

A double categorical
look at different
flavours of factorisation
system

Virtual double cats
workshop

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Plan

① From factorisation
systems to double
cats of maps .

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- ② Orthogonality & factorisation for double cats of maps

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- ① From factorisation systems to double cats of maps .
- ② Orthogonality & factorisation for double cats of maps
- ③ Algebraic weak Fact. systems

① Factorisation systems

- \mathcal{C} a cat, $\mathcal{E}, \mathcal{M} \subseteq \text{Mor}(\mathcal{C})$

Eg. (Surjections, injections)
in Set or algebraic cat.

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$$A \xrightarrow{F} B$$
$$\mathcal{E} \ni e \downarrow c \nearrow m \in \mathcal{M}$$

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 \mathcal{E} 's & \mathcal{M} 's though?

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They are orthogonal.

Orthogonality

Say $\mathcal{E} \perp M$ if



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Say $\mathcal{E} \perp \mathcal{M}$ if



- $\mathcal{E}^\perp = \{g \in \mathcal{C} : \mathcal{E} \perp g\}$
- $\perp \mathcal{M} = \{f \in \mathcal{C} : f \perp \mathcal{M}\}$

Orthogonality

Say $\varepsilon \perp M$ if

Axiom of orthogonality

$$\therefore \mathcal{E} = \perp \mathcal{N} \text{ & } \mathcal{N} = \mathcal{E}^{\perp}$$

i.e. $f \in E$ iff it has unique
left lifting prop. wrt morphs
of M , & $f \in M$ iff ...

- Factorisation system (\mathcal{E}, \mathcal{M})
satisfies axioms of
1) factorisation & 2) orthogonality.

- Factorisation system (\mathcal{E}, \mathcal{M}) satisfies axioms of 1) factorisation & 2) orthogonality.
- Changing to weak orthogonality



we get weak factorisation systems.

- Call these OFS / WFS.

- In OFS/WFS (\mathcal{E}, \mathcal{M})
both classes closed under
composition & contain ids
 \Rightarrow wide subcat
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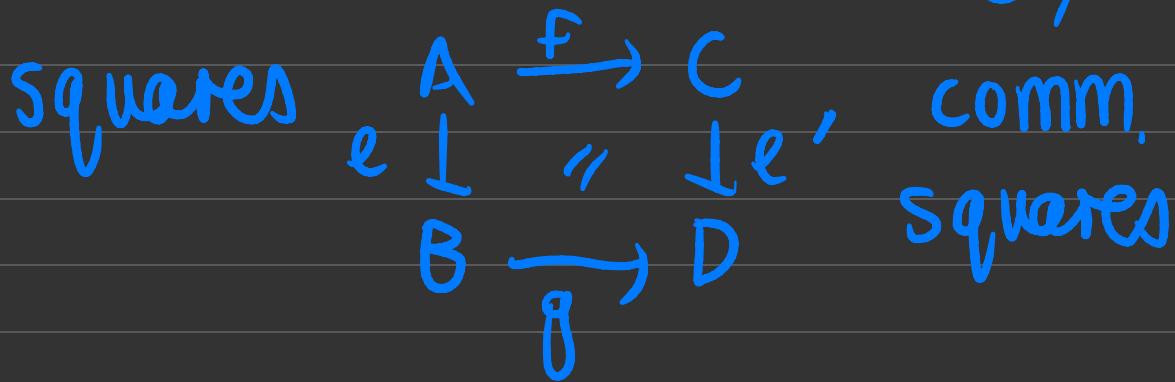
$$\mathcal{E} \longrightarrow \text{Arr}(\mathcal{E})$$

of arrow category

- Both cats assoc. to \mathcal{E} (or \mathcal{M}) are useful.
Can we put them in 1 structure?

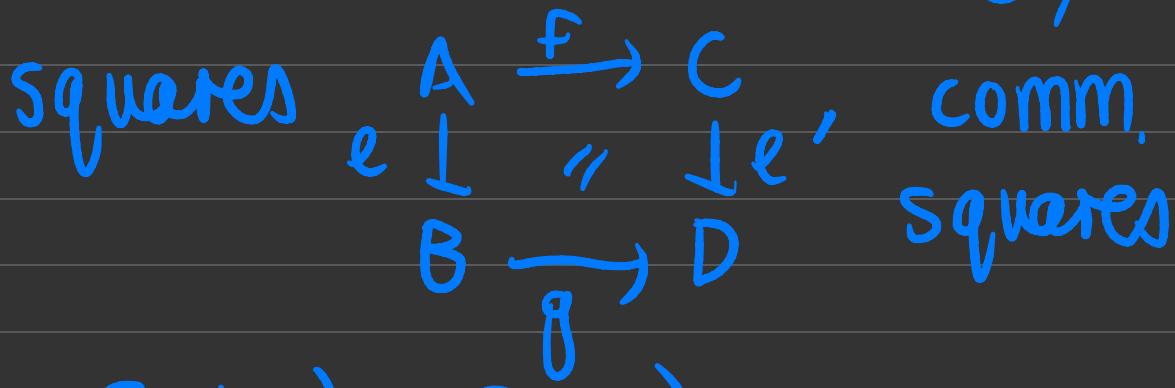
Yes - into a double cat

- $D(\mathcal{E})$: same obs & horizontal arrows as \mathcal{C} , vertical arrows as in \mathcal{E} ,



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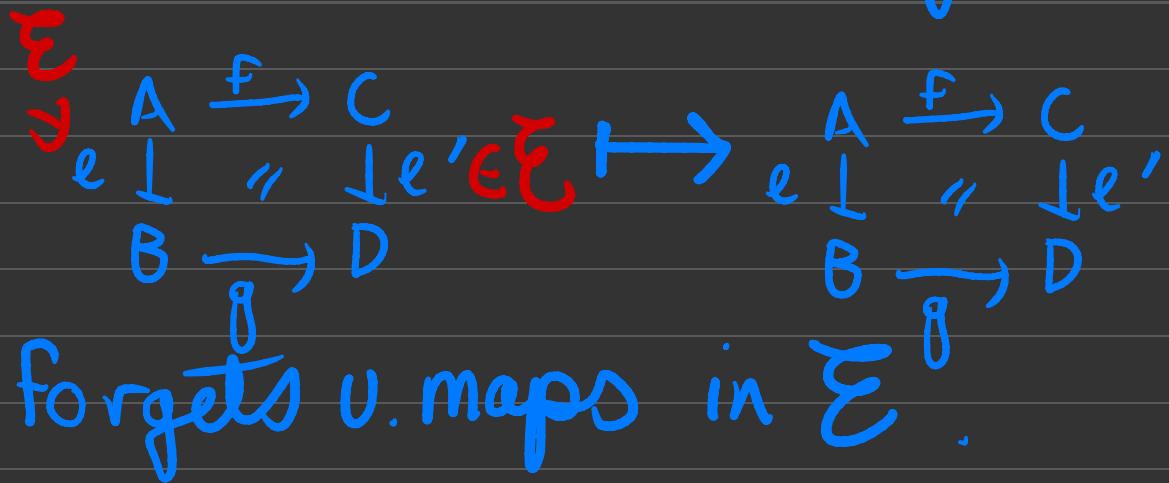
- $D(\mathcal{E})$: same obs & horizontal arrows as \mathcal{C} , vertical arrows as in \mathcal{E} ,



- $S_q(\mathcal{C}) = D(\mathcal{C})$ -
horizontal, vert. arrows
as in \mathcal{C} , all squares

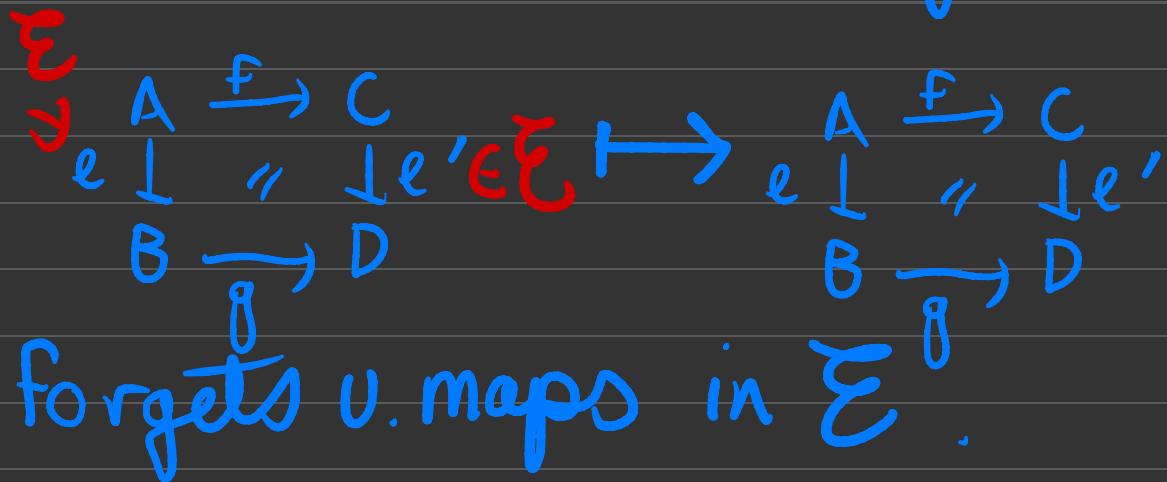
Forgetful double fun

$$U: D(\mathcal{E}) \longrightarrow Sq(\mathcal{C})$$



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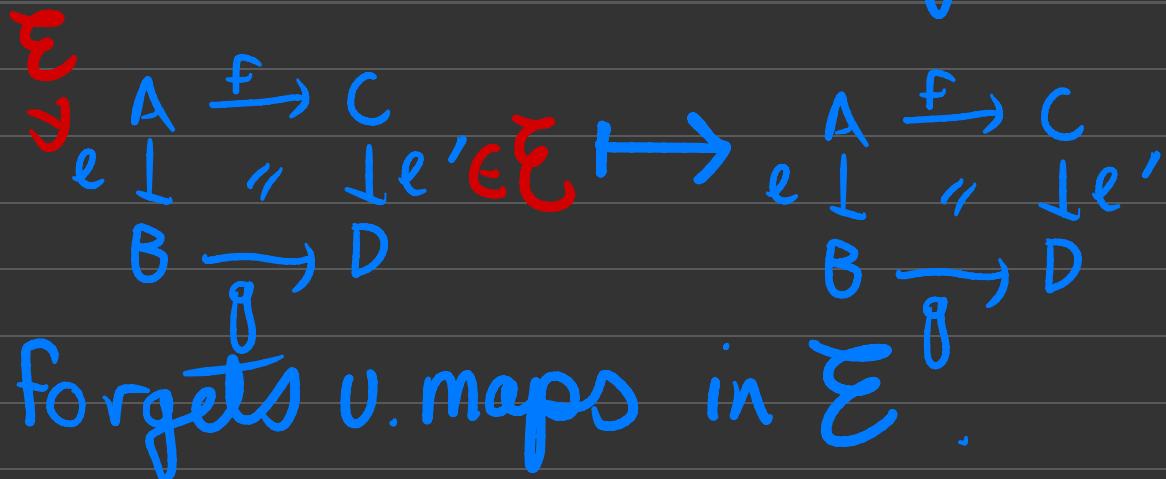
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Forgetful double fun

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- Encodes $C_{\mathcal{E}} \hookrightarrow \mathcal{C}$ via cats of vert. morphs
- Encodes $\mathcal{E} \hookrightarrow \text{Mor}(\mathcal{C})$ via cats of vert arrows & squares.

Double cats of maps

- Think of $U: \mathcal{A} \rightarrow S_q(\mathcal{C})$ as double cat of maps over \mathcal{C}

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- Think of $U: \mathcal{A} \rightarrow \text{Sq}(\mathcal{C})$ as double cat of maps over \mathcal{C}
- In practice, U usually id. on horiz. cats,
faithful on squares -
concrete double cat of
maps.
- Eg. $U: \mathcal{D}(\mathcal{E}) \rightarrow \text{Sq}(\mathcal{C})$

Examples

① $D(Surj) \rightarrow Sq(Set)$



i.e. double cat of surjections

Examples

① $D(\text{Surj}) \rightarrow Sg(\text{Set})$



i.e. double cat of surjection

Encodes property
(being surjection)
but can encode
structure too.

Examples

② $U : \text{SplEpi} \rightarrow \text{Sq}(\text{Set})$

$U.\text{arrows} : A \xrightarrow{(F, u)} B$
section of F

squares : $\begin{array}{ccc} A & \xrightarrow{f} & C \\ u \uparrow \downarrow F & & v \uparrow \downarrow g \\ B & \xrightarrow{s} & D \end{array}$

- Vert. comp : compose
sections

Examples

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Maps w' structure -
 \mathcal{U} faithful ✓ (full ✗) on squares

Examples

- On Cat, conc. double cats of - adjunctions
 - equivalences
 - fibrations &
- any flavour of map

Examples

- On Cat , conc. double cats of
 - adjunctions
 - equivalences
 - fibrations &
- any flavour of map
- In connection with hott - structured
fibrations
on simplicial/cubical sets.

Useful guiding example

① $\text{SplRef} \rightarrow \text{Sq}((\text{Cat}))$

vert. maps "split reflection":

$$A \xrightleftharpoons[\perp]{F} B$$

$$+ (f + u, n : l \Rightarrow uf, \epsilon = \text{id})$$

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② $\text{SplFib} \rightarrow \text{Sq}(\text{Cat})$

vert maps "split fibrations"

So far :

- replaced classes of maps

$\mathcal{E}, \mathcal{M} \subseteq \text{Mor}(\mathcal{C})$ by

double cats of maps

$$\mathbb{U} \xrightarrow{u} \text{Sq}(\mathcal{C}) \xleftarrow{v} \mathbb{IR}$$

So Far :

- replaced classes of maps
 $\mathcal{E}, \mathcal{M} \subseteq \text{Mor}(\mathcal{C})$ by
double cat's of maps
- $\mathbb{L} \xrightarrow{u} \text{Sq}(\mathcal{C}) \xleftarrow{v} \mathbb{R}$
- Can we understand
"fact. systems" in this
setting approp. to
structured maps ?

Orthogonality via double cats ?

- Given OFS $(\mathcal{E}, \mathcal{M})$
consider

$$D(\mathcal{E}) \xrightarrow{u} S_g(\mathcal{E}) \xleftarrow{v} D(\mathcal{M}).$$

Orthogonality via double cats?

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consider

$$D(\mathcal{E}) \xrightarrow{u} S_0(\mathcal{E}) \xleftarrow{v} D(\mathcal{M}).$$

- Given $f \in D(\mathcal{E}), g \in D(\mathcal{M})$ vert.

$$\begin{array}{ccc} & & \\ & & \\ f & \xrightarrow{u} & V_C \\ u_A & & \\ & \downarrow & \\ & \exists! & \\ & \nearrow & \\ u_B & & \\ & \xrightarrow{s} & \\ & & V_D \\ & & g \end{array}$$

Thoughts

- For structured maps,
unique diagonals too
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→ Structured liftings
satisfying compatibilities

Lifting operations

Consider $L \xrightarrow{u} Sq(c) \xleftarrow{v} IR$

(L, IR) -lifting operation Ψ

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consists of :
@ $f \in L, g \in IR$ vertical
& each square

$$UA \xrightarrow{r} VC$$

$$\begin{array}{ccc} UF & \perp & \downarrow Vg \\ UB & \xrightarrow{s} & VD \end{array}$$

Lifting operations

Consider $L \xrightarrow{u} S_q(e) \xleftarrow{v} IR$
 (L, IR) -lifting operation Ψ

consists of :
@ $f \in L, g \in IR$ vertical
& each square

$UA \xrightarrow{r} VC$ a diagonal
 $UF \perp \xrightarrow{\Psi(g(r,s))} = \downarrow Vg$ fillet
 $UB \xrightarrow{s} VD$ such that

The fillers

$$\begin{array}{ccc} \text{UA} & \xrightarrow{r} & \text{VC} \\ \text{uf} \perp & \xrightarrow{\text{right}} & \perp \text{vg} \\ \text{UB} & \xrightarrow{s} & \text{VD} \end{array}$$

are compatible with
double category
structure on
LL & IR;

① φ is natural in squares:

(II) At $\alpha: F' \rightarrow F$ a square

$$\begin{array}{ccccc} Ua & \xrightarrow{(U\alpha)} & Ua & \xrightarrow{r} & Vc \\ uf \downarrow & & \downarrow uf & \nearrow \varphi & \downarrow Vg = uf \downarrow \\ Ub & \xrightarrow{(U\alpha)} & Ub & \xrightarrow{s} & Vd \\ & & & & Ub \xrightarrow{s.U\alpha} Vd \end{array}$$

(Ir) & sim. on the right;

① φ is natural in squares:

(1l) At $\alpha: F' \rightarrow F$ a square

$$\begin{array}{ccccc} Ua & \xrightarrow{\text{(U}\alpha\text{)}} & Ua & \xrightarrow{r} & Vc \\ u_f \downarrow & & \downarrow u_f & \nearrow \varphi & \downarrow v_g = u_f' \\ Ub & \xrightarrow{\text{(U}\alpha\text{)}} & Ub & \xrightarrow{s} & Vd \\ & & & & \downarrow s \cdot u_\alpha \end{array}$$

(1r) & sim. on the right;

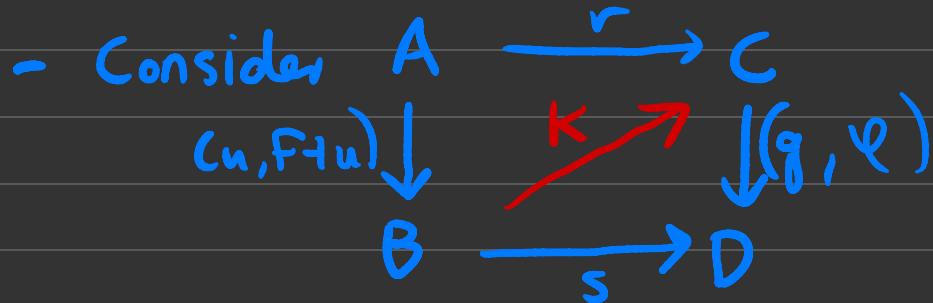
② φ respects vertical comp.:

$$\begin{array}{ccc} (2l) Ua_1 & \xrightarrow{r} & Vc \\ u_{f_1} \downarrow & \nearrow \varphi_{12} & \\ Ua_2 & & \\ u_{f_2} \downarrow & \nearrow \varphi_{23} & \\ Ua_3 & \xrightarrow{s} & Vd \end{array} \quad \begin{array}{ccc} Ua_1 & \xrightarrow{r} & Vc \\ u_{f_1} \downarrow & \nearrow \varphi_{12} & \\ Ua_2 & \xrightarrow{s} & Vd \\ u_{f_2} \downarrow & \nearrow \varphi_{23} & \\ Ua_3 & \xrightarrow{s} & Vd \end{array}$$

(2r) & on the right.

Example

- $(\text{LL}, \text{IR}) = (\text{split reflections}, \text{split fibrations})$



Example

• $(\text{LR}, \text{IR}) = (\text{split reflections}, \text{split fibrations})$



- $f+u$ w' unit $b \xrightarrow{n_b} uFb$

so $sb \xrightarrow{s n_b} suFb = grFb$

Example

- $(\text{LR}, \text{IR}) = (\text{split reflections}, \text{split fibrations})$

- Consider $A \xrightarrow{r} C$

$$\begin{array}{ccc} & & r \\ (u, f+u) \downarrow & K \nearrow & \downarrow (g, \varphi) \\ B & \xrightarrow{s} & D \end{array}$$

- $f+u$ w' unit $b \xrightarrow{\pi b} uFb$

$$\text{so } sb \xrightarrow{s\pi b} suf b = grFb$$

$$\uparrow g$$

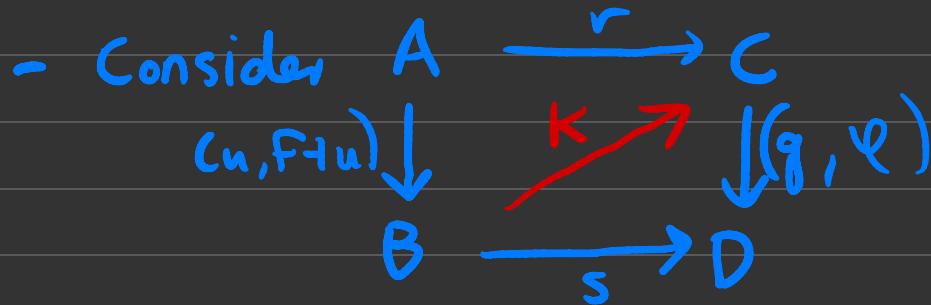
$$kb \longrightarrow ufb$$

$$\varphi(grfb, s\pi b)$$

given lifting
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Example

• $(\text{LL}, \text{IR}) = (\text{split reflections}, \text{split fibrations})$



- $f \dashv u$ w' unit $b \xrightarrow{\text{snb}} uFb$

so $sb \xrightarrow{\text{snb}} suf b = grFb$

$\uparrow g$ given lifting
 $kb \xrightarrow{\psi_{grFb, snb}} ufb$ of split fib.

- Object part of lifting operation - exercise!

I call a triple
 $\text{LL} \xrightarrow{u} \text{Sq}(\mathcal{C})$, $\text{IR} \xrightarrow{v} \text{Sq}(\mathcal{C})$
& (LL, IR) -lifting op. φ
a lifting structure

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Def) An algebraic weak fact. system is a lifting structure $(\text{LL}, \varphi, \text{IR})$ sat. axioms of orthogonality & factorisation.

Axiom of orthogonality

(LL, IR)-liftings $\varphi \sim$

① $\text{IR} \xrightarrow{\varphi} \underline{\text{RLP(LL)}}$ where
 RLP(LL) has vert. arrows
 (g, φ) where $\begin{array}{ccc} UA & \xrightarrow{r} & C \\ \text{UF} \perp \varphi & \nearrow \varphi^{-1}(r,s) & \downarrow g \in \mathcal{C} \\ UB & \xrightarrow{s} & D \end{array}$
 compat. w' squares & v. comp.
 in LL

② $\text{LL} \xrightarrow{\varphi e} \underline{\text{LLP(IR)}}$
 where LLP(IR) dually defined

(L, φ, R) satisfies

axiom of orthogonality

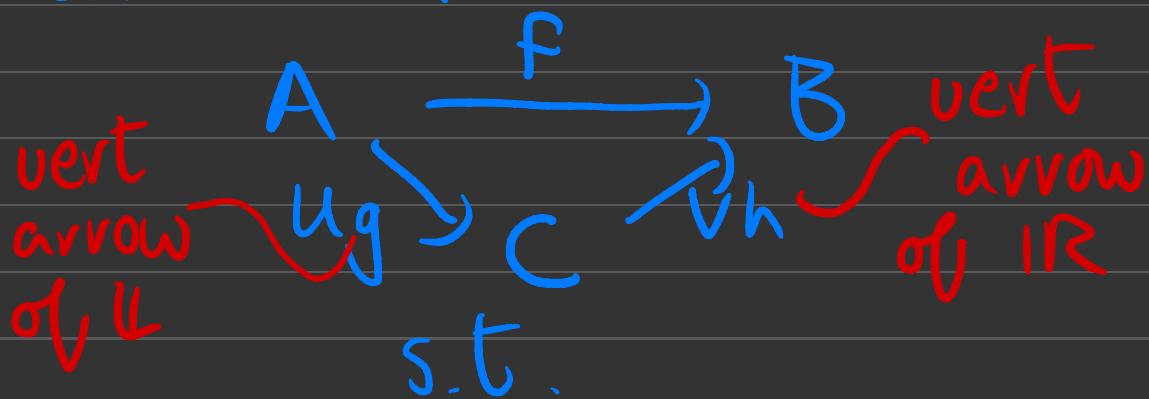
if $\varphi_L : L \rightarrow L P^A(R)$
& $\varphi_R : R \rightarrow R P(L)$
are invertible.

Axiom of Factorisation

Lifting str $(\mathcal{L}, \varphi, \text{IR})$
satisfies axiom

if each $A \xrightarrow{f} B \in \mathcal{C}$

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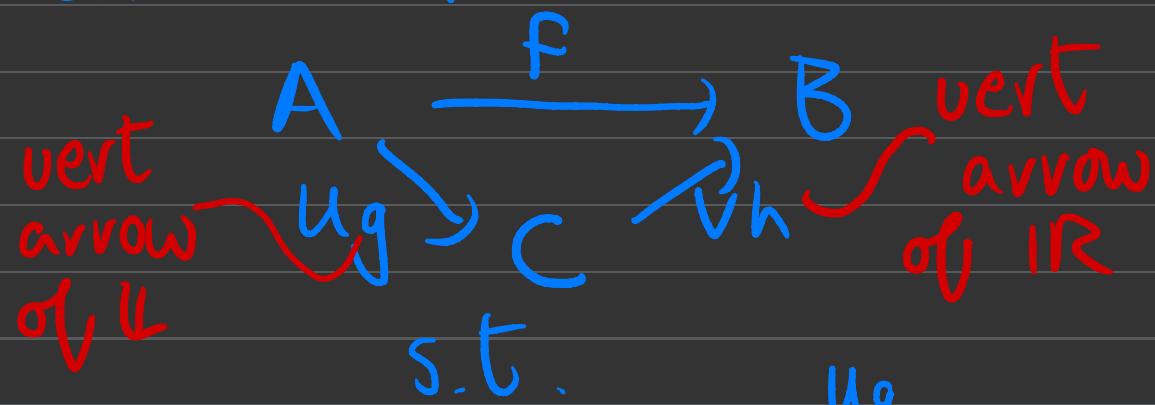


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$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ u_g \downarrow & & \downarrow v_h \\ C & \xrightarrow{g} & B \end{array}$$

U-universal
in $\text{Mor}(\mathcal{C})$

$$\begin{array}{ccc} A & \xrightarrow{u_g} & C \\ f \downarrow & & \downarrow v_h \\ B & \xrightarrow{g} & B \end{array}$$

V-universal
in $\text{Mor}(\mathcal{C})$

Example

(SplRef, \sqsubseteq , SplFils)
is awfs on Cat -



comma cat:
obs ($b \rightarrow f_a, a$)

Awfs classically

- Awfs on \mathcal{C} involves:
 - funct. fact $\mathcal{C}^{\perp} \xrightarrow{K} \mathcal{C}^{\perp\perp}$
 - comonad & monad L, R on \mathcal{C}^{\perp} related by a distrib. law.

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- Making small ob. argument converge (Garner)

Applications

-Homotopy Theory (Riehl ...)

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- Higher cats (Nikolaus, Gepner)

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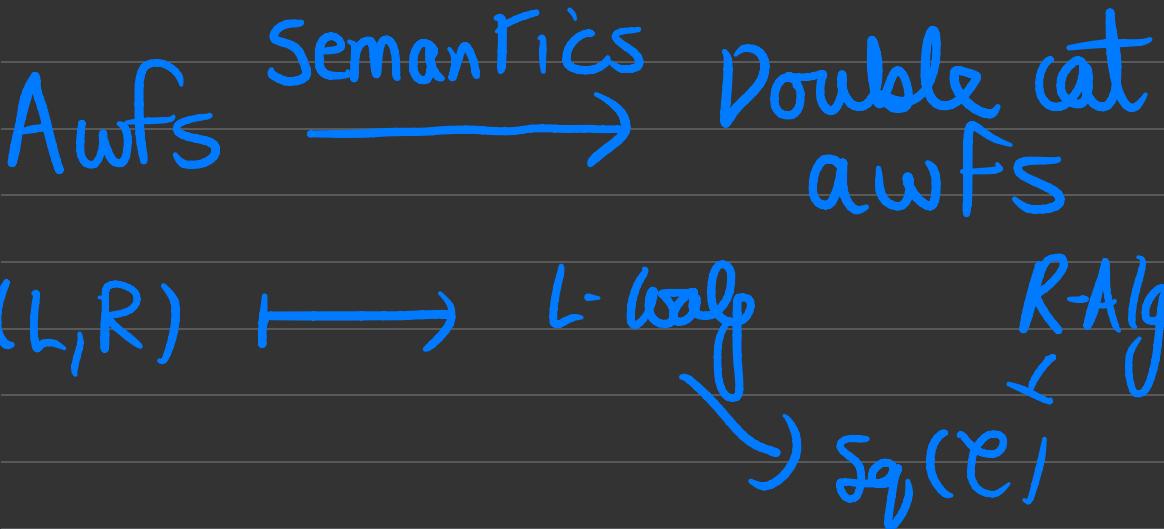
- Homotopy Theory (Riehl ...)
- Higher cats (Nikolaus, Gepner ...)
- Models of hott (Coquand, Gambino, van den Berg ...)

Value of double cats

- Equiv. to double cat. def
(JB22)

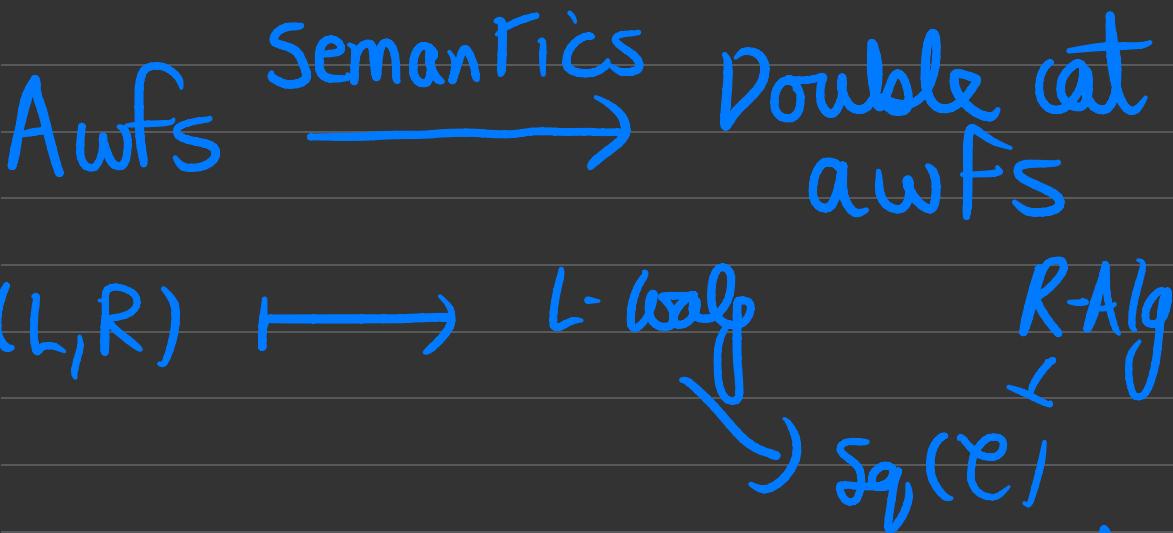
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Awfs in fact determined
by left or right
double cat off mass
(Recognition thru BG16)

- Constructions
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often hard - lots of
structure,
 - Much easier to do
construction on semantic
(double-cat) side &
apply recognition theorem
- See BG16.

MORAL

Whenever you are
struggling with a
problem about awfs,

think

double-categorically

Some refs

Grandis-Tholen :

Natural weak fct. systems

Garner : Understanding
small ob. arg.

Riehl - Algebraic
model structures

Bourke-Garner : Awfs I -
accessible awfs

Bourke : An orthogonal
approach to awfs

Is this madness?

$\langle \text{L} \rightarrow \text{S} \times \text{C} \rangle \leftarrow_{\text{V}} \text{IR}$

(L, IR) - lifting op :

$$\text{UA} \xrightarrow{\text{v}} \text{VC}$$

$$\text{UA} \xrightarrow{\text{v}} \text{VC}$$

$$\text{uf} \perp A \perp \text{vg} = \text{uf} \perp \varphi, A \downarrow \perp$$

$$\text{UB} \xrightarrow{\text{s}} \text{VD}$$

$$\text{UB} \xrightarrow{\varphi_A} \text{VC}$$

$$\perp \varphi_2 A \perp \text{vg}$$

$$\text{UB} \xrightarrow{\text{s}} \text{VD}$$

"Awfs"-like structures
on more gen. double cats?