Operads as double functors

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Torino

Virtual Double Categories Workshop
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Table of Contents

Motivations

Aim

A more natural approach to colored operads
(symmetric multicategories).

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Achievement

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It this approach, double categories play a pivotal role.

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Technical tools

It this approach, double categories play a pivotal role.

Byproduct

This is how I learned to love double categories.

Motivations

The operad of sets

- Objects are sets.
- Arrows $f : X_1; \cdots ; X_n \to Y$
  are maps which take a list of elements
  $x_1; \cdots ; x_n$ (with $x_i \in X_i$)
  and give an element $y \in Y$.

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  $x_1; \cdots ; x_n$ (with $x_i \in X_i$)
  and give an element $y \in Y$.

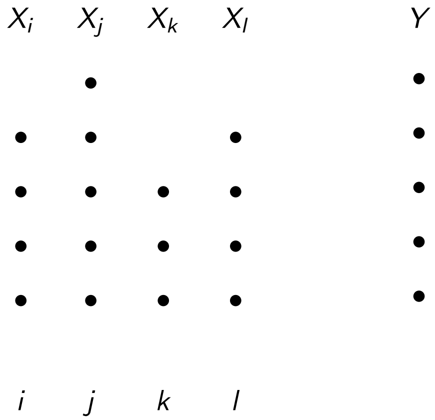
But order doesn't really matter...

Motivations

More naturally

- Objects are sets.
- Arrows $f : (X_i)_{i \in A} \to Y$
  are maps which take a family of elements
  $(x_i)_{i \in A}$ (with $x_i \in X_i$)
  and give an element $y \in Y$.

Motivations



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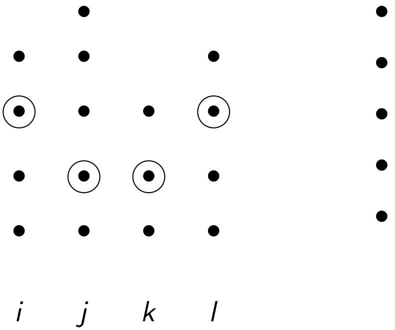

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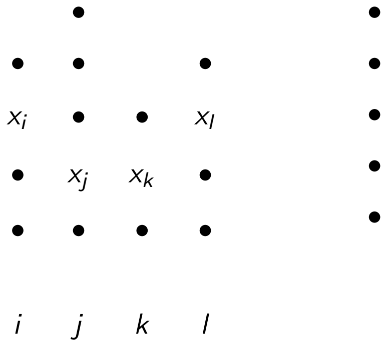

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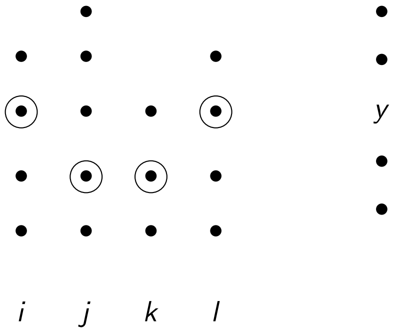

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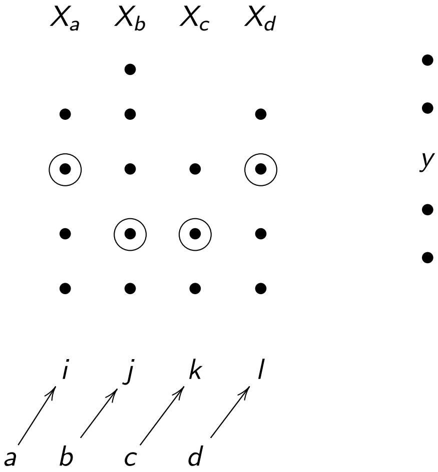

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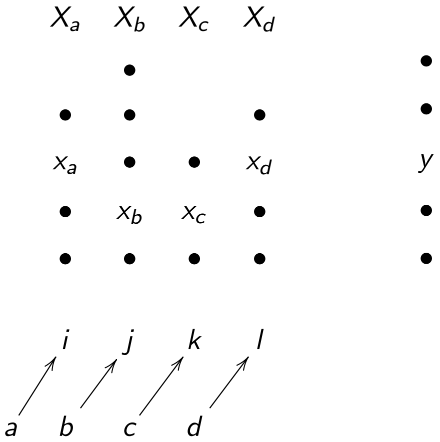
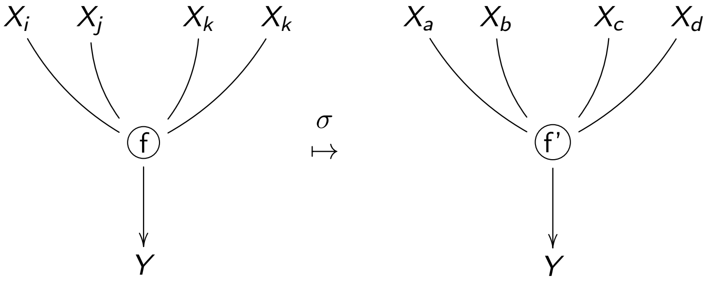


Motivations

The maps f and f' are the same, up to the indexing of domains.



But indexing is necessary for composing arrows.

Motivations

Reindexing

Given a bijection $\sigma : B \to A$,

- for any family of sets indexed by A, we get by composition a family (essentially the same) indexed by B.
- for any arrow from first family as domain, we get an arrow (essentially the same) from the reindexed family.

Motivations

We will see that considering families of arrows
(instead of single arrows)
we so get a double category whose cells
expresses symmetry of maps with respect to reindexing.

$$\begin{tikzcd}
(X'_j)_{j\in B} \arrow[d, "f'"'] & (X_i)_{i\in A} \arrow[l, "\sigma^*"'] \arrow[d, "f"] \\
(Y'_s)_{s\in D} & (Y_t)_{t\in C} \arrow[l, "\rho^*"']
\end{tikzcd}$$

Motivations

Main idea

To properly understand operads,
we need a framework allowing to express symmetry of arrows
and yet retaining the possibility of composing them.

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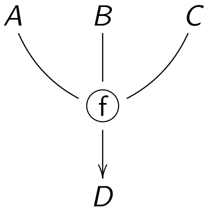
To properly understand operads,
we need a framework allowing to express symmetry of arrows
and yet retaining the possibility of composing them.

Double categories provide this framework.

Table of Contents

Operads (arity)

A ternary arrow



Operads (arity)

An unary arrow

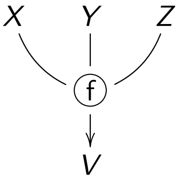
$$\begin{tikzcd} A \arrow[d, no head] \\ f \arrow[d] \\ D \end{tikzcd}$$

Operads (arity)

A nullary arrow

$$\begin{tikzcd} f \arrow[d] \\ D \end{tikzcd}$$

Operads (composition)



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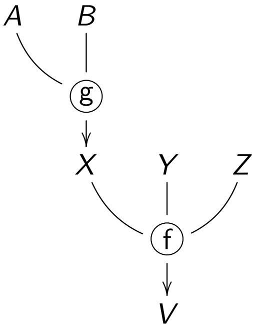

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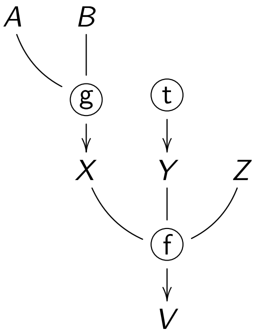

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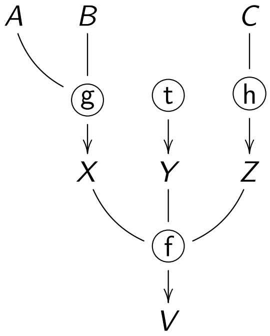

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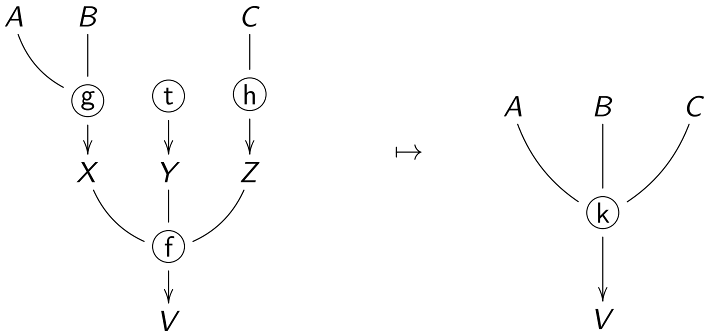
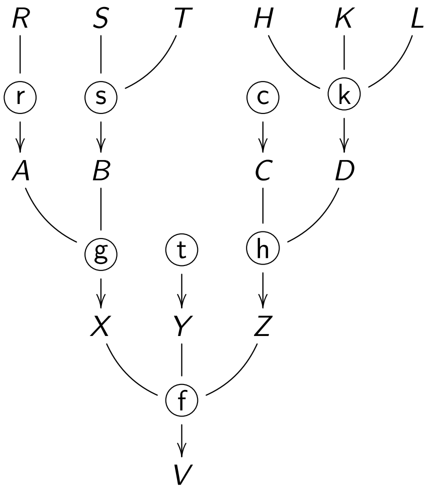


Operads (associativity)



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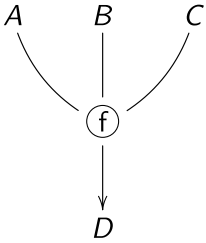

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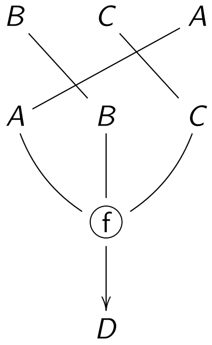

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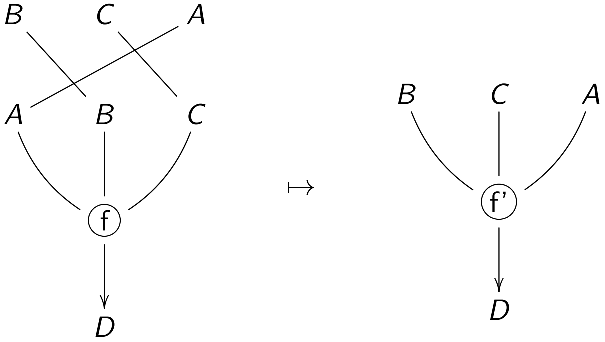
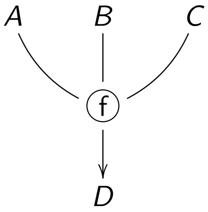


Operads: Classical approach

The domain of an arrow is a list $X : \mathsf{n} \to \mathcal{O}_0$
of objects in $\mathcal{O}$



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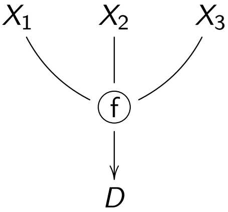

Operads: Classical approach

Arrows can be transported along permutations σ
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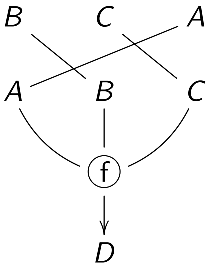

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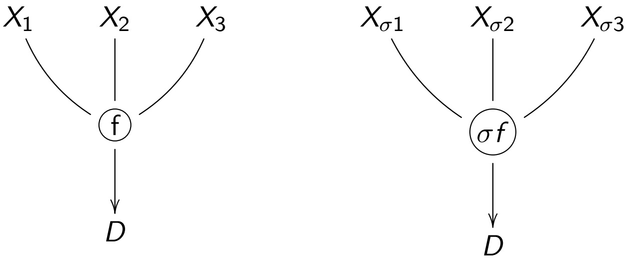

Operads: Classical approach

Axioms

- Composition and associativity.
- Permutations act on arrows.
- The action is compatible with composition.

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- Permutations act on arrows.
- The action is compatible with composition.

When made explicit, these conditions assume
a rather unwieldy form (involving for instance block permutations)
showing clearly the drawback of the skeletal choice for indexing.

Operads: Examples

Any symmetric monoidal category gives an operad $\mathcal{O}$,
whose arrows $f : X_1; \cdots ; X_n \to Y$
are arrows $f : X_1 \otimes \cdots \otimes X_n \to Y$.

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In particular, one can consider a cartesian monoidal category.

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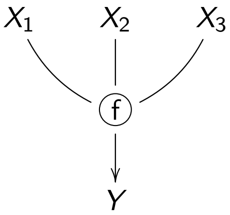
In particular, one can consider a cartesian monoidal category.

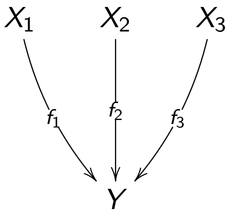
Starting with $(\mathrm{Set}, \times, 1)$ we get the operad of sets.

Operads: Examples

Starting with a cocartesian monoidal category $(\mathcal{C}, +, 0)$ we get the sequential operad $\mathcal{C}_{\blacktriangleright}$ whose maps are sequences of concurrent arrows in $\mathcal{C}$ (discrete cocones).

Of course, one can consider $\mathcal{C}_{\blacktriangleright}$ for any category $\mathcal{C}$.





Operads: Aim

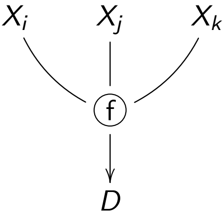
Non-skeletal operads

These examples point toward a more natural notion of operad:
domain of maps are families indexed by arbitrary finite sets
(rather than by sets in a skeleton $\mathbb{N}$ of Set_f)
and the reindexings of these families act on arrows as well.

Table of Contents

Operads: non-skeletal approach

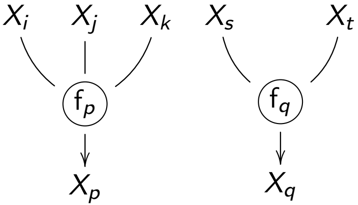
The domain of an arrow is an arbitrary family
$X : A \to \mathcal{O}_0$ of objects



The order of the objects in the domain is fictitious,
we should think of it as floating in a three dimensional space.

Operads: non-skeletal approach

Taking in account composition,
we need to consider families of arrows.



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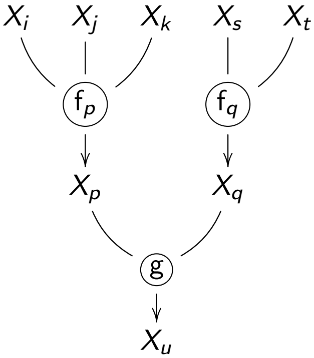
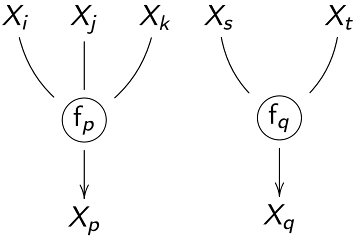


Operads: non-skeletal approach

Any family of arrows has an underlying mapping



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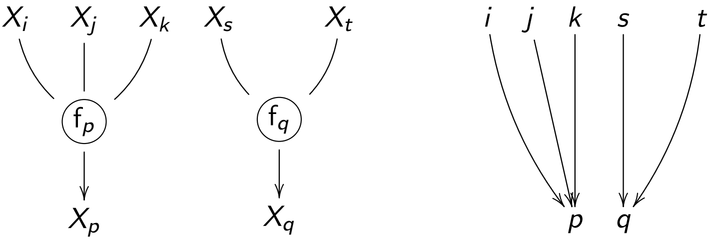

Operads: non-skeletal approach

Question

So, what do we get by considering families of arrows in a non-skeletal operad $\mathcal{O}$?

Answer

They form a category $\mathcal{D}_{\mathcal{O}}$ over finite sets: the functor $d : \mathcal{D}_{\mathcal{O}} \to \mathsf{Set}_f$ keeps track of the indexing of objects and maps.

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The category $\mathcal{D}_{\mathcal{O}}$, in its skeletal form, appears in the literature under several names, such as "operator" or "envelope" category of $\mathcal{O}$, or the free PROP generated by $\mathcal{O}$.

Operads: non-skeletal approach

Question

What further structure is inherited by $\mathcal{D}_{\mathcal{O}}$
from the operad structure $\mathcal{O}$?

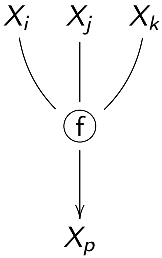
Answer

Its maps (families of arrows in $\mathcal{O}$)
can be reindexed along pullbacks in Set_f.

Reindexing along pullbacks

For instance, we can reindex a single arrow or a family of arrows along pullbacks whose horizontal sides are isomorphisms.

We so obtain the same arrow (up to indexing).



$$\begin{tikzcd}
i \arrow[dddr, bend right] & & \\
& j \arrow[dd] & \\
& & k \arrow[dl, bend left] \\
& p &
\end{tikzcd}$$

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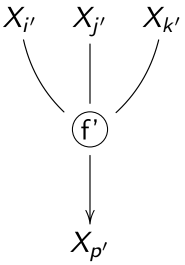
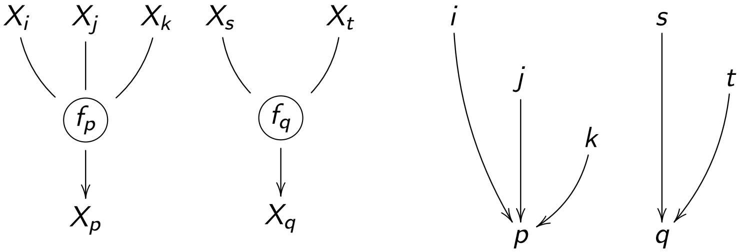


$$
\begin{tikzcd}
i' \arrow[rr] \arrow[dddr, bend right] & & i \arrow[dddr, bend right=10] & & \\
 & j' \arrow[rr] \arrow[dd] & & j \arrow[dd] & \\
 & & k' \arrow[rr] \arrow[dl, bend left] & & k \arrow[dl, bend left] \\
 & p' \arrow[rr] & & p &
\end{tikzcd}
$$

Reindexing along pullbacks

Or we can reindex along injective mappings
to pick up just some arrows of the family.



Reindexing along pullbacks

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$$
\begin{tikzcd}
 & i \arrow[dddr, bend right] & & & s \arrow[ddd] & \\
 & & j \arrow[dd] & & & t \arrow[ddl, bend left] \\
 & & & k \arrow[dl, bend left] & & \\
p' \arrow[rr] & & p & & q &
\end{tikzcd}
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& & k' \arrow[rr] \arrow[dl, bend left] & & k \arrow[dl, bend left] & & \\
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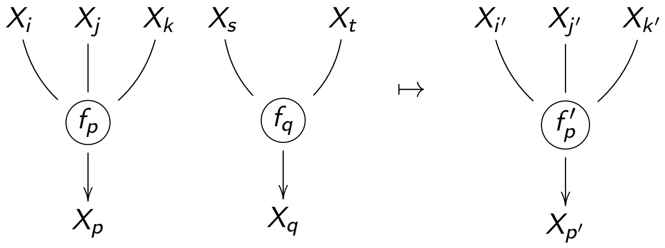
Reindexing along pullbacks



Reindexing along pullbacks

Or we can reindex along more general mappings
to obtain copies of some of the arrows in a family.

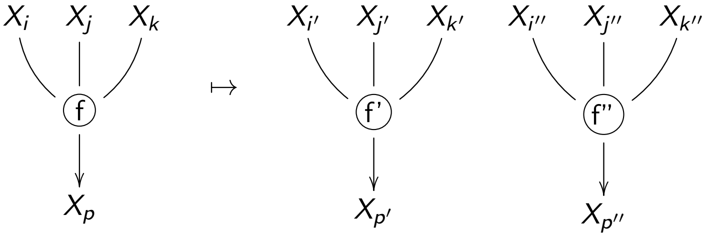


Table of Contents

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Answer

- They form a category over finite sets $d : \mathcal{D}_{\mathcal{O}} \to \mathsf{Set}_f$.
- For any pullback in Set_f there is a reindexing of objects and of maps over it.
- The reindexing is compatible with composition.

Here they come double categories

For any pullback in Set_f there is a reindexing over it.

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[d, "df"] \\ B \end{tikzcd}$$

Here they come double categories

For any pullback in Set_f there is a reindexing over it.

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, "s"] \arrow[d, "k"'] \arrow[dr, phantom, "\mathrm{pb}"] & A \arrow[d, "df"] \\ B' \arrow[r, "t"] & B \end{tikzcd}$$

Here they come double categories

For any pullback in Set_f there is a reindexing over it.

$$\begin{tikzcd} s^*X \arrow[r, dashed] & X \arrow[d, "f"] \\ t^*Y \arrow[r, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, "s"] \arrow[d] & A \arrow[d, "df"] \\ B' \arrow[r, "t"] & B \end{tikzcd}$$

the dashed arrows indicate that s^*X and t^*Y are the reindexing of the families X and Y along s and t.

Here they come double categories

For any pullback in Set_f there is a reindexing over it.

$$\begin{tikzcd} s^*X \arrow[r, dashed] \arrow[d, dotted, "f'"'] & X \arrow[d, "f"] \\ t^*Y \arrow[r, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, "s"] \arrow[d, "df'=k"'] & A \arrow[d, "df"] \\ B' \arrow[r, "t"] & B \end{tikzcd}$$

the vertical dotted arrow is uniquely determined.

Here they come double categories

The reindexing is compatible with composition:
reindexing squares can be composed (also) vertically.

Here they come double categories

The reindexing is compatible with composition:
reindexing squares can be composed (also) vertically.

The reindexing squares are the cells of a double category $\mathbb{D}_{\mathcal{O}}$ over $\mathbb{P}\mathrm{b}\,(\mathrm{Set}_{\mathrm{f}})$, the double category of pullbacks in finite sets.

The double category for an operad

The double category $\mathbb{D}_{\mathcal{O}}$

- Objects are finite families $A \to \mathcal{O}_0$ of objects in $\mathcal{O}$.

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- Objects are finite families $A \to \mathcal{O}_0$ of objects in $\mathcal{O}$.
- Vertical arrows (proarrows) are the maps of $\mathcal{D}$, that is families of arrows in $\mathcal{O}$.
- Horizontal arrows are the arrows of the discrete fibration $\mathcal{O}_0^A$, the family fibration on the set $\mathcal{O}_0$.

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- Objects are finite families $A \to \mathcal{O}_0$ of objects in $\mathcal{O}$.
- Vertical arrows (proarrows) are the maps of $\mathcal{D}$, that is families of arrows in $\mathcal{O}$.
- Horizontal arrows are the arrows of the discrete fibration $\mathcal{O}_0^A$, the family fibration on the set $\mathcal{O}_0$.
- Cells are given by reindexing of (families of) arrows.

Operads as discrete fibrations

The reindexing of objects (along mappings)
and of maps (along pullbacks)
can be expressed by saying that the double functor
$d : \mathbb{D}_{\mathcal{O}} \to \mathbb{P}\mathrm{b}\,(\mathrm{Set}_{\mathrm{f}})$ is a discrete double fibration.

discrete double fibration (Lambert, 2021)

That is, both the components
$d_0 : \mathbb{D}_0 \to \mathrm{Set}_f$ and $d_1 : \mathbb{D}_1 \to \mathrm{PbSet}_f$
are discrete fibrations.

Operads as discrete fibrations

To assure that a double discrete fibration $d : \mathbb{D} \to \mathbb{P}\mathrm{b}\left(\mathrm{Set_f}\right)$
comes indeed from an operad $\mathcal{O}$,
it should satisfy the following glueing condition:

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Glueing condition for objects

If X and Y are objects in $\mathbb{D}$ over A and B respectively,
there is a unique object Z over a sum $C = A + B$ in Set_f
which restricts to X and Y along injections.

Operads as discrete fibrations

To assure that a double discrete fibration $d : \mathbb{D} \to \mathbb{P}\mathrm{b}\,(\mathrm{Set_f})$
comes indeed from an operad $\mathcal{O}$,
it should satisfy the following glueing condition:

Glueing condition for objects

If X and Y are objects in $\mathbb{D}$ over A and B respectively,
there is a unique object Z over a sum $C = A + B$ in Set_f
which restricts to X and Y along injections.

Glueing condition for maps

If f and g are maps over s and t respectively,
there is a unique map h over a sum $r = s + t$ in Set_f^2
which restricts to f and g along injections
(which are pullbacks in Set_f).

Operads as discrete fibrations

The glueing condition for objects
assures that the horizontal part d^h of $d : \mathbb{D} \to \mathbb{P}\mathrm{b}\,(\mathrm{Set_f})$
is indeed the family fibration on $\mathcal{O}_0$
(where $\mathcal{O}_0$ is the fiber over a terminal set).

The glueing condition for maps
assures that a proarrow in $\mathbb{D}$ (that is, an object in $\mathbb{D}_1$)
is indeed a family of “single arrows”, that is of proarrows
with the codomain indexed by a terminal set.

Operads as discrete fibrations

We so arrive to our definition of operad.

Operads as discrete fibrations

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Non-skeletal notion of operad

An operad is a double discrete fibration $d : \mathbb{D} \to \mathbb{P}\mathrm{b}\,(\mathrm{Set_f})$ satisfying the glueing condition.

Operads as discrete fibrations

We so arrive to our definition of operad.

Non-skeletal notion of operad

An operad is a double discrete fibration $d : \mathbb{D} \to \mathbb{P}\mathrm{b}\,(\mathrm{Set}_{\mathrm{f}})$ satisfying the glueing condition.

Note that $\mathbb{D}$ is a strict double category,
and that $d : \mathbb{D} \to \mathbb{P}\mathrm{b}\,(\mathrm{Set}_{\mathrm{f}})$ is a strict double functor.

Operads as discrete fibrations

This notion of non-skeletal operad
is essentially equivalent to the classical one.

Operads as discrete fibrations

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is essentially equivalent to the classical one.

Indeed, one can define morphisms $\mathcal{O} \to \mathcal{O}'$ of non-skeletal operads
as double functors $\mathbb{D}_{\mathcal{O}} \to \mathbb{D}_{\mathcal{O}'}$ over Set_f.
The category of non-skeletal operads is then equivalent
to the category of classical operads.

Operads as discrete fibrations

Note for example that the compatibility condition
between permutations action and composition
in the classical definition of operads,
becomes now simply an instance vertical composition of cells:

$$
\begin{tikzcd}
s^*X \arrow[r, dashed] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
t^*Y \arrow[r, dashed] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
Z \arrow[r, dashed] & Z
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A' \arrow[r, "s"] \arrow[d] & A \arrow[d] \\
B' \arrow[r, "t"] \arrow[d] & B \arrow[d] \\
1 \arrow[r] & 1
\end{tikzcd}
$$

Operads as discrete fibrations

Confronting two ways of expressing compatibility with composition (figure from Leinster book).

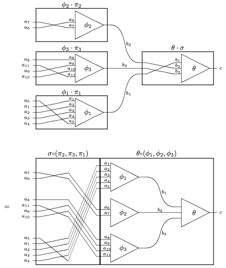

Figure 2-H: Symmetric multicategory axiom

Operads as discrete fibrations

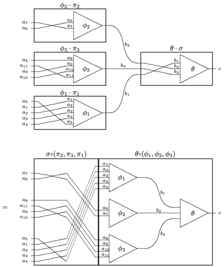

Figure 2-H: Symmetric multicategory axiom

$$
\begin{tikzcd}
s^*X \arrow[r, dashed] \arrow[d, "f'"] \arrow[dd, bend right=60, "(gf)'"'] & X \arrow[d, "f"] \\
t^*Y \arrow[r, dashed] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
Z \arrow[r, dashed] & Z
\end{tikzcd}
$$

Table of Contents

Operads as double functors

Double Grothendieck correspondence (Lambert 2021, Paré 2011)

Similarly to the classical case,
double discrete fibrations $d : \mathbb{D} \to \mathbb{A}$ correspond to
lax functors $F : \mathbb{A}^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$
to the (non-strict) double category of mappings and spans.

Operads as double functors

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lax functors $F : \mathbb{A}^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$
to the (non-strict) double category of mappings and spans.

Universal property of the monoid construction (Cruttwell & Shulman 2010)

Since the monoid construction on $\mathbb{S}\mathrm{pan}$ gives $\mathbb{C}\mathrm{at}$,
the double category of functors and profunctors,
lax functors $F : \mathbb{A}^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$ correspond to
normal lax functors $F' : \mathbb{A}^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$.

Operads as double functors

Thus, given an non-skeletal operad

$$d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}}$$

there are corresponding lax functors

$$F_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$$

$$F'_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$$

Operads as double functors

Furthermore it is easy to see that the glueing condition on $d_{\mathcal{O}}$ corresponds to the fact that $F_{\mathcal{O}}$ and $F'_{\mathcal{O}}$ preserve products.

Recall that a double category $\mathbb{A}$ has products
if $\mathbb{A}_0$ and $\mathbb{A}_1$ both have products,
preserved by the source and target functors $\mathbb{A}_1 \to \mathbb{A}_0$.

And that a double functor $F : \mathbb{A} \to \mathbb{B}$ preserves products
if $F_0 : \mathbb{A}_0 \to \mathbb{B}_0$ and $F_1 : \mathbb{A}_1 \to \mathbb{B}_1$
both preserve products.

Operads as double functors

Furthermore it is easy to see that the glueing condition on $d_{\mathcal{O}}$ corresponds to the fact that $F_{\mathcal{O}}$ and $F'_{\mathcal{O}}$ preserve products.

Products in $(\mathbb{P}\mathrm{b}\,\mathrm{Set}_{\mathrm{f}})^{\mathrm{op}}$ are sums in $\mathbb{P}\mathrm{b}\,\mathrm{Set}_{\mathrm{f}}$,
that is pair of commuting squares whose horizontal sides are sums in Set_f (since Set_f is extensive).

$$
\begin{tikzcd}
A_1 \arrow[r, "i"] \arrow[d, "s"'] & A_1 + A_2 \arrow[d, "s+t"] & A_2 \arrow[l, "j"'] \arrow[d, "t"] \\
B_1 \arrow[r, "i"'] & B_1 + B_2 & B_2 \arrow[l, "j"]
\end{tikzcd}
$$

What is an operad?

Summarizing

A (non-skeletal) operad $\mathcal{O}$
can be defined in three equivalent ways:

1. A double discrete fibration with glueing
   $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}}$.
2. A product-preserving lax functor
   $F_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$.
3. A product-preserving normal lax functor
   $F'_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$.

What is an operad?

Summarizing

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can be defined in three equivalent ways:

1. A double discrete fibration with glueing
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3. A product-preserving normal lax functor
$F'_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$.

Each definition gives a different point of view
best suited to treat some aspects of operads.

Operads as double functors (explicitly)

The functor $F_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$
takes a set $A \in \mathsf{Set}_f$ to the set $\mathcal{O}_0^A$,
and a mapping $t : A \to B$ to the span
whose vertex is formed by all families of arrows over t
and whose legs are given by domain and codomain.

Operads as double functors (explicitly)

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The functor $F'_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$
takes a set $A \in \mathsf{Set}_f$ to the category $\mathcal{O}_1^A$,
(where $\mathcal{O}_1$ is the category of unary arrows in $\mathcal{O}$)
and a mapping $t : A \to B$ to the profunctor $\overline{t}$ such that
$\overline{t}(X, Y)$ is formed by all families of arrows $f : X \to Y$ over t.

Table of Contents

Special operads

Given an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}}$,
the horizontal part $d^h_{\mathcal{O}} : \mathbb{D}_0 \to \mathsf{Set}_f$ is forced to be
the discrete family fibration on the set $\mathcal{O}_0$
(by the glueing or product preserving condition).

Special operads

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Thus, the character of $\mathcal{O}$ is in a sense determined
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Special operads

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by the vertical part $d^v_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}_f$.

A similar consideration holds for the
lax double functor form $F'_{\mathcal{O}}$ of the operad $\mathcal{O}$:
their character is determined by the vertical part,
that is the lax functor of bicategories $(F')^v_{\mathcal{O}} : \mathsf{Set}_f \to \mathbb{P}\mathrm{rof}$.

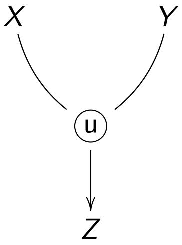
Symmetric monoidal categories

The vertical part $d^v_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}_f$ is an opfibration
if and only if $\mathcal{O}$ has tensor products.
That is, it is a symmetric monoidal category
in its universal form
(the representable multicategories of Hermida and Leinster).

Symmetric monoidal categories

Universal arrows

The opcartesian arrows for $d^{v}_{\mathcal{O}}$ are
the universal arrows defining tensor products.



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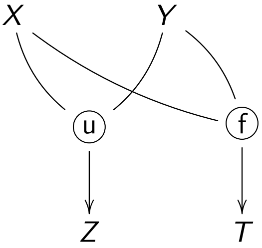

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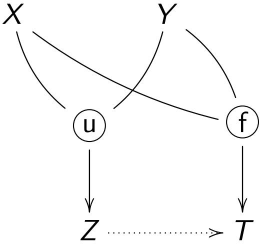

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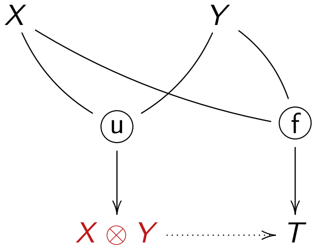

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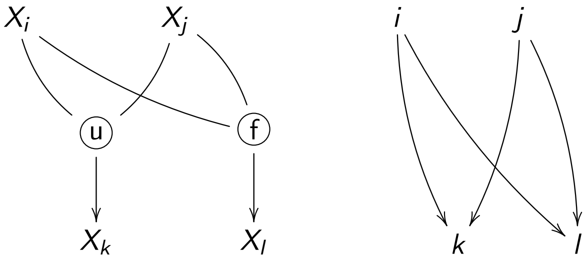

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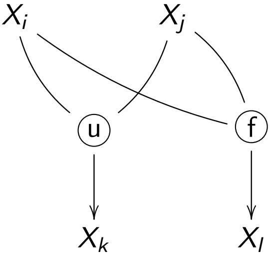

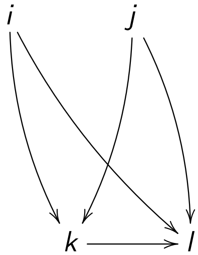

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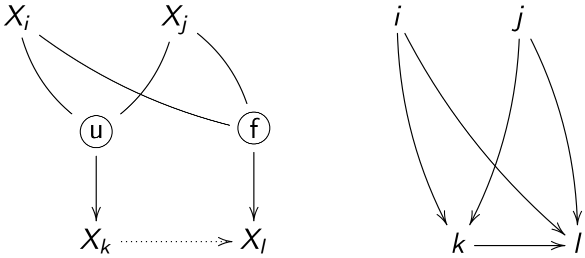


Symmetric monoidal categories as double functors

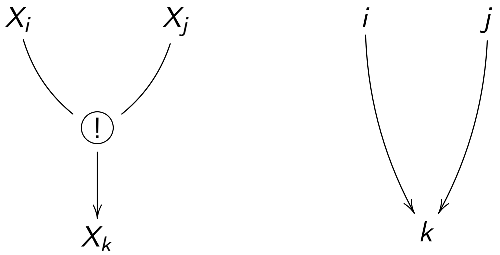
An operad $F_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$,
is a symmetric monoidal category
if and only if its vertical part $F^{v}_{\mathcal{O}} : \mathsf{Set}_f \to \mathsf{Prof}$,
(in general, a lax functor of bicategories)
lands in representable profunctors.

Commutative monoids

The vertical part $d^v_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}_f$ is a discrete opfibration if and only if $\mathcal{O}$ is a commutative monoid.
That is, it is a discrete symmetric monoidal category.

Commutative monoids

There is exactly one arrow out of any family of objects giving the product of the family.



Commutative monoids

There is exactly one arrow out of any family of objects giving the product of the family.

$$
\begin{tikzcd}
X_i \arrow[dr, dash] & & X_j \arrow[dl, dash] \\
& ! \arrow[d] & \\
& \color{red}{X_i \cdot X_j} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
i \arrow[dr, bend right] & & j \arrow[dl, bend left] \\
& k &
\end{tikzcd}
$$

Commutative monoids

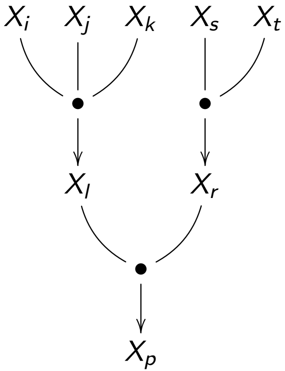
Thus, in this framework a commutative monoid
on the set $\mathcal{O}_0$ consists of

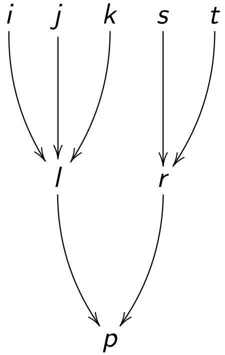
1. A "multiplication" $mX \in \mathcal{O}_0$
   of any finite family of elements $X : A \to \mathcal{O}_0$.
2. Multiplication is closed with respect to composition
   and stable with respect to reindexing.

Commutative monoids

Associativity and commutativity

Multiplication is closed with respect to composition
and stable with respect to reindexing.





Commutative monoids

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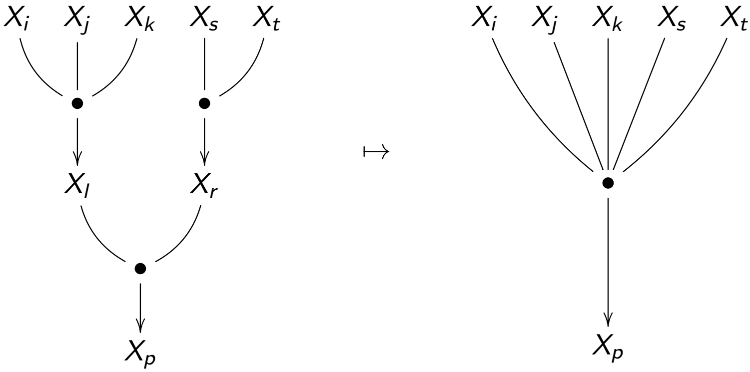

Commutative monoids

Equivalently, a commutative monoid consists of
a discrete fibration $d : \mathbb{D} \to \mathsf{Set}_f$,
which is the family fibration on the set $\mathcal{O}_0$
and a discrete opfibration $d' : \mathbb{D}' \to \mathsf{Set}_f$
with the same objects and wich are compatible:

Transporting an object X along any of the two possible paths over a pullback in the base one gets the same object W.

Commutative monoids

Transporting an object X along any of the two possible paths over a pullback in the base one gets the same object W.

$$X$$

$$\begin{tikzcd}
C \arrow[r] \arrow[dr] & A \arrow[dr] & \\
& D \arrow[r] & B
\end{tikzcd}$$

Commutative monoids

Transporting an object X along any of the two possible paths over a pullback in the base one gets the same object W.

$$
\begin{tikzcd}
 & X \arrow[dr] & \\
 & & Y \\
C \arrow[r] \arrow[dr] & A \arrow[dr] & \\
 & D \arrow[r] & B
\end{tikzcd}
$$

Commutative monoids

Transporting an object X along any of the two possible paths over a pullback in the base one gets the same object W.

$$
\begin{tikzcd}
 & X \arrow[dr] & \\
 & W \arrow[r] & Y \\
C \arrow[r] \arrow[dr] & A \arrow[dr] & \\
 & D \arrow[r] & B
\end{tikzcd}
$$

Commutative monoids

Transporting an object X along any of the two possible paths over a pullback in the base one gets the same object W.

$$\begin{tikzcd}
 & X & \\
 & & \\
C \arrow[r] \arrow[dr] & A \arrow[dr] & \\
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\end{tikzcd}$$

Commutative monoids

Transporting an object X along any of the two possible paths over a pullback in the base one gets the same object W.

$$\begin{tikzcd} Z \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r] \arrow[dr] & A \arrow[dr] & \\ & D \arrow[r] & B \end{tikzcd}$$

Commutative monoids

Transporting an object X along any of the two possible paths over a pullback in the base one gets the same object W.

$$\begin{tikzcd}
Z \arrow[r] \arrow[dr] & X \\
 & W'
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r] \arrow[dr] & A \arrow[dr] & \\
 & D \arrow[r] & B
\end{tikzcd}$$

Commutative monoids

Transporting an object X along any of the two possible paths over a pullback in the base one gets the same object W.

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & X \arrow[dr] & \\ & W' = W \arrow[r] & B \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r] \arrow[dr] & A \arrow[dr] & \\ & D \arrow[r] & B \end{tikzcd}$$

Commutative monoids as double functors

Equivalently, a commutative monoid consists of a (strict) product-preserving double functor $(\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{S}\mathrm{q}\,\mathsf{Set}$.

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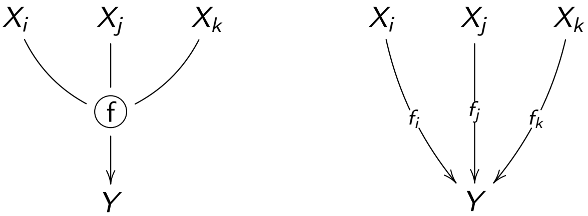
Operads are a lax notion of commutative monoid.

Sequential operads

The vertical part $d^v_{\mathcal{O}} : \mathcal{D} \to \mathrm{Set}_f$ is a fibration
if and only if $\mathcal{O}$ is a sequential operad.



Sequential operads

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The cartesian arrow in $\mathcal{D}$ are those made up
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Sequential operads

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The cartesian arrow in $\mathcal{D}$ are those made up
of identities (or isomorphims) in $\mathcal{C}$.

Cartesian arrows form a "central monoid" in the operad,
which characterize abstractly sequential operads (Pisani 2014).

Cocartesian monoidal categories

The vertical part $d^{v}_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}_f$ is a bifibration
if and only if $\mathcal{O}$ is both monoidal and sequential.
That is, $\mathcal{O}$ is a cocartesian monoidal category
(since universal arrows are colimiting cones).

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Copying and deleting

The well-known characterization of cartesian monoidal categories
is a manifestation of (the dual of) the above fact:
the "copying-deleting" arrows are the cartesian maps of $d^{v}_{\mathcal{O}}$.

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Caution

The term "cartesian" is overworked:
cartesian arrow (of a fibration), cartesian monoidal category, cartesian operad (to be considered later on)...

Exponentiable operads

An operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}}$, is exponentiable
if and only if its vertical part $d^{v}_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}_f$
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That is, if and only if $d^{v}_{\mathcal{O}}$ is a Conduché fibration
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These include fibrations and opfibration, so that
symmetric monoidal categories and sequential operad
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Exponentiable operads coincide with
promonoidal multicategories.

Exponentiable operads as double functors

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(in general, a lax functor of bicategories)
is a pseudofunctor.

Unary operads

The vertical part $d^v_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}_f$
lands in bijective mappings
if and only if $\mathcal{O}$ is unary
that is, all its arrows have arity one.

Table of Contents

Changing the base

Till now, we have presented a non-skeletal approach to operads.
The main advantages are:

- It avoids the introduction of spurious orders, rendering neater the notion.
- We can exploit the language of double categories, to capture in a smooth way various classes of operads and to highlight their connections.

Changing the base

Now, we briefly review a possible generalization, obtained by replacing the base category Set_f with another category $\mathcal{S}$.

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Infintary operads

1. A double discrete fibration with glueing $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}$.
2. A product-preserving lax functor $F_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set})^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$.
3. A product-preserving normal lax functor $F'_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$.

Changing the base: example

Consider a category $\mathcal{C}$ and the family fibration $d : \mathrm{Fam}\,\mathcal{C} \to \mathsf{Set}$ given by $\mathcal{C}^A$; $A \in \mathsf{Set}$ is the vertical part of an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}$ (the infinitary sequential operad on $\mathcal{C}$).

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If $\mathcal{C}$ has small sums the family fibration $d : \mathrm{Fam}\,\mathcal{C} \to \mathsf{Set}$ is a bifibration.

We thus have a notion of infinitary monoidal category, namely, an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}$ such that the vertical part $d^{v}_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}$ is an opfibration.

Changing the base: example

Of course, we also have a notion of infinitary commutative monoid, namely, an operad $d_{\mathcal{O}}$ on Set such that the vertical part d^v is a discrete opfibration.

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And taking isomorphism classes of an infinitary monoidal category one gets an infinitary commutative monoid.

Changing the base: example

Of course, we also have a notion of infinitary commutative monoid, namely, an operad $d_{\mathcal{O}}$ on Set such that the vertical part d^v is a discrete opfibration.

And taking isomorphism classes of an infinitary monoidal category one gets an infinitary commutative monoid.

This is a way to make it precise the idea that universal sums or products can be "decategorified" to give algebraic structures, not only in the finite case.

Changing the base: example

More generally, we have a notion of monoidal category on $\mathcal{S}$, namely, an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathcal{S}$ such that the vertical part is an opfibration, and such that opcartesian arrows are stable with respect to reindexing.

This sort of Beck condition is necessary to assure that, also in this general case, by taking isomorphism classes one gets a commutative monoid on $\mathcal{S}$.

We now show how also the notion of cartesian operad
can be developed relatively to any category $\mathcal{S}$ is with pullbacks.

Cartesian operads

Idea 1

The notion of cartesian operad (or cartesian multicategory) is aimed to fill the missing term in the equality

operads :: symmetric monoidal $=$?? :: cartesian monoidal

Cartesian operads

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operads :: symmetric monoidal = ?? :: cartesian monoidal

Thus, one minimum requirement is:
representable cartesian operads = cartesian monoidal categories.
That is, if a cartesian operad $\mathcal{O}$ has tensor products, these are cartesian (that is, universal) products.

Cartesian operads

Idea 2

Cartesian operads are operads $\mathcal{O}$
with an adjunctive structure which makes it possible
weakening and contraction of variables.

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Cartesian operads are operads $\mathcal{O}$
with an adjunctive structure which makes it possible
weakening and contraction of variables.

Cartesian operads are a notion of algebraic theory
alternative to (and more flexible than) Lawvere theories.

Cartesian operads

weakening and contraction

For instance, in the operad of sets,
a map $f : X, Y, X \to T$ gives another map $f' : Y, Z, X \to T$
by the rule $f'(y, z, x) = f(x, y, x)$
which introduces the extra variable z (weakening)
and duplicate the variable x (contraction).

Cartesian operads

weakening and contraction

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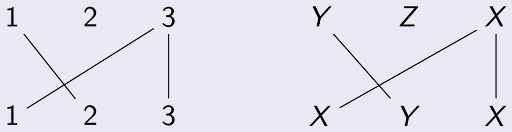
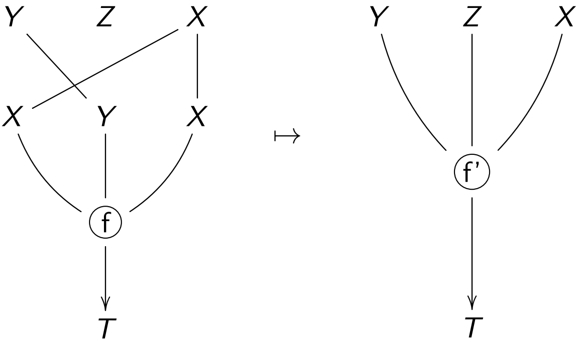
weakening and contraction

The map f' is then obtained by f
covariantly along the reindexing of the domain



Cartesian operads : “contraction” and “weakening”

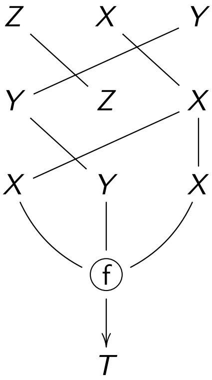
Reindexing arrows act covariantly on maps.



Cartesian operads

Reindexing arrows act on maps.

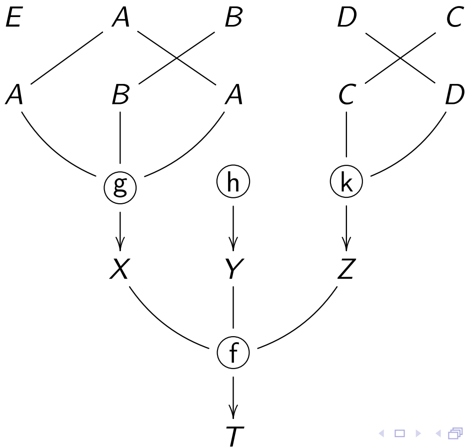
This is unambiguous:



Cartesian operads

The action is compatible with composition from below.

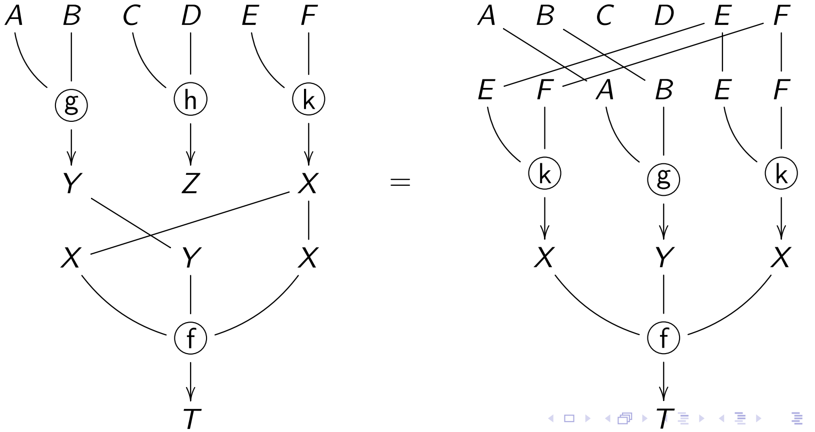
This is unambiguous:



Cartesian operads

Combing

The action is compatible with composition from above.



Cartesian operads

Cartesian operads

Let $\mathcal{S}$ be a category with pullbacks. A cartesian operad on $\mathcal{S}$ is an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathcal{S}$,
such that $\mathbb{D}$ has, in addition to ordinary cells, also
triangular cells, formed by two proarrows and an arrow.

Triangular cells (giving covariant reindexing)

$$
\begin{tikzcd}
X \arrow[rr, dashed, "t"] \arrow[dr, "f"'] \arrow[rr, phantom, shift right=6ex, "\tau"] & & Y \arrow[dl, "f'"] \\
& Z &
\end{tikzcd}
$$

Cartesian operads

Cartesian operads

A cartesian operad on $\mathcal{C}$ is an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathcal{C}$,
such that $\mathbb{D}$ has, in addition to ordinary cells,
also triangular cells satisfying the conditions

- Maps in $\mathcal{D}$ (proarrows) can be covariantly reindexed along commutative triangles in $\mathcal{C}$.
- Triangular cells compose horizontally and with proarrows out of them.
- Triangular cells can be pasted with square cells.
- Triangular cells are stable with respect to reindexing.

Cartesian operads

Covariant reindexing of maps

Given a proarrow $f : t^*Y \to Z$ in $\mathcal{D}$,
and a commutative triangle in $\mathcal{S}$ completing df and t,
there is a unique extension to a triangular cell over it:

$$\begin{tikzcd} t^*Y \arrow[rr, dashed] \arrow[dr, "f"'] & & Y \\ & Z & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "t"] \arrow[dr, "df"'] & & B \\ & C & \end{tikzcd}$$

Cartesian operads

Covariant reindexing of maps

Given a proarrow $f : t^*Y \to Z$ in $\mathcal{D}$,
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& Z &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[rr, "t"] \arrow[dr, "df"'] & & B \arrow[dl, "s"] \\
& C &
\end{tikzcd}
$$

Cartesian operads

Covariant reindexing of maps

Given a proarrow $f : t^*Y \to Z$ in $\mathcal{D}$,
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$$
\begin{tikzcd}
t^*Y \arrow[rr, dashed] \arrow[dr, "f"'] \arrow[dr, phantom, shift left=6ex, "\mathcal{T}"] & & Y \arrow[dl, dotted, "f'"] \\
& Z &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[rr, "t"] \arrow[dr, "df"'] & & B \arrow[dl, "s=df'"] \\
& C &
\end{tikzcd}
$$

Cartesian operads

A triangular cell can be pasted with a square cell,
giving a triangular cell.

Pasting

$$
\begin{tikzcd}
t^*X \arrow[rr, dashed] \arrow[d, "h"'] \arrow[drr, phantom, "\alpha"] & & X \arrow[d, "f"] \\
s^*Y \arrow[rr, dashed] \arrow[dr, bend right, "k"'] \arrow[rr, phantom, "\tau", shift right=25] & & Y \arrow[dl, bend left, "g"] \\
& Z &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
t^*X \arrow[rr, dashed] \arrow[dr, bend right, "kh"'] \arrow[rr, phantom, "\tau'", shift right=40] & & Y' \arrow[dl, bend left, "gf"] \\
& Z &
\end{tikzcd}
$$

Cartesian operads

Triangular cells are stable with respect to reindexing.

Stability

$$
\begin{tikzcd}
& Y' \arrow[rr, dashed] \arrow[dd] & & Y \arrow[dd] \\
X' \arrow[ur, dashed] \arrow[rr, dashed] \arrow[dr] & & X \arrow[ur, dashed] \arrow[dr] & \\
& Z' \arrow[rr, dashed] & & Z
\end{tikzcd}
$$

Algebraic products

Given a cartesian operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathcal{S}$,
an object $X \in \mathbb{D}$ and a map $t : dX \to J$ in $\mathcal{S}$,
an algebraic product for X along t
is an object $P \in \mathbb{D}$ over J along with a vertical map
$\pi : t^*P \to X$ and a map $u : X \to P$ over t...

$$
\begin{tikzcd}
t^*P \arrow[d, "\pi"'] \arrow[dr, dashed] & \\
X \arrow[r, "u"'] & P
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
dX \arrow[d, "\mathrm{id}"'] \arrow[dr, "t"] & \\
dX \arrow[r, "t"'] & J
\end{tikzcd}
$$

Algebraic products

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$\pi : t^*P \to X$ and a map $u : X \to P$ over t...

...such that the following are both triangular cells:

$$
\begin{tikzcd}
t^*P \arrow[r, dashed] \arrow[d, "\pi"'] & P \arrow[ddl, bend left, "\mathrm{id}"] \\
X \arrow[d, bend right, "u"'] & \\
P &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, dashed, "\Delta"] \arrow[ddr, bend right, "\mathrm{id}"'] & h^*X \arrow[d, "t^*u"] \\
& t^*P \arrow[d, bend left, "\pi"] \\
& X
\end{tikzcd}
$$

Main result

Main result for cartesian operads

For a cartesian operad $\mathcal{O}$ on $\mathcal{S}$, the following are equivalent:

1. $\mathcal{O}$ has algebraic products.
2. $\mathcal{O}$ has universal products.
3. $\mathcal{O}$ is monoidal (representable).

This result indicates that we have indeed captured a proper notion of cartesian operad.

Further evidence

One can also generalize results such as the following:

Cartesian + Sequential = Semiadditive (Pisani 2014)

Cartesian structures on sequential operads
correspond to enrichments of the underlying category
in the category of commutative monoids.

In the present context, objects are to be intended as sections $x : \mathcal{C} \to \mathbb{D}^h$ of $d^h_{\mathcal{O}}$, and the commutative monoid $\mathcal{O}(x; y)$ is a commutative monoid on $\mathcal{S}$ in the generalized sense.

Commuting operations

Internal operations

A map $f : X \to Y$ in an operad $d_{\mathcal{O}}$ on $\mathcal{S}$
is an internal operation if $X = (df)^* Y$,
that is, it is parallel to the horizontal lifting from Y of its image.

$$X \xrightarrow{f} Y$$

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$$X \xrightarrow{f} Y \qquad dX \xrightarrow{df} dY$$

Classically (for operads over Set_f),
these are indeed the internal operations
$f : X; \cdots ; X \to X$ (or families of such maps).

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$$X \underset{}{\overset{f}{\rightrightarrows}} Y \qquad dX \xrightarrow{df} dY$$

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$f : X; \cdots ; X \to X$ (or families of such maps).

Commuting internal operation

Commuting internal operations

Then one has the following commutativity notion:
two internal operations f and g, with the same codomain, commute if the square below commutes in $\mathcal{D}$: $fg' = gf'$.

$$\begin{tikzcd}
W \arrow[r, dotted, "g'"] \arrow[r, dashed, shift right] \arrow[d, dotted, "f'"'] \arrow[d, dashed, shift left] & X \arrow[d, "f"'] \arrow[d, dashed, shift left] \\
Z \arrow[r, "g"] \arrow[r, dashed, shift right] & Y
\end{tikzcd}$$

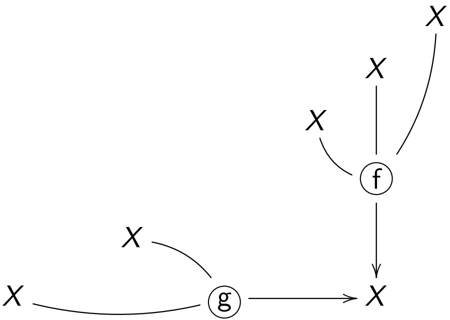
(The dotted maps f' and g' are obtained by the reindexing axiom, applied once "horizontally" and once "vertically".
The notion does not depend on the choice of the pullback.)

Commuting internal operation

Classically (for operads over Set_f),
one gets the usual notion, expressed in an elegant way.

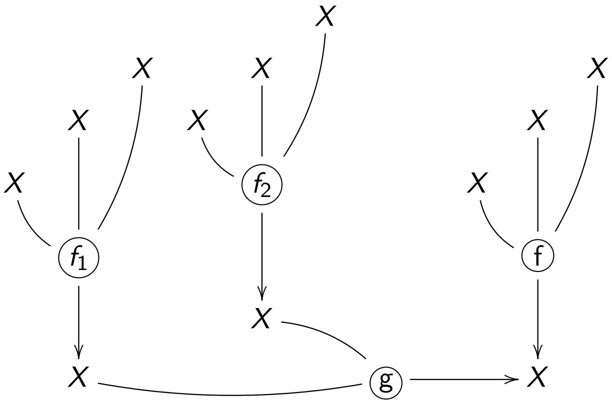
Commuting internal operation

Two internal operations with the same codomain.



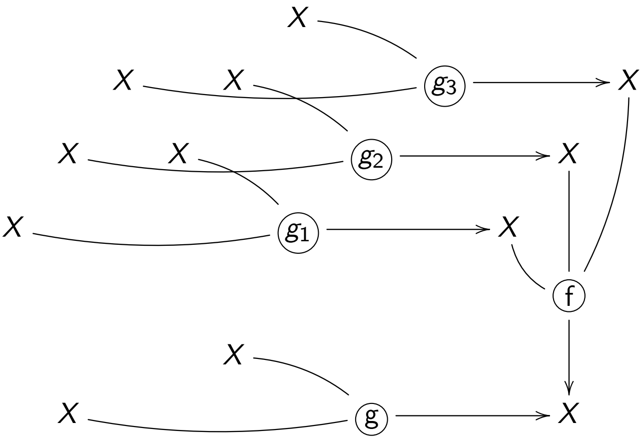
Commuting internal operation

Reindexing f along the arrow parallel to g.



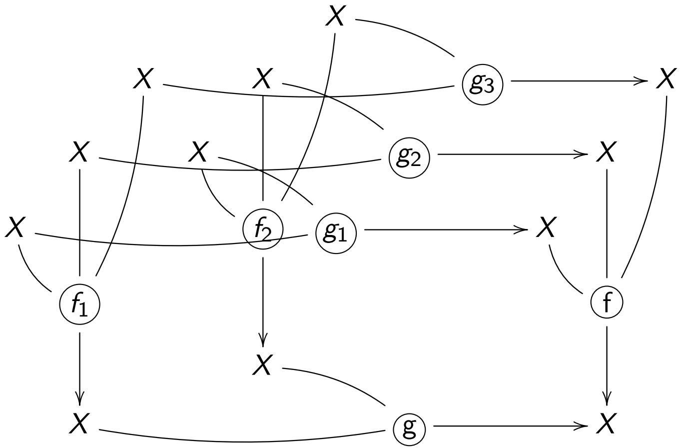
Commuting internal operation

Reindexing g along the arrow parallel to f.



Commuting internal operation

The two reindexing can be composed and may give the same result.



Commuting internal operation

One can prove, in the general setting,
sort of Hilton-Eckman arguments.

Table of Contents

Fibration as double discrete fibrations

Decoupled fibrations

The present approach to operads
points also toward an idea of decoupled fibration.
Indeed, an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathcal{S}$ can be seen
as a (split) fibration where the (chosen) cartesian arrows
are separated from the other arrows (proarrows).

Fibration as double discrete fibrations

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Indeed, an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathcal{S}$ can be seen
as a (split) fibration where the (chosen) cartesian arrows
are separated from the other arrows (proarrows).

Indeed, we have the following result:

Split fibrations $d : \mathcal{D} \to \mathcal{S}$
are discrete double fibrations $d : \mathbb{D} \to \mathbb{S}\mathrm{q}\, \mathcal{S}$
such that $\mathbb{D}$ has conjoints preserved by d.

Fibration as double discrete fibrations

From the point of view of lax functors, we have:

A lax functor $F : \mathbb{S}\mathrm{q}\,\mathcal{S} \to \mathbb{C}\mathrm{at}$ is a fibration
if and only if it preserves conjoints.

Which amounts to saying that the vertical part F^v
is determined by the horizontal part:
$F^v(f)$ is the profunctor represented by $F^h(f)$.

To explore

Another promising development is considering operads on double categories which are more "relations-like", for instance cospans in Set_f.

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Another promising development is considering operads on double categories which are more "relations-like", for instance cospans in Set_f.

In this case, it seems appropriate to consider the double category of summand squares rather than that pullback squares.
There is no difference in extensive categories.