Double-Categorical Compositional Rewriting Theory

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Virtual Double Categories Workshop

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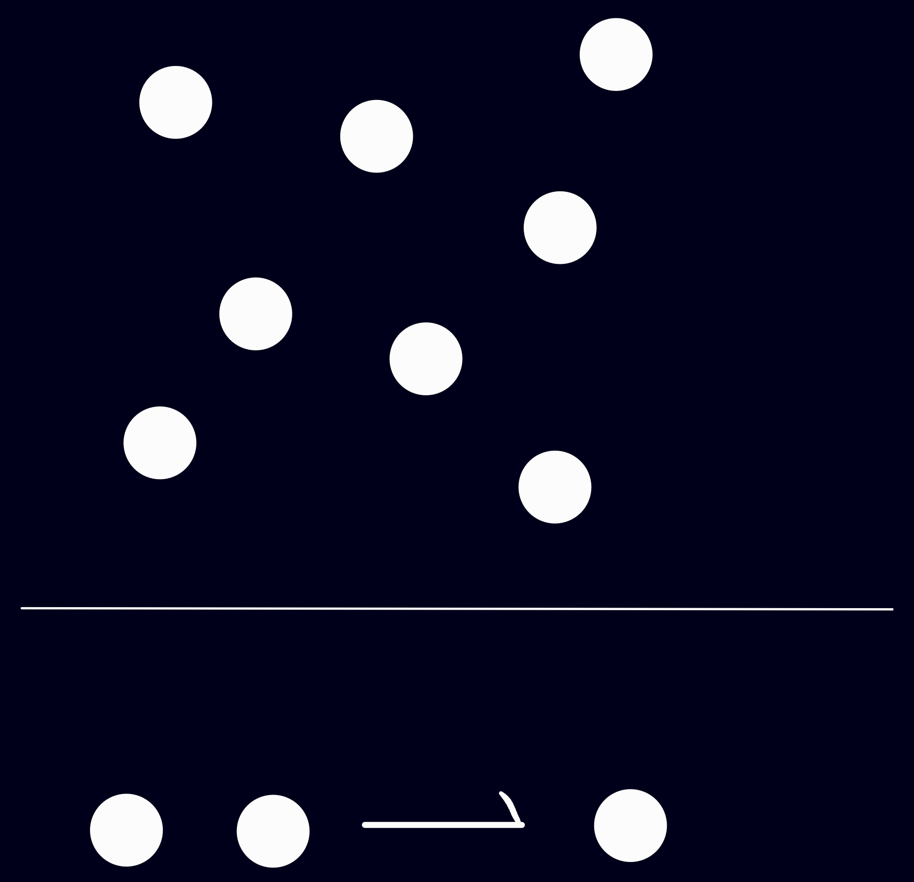
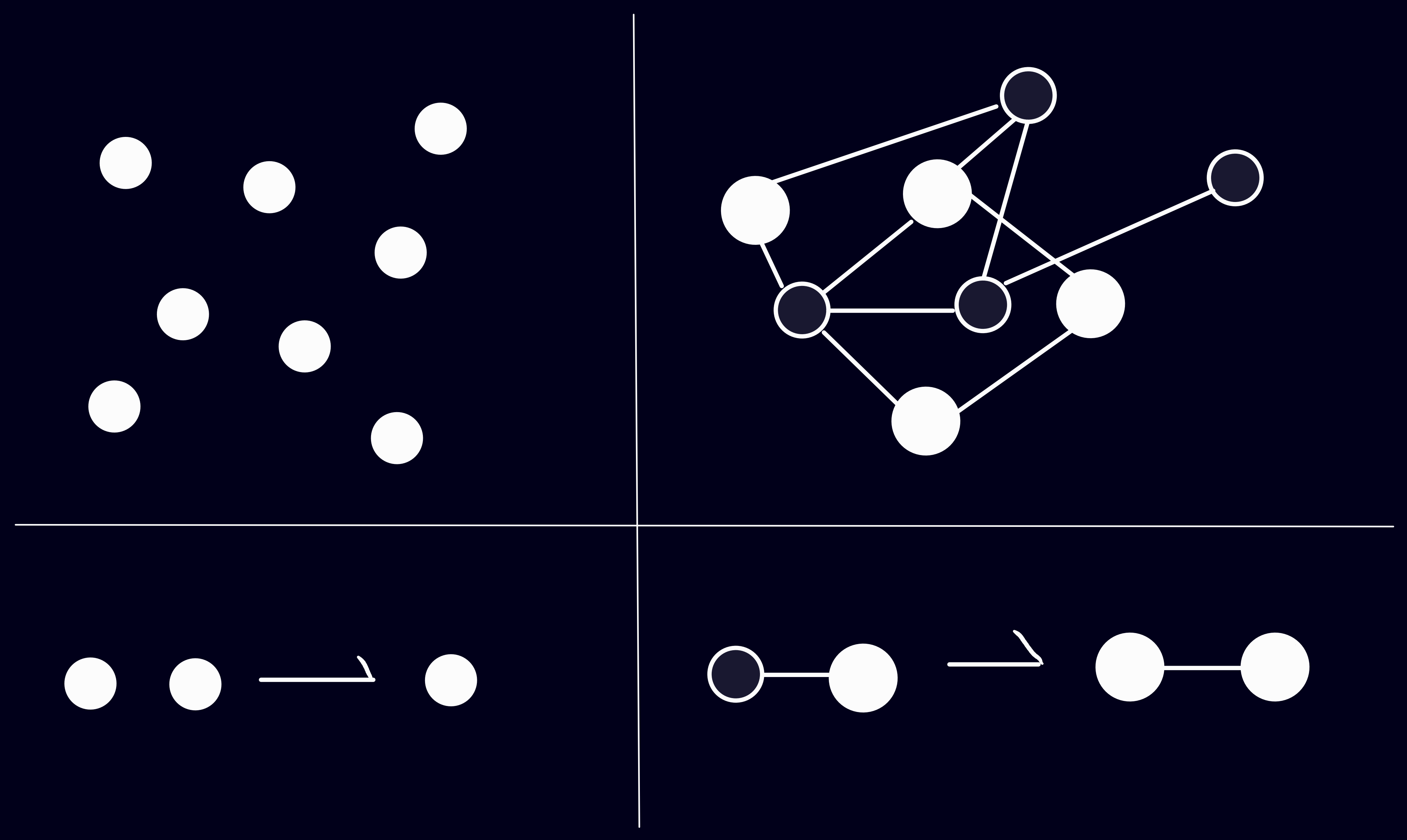
Based upon joint work with

- P.-A. Melliès & N. Zeilberger
- R. Harmer & J. Krivine (2204.07175)

① MOTIVATION



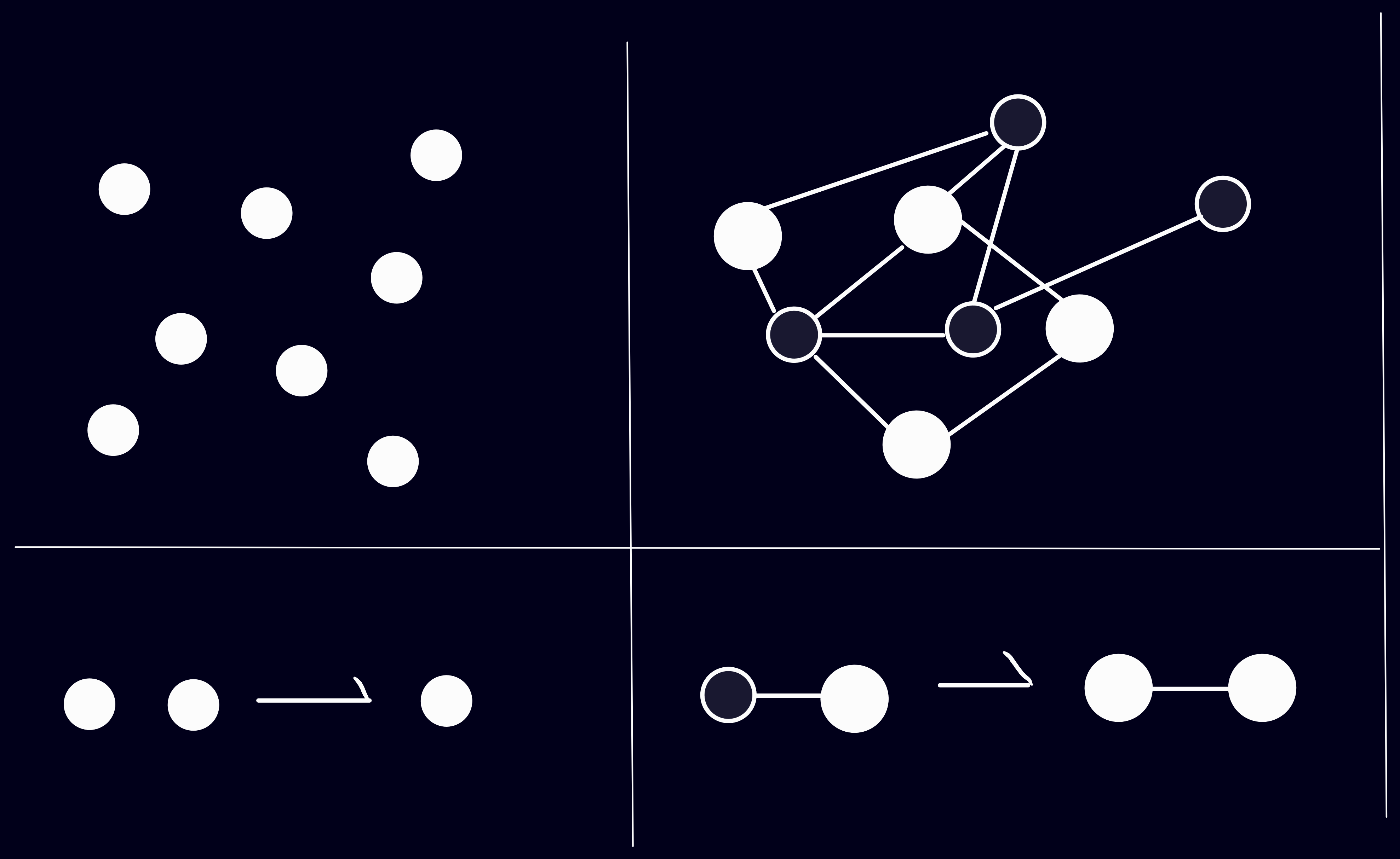
① MOTIVATION

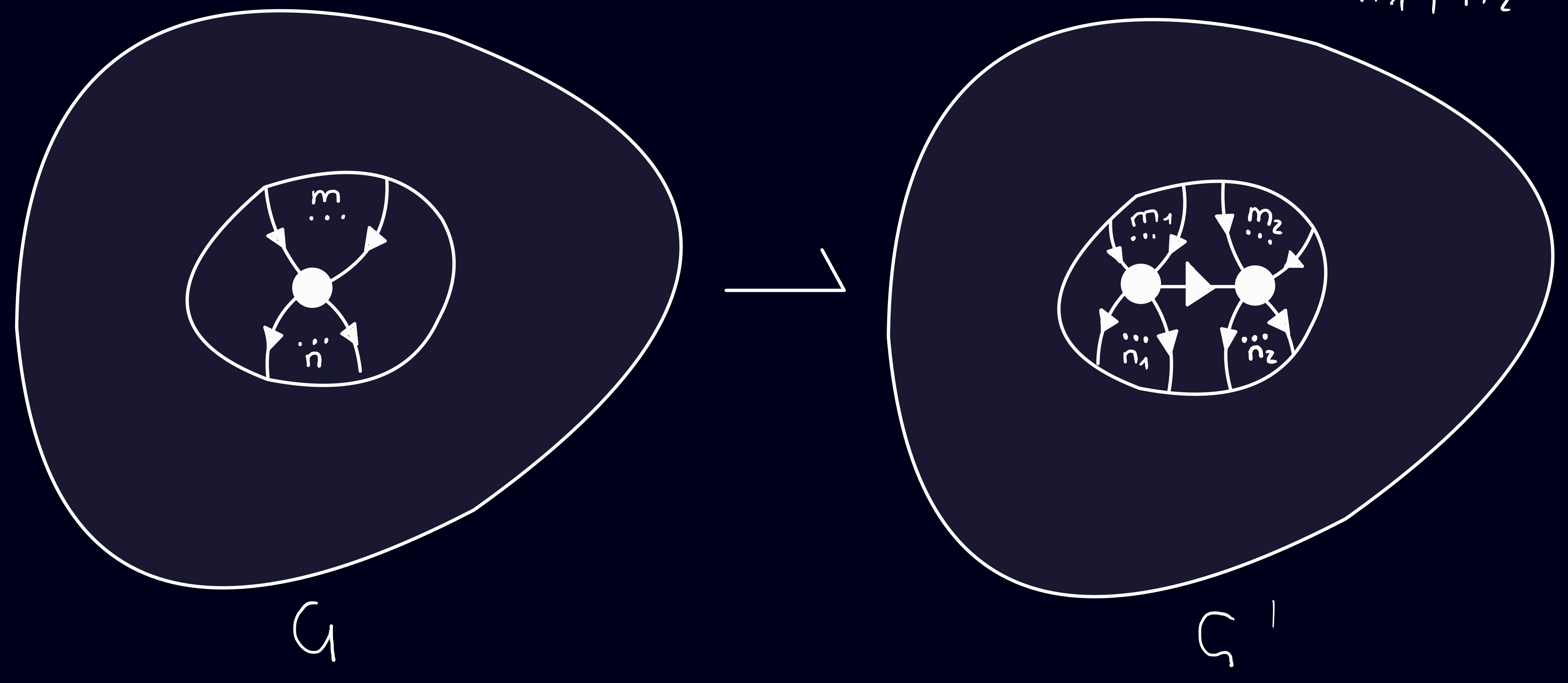


① MOTIVATION

$m_1 + m_2 = m$

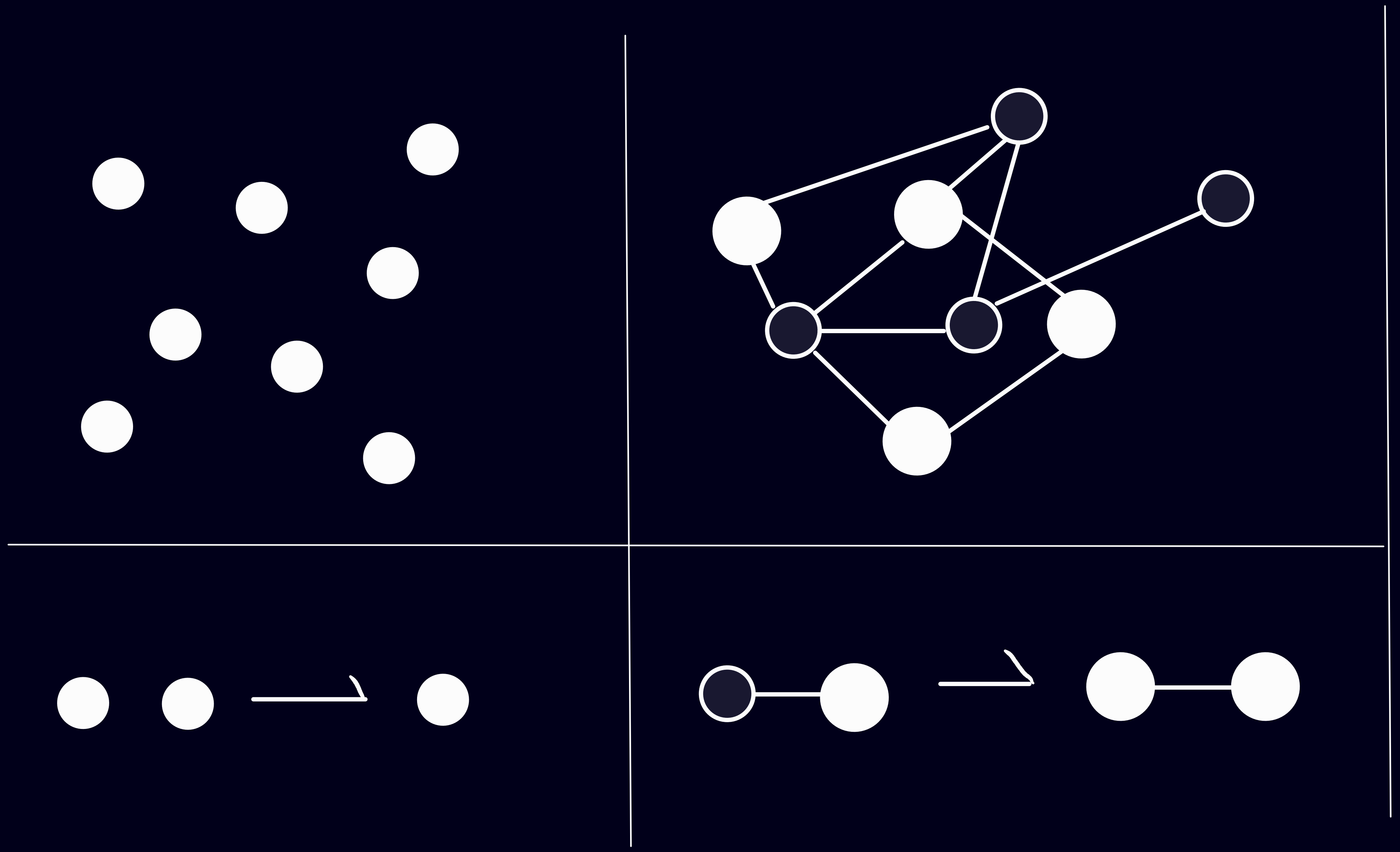
$n_1 + n_2 = n$

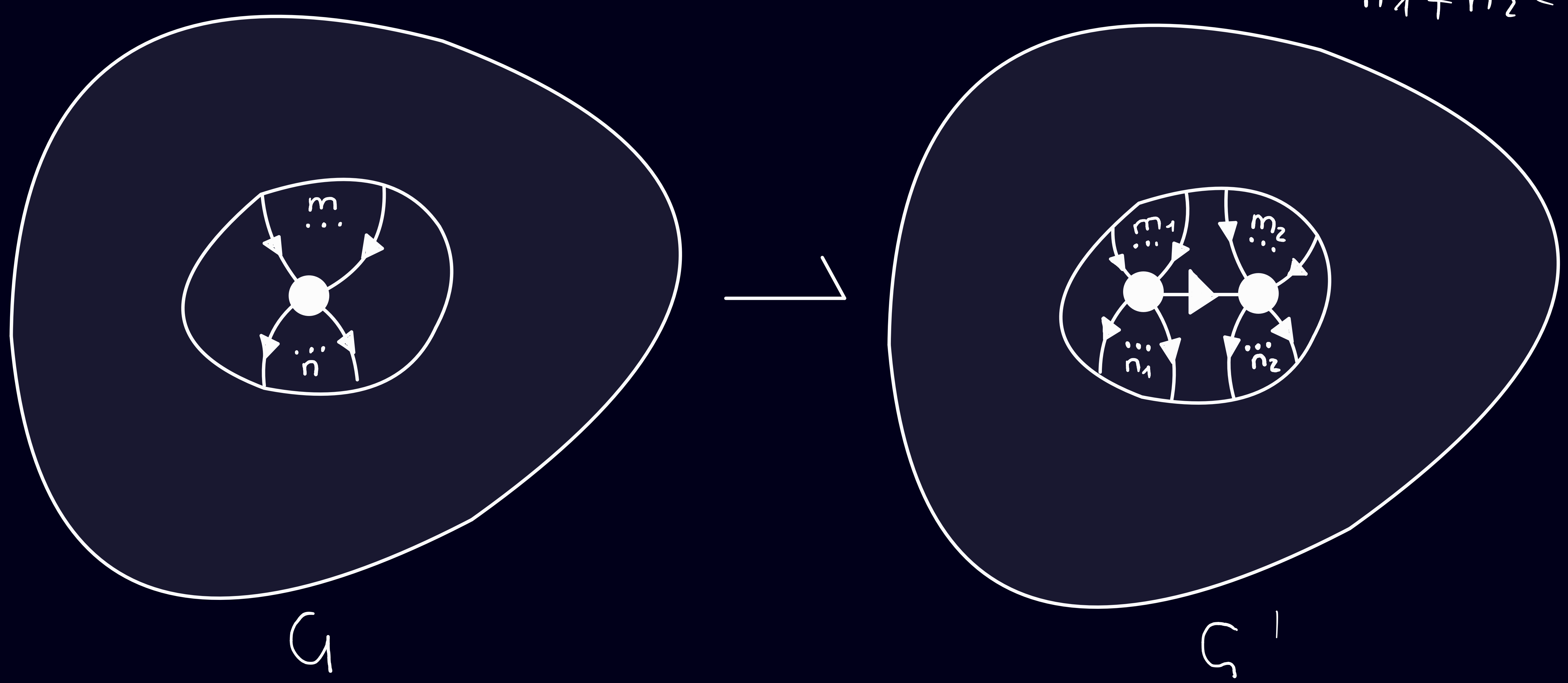




① MOTIVATION

$m_1 + m_2 = m$
$n_1 + n_2 = n$





↪ ALL FORMALIZABLE IN DOUBLE-PUSHOUT (DPO) SEMANTICS:

$$
\begin{tikzcd}
O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
r_\alpha(X) & X \arrow[l]
\end{tikzcd}
:=
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\
r_\alpha(X) & K_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X
\end{tikzcd}
$$

PO — PUSHOUT

② MOTIVATION: FORMALIZATION OF REWRITING + COMBINATORICS!

- INSPIRATION FROM COMBINATORIAL SPECIES THEORY:

(i) $\hat{x}(x^n) = x^{n+1}$, $\quad \frac{d}{dx}(x^n) = (n)_1 x^{n-1} \equiv \begin{cases} 0, & n=0 \\ n x^{n-1}, & \text{else} \end{cases}$

$\hookrightarrow$ (ii) $x^n = \hat{x}^n(1)$

② MOTIVATION: FORMALIZATION OF REWRITING + COMBINATORICS!

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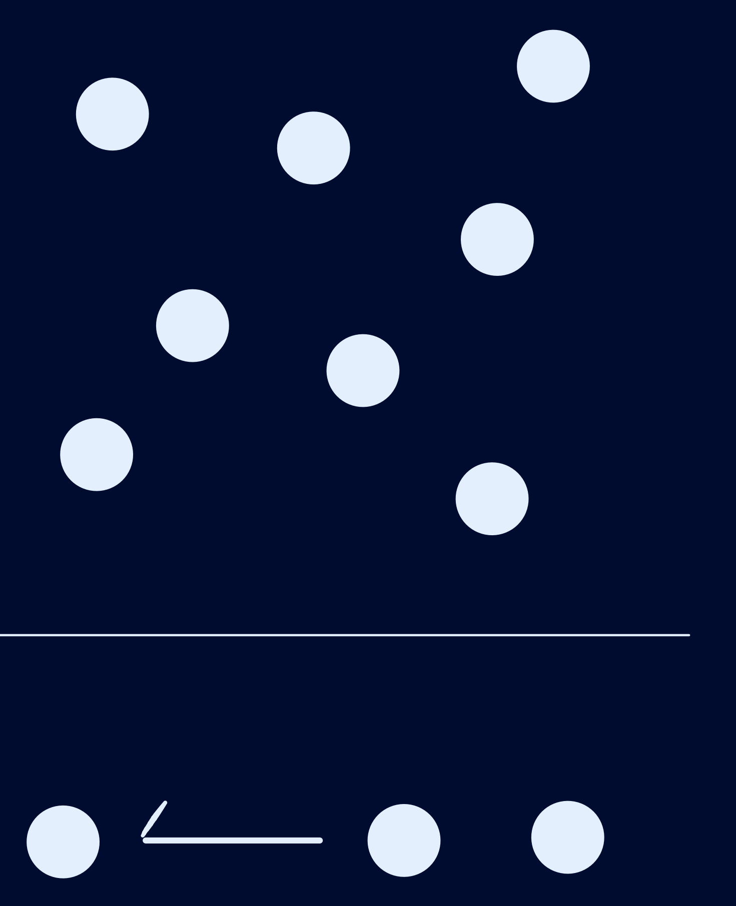
↳ (ii) $x^n = \hat{x}^n(1)$

$$\hat{x}^p \left(\frac{d}{dx}\right)^q (x^n) = (n)_q \, x^{n-q+p}$$

\# OF WAYS TO REMOVE q ELEMENTS FROM A SET OF n ELEMENTS



$$\hat{x}\left(\frac{d}{dx}\right)^2 x^n = (n)_2 \, x^{n-1}$$

② Motivation: Formalization of Rewriting + Combinatorics!

- Inspiration from combinatorial species theory:

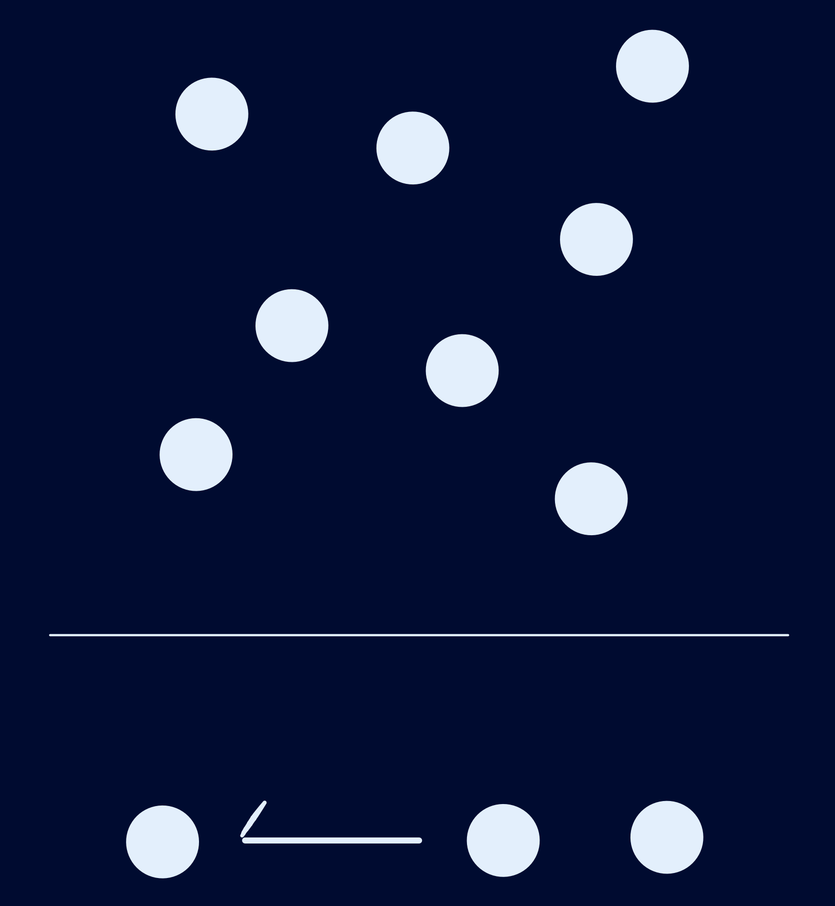
(i) $\hat{X}(x^n) = x^{n+1}$, $\quad \frac{d}{dx}(x^n) = (n)_1 x^{n-1} \equiv \begin{cases} 0, & n=0 \\ n x^{n-1}, & \text{else} \end{cases}$

↪ (ii) $x^n = \hat{x}^n(1)$

$$\hat{X}^p \left(\frac{d}{dx}\right)^q (x^n) = (n)_q \, x^{n-q+p}$$

# of ways to remove q elements from a set of n elements

(iii) $$\hat{X}^p \left(\frac{d}{dx}\right)^q \hat{X}^r \left(\frac{d}{dx}\right)^s = \sum_{k \geq 0} \underbrace{\binom{q}{k} k! \binom{r}{k}}_{\in \mathbb{Z}_{\geq 0}} \hat{X}^{p+r-k} \left(\frac{d}{dx}\right)^{q+s-k}$$



$$\hat{X}\left(\frac{d}{dx}\right)^2 x^n = (n)_2 \, x^{n-1}$$

② MOTIVATION: FORMALIZATION OF REWRITING + COMBINATORICS!

- INSPIRATION FROM COMBINATORIAL SPECIES THEORY:

(i) $\hat{x}(x^n) = x^{n+1}$, $\quad \frac{d}{dx}(x^n) = (n)_1 x^{n-1} \equiv \begin{cases} 0, & n=0 \\ n x^{n-1}, & \text{else} \end{cases}$

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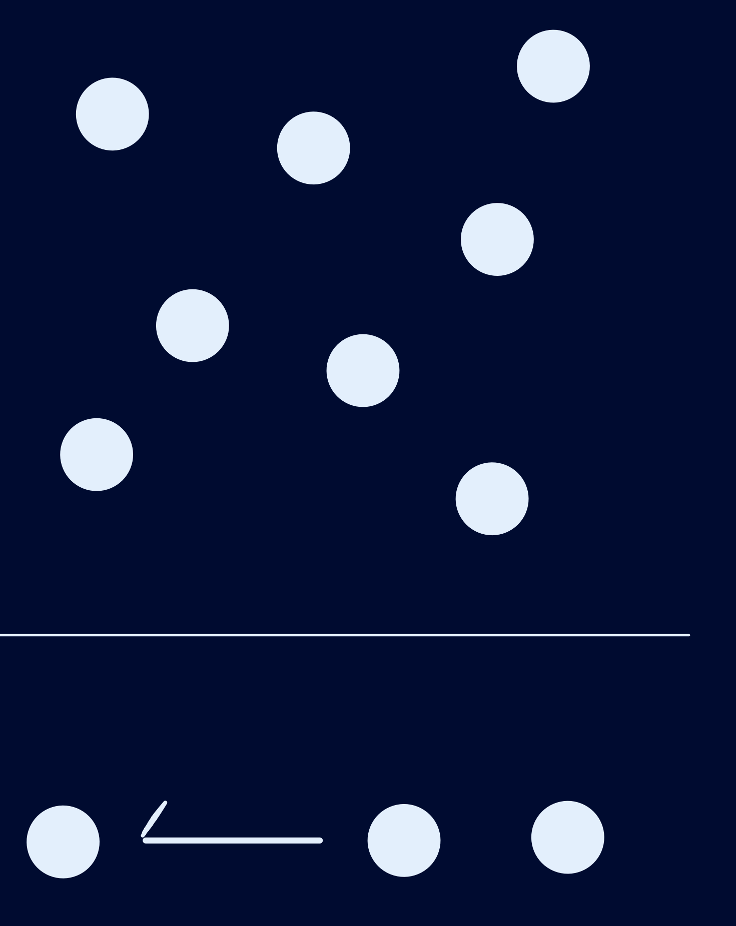
$$\hat{x}^p \left(\frac{d}{dx}\right)^q (x^n) = (n)_q\, x^{n-q+p}$$

OF WAYS TO REMOVE q ELEMENTS FROM A SET OF n ELEMENTS

(iii) $$\hat{x}^p \left(\frac{d}{dx}\right)^q \hat{x}^r \left(\frac{d}{dx}\right)^s = \sum_{k\geq 0} \underbrace{\binom{q}{k} k! \binom{r}{k}}_{\in \mathbb{Z}_{\geq 0}} \hat{x}^{p+r-k} \left(\frac{d}{dx}\right)^{q+s-k}$$



$$\hat{x}\left(\frac{d}{dx}\right)^2 x^n = (n)_2\, x^{n-1}$$

GOAL:

(i) "$\rho(\delta(r))|X\rangle := \sum_\alpha |r_\alpha(X)\rangle = \sum_Y \overbrace{\mu_X^Y}^{\in \mathbb{Z}_{\geq 0}} |Y\rangle$"

(ii) "$|X\rangle = \rho(\delta(X \leftarrow \emptyset))|\emptyset\rangle$"

(iii) "$\rho(\delta(r_2))\rho(\delta(r_1)) = \sum_{r_u} \underbrace{\mu_{r_1,r_2}^{r_u}}_{\in \mathbb{Z}_{\geq 0}} \rho(\delta(r_u))$"

(iv) "$\rho(\delta(r_2))\rho(\delta(r_1)) = \rho(\delta(r_2) \odot \delta(r_1))$"

$\odot$ – RULE ALGEBRA PRODUCT

③ Conceptual Obstacle: Essential Uniqueness of Universal Constructions

Recap:

$$
\begin{tikzcd}
O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
r_\alpha(X) & X \arrow[l]
\end{tikzcd}
:=
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\
r_\alpha(X) & K_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X
\end{tikzcd}
$$

PO — Pushout

Definition:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\text{ is a PO } :\Leftrightarrow \forall
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D & \\
& & X
\end{tikzcd}
:
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed, "\exists!"] & \\
& & X
\end{tikzcd}
$$

$$
\hookrightarrow
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\wedge
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\
C \arrow[r] & D'
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed, "\exists!", "\cong"'] & \\
& & D'
\end{tikzcd}
$$

Example: (in $\underline{\text{FinSet}}$)

$$
\begin{tikzcd}
I \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\
C \arrow[r, hook] & \text{"}B \cup_I C\text{"}
\end{tikzcd}
$$

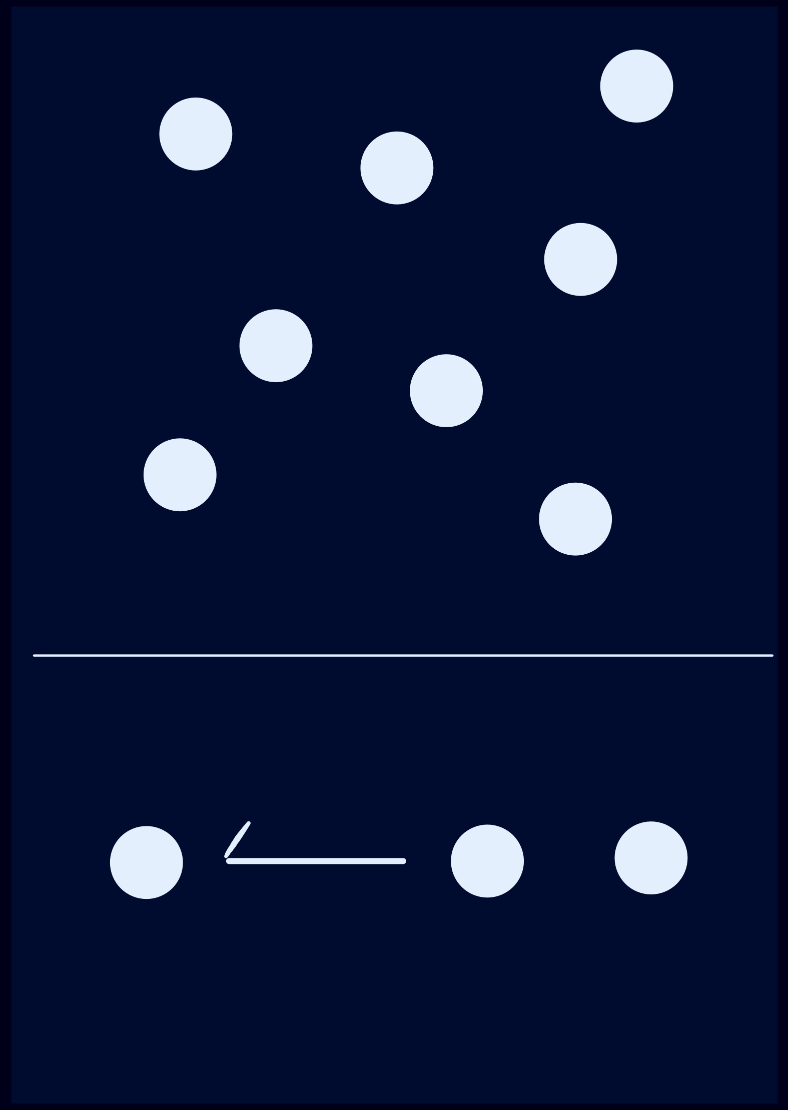
③ Conceptual Obstacle: Essential Uniqueness of Universal Constructions

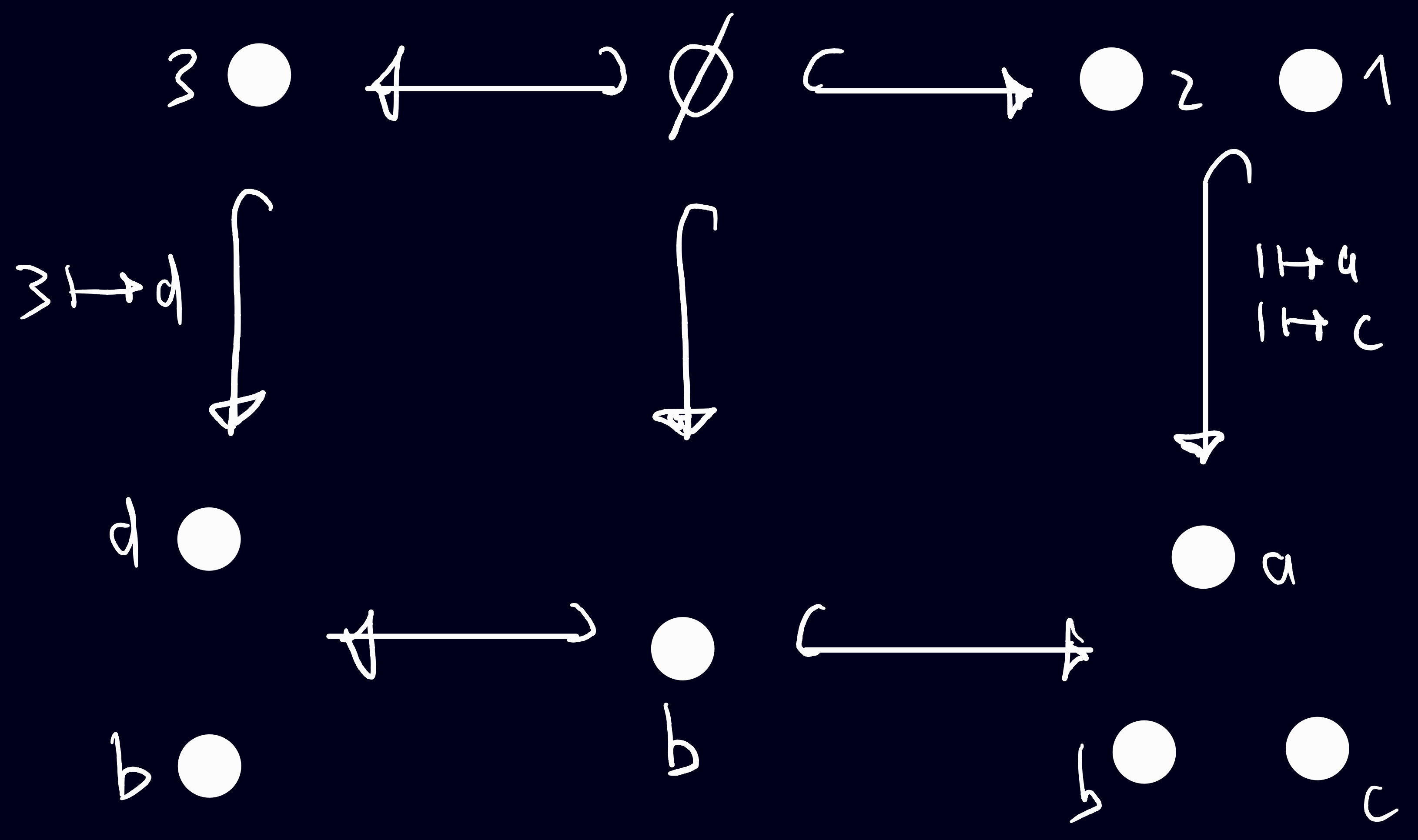
Recap:

$$\begin{tikzcd} O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\ r_\alpha(X) & X \arrow[l] \end{tikzcd} \quad := \quad \begin{tikzcd} O \arrow[d, "n"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\ r_\alpha(X) & K_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X \end{tikzcd}$$

PO — Pushout

Example:





In $\underline{\mathrm{FinSet}}$:

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d, "f"] \\ C \arrow[r, hook] & \text{"}B \cup_I C\text{"} \end{tikzcd}$$

④ ANSATZ: CATEGORIFICATION

GOAL:

(i) " $\rho(\delta(r))|X\rangle := \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{Y} \overbrace{\mu_X^Y}^{\in \mathbb{Z}_{\geq 0}} |Y\rangle$ "

(ii) " $|X\rangle = \rho(\delta(X \leftarrow \emptyset))|\emptyset\rangle$ "

(iii) " $\rho(\delta(r_2))\rho(\delta(r_1)) = \sum_{r_u} \underbrace{\mu_{r_1,r_2}^{r_u}}_{\in \mathbb{Z}_{\geq 0}} \rho(\delta(r_u))$ "

(iv) " $\rho(\delta(r_2))\rho(\delta(r_1)) = \rho(\delta(r_2) \odot \delta(r_1))$ "

$\odot$ – RULE ALGEBRA PRODUCT

I. FORMALIZE
$$\begin{tikzcd} O \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d] \\ r_\alpha(X) & X \arrow[l] \end{tikzcd}$$
AS 2-CELLS IN A DOUBLE CATEGORY

II. FORMALIZE $\mathbb{Z}_{\geq 0}$-COEFFICIENTS AS CARDINALITIES (OF SUITABLE SETS...)

METHODS: DOUBLE CATEGORIES, PRESHEAVES, FIBRATIONS, COENDS, MULTISUMS ...

⑤ DEFINITION: A DOUBLE CATEGORY $\mathbb{D}$ IS A (PSEUDO) INTERNAL CATEGORY IN $\underline{CAT}$

$$
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "u"] & \mathbb{D}_1 \arrow[l, bend right, "S"'] \arrow[l, bend left, "T"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\diamond_h"']
\end{tikzcd}
$$

$$
\mathbb{D}_0 : \quad
\begin{tikzcd}
X \arrow[d, "f"] \\
Y
\end{tikzcd}
$$

"0-cells" – objects of $\mathbb{D}_0$

"vertical morphisms" – morphisms of $\mathbb{D}_0$

$$
\mathbb{D}_1 : \quad
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
$$

"horizontal morphisms" – objects of $\mathbb{D}_1$

"2-cells" – morphisms of $\mathbb{D}_1$

5 DEFINITION: A DOUBLE CATEGORY $\mathbb{D}$ IS A (PSEUDO) INTERNAL CATEGORY IN $\underline{CAT}$

$$
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "u"] & \mathbb{D}_1 \arrow[l, bend right, "S"'] \arrow[l, bend left, "T"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\diamond_h"']
\end{tikzcd}
$$

$\mathbb{D}_0$:
$$
\begin{tikzcd}
X \arrow[d, "f"] \\
Y
\end{tikzcd}
$$

"0-cells" – objects of $\mathbb{D}_0$

"vertical morphisms" – morphisms of $\mathbb{D}_0$

$$
\begin{tikzcd}
O \arrow[d, "n"'] \\
Y
\end{tikzcd}
\quad \xleftarrow{\ T\ } \quad
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
\quad \xmapsto{\ S\ } \quad
\begin{tikzcd}
I \arrow[d, "m"] \\
X
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[d, "f"'] \\
B
\end{tikzcd}
\quad \xmapsto{\ u\ } \quad
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow id"] & A \arrow[l, "id_A"'] \arrow[d, "f"] \\
B & B \arrow[l, "id_B"]
\end{tikzcd}
$$

$\mathbb{D}_1$:
$$
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
$$

"horizontal morphisms" – objects of $\mathbb{D}_1$

"2-cells" – morphisms of $\mathbb{D}_1$

⑤ DEFINITION: A DOUBLE CATEGORY $\mathbb{D}$ IS A (PSEUDO) INTERNAL CATEGORY IN $\underline{CAT}$

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "u"] & \mathbb{D}_1 \arrow[l, bend right, "S"'] \arrow[l, bend left, "T"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\diamond_h"']
\end{tikzcd}$$

$\mathbb{D}_0$:
$$\begin{tikzcd}
X \arrow[d, "f"] \\
Y
\end{tikzcd}$$

"0-cells" – objects of $\mathbb{D}_0$

"vertical morphisms" – morphisms of $\mathbb{D}_0$

$$\begin{tikzcd}
O \arrow[d, "n"'] \\
Y
\end{tikzcd}
\xleftarrow{\;T\;}\!\!\shortmid
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
\shortmid\!\!\xrightarrow{\;S\;}
\begin{tikzcd}
I \arrow[d, "m"] \\
X
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, "f"'] \\
B
\end{tikzcd}
\shortmid\!\!\xrightarrow{\;U\;}
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow id"] & A \arrow[l, "id_A"'] \arrow[d, "f"] \\
B & B \arrow[l, "id_B"]
\end{tikzcd}$$

$\mathbb{D}_1$:
$$\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}$$

"horizontal morphisms" – objects of $\mathbb{D}_1$

"2-cells" – morphisms of $\mathbb{D}_1$

$$\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
O' & I' \arrow[l, "r'"]
\end{tikzcd}
\;\diamond_v\;
\begin{tikzcd}
O' \arrow[d, "n'"'] \arrow[dr, phantom, "\Downarrow\alpha'"] & I' \arrow[l, "r'"'] \arrow[d, "m'"] \\
O'' & I'' \arrow[l, "r''"]
\end{tikzcd}
:=
\begin{tikzcd}
O \arrow[d, "n' \circ n"'] \arrow[dr, phantom, "\Downarrow \alpha' \circ \alpha"] & I \arrow[l, "r"'] \arrow[d, "m' \circ m"] \\
O'' & I'' \arrow[l, "r''"]
\end{tikzcd}$$

"vertical composition"

– composition in $\mathbb{D}_1$

5 DEFINITION: A DOUBLE CATEGORY $\mathbb{D}$ IS A (PSEUDO) INTERNAL CATEGORY IN $\underline{CAT}$

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[rr, "u"] & & \mathbb{D}_1 \arrow[ll, bend right, "S"'] \arrow[ll, bend left, "T"] & & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[ll, "\diamond_h"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D}_0 : & X \arrow[d, "f"] \\
& Y
\end{tikzcd}$$

"0-cells" — objects of $\mathbb{D}_0$

"vertical morphisms" — morphisms of $\mathbb{D}_0$

$$\begin{tikzcd}
O \arrow[d, "n"'] & & O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] & & I \arrow[d, "m"] \\
Y & & Y & X \arrow[l, "s"] & & X
\end{tikzcd}
\qquad \xleftarrow{\ T\ } \text{ (first to second)}, \quad \xrightarrow{\ S\ } \text{ (second to third)}$$

$$\begin{tikzcd}
A \arrow[d, "f"'] & \overset{u}{\longmapsto} & A \arrow[d, "f"'] & A \arrow[l, "id_A"'] \arrow[d, "f"] \arrow[dl, phantom, "\Downarrow id"] \\
B & & B & B \arrow[l, "id_B"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D}_1 : & O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
& Y & X \arrow[l, "s"]
\end{tikzcd}$$

"horizontal morphisms" — objects of $\mathbb{D}_1$

"2-cells" — morphisms of $\mathbb{D}_1$

$$\begin{tikzcd}
O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
O' & I' \arrow[l, "r'"]
\end{tikzcd}
\ \diamond_v\
\begin{tikzcd}
O' \arrow[d, "n'"'] & I' \arrow[l, "r'"'] \arrow[d, "m'"] \arrow[dl, phantom, "\Downarrow\alpha'"] \\
O'' & I'' \arrow[l, "r''"]
\end{tikzcd}
\ :=\
\begin{tikzcd}
O \arrow[d, "n' \circ n"'] & I \arrow[l, "r"'] \arrow[d, "m' \circ m"] \arrow[dl, phantom, "\Downarrow\alpha' \circ \alpha"] \\
O'' & I'' \arrow[l, "r''"]
\end{tikzcd}$$

"vertical composition"

— composition in $\mathbb{D}_1$

$$\begin{tikzcd}
C \arrow[d, "c"'] & B \arrow[l, "r_2"'] \arrow[d, "b"] \arrow[dl, phantom, "\Downarrow\alpha_2"] \\
C' & B' \arrow[l, "s_2"]
\end{tikzcd}
\ \diamond_h\
\begin{tikzcd}
B \arrow[d, "b"'] & A \arrow[l, "r_1"'] \arrow[d, "a"] \arrow[dl, phantom, "\Downarrow\alpha_1"] \\
B' & A' \arrow[l, "s_1"]
\end{tikzcd}
\ :=\
\begin{tikzcd}
C \arrow[d, "c"'] & A \arrow[l, "r_2 \diamond_h r_1"'] \arrow[d, "a"] \arrow[dl, phantom, "\Downarrow\alpha_2 \diamond_h \alpha_1"] \\
C' & A' \arrow[l, "s_2 \diamond_h s_1"]
\end{tikzcd}$$

"horizontal composition" ← IN GENERAL ONLY WEAKLY ASSOCIATIVE!

⑥ DEFINITION: A **PRESENTATION** OF A DOUBLE CATEGORY $\mathbb{D}$ IS A FAMILY $(h_n)_{n\geq 0}$ OF FUNCTORS $h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$, WHERE $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \cdots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$,

$$h_0 := U, \quad h_1 := id, \quad h_2(-_2, -_1) := -_2 \diamond_h -_1,$$

$$\forall n \geq 2: \ h_{n+1}(-_{n+1}, \ldots, -_1) \cong h_2(-_{n+1}, h_n(-_n, \ldots, -_1))$$

- NOTATIONAL CONVENTION:

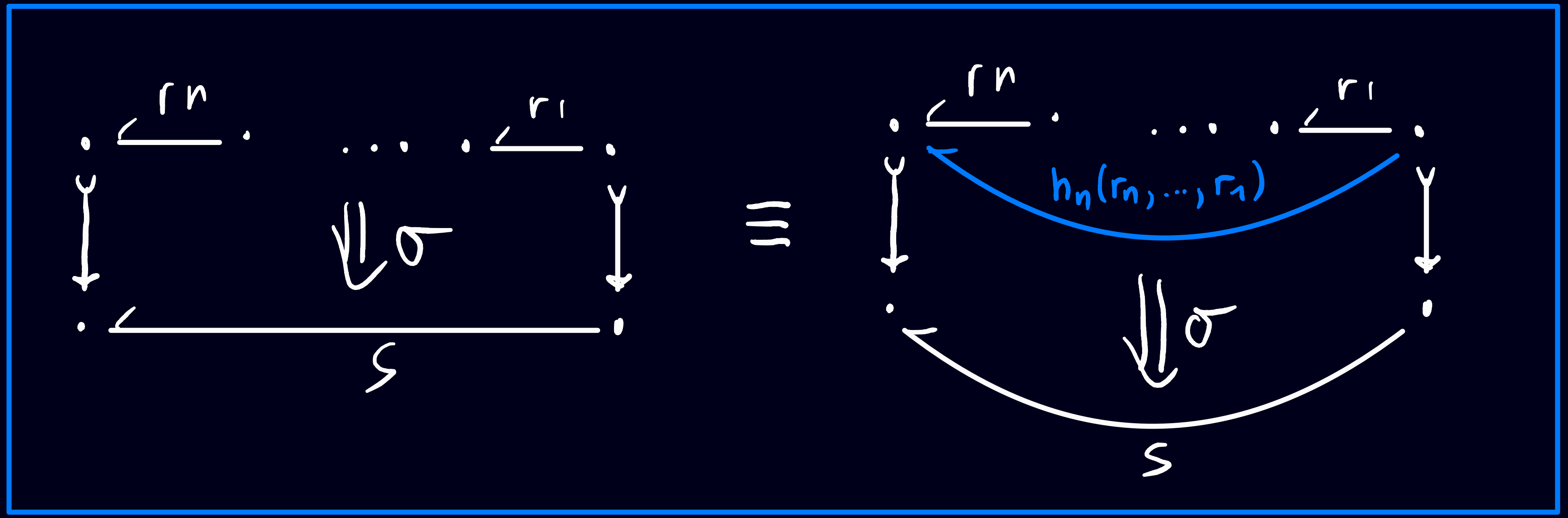
$$\begin{tikzcd} \cdot \arrow[d] & \cdot \arrow[l, "r_n"'] & \cdots & \cdot & \cdot \arrow[l, "r_1"'] \arrow[d] \\ \cdot & & \Downarrow\sigma & & \cdot \arrow[llll, "s"] \end{tikzcd} \equiv \begin{tikzcd} \cdot \arrow[d] & \cdot \arrow[l, "r_n"'] & \cdots & \cdot & \cdot \arrow[l, "r_1"'] \arrow[d] \arrow[llll, bend left, "h_n(r_n, \ldots, r_1)"] \\ \cdot & & \Downarrow\sigma & & \cdot \arrow[llll, bend left, "s"] \end{tikzcd}$$

⑥ DEFINITION: A **PRESENTATION** OF A DOUBLE CATEGORY $\mathbb{D}$ IS A FAMILY $(h_n)_{n\geq 0}$ OF FUNCTORS $h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$, WHERE $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \cdots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$,

$$h_0 := U, \quad h_1 := id, \quad h_2(-_2,-_1) := -_2 \circ_h -_1,$$

$$\forall n\geq 2: \; h_{n+1}(-_{n+1},\dots,-_1) \cong h_2(-_{n+1}, h_n(-_n,\dots,-_1))$$

- NOTATIONAL CONVENTION:

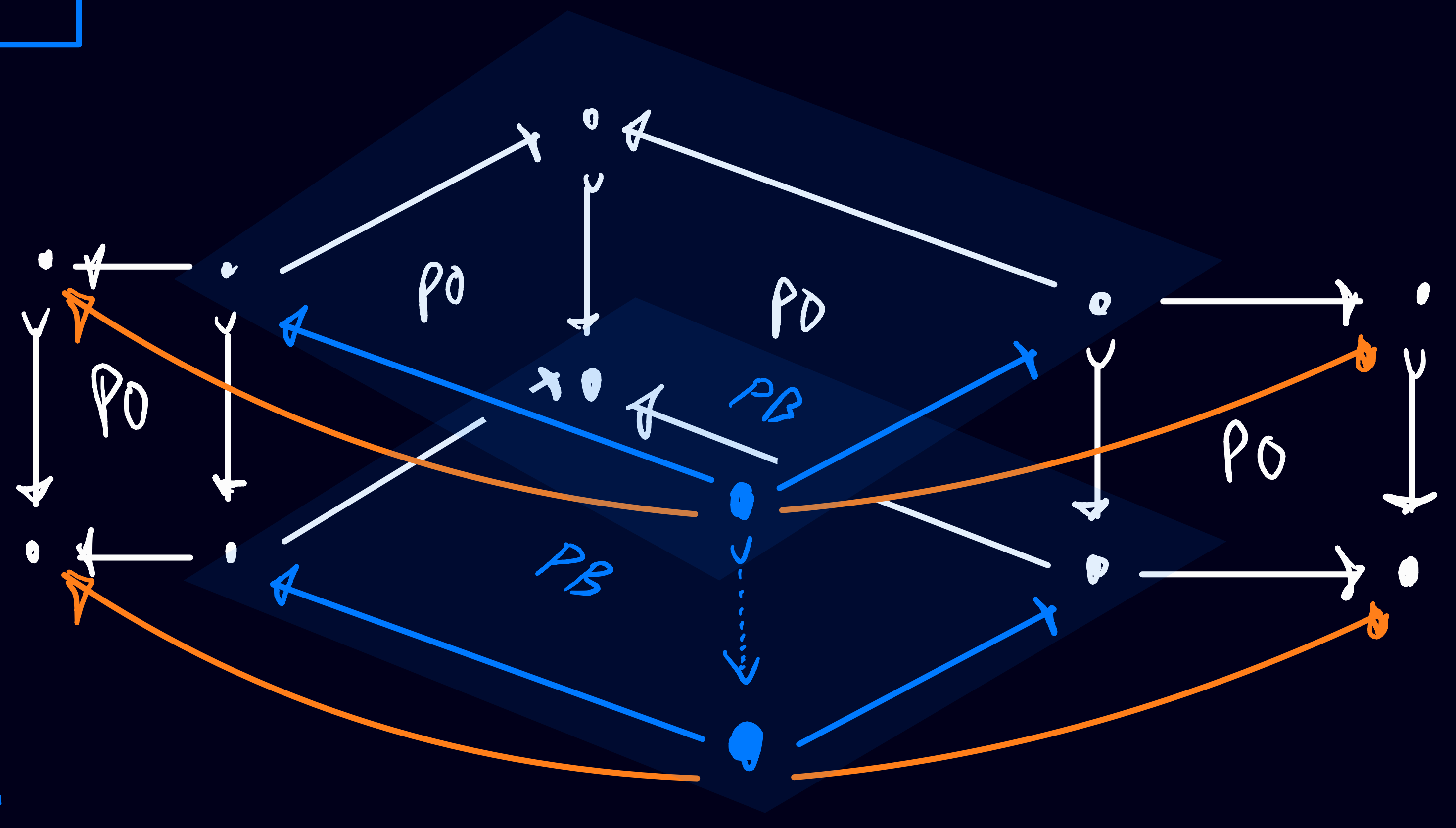


- EXAMPLE:

$$\begin{tikzcd} O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\ Y & X \arrow[l, "s"] \end{tikzcd} := \begin{tikzcd} O \arrow[d, "n"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\ Y & K_s \arrow[l, "o_s"] \arrow[r, "i_s"'] & X \end{tikzcd}$$

↝ HORIZONTAL COMPOSITION:

≙ CHOICE OF PULLBACKS (PBs)!



⑦ KEY CONCEPT: (COVARIANT) PRESHEAVES $F : \mathbb{D}_1 \to \underline{\mathbf{Set}}$

↳ IDEA: $\forall r \in \mathbb{D}_1 : \quad \hat{\Delta}_r := \mathbb{D}_1(r, -)$

↳ $\left|\hat{\Delta}_r(Y \xleftarrow{s} X)\right| = \left| \left\{ \begin{array}{ccc} O & \xleftarrow{r} & I \\ n \downarrow & \Downarrow\alpha & \downarrow m \\ Y & \xleftarrow[s]{} & X \end{array} \in \mathbb{D}_1 \right\} \right| \propto$ "# ways to rewrite X into Y along $Y \xleftarrow{s} X$ with rule $O \xleftarrow{r} I$"

▸ BUT: we want "$\rho(\delta(r))|X\rangle = \underbrace{\rho(\delta(r))\,\rho(\delta(\overset{?}{\overbrace{X \leftarrow \emptyset}}))|\overset{?}{\widetilde{\emptyset}}\rangle}_{?} = \sum_\alpha |r_\alpha(X)\rangle = \sum_Y \underbrace{\overbrace{M_{r,X}{}^{Y}}^{\in \mathbb{Z}_{\geq 0}}}_{?} |Y\rangle$"

⑦ KEY CONCEPT: (COVARIANT) PRESHEAVES $F : \mathbb{D}_1 \to \underline{\mathbf{Set}}$

↳ $\underline{\text{IDEA}}$: $\forall\, r \in \mathbb{D}_1 : \ \hat{\Delta}_r := \mathbb{D}_1(r, -)$

↳ $\left|\hat{\Delta}_r\left(Y \xleftarrow{s} X\right)\right| = \left|\left\{ \begin{tikzcd} O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\ Y & X \arrow[l, "s"] \end{tikzcd} \in \mathbb{D}_1 \right\}\right| \propto$ "# ways to rewrite X into Y along $Y \xleftarrow{s} X$ with rule $O \xleftarrow{r} I$"

▸ BUT: we want "$\underbrace{\rho(\delta(r))\,\rho(\delta(\overbrace{X \leftarrow \emptyset}^{?}))}_{?} |\overset{?}{\widetilde{\emptyset}}\rangle = \sum_\alpha |r_\alpha(X)\rangle = \sum_Y \underbrace{\overbrace{M_{r,X}^{\;Y}}^{\in \mathbb{Z}_{\geq 0}}}_{?} |Y\rangle$"

▸ $\underline{\text{ASSUMPTION}}$: $\mathbb{D}_0$ HAS A STRICT INITIAL OBJECT $\emptyset$ (i.e., $\forall\, X \in \mathbb{D}_0 : \exists!\ \emptyset \rightarrowtail X \ \wedge\ \forall\, X \xrightarrow{f} \emptyset : X = \emptyset$),

AND SUCH THAT (i) $\forall\, X \in \mathbb{D}_0 : \exists!\ (X \leftarrow \emptyset) \in ob(\mathbb{D}_1) \ \wedge\ \exists!\ (\emptyset \leftarrow X) \in ob(\mathbb{D}_1)$

(ii) $\forall \begin{smallmatrix} X \\ f\downarrow \\ Y \end{smallmatrix} \in \mathbb{D}_0 : \left|\left\{ \begin{tikzcd} X \arrow[d, "f"'] & \emptyset \arrow[l] \arrow[d, equal] \arrow[dl, phantom, "\Downarrow\alpha"] \\ Y & \emptyset \arrow[l] \end{tikzcd} \in \mathbb{D}_1 \right\}\right| \leq 1 \ \wedge\ \left|\left\{ \begin{tikzcd} \emptyset \arrow[d, equal] & X \arrow[l] \arrow[d, "f"] \arrow[dl, phantom, "\Downarrow\beta"] \\ \emptyset & Y \arrow[l] \end{tikzcd} \in \mathbb{D}_1 \right\}\right| \leq 1$

⑧ DEFINITION: A COEND FOR A FUNCTOR $F : \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathbf{Set}}$

IS DEFINED AS

$$\int^{C \in \mathcal{C}} F(C,C) = \left(\coprod_{C \in \mathcal{C}} F(C,C) \right) \Big/ \sim$$

with: $\underbrace{(C,x)}_{F(C,C)} \sim \underbrace{(C',x')}_{F(C',C')} :\Longleftrightarrow \exists\, C \xrightarrow{\gamma} C',\ y \in F(C',C) : x = F(\gamma, id)y \ \wedge\ x' = F(id,\gamma)y$

⑧ DEFINITION: A COEND FOR A FUNCTOR $F:\mathcal{C}^{op}\times\mathcal{C}\to\underline{Set}$

IS DEFINED AS
$$\int^{C\in\mathcal{C}} F(C,C) = \left(\coprod_{C\in\mathcal{C}} F(C,C)\right)\Big/\sim$$

with: $(C,\overset{F(C,C)}{\overset{\Downarrow}{x}})\sim(C',\overset{F(C',C')}{\overset{\Downarrow}{x'}}) :\Longleftrightarrow \exists\, C\xrightarrow{\gamma}C',\ y\in F(C',C):\ x=F(\gamma,id)y \ \wedge\ x'=F(id,\gamma)y$

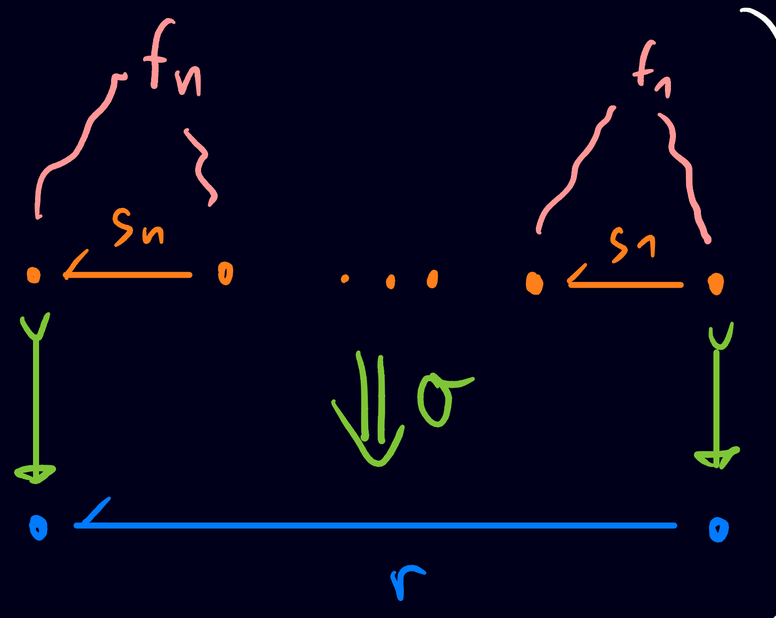
KEY CONCEPT: CONVOLUTION PRODUCTS OF PRESHEAVES $F_n,\dots,F_1:\mathbb{D}_1\to\underline{Set}$

- $\mathbb{D}_1(h_n(-),r):\mathbb{D}_n^{op}\to\underline{Set}$
- $\mathbb{F}_n := F_n\times\dots\times F_1:\mathbb{D}_n\to\underline{Set}$

$$(F_n*\cdots*F_1) := r\mapsto \int^{S=(S_n,\dots,S_1)\in\mathbb{D}_n} \mathbb{D}_1(h_n(S),r)\times\mathbb{F}_n(S)$$

$(\cong \mathrm{Lan}_{h_n}(\mathbb{F}_n))$

$$= \left\{ (S,(\sigma,f)) \;\middle|\; \begin{array}{l} S\in\mathbb{D}_n \\ \sigma\in\mathbb{D}_1(h_n(S),r) \\ f\in\mathbb{F}_n(S)\end{array}\right\}\Big/\sim \;\equiv\; \left\{ \right.$$



$\left.\right\}/\sim$

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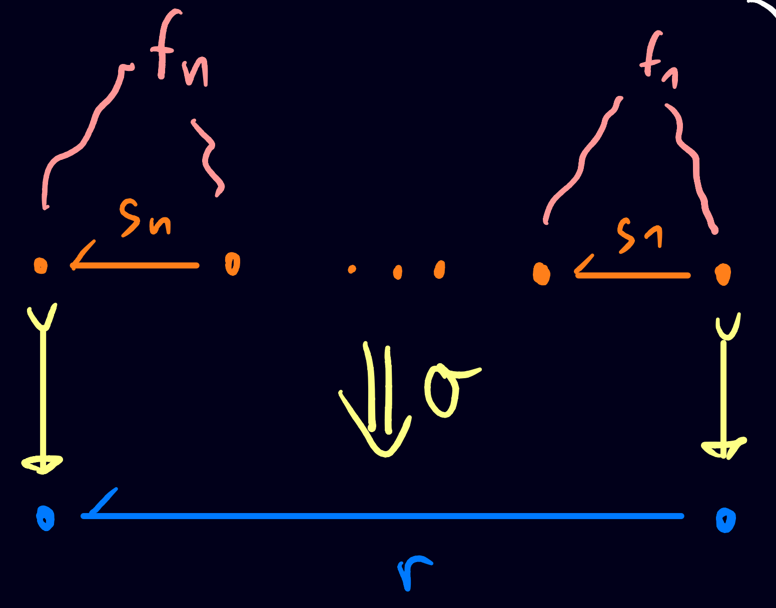
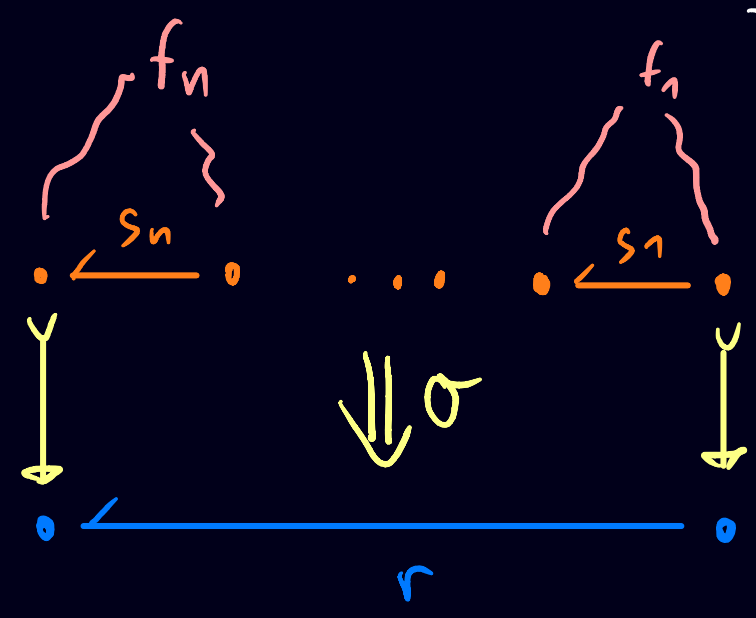
$$(F_n * \cdots * F_1)(r) = \left\{ (S,(\sigma,f)) \;\middle|\; \begin{array}{l} S \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(S), r) \\ f \in \mathbb{F}_n(S) \end{array} \right\} \Big/ \sim \;\equiv\; \left\{ \right.$$



$$\left. \right\} \Big/ \sim$$

- $(S,(\sigma,f)) \sim (S',(\sigma',f')) :\Leftrightarrow \exists\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ (\tau,g) \in \mathbb{D}_1(h_n(S'), r) \times \mathbb{F}_n(S):$

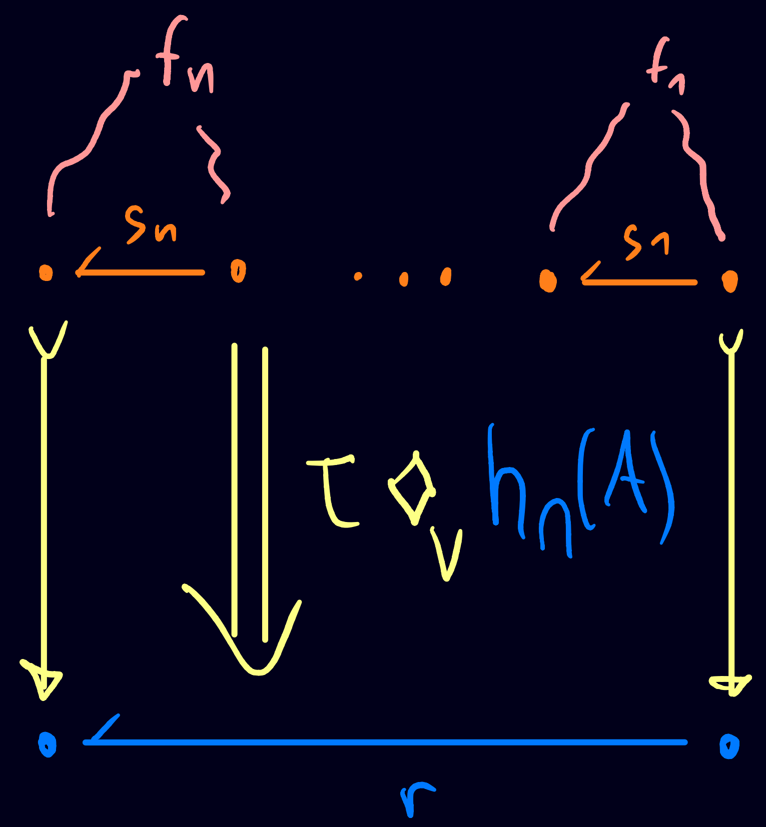
$$(\sigma,f) = (\mathbb{D}_1(h_n(A), r)\tau,\ g) \;\wedge\; (\sigma',f') = (\tau,\ \mathbb{F}_n(A)g)$$

⑨ $(F_n * \cdots * F_1)(r) = \left\{ (S,(\sigma,f)) \;\middle|\; \begin{array}{l} S \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(S), r) \\ f \in \mathbb{F}_n(S) \end{array} \right\} \Big/ \sim \;\equiv\; \Bigg\{$  $\Bigg\} \Big/ \sim$

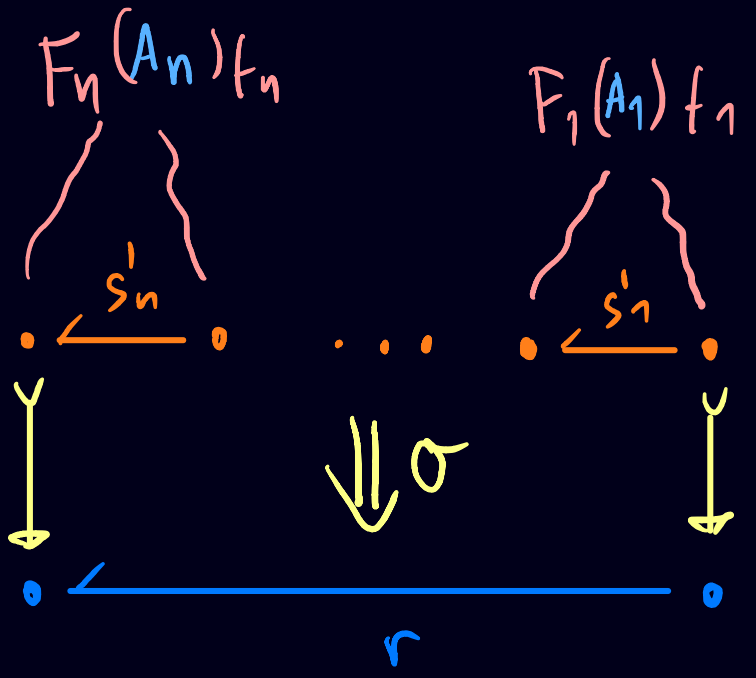
$\cdot\ (S,(\sigma,f)) \sim (S',(\sigma',f')) :\Leftrightarrow \exists\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ (\tau,g) \in \mathbb{D}_1(h_n(S'), r) \times \mathbb{F}_n(S):$

$$(\sigma,f) = \big(\mathbb{D}_1(h_n(A), r)\tau,\ g\big) \;\wedge\; (\sigma',f') = \big(\tau,\ \mathbb{F}_n(A)g\big)$$

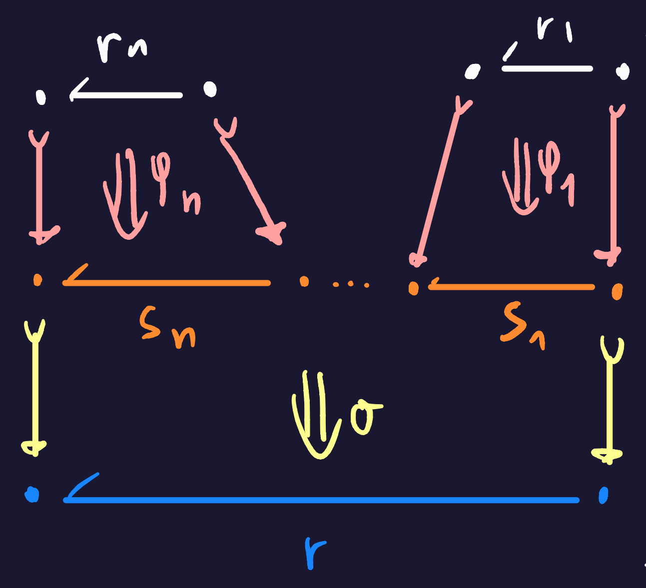
↳ $\forall\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1:$



$\sim$



EXAMPLE: $\hat{\Delta}_{r_j} := \mathbb{D}_1(r_j, -) : \mathbb{D}_1 \to \underline{Set}$ $(j = 1, \ldots, n)$

↳ $(\hat{\Delta}_{r_n} * \cdots * \hat{\Delta}_{r_1})(r) = \Bigg\{$  $\Bigg\} \Big/ \sim$

⑩ Key concept: Fibrational structures

- Definition: A functor $G:\mathcal{E}\to\mathcal{B}$ is a **Grothendieck opfibration** iff

$$\forall\ \begin{tikzcd} e \arrow[d, mapsto, "G"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd} \ :\ \exists\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, mapsto, "G"'] & e' \arrow[d, mapsto, "G"] \\ b \arrow[r, "f"'] & b' \end{tikzcd} \ :\ \forall\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, mapsto, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, mapsto, "G"'] & e'' \arrow[d, mapsto, "G"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd} \ :\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, mapsto, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, mapsto, "G"'] \arrow[r, dashed, "\beta", "\exists!"'] & e'' \arrow[d, mapsto, "G"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g=G(\beta)"'] & b'' \end{tikzcd}$$

⑩ KEY CONCEPT: FIBRATIONAL STRUCTURES

- DEFINITION: A functor $G:\mathcal{E}\to\mathcal{B}$ is a GROTHENDIECK OPFIBRATION iff

$$
\forall\;
\begin{tikzcd}
e \arrow[d, "G"'] & \\
b \arrow[r, "f"'] & b'
\end{tikzcd}
\;:\;
\exists\;
\begin{tikzcd}
e \arrow[r, "\gamma(f)"] \arrow[d, "G"'] & e' \arrow[d, "G"] \\
b \arrow[r, "f"'] & b'
\end{tikzcd}
\;:\;
\forall\;
\begin{tikzcd}
e \arrow[r, "\gamma(f)"] \arrow[d, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, "G"] & e'' \arrow[d, "G"] \\
b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b''
\end{tikzcd}
\;:\;
\begin{tikzcd}
e \arrow[r, "\gamma(f)"] \arrow[d, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, "G"] \arrow[r, dashed, "\beta", "\exists!"'] & e'' \arrow[d, "G"] \\
b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g=G(\beta)"'] & b''
\end{tikzcd}
$$

- DEFINITION: A functor $S:\mathcal{E}\to\mathcal{B}$ is a STREET OPFIBRATION iff

$$
\forall\;
\begin{tikzcd}
e \arrow[d, "S"'] & \\
b \arrow[r, "f"'] & b'
\end{tikzcd}
\;:\;
\exists\;
\begin{tikzcd}
e \arrow[rr, "\sigma(f)"] \arrow[d, "S"'] & & \tilde{e} \arrow[d, "S"] \\
b \arrow[r, "f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b}
\end{tikzcd}
\;:\;
\forall\;
\begin{tikzcd}
e \arrow[rr, "\sigma(f)"] \arrow[d, "S"'] \arrow[rrr, bend left, "\varepsilon"] & & \tilde{e} \arrow[d, "S"] & e'' \arrow[d, "S"] \\
b \arrow[r, "f"'] \arrow[rrr, bend right, "S(\varepsilon)=g\circ\varphi_f\circ f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \arrow[r, "g"'] & b''
\end{tikzcd}
\;:\;
\begin{tikzcd}
e \arrow[rr, "\sigma(f)"] \arrow[d, "S"'] \arrow[rrr, bend left, "\varepsilon"] & & \tilde{e} \arrow[d, "S"] \arrow[r, dashed, "\beta"] & e'' \arrow[d, "S"] \\
b \arrow[r, "f"'] \arrow[rrr, bend right, "S(\varepsilon)=g\circ\varphi_f\circ f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \arrow[r, "g=S(\beta)"'] & b''
\end{tikzcd}
$$

(11) DEFINITION: A FUNCTOR $M:\mathcal{E}\to\mathcal{B}$ IS A MULTI-OPFIBRATION IFF

$$
\forall\;
\begin{tikzcd}
e \arrow[d, mapsto, "M"'] & \\
b \arrow[r, "f"'] & b'
\end{tikzcd}
\;:\;
\exists
\left\{
\begin{tikzcd}
e \arrow[r, "\mu_j(f)"] \arrow[d, mapsto, "M"'] & e_j \arrow[d, mapsto, "M"] \\
b \arrow[r, "f"'] & b'
\end{tikzcd}
\right\}_{j\in J_{e,f}}
\;:\;
\forall\;
\begin{tikzcd}
e \arrow[rr, bend left, "\varepsilon"] \arrow[d, mapsto, "M"'] & & e'' \arrow[d, mapsto, "M"] \\
b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b''
\end{tikzcd}
\;:\;
\exists j\in J_{e,f}:\;
\begin{tikzcd}
e \arrow[r, "\mu_j(f)"'] \arrow[rr, bend left, "\varepsilon"] \arrow[d, mapsto, "M"'] & e'_j \arrow[r, dashed, "\beta_j", "\exists!"'] \arrow[d, mapsto, "M"] & e'' \arrow[d, mapsto, "M"] \\
b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g=M(\beta_j)"'] & b''
\end{tikzcd}
$$

$$
\wedge\;\forall\;
\begin{tikzcd}
e \arrow[r, "x"] \arrow[rr, bend left, "\varepsilon"] \arrow[d, mapsto, "M"'] & X \arrow[r, "\gamma"] \arrow[d, mapsto, "M"] & e'' \arrow[d, mapsto, "M"] \\
b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b''
\end{tikzcd}
\;:\;
\begin{tikzcd}
e \arrow[r, "\mu_j(f)"'] \arrow[rr, bend left, "\varepsilon"] \arrow[dd, mapsto, "M"'] \arrow[dr, bend right, "x"'] & e'_j \arrow[r, dashed, "\beta_j"] \arrow[d, dashed, "\exists!"] \arrow[dd, mapsto, "M", near start] & e'' \arrow[dd, mapsto, "M"] \\
 & X \arrow[ur, bend right, "\gamma"'] \arrow[d, mapsto, "M"] & \\
b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b''
\end{tikzcd}
$$

12 DEFINITION: A FUNCTOR $R:\mathcal{E}\to\mathcal{B}$ IS A RESIDUAL MULTI-OPFIBRATION IFF

$$\forall\ \begin{tikzcd} & e \arrow[d, maps to, "R"'] \\ & b \arrow[r, "f"'] & b' \end{tikzcd} \ :\ \exists \left\{ \begin{tikzcd} e \arrow[r, "\varphi_j(f)"] \arrow[d, maps to, "R"'] & e'_j \arrow[d, maps to] \\ b \arrow[r, "f"'] & b' \arrow[r, "f x_j"'] & b_j \end{tikzcd} \right\}_{j\in J_{e,f}}$$

"residue"

$$:\ \forall\ \begin{tikzcd} e \arrow[rrr, bend left, "\varepsilon"] \arrow[d, maps to, "R"'] & & & e'' \arrow[d, maps to] \\ b \arrow[r, "f"'] & b' \arrow[rr, "g"'] & & b'' \end{tikzcd} \quad R(\varepsilon)=g\circ f$$

$$:\ \exists\, j\in J_{e,f}:\ \begin{tikzcd} e \arrow[rr, "\varphi_j(f)"] \arrow[rrr, bend left, "\varepsilon"] \arrow[d, maps to, "R"'] & & e'_j \arrow[r, dashed, "\exists!\ \beta_j"] \arrow[d, maps to, "R"] & e'' \arrow[d, maps to, "R"] \\ b \arrow[r, "f"'] \arrow[rrr, bend right, "R(\varepsilon)=g\circ f"'] & b' \arrow[r, "f x_j"] \arrow[rr, "g"', bend right=10] & b_j \arrow[r, "h_j"] & b'' \end{tikzcd}$$

$$\wedge\ \forall\ \begin{tikzcd} e \arrow[r, "\chi"] \arrow[rr, bend left, "\varepsilon=\chi\circ\gamma"] \arrow[d, maps to, "R"'] & \tilde{e} \arrow[r, "\gamma"] \arrow[d, maps to, "R"] & e'' \arrow[d, maps to, "R"] \\ & c \arrow[dr, "y"] & \\ b \arrow[r, "f"'] \arrow[rr, bend right, "R(\varepsilon)=g\circ f"'] & b' \arrow[u, "x"] \arrow[r, "g"'] & b'' \end{tikzcd}$$

$$:\ \exists!\ e'_j \xrightarrow{\varphi_j} \tilde{e}\ :\ \begin{tikzcd} e \arrow[r, "\varphi_j(f)"] \arrow[rr, bend right, "\chi"'] \arrow[rrr, bend left, "\varepsilon"] \arrow[d, maps to, "R"'] & e'_j \arrow[rr, dashed, "\beta_j"] \arrow[d, maps to, "R"] \arrow[r, "\varphi_j"] & \tilde{e} \arrow[r, "\gamma"] \arrow[dd, maps to, "R"] & e'' \arrow[d, maps to, "R"] \\ b \arrow[r, "f"'] \arrow[rrr, bend right=60, "R(\varepsilon)=g\circ f"'] & b' \arrow[r, "f x_j"] \arrow[rr, "g"] \arrow[dr, "x"'] & b_j \arrow[d, "R(\varphi_j)"] & b'' \\ & & c \arrow[ur, "y"'] & \end{tikzcd}$$

Nicolas Behr, *Virtual Double Categories Workshop*, November 30, 2022

13 DEFINITION: LET $X : \mathcal{E} \to \mathcal{B}$ BE AN X-OPFIBRATION ($X \in \{G, S, M, R\}$).

THEN A CLEAVAGE FOR X IS DEFINED AS A CHOICE OF REPRESENTATIVE FOR EACH X-OPCARTESIAN LIFTING:

$$
G^{*}\left(\begin{tikzcd} e \arrow[d, "G"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \begin{tikzcd} e \arrow[r, "G^{*}(f)"] \arrow[d, "G"'] & b \arrow[d, "G"] \\ b \arrow[r, "f"'] & b \end{tikzcd}
$$

$$
S^{*}\left(\begin{tikzcd} e \arrow[d, "s"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \begin{tikzcd} e \arrow[rr, "\sigma^{*}(f)"] \arrow[d, "s"'] & & \tilde{e} \arrow[d, "s"] \\ b \arrow[r, "f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \end{tikzcd}
$$

$$
M^{*}\left(\begin{tikzcd} e \arrow[d, "M"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \left\{\begin{tikzcd} e \arrow[r, "\mu_j^{*}(f)"] \arrow[d, "M"'] & e'_j \arrow[d, "m"] \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right\}_{j \in \boxed{\mathcal{Z}^{*}_{e;f}}}
$$

$$
R^{*}\left(\begin{tikzcd} e \arrow[d, "R"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \left\{\begin{tikzcd} e \arrow[rr, "\rho_j^{*}(f)"] \arrow[d, "R"'] & & e'_j \arrow[d, "R"] \\ b \arrow[r, "f"'] & b' \arrow[r, "f x_j"'] & b'_j \end{tikzcd}\right\}_{j \in \boxed{\mathcal{Z}^{*}_{e;f}}}
$$

ONE REPRESENTATIVE PER EQUIVALENCE CLASS IN $\mathcal{Z}_{e;f}$!

14 EMPIRICAL RESULT: $\mathbb{D}$ FOR COMPOSITIONAL* REWRITING SEMANTICS

*2204.07175

- $h_2 = \diamond_h : \mathbb{D}_2 \longrightarrow \mathbb{D}_1$ IS A "GLOBULAR" STREET OPFIBRATION, i.e.,

$$\forall \begin{tikzcd} R=(r_2,r_1) \arrow[d, "h_2"'] & \\ r \arrow[r, "\alpha"'] & s \end{tikzcd} : \exists \begin{tikzcd} R \arrow[rr, "A"] \arrow[d, "h_2"'] & & T \arrow[d, "h_2"] \\ r \arrow[r, "\alpha"'] & s \arrow[r, "\cong", "\varphi_\alpha"'] & t \end{tikzcd} : S(\varphi_\alpha) = id_{S(s)} \wedge T(\varphi_\alpha) = id_{T(s)} \wedge$$

(STREET OPFIBRATION CONDITIONS)

$$\forall \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l, "r_2"'] \arrow[d, phantom, "\Downarrow\alpha"] & \bullet \arrow[l, "r_1"'] \arrow[d] \\ \bullet & & \bullet \arrow[ll, "s"] \end{tikzcd} : \exists \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l, "r_2"'] \arrow[d] \arrow[dl, phantom, "\Downarrow A_2"] & \bullet \arrow[l, "r_1"'] \arrow[d] \arrow[dl, phantom, "\Downarrow A_1"] \\ \bullet & \bullet \arrow[l, "t_2"'] \arrow[d, phantom, "\cong \Downarrow \varphi_\alpha^{-1}"] & \bullet \arrow[l, "t_1"'] \arrow[ll, bend left=60, "s"] \\ & & \end{tikzcd} : \varphi_\alpha^{-1} \diamond_v (A_2 \diamond A_1) = \alpha$$

"GLOBULAR" ISOMORPHISM

By INDUCTION ON n, ONE FINDS THAT

$\forall n \geq 2: \; h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$ ARE "GLOBULAR" STREET OPFIBRATIONS

15

$\mathbb{D}_0$ HAS MULTI-SUMS:

$$\forall (A,B)\in \mathbb{D}_0\times\mathbb{D}_0 : \exists \left\{ A \overset{a_j}{\rightarrowtail} M_j \overset{b_j}{\leftarrowtail} B \right\}_{j\in J_{(A,B)}} : \forall \begin{tikzcd} A \arrow[dr, tail, bend right, "x_A"'] & & B \arrow[dl, tail, bend left, "x_B"] \\ & X & \end{tikzcd} : \exists j \in J_{(A,B)} : \begin{tikzcd} A \arrow[r, tail, "a_j"] \arrow[dr, tail, bend right, "x_A"'] & M_j \arrow[d, dashed, tail, "\exists!\, m_j"] & B \arrow[l, tail, "b_j"'] \arrow[dl, tail, bend left, "x_B"] \\ & X & \end{tikzcd}$$

$$\wedge\ \forall \begin{tikzcd} A \arrow[dr, tail, "y_A"] \arrow[ddr, tail, bend right, "x_A"'] & & B \arrow[dl, tail, "y_B"'] \arrow[ddl, tail, bend left, "x_B"] \\ & Y \arrow[d, tail, "y_X"] & \\ & X & \end{tikzcd} : \exists!\ M_j \overset{m}{\rightarrowtail} Y : \begin{tikzcd} A \arrow[r, tail, "a_j"] \arrow[dr, tail, "y_A"] \arrow[ddr, tail, bend right, "x_A"'] & M_j \arrow[d, dashed, tail, "m"] & B \arrow[l, tail, "b_j"'] \arrow[dl, tail, "y_B"'] \arrow[ddl, tail, bend left, "x_B"] \\ & Y \arrow[d, tail, "y_X"] & \\ & X & \end{tikzcd}$$

DEFINITION: CLEAVAGE FOR MULTI-SUMS:

$$\forall (A,B)\in \mathbb{D}_0\times\mathbb{D}_0 : ms(A,B) = \left\{ A \overset{a_j}{\rightarrowtail} M_j \overset{b_j}{\leftarrowtail} B \right\}_{J^{\circ}_{(A,B)}}$$

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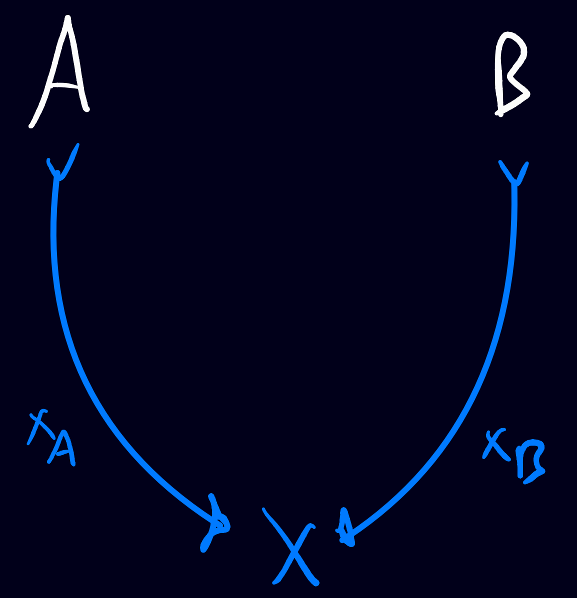
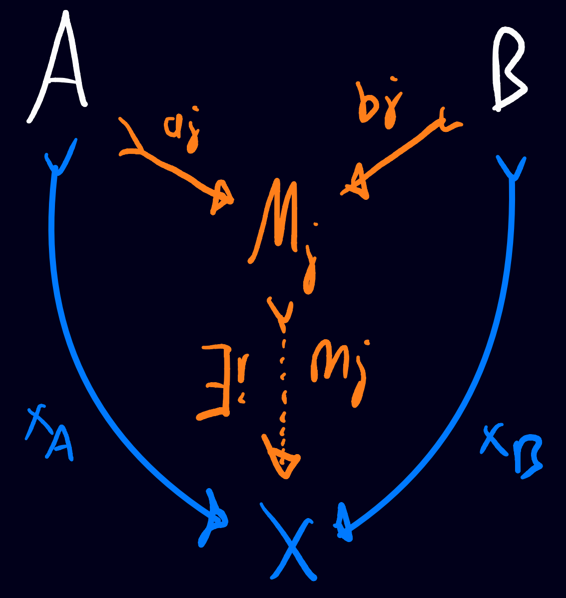
$\mathbb{D}_0$ HAS MULTI-SUMS:

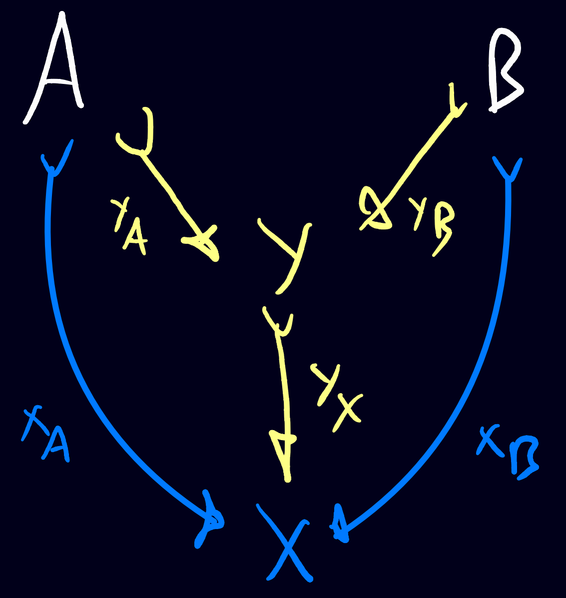
$\forall\ (A,B) \in \mathbb{D}_0 \times \mathbb{D}_0 : \exists \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{j \in J_{(A,B)}} : \forall$



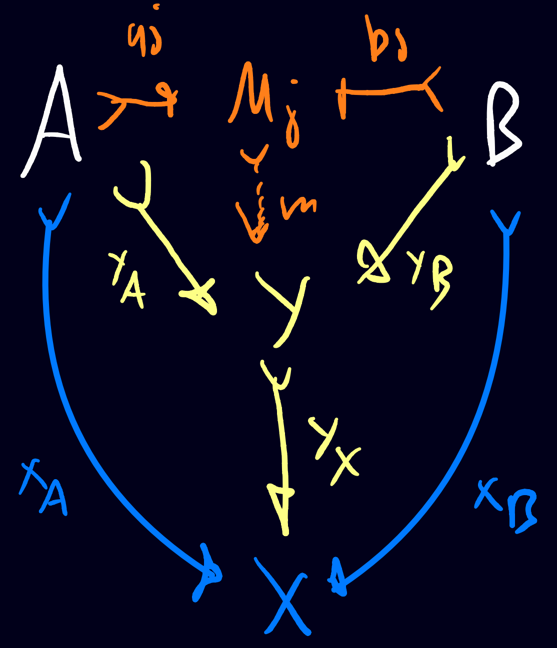
$: \exists j \in J_{(A,B)} :$



$\wedge\ \forall$



$: \exists!\ M_j \xrightarrow{m} Y :$

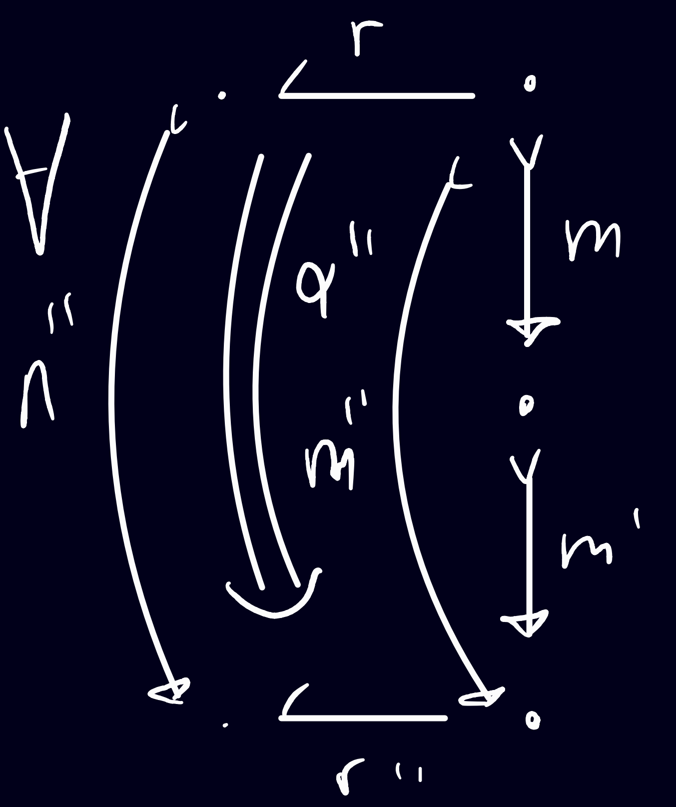


DEFINITION: CLEAVAGE FOR MULTI-SUMS:

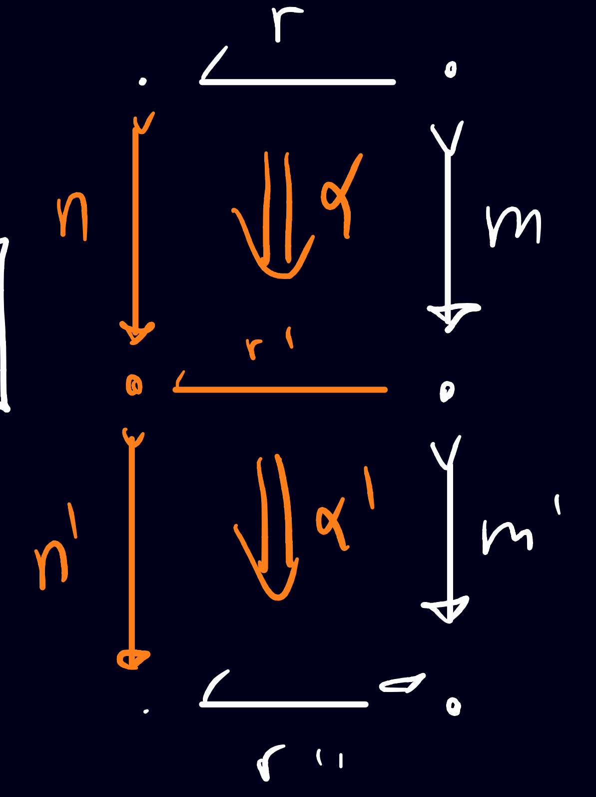
$\forall (A,B) \in \mathbb{D}_0 \times \mathbb{D}_0 : ms(A,B) = \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{J^{\circ}_{(A,B)}}$

$S : \mathbb{D}_1 \to \mathbb{D}_0$ IS A MULTI-OPFIBRATION

$\forall$



$: \exists$

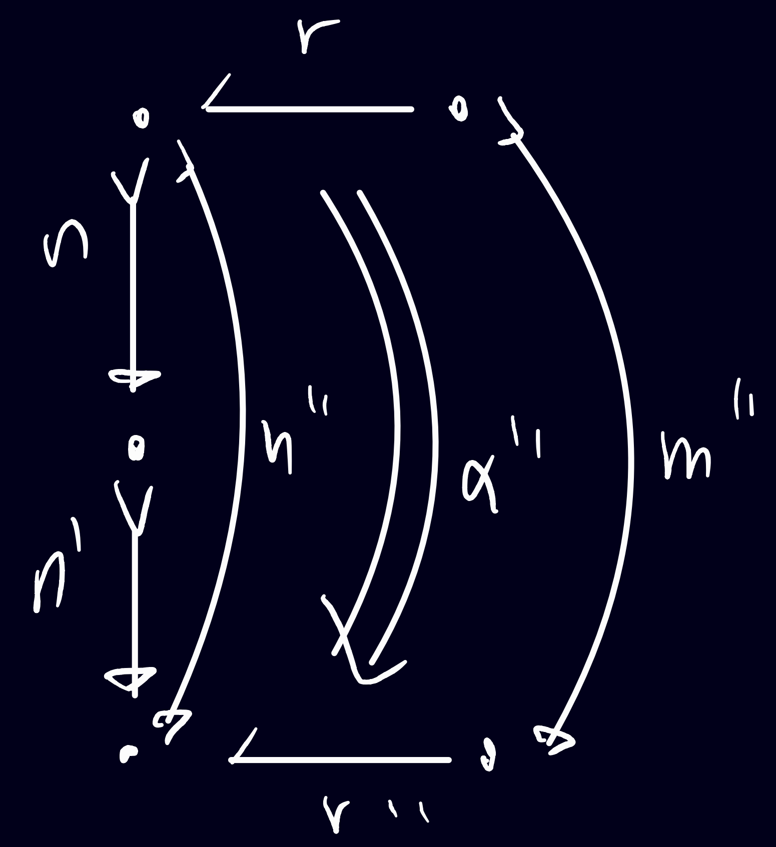


$: n' \circ n = n''$

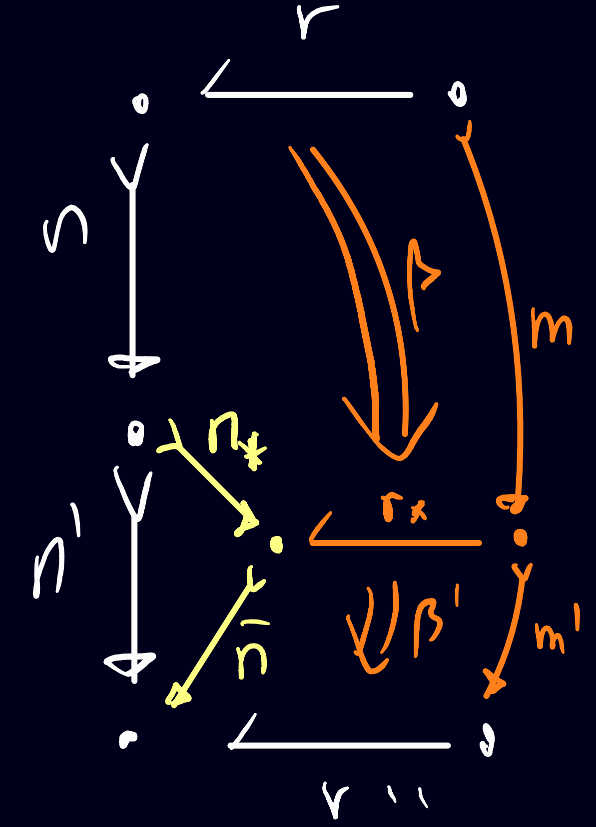
$\wedge\ \alpha' \circ_v \alpha = \alpha''$

$T : \mathbb{D}_1 \to \mathbb{D}_0$ IS A RESIDUAL MULTI-OPFIBRATION

$\forall$



$: \exists$



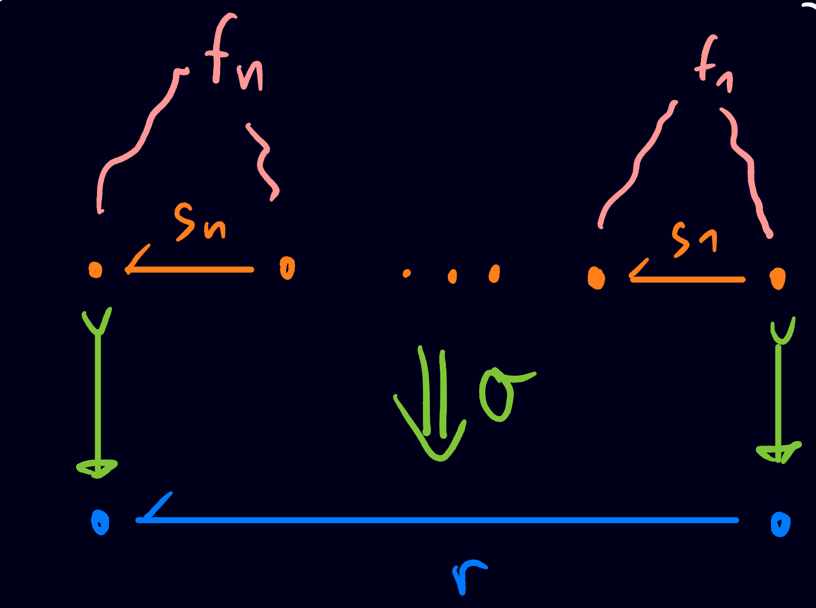
$: \bar{n} \circ n_x = n'$

$\wedge\ m' \circ m = m''$

$\wedge\ \beta' \circ_v \beta = \beta''$

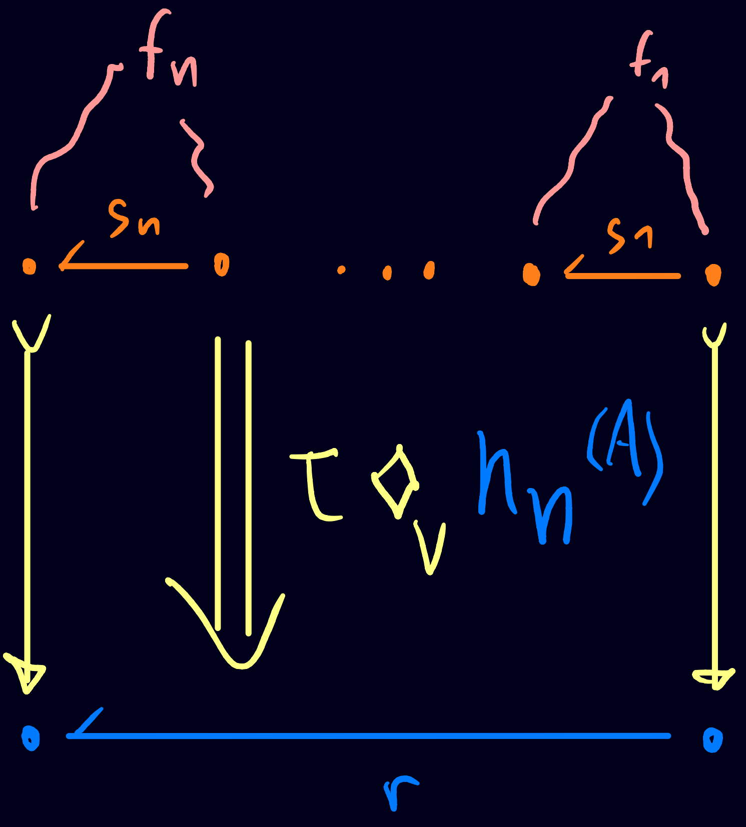
16 CONVOLUTION PRODUCTS REVISITED

RECAP: $$(F_n * \cdots * F_1)(r) := \int^{S=(S_n,\dots,S_1)\in \mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S) = \left\{ \vphantom{\int} \right.$$

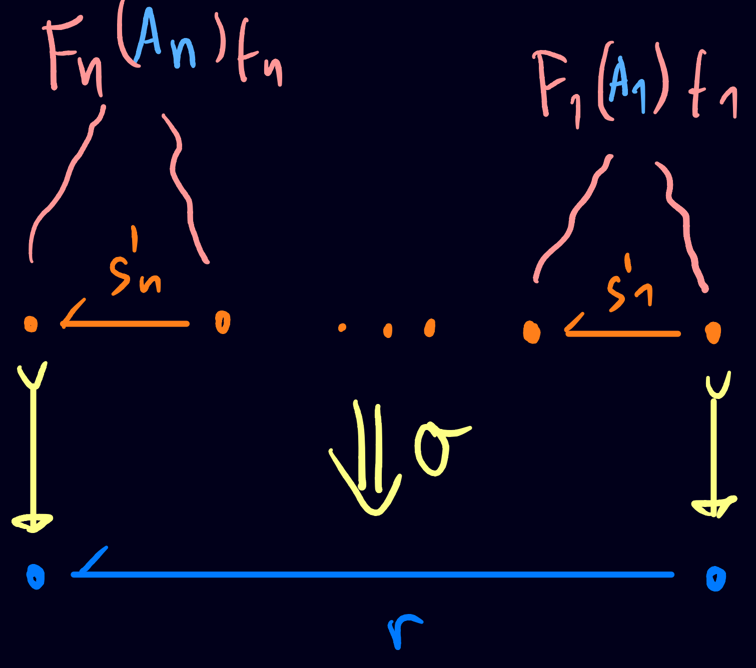


$\left. \vphantom{\int} \right\} / \sim$

$\forall S \xrightarrow{A} S' \in \mathbb{D}_n,$
$h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:

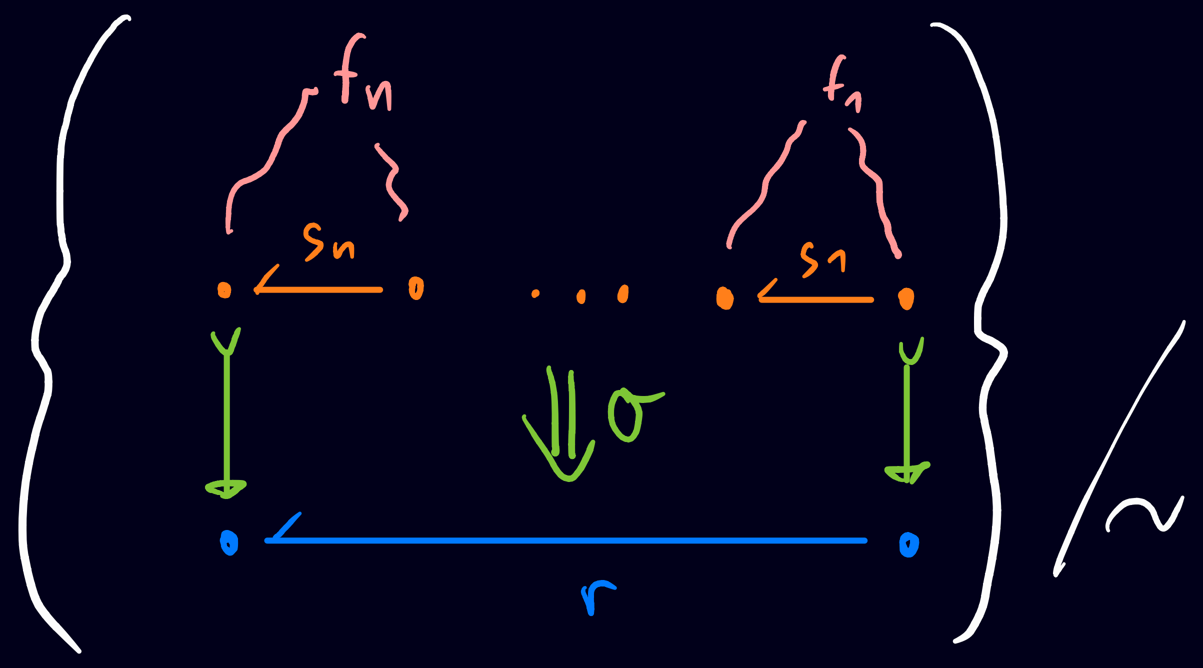


$\sim$

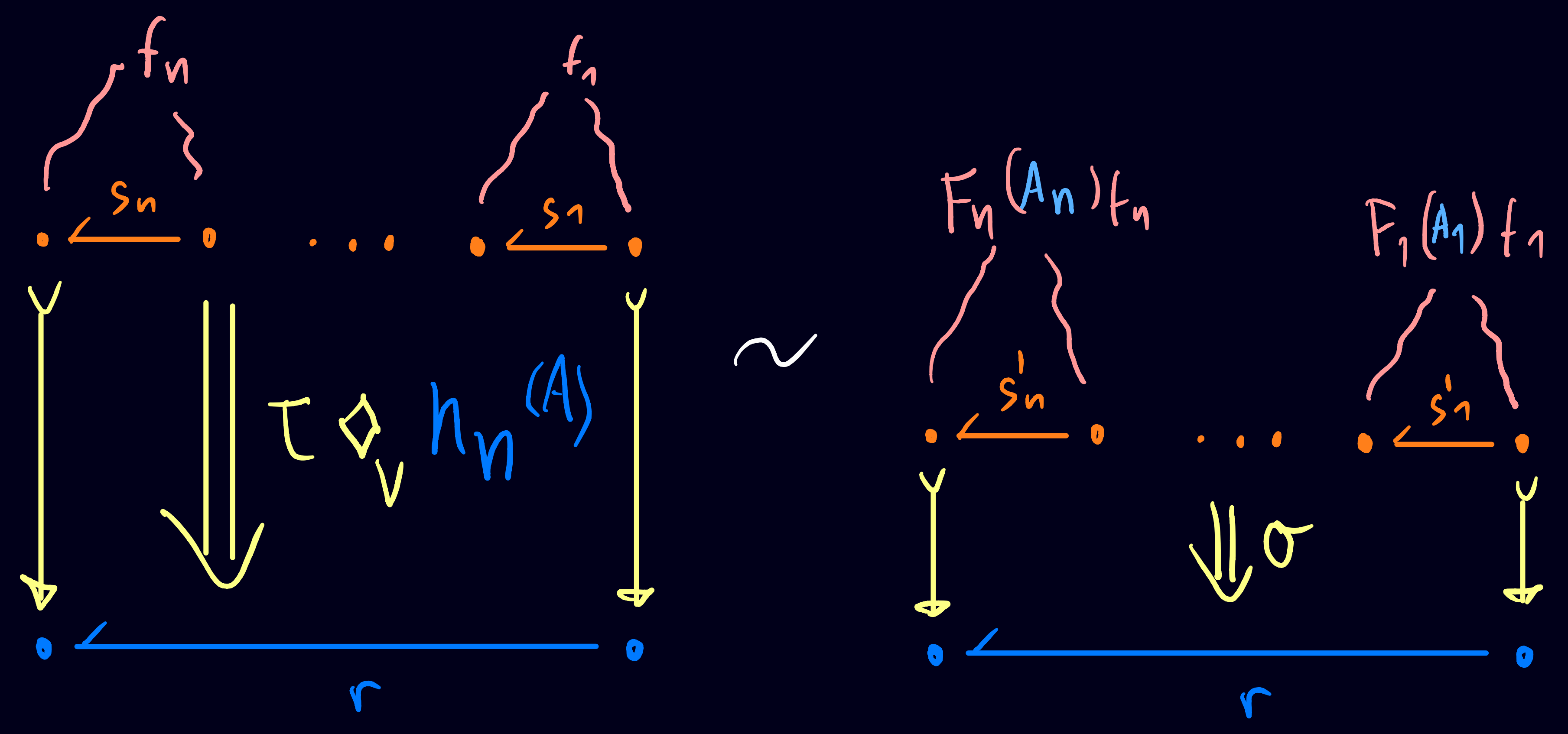


(16) CONVOLUTION PRODUCTS REVISITED

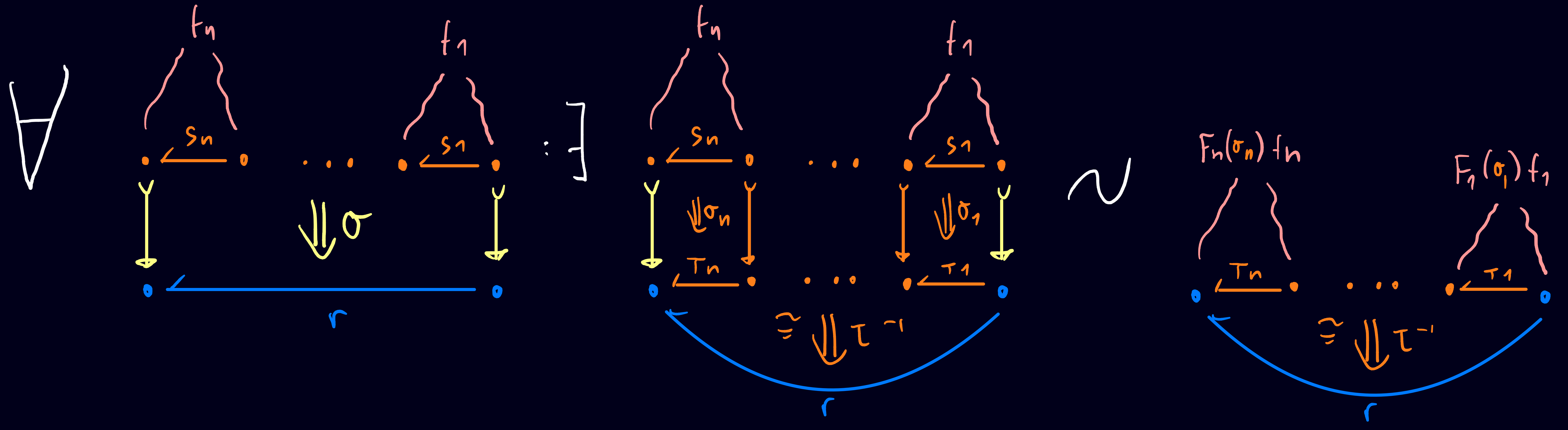
RECAP: $(F_n * \cdots * F_1)(r) := \int^{S=(S_n,\dots,S_1)\in \mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S) = \{ \cdots \}/\sim$



$\forall\ S \xrightarrow{A} S' \in \mathbb{D}_n$,
$h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:

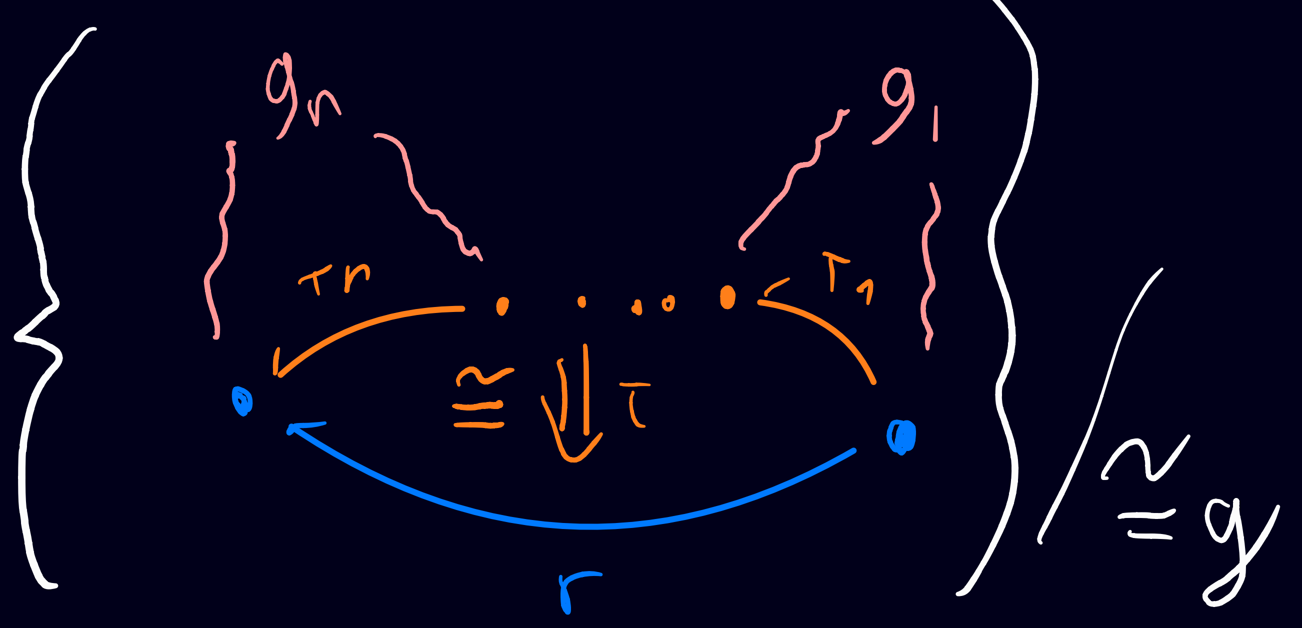


NOW: $h_n : \mathbb{D}_n \to \mathbb{D}_1$ IS A "GLOBULAR" STREET OPFIBRATION $\Rightarrow$

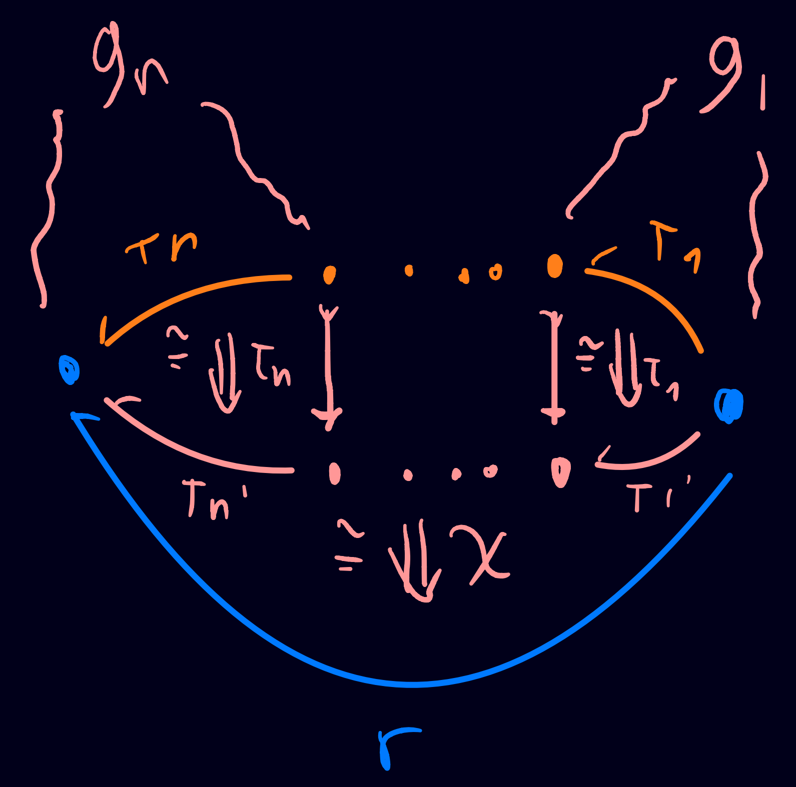


17

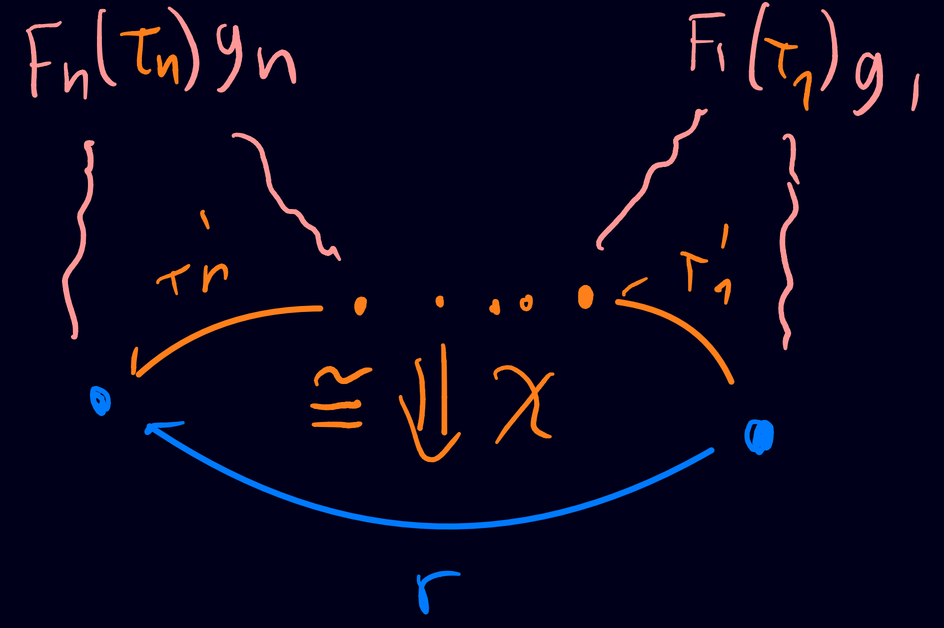
↳ $(F_n * \dots * F_1)(r) \cong$



WHERE

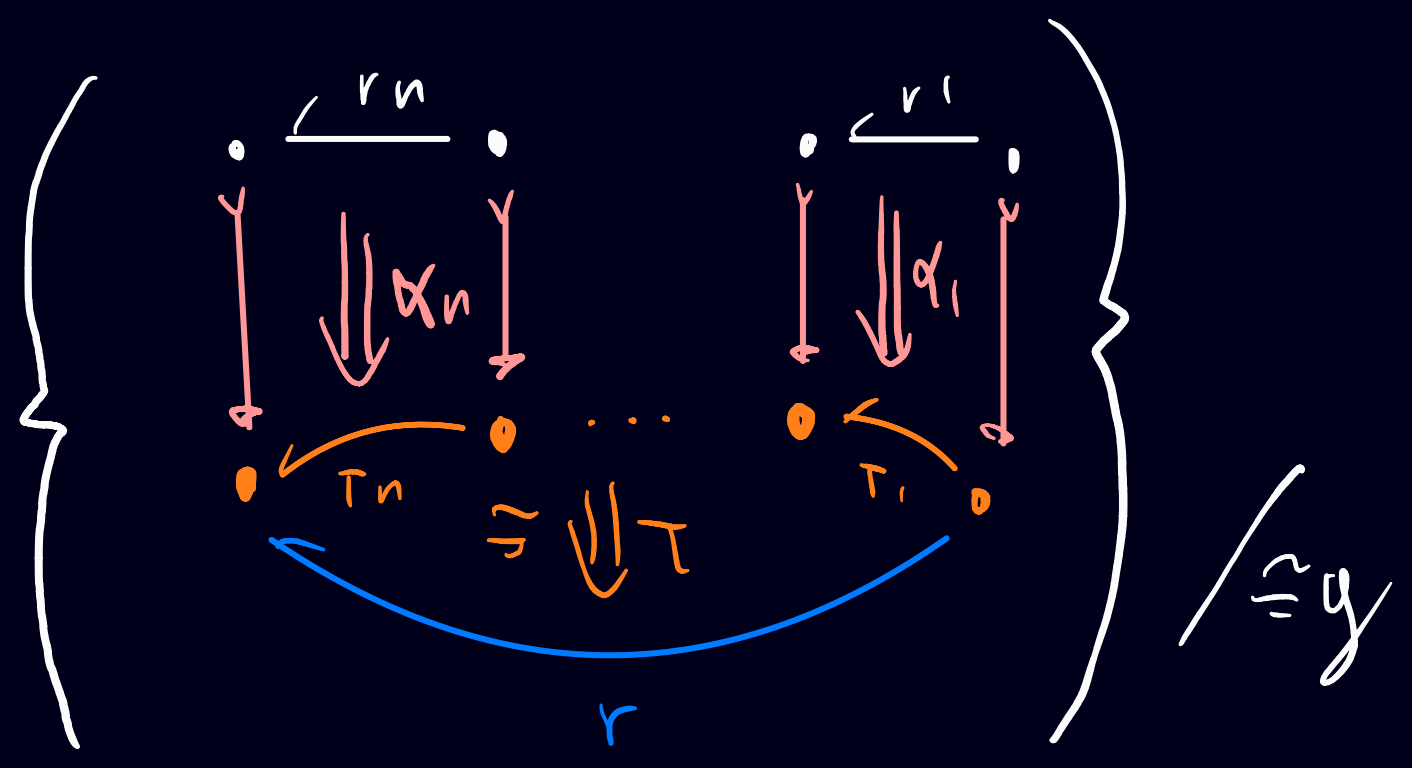


$\cong_g$



EXAMPLE: FOR $\hat{\Delta}_{r_j} := \mathbb{D}_1(r_j, -)$ $(j = 1, \dots, n)$

$(\hat{\Delta}_{r_n} * \dots * \hat{\Delta}_{r_1})(r) \cong$



✓

18

KEY RESULT: WEAK ASSOCIATIVITY OF $*$

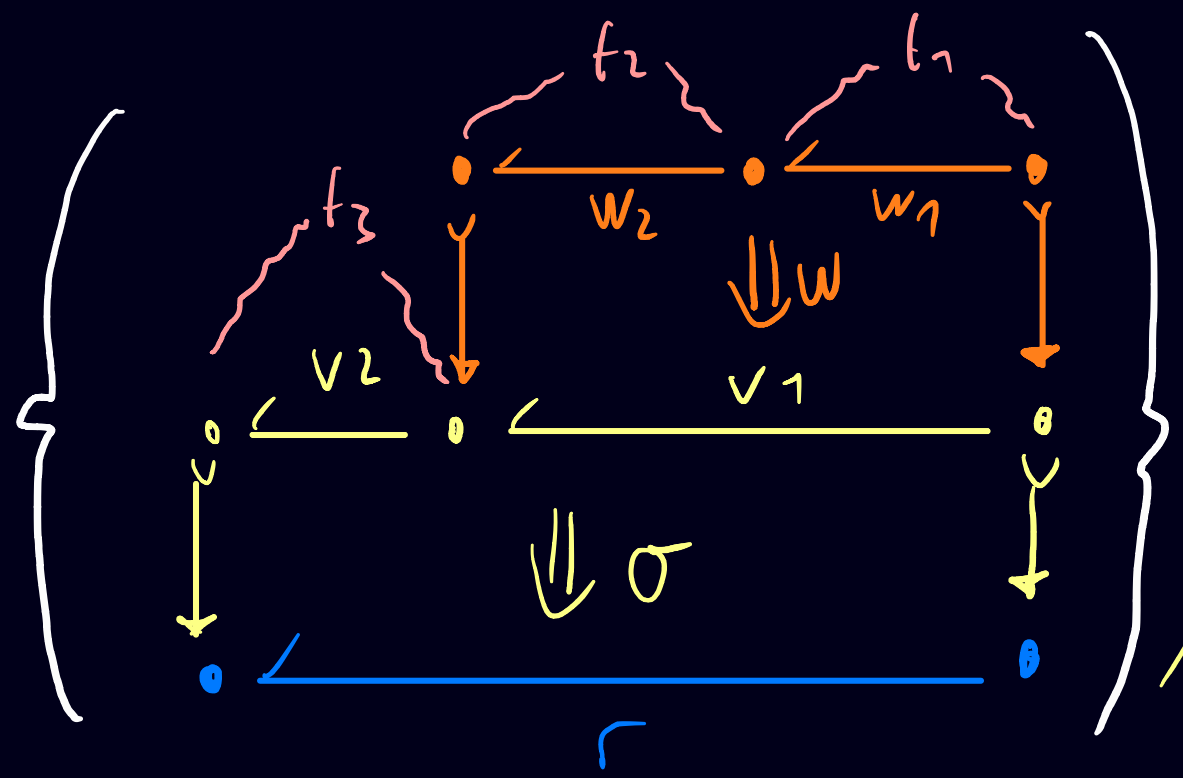
$$\forall\, F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \mathbb{D}_0\,,\ r \in \mathbb{D}_1 : \quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

(18) KEY RESULT: WEAK ASSOCIATIVITY OF $*$

$$\forall\, F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \mathbb{D}_0\,,\ r \in \mathbb{D}_1 :\quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

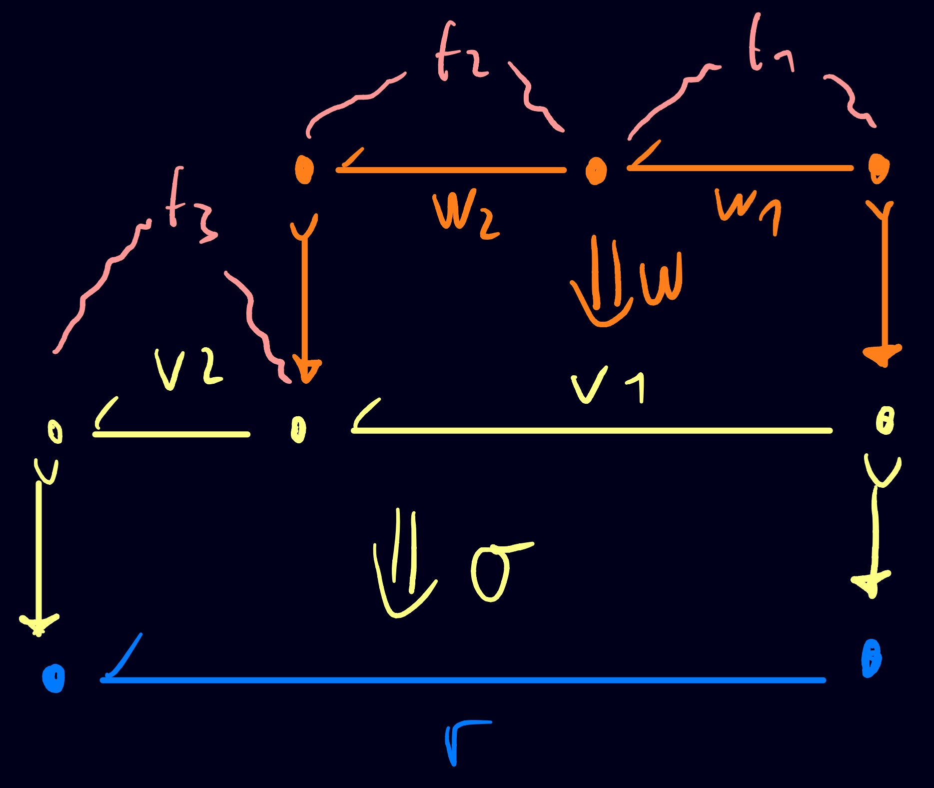
PROOF (SKETCH):

$F_3 * (F_2 * F_1)(r) =$

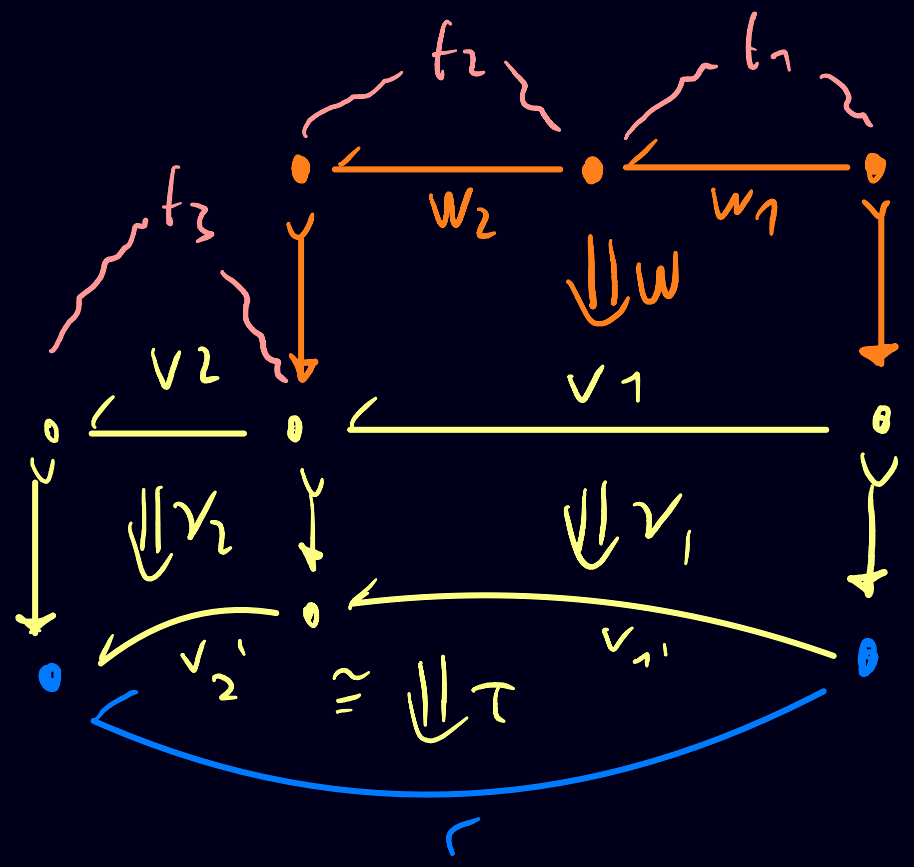


$/\!\sim_v\ /\!\sim_w$

$\forall$



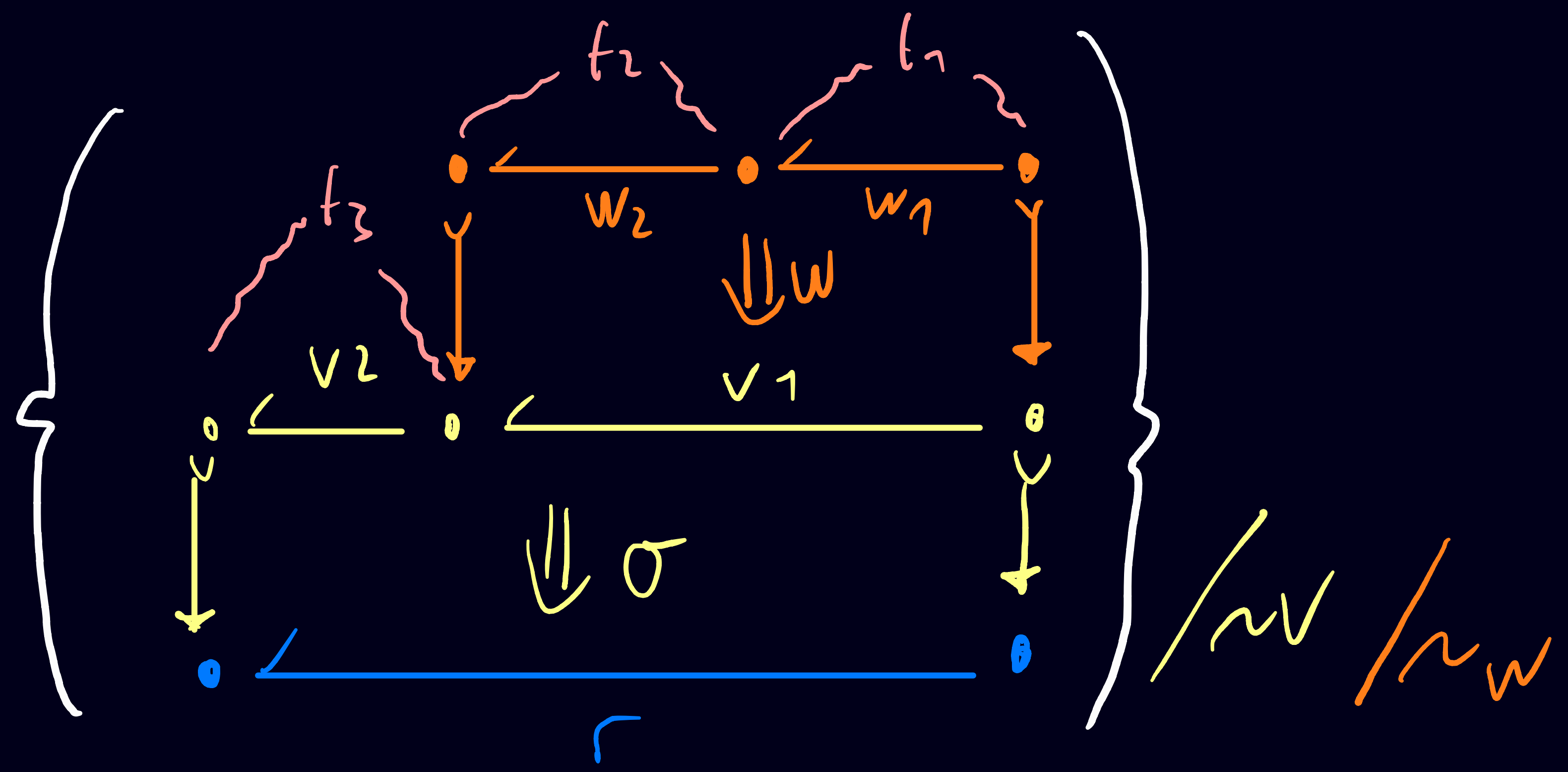
$: \exists$

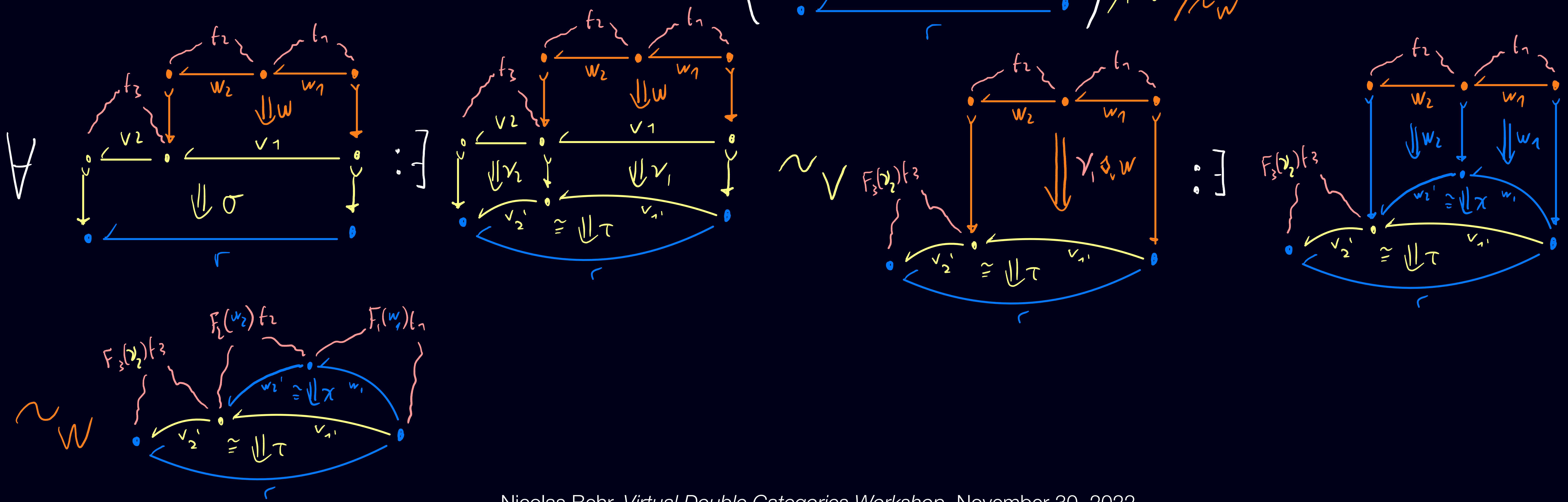


(18) KEY RESULT: WEAK ASSOCIATIVITY OF $*$

$\forall F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \mathbb{D}_0,\ r \in \mathbb{D}_1 :\quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$

PROOF (SKETCH):

$F_3 * (F_2 * F_1)(r) =$ 

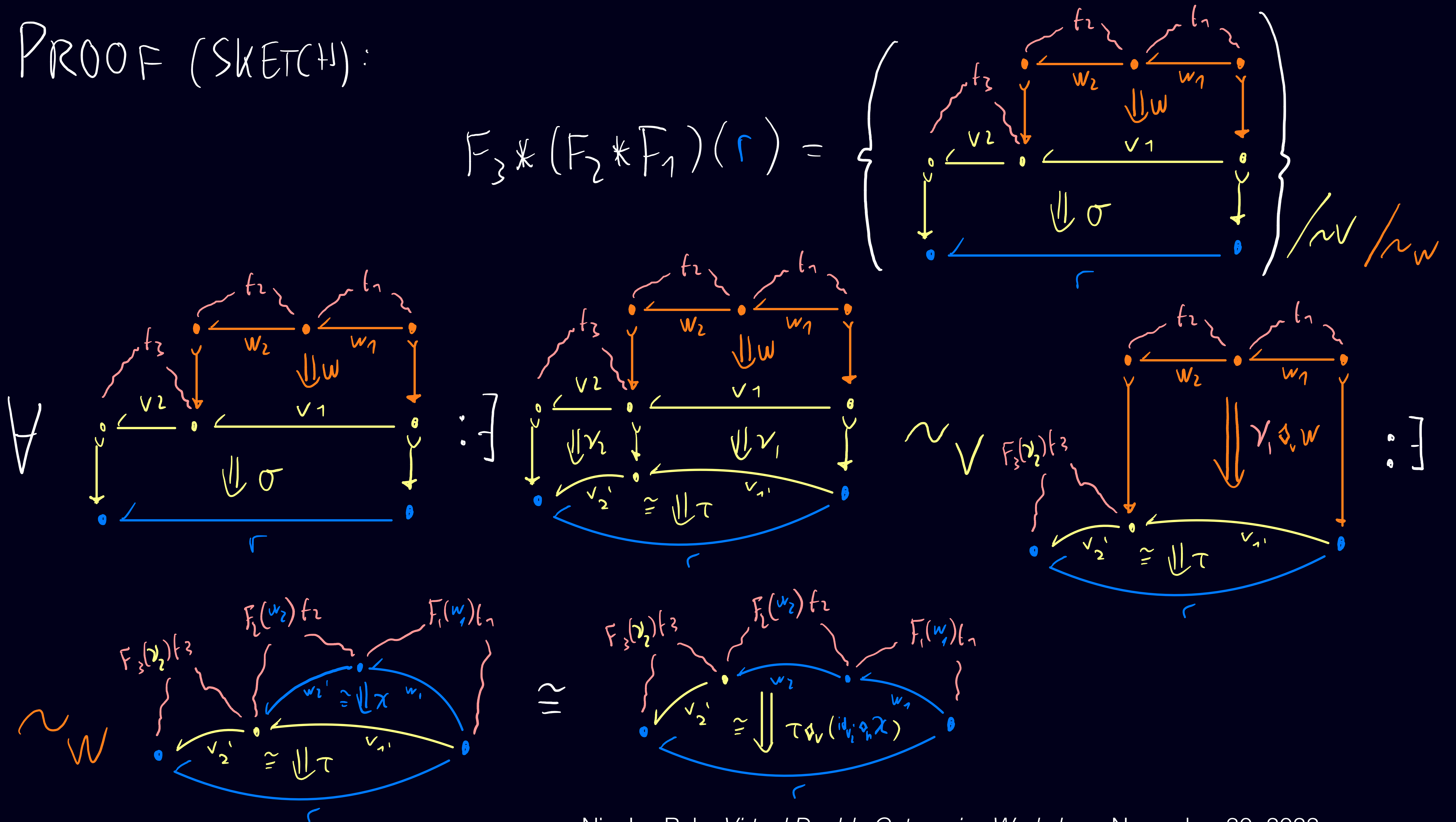


18 KEY RESULT: WEAK ASSOCIATIVITY OF $*$

$\forall F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \mathbb{D}_0,\ r \in \mathbb{D}_1 :\quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$

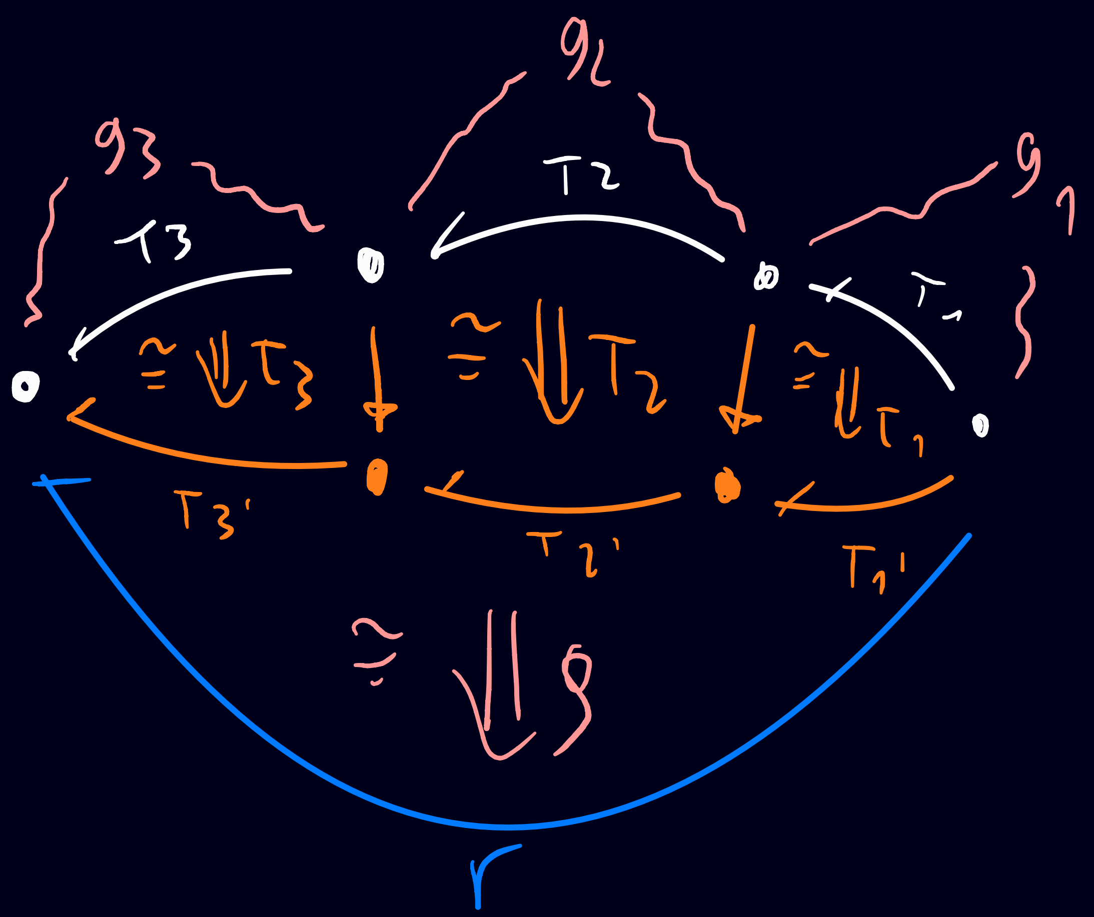
PROOF (SKETCH):

$F_3 * (F_2 * F_1)(r) =$

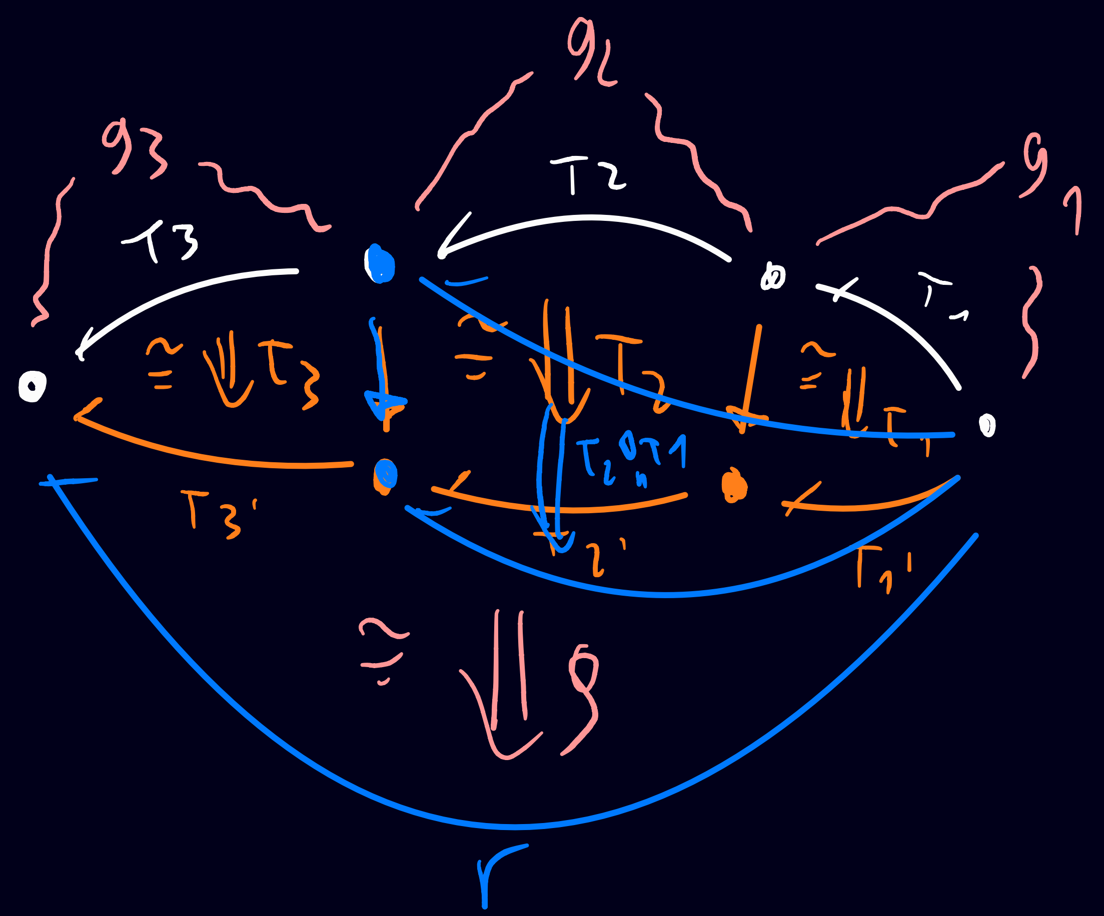


19

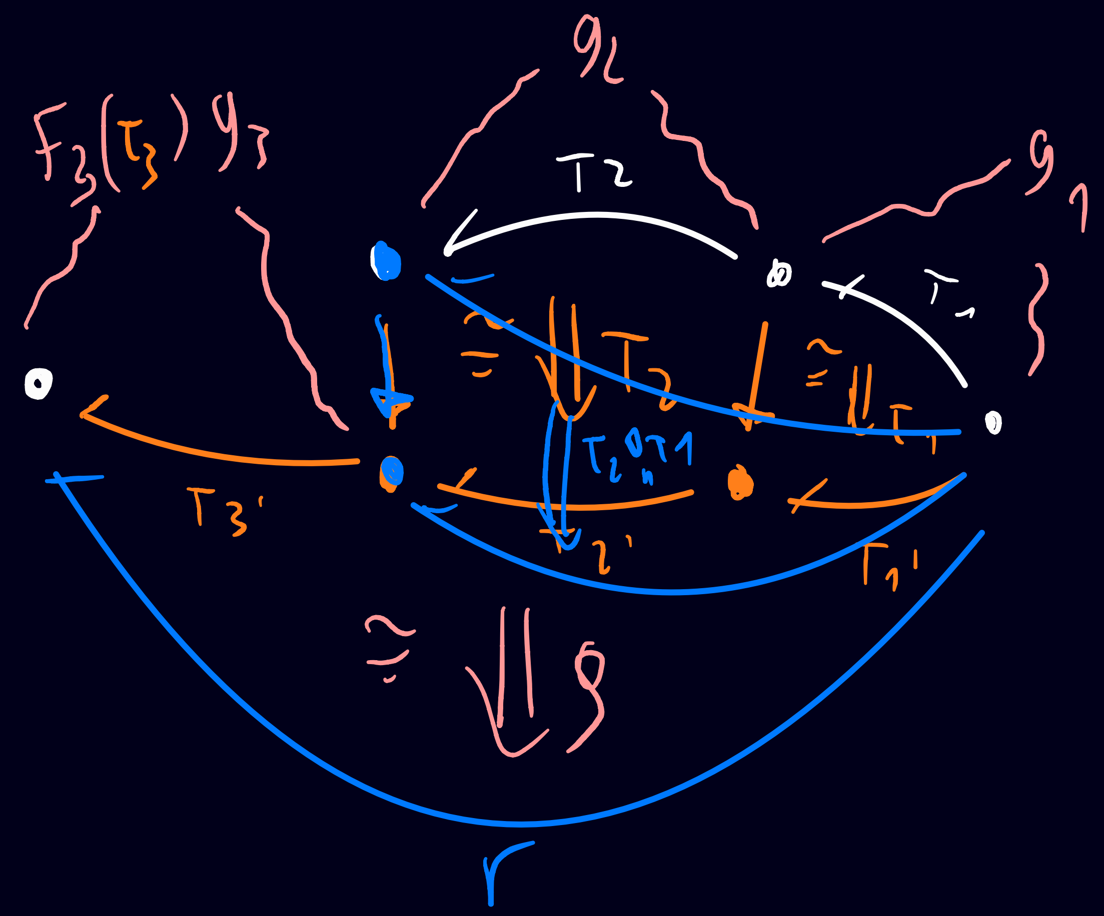
$\cdot\ \forall$



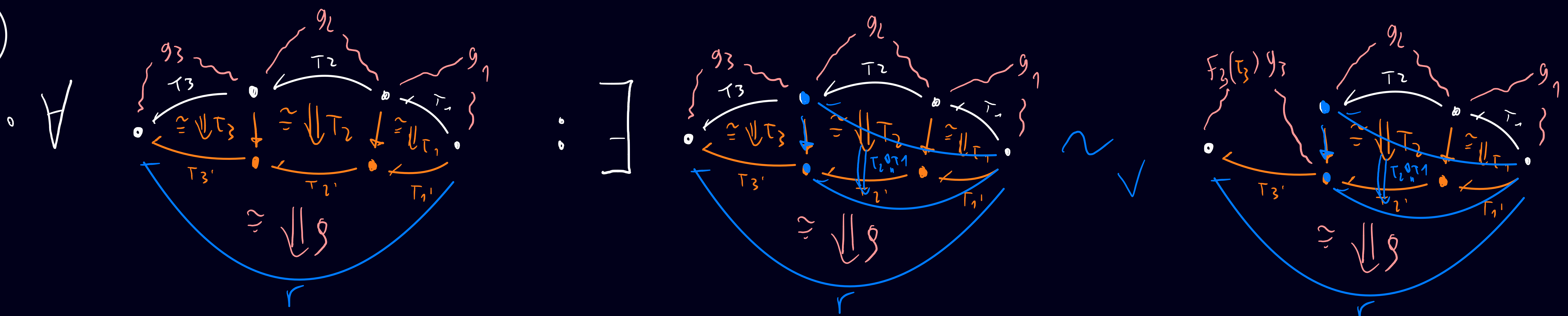
$:\ \exists$

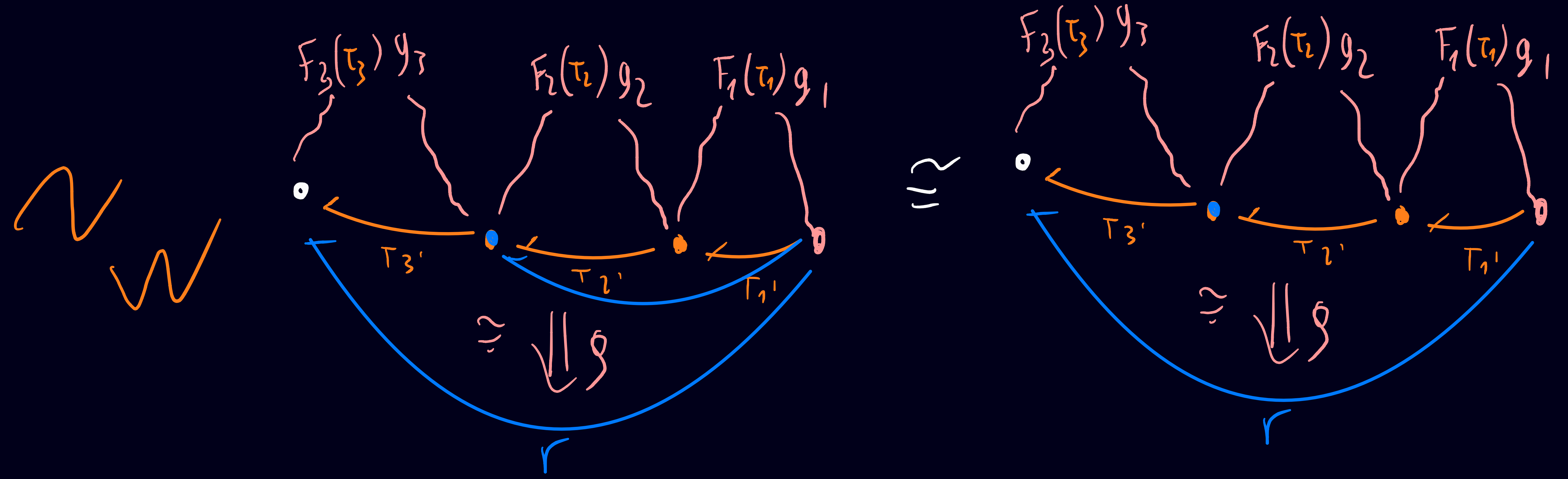


$\sim\ \checkmark$

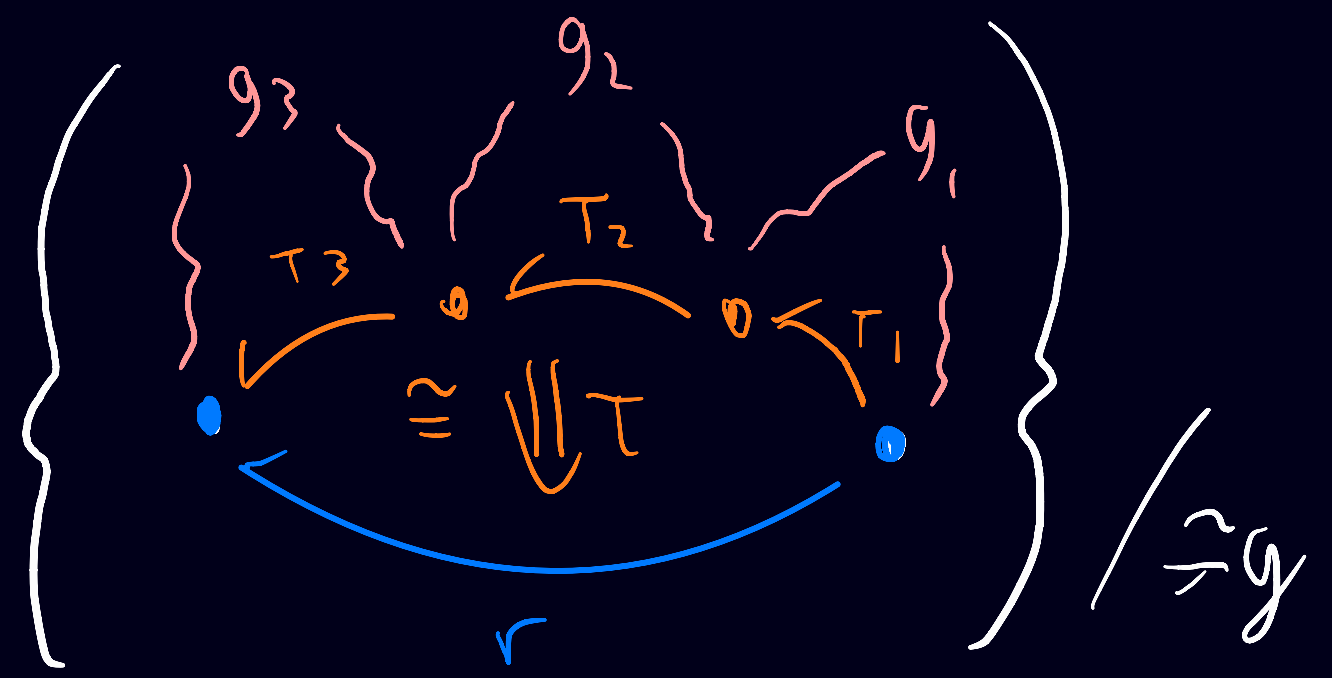


19





$\hookrightarrow \; F_3 * (F_2 * F_1)(r) \cong$



$= (F_3 * F_2 * F_3)(r)$

$\square$

20 FINAL INGREDIENT: CATEGORIFICATION OF RULE ALGEBRA

CLAIM:

$$(\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1})(r) \cong \left\{ \begin{tikzcd} O_2 \arrow[d, tail] & I_2 \arrow[l, "r_2"'] \arrow[dr, tail] \arrow[l, phantom, shift right=4, "\Downarrow\alpha_2"] & & O_1 \arrow[dl, tail'] & I_1 \arrow[l, "r_1"'] \arrow[d, tail] \arrow[l, phantom, shift right=4, "\Downarrow\alpha_1"] \\ \bullet & & \bullet \arrow[ll, "r_2'"'] \arrow[d, phantom, "\cong\Downarrow\rho"] & & \bullet \arrow[ll, "r_1'"'] \arrow[llll, bend left, "r"] \\ & & \; & & \end{tikzcd} \right\} \Big/ \cong_y$$

$$\cong \coprod_{(O_2 \rightarrowtail M_j \leftarrowtail I_1) \in ms(O_2, I_1)} \; \coprod_{\beta_{jk} \in T^*(O_1 \rightarrowtail M_j)} \; \coprod_{\gamma_{jk} \in S^*(I_2 \rightarrowtail M_j \xrightarrow{x_k} N_k)} \left\{ \begin{tikzcd} \bullet \arrow[dd, tail] & \bullet \arrow[l, "r_2"'] \arrow[d] \arrow[dd, phantom, "\Downarrow\gamma_{jk}"] & & \bullet \arrow[dl] & \bullet \arrow[l, "r_1"'] \arrow[dd, tail] \arrow[dd, phantom, shift right=8, "\Downarrow\beta_{jk}"] \\ & & M_j \arrow[d, tail, "x_k"] & & \\ \bullet \arrow[d, tail] & & N_k \arrow[ll, "s_2"'] \arrow[d, phantom, "\Downarrow\rho"] & & \bullet \arrow[ll, "s_1"'] \arrow[d, tail] \\ \bullet & & \; & & \bullet \arrow[llll, "r"] \end{tikzcd} \right\}$$

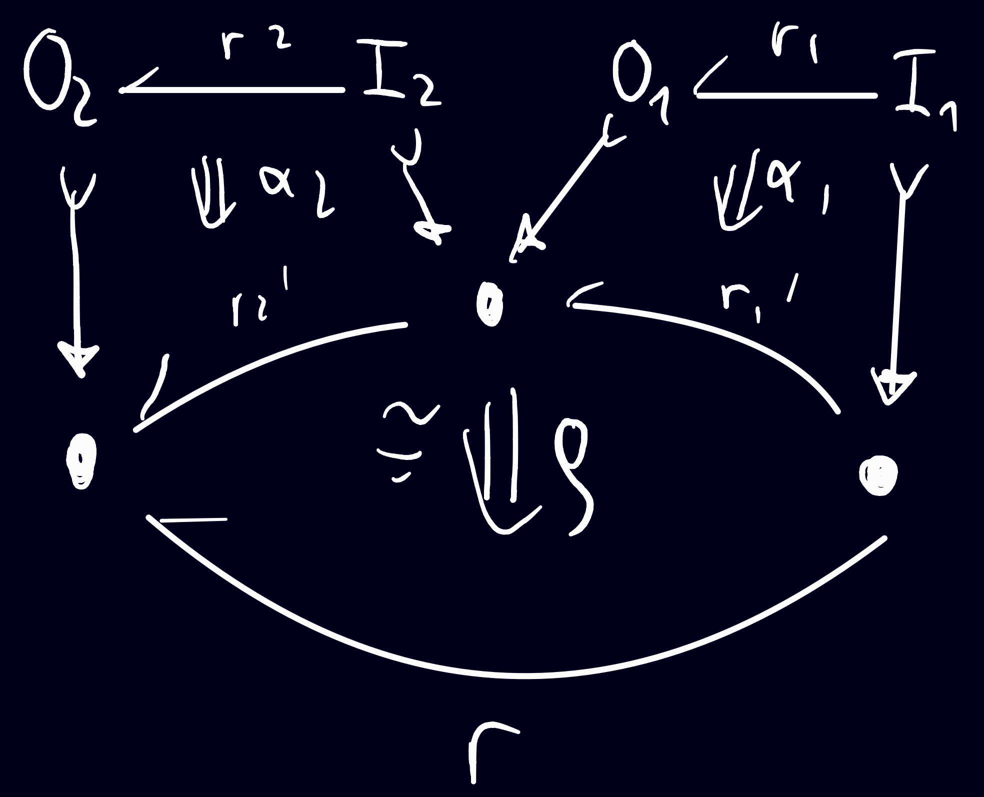
$$=: \hat{\Delta}_{r_2 \circ r_1}(r)$$

21 PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

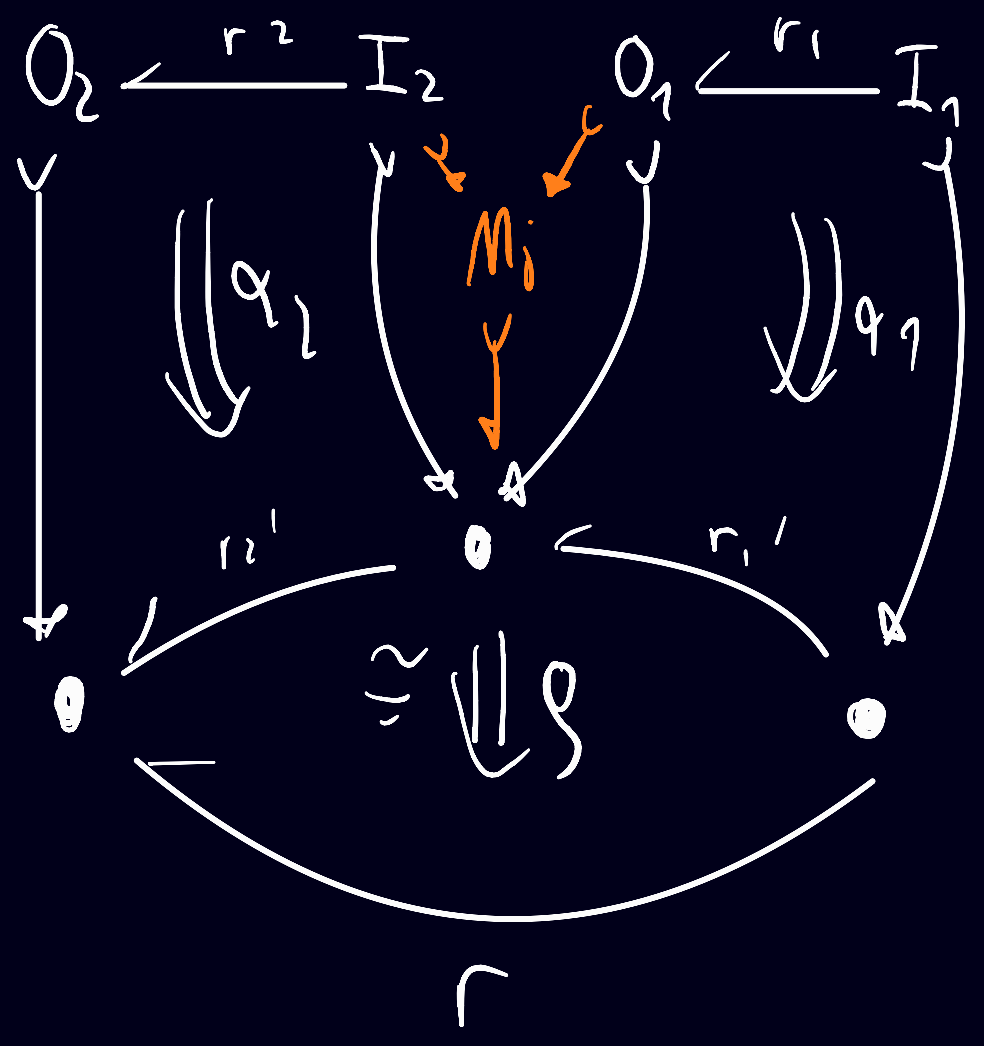
$$
\forall
\begin{tikzcd}
O_2 \arrow[d, tail] & I_2 \arrow[l, "r_2"'] \arrow[dr, tail] \arrow[dl, phantom, "\Downarrow\alpha_2"] & & O_1 \arrow[dl, tail] & I_1 \arrow[l, "r_1"'] \arrow[d, tail] \arrow[dl, phantom, "\Downarrow\alpha_1"] \\
\bullet & & \bullet \arrow[ll, "r_2'"'] \arrow[d, phantom, "\cong\Downarrow\rho"] & & \bullet \arrow[ll, "r_1'"] \arrow[llll, bend left=40, "r"] \\
& & {} & &
\end{tikzcd}
:\ \exists!
\begin{tikzcd}
O_2 \arrow[dd, tail] & I_2 \arrow[l, "r_2"'] \arrow[ddr, tail] \arrow[dr, tail] \arrow[ddl, phantom, "\Downarrow\alpha_2"] & & O_1 \arrow[ddl, tail] \arrow[dl, tail] & I_1 \arrow[l, "r_1"'] \arrow[dd, tail] \arrow[ddl, phantom, "\Downarrow\alpha_1"] \\
& & M_i \arrow[d, tail] & & \\
\bullet & & \bullet \arrow[ll, "r_2'"'] \arrow[d, phantom, "\cong\Downarrow\rho"] & & \bullet \arrow[ll, "r_1'"] \arrow[llll, bend left=40, "r"] \\
& & {} & &
\end{tikzcd}
$$

(21) PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

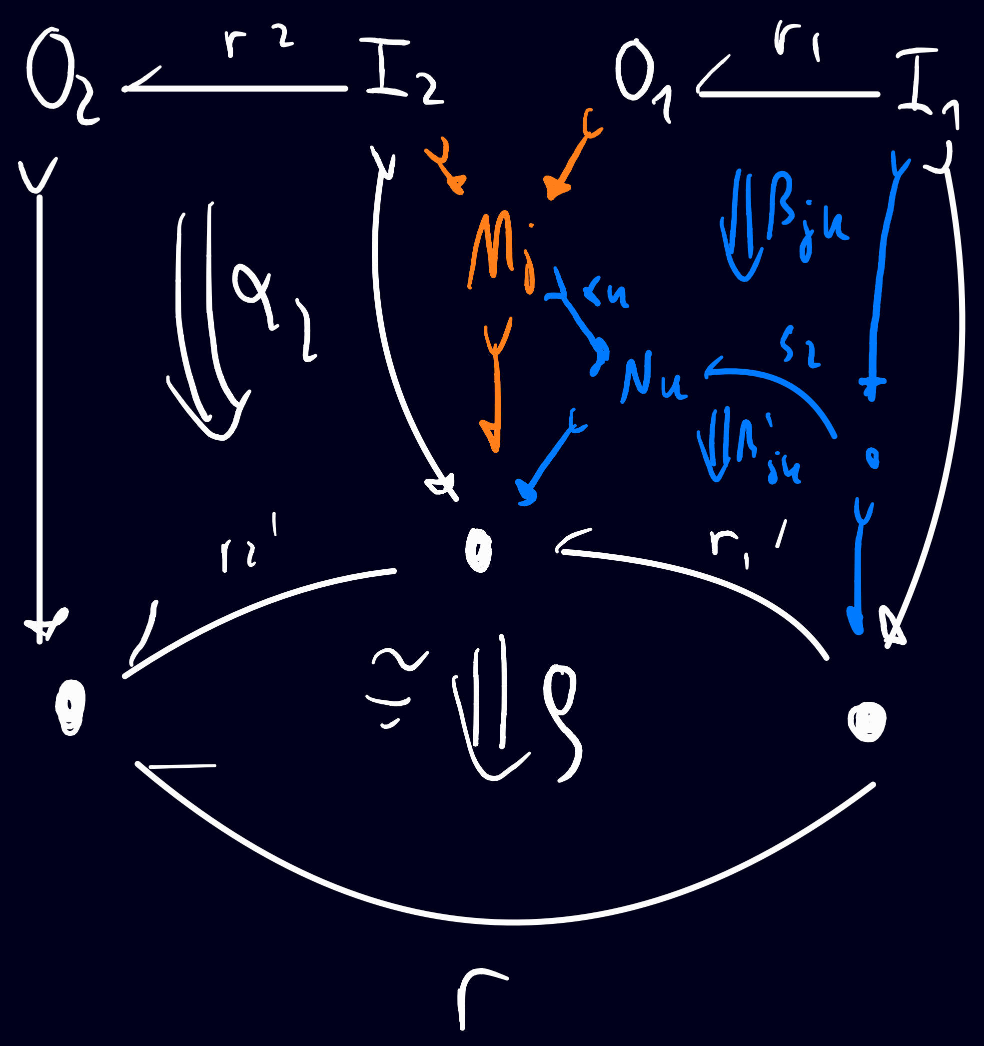
$\forall$



: $\exists!$

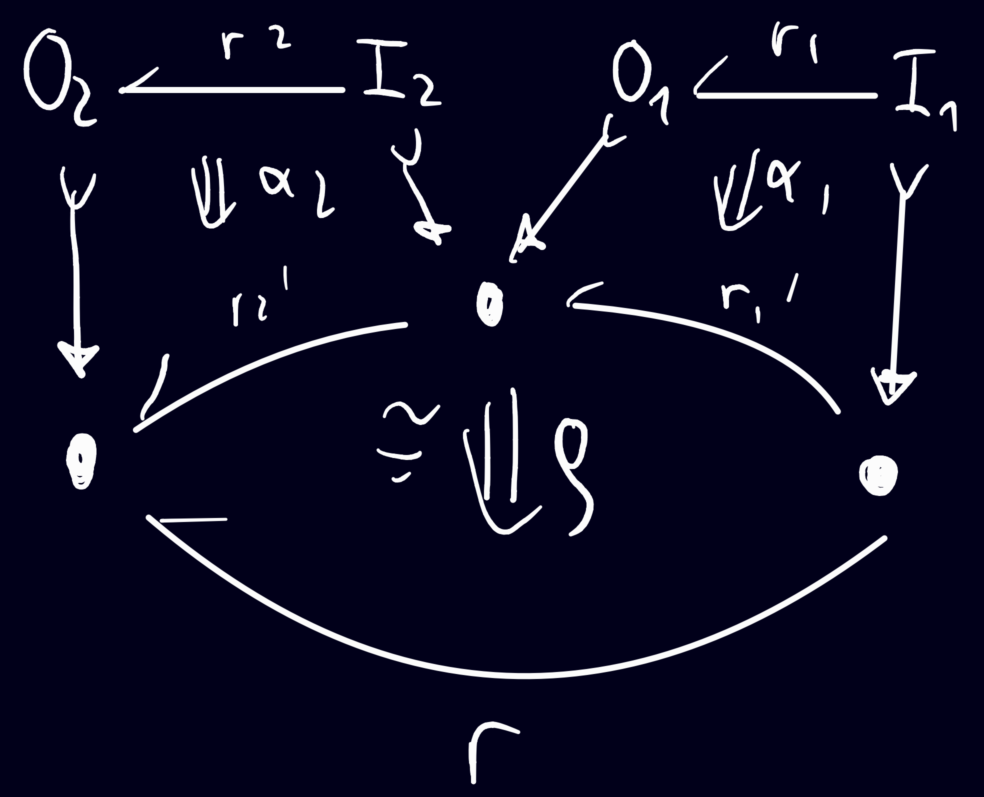


: $\exists!$

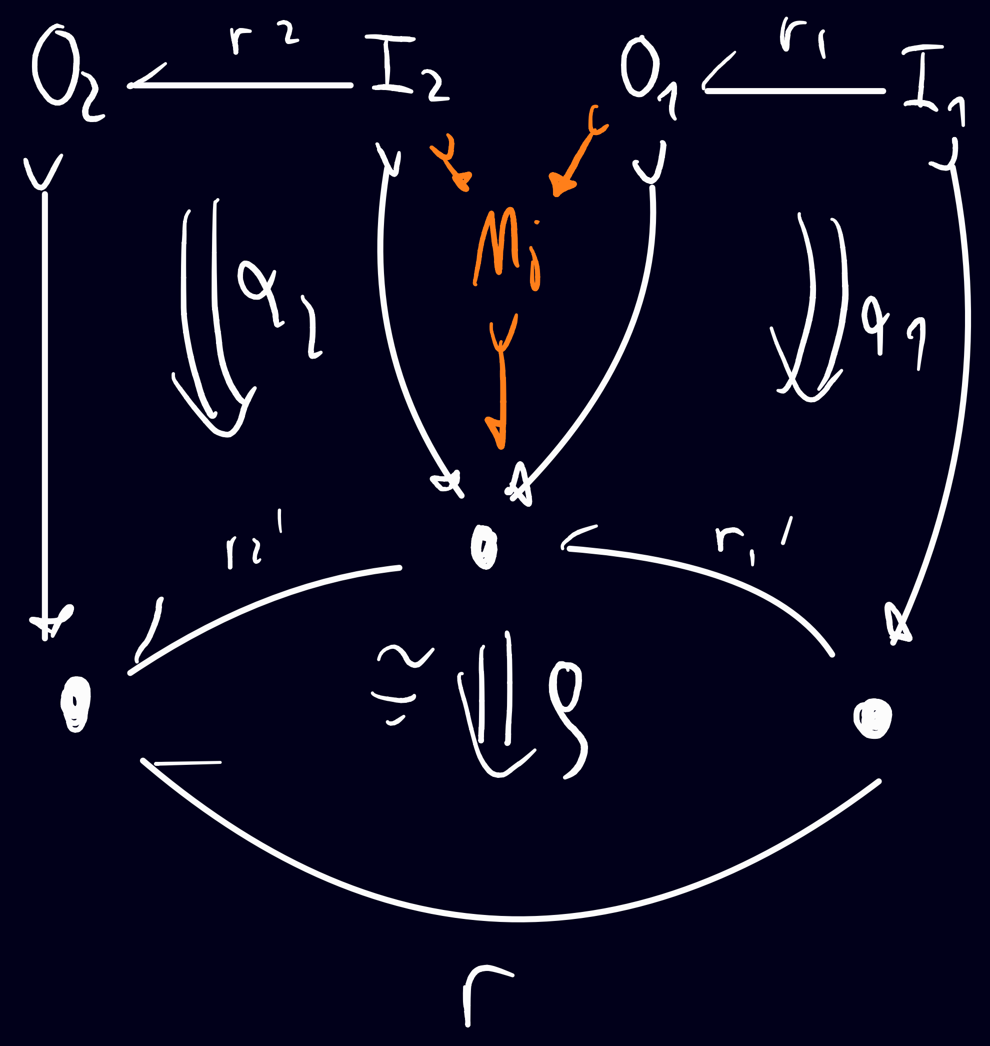


21 PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

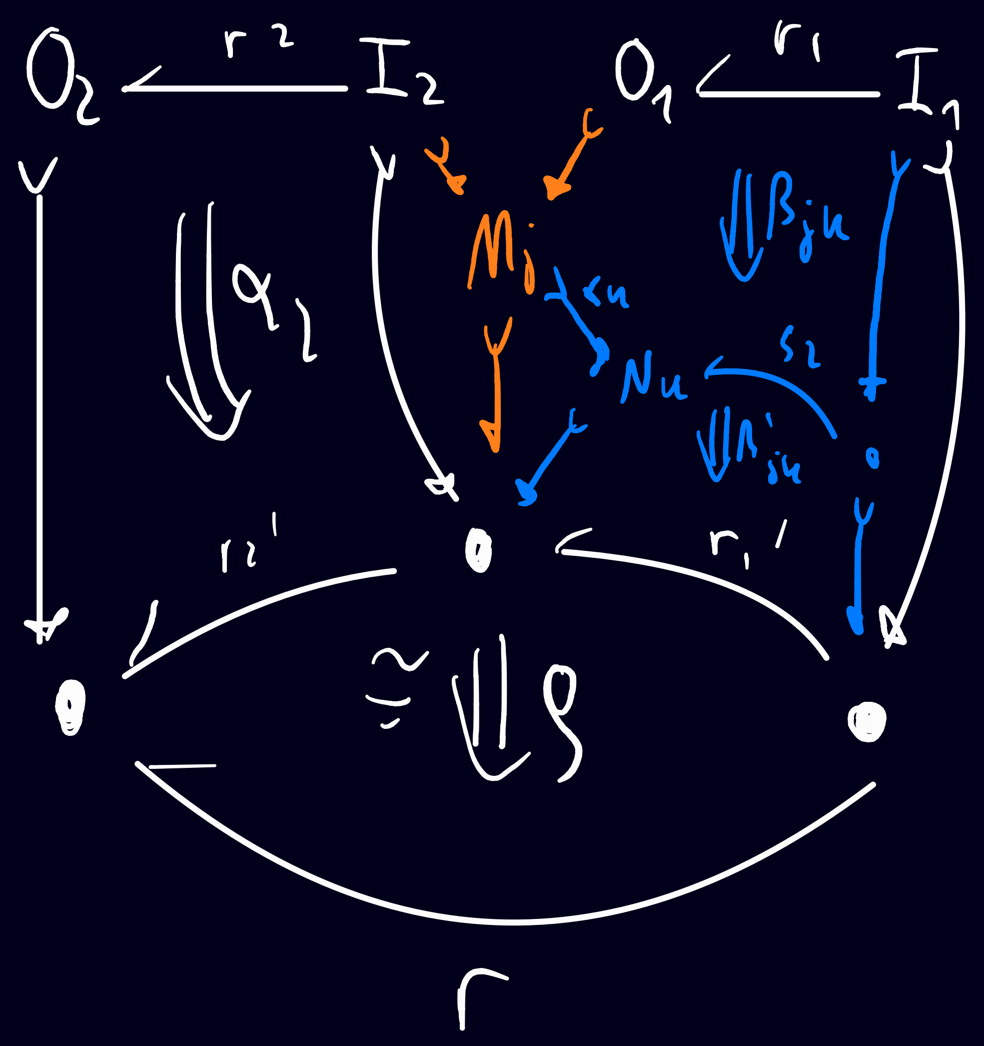
$\forall$



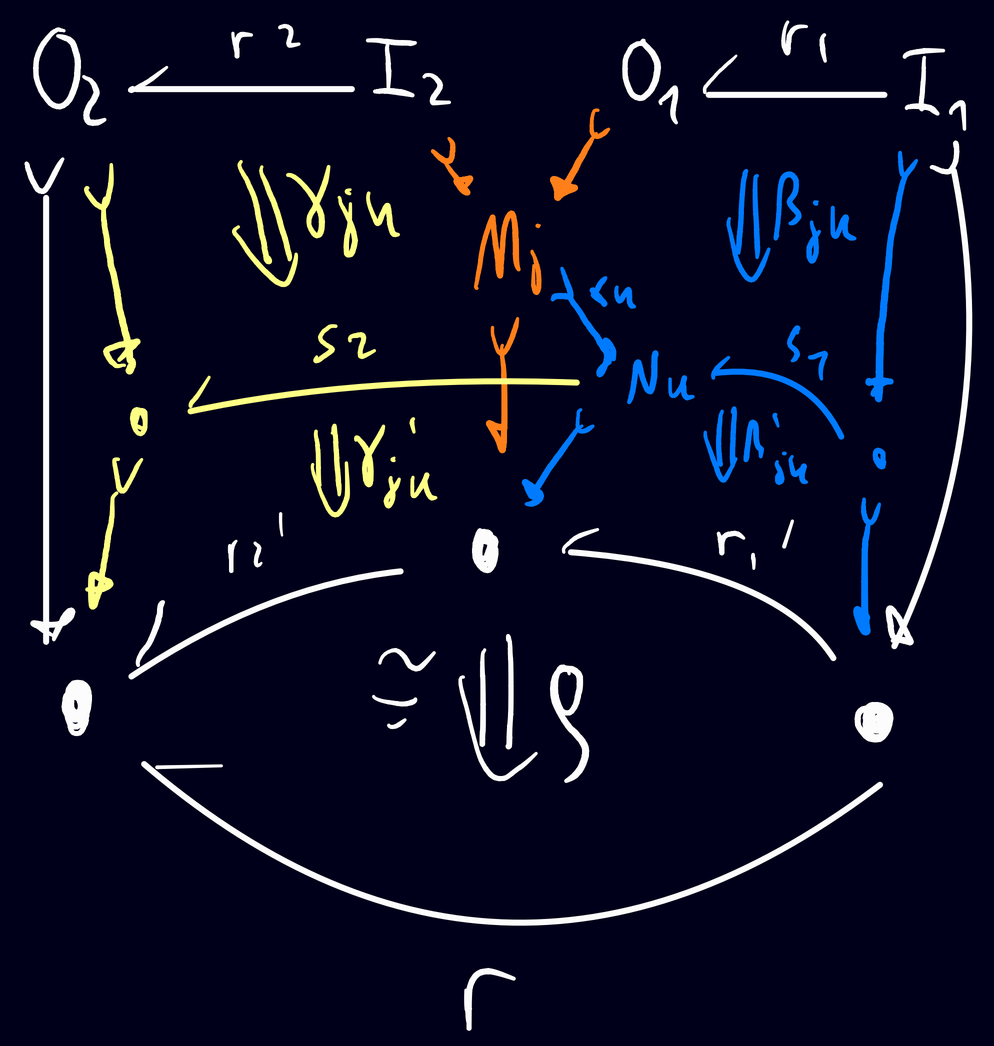
: $\exists!$



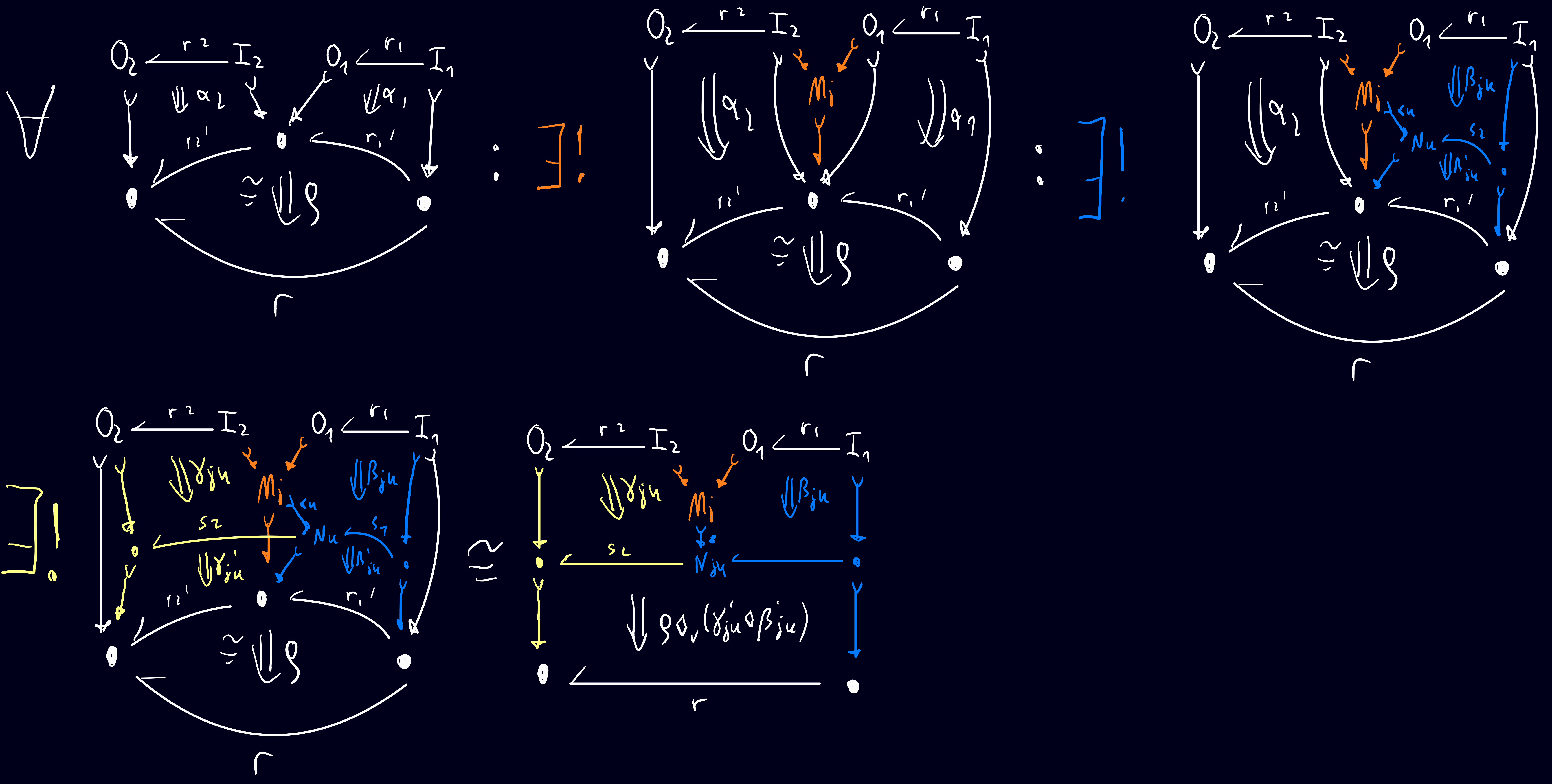
: $\exists!$



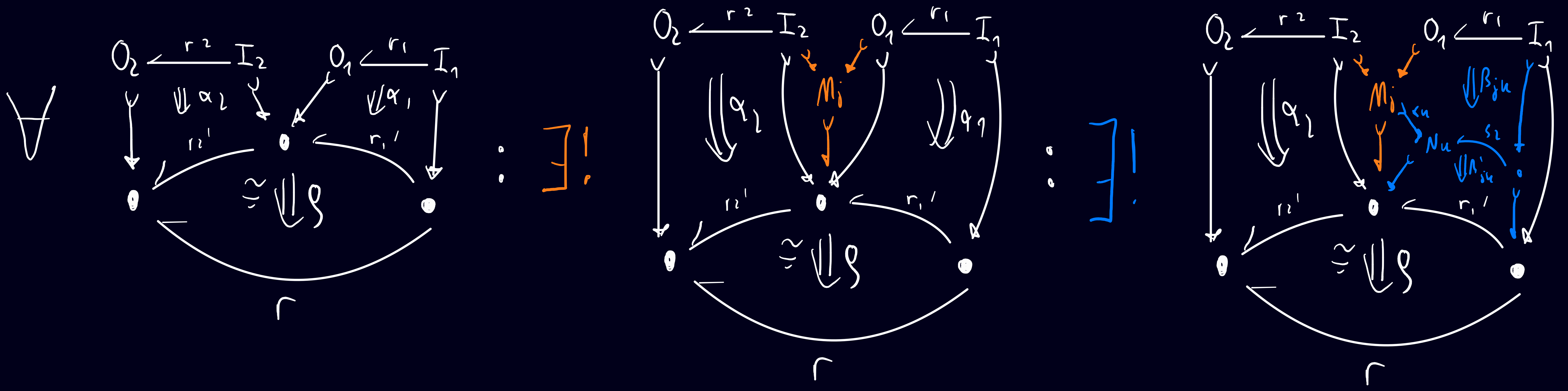
$\exists!$

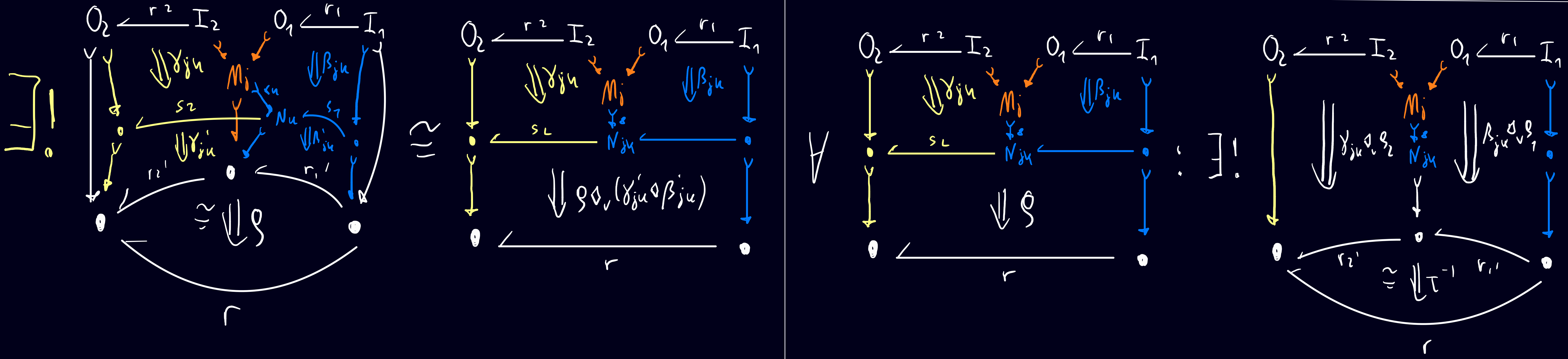


(21) PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:



(21) PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

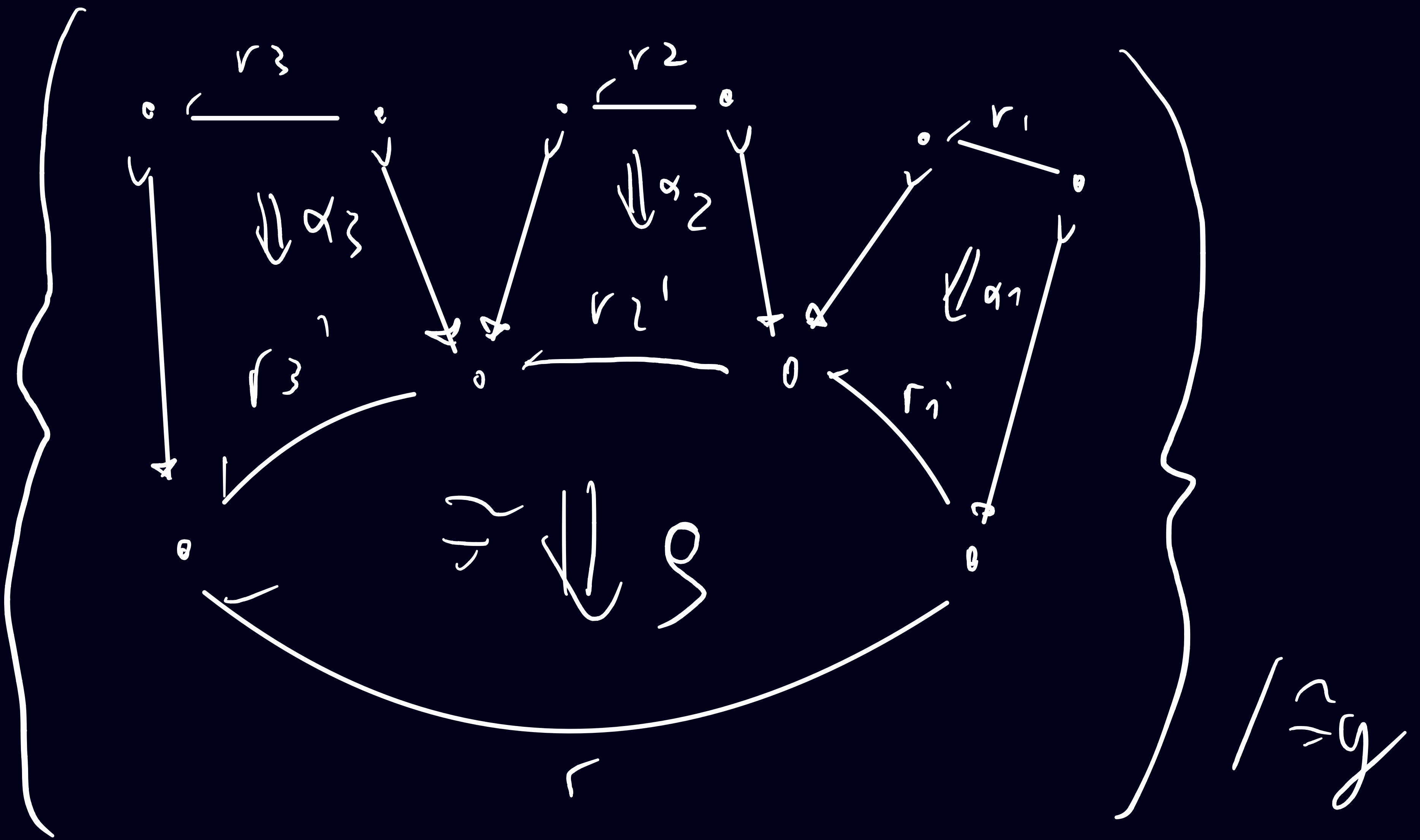




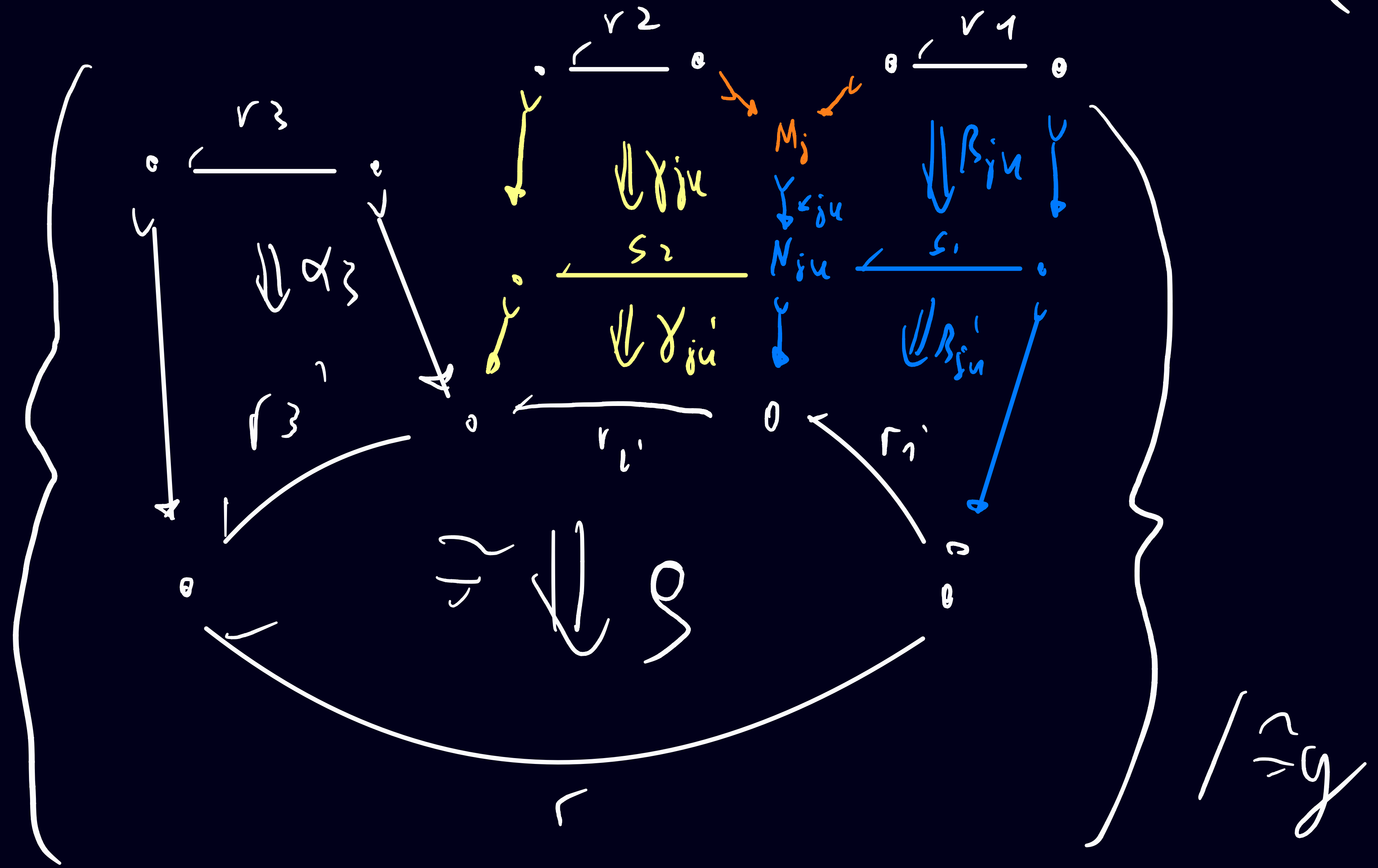
(22) CLAIM: $\hat{\Delta}_{r_3 \odot (r_2 \odot r_1)}(r) \cong \hat{\Delta}_{(r_3 \odot r_2) \odot r_1}$

PROOF (SKETCH):

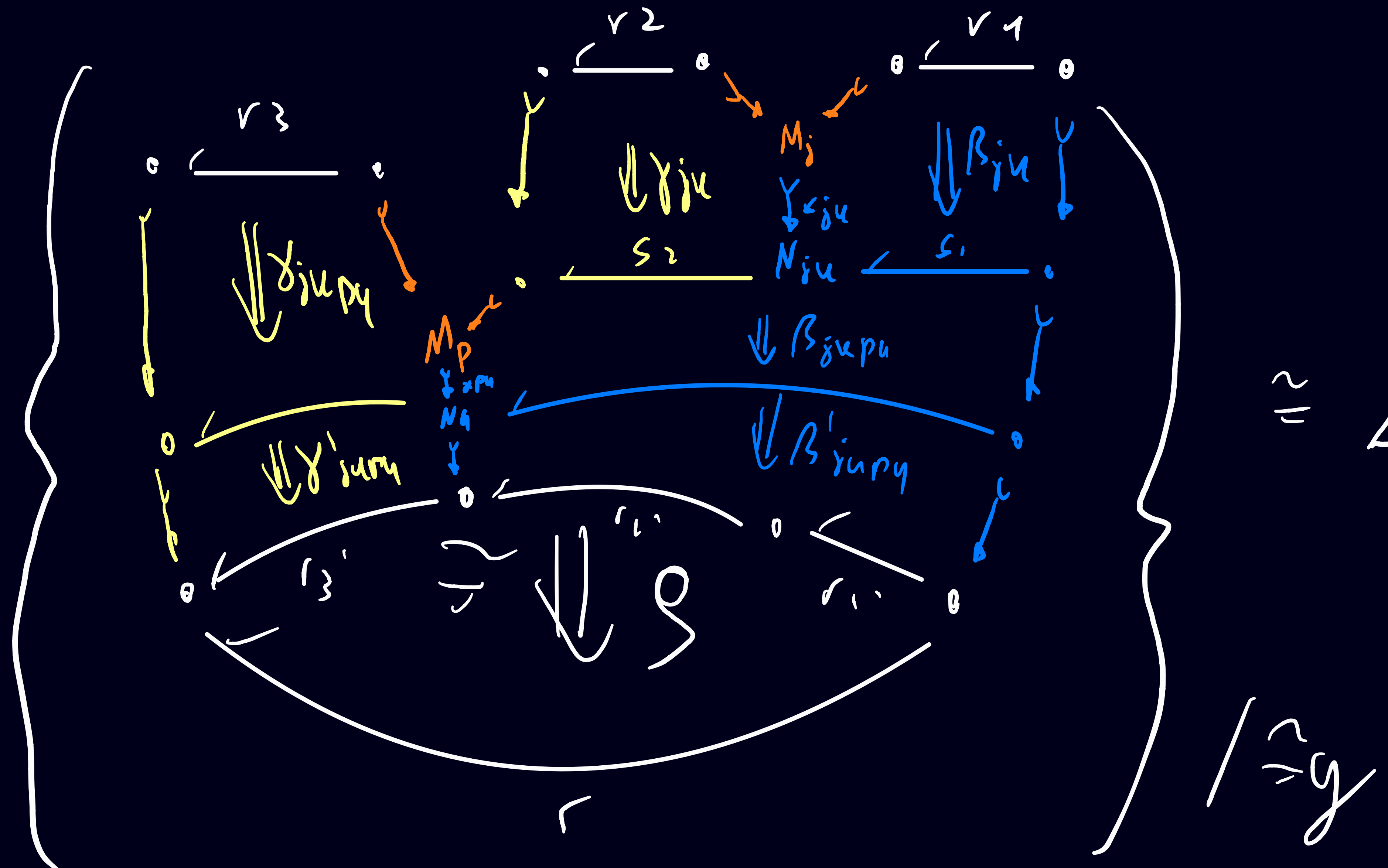
$(\hat{\Delta}_{r_3} * \hat{\Delta}_{r_2} * \hat{\Delta}_{r_1})(r) \cong$



$\cong$

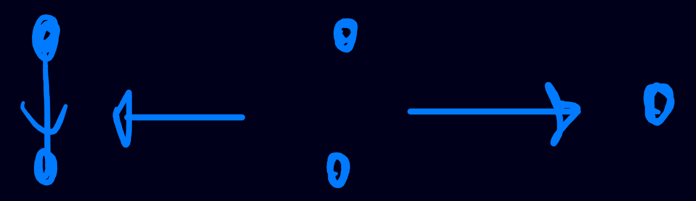


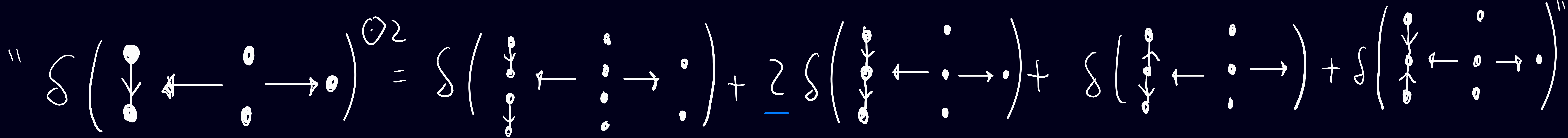
$\cong$



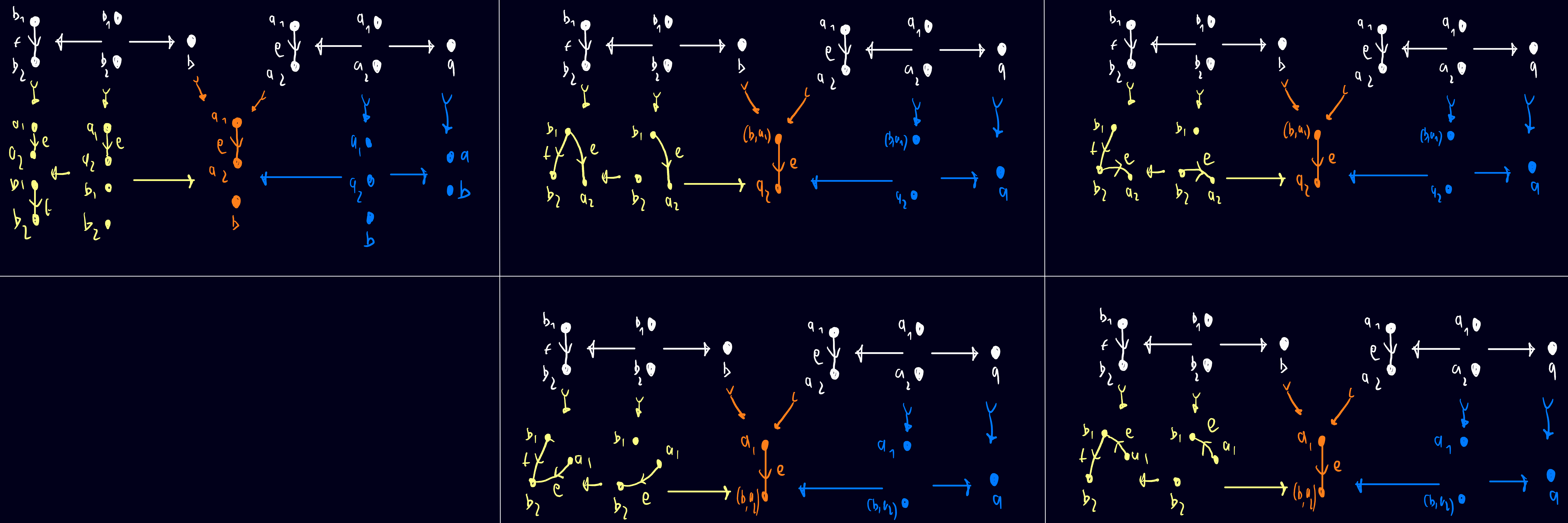
$\cong \hat{\Delta}_{r_3 \odot (r_2 \odot r_1)}(r)$

$\square$

23 EXAMPLE: SELF-COMPOSITIONS OF THE REWRITING RULE 

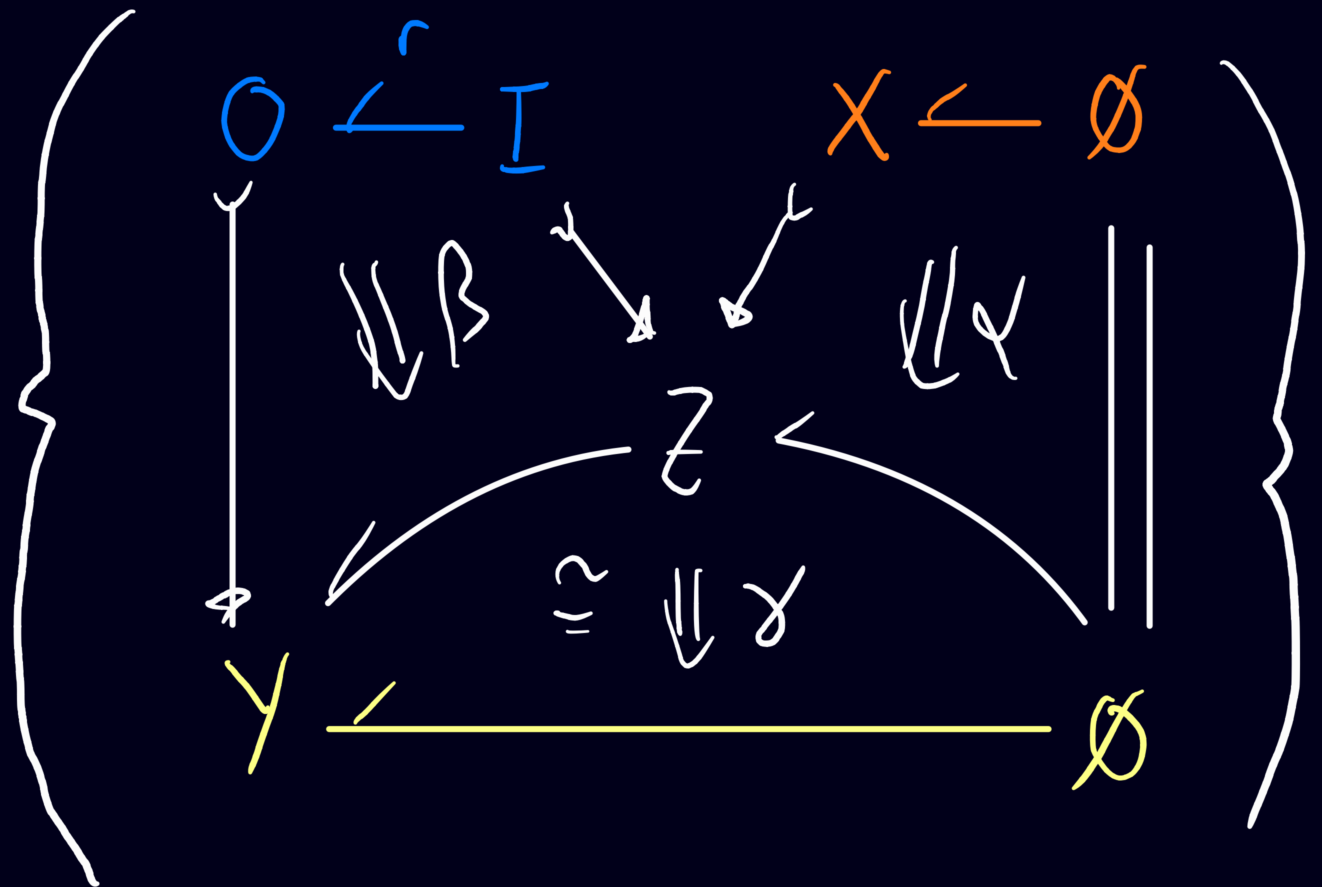
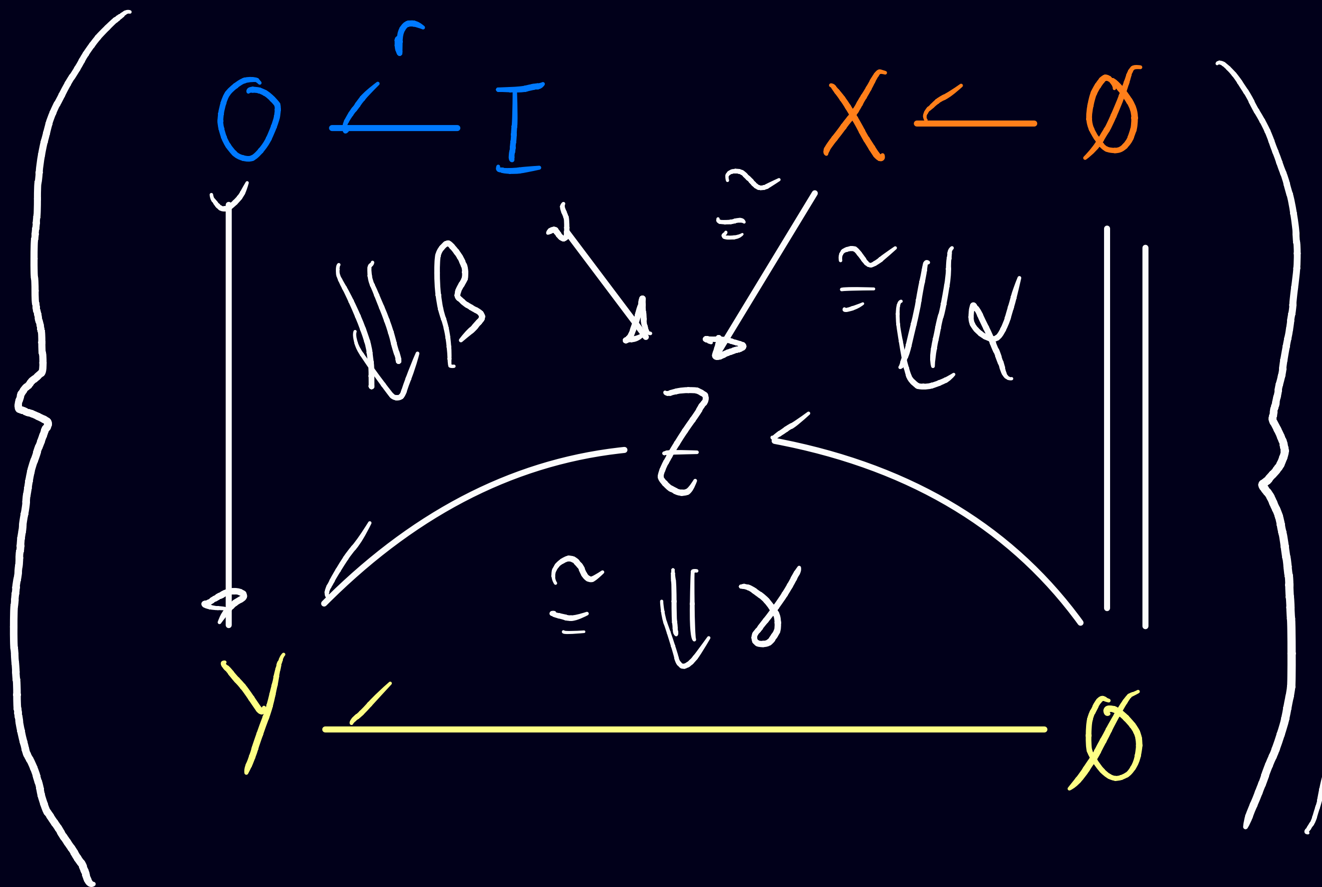


5 CONTRIBUTIONS TO $\hat{A}_{r_2 \circ r_1}$:

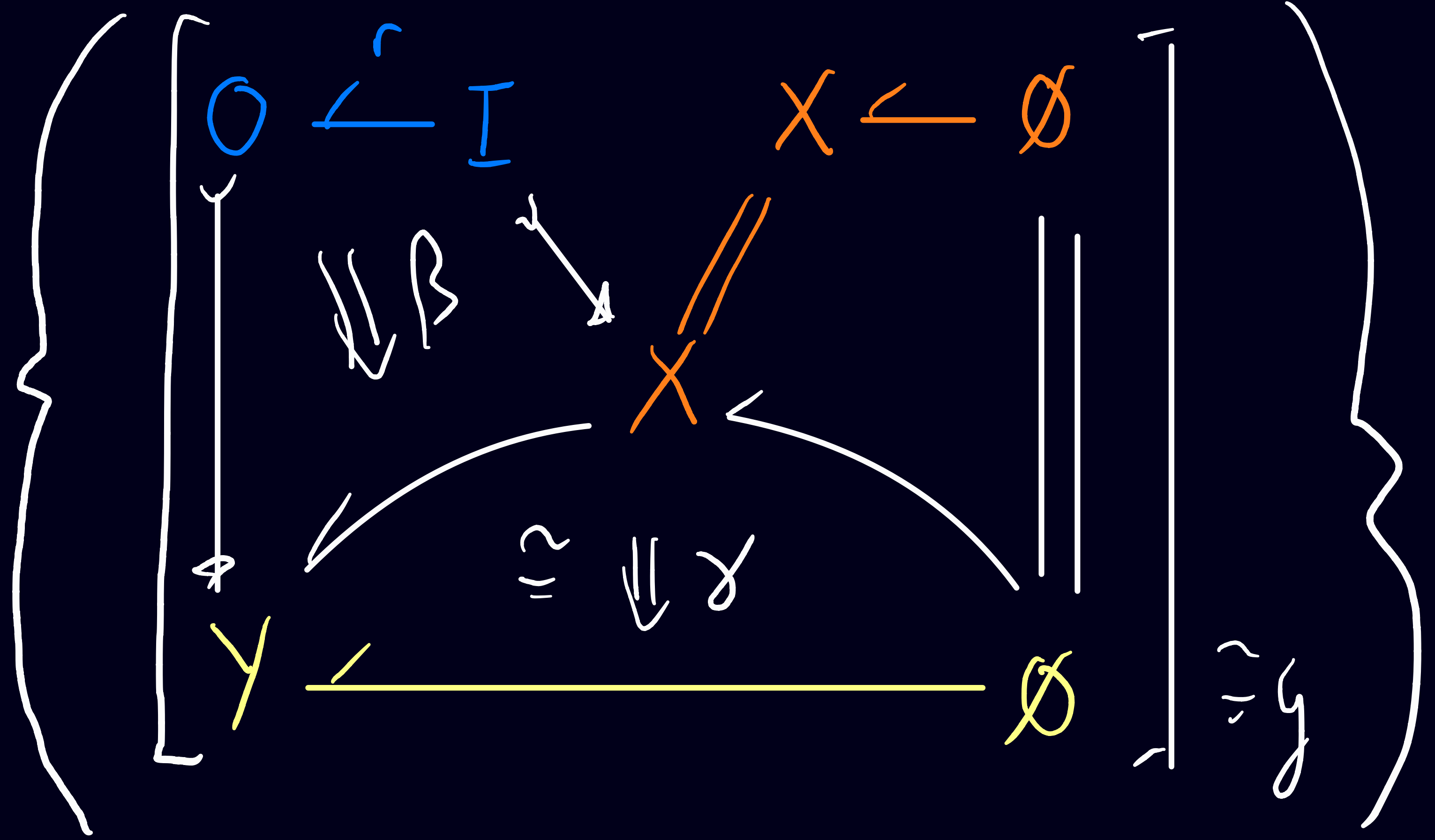
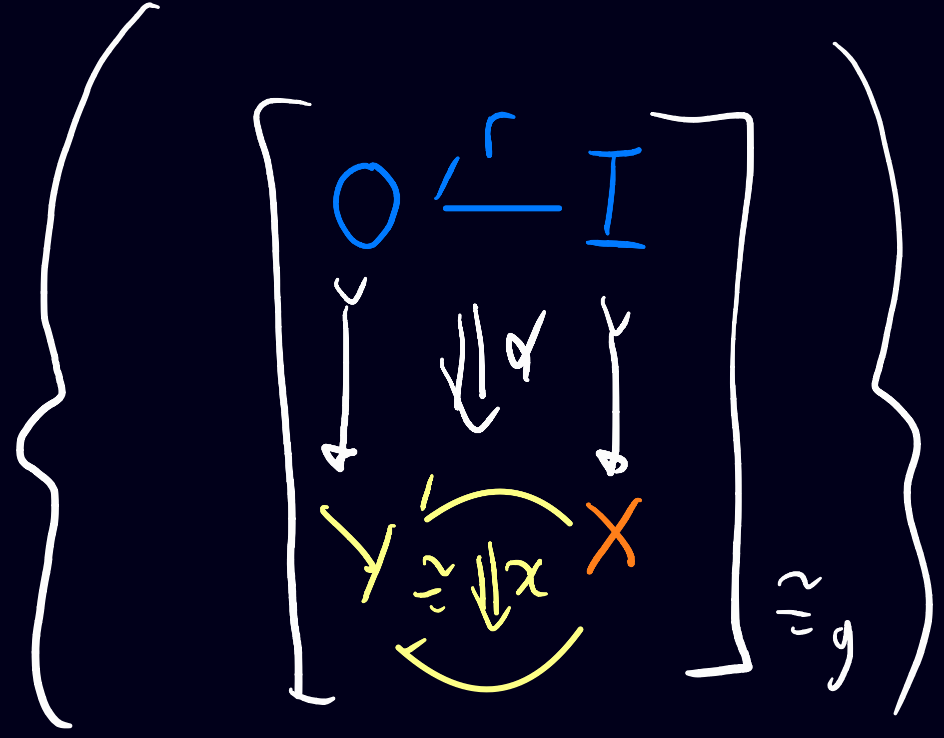


(24) Counting rewriting sequences

- "$\rho(\mathbb{S}(r))|X\rangle = \rho(\mathbb{S}(r))\rho(\mathbb{S}(X \leftarrow \emptyset))|\emptyset\rangle = \sum_{\alpha} \rho(\mathbb{S}(r_{\alpha}(X) \leftarrow \emptyset))|\emptyset\rangle$"

$(\hat{\Delta}_r * \hat{\Delta}_{X \leftarrow \emptyset})(Y \leftarrow \emptyset) \cong$  $/_{\cong_g}$ $\cong$  $/_{\cong_g}$

$\mathbb{S}$ is a MOPF $\Rightarrow$ "lifts" isos!

$\cong$  $\cong$  ✓

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Outlook

- "# of ways to rewrite X via applying rule r":

$$\int^{Y\in\mathbb{D}_0}\left(\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset}\right)(Y\leftarrow\emptyset)\cong\coprod_{\alpha\in\mathbb{D}_1}\left\{\begin{tikzcd} O \arrow[d, tail] \arrow[r, phantom, ""] & I \arrow[l, "r"'] \arrow[d, tail] \\ Y \arrow[r, phantom, ""] & X \arrow[l, "s"] \arrow[ul, phantom, "\Downarrow\alpha"] \end{tikzcd}\right\}\Big/\sim_Y$$

WHERE: $\forall$

$$\begin{tikzcd} Y \arrow[d, "\cong"'] & \\ Y' & X \arrow[ul, bend right, "s"'] \arrow[l, bend left, "s'"] \arrow[ull, phantom, "\Downarrow\gamma"] \end{tikzcd} \quad : \quad \begin{tikzcd} O \arrow[d, tail] & I \arrow[l, "r"'] \arrow[d, tail] \\ Y & X \arrow[l, "s"] \arrow[ul, phantom, "\Downarrow\alpha"] \end{tikzcd} \sim_Y \begin{tikzcd} O \arrow[d, tail] & I \arrow[l, "r"'] \arrow[d, tail] \\ Y' & X \arrow[l, "s'"] \arrow[ul, phantom, "\Downarrow\gamma\odot_v\alpha"] \end{tikzcd}$$

25 OUTLOOK

▸ "# OF WAYS TO REWRITE X VIA APPLYING RULE r":

$$\int^{Y\in\mathbb{D}_0}\left(\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset}\right)(Y\leftarrow\emptyset)\;\cong\;\coprod_{\alpha\in\mathbb{D}_1}\left\{\begin{tikzcd} O \arrow[d] & I \arrow[l, "r"'] \arrow[d] \\ Y \arrow[r, phantom, "\Uparrow\alpha"', pos=0.5] & X \arrow[l, "s"] \end{tikzcd}\right\}\Big/\sim_Y$$

WHERE: $\forall$
$$\begin{tikzcd} Y \arrow[d, "\cong"'] \arrow[dr, phantom, "\Downarrow\gamma"] & \\ Y' & X \arrow[ul, bend right, "s"'] \arrow[l, bend left, "s'"] \end{tikzcd} \;:\; \begin{tikzcd} O \arrow[d] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d] \\ Y & X \arrow[l, "s"] \end{tikzcd} \sim_Y \begin{tikzcd} O \arrow[d] \arrow[r, phantom, "\Downarrow\gamma\circ\alpha"] & I \arrow[l, "r"'] \arrow[d] \\ Y' & X \arrow[l, "s'"] \end{tikzcd}$$

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(25) OUTLOOK

- "# OF WAYS TO REWRITE X VIA APPLYING RULE r":

$$\int^{Y\in\mathbb{D}_0} (\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset})(Y\leftarrow\emptyset) \cong \coprod_{\alpha\in\mathbb{D}_1}\left\{\begin{tikzcd} O \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d] \\ Y & X \arrow[l, "s"] \end{tikzcd}\right\}\Big/\sim_Y$$

WHERE: $\forall$ $\begin{tikzcd} Y \arrow[d, "\cong"'] \arrow[dr, phantom, "\Downarrow\gamma"] & & X \arrow[ll, bend right, "s"'] \arrow[dll, bend left, "s'"] \\ Y' & & \end{tikzcd}$: $\begin{tikzcd} O \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d] \\ Y & X \arrow[l, "s"] \end{tikzcd} \sim_Y \begin{tikzcd} O \arrow[d] \arrow[dr, phantom, "\Downarrow\gamma\circ\alpha"] & I \arrow[l, "r"'] \arrow[d] \\ Y' & X \arrow[l, "s'"] \end{tikzcd}$

THANK YOU!

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