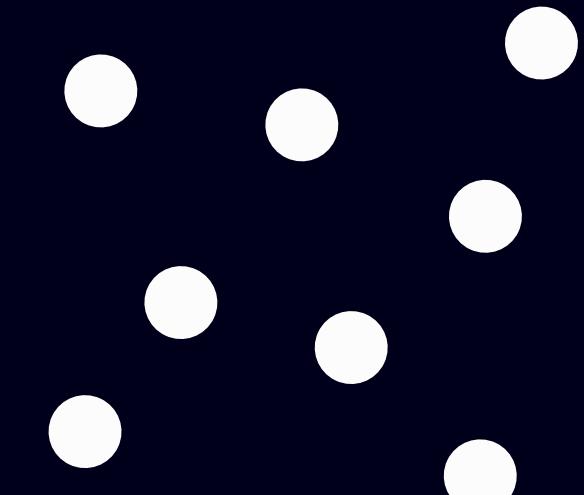
VIRTUAL DOUBLE CATEGORIES WORKSHOP

NOVEMBER 30, 2022

BASED UPON JOINT WORK WITH

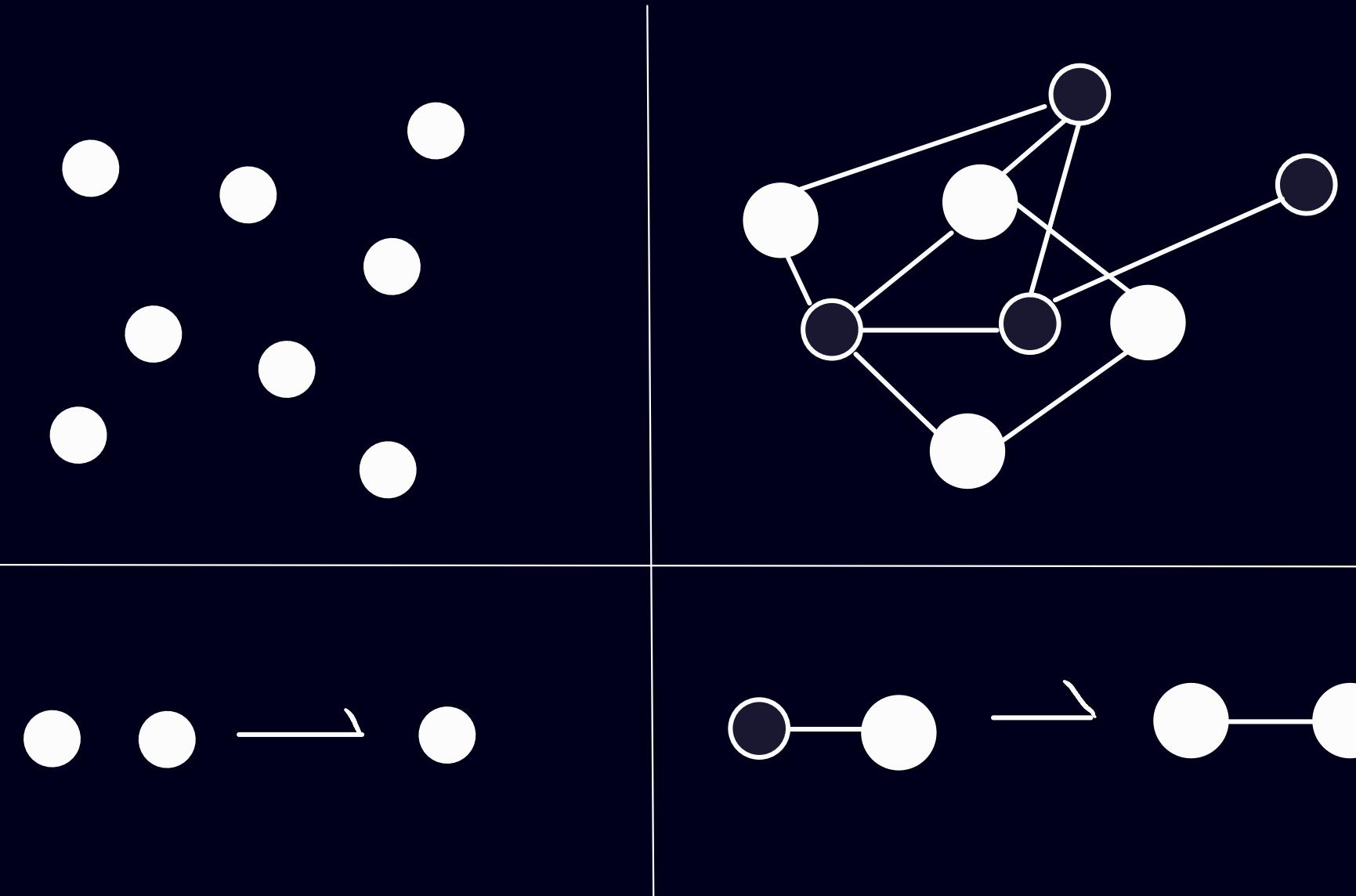
- P.-A. MELLIÈS & N. ZEILBERGER
- R. HARMER & J. KRIVINE (2204.07175)

# ① MOTIVATION



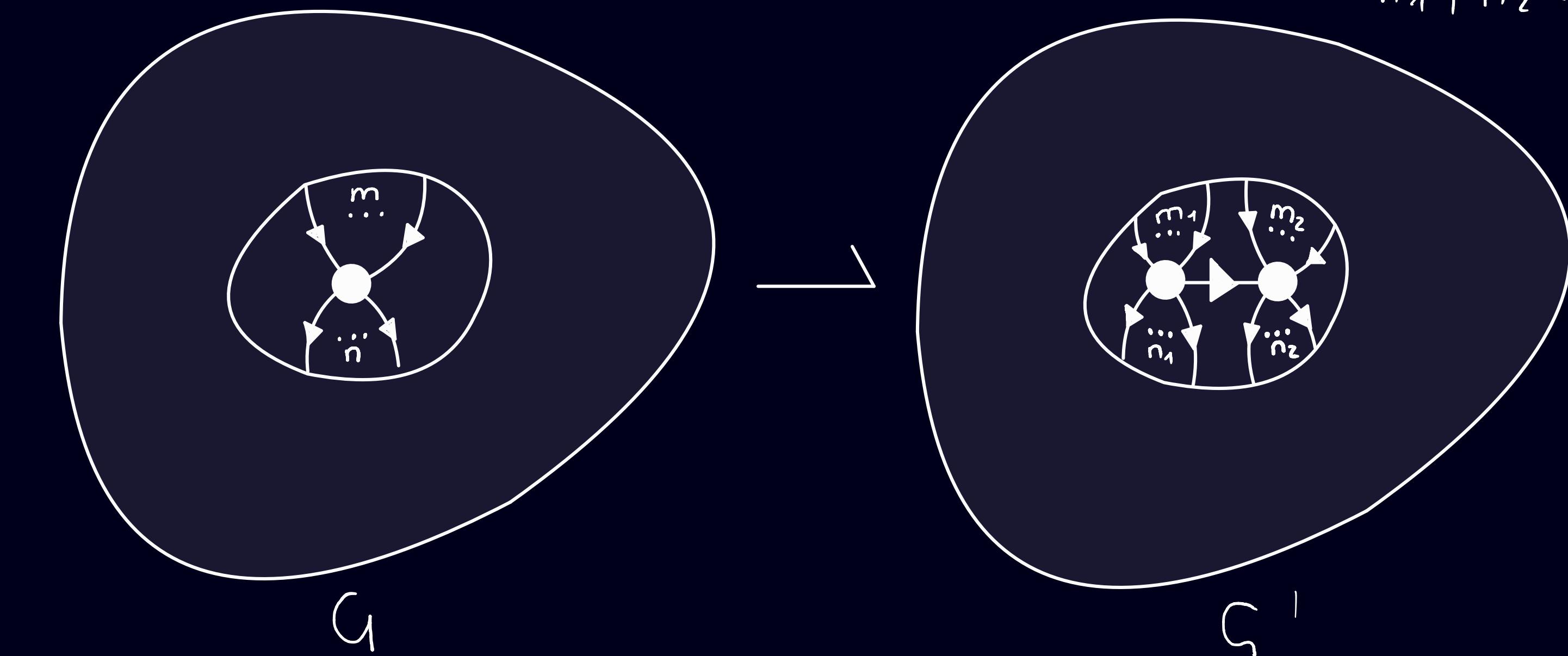
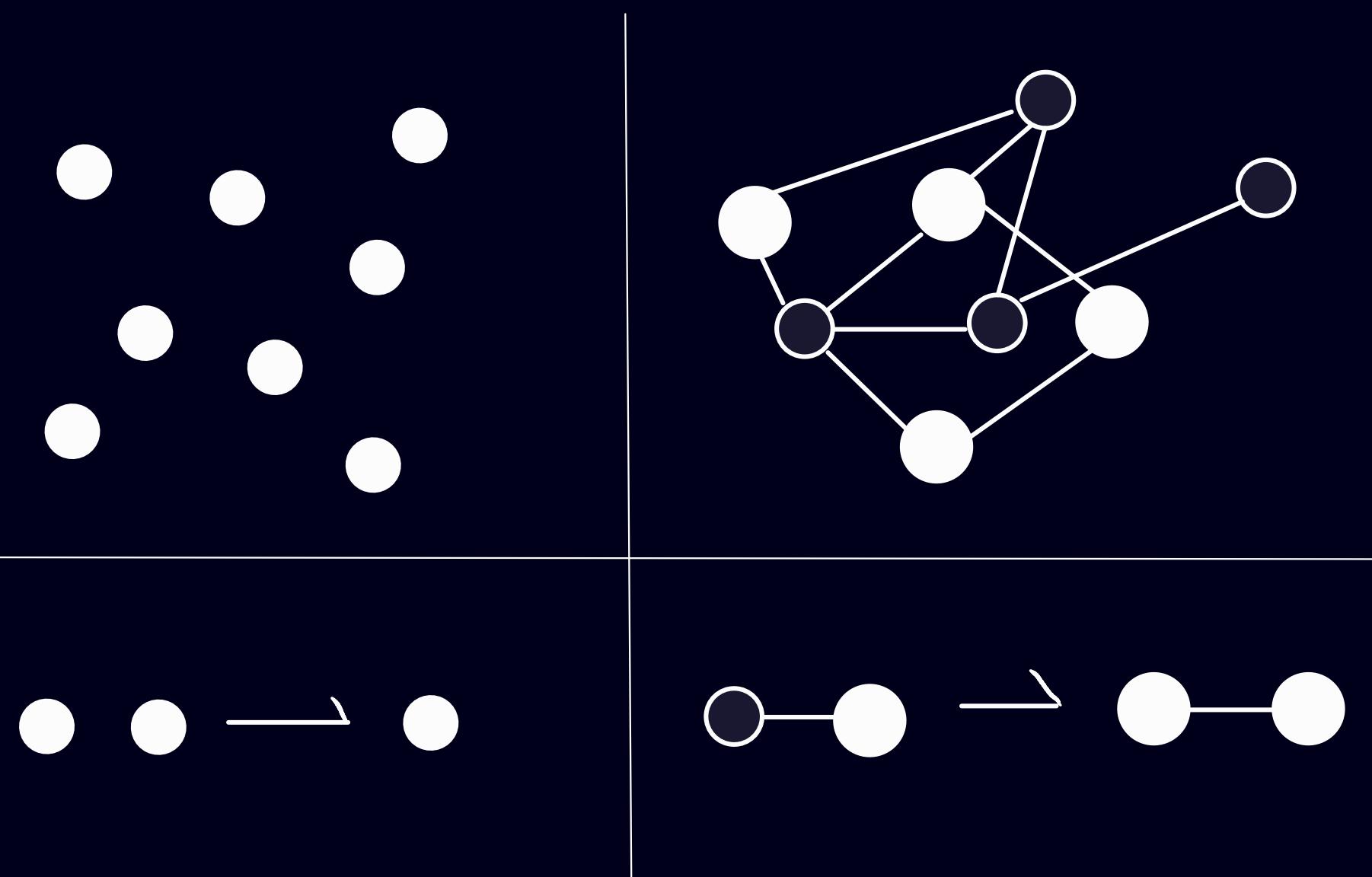
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# MOTIVATION



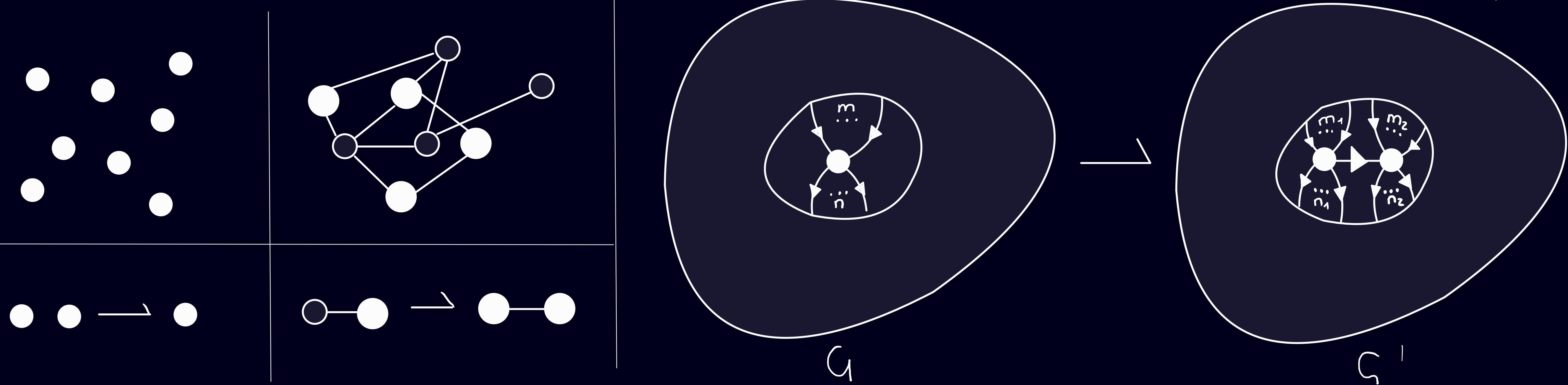
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## MOTIVATION



$$\begin{aligned}m_1 + m_2 &= m \\n_1 + n_2 &= n\end{aligned}$$

# ① MOTIVATION



↳ ALL FORMALIZABLE IN DOUBLE - PUSHOUT (DPO) SEMANTICS :

$$\begin{array}{ccc}
 \begin{array}{c} O \xleftarrow{r} I \\ n \downarrow \Downarrow \alpha \downarrow m \\ r_\alpha(x) \longleftarrow X \end{array} & := & \begin{array}{c} O \xleftarrow{\text{or}} K_r \xrightarrow{\text{ir}} I \\ n \downarrow \text{PO} \downarrow k_\alpha \text{PO} \downarrow m \\ r_\alpha(x) \xleftarrow{o_\alpha} K_\alpha \xrightarrow{i_\alpha} X \end{array} \\
 & & \text{PO} - \text{PUSHOUT}
 \end{array}$$

## ② MOTIVATION: FORMALIZATION OF REWRITING + COMBINATORICS!

► INSPIRATION FROM COMBINATORIAL SPECIES THEORY:

$$(i) \hat{X}(x^n) = x^{n+1}, \quad \frac{d}{dx}(x^n) = (n)_1 x^{n-1} = \begin{cases} 0, & n=0 \\ nx^{n-1}, & \text{else} \end{cases}$$

$$\hookrightarrow (ii) X^n = \hat{x}^n(1)$$

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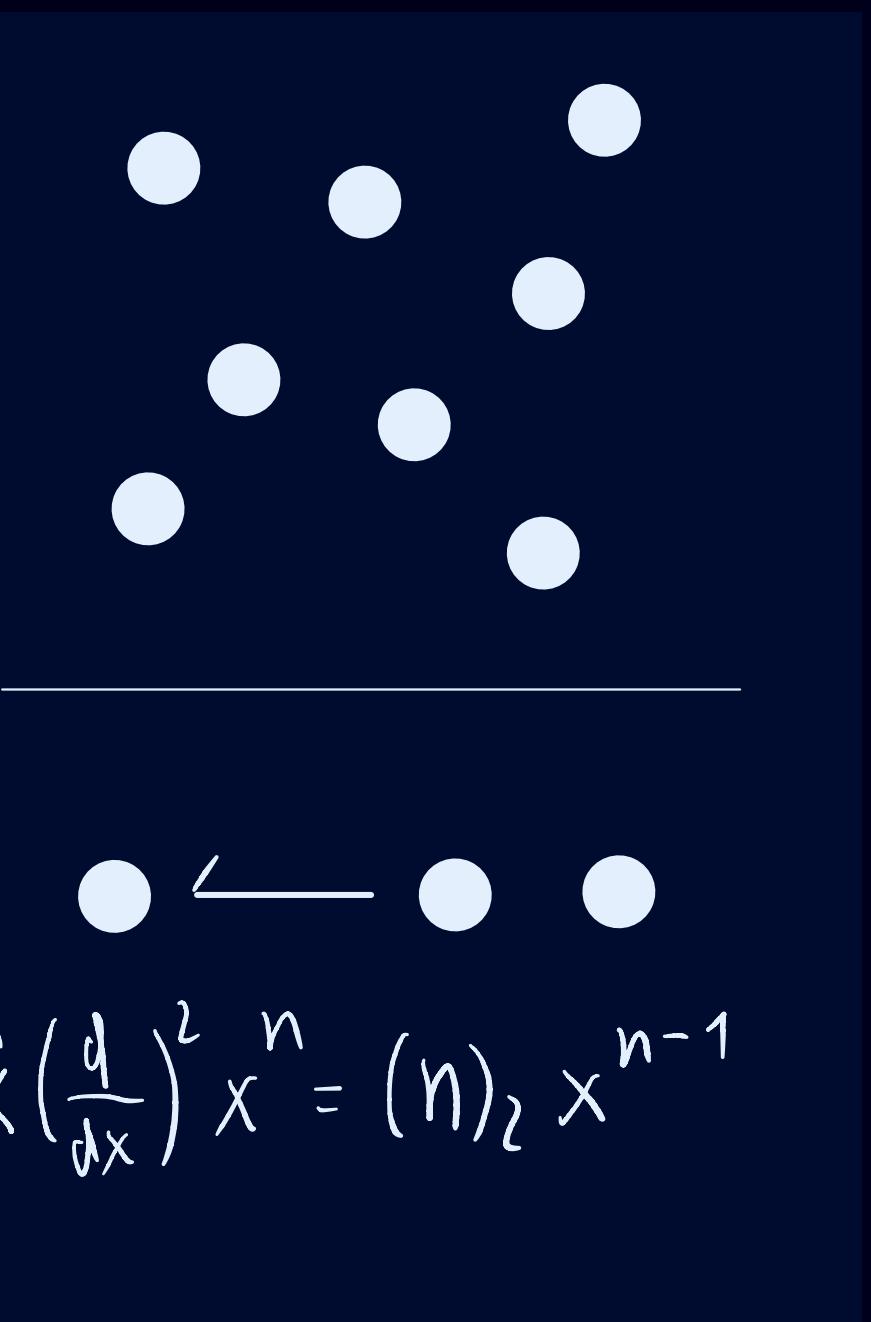
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$$(i) \hat{X}(x^n) = x^{n+1}, \quad \frac{d}{dx}(x^n) = (n)_1 x^{n-1} \equiv \begin{cases} 0, & n=0 \\ nx^{n-1}, & \text{else} \end{cases}$$

$$\hookrightarrow (ii) X^n = \hat{x}^n(1)$$

$$\hat{X}^p \left( \frac{d}{dx} \right)^q (x^n) = \overbrace{(n)_q}^{\# \text{ OF WAYS TO REMOVE } q \text{ ELEMENTS FROM A SET OF } n \text{ ELEMENTS}} x^{n-q+p}$$

# OF WAYS TO REMOVE  
q ELEMENTS FROM A  
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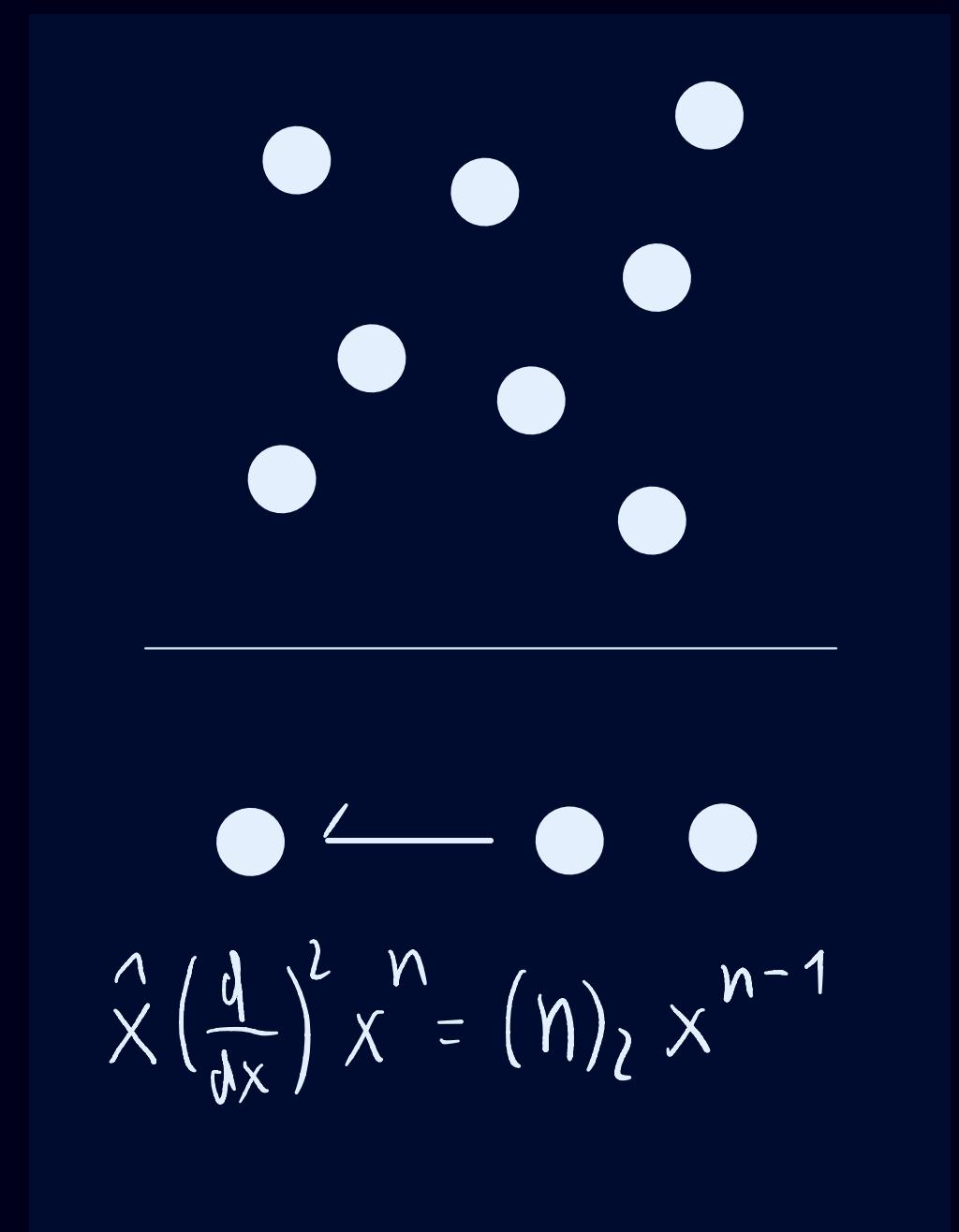
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$$(iii) \hat{X}^P \left( \frac{d}{dx} \right)^q \hat{X}^r \left( \frac{d}{dx} \right)^s = \sum_{k \geq 0} \underbrace{\binom{q}{k} k!}_{\in \mathbb{Z}_{\geq 0}} \binom{r}{k} \hat{X}^{P+r-k} \left( \frac{d}{dx} \right)^{q+s-k}$$



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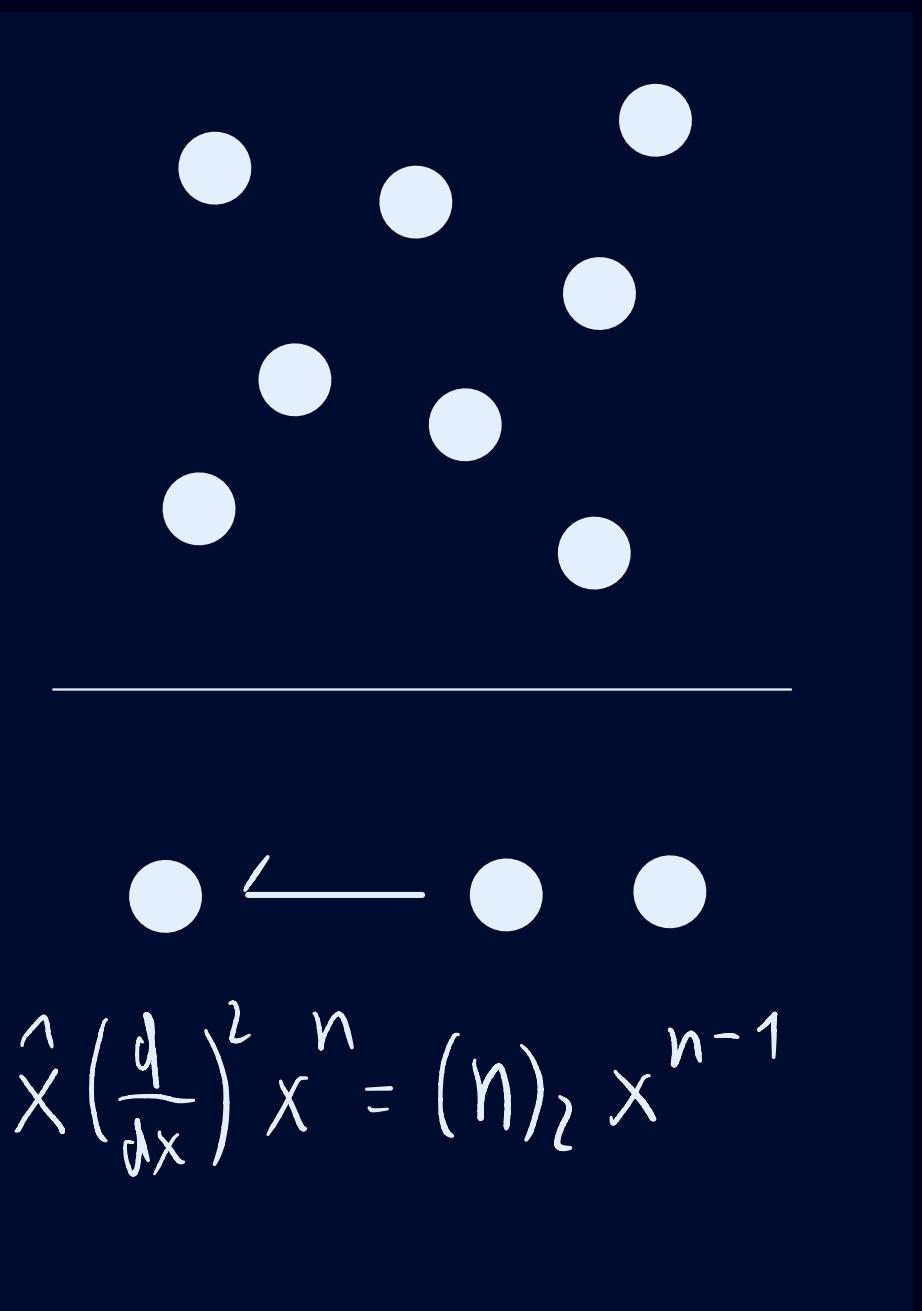
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$$\hat{X} \left( \frac{d}{dx} \right)^2 x^n = (n)_2 x^{n-1}$$

GOAL: (i) " $\mathcal{G}(\delta(r))|X\rangle := \sum_{\alpha} |\Gamma_{\alpha}(x)\rangle = \sum_{\gamma} \underbrace{m_x^{\gamma}}_{\in \mathbb{Z}_{\geq 0}} |y\rangle$ " (ii) " $|X\rangle = \mathcal{G}(\delta(X \leftarrow \emptyset))|\emptyset\rangle$ "

(iii) " $\mathcal{G}(\delta(r_2))\mathcal{G}(\delta(r_1)) = \sum_{r_u} \underbrace{m_{r_1, r_2}^{r_u}}_{\in \mathbb{Z}_{\geq 0}} \mathcal{G}(\delta(r_u))$ " (iv) " $\mathcal{G}(\delta(r_2))\mathcal{G}(\delta(r_1)) = \mathcal{G}(\delta(r_2) \odot \delta(r_1))$ "

### ③ CONCEPTUAL OBSTACLE : ESSENTIAL UNIQUENESS OF UNIVERSAL CONSTRUCTIONS

RECAP:

$$\begin{array}{ccc} O \xleftarrow{r} I & & O \xleftarrow{\text{or}} K_r \xrightarrow{i_r} I \\ n \downarrow \quad \downarrow \alpha \quad \downarrow m & := & n \downarrow \quad PO \downarrow k_\alpha \quad PO \downarrow m \\ r_\alpha(x) \leftarrow X & & r_\alpha(x) \xleftarrow{o_\alpha} K_\alpha \xrightarrow{i_\alpha} X \end{array}$$

PO — PUSHOUT

DEFINITION:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a PO : $\Leftrightarrow$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \\ & \searrow & \downarrow \\ & & X \end{array}$$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \\ & \searrow & \downarrow \\ & & X \end{array}$$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \text{PO} & \downarrow \\ C & \longrightarrow & D \end{array} \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \text{PO} & \downarrow \\ C & \longrightarrow & D' \end{array} \Rightarrow \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \\ & \searrow & \downarrow \\ & & D' \end{array}$$

EXAMPLE:  
(in FinSet)

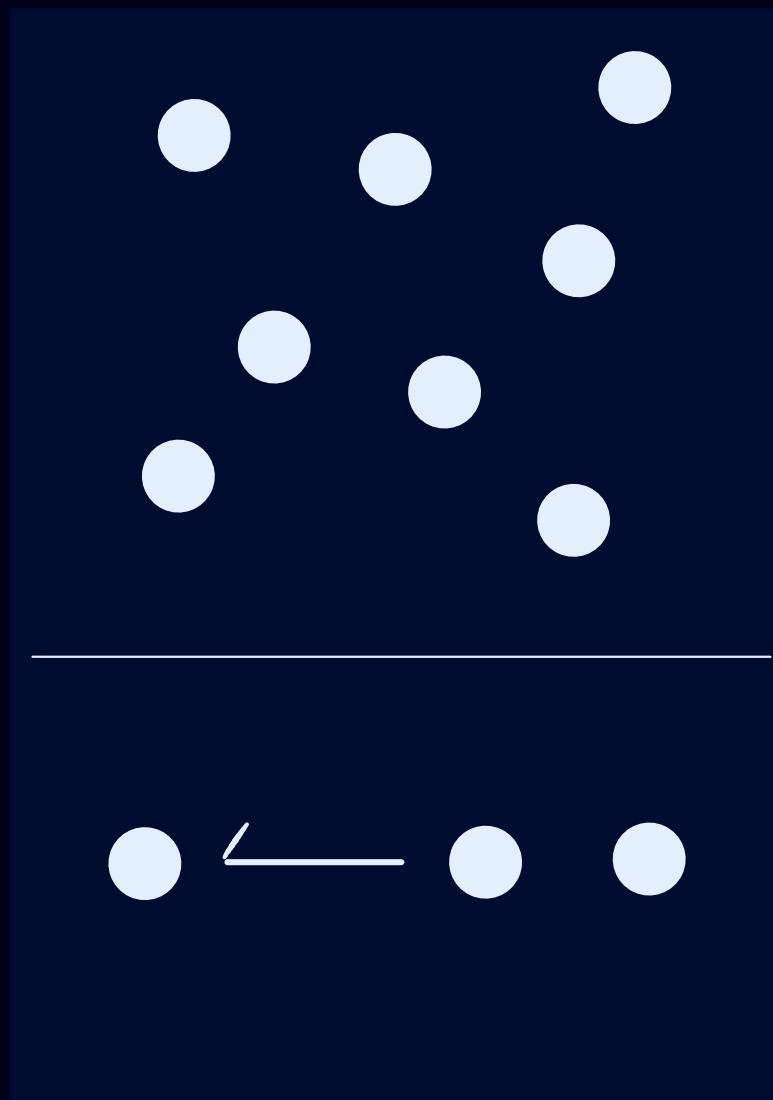
$$\begin{array}{ccc} I & \hookrightarrow & B \\ \downarrow & \text{PO} & \downarrow \\ C & \hookrightarrow & "B \cup_I C" \end{array}$$

### ③ CONCEPTUAL OBSTACLE : ESSENTIAL UNIQUENESS OF UNIVERSAL CONSTRUCTIONS

RECAP:

$$\begin{array}{ccc} O & \xleftarrow{r} & I \\ n \downarrow & \Downarrow \alpha & \downarrow m \\ r_\alpha(X) & \xleftarrow{\quad} & X \end{array} \quad ;= \quad \begin{array}{ccc} O & \xleftarrow{\text{or}} & K_r \xrightarrow{i_r} I \\ n \downarrow & & \downarrow k_\alpha \\ PO & & PO \\ \downarrow & & \downarrow m \\ r_\alpha(X) & \xleftarrow{o_\alpha} & K_\alpha \xrightarrow{i_\alpha} X \end{array} \quad \text{PO} - \text{PUSHOUT}$$

EXAMPLE:



$$\begin{array}{ccccc} 3 & \bullet & \not\rightarrow & \emptyset & \subset \rightarrow & \bullet_2 & \bullet_1 \\ & \downarrow 3 \mapsto d & & \downarrow & & \downarrow & & \downarrow 1 \mapsto a \\ d & \bullet & \not\rightarrow & & & \not\rightarrow & a & \end{array}$$

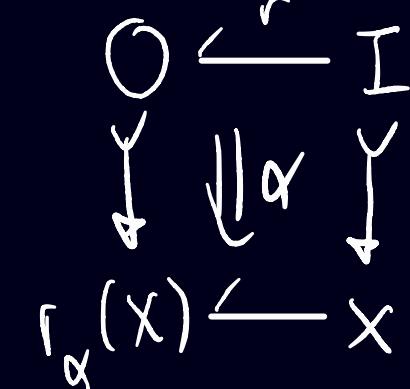
$$\begin{array}{ccccc} & & & & b \\ & \not\rightarrow & & \not\rightarrow & c \\ b & \bullet & & \bullet & c \end{array}$$

IN FinSet:

$$\begin{array}{ccc} I & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \hookrightarrow & "BU_I C" \end{array}$$

## ④ ANSATZ: CATEGORIFICATION

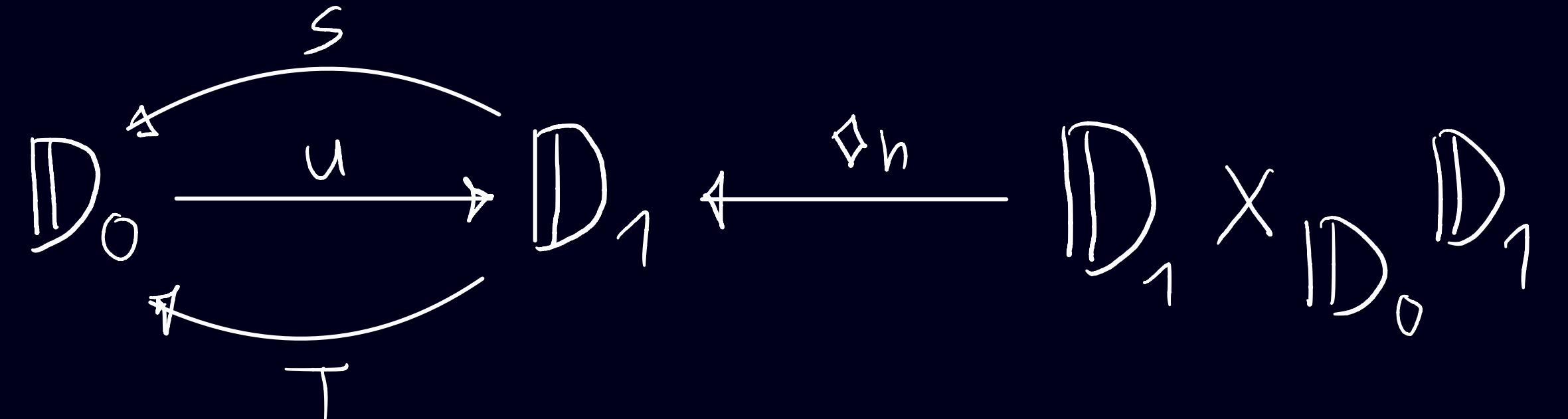
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 (iii) " $\mathcal{G}(\mathcal{S}(r_2))\mathcal{G}(\mathcal{S}(r_1)) = \sum_{r_u} \underbrace{N_{r_1, r_2}^{r_u}}_{\in \mathbb{Z}_{\geq 0}} \mathcal{G}(\mathcal{S}(r_u))$ " (iv) " $\mathcal{G}(\mathcal{S}(r_2))\mathcal{G}(\mathcal{S}(r_1)) = \mathcal{G}(\mathcal{S}(r_2) \odot \mathcal{S}(r_1))$ "  
 ○ - RULE ALGEBRA PRODUCT

I. FORMALIZE  AS Z-CELLS IN A DOUBLE CATEGORY

II. FORMALIZE  $\mathbb{Z}_{\geq 0}$ -COEFFICIENTS AS CARDINALITIES (OF SUITABLE SETS...)

METHODS: DOUBLE CATEGORIES, PRESHEAVES, FIBRATIONS, COENDS, MULTISUMS ...

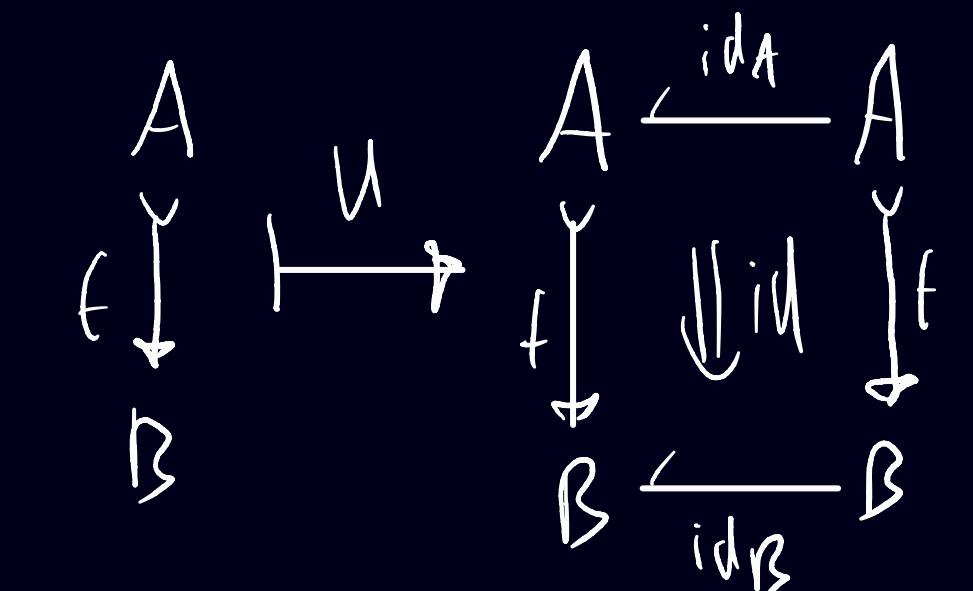
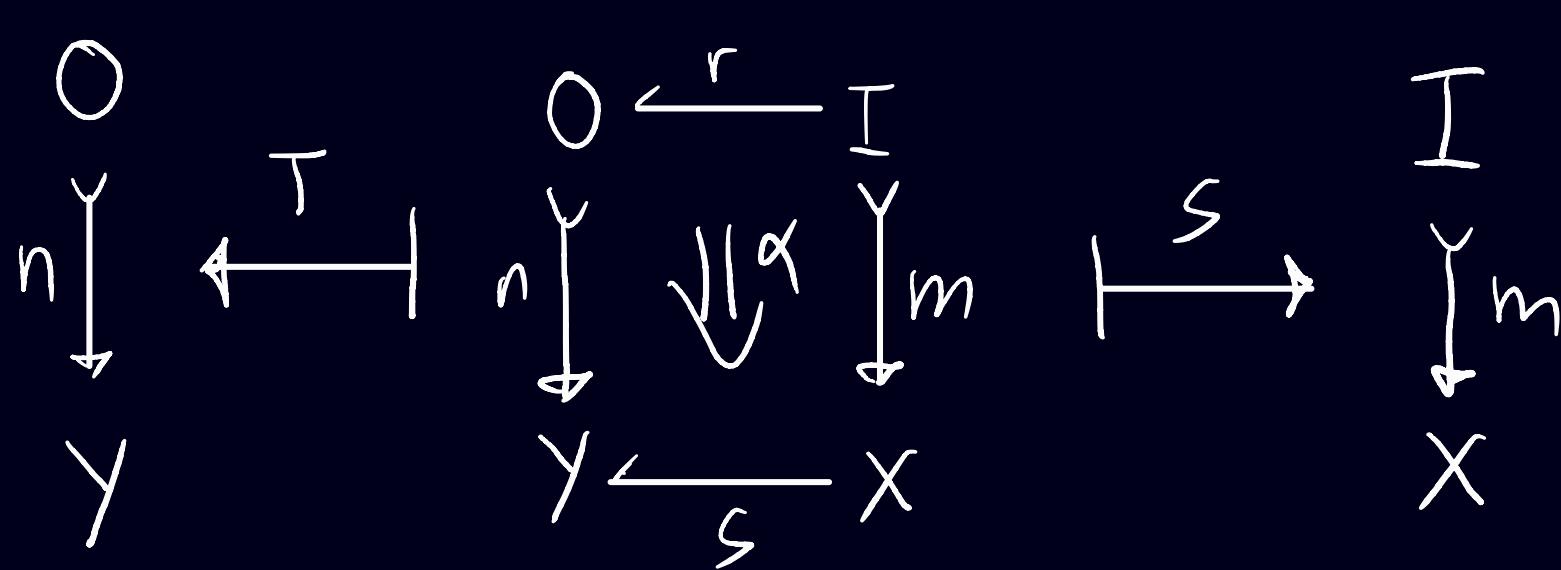
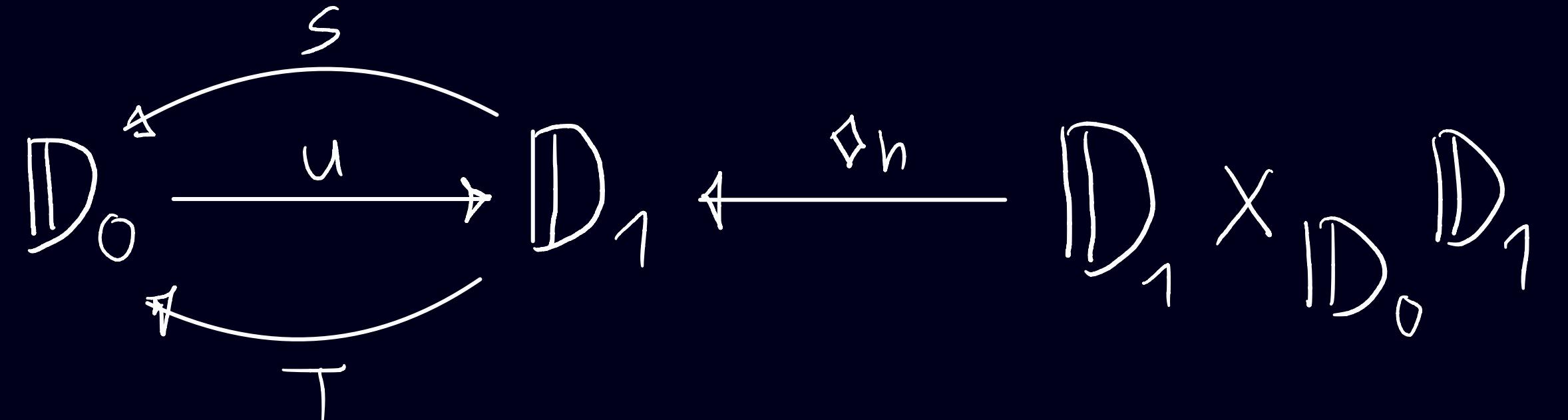
5 DEFINITION: A DOUBLE CATEGORY  $\mathbb{D}$  IS A (PSEUDO) INTERNAL CATEGORY IN CAT



$\mathbb{D}_0 :$  "0-cells" - objects of  $\mathbb{D}_0$   
 "vertical morphisms" - morphisms of  $\mathbb{D}_0$

$\mathbb{D}_1 :$  "horizontal morphisms" - objects of  $\mathbb{D}_1$   
 "2-cells" - morphisms of  $\mathbb{D}_1$

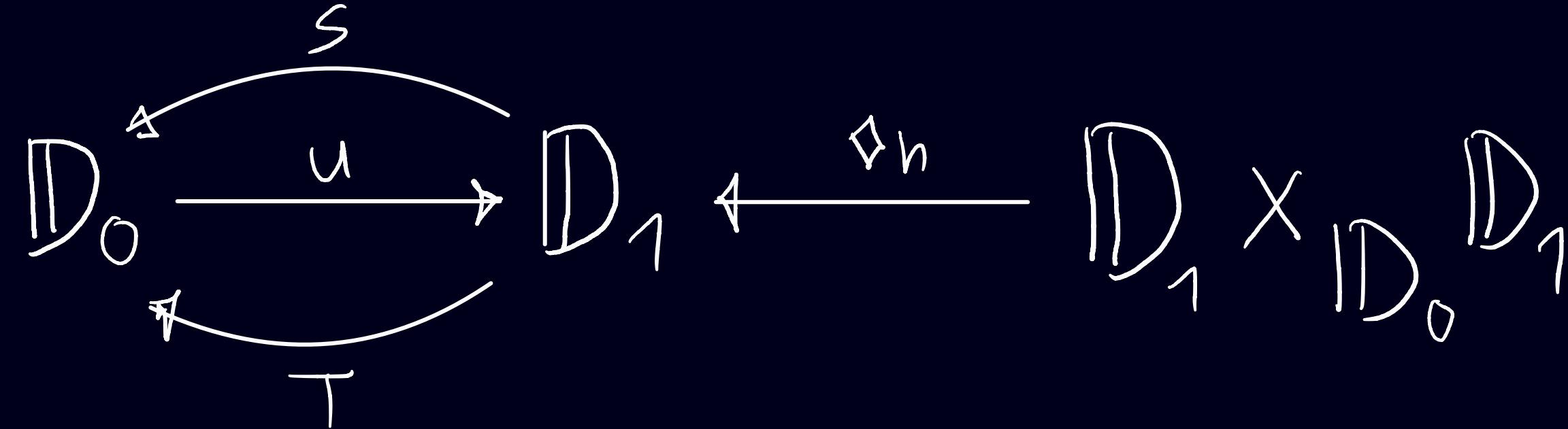
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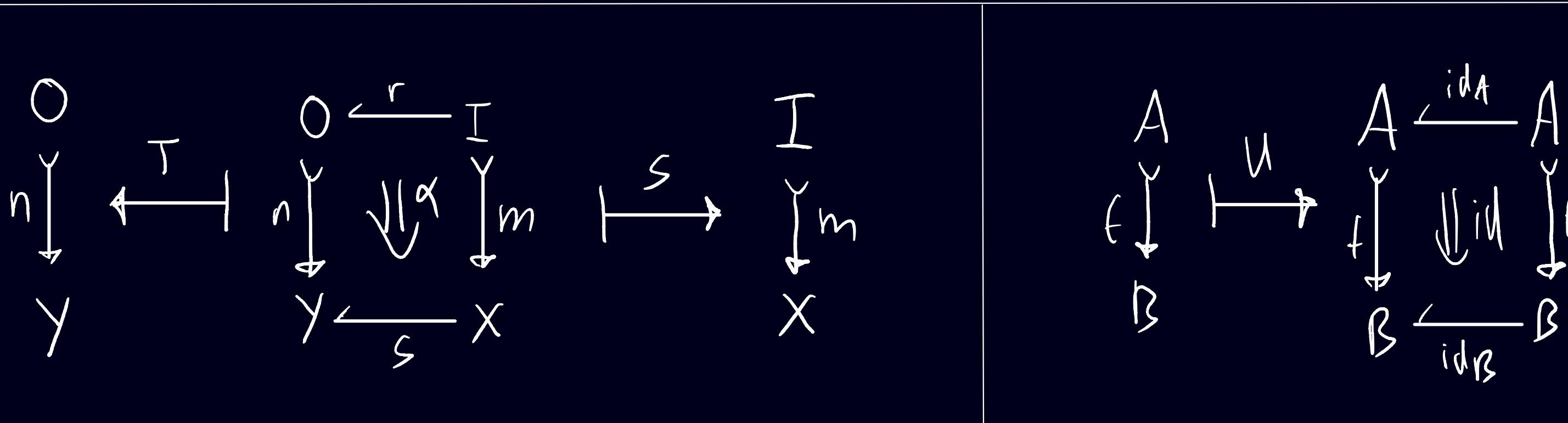
$\mathbb{D}_0:$   $\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$  "0-cells" - objects of  $\mathbb{D}_0$   
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$\mathbb{D}_1:$   $\begin{array}{c} O \xleftarrow{r} I \\ \Downarrow \alpha \\ m \end{array}$  "horizontal morphisms"  
 - objects of  $\mathbb{D}_1$   
 $\begin{array}{c} Y \xleftarrow{s} X \\ \Downarrow \beta \\ x \end{array}$  "2-cells" - morphisms of  $\mathbb{D}_1$

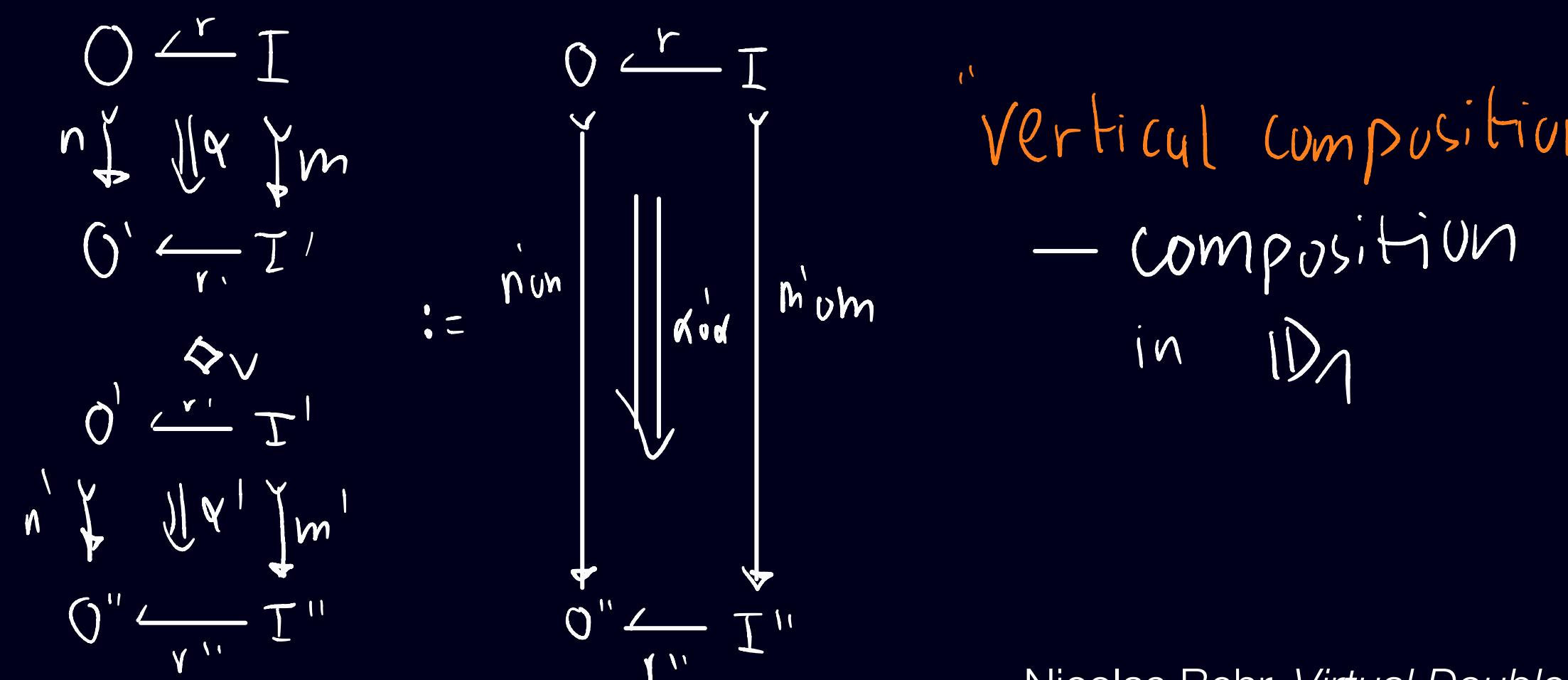
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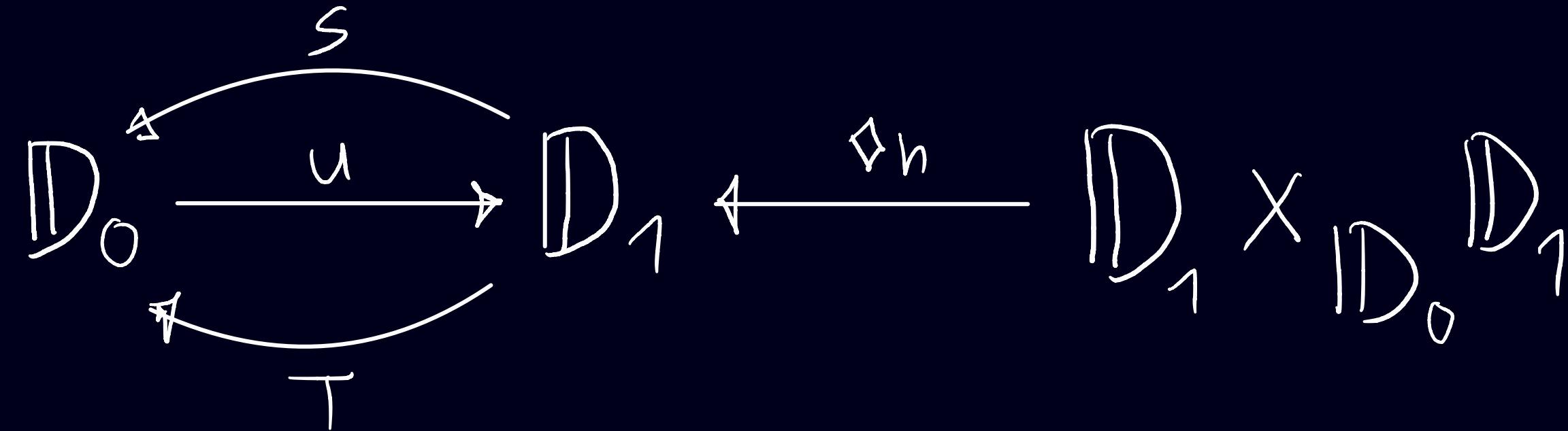


$\mathbb{D}_1$ :  
 "horizontal morphisms"  
 - objects of  $\mathbb{D}_1$   
 "2-cells" - morphisms of  $\mathbb{D}_1$

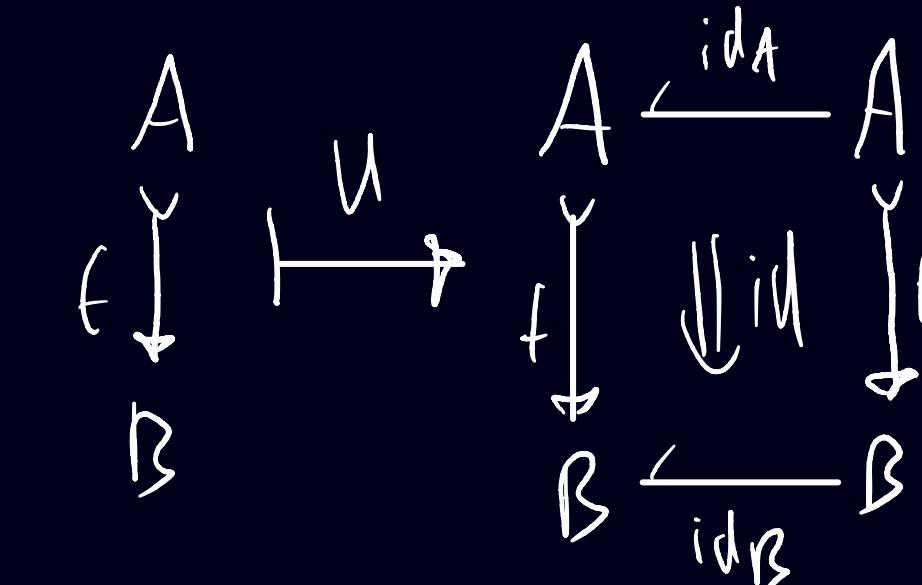
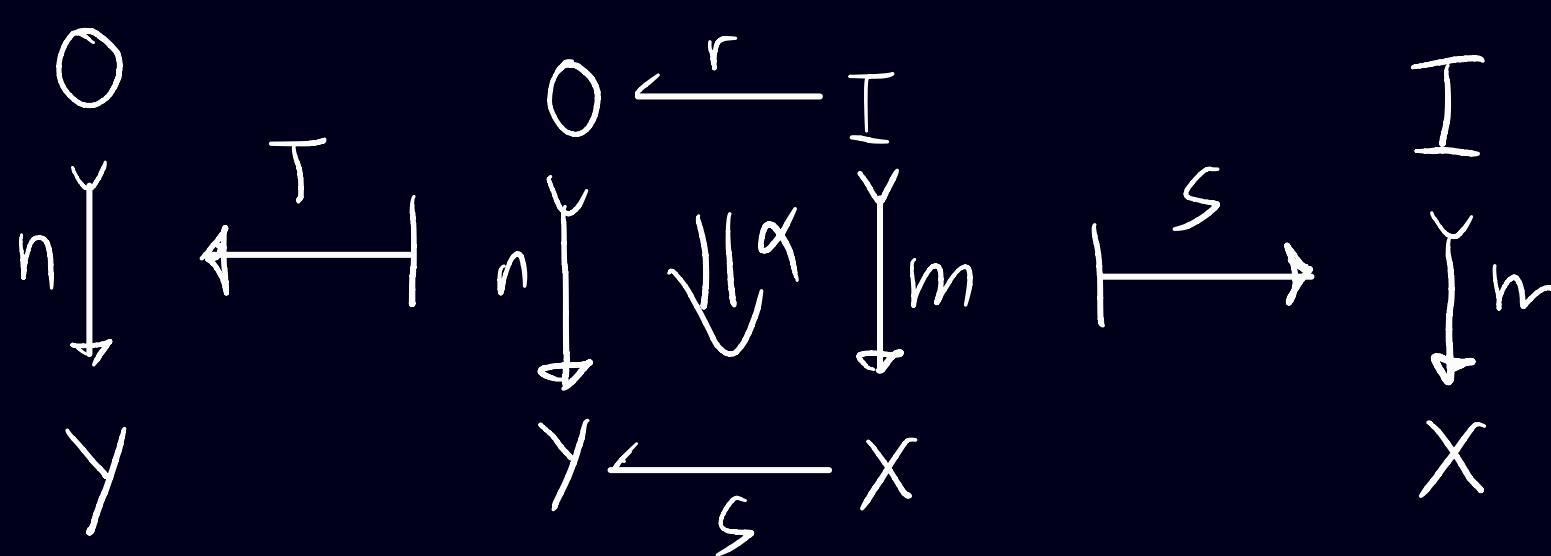


"Vertical composition"  
 - composition  
 in  $\mathbb{D}_1$

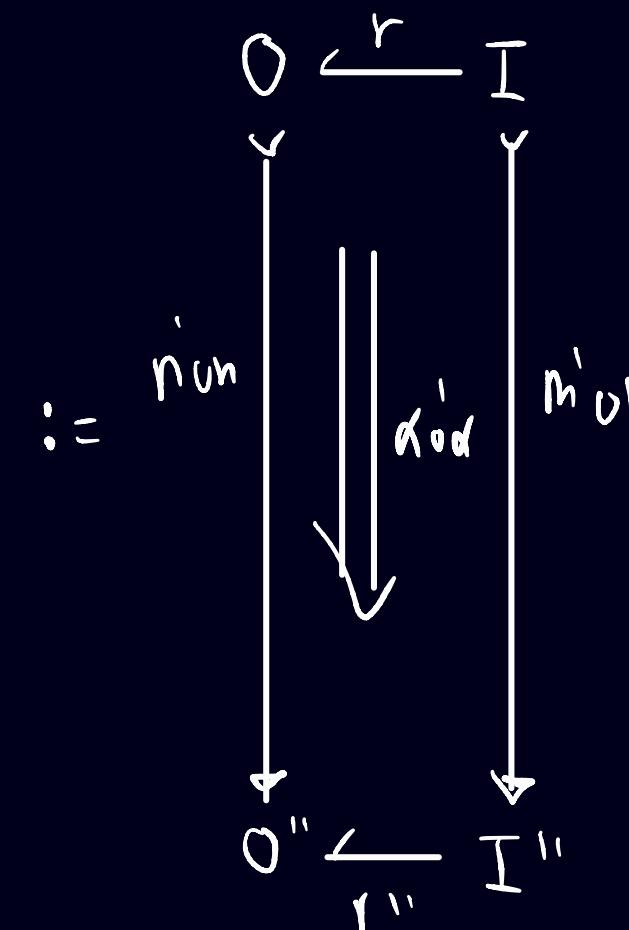
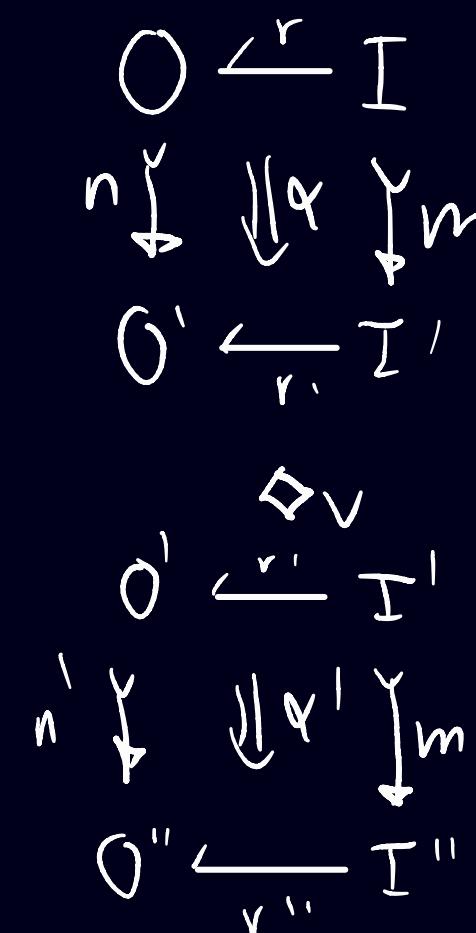
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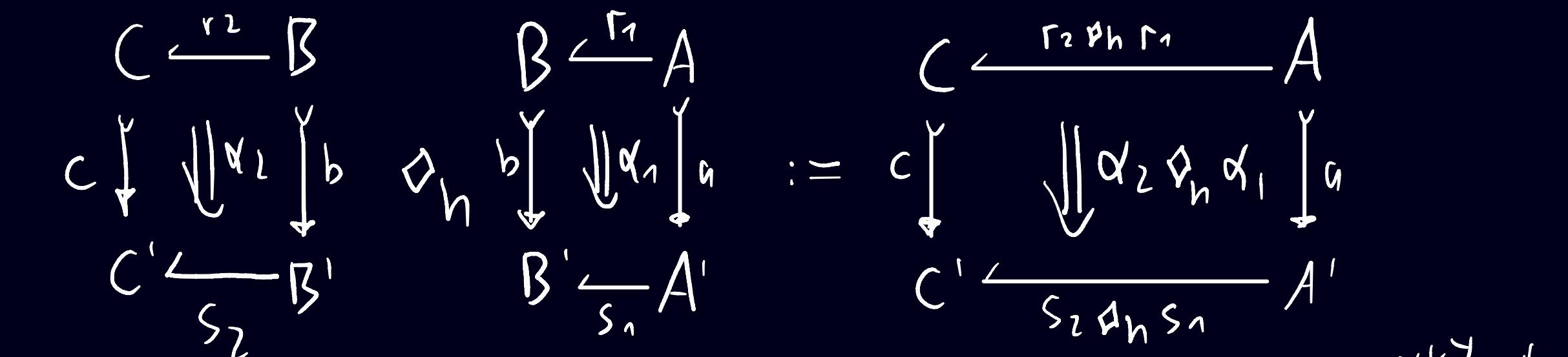
$\mathbb{D}_0$ :  $\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$  "0-cells" - objects of  $\mathbb{D}_0$   
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$\mathbb{D}_1$ :  $\begin{array}{c} O \xleftarrow{r} I \\ \Downarrow \alpha \Downarrow \\ m \end{array}$  "horizontal morphisms"  
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"Vertical composition"  
                             - composition  
                             in  $\mathbb{D}_1$



"horizontal composition"  $\rightsquigarrow$  IN GENERAL UNLAWFUL  
                             WEAKLY ASSOCIATIVE!

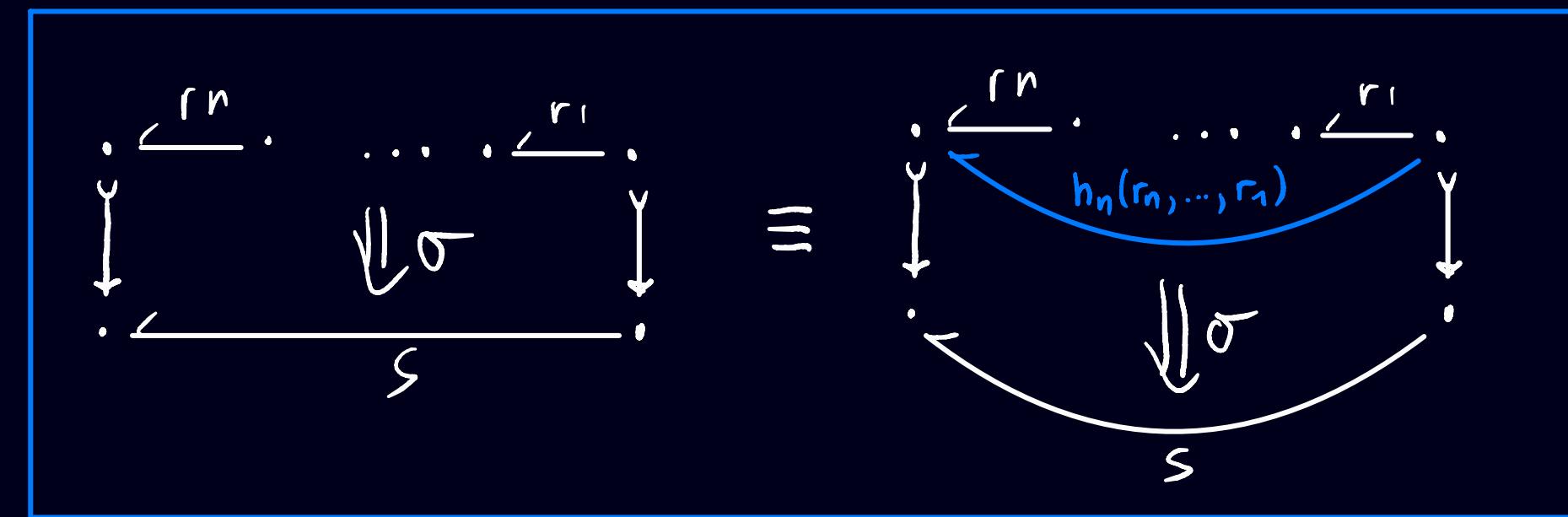
⑥ DEFINITION: A PRESENTATION OF A DOUBLE CATEGORY  $\mathbb{D}$  IS A FAMILY  $(h_n)_{n \geq 0}$

OF FUNCTORS  $h_n: \mathbb{D}_n \rightarrow \mathbb{D}_1$ , WHERE  $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \dots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$ ,

$$h_0 := U, \quad h_1 := id, \quad h_2(-_2, -_1) := -_2 \diamond_h -_1,$$

$$\forall n \geq 2: h_{n+1}(-_{n+1}, \dots, -_1) \cong h_2(-_{n+1}, h_n(-_n, \dots, -_1))$$

NOTATIONAL CONVENTION:



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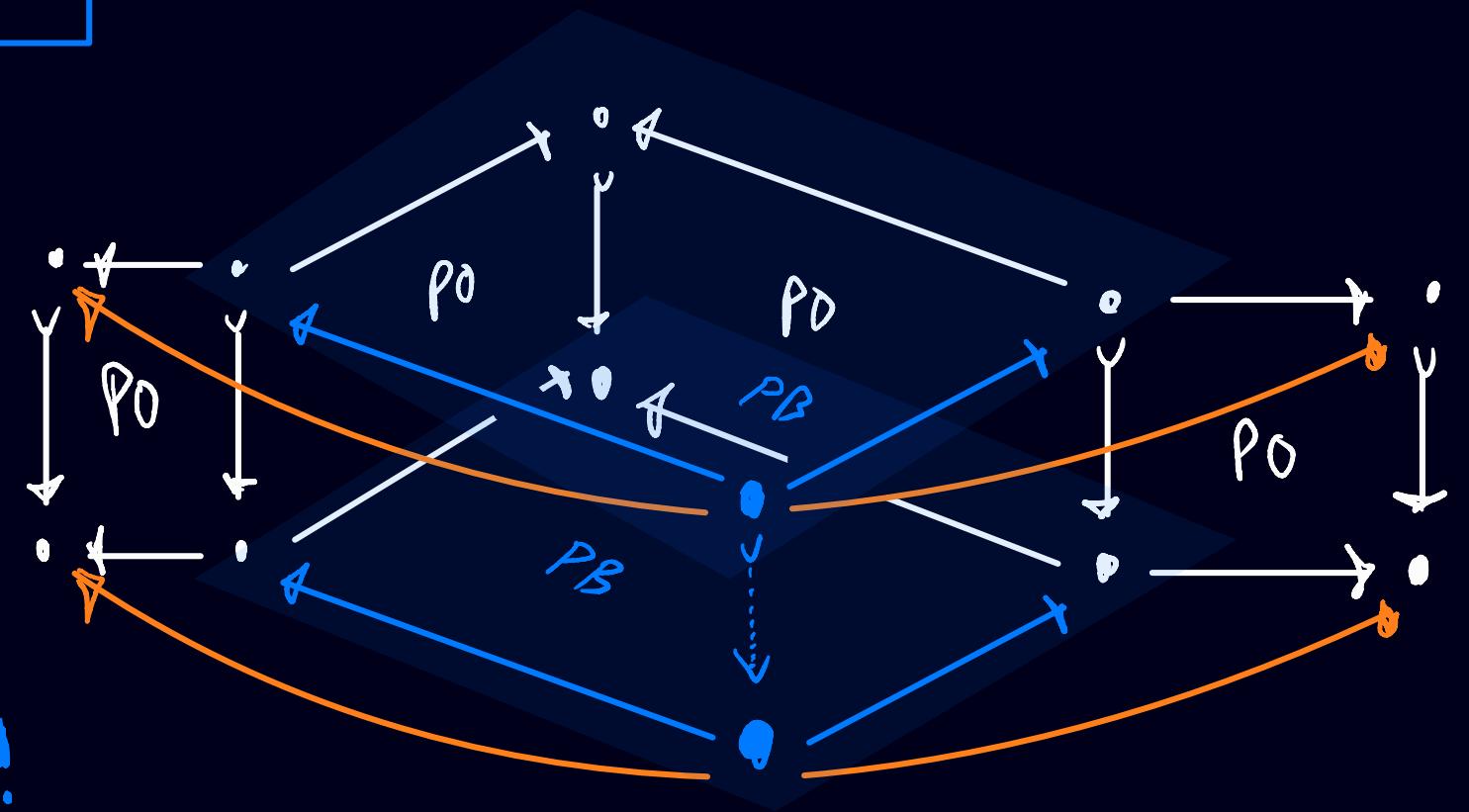
NOTATIONAL CONVENTION:

EXAMPLE:

$$\begin{array}{ccc} O & \xleftarrow{r} & I \\ n \downarrow \psi \times \downarrow m & = & n \downarrow p_O \xleftarrow{K_r} p_O \xrightarrow{i_r} I \\ Y & \xleftarrow{s} & X \end{array}$$

HORIZONTAL  
COMPOSITION:

$\cong$  CHOICE OF  
PULLBACKS ( $P_B$ )!



# 7 KEY CONCEPT: (COVARIANT) PRESHEAVES $F: \mathbb{D}_1 \rightarrow \underline{\text{Set}}$

↳ IDEA:  $\forall r \in \mathbb{D}_1 : \hat{\Delta}_r := \mathbb{D}_1(r, -)$

$$\hookrightarrow |\hat{\Delta}_r(y \xleftarrow{s} x)| = \left| \left\{ \begin{array}{c} o \xleftarrow{r} I \\ \downarrow y \quad \downarrow m \in \mathbb{D}_1 \\ y \xleftarrow{s} x \end{array} \right\} \right| \propto \text{"#ways to rewrite } x \text{ into } y \text{ along } y \xleftarrow{s} x \text{ with rule } o \xleftarrow{r} I \text{"}$$

↳ BUT: we want "  $\oint(\delta(r)) |x\rangle = \underbrace{\oint(\delta(r)) \oint(\delta(x \xleftarrow{\emptyset}))}_{?} |\emptyset\rangle = \sum_{\alpha} |\Gamma_{\alpha}(x)\rangle = \sum_y \underbrace{M_{r,y}}_{\in \mathbb{Z}_{\geq 0}} |y\rangle$ "

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$$\Rightarrow \text{BUT: we want " } g(\delta(r)) |x\rangle = \underbrace{g(\delta(r)) g(\delta(x \xleftarrow{\emptyset} \emptyset))}_{?} |\emptyset\rangle = \sum_{\alpha} |\Gamma_{\alpha}(x)\rangle = \sum_y \underbrace{M_{r,y}}_{?} |y\rangle \text{ "}$$

↳ ASSUMPTION:  $\mathbb{D}_0$  HAS A STRICT INITIAL OBJECT  $\emptyset$  (i.e.,  $\forall x \in \mathbb{D}_0: \exists! \emptyset \rightarrow x \wedge \forall x \xrightarrow{f} \emptyset: X = \emptyset$ ),

AND SUCH THAT (i)  $\forall x \in \mathbb{D}_0: \exists! (x \xleftarrow{\emptyset}) \in \text{ob}(\mathbb{D}_1) \wedge \exists! (\emptyset \xleftarrow{x}) \in \text{ob}(\mathbb{D}_1)$

$$(ii) \forall \begin{array}{c} x \\ \downarrow y \end{array} \in \mathbb{D}_0: \left| \left\{ \begin{array}{c} x \xleftarrow{\emptyset} \\ \downarrow y \end{array} \in \mathbb{D}_1 \right\} \right| \leq 1 \wedge \left| \left\{ \begin{array}{c} \emptyset \xleftarrow{x} \\ \downarrow y \end{array} \in \mathbb{D}_1 \right\} \right| \leq 1$$

⑧ DEFINITION: A COEND FOR A FUNCTOR  $F: \mathcal{C}^{\text{OP}} \times \mathcal{C} \rightarrow \underline{\text{Set}}$

IS DEFINED AS  $\int^{C \in \mathcal{C}} F(C, C) = \left( \coprod_{C \in \mathcal{C}} F(C, C) \right) / \sim$

$$F(C, C) \quad F(C', C')$$

$$\Downarrow \quad \Downarrow$$

with:  $((C, x) \sim (C', x')) \Leftrightarrow \exists \gamma: C \rightarrow C' \text{, } y \in F(C', C) : x = F(\gamma, \text{id})y \wedge x' = F(\text{id}, \gamma)y$

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KEY CONCEPT: CONVOLUTION PRODUCTS OF PRESHEAVES  $F_n, \dots, F_1: \mathbb{D}_n \rightarrow \underline{\text{Set}}$

$$\begin{aligned} (F_n * \dots * F_1) := \Gamma &\mapsto \int^{S=(S_n, \dots, S_1) \in \mathbb{D}_n} \mathbb{D}_1(h_n(S), \Gamma) \times F_n(S) \\ (\simeq \text{Lan}_{h_n}(F_n)) \\ &= \left\{ (S, (\sigma, f)) \mid \begin{array}{l} S \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(S), \Gamma) \\ f \in F_n(S) \end{array} \right\} / \sim \\ &\quad \cdot \mathbb{D}_1(h_n(-), \Gamma): \mathbb{D}_n^{\text{OP}} \rightarrow \underline{\text{Set}} \\ &\quad \cdot F_n := F_n \times \dots \times F_1: \mathbb{D}_n \rightarrow \underline{\text{Set}} \\ &\quad \equiv \left\{ \begin{array}{c} f_n \\ \vdots \\ s_n \end{array} \dots \begin{array}{c} f_1 \\ \vdots \\ s_1 \end{array} \end{array} \right\} / \sim \end{aligned}$$

g

$$(F_n * \dots * F_1)(r) = \left\{ (\zeta, (\sigma, f)) \mid \begin{array}{l} \zeta \in D_n \\ \sigma \in D_1(h_n(\zeta), r) \\ f \in F_n(\zeta) \end{array} \right\} / \sim \equiv \left\{ \begin{array}{c} \text{Diagram showing } \zeta \text{ as a sequence of points } s_n, \dots, s_1 \\ \text{with arrows } f_n, \dots, f_1 \text{ pointing left} \\ \text{and a double arrow } \Downarrow \sigma \text{ below} \end{array} \right\} / \sim$$

$\cdot (\zeta, (\sigma, f)) \sim (\zeta', (\sigma', f')) \Leftrightarrow \exists \zeta \xrightarrow{A} \zeta' \in D_n, (\tau, g) \in D_1(h_n(\zeta'), r) \times F_n(\zeta) :$

$$(\sigma, f) = (D_1(h_n(A), r) \tau, g) \wedge (\sigma', f') = (\tau, F_n(A) g)$$

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$$(F_n * \dots * F_1)(\Gamma) = \left\{ (\mathcal{S}, (\sigma, f)) \mid \begin{array}{l} \mathcal{S} \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(\mathcal{S}), \Gamma) \\ f \in F_n(\mathcal{S}) \end{array} \right\} / \sim \equiv \left\{ \begin{array}{c} \text{Diagram showing } \mathcal{S} \text{ as a } n\text{-ary tree with } f_n \text{ at root } \mathcal{S} \\ \text{with children } s_n, \dots, s_1 \\ \text{and } \sigma \text{ below it} \end{array} \right\} / \sim$$

$\cdot (\mathcal{S}, (\sigma, f)) \sim (\mathcal{S}', (\sigma', f')) \Leftrightarrow \exists S \xrightarrow{A} S' \in \mathbb{D}_n, (\tau, g) \in \mathbb{D}_1(h_n(S'), \Gamma) \times F_n(S) :$

$$(\sigma, f) = (\mathbb{D}_1(h_n(A), \Gamma) \tau, g) \wedge (\sigma', f') = (\tau, F_n(A) g)$$

$\Leftarrow \forall S \xrightarrow{A} S' \in \mathbb{D}_n, h_n(S') \xrightarrow{\tau} \Gamma \in \mathbb{D}_1$  :

EXAMPLE:  $\hat{\Delta}_{r_j} := \mathbb{D}_1(r_j, -) : \mathbb{D}_1 \rightarrow \underline{\text{Set}} \quad (j=1, \dots, n)$

$$\Leftarrow (\hat{\Delta}_{r_n} * \dots * \hat{\Delta}_{r_1})(\Gamma) = \left\{ \begin{array}{c} \text{Diagram showing } \mathcal{S} \text{ as a } n\text{-ary tree with } r_n \text{ at root } \mathcal{S} \\ \text{with children } \downarrow \varphi_n, \dots, \downarrow \varphi_1 \\ \text{and } \sigma \text{ below it} \end{array} \right\} / \sim$$

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# KEY CONCEPT: FIBRATIONAL STRUCTURES

• DEFINITION: A FUNCTOR  $G: \mathcal{E} \rightarrow \mathcal{B}$  IS A GROTHENDIECK OPFIBRATION IFF

$$\forall G \begin{array}{c} e \\ \downarrow \\ b \xrightarrow{f} b' \end{array} : \boxed{\begin{array}{c} e \xrightarrow{\gamma(t)} e' \\ \downarrow G \\ b \xrightarrow{f} b' \end{array}} \quad \forall G \begin{array}{c} e \xrightarrow{\gamma(t)} e' \\ \downarrow G \\ b \xrightarrow{f} b' \end{array} : \quad \forall G \begin{array}{c} e \xrightarrow{\gamma(t)} e' \\ \downarrow G \\ b \xrightarrow{f} b' \xrightarrow{g} b'' \end{array}$$

$$\begin{array}{c} \mathcal{E} \\ \nearrow \gamma(t) \quad \searrow \\ e & \xrightarrow{\gamma(t)} & e' \\ \downarrow G & & \downarrow G \\ b & \xrightarrow{f} & b' \\ & & \searrow g \\ & & b'' \end{array}$$

$G(\varepsilon) = g \circ f$

$$\begin{array}{c} \mathcal{E} \\ \nearrow \gamma(t) \quad \searrow \\ e & \xrightarrow{\gamma(t)} & e' \\ \downarrow G & & \downarrow G \\ b & \xrightarrow{f} & b' \\ & & \xrightarrow{g} b'' \\ & & \searrow g = G(\beta) \\ & & b'' \end{array}$$

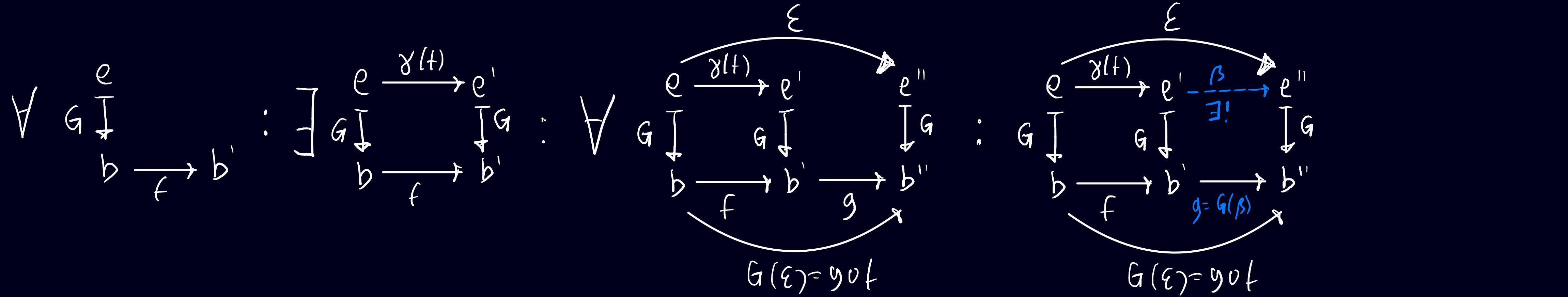
$\exists! \beta : e' \dashrightarrow e''$

10

# KEY CONCEPT: FIBRATIONAL STRUCTURES

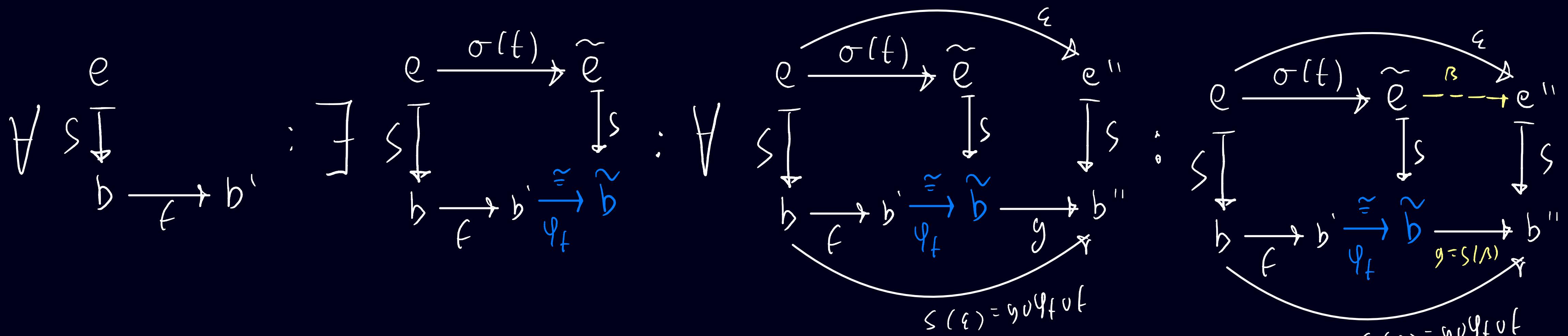
DEFINITION: A FUNCTOR  $G: \mathcal{E} \rightarrow \mathcal{B}$  IS A GROTHENDIECK OPFIBRATION IFF

$$\forall G \downarrow \begin{array}{c} e \\ \downarrow \\ b \xrightarrow{f} b' \end{array} : \exists G \downarrow \begin{array}{c} e \xrightarrow{\gamma(t)} e' \\ \downarrow \\ b \xrightarrow{f} b' \end{array}$$



DEFINITION: A FUNCTOR  $S: \mathcal{E} \rightarrow \mathcal{B}$  IS A STREET OPFIBRATION IFF

$$\forall S \downarrow \begin{array}{c} e \\ \downarrow \\ b \xrightarrow{f} b' \end{array} : \exists S \downarrow \begin{array}{c} \tilde{e} \xrightarrow{\sigma(t)} \tilde{e}' \\ \downarrow \\ b \xrightarrow{f} b' \xrightarrow{\cong} \tilde{b} \end{array}$$



11 DEFINITION: A FUNCTOR  $M: \mathcal{E} \rightarrow \mathcal{B}$  IS A MULTI-OPIFIBRATION IFF

$$\forall M \begin{array}{c} e \\ \downarrow \\ b \end{array} \xrightarrow{f} \begin{array}{c} b' \\ \downarrow \\ b'' \end{array} : \exists \left\{ \begin{array}{c} e \xrightarrow{\mu_i(f)} e_j \\ \downarrow \\ b \xrightarrow{f} b' \end{array} \right\}_{j \in J_{e,f}}$$

$$\forall M \begin{array}{c} e \\ \downarrow \\ b \end{array} \xrightarrow{f} \begin{array}{c} b' \\ \xrightarrow{g} \\ b'' \end{array} : \exists j \in J_{e,f} : M \begin{array}{c} e \\ \xrightarrow{\epsilon} \\ e'' \end{array} \quad M(\epsilon) = g \circ f$$

$$\begin{array}{c} e \\ \downarrow \\ b \end{array} \xrightarrow{f} \begin{array}{c} b' \\ \xrightarrow{g=M(\beta_j)} \\ b'' \end{array} : \begin{array}{c} e \xrightarrow{\mu_j(f)} e_j \\ \downarrow \\ b \xrightarrow{f} b' \end{array} \xrightarrow{\beta_j} \begin{array}{c} e \\ \downarrow \\ e'' \end{array} \quad M(\epsilon) = g \circ f$$

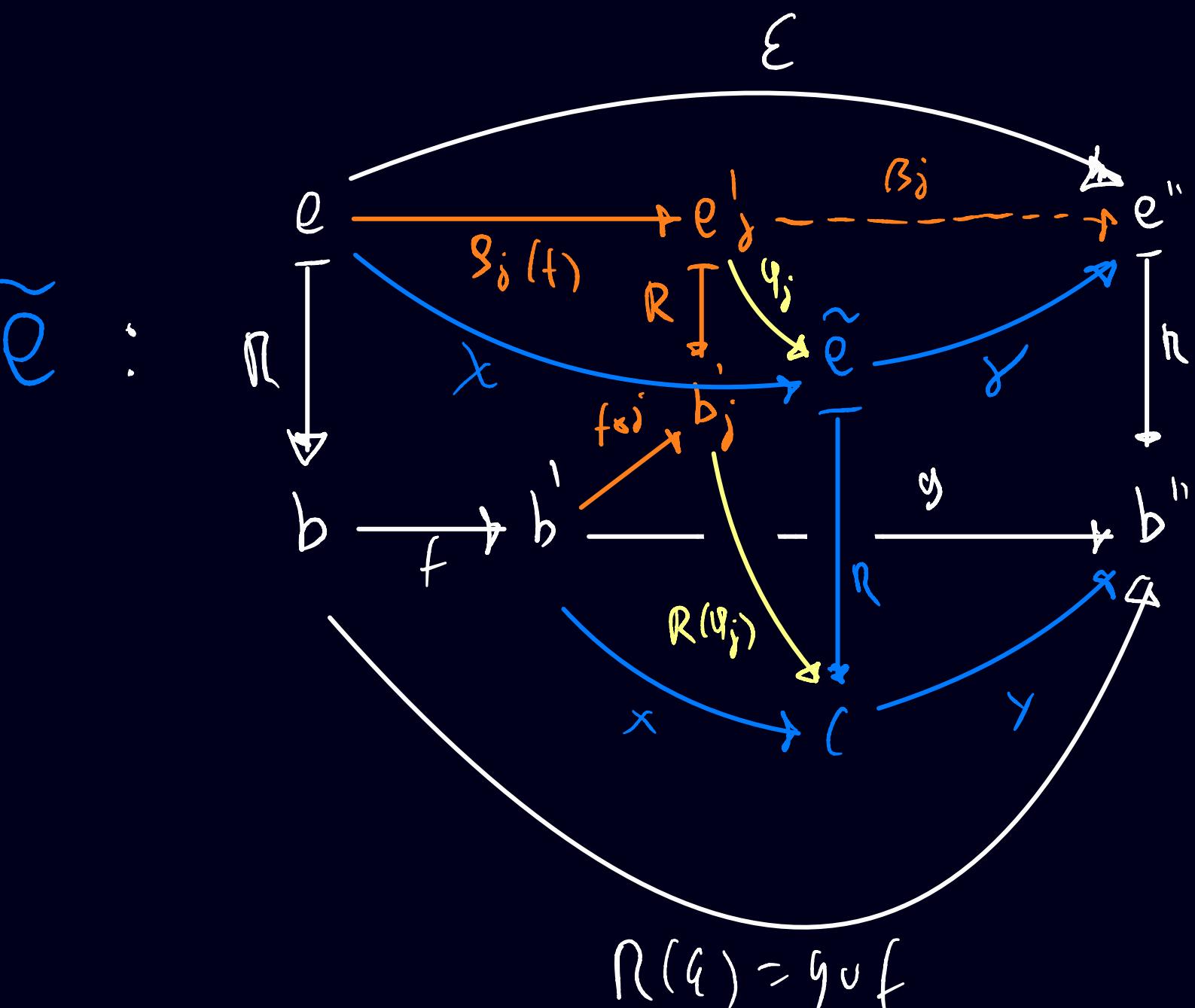
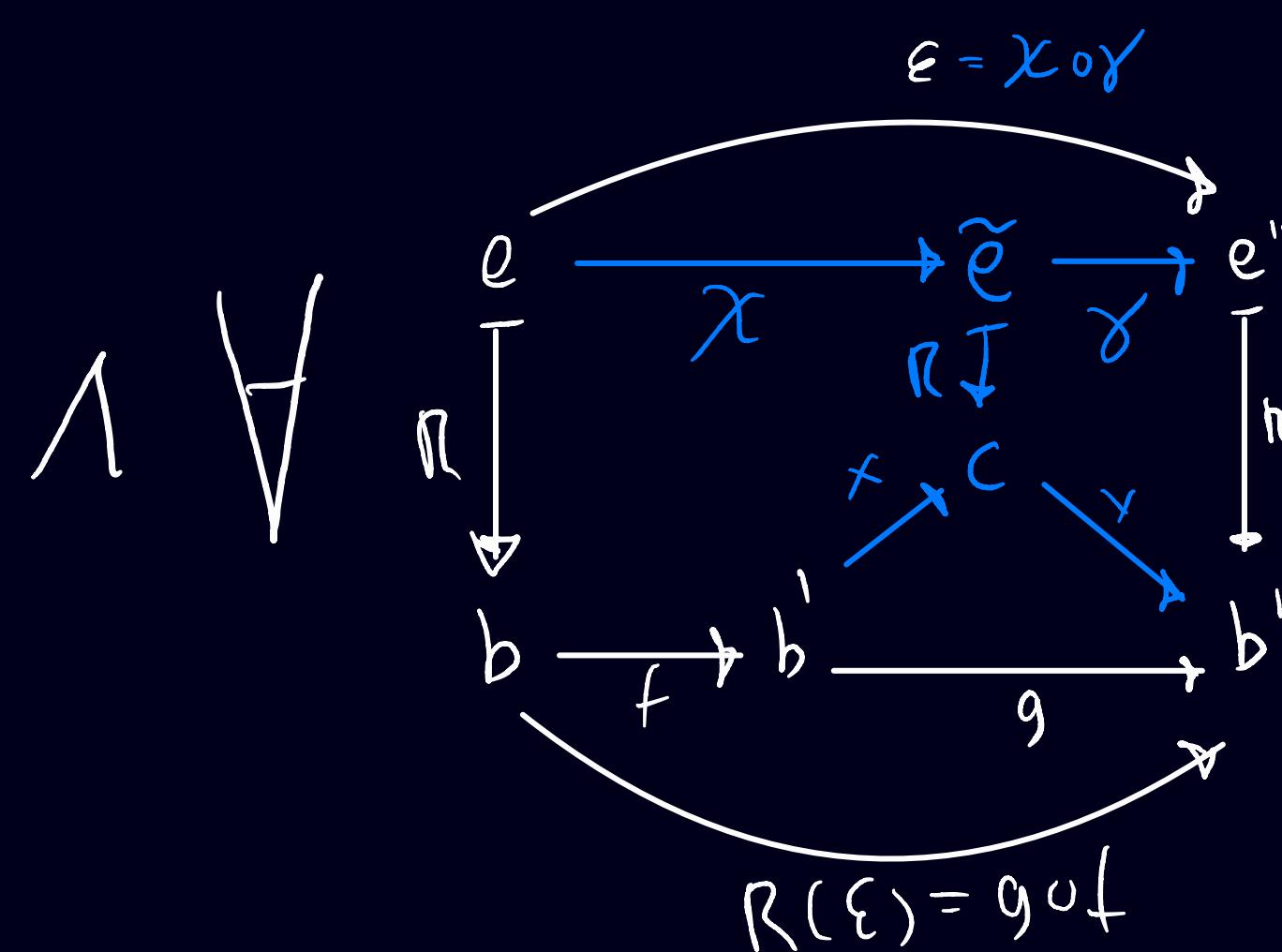
$$\wedge \forall M \begin{array}{c} e \\ \downarrow \\ b \end{array} \xrightarrow{f} \begin{array}{c} b' \\ \xrightarrow{g} \\ b'' \end{array} : M(\epsilon) = g \circ f$$

$$\begin{array}{c} e \\ \downarrow \\ b \end{array} \xrightarrow{f} \begin{array}{c} b' \\ \xrightarrow{g} \\ b'' \end{array} : \begin{array}{c} e \xrightarrow{\mu_j(f)} e_j \\ \downarrow \\ b \xrightarrow{f} b' \end{array} \xrightarrow{\beta_j} \begin{array}{c} e \\ \downarrow \\ e'' \end{array} \quad M(\epsilon) = g \circ f$$

12 DEFINITION: A FUNCTOR  $R: \mathcal{E} \rightarrow \mathcal{B}$  IS A RESIDUAL MULTI-OPIFIBRATION IFF

The diagram illustrates the definition of a residual multi-opfibration. It features several commutative squares and curved arrows:

- Left Square:**  $e \xrightarrow{R} b$  (vertical) and  $b \xrightarrow{f} b'$  (horizontal).
- Right Square:**  $e'' \xrightarrow{R} b''$  (vertical) and  $b' \xrightarrow{g} b''$  (horizontal).
- Top Square:**  $e \xrightarrow{\epsilon} e''$  (vertical) and  $b \xrightarrow{f_*} b'$  (horizontal).
- Bottom Square:**  $b \xrightarrow{f} b'$  (vertical) and  $b' \xrightarrow{g} b''$  (horizontal).
- Curved Arrows:**
  - $e \xrightarrow{\epsilon} e''$
  - $b \xrightarrow{f_*} b'$
  - $b' \xrightarrow{g} b''$
  - $b \xrightarrow{f} b \xrightarrow{f_*} b' \xrightarrow{g} b''$  (a curved arrow from  $b$  to  $b''$  passing through  $b'$ )
- Residues:**
  - A box labeled "residue" contains the diagram  $b \xrightarrow{f} b' \xrightarrow{b_j} b_j$  with  $f \star_j$  written below it.
  - A bracket labeled  $\exists j \in \{e\}_f$  points to the residue diagram.
  - A bracket labeled  $\exists j \in \{e\}_f$  points to the bottom square  $b \xrightarrow{f} b' \xrightarrow{g} b''$ .
- Equations:**
  - $R(\epsilon) = g \circ f$
  - $R(f_*) = g \circ f$



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DEFINITION: LET  $X: \mathcal{E} \rightarrow \mathcal{B}$  BE AN  $X$ -OPFIBRATION ( $X \in \{G, S, M, R\}$ ) .

THEN A CLEAVAGE FOR  $X$  IS DEFINED AS A CHOICE OF REPRESENTATIVE FOR EACH  $X$ -OPCARTESIAN LIFTING :

$$G^*(e \downarrow_{b \xrightarrow{f} b'}) := \begin{array}{c} e \xrightarrow{G(f)} b \\ \downarrow \quad \downarrow G \\ b \xrightarrow{f} b' \end{array}$$

$$S^*(s \downarrow_{b \xrightarrow{f} b'}) := \begin{array}{c} s \xrightarrow{\sigma^*(f)} \tilde{e} \\ \downarrow \quad \downarrow S \\ b \xrightarrow{f} b' \xrightarrow{\tilde{f}} \tilde{b} \end{array}$$

$$M^*(e \downarrow_{b \xrightarrow{f} b'}) := \left\{ \begin{array}{c} e \xrightarrow{M_i(f)} e'_i \\ \downarrow \quad \downarrow M \\ b \xrightarrow{f} b' \end{array} \right\}_{i \in \mathcal{I}_{e; f}}$$

$$R^*(e \downarrow_{b \xrightarrow{f} b'}) := \left\{ \begin{array}{c} e \xrightarrow{R_j(f)} e'_j \\ \downarrow \quad \downarrow R \\ b \xrightarrow{f} b' \xrightarrow{f \times_j} b'_j \end{array} \right\}_{j \in \mathcal{J}_{e; f}}$$

ONE REPRESENTATIVE  
PER EQUIVALENCE CLASS IN  $\mathcal{J}_{e; f}$  !

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# EMPIRICAL RESULT: $\mathbb{D}$ FOR COMPOSITIONAL REWRITING SEMANTICS

\* 2204.07.17S

- $h_2 = \Phi_n : \mathbb{D}_2 \rightarrow \mathbb{D}_1$  IS A "GLOBULAR" STREET OPFIBRATION, i.e.,

$$\forall R = (r_2, r_1) \text{ s.t. } h_2 : \begin{array}{c} R \xrightarrow{A} T \\ \downarrow h_2 \\ r \xrightarrow{\alpha} s \xrightarrow{\varphi_\alpha} t \end{array} : \quad \begin{array}{c} \exists h_2 : S(\varphi_\alpha) = \text{id}_{S(s)} \wedge T(\varphi_\alpha) = \text{id}_{T(t)} \\ \text{(STREET OPFIBRATION / CONDITIONS /)} \end{array}$$

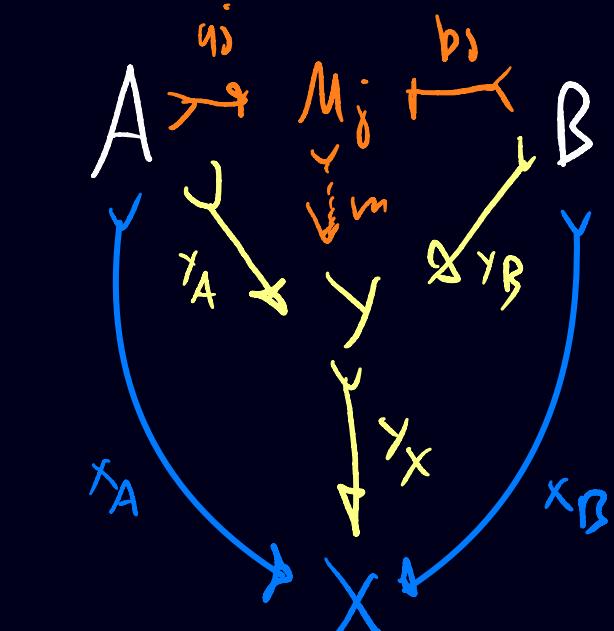
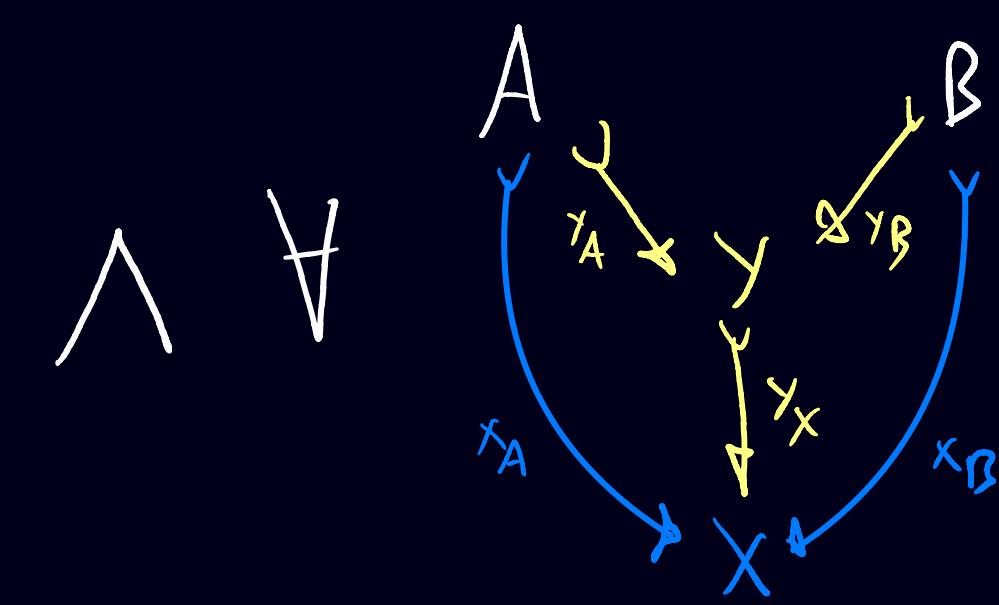
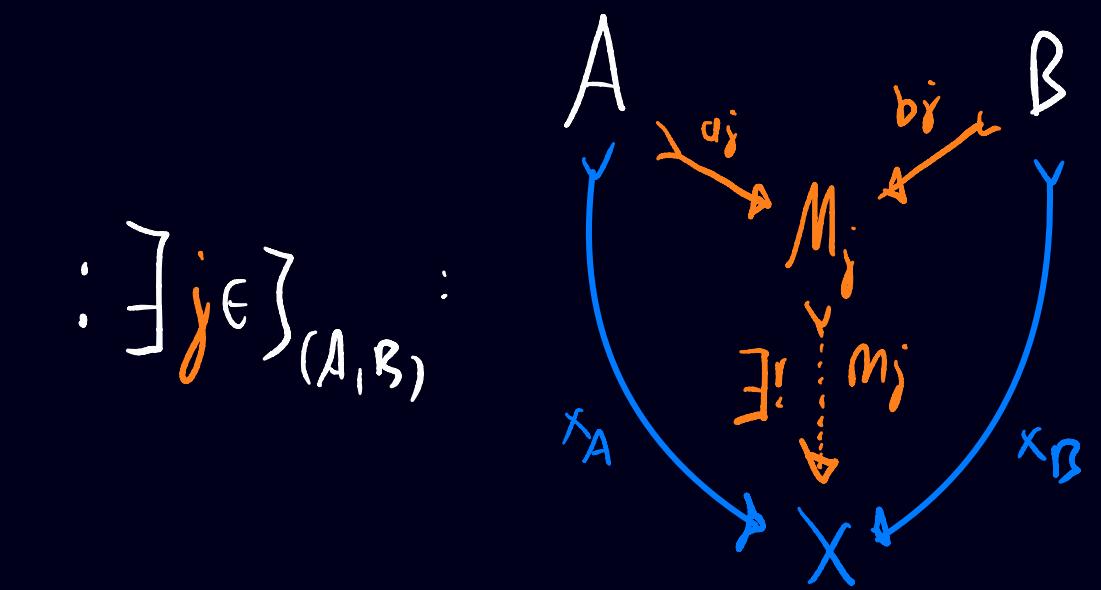
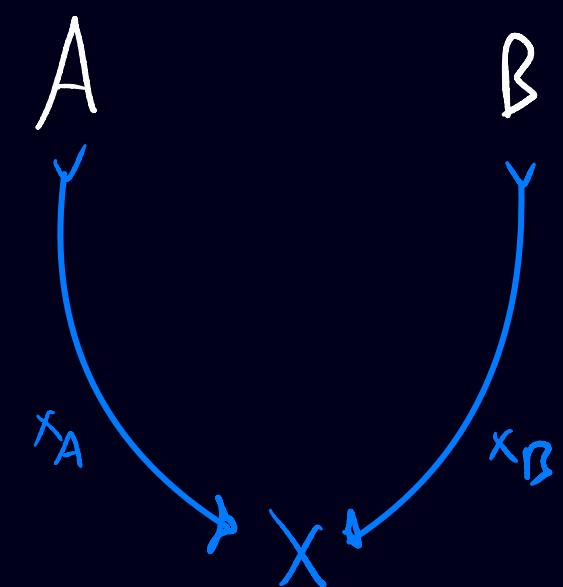
$$\forall \begin{array}{c} \cdot \xleftarrow{r_2} \cdot \xleftarrow{r_1} \\ \Downarrow \alpha \\ \cdot \xleftarrow{s} \end{array} : \quad \begin{array}{c} \exists \begin{array}{c} \cdot \xleftarrow{r_2} \cdot \xleftarrow{r_1} \\ \Downarrow A_2 \quad \Downarrow A_1 \\ \cdot \xleftarrow{t_2} \xleftarrow{\varphi_\alpha^{-1}} \cdot \xleftarrow{t_1} \end{array} \\ \Downarrow \varphi_\alpha^{-1} \circ (A_2 \diamond A_1) = \alpha \end{array}$$

"GLOBULAR"  
ISOMORPHISM

By INDUCTION ON  $n$ ,  
ONE FINDS THAT

$\forall n \geq 2 : h_n : \mathbb{D}_n \rightarrow \mathbb{D}_1$   
ARE "GLOBULAR"  
STREET OPFIBRATIONS

15

 $\mathbb{D}_0$  HAS MULTI-SUMS:
 $\forall (A, B) \in \mathbb{D}_0 \times \mathbb{D}_0 : \exists \left\{ \begin{array}{c} A \\ \xrightarrow{a_j} M_j \\ \xrightarrow{b_j} B \end{array} \right\}_{j \in \mathcal{J}_{(A, B)}} :$ 


DEFINITION: CLEAVAGE FOR MULTI-SUMS:

$$\forall (A, B) \in \mathbb{D}_0 \times \mathbb{D}_0 : ms(A, B) = \left\{ \begin{array}{c} A \\ \xrightarrow{a_j} M_j \\ \xrightarrow{b_j} B \end{array} \right\}_{j \in \mathcal{J}_{(A, B)}}$$

15

 $\mathbb{D}_0$  HAS MULTI-SUMS:

$$\forall (A, B) \in \mathbb{D}_0 \times \mathbb{D}_0 : \exists \left\{ \begin{array}{c} A \\ \xrightarrow{a_j} M_j \\ \xrightarrow{b_j} B \end{array} \right\}_{j \in \mathcal{I}_{(A, B)}} :$$

$$\forall \begin{array}{c} A \\ \downarrow x_A \\ X \\ \downarrow x_B \\ B \end{array} : \exists j \in \mathcal{I}_{(A, B)} :$$

$$\begin{array}{ccc} A & \xrightarrow{a_j} & M_j \\ \downarrow x_A & \nearrow m_j & \downarrow b_j \\ \exists ! m_j & & \downarrow x_B \\ & \nearrow b_j & \\ B & \xleftarrow{b_j} & B \end{array}$$

$$\wedge \forall \begin{array}{c} A \\ \xrightarrow{x_A} Y \\ \downarrow y_A \\ X \\ \downarrow y_X \\ \xrightarrow{x_B} B \end{array} : \exists ! M_j \xrightarrow{m} Y :$$

$$\begin{array}{ccc} A & \xrightarrow{a_j} & M_j \\ \downarrow x_A & \nearrow m & \downarrow b_j \\ \exists ! m & & \downarrow y_B \\ & \nearrow y_B & \\ B & \xleftarrow{b_j} & B \end{array}$$

DEFINITION: CLEAVAGE FOR MULTI-SUMS:

$$\forall (A, B) \in \mathbb{D}_0 \times \mathbb{D}_0 : ms(A, B) = \left\{ \begin{array}{c} A \\ \xrightarrow{a_j} M_j \\ \xrightarrow{b_j} B \end{array} \right\}_{j \in \mathcal{I}_{(A, B)}}$$

 $S: \mathbb{D}_1 \rightarrow \mathbb{D}_0$  IS A MULTI-OPFIBRATION

$$\forall \begin{array}{c} n \\ \downarrow r \\ \alpha'' \\ \downarrow m \\ m' \end{array} : \exists \begin{array}{c} n' \\ \downarrow r' \\ \alpha' \\ \downarrow m' \\ m'' \end{array} : n' \circ h = n'' \wedge \alpha' \circ_{v\alpha} \alpha'' = \alpha''$$

$$\forall \begin{array}{c} n \\ \downarrow r \\ \alpha'' \\ \downarrow m \\ m' \end{array} : \exists \begin{array}{c} n' \\ \downarrow r' \\ \alpha' \\ \downarrow m' \\ m'' \end{array} : n \circ n' = n' \wedge \alpha \circ_{v\alpha} \alpha' = \alpha'$$

 $T: \mathbb{D}_1 \rightarrow \mathbb{D}_0$  IS A RESIDUAL MULTI-OPFIBRATION

$$\forall \begin{array}{c} n \\ \downarrow r \\ \alpha'' \\ \downarrow m \\ m' \end{array} : \exists \begin{array}{c} n' \\ \downarrow r' \\ \alpha' \\ \downarrow m' \\ m'' \end{array} : n \circ n' = n' \wedge \alpha \circ_{v\alpha} \alpha' = \alpha'$$

$$\forall \begin{array}{c} n \\ \downarrow r \\ \alpha'' \\ \downarrow m \\ m' \end{array} : \exists \begin{array}{c} n' \\ \downarrow r' \\ \beta' \\ \downarrow m' \\ m'' \end{array} : n \circ n' = n' \wedge m \circ m' = m'' \wedge \beta' \circ_{v\beta} \beta'' = \beta''$$

# 16 CONVOLUTION PRODUCTS REVISITED

RECAP:

$$(F_n * \dots * F_1)(\Gamma) := \left\{ \begin{array}{l} S = (S_n, \dots, S_1) \in \mathbb{D}_n \\ \mathbb{D}_1(h_n(S), \Gamma) \times F_n(S) \end{array} \right. = \left\{ \begin{array}{c} f_n \\ \vdots \\ \dots \\ \vdots \\ f_1 \end{array} \right\} \xrightarrow{\sigma} \Gamma$$

$$\forall S \xrightarrow{A} S' \in \mathbb{D}_n, \quad h_n(S') \xrightarrow{\tau} \Gamma \in \mathbb{D}_1 : \quad \sim$$

# 16 CONVOLUTION PRODUCTS REVISITED

RECAP:

$$(F_n * \dots * F_1)(\Gamma) := \left\{ \begin{array}{l} S = (S_n, \dots, S_1) \in \mathbb{D}_n \\ \mathbb{D}_1(h_n(S), \Gamma) \times F_n(S) \end{array} \right. = \left\{ \begin{array}{c} f_n \\ \vdots \\ \dots \\ \vdots \\ f_1 \end{array} \right\} \xrightarrow{\sigma} \Gamma$$

$$\forall S \xrightarrow{A} S' \in \mathbb{D}_n, \quad h_n(S') \xrightarrow{\tau} \Gamma \in \mathbb{D}_1 : \quad \begin{array}{c} f_n \\ \vdots \\ \dots \\ \vdots \\ f_1 \end{array} \xrightarrow{\sim} \begin{array}{c} f_n \\ \vdots \\ \dots \\ \vdots \\ f_1 \end{array}$$

$$h_n(A) \xrightarrow{\sim} \begin{array}{c} f_n \\ \vdots \\ \dots \\ \vdots \\ f_1 \end{array}$$

Now:  $h_n: \mathbb{D}_n \rightarrow \mathbb{D}_1$  is a "GLOBULAR" STREET OPFIBRATION  $\Rightarrow$

$$\begin{array}{c} f_n \\ \vdots \\ \dots \\ \vdots \\ f_1 \end{array} \xrightarrow{\sim} \begin{array}{c} f_n \\ \vdots \\ \dots \\ \vdots \\ f_1 \end{array} \xrightarrow{\sim} \begin{array}{c} f_n \\ \vdots \\ \dots \\ \vdots \\ f_1 \end{array} \xrightarrow{\sim} \begin{array}{c} f_n \\ \vdots \\ \dots \\ \vdots \\ f_1 \end{array}$$

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$$(F_n * \dots * F_1)(r) \cong \left\{ \begin{array}{c} \text{Diagram showing } g_n \text{ and } g_1 \text{ with arrows } \tau_n \text{ and } \tau_1, \text{ and a blue curved arrow } r \text{ connecting them.} \\ \cong \downarrow \tau \\ \cong g \end{array} \right\}$$

WHERE

$$\left\{ \begin{array}{c} \text{Diagram showing } g_n \text{ and } g_1 \text{ with arrows } \tau_n \text{ and } \tau_1, \text{ and a blue curved arrow } r \text{ connecting them.} \\ \cong \downarrow \tau_n \\ \cong \downarrow \tau_1 \\ \cong \downarrow \chi \\ \cong g \end{array} \right\} \quad \left\{ \begin{array}{c} F_n(\tau_n)g_n \\ F_1(\tau_1)g_1 \\ \text{Diagram showing } g_n \text{ and } g_1 \text{ with arrows } \tau_n \text{ and } \tau_1, \text{ and a blue curved arrow } r \text{ connecting them.} \\ \cong \downarrow \chi \\ \cong g \end{array} \right\}$$

EXAMPLE: FOR  $\hat{\Delta}_{r_j} := \text{ID}_j(r_j, -)$  ( $j=1, \dots, n$ )

$$(\hat{\Delta}_{r_n} * \dots * \hat{\Delta}_{r_1})(r) \cong \left\{ \begin{array}{c} \text{Diagram showing } r_n \text{ and } r_1 \text{ with arrows } \alpha_n \text{ and } \alpha_1, \text{ and a blue curved arrow } r \text{ connecting them.} \\ \downarrow \alpha_n \\ \downarrow \alpha_1 \\ \cong \downarrow \tau \\ \cong g \end{array} \right\}$$

✓

⑯ KEY RESULT: WEAK ASSOCIATIVITY OF \*

$$\forall F_3, F_2, F_1 : \mathbb{D}_1 \rightarrow \underline{\text{Set}}, \quad r \in \mathbb{D}_1 : \quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

18 KEY RESULT: WEAK ASSOCIATIVITY OF \*

$$\forall F_3, F_2, F_1 : \mathbb{D}_1 \rightarrow \underline{\text{Set}}, r \in \mathbb{D}_1 : F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

PROOF (SKETCH):

$$F_3 * (F_2 * F_1)(r) = \left\{ \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ with nodes } t_3, w_2, w_1, v_2, v_1, l_2, l_1, \text{ and morphisms } f_3, f_2, f_1, \text{ with } \Downarrow w \text{ and } \Downarrow \sigma. \\ \Downarrow \sigma \text{ is labeled } r. \end{array} \right\} / \sim_V / \sim_W$$

$$\forall \left[ \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ with nodes } t_3, w_2, w_1, v_2, v_1, l_2, l_1, \text{ and morphisms } f_3, f_2, f_1, \text{ with } \Downarrow w \text{ and } \Downarrow \sigma. \\ \Downarrow \sigma \text{ is labeled } r. \end{array} \right] : \exists \left[ \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ with nodes } t_3, w_2, w_1, v_2, v_1, l_2, l_1, \text{ and morphisms } f_3, f_2, f_1, \text{ with } \Downarrow w \text{ and } \Downarrow \sigma. \\ \Downarrow \sigma \text{ is labeled } r. \end{array} \right] \approx \Downarrow \tau$$

$$\sim_V \sqrt{F_3(\gamma_2) t_3} \Downarrow \gamma_1 \Downarrow w \Downarrow \tau \sim_W$$

18 KEY RESULT: WEAK ASSOCIATIVITY OF \*

$$\forall F_3, F_2, F_1 : \mathbb{D}_1 \rightarrow \underline{\text{Set}}, \Gamma \in \mathbb{D}_1 : F_3 * (F_2 * F_1)(\Gamma) \cong (F_3 * F_2 * F_1)(\Gamma) \cong (F_3 * F_2) * F_1(\Gamma)$$

PROOF (SKETCH):

$$F_3 * (F_2 * F_1)(\Gamma) = \left\{ \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \text{ with a commutative square and a diagonal } \Downarrow \sigma. \\ \text{The diagram is enclosed in curly braces.} \end{array} \right\} / \sim_V / \sim_W$$

$$\forall \left[ \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \text{ with a commutative square and a diagonal } \Downarrow \sigma. \\ \text{The diagram is enclosed in brackets.} \end{array} \right] \sim_V$$

$$\sim_V F_3(\gamma_2)t_3 \left[ \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \text{ with a commutative square and a diagonal } \Downarrow \sigma. \\ \text{The diagram is enclosed in brackets.} \end{array} \right] \sim_V$$

$$\sim_W F_3(\gamma_2)t_3 \left[ \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \text{ with a commutative square and a diagonal } \Downarrow \sigma. \\ \text{The diagram is enclosed in brackets.} \end{array} \right] \sim_W$$

$$\sim_W \left[ \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \text{ with a commutative square and a diagonal } \Downarrow \sigma. \\ \text{The diagram is enclosed in brackets.} \end{array} \right]$$

18 KEY RESULT: WEAK ASSOCIATIVITY OF \*

$$\forall F_3, F_2, F_1 : \mathbb{D}_1 \rightarrow \underline{\text{Set}}, \Gamma \in \mathbb{D}_1 : F_3 * (F_2 * F_1)(\Gamma) \cong (F_3 * F_2 * F_1)(\Gamma) \cong (F_3 * F_2) * F_1(\Gamma)$$

PROOF (SKETCH):

$$F_3 * (F_2 * F_1)(\Gamma) = \left\{ \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \sigma, \omega. \\ \text{The diagram consists of three nodes at the top: } t_3, t_2, t_1. \\ \text{Horizontal arrows: } t_3 \xrightarrow{w_2} t_2 \xrightarrow{w_1} t_1. \\ \text{Vertical arrows: } t_3 \downarrow v_2 \rightarrow t_2, t_2 \downarrow v_1 \rightarrow t_1. \\ \text{Diagonal arrows: } t_3 \downarrow \omega \rightarrow t_1. \\ \text{Bottom node: } \Gamma. \end{array} \right\} / \sim_V / \sim_W$$

$$\forall \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \sigma, \omega. \\ \text{The diagram consists of three nodes at the top: } t_3, t_2, t_1. \\ \text{Horizontal arrows: } t_3 \xrightarrow{w_2} t_2 \xrightarrow{w_1} t_1. \\ \text{Vertical arrows: } t_3 \downarrow v_2 \rightarrow t_2, t_2 \downarrow v_1 \rightarrow t_1. \\ \text{Diagonal arrows: } t_3 \downarrow \omega \rightarrow t_1. \\ \text{Bottom node: } \Gamma. \end{array}$$

$$\vdots \quad \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \sigma, \omega. \\ \text{The diagram consists of three nodes at the top: } t_3, t_2, t_1. \\ \text{Horizontal arrows: } t_3 \xrightarrow{w_2} t_2 \xrightarrow{w_1} t_1. \\ \text{Vertical arrows: } t_3 \downarrow v_2 \rightarrow t_2, t_2 \downarrow v_1 \rightarrow t_1. \\ \text{Diagonal arrows: } t_3 \downarrow \omega \rightarrow t_1. \\ \text{Bottom node: } \Gamma. \end{array}$$

$$\sim_V \quad \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \sigma, \omega. \\ \text{The diagram consists of three nodes at the top: } t_3, t_2, t_1. \\ \text{Horizontal arrows: } t_3 \xrightarrow{w_2} t_2 \xrightarrow{w_1} t_1. \\ \text{Vertical arrows: } t_3 \downarrow v_2 \rightarrow t_2, t_2 \downarrow v_1 \rightarrow t_1. \\ \text{Diagonal arrows: } t_3 \downarrow \omega \rightarrow t_1. \\ \text{Bottom node: } \Gamma. \end{array}$$

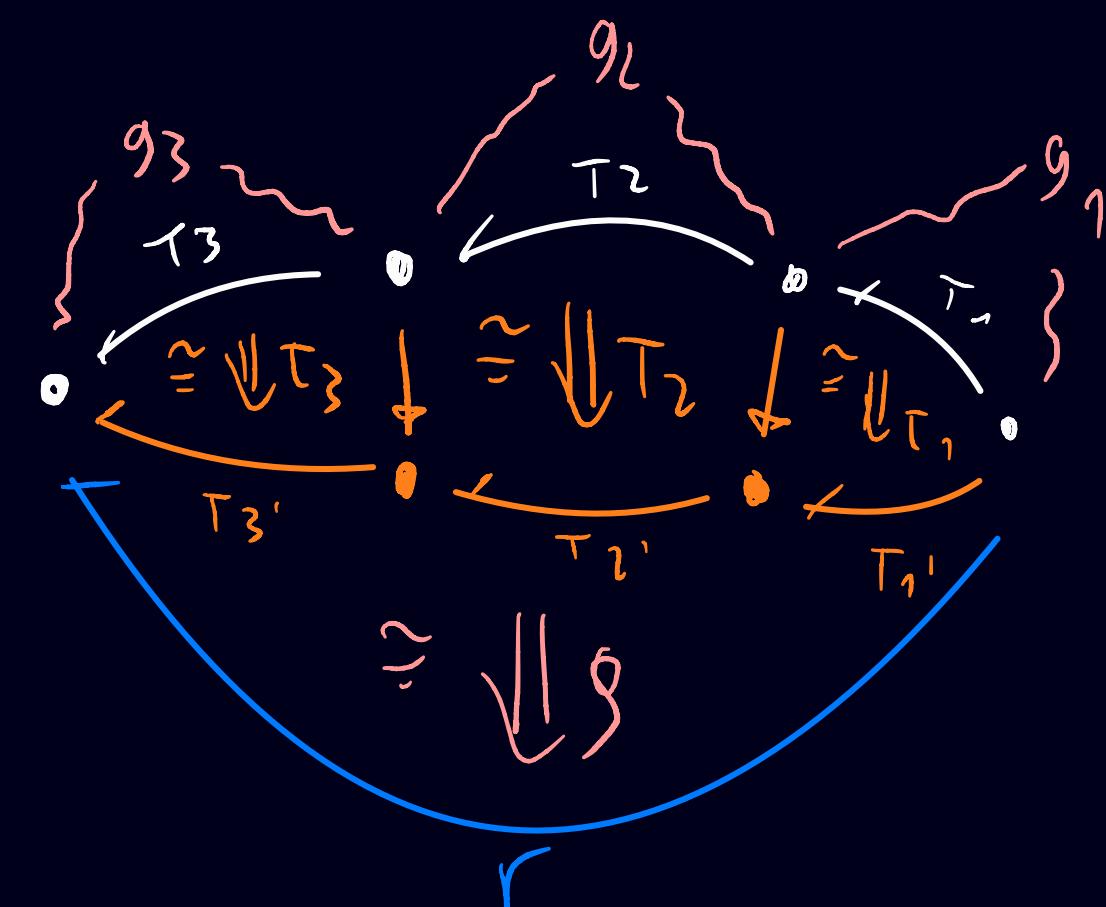
$$\sim_W \quad \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \sigma, \omega. \\ \text{The diagram consists of three nodes at the top: } t_3, t_2, t_1. \\ \text{Horizontal arrows: } t_3 \xrightarrow{w_2} t_2 \xrightarrow{w_1} t_1. \\ \text{Vertical arrows: } t_3 \downarrow v_2 \rightarrow t_2, t_2 \downarrow v_1 \rightarrow t_1. \\ \text{Diagonal arrows: } t_3 \downarrow \omega \rightarrow t_1. \\ \text{Bottom node: } \Gamma. \end{array}$$

$$\sim_W \quad \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \sigma, \omega. \\ \text{The diagram consists of three nodes at the top: } t_3, t_2, t_1. \\ \text{Horizontal arrows: } t_3 \xrightarrow{w_2} t_2 \xrightarrow{w_1} t_1. \\ \text{Vertical arrows: } t_3 \downarrow v_2 \rightarrow t_2, t_2 \downarrow v_1 \rightarrow t_1. \\ \text{Diagonal arrows: } t_3 \downarrow \omega \rightarrow t_1. \\ \text{Bottom node: } \Gamma. \end{array}$$

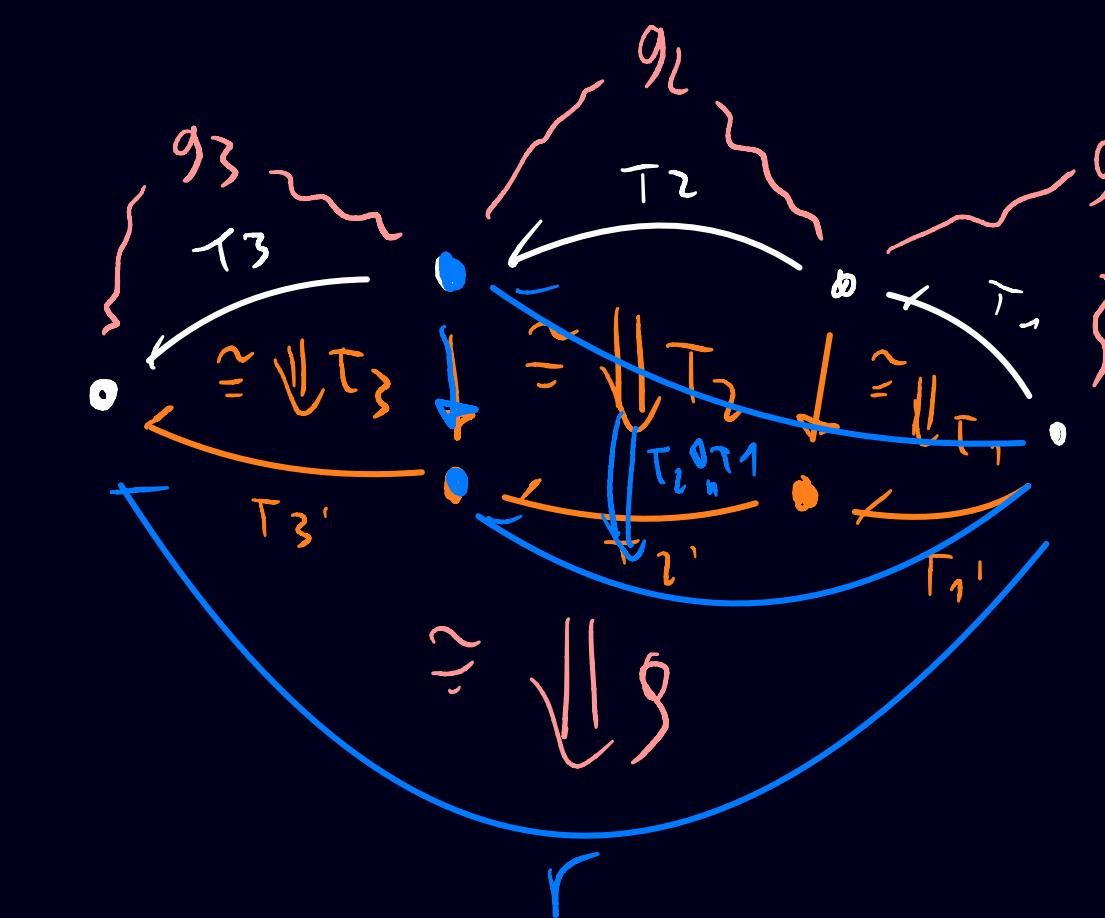
$$\cong \quad \begin{array}{c} \text{Diagram showing } F_3 * (F_2 * F_1) \text{ as a double category with objects } t_3, t_2, t_1 \text{ and morphisms } w_2, w_1, v_2, v_1, \sigma, \omega. \\ \text{The diagram consists of three nodes at the top: } t_3, t_2, t_1. \\ \text{Horizontal arrows: } t_3 \xrightarrow{w_2} t_2 \xrightarrow{w_1} t_1. \\ \text{Vertical arrows: } t_3 \downarrow v_2 \rightarrow t_2, t_2 \downarrow v_1 \rightarrow t_1. \\ \text{Diagonal arrows: } t_3 \downarrow \omega \rightarrow t_1. \\ \text{Bottom node: } \Gamma. \end{array}$$

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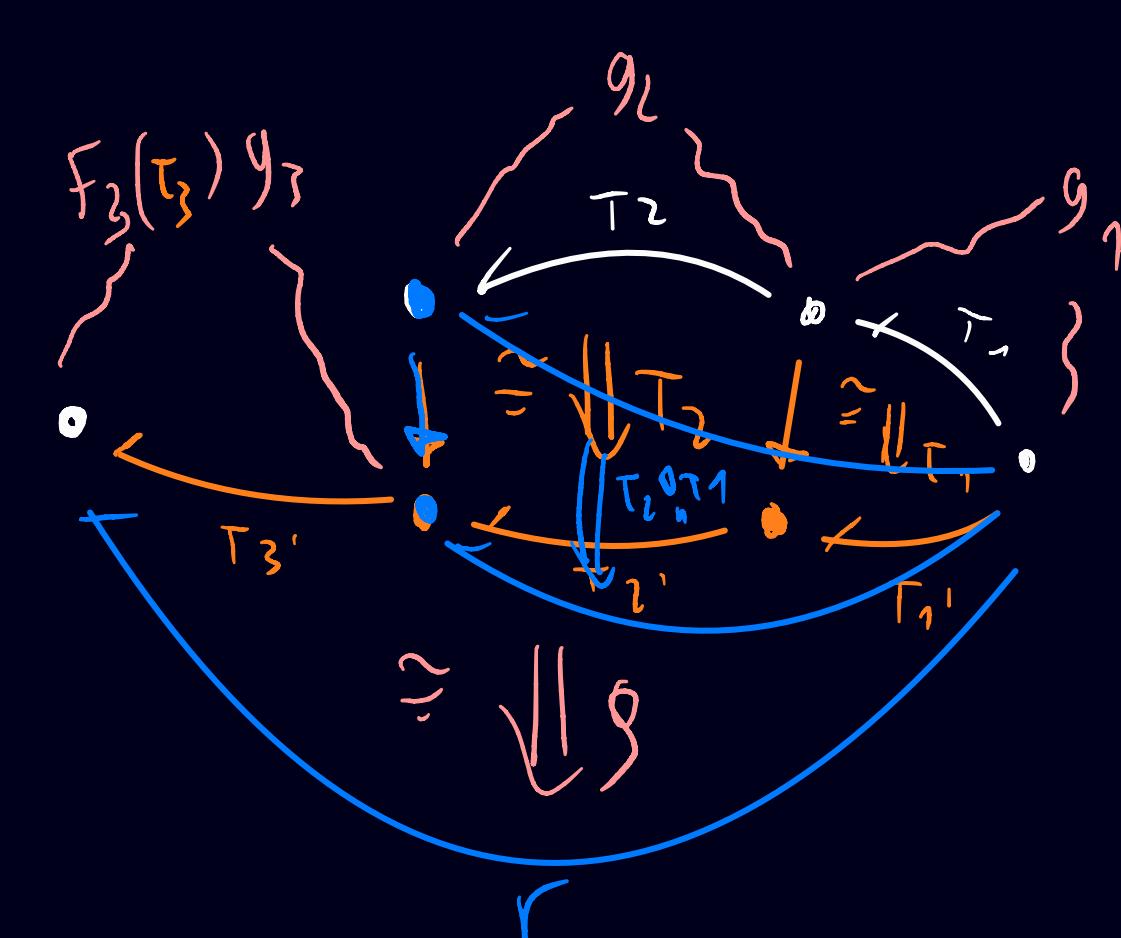
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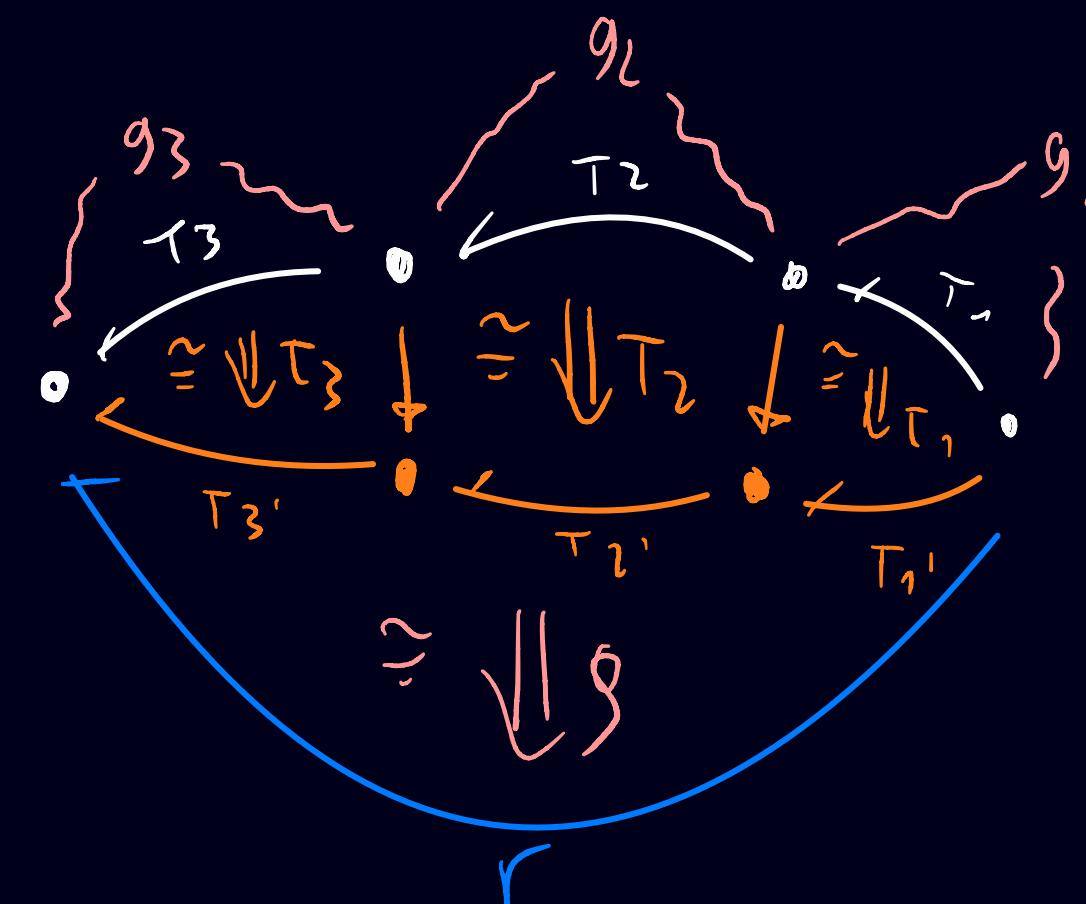


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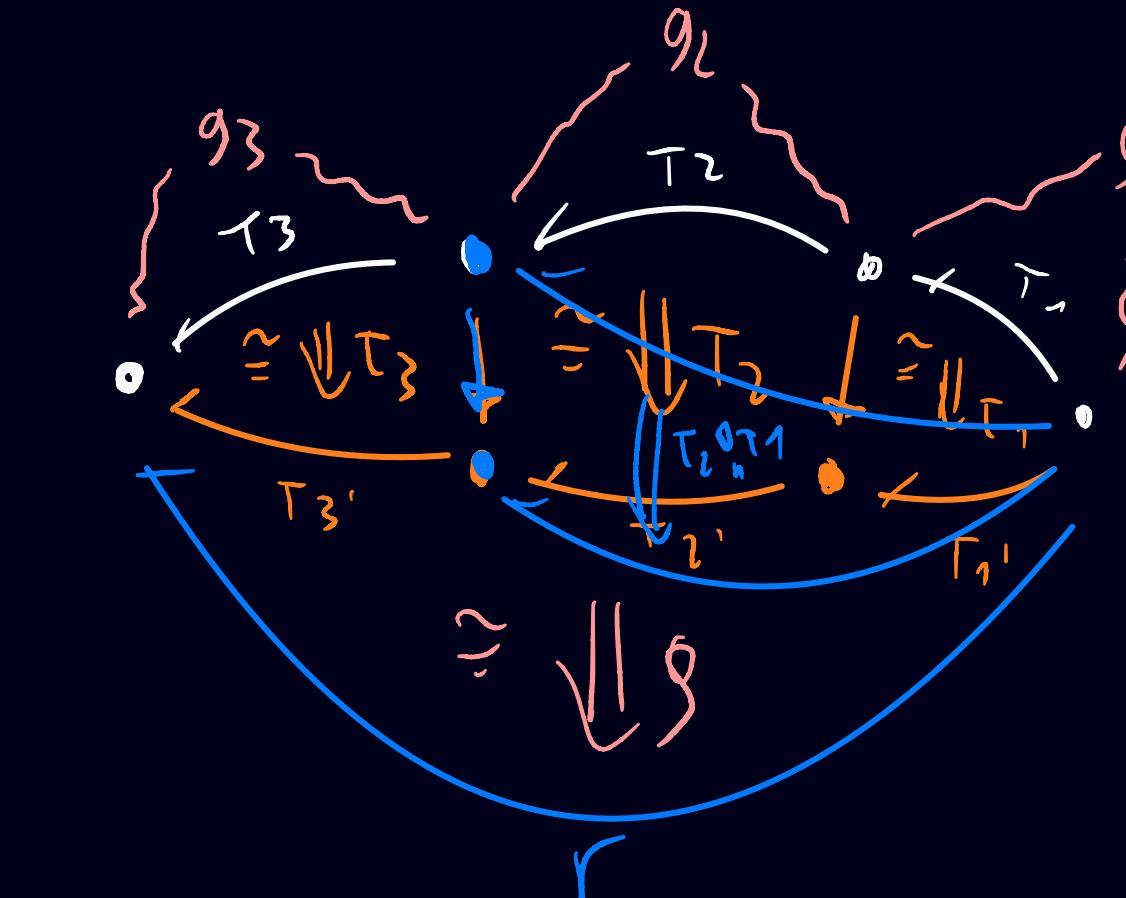


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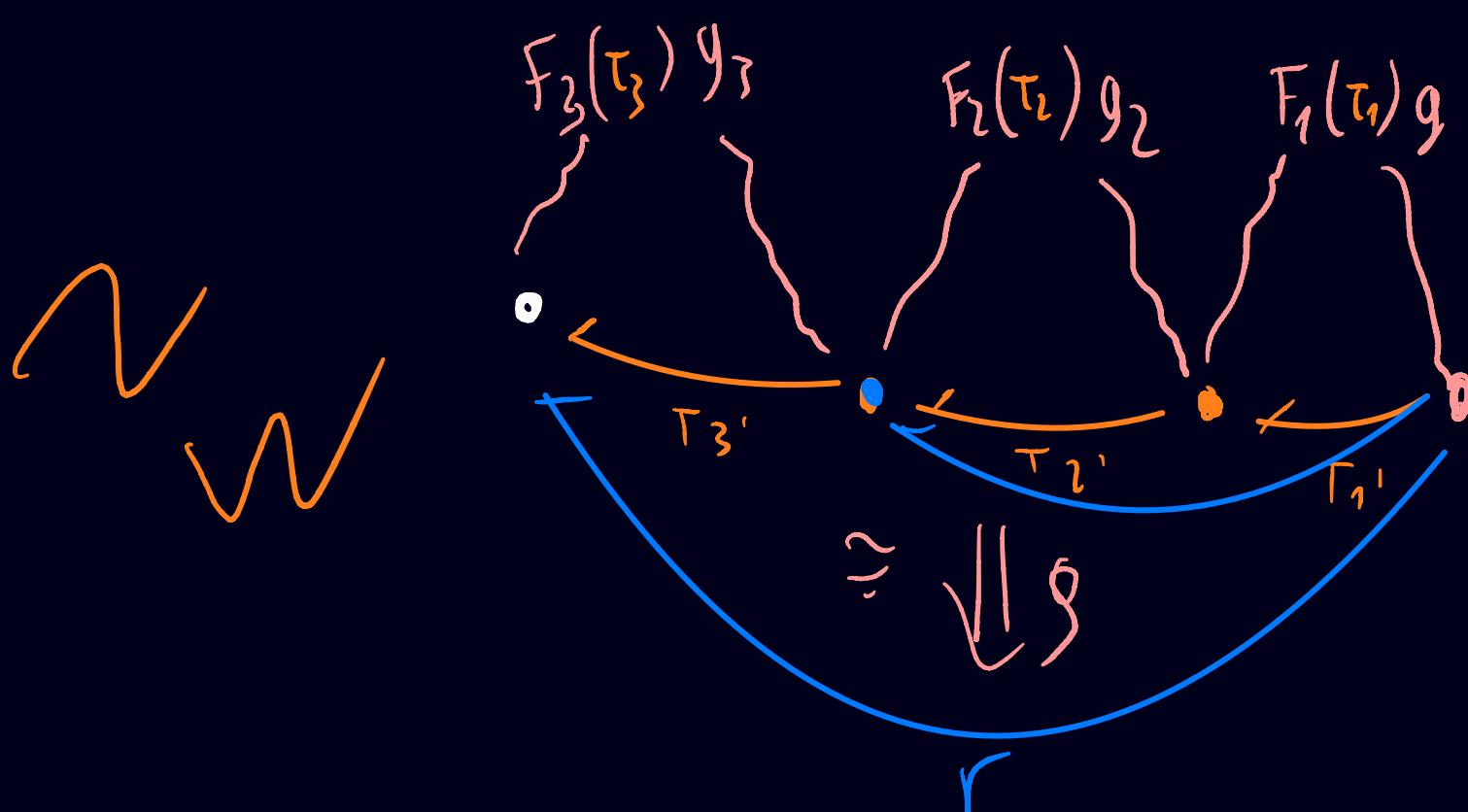
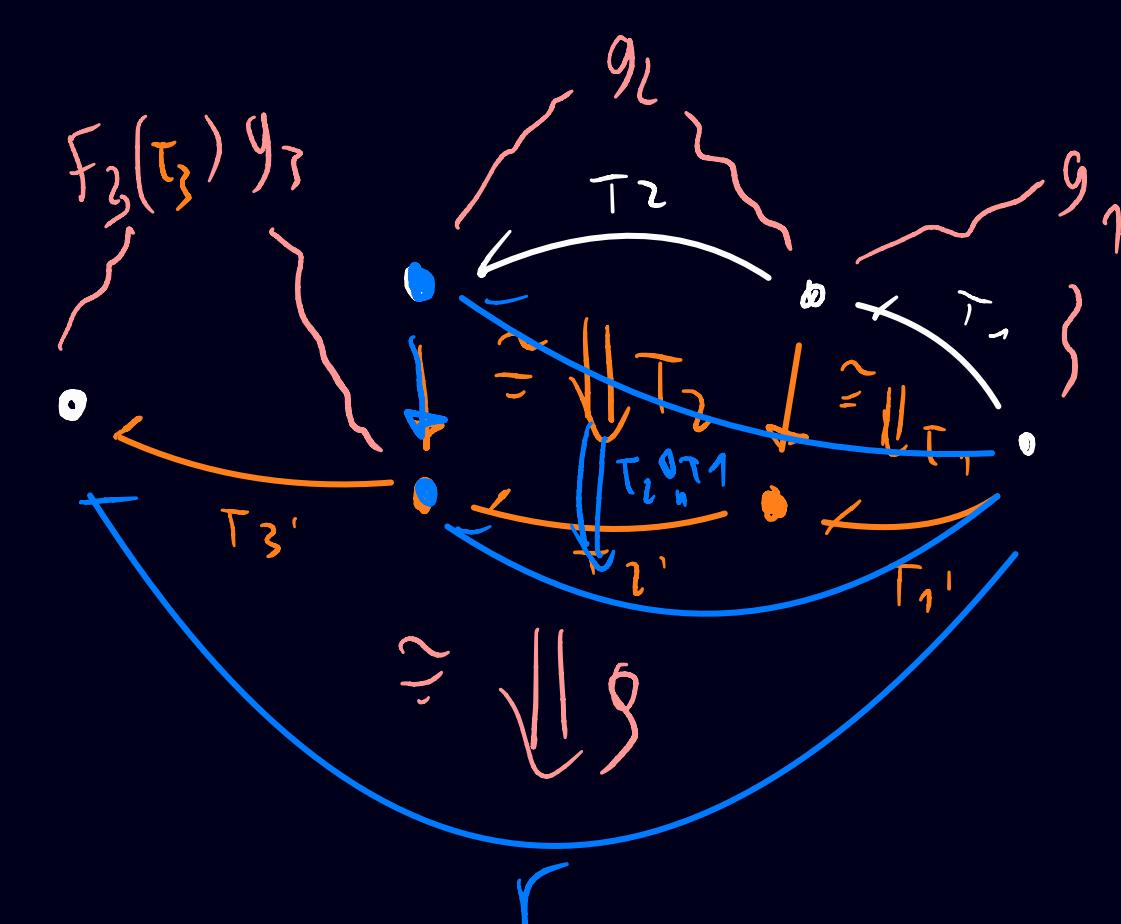
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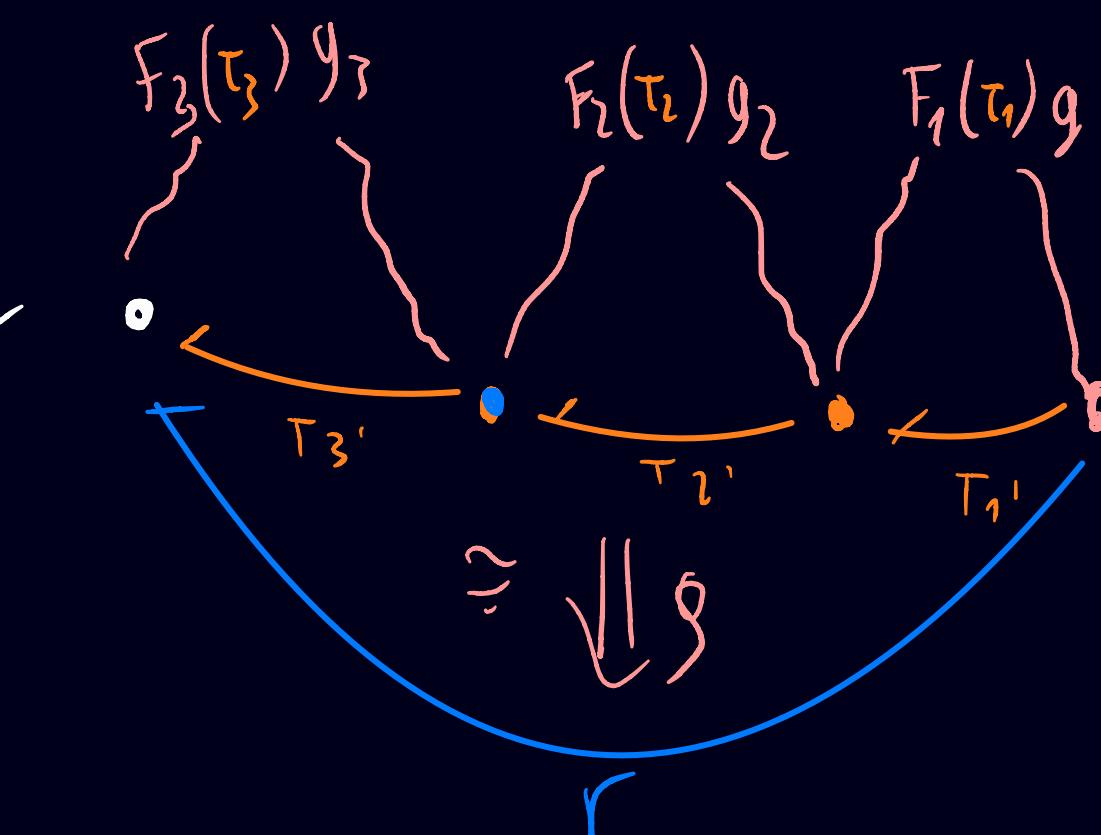
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$$\hookrightarrow F_3 * (F_2 * F_1)(r) \stackrel{\approx}{=} \left\{ \begin{array}{c} g_3 \\ \swarrow T_3 \\ \approx \Downarrow \beta \\ \searrow T_2 \\ g_2 \\ \swarrow T_2 \\ \approx \Downarrow \beta \\ \searrow T_1 \\ g_1 \end{array} \right\} / \approx \beta = (F_3 * F_2 * F_3)(r)$$

□

20 FINAL INGREDIENT: CATEGORIFICATION OF RULE ALGEBRA

CLAIM:  $(\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1})(r) \cong \left\{ \begin{array}{c} O_2 \xleftarrow{r_2} I_2 \\ \Downarrow \alpha_2 \\ \Downarrow \beta \\ O_1 \xleftarrow{r_1} I_1 \\ \Downarrow \alpha_1 \\ \Downarrow \beta \\ r \end{array} \right\} / \cong_y$

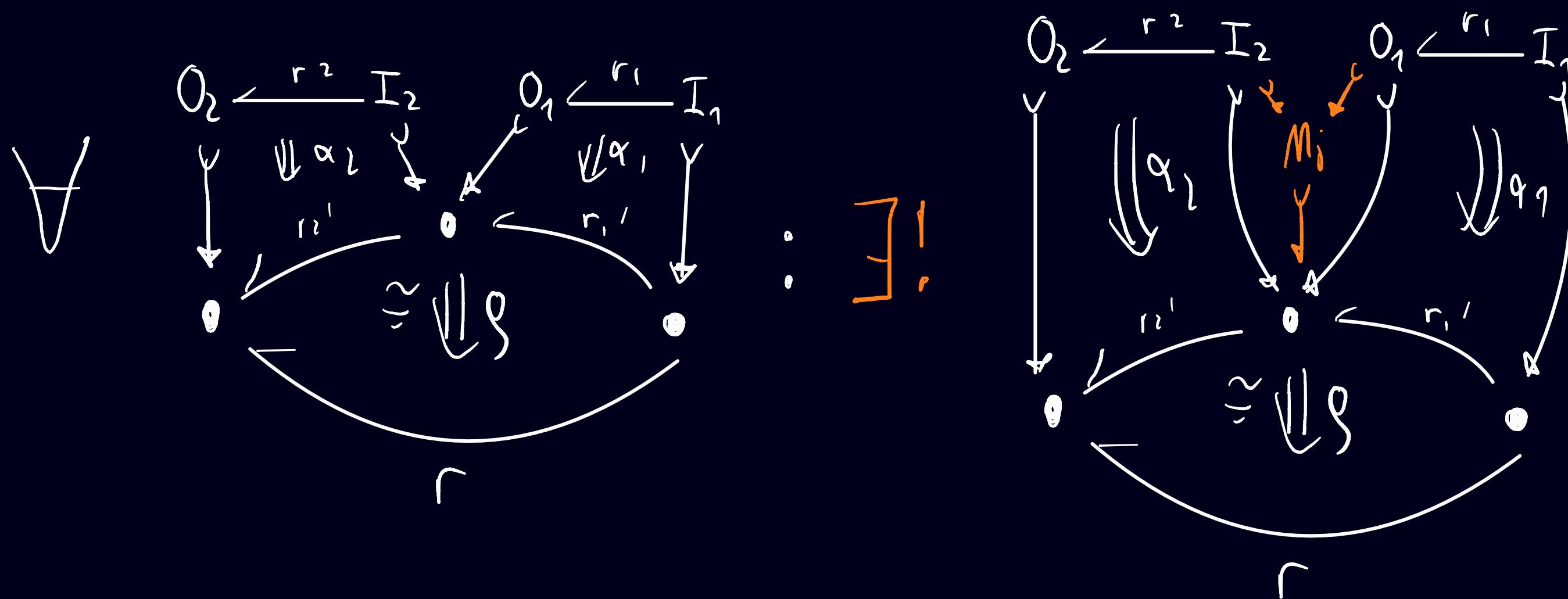
$$\cong \frac{\text{---}}{(O_2 \rightarrow M_j \leftarrow I_1)} \quad \beta_{j,k} \in T^*(O_1 \rightarrow M_j) \quad \gamma_{j,k} \in S^*(I_2 \rightarrow M_j \xrightarrow{*n} N_k)$$

$\in \text{MS}(O_2, I_1)$

$$\left\{ \begin{array}{c} \cdot \xleftarrow{r_2} \cdot \\ \Downarrow \gamma_{j,k} \\ M_j \\ \Downarrow \beta_{j,k} \\ \cdot \xleftarrow{r_1} \cdot \\ \Downarrow \beta_{j,k} \\ N_k \\ \Downarrow \gamma_{j,k} \\ \cdot \xleftarrow{r} \cdot \end{array} \right\}$$

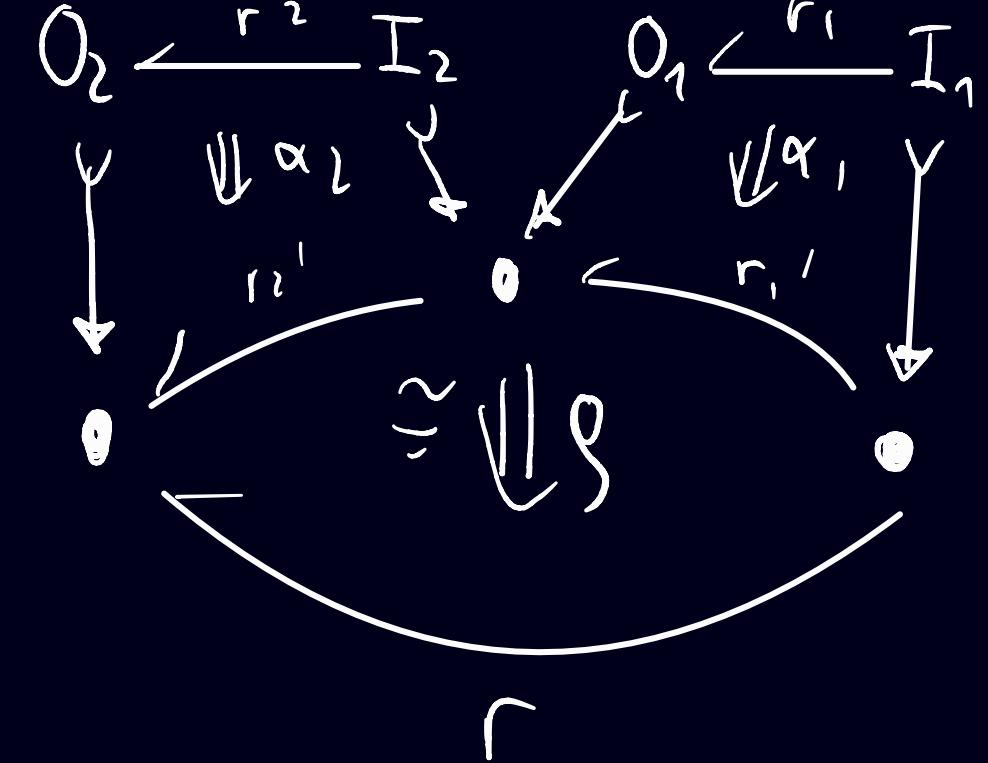
$$= : \hat{\Delta}_{r_2 \circ r_1}(r)$$

21 PROOF (SKETCH) : ASSUMING CHOSEN CLEAVAGES :

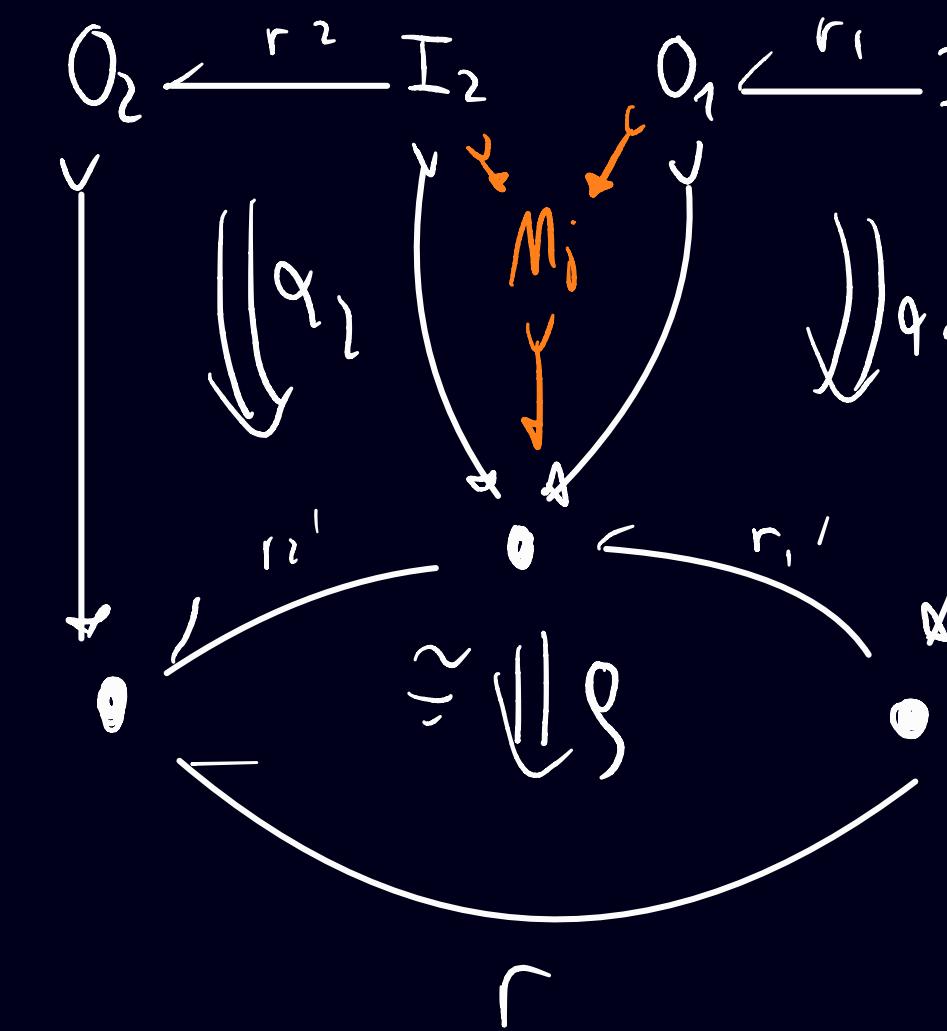


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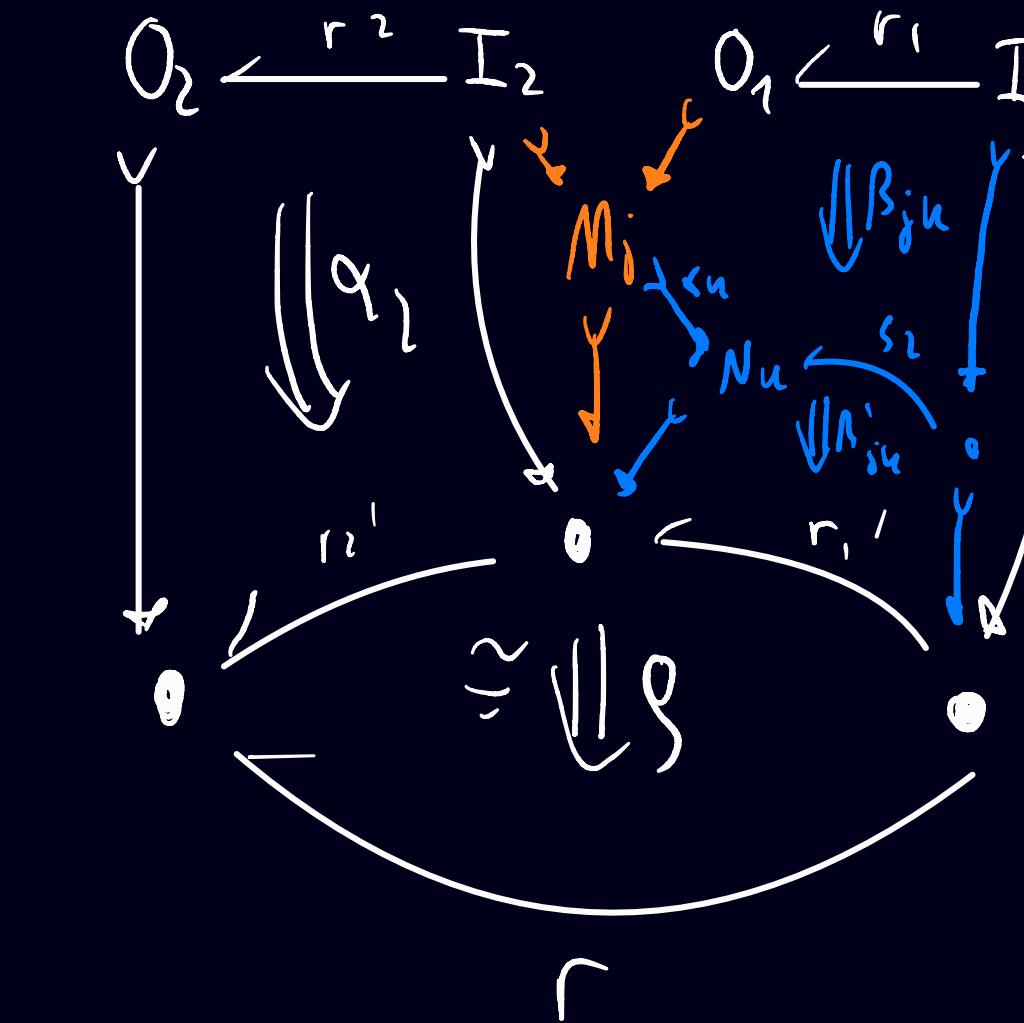
$\forall$



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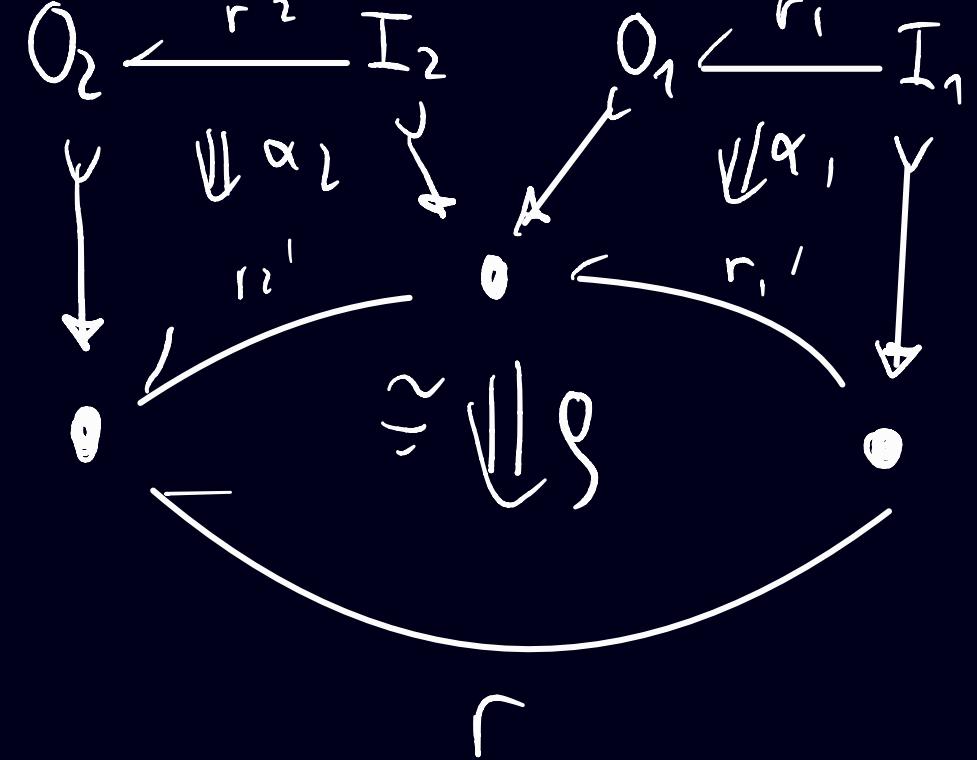


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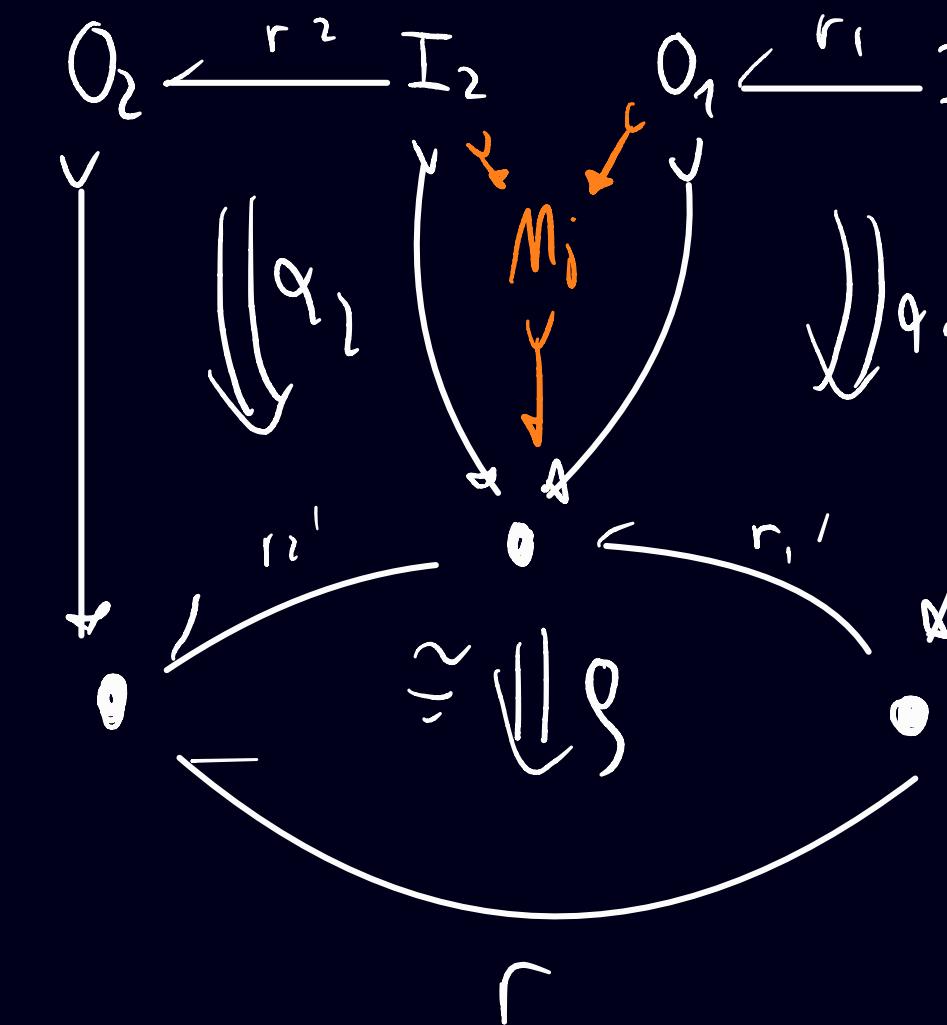


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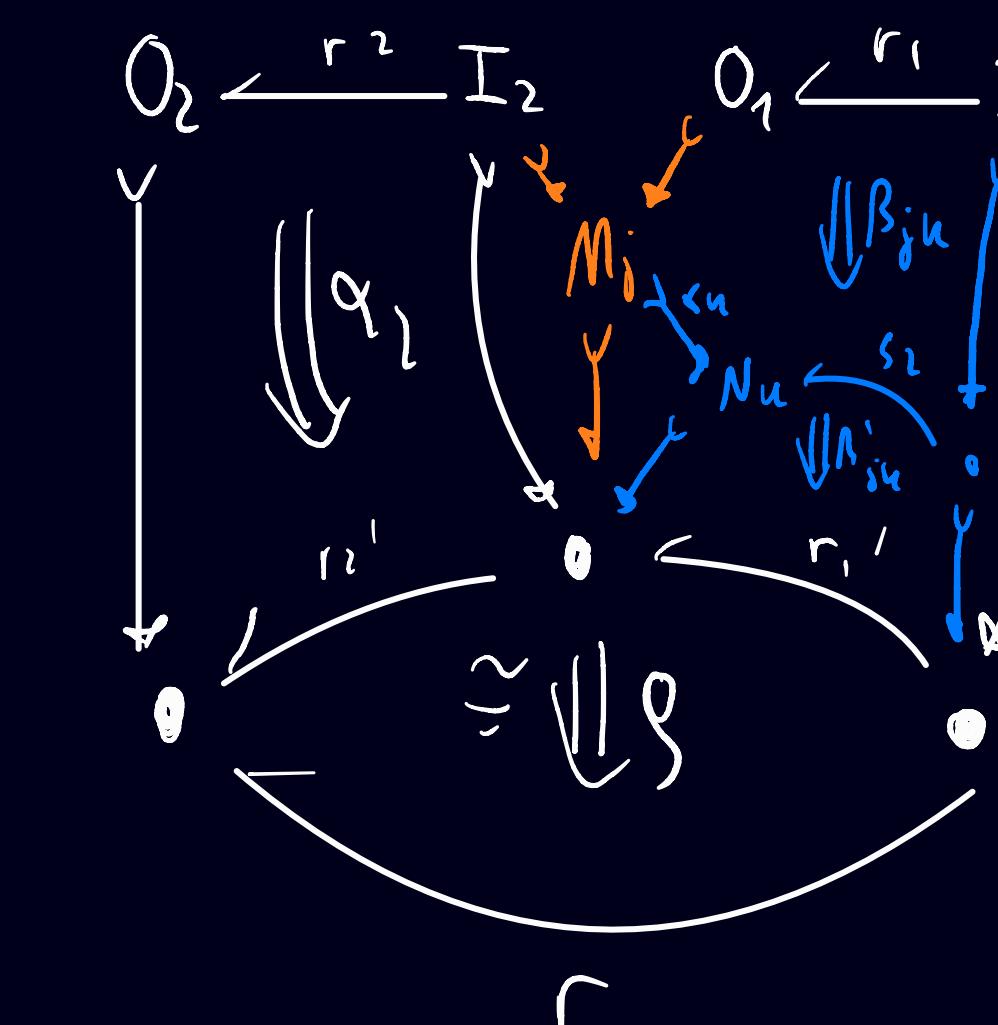
$\forall$



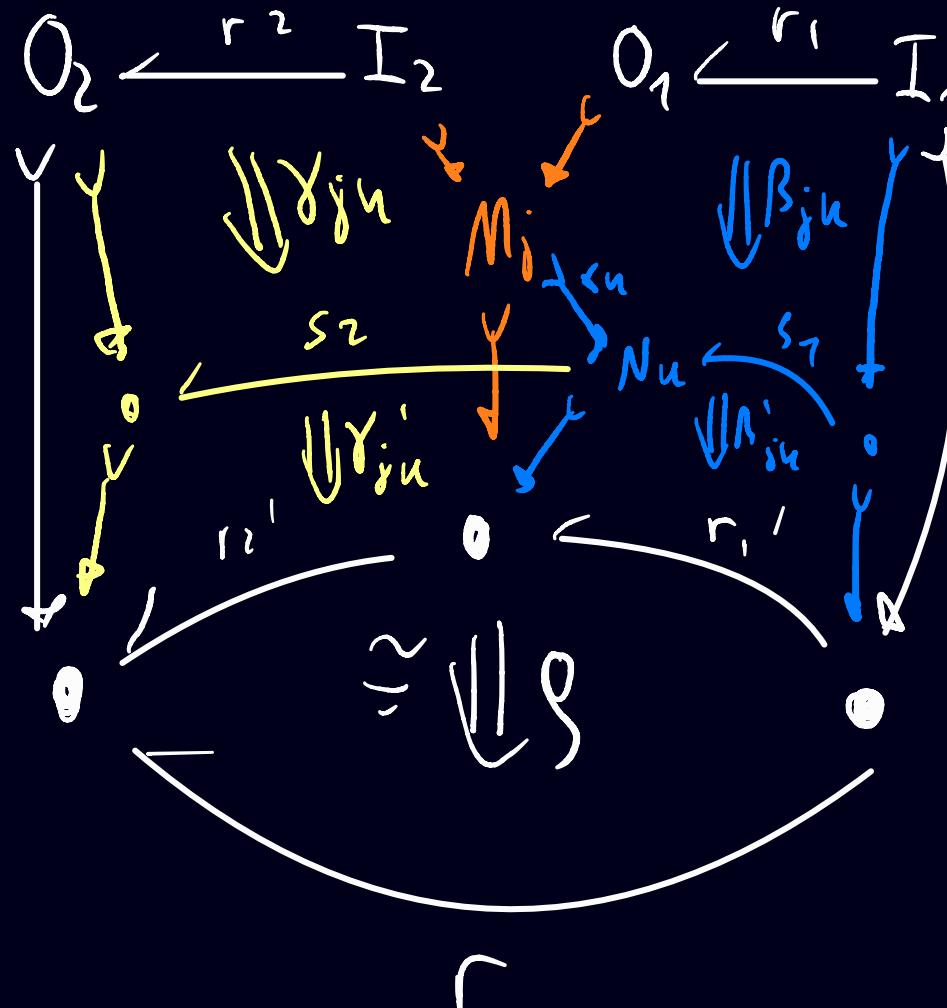
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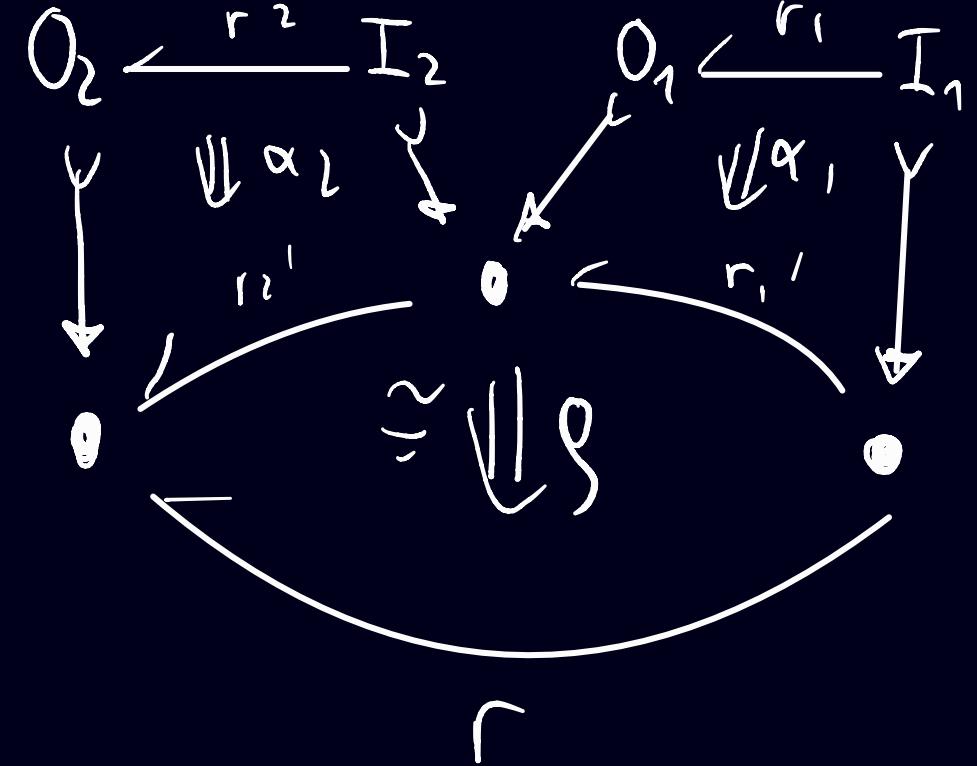


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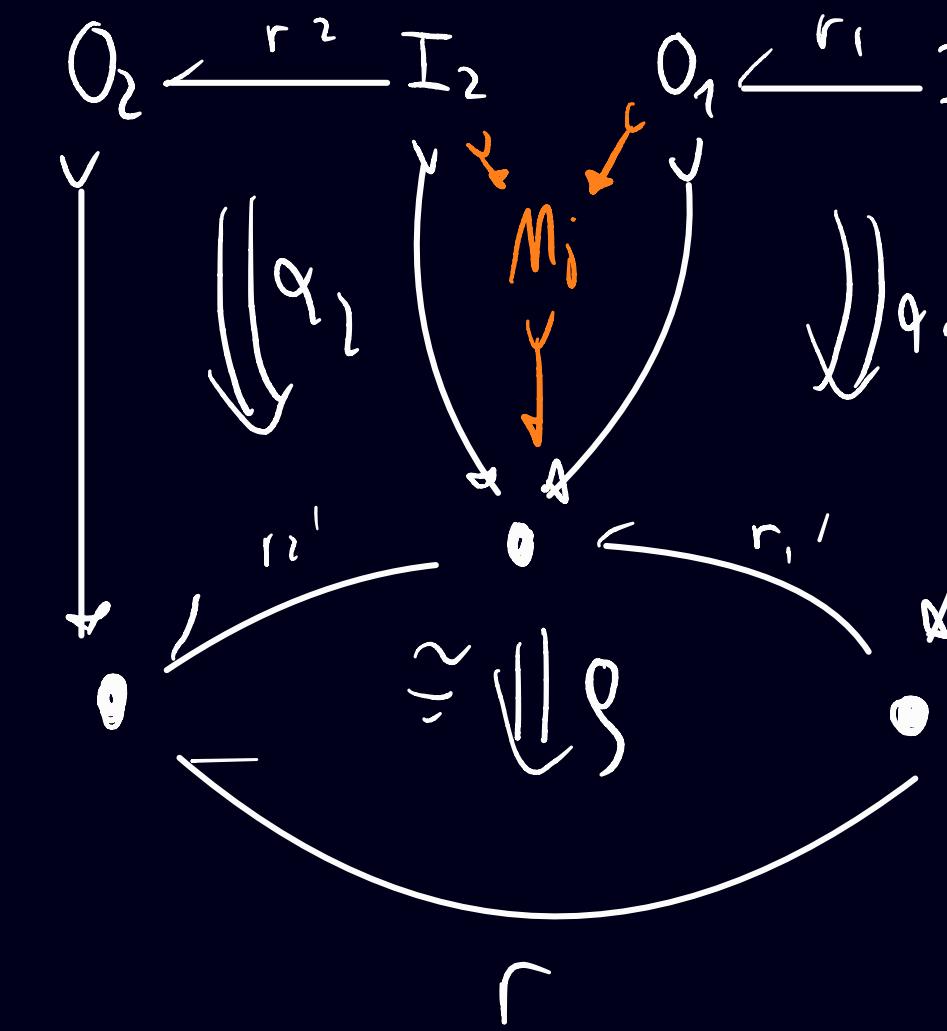


21 PROOF (SKETCH) : ASSUMING CHOSEN CLEAVAGES :

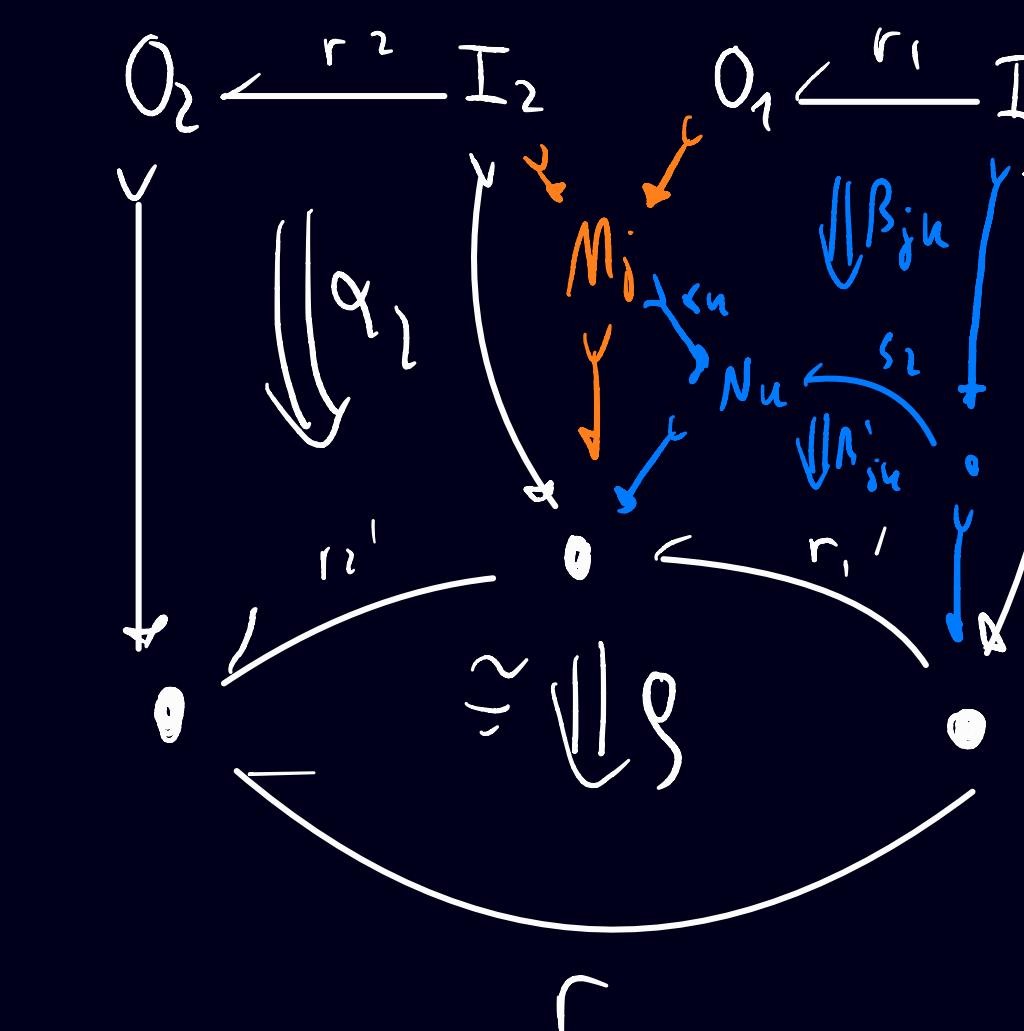
$\forall$



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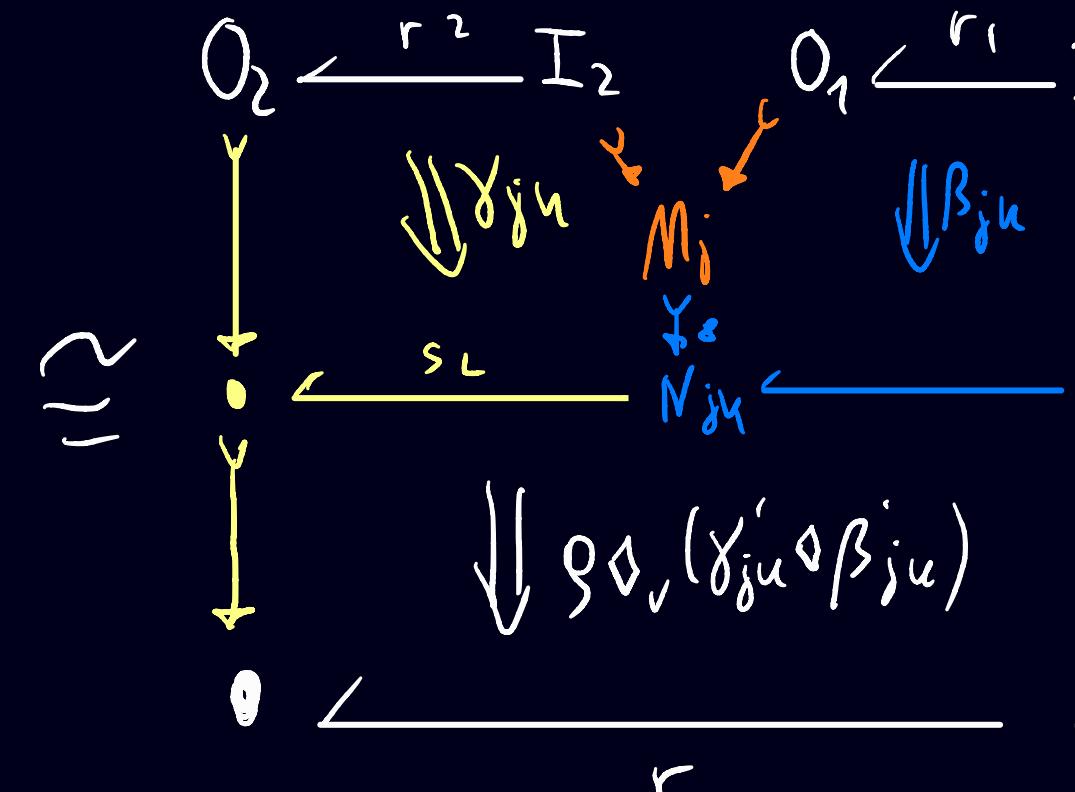
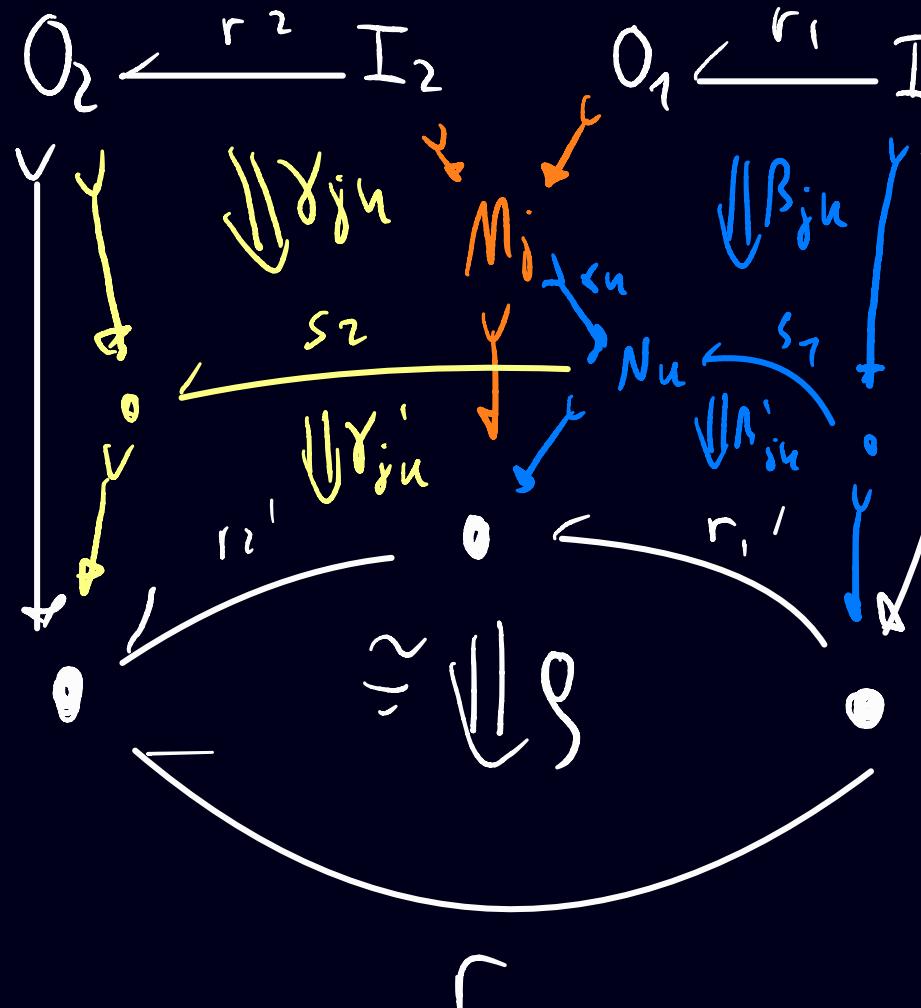


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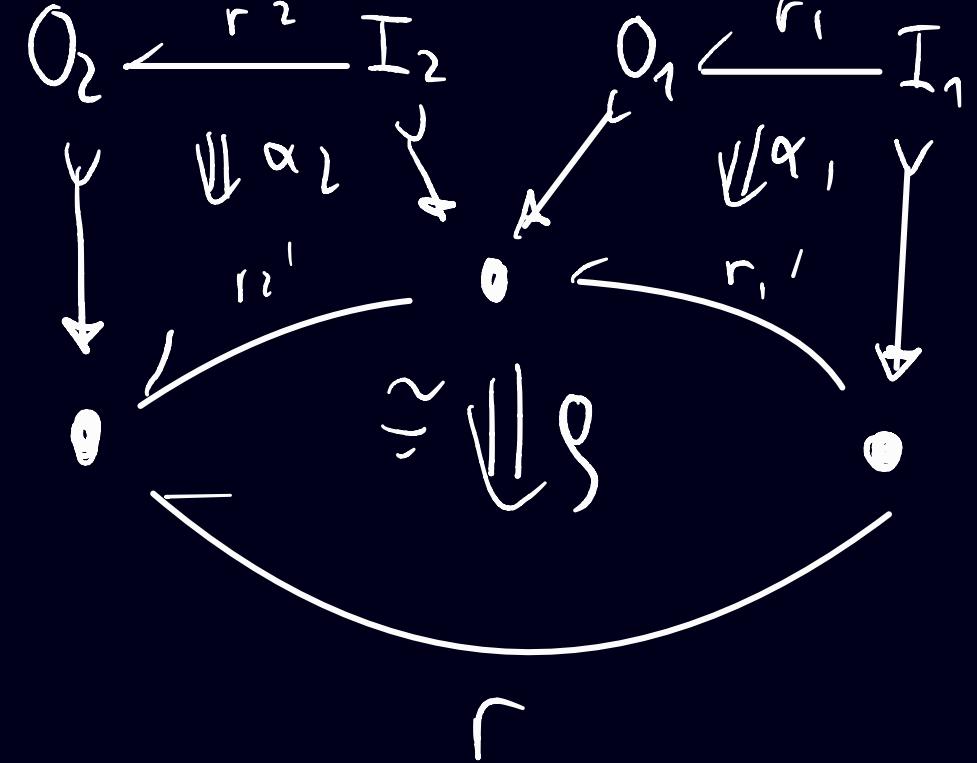
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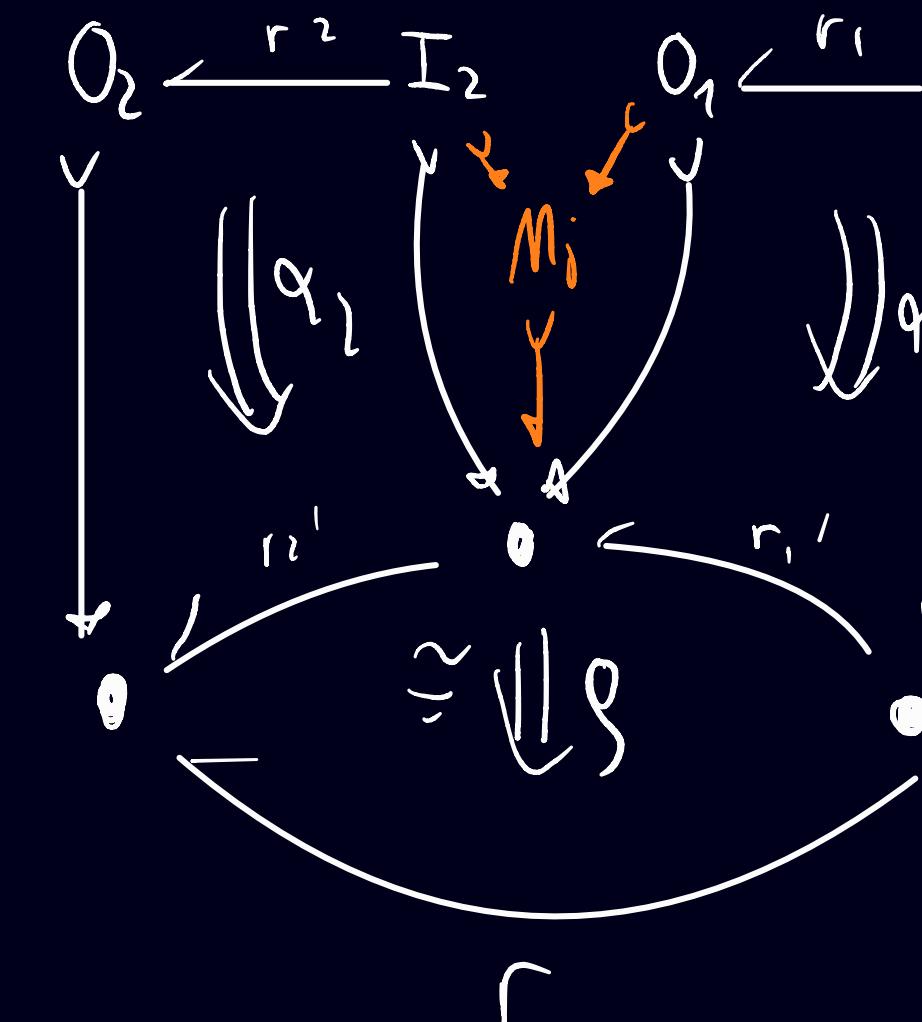
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21 PROOF (SKETCH) : ASSUMING CHOSEN CLEAVAGES :

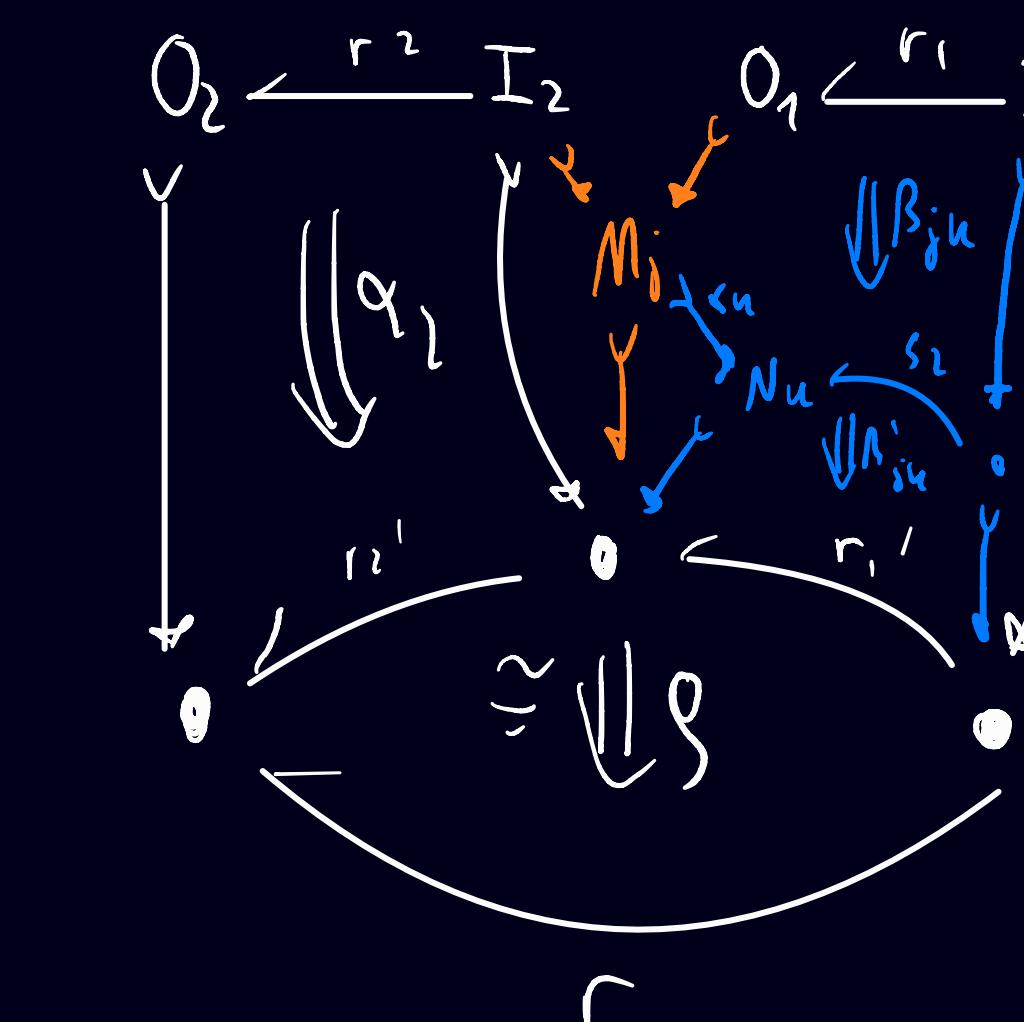
$\forall$



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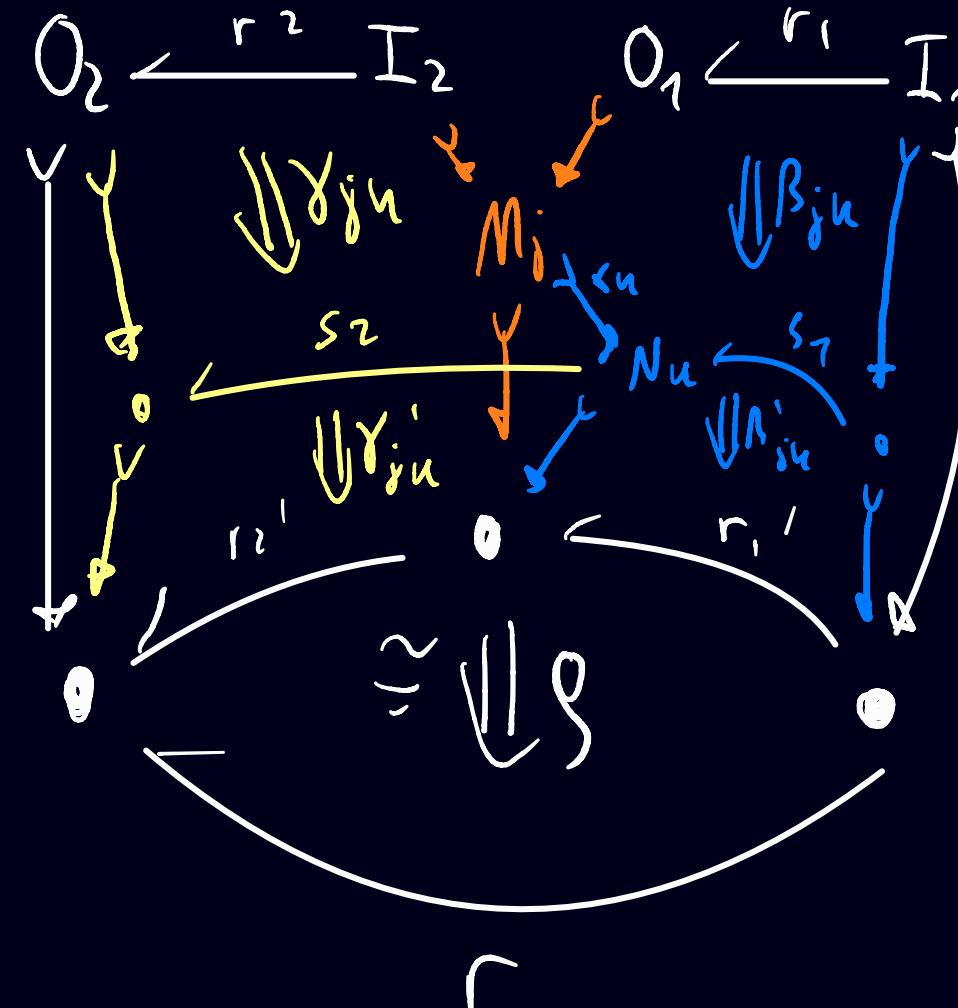


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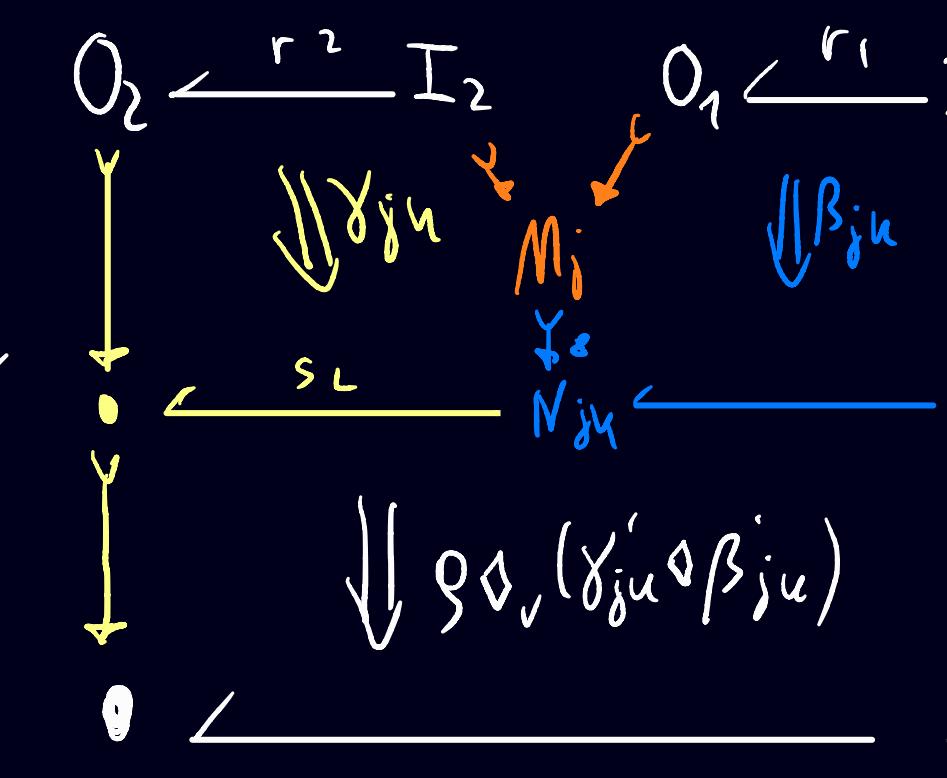


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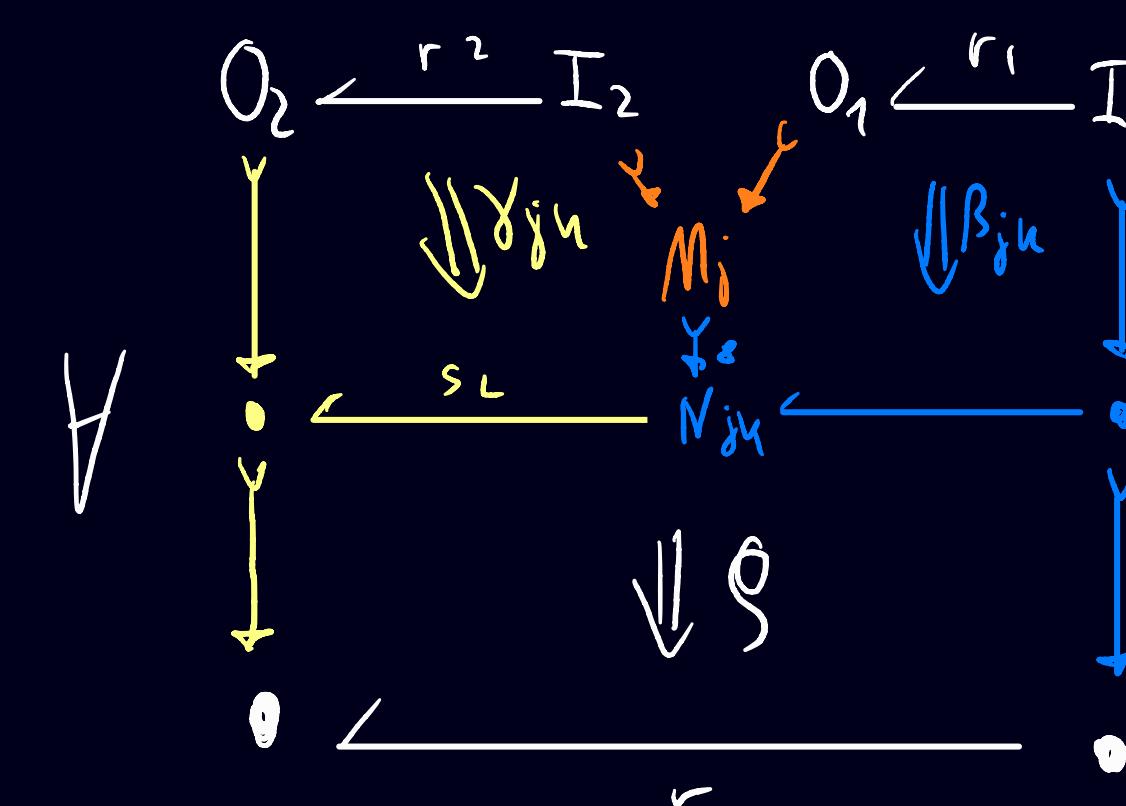
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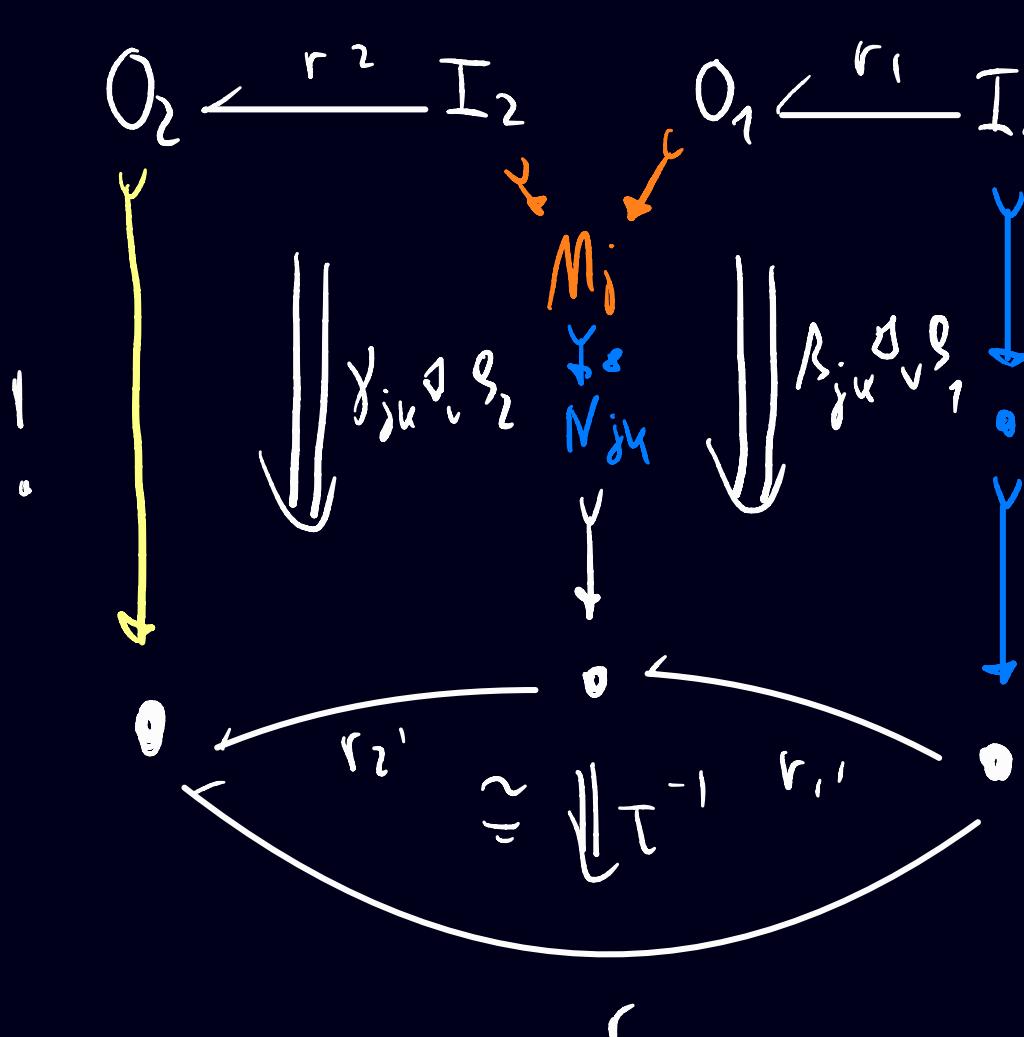
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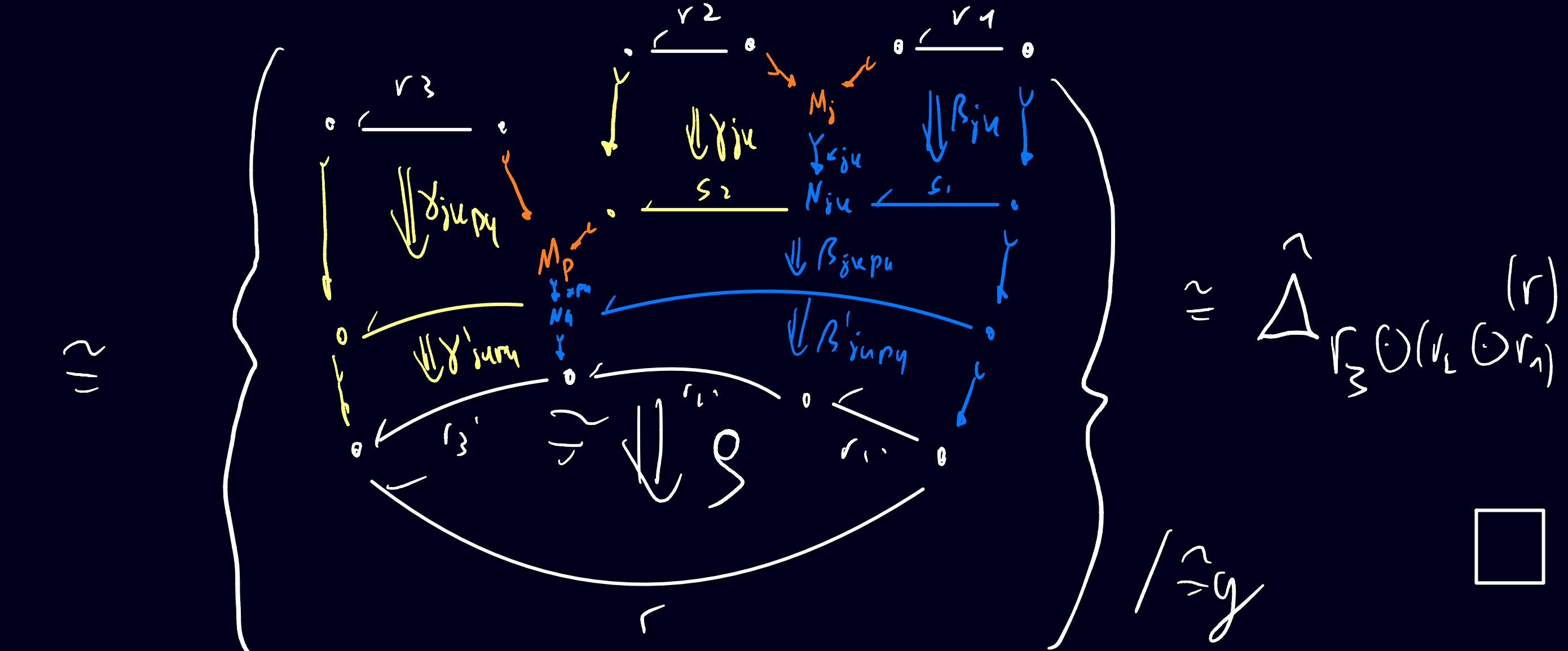
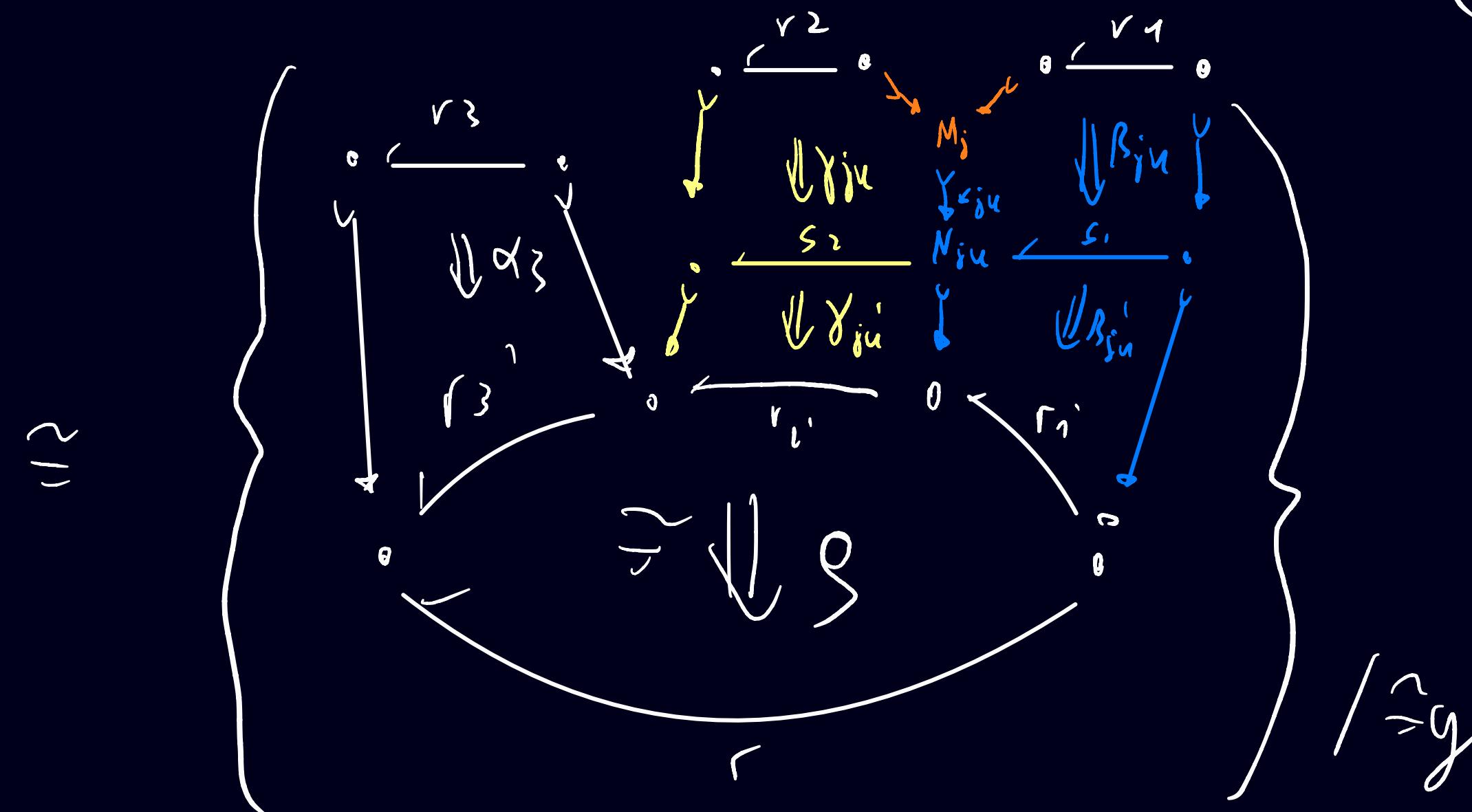
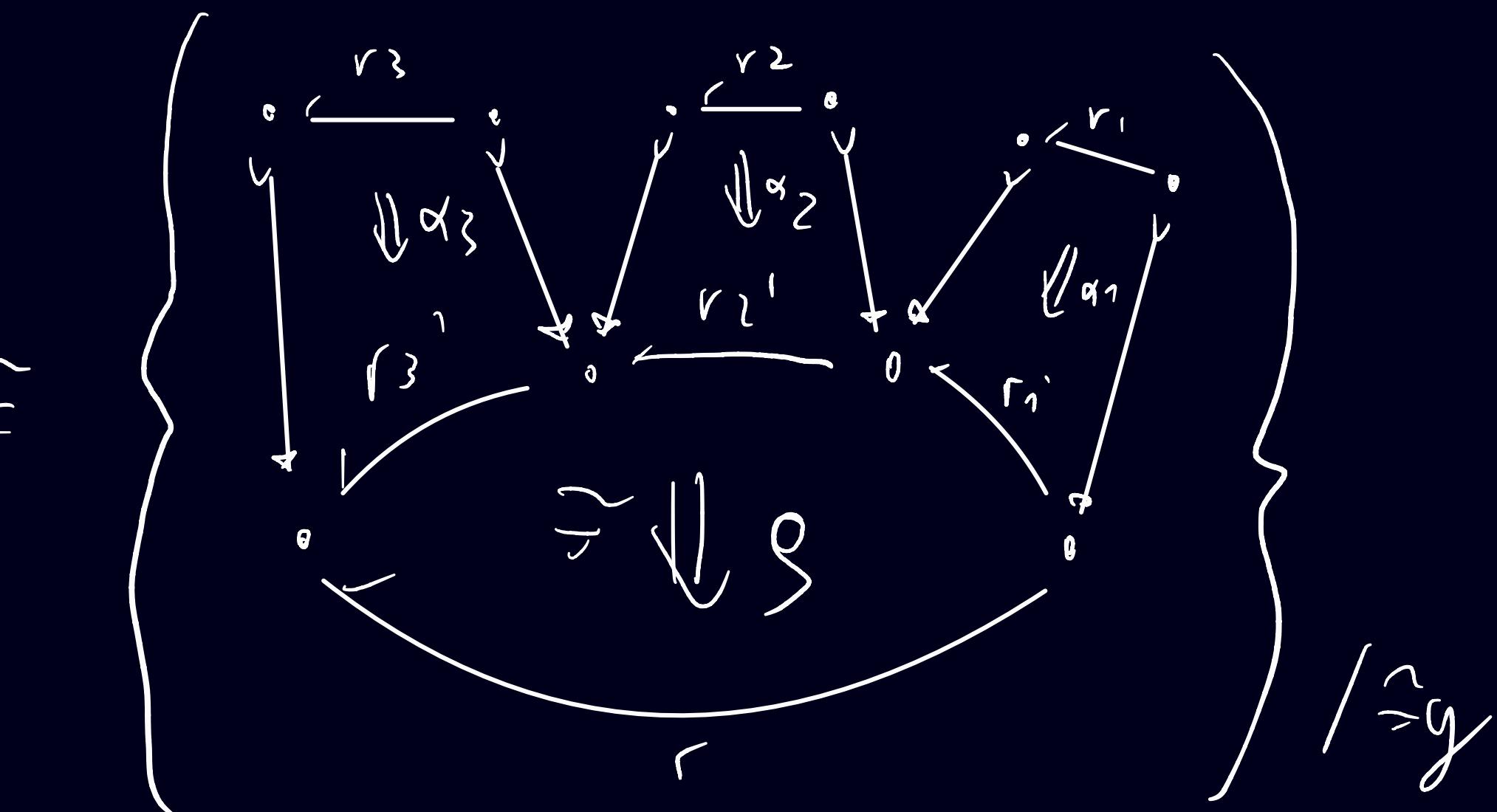


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22 CLAIM:  $\widehat{\Delta}_{\Gamma_3 \circ (\Gamma_2 \circ \Gamma_1)}(r) \cong \widehat{\Delta}_{(\Gamma_3 \circ \Gamma_2) \circ \Gamma_1}(r)$

PROOF (SKETCH):

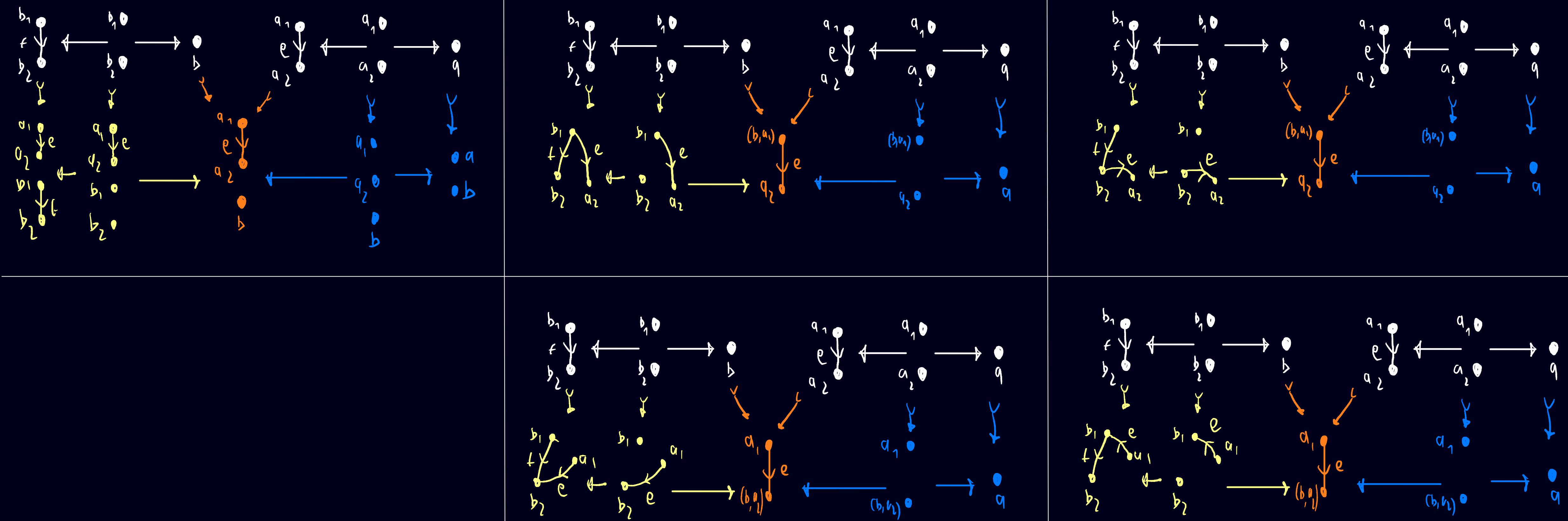
$$(\widehat{\Delta}_{\Gamma_3} * \widehat{\Delta}_{\Gamma_2} * \widehat{\Delta}_{\Gamma_1})(r) \cong$$



23

## EXAMPLE : SELF-COMPOSITIONS OF THE REWRITING RULE

$$\text{``} S \left( \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right) \leftarrow \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \rightarrow \bullet \right)^{\circ 2} = S \left( \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right) \leftarrow \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \rightarrow \bullet \right) + 2 S \left( \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right) \leftarrow \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \rightarrow \bullet \right) + S \left( \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right) \leftarrow \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \rightarrow \bullet \right) + S \left( \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right) \leftarrow \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \rightarrow \bullet \right)$$

5 CONTRIBUTIONS TO  $\hat{A}_{r_2, 0 r_1}$ :

24 COUNTING REWRITING SEQUENCES

$$\text{``} g(S(r)) | X \text{''} = g(S(r))g(S(X \leftarrow \emptyset)) | \emptyset = \sum_{\alpha} g(S(\Gamma_\alpha(X) \leftarrow \emptyset)) | \emptyset \text{''}$$

$$(\hat{\Delta}_r * \hat{\Delta}_{X \leftarrow \emptyset})(Y \leftarrow \emptyset) \cong \left\{ \begin{array}{c} O \xleftarrow{I} \quad X \leftarrow \emptyset \\ \downarrow \beta \quad \downarrow \chi \\ Z \leftarrow \emptyset \\ \cong \downarrow \gamma \\ Y \xrightarrow{\cong} \emptyset \end{array} \right\} / \cong_g \quad \cong \left\{ \begin{array}{c} O \xleftarrow{I} \quad X \leftarrow \emptyset \\ \downarrow \beta \quad \cong \downarrow \chi \\ Z \leftarrow \emptyset \\ \cong \downarrow \gamma \\ Y \xrightarrow{\cong} \emptyset \end{array} \right\} / \cong_g$$

$\hookrightarrow$  is a MOPF  $\Rightarrow$  "lifts" is os!

$$\cong \left\{ \begin{array}{c} O \xleftarrow{I} \quad X \leftarrow \emptyset \\ \downarrow \beta \quad \downarrow \chi \\ Z \leftarrow \emptyset \\ \cong \downarrow \gamma \\ Y \xrightarrow{\cong} \emptyset \end{array} \right\} = \left\{ \begin{array}{c} O \xleftarrow{I} \\ \downarrow \alpha \\ Y \xrightarrow{\cong} \emptyset \end{array} \right\} / \cong_g \quad \checkmark$$

## OUTLOOK

► " # OF WAYS TO REWRITE  $X$  VIA APPLYING RULE  $r$  ":

$$\int_{Y \in \text{ID}_0} (\hat{\Delta}_r * \hat{\Delta}_{X \leftarrow \emptyset})(Y \leftarrow \emptyset) \simeq \coprod_{\alpha \in \text{ID}_1} \left\{ \begin{array}{c} O \xleftarrow{r} I \\ \downarrow \alpha \\ Y \xleftarrow{s} X \end{array} \right\} / \sim_Y$$

WHERE :  $\forall \begin{array}{c} Y \leftarrow s \\ \downarrow \simeq \downarrow \gamma \\ Y' \leftarrow s' \end{array} X : \begin{array}{c} O \xleftarrow{r} I \\ \downarrow \alpha \\ Y \xleftarrow{s} X \end{array} \sim_Y \begin{array}{c} O \xleftarrow{r} I \\ \downarrow \alpha \\ Y' \xleftarrow{s'} X \end{array}$

## OUTLOOK

► " # OF WAYS TO REWRITE  $X$  VIA APPLYING RULE  $r$  ":

$$\int_{Y \in \text{ID}_0} (\hat{\Delta}_r * \hat{\Delta}_{X \leftarrow \emptyset})(Y \leftarrow \emptyset) \equiv \coprod_{\alpha \in \text{ID}_1} \left\{ \begin{array}{c} O \xleftarrow{r} I \\ \downarrow \alpha \\ Y \xleftarrow{s} X \end{array} \right\} / \sim_Y$$

WHERE :  $\forall \begin{array}{c} Y \leftarrow s \\ \downarrow \sim \Downarrow \gamma \\ X \end{array} : \begin{array}{c} O \xleftarrow{r} I \\ \downarrow \alpha \\ Y \xleftarrow{s} X \end{array} \sim_Y \begin{array}{c} O \xleftarrow{r} I \\ \downarrow \sim \Downarrow \gamma \\ Y' \xleftarrow{s'} X \end{array}$

► STARTING MARCH 2023: ANR PROJECT COREACT

[coreact.wiki](http://coreact.wiki)

Cog-based Rewriting: towards Executable Applied Category Theory

- "# OF WAYS TO REWRITE  $X$  VIA APPLYING RULE  $r$ :

$$\sum_{Y \in ID_0} (\hat{\Delta}_r * \hat{\Delta}_{X \leftarrow \emptyset})(Y \leftarrow \emptyset) \equiv \coprod_{\alpha \in ID_1} \left\{ \begin{array}{c} O \xleftarrow{r} I \\ \downarrow \alpha \downarrow \\ Y \xleftarrow{s} X \end{array} \right\} / \sim_Y$$

THANK  
YOU!

WHERE :  $\forall \begin{array}{c} Y \xleftarrow{s} \\ \downarrow \sim \downarrow \\ Y' \xleftarrow{s'} \end{array} X : \begin{array}{c} O \xleftarrow{r} I \\ \downarrow \alpha \downarrow \\ Y \xleftarrow{s} X \end{array} \sim_Y \begin{array}{c} O \xleftarrow{r} I \\ \downarrow \alpha \downarrow \\ Y' \xleftarrow{s'} X \end{array}$

- STARTING MARCH 2023: ANR PROJECT COREACT

[coreact.wiki](http://coreact.wiki)

Coq-based Rewriting: towards Executable Applied Category Theory