Gray-type monoidal product and Bifunctor Theorem for double categories

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Virtual Double Categories Workshop

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- Bifunctor Theorem for (lax-hop) double categories
  - (no) Gray-type tensor product on (strict-lax) double categories
  - Bifunctor Theorem
  - "(Un)currying" 2-functors
  - application to monads in double categories

Double categories

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C_0: 0-cells and 1v-cells, C_1: 1h-cells and 2-cells.

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- natural transformations (2-cells in Cat_2)

$$\alpha : c \otimes (id_{C_1} \times_{C_0} c) \Rightarrow c \otimes (c \times_{C_0} id_{C_1})$$

$$\lambda : c \otimes (u \times_{C_0} id_{C_1}) \Rightarrow id_{C_1}$$

$$\rho : c \otimes (id_{C_1} \times_{C_0} u) \Rightarrow id_{C_1}$$

which satisfy a pentagon and a triangle.

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$$\begin{tikzcd} B \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\boxed{\alpha}"] & A \arrow[d, "g"] \\ B' \arrow[r, "N"] & A' \end{tikzcd}$$

$\alpha : M \longrightarrow N$ A-B-bimodule morphism

$$a \cdot n \cdot b := g(a) \cdot n \cdot f(b)$$

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- Each bicategory can be embedded into a pseudodouble category.

• Every pseudodouble category is double-equivalent to a double category [Grandis-Paré: "Limits in double categories" (1999)].

Functors of double categories

- (strict) double functors,
- pseudo double functors,
- (co)lax double functors.

$$\begin{tikzcd} \cdot \arrow[r, "F(f)"] \arrow[d, "="'] & \cdot \arrow[r, "F(g)"] & \cdot \arrow[d, "="] \\ \cdot \arrow[rr, "F(gf)"'] & & \cdot \end{tikzcd}$$

Why

the 2-category $\mathrm{Mnd}(\mathcal{K})$

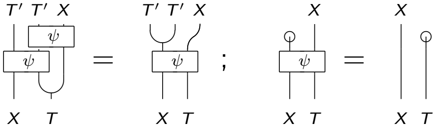
of 2-monads

isn't enough

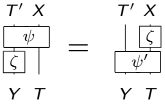
The 2-category Mnd($\mathcal{K}$) of monads in $\mathcal{K}$

<u>0-cells:</u>
2-monads $(\mathcal{A},\ T\colon \mathcal{A} \to \mathcal{A},\ \mu_T\colon TT \to T,\ \eta_T\colon \mathrm{Id}_{\mathcal{A}} \to T)$

<u>1-cells:</u> pairs $(X, \psi) : (\mathcal{A}, T) \to (\mathcal{A}', T')$ where $X : \mathcal{A} \to \mathcal{A}'$ is a 1-cell and $\psi : T'X \Rightarrow XT$ a 2-cell s.t.



<u>2-cells:</u> $(X, \psi) \Rightarrow (Y, \psi')$ are given by 2-cells $\zeta : X \to Y$ in $\mathcal{K}$ satisfying:



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BUT:

• monad morphisms between monads on the bicategories $Span_d(\mathcal{V})$ and $\mathcal{V}$-Mat are not functors of categories internal in $\mathcal{V}$, resp. of categories enriched over $\mathcal{V}$.

The double category $\mathrm{Mnd}(\mathbb{D})$ of (double) monads

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This allows to **describe mathematical structures** and **morphisms between them** as **monads** and **vertical monad maps in appropriate double categories**.

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1v-cells:

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and the interchange does not hold in general on lax transformations (strictness of transformations is needed).

Gray tensor product on 2-Cat by generators and relations

One looks for a 2-category $\mathcal{A} \otimes \mathcal{B}$ s.t.:

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Bifunctor Theorem for (lax-hop) double categories

Towards the Gray tensor product on (strict-lax) double categories

[B. Femić: " Bifunctor Theorem and Gray monoidal structure for double categories with lax double functors"]

We define $[\![\mathbb{A}, \mathbb{B}]\!]$ (as a candidate for inner-hom in Dbl^{st}_{lx})

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- Hence, there is an isomorphism of sets:

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Bifunctor Theorem: 1- and 2-categories

[MacLane:]
Given functors $L_C : \mathcal{B} \longrightarrow \mathcal{D}$ and $M_B : \mathcal{C} \longrightarrow \mathcal{D}$ so that
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$\mathrm{Lax}^{u-d}_{hop}(\mathbb{A} \times \mathbb{B}, \mathbb{C})$: *unital and decomposable* lax double functors (ι and $\gamma_{(1_A, g),(f, 1_{B'})}$ are invertible).

$\mathcal{F}$ restricts to 2-equivalences:

$$\mathcal{F}' : q\text{-}\mathrm{Lax}^{ns-u}_{hop}(\mathbb{A} \times \mathbb{B}, \mathbb{C}) \xrightarrow{\simeq} \mathrm{Lax}^{u-d}_{hop}(\mathbb{A} \times \mathbb{B}, \mathbb{C}).$$

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composing with $\mathcal{F}'$ one gets a **currying** 2-functor which is a 2-equivalence:

$$\mathsf{Lax}^{u\text{-}d}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C}) \simeq \mathsf{Lax}^{u}_{hop}(\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!]^{ns\text{-}u})$$

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Bifunctor Theorem as a generalization of Beck's result on the composition of monads:

$$\begin{tikzcd}
q\text{-}\mathsf{Lax}_{hop}(* \times *, \mathbb{D}) \arrow[r, "\mathcal{F}"] \arrow[d, "\cong"'] & \mathsf{Lax}_{hop}(*, \mathbb{D}) \arrow[d, "\cong"] \\
\mathsf{Mnd}(\mathsf{Mnd}(\mathcal{H}(\mathbb{D}))) \arrow[r, "\mathrm{Comp}(\mathcal{H}(\mathbb{D}))"'] & \mathsf{Mnd}(\mathcal{H}(\mathbb{D}))
\end{tikzcd}$$