

Some things about double categories

Robert Paré

Virtual Double Category Workshop

November 28, 2022

Double categories

(Ehresmann) A *double category* is a category object in **Cat**

$$\mathbb{A}: \quad \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \begin{array}{c} \xRightarrow{p_1} \\ \xRightarrow{\bullet} \\ \xRightarrow{p_1} \end{array} \mathbf{A}_1 \begin{array}{c} \xRightarrow{d_0} \\ \xleftarrow{\text{id}} \\ \xRightarrow{d_1} \end{array} \mathbf{A}_0$$

$$\begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{p_2} & \mathbf{A}_1 \\ \downarrow p_1 & \boxed{\text{Pb}} & \downarrow d_0 \\ \mathbf{A}_1 & \xrightarrow{d_1} & \mathbf{A}_0 \end{array} \qquad \begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{\mathbf{A}_1 \times_{\mathbf{A}_0} \bullet} & \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \\ \downarrow \bullet \times_{\mathbf{A}_0} \mathbf{A}_1 & \boxed{\text{Assoc}} & \downarrow \bullet \\ \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{\bullet} & \mathbf{A}_\bullet \end{array}$$

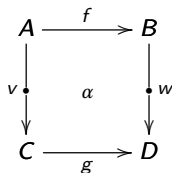
Double functor

$$\begin{array}{ccccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \begin{array}{c} \xRightarrow{\quad} \\ \xRightarrow{\quad} \\ \xRightarrow{\quad} \end{array} & \mathbf{A}_1 & \begin{array}{c} \xRightarrow{\quad} \\ \xRightarrow{\quad} \\ \xRightarrow{\quad} \end{array} & \mathbf{A}_0 \\ \downarrow F_1 \times_{F_0} F_1 & & \downarrow F_1 & & \downarrow F_0 \\ \mathbf{B}_1 \times_{\mathbf{B}_0} \mathbf{B}_1 & \begin{array}{c} \xRightarrow{\quad} \\ \xRightarrow{\quad} \\ \xRightarrow{\quad} \end{array} & \mathbf{B}_1 & \begin{array}{c} \xRightarrow{\quad} \\ \xRightarrow{\quad} \\ \xRightarrow{\quad} \end{array} & \mathbf{B}_0 \end{array}$$

Think inside the box

$$\mathbb{A} : \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \rightrightarrows \mathbf{A}_1 \xrightleftharpoons[\text{id}]{\quad} \mathbf{A}_0$$

- Objects of \mathbf{A}_0 are *objects* of \mathbb{A}
- Morphisms of \mathbf{A}_0 are *horizontal arrows* of \mathbb{A}
- Objects of \mathbf{A}_1 are *vertical arrows* of \mathbb{A}
- Morphisms of \mathbf{A}_1 are *double cells* of \mathbb{A}

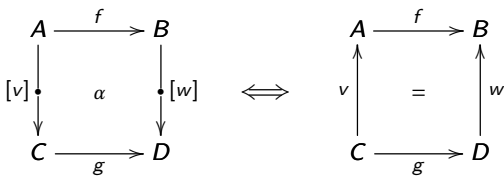


A double category is a category with two kinds of morphisms, suitably related

Opposite

A an arbitrary category

$$(\square \mathbf{A})^{co}$$



The diagram illustrates the relationship between a commutative square in a category \mathbf{A} and its corresponding square in the opposite category \mathbf{A}^{co} . On the left, a square in \mathbf{A} has vertices A (top-left), B (top-right), C (bottom-left), and D (bottom-right). Horizontal arrows are $f: A \rightarrow B$ and $g: C \rightarrow D$. Vertical arrows are $[v]: A \rightarrow C$ and $[w]: B \rightarrow D$. The square is labeled α . On the right, the corresponding square in \mathbf{A}^{co} has the same vertices but with reversed vertical arrows: $v: C \rightarrow A$ and $w: D \rightarrow B$. The horizontal arrows remain $f: A \rightarrow B$ and $g: C \rightarrow D$. The square is labeled $=$. A double-headed arrow \iff connects the two squares.

Student duality

\mathbf{A} a regular category

$\mathbf{Rel}(\mathbf{A})$

$$\begin{array}{ccc}
 A & \xrightarrow{f} & C \\
 R \downarrow & \alpha & \downarrow S \\
 B & \xrightarrow{g} & D
 \end{array}
 \quad \text{iff} \quad
 \begin{array}{ccc}
 R & \xrightarrow{\alpha} & S \\
 \downarrow & & \downarrow \\
 A \times B & \xrightarrow{f \times g} & C \times D
 \end{array}$$

Proposition

There is a double functor $\square \mathbf{A}^{co} \rightarrow \mathbf{Rel}(\mathbf{A})$ which is the identity on objects and horizontal arrows, faithful on vertical arrows and full and faithful on cells

$$\begin{array}{c} A \\ \downarrow [v] \bullet \\ B \end{array}
 \longleftrightarrow
 B \xrightarrow{v} A
 \quad \longmapsto \quad
 \begin{array}{c} B \\ \downarrow \langle v, 1_B \rangle \\ A \times B \end{array}
 \longleftrightarrow
 \begin{array}{c} A \\ \downarrow \bullet v^* \\ B \end{array}$$

Companions

v *companion* to f

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \xrightarrow{f} B \\
 \parallel & \psi & \downarrow v \quad \chi \quad \parallel \\
 A & \xrightarrow{f} & B \xlongequal{\quad} B
 \end{array}
 =
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \parallel & \text{id}_f & \parallel \\
 A & \xrightarrow{f} & B
 \end{array}$$

$$\chi\psi = \text{id}_f$$

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \parallel & \psi & \downarrow v \\
 A & \xrightarrow{f} & B \\
 v \downarrow & \chi & \parallel \\
 B & \xlongequal{\quad} & B
 \end{array}
 =
 \begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 v \downarrow & 1_v & \downarrow v \\
 B & \xlongequal{\quad} & B
 \end{array}$$

$$\chi \circ \psi = 1_v$$

Proposition

- (1) If f has a companion it's unique up to isomorphism: write $v = f_*$
- (2) $(1_A)_* \cong \text{id}_A$
- (3) $(gf)_* \cong g_* f_*$

Conjoints

w is *conjoint* to f

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \parallel & \alpha & \downarrow w \\ A & \xrightarrow{f} & B \end{array} \quad \begin{array}{c} B \\ \parallel \\ B \end{array} \quad \begin{array}{c} \beta \\ \downarrow \\ A \end{array} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \parallel & \text{id}_f & \parallel \\ A & \xrightarrow{f} & B \end{array}$$

$$\beta\alpha = \text{id}_f$$

$$\begin{array}{ccc} B & \xrightarrow{\quad} & B \\ \downarrow w & \beta & \parallel \\ A & \xrightarrow{f} & B \\ \parallel & \alpha & \downarrow w \\ A & \xrightarrow{\quad} & A \end{array} \quad = \quad \begin{array}{ccc} B & \xrightarrow{\quad} & B \\ \downarrow w & 1_B & \downarrow w \\ A & \xrightarrow{\quad} & A \end{array}$$

$$\alpha \circ \beta = 1_w$$

- Unique up to iso: write $w = f^*$
- $1_A^* \cong \text{id}_A$
- $(gf)^* \cong f^*g^*$

Adjoint

$$\begin{array}{c}
 A \equiv A \equiv A \\
 \downarrow v \quad = \quad \downarrow v \\
 B \equiv B \quad \epsilon \\
 \downarrow w \\
 \eta \quad A \equiv A \\
 \downarrow v \quad = \quad \downarrow v \\
 B \equiv B \equiv B
 \end{array}
 =
 \begin{array}{c}
 A \equiv A \\
 \downarrow v \quad = \quad \downarrow v \\
 B \equiv B \\
 \\
 B \equiv B \\
 \downarrow w \quad = \quad \downarrow w \\
 A \equiv A
 \end{array}
 =
 \begin{array}{c}
 B \equiv B \equiv B \\
 \downarrow w \quad = \quad \downarrow w \\
 \eta \quad A \equiv B \\
 \downarrow v \\
 B \equiv B \quad \epsilon \\
 \downarrow w \quad = \quad \downarrow w \\
 A \equiv A \equiv A
 \end{array}$$

$w \dashv v$

Companions, conjoints, adjoints

Theorem

Any two of the following conditions imply the third:

(1) $v = f_*$

(2) $w = f^*$

(3) $v \dashv w$

Theorem

In $\mathbb{R}el(\mathbf{A})$

(1) *Every f has a companion: $f_* = (A \xrightarrow{\langle 1_A, f \rangle} A \times B)$*

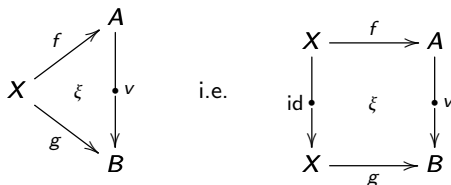
(2) *Every f has a conjoint: $f^* = (A \xrightarrow{\langle f, 1_A \rangle} B \times A)$*

(3) *Every adjoint pair $R \dashv S$ is of the form $f_* \dashv f^*$*

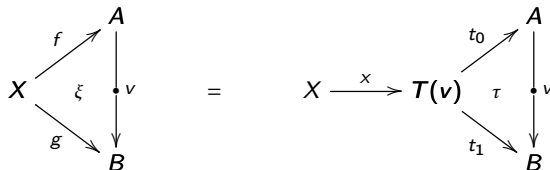
Say $\mathbb{R}el(\mathbf{A})$ is *Cauchy*

Tabulators

The *tabulator* of v is a universal cell τ



$$\forall \xi \exists ! x (\xi = \tau x)$$

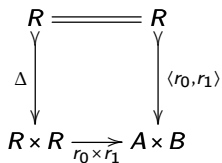
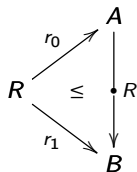


$T(v)$ is *effective* if t_1 has a companion, t_0 has a conjoint and $v \cong t_{1*} \bullet t_0^*$

Tabulating relations

Proposition

$\mathbb{R}el(\mathbf{A})$ has effective tabulators

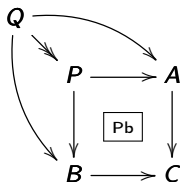


Double functors on relations

Theorem

Double functors $\mathbf{Rel}(\mathbf{A}) \rightarrow \mathbf{Rel}(\mathbf{B})$ “are” functors $\mathbf{A} \rightarrow \mathbf{B}$ which preserve quasi-pullbacks

Quasi-pullback Q



Transformations

Doub = **Cat**(**Cat**) is cartesian closed, so **Doub**(\mathbb{A}, \mathbb{B}) is a double category

A *horizontal transformation* $t: F \multimap G$

- $\forall A$ a horizontal arrow $tA: FA \multimap GA$
- $\forall v: A \multimap \bullet \multimap A'$ a cell

$$\begin{array}{ccc}
 FA & \xrightarrow{tA} & GA \\
 Fv \downarrow & & \downarrow Gv \\
 & tv & \\
 FA' & \xrightarrow{tA'} & GA'
 \end{array}$$

- Horizontally natural

$$\begin{array}{ccccc}
 FA & \xrightarrow{tA} & GA & \xrightarrow{Gf} & GC \\
 Fv \downarrow & & \downarrow Gv & G\alpha & \downarrow Gw \\
 FA' & \xrightarrow{tA'} & GA' & \xrightarrow{Gg} & GC'
 \end{array}
 =
 \begin{array}{ccccc}
 FA & \xrightarrow{Ff} & FC & \xrightarrow{tC} & GC \\
 Fv \downarrow & F\alpha & \downarrow Fw & tw & \downarrow Gw \\
 FA' & \xrightarrow{Fg} & FC' & \xrightarrow{tC'} & GC'
 \end{array}$$

Transformations (continued)

- Vertically functorial

$$\begin{array}{ccc}
 FA & \xrightarrow{tA} & GA \\
 \downarrow F \text{id}_A & \bullet & \downarrow t(\text{id}_A) \\
 FA & \xrightarrow{tA} & GA \\
 & \bullet & \downarrow G \text{id}_A \\
 & & GA
 \end{array}
 =
 \begin{array}{ccc}
 FA & \xrightarrow{tA} & GA \\
 \downarrow \text{id}_{FA} & \bullet & \downarrow \text{id}_{tA} \\
 GA & \xrightarrow{tA} & GA \\
 & \bullet & \downarrow \text{id}_{GA} \\
 & & GA
 \end{array}$$

$$\begin{array}{ccc}
 FA & \xrightarrow{tA} & GA \\
 \downarrow Fv & \bullet & \downarrow tv \\
 FA' & \xrightarrow{tA'} & GA' \\
 \downarrow Fv' & \bullet & \downarrow tv' \\
 FA'' & \xrightarrow{tA''} & GA''
 \end{array}
 =
 \begin{array}{ccc}
 FA & \xrightarrow{tA} & GA \\
 \downarrow F(v' \bullet v) & \bullet & \downarrow t(v' \bullet v) \\
 FA'' & \xrightarrow{tA''} & GA''
 \end{array}$$

Vertical transformations and cells

A **vertical transformation** $u: F \multimap H$ is the transpose notion (switch horizontal and vertical)

- $\forall A$ a vertical arrow $uA: FA \multimap HA$
- $\forall f: A \multimap A'$ a cell

$$\begin{array}{ccc}
 FA & \xrightarrow{Ff} & FA' \\
 uA \downarrow & uf & \downarrow uA' \\
 HA & \xrightarrow{Hf} & HA'
 \end{array}$$

- Vertically natural
- Horizontally functorial

A **double cell** assigns to each object A a cell vA

$$\begin{array}{ccc}
 F & \xrightarrow{t} & G \\
 u \downarrow & v & \downarrow u' \\
 H & \xrightarrow{t'} & K
 \end{array}
 \qquad
 \begin{array}{ccc}
 FA & \xrightarrow{tA} & GA \\
 uA \downarrow & vA & \downarrow u'A \\
 HA & \xrightarrow{t'A} & KA
 \end{array}$$

satisfying two conditions – horizontal and vertical naturality

Transformations for $\mathbb{R}el$

$F, G: \mathbf{A} \longrightarrow \mathbf{B}$ quasi-pullback preserving functors
 $\Phi, \Psi: \mathbb{R}el(\mathbf{A}) \longrightarrow \mathbb{R}el(\mathbf{B})$ their extensions to $\mathbb{R}el$

Theorem

- (1) Horizontal transformations $\Phi \longrightarrow \Psi$ are in natural bijection with natural transformations $F \longrightarrow G$
- (2) Vertical transformations $\Phi \longrightarrow \bullet \longrightarrow \Psi$ are in natural bijection with relations

$$V \rightrightarrows F \times G$$

in the category $QPB(\mathbf{A}, \mathbf{B})$ of quasi-pullback preserving functors and quasi-cartesian natural transformations

Question: Is $QPB(\mathbf{A}, \mathbf{B})$ a regular category?

Kleisli

$\mathbb{T} = (T, \eta, \mu)$ is a monad on \mathbf{A}

We get a double category $\mathbb{Kl}(\mathbb{T})$

$$\begin{array}{ccc}
 A \xrightarrow{f} B & & A \xrightarrow{f} B \\
 [v] \downarrow \quad \alpha \quad \downarrow [w] & \longleftrightarrow & v \downarrow \quad = \quad \downarrow w \\
 C \xrightarrow{g} D & & TC \xrightarrow{Tg} TD
 \end{array}$$

- Every horizontal arrow $f: A \longrightarrow B$ has a companion

$$\begin{array}{ccc}
 A & & A \\
 f_* \downarrow & \longleftrightarrow & \downarrow f \\
 B & & B \\
 & & \downarrow \eta B \\
 & & TB
 \end{array}
 \quad (f_* = [\eta B \cdot f])$$

- $f: A \longrightarrow B$ has a conjoint iff $T(f)$ iso

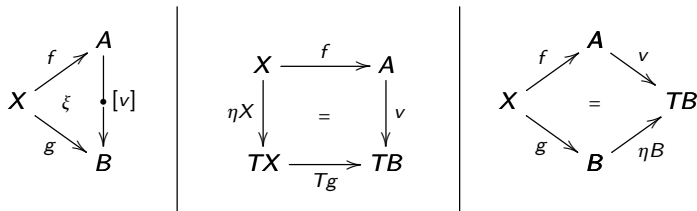
$$\begin{array}{ccc}
 B & & B \\
 f^* \downarrow & \longleftrightarrow & \downarrow \eta B \\
 A & & TB \\
 & & \downarrow (Tf)^{-1} \\
 & & TA
 \end{array}
 \quad (f^* = [(Tf)^{-1} \cdot \eta B])$$

Tabulating Kleisli

Proposition

$\mathbb{K}l(T)$ has tabulators iff \mathbf{A} has pullbacks along ηA 's

Proof.



- The tabulators are not effective

Double functors on $\mathbb{K}1$

Theorem

Double functors $\mathbb{K}1(\mathbb{T}) \rightarrow \mathbb{K}1(\mathbb{S})$ correspond to monad morphisms $\mathbb{T} \rightarrow \mathbb{S}$

Morphism of monads:

$(\Psi, \psi): \mathbb{T} \rightarrow \mathbb{S} \quad (\mathbb{T} = (\mathbf{A}, T, \eta, \mu), \mathbb{S} = (\mathbf{B}, S, \kappa, \nu))$

$$\begin{array}{ccc}
 \mathbf{A} & \xrightarrow{\Psi} & \mathbf{B} \\
 T \downarrow & \psi \nearrow & \downarrow S \\
 \mathbf{A} & \xrightarrow{\Psi} & \mathbf{B}
 \end{array}$$

$$\begin{array}{ccc}
 & \Psi & \\
 \Psi\eta \swarrow & & \searrow \kappa\Psi \\
 \Psi T & \xrightarrow{\psi} & S\Psi
 \end{array}$$

$$\begin{array}{ccccc}
 \Psi T T & \xrightarrow{\psi T} & S\Psi T & \xrightarrow{S\psi} & SS\Psi \\
 \Psi\mu \searrow & & & & \swarrow \nu\Psi \\
 & \Psi T & \xrightarrow{\psi} & S\Psi &
 \end{array}$$

Transformations of monad morphisms

$$(\Phi, \phi), (\Psi, \psi): \mathbb{T} \longrightarrow \mathbb{S}$$

A *Street 2-cell* $t: (\Phi, \phi) \rightrightarrows (\Psi, \psi)$ is

- a natural transformation $t: \Phi \rightrightarrows \Psi$
- satisfying

$$\begin{array}{ccc} \Phi T & \xrightarrow{tT} & \Psi T \\ \phi \downarrow & = & \downarrow \psi \\ S\Phi & \xrightarrow{St} & S\Psi \end{array}$$

Other transformations

A *Lack-Street 2-cell* $u: (\Phi, \phi) \longrightarrow (\Psi, \psi)$ is

- a natural transformation $u: \Phi \longrightarrow S\Psi$
- satisfying

$$\begin{array}{ccc} \Phi T & \xrightarrow{\phi} & S\Phi \\ \downarrow uT & & \downarrow Su \\ S\Psi T & \xrightarrow{S\psi} & SS\Psi \xrightarrow{\nu\Psi} S\Psi \\ & & \downarrow \nu\Psi \\ & & S\Psi \end{array}$$

Theorem

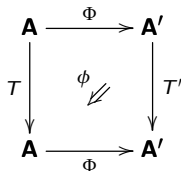
Let (Φ, ϕ) and (Ψ, ψ) be monad morphisms $\mathbb{T} \longrightarrow \mathbb{S}$ giving rise to double functors $\overline{\Phi}, \overline{\Psi}: \mathbb{Kl}(\mathbb{T}) \longrightarrow \mathbb{Kl}(\mathbb{S})$. Then

(1) horizontal transformations $\overline{\Phi} \longrightarrow \overline{\Psi}$ correspond to Street 2-cells $(\Phi, \phi) \longrightarrow (\Psi, \psi)$

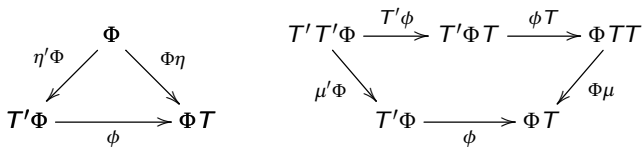
(2) vertical transformations $\overline{\Phi} \longrightarrow \bullet \longrightarrow \overline{\Psi}$ correspond to Lack-Street 2-cells $(\Phi, \phi) \longrightarrow \bullet \longrightarrow (\Psi, \psi)$

Lax morphisms of monads

- A *lax morphism of monads* (Φ, ϕ)

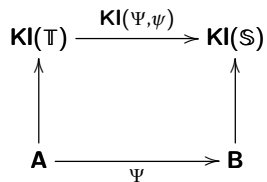
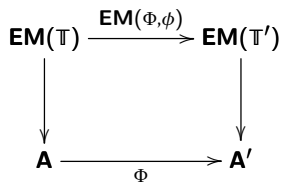


satisfying



Lax vs oplax

- (Ψ, ψ) was oplax



Lax and oplax together at last

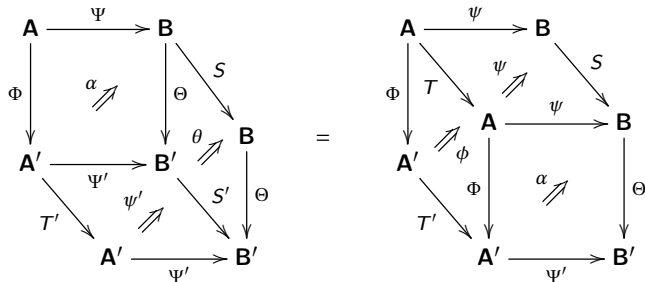
The double category \mathbf{Monad}

$$\begin{array}{ccc}
 \mathbb{T} & \xrightarrow{(\Phi, \phi)} & \mathbb{T}' \\
 (\Psi, \psi) \downarrow \bullet & \alpha & \downarrow \bullet (\Psi', \psi') \\
 \mathbb{S} & \xrightarrow{(\Theta, \theta)} & \mathbb{S}'
 \end{array}$$

$$\Psi' \Phi \xrightarrow{\alpha} \Theta \Psi$$

$$\begin{array}{ccccc}
 & \Psi' \Phi T & \xrightarrow{\alpha T} & \Theta \Psi T & \\
 \Psi' T' \Phi & \nearrow \Psi' \phi & & \searrow \Theta \psi & \\
 & S' \Psi' \Phi & \xrightarrow{S' \alpha} & S' \Theta \Psi & \\
 & \searrow \psi' \Phi & & \nearrow \theta \Psi &
 \end{array}$$

Fear of hexagons



Properties of Monad

Theorem

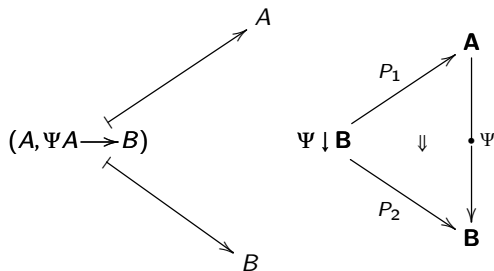
- (1) (Φ, ϕ) has a companion iff ϕ is iso*
- (2) (Φ, ϕ) has a conjoint iff Φ has a left adjoint*
- (3) $\mathbb{M}nd$ has tabulators and they are effective*

The tabulator

The tabulator of $(\Psi, \psi): (\mathbf{A}, T, \eta, \mu) \longrightarrow (\mathbf{B}, S, \kappa, \nu)$ is given by the comma category $\Psi \downarrow \mathbf{B}$ with monad

$$\Psi \downarrow \mathbf{B} \xrightarrow{T \downarrow S} \Psi \downarrow \mathbf{B}$$

$$(A, \Psi A \xrightarrow{b} B) \longmapsto (TA, \Psi TA \xrightarrow{\psi^A} S\Psi A \xrightarrow{Sb} SB)$$



Eilenberg-Moore for a change

A lax morphism $(\Phi, \phi): \mathbb{T} \rightrightarrows \mathbb{T}'$ gives an algebraic functor over Φ

$$(TA \xrightarrow{\alpha} A) \mapsto (T'\Phi A \xrightarrow{\phi A} \Phi TA \xrightarrow{\Phi a} \Phi A)$$

$$\begin{array}{ccc} \mathbf{EM}(\mathbb{T}) & \xrightarrow{\mathbf{EM}(\Phi, \phi)} & \mathbf{EM}(\mathbb{T}') \\ \downarrow & & \downarrow \\ \mathbf{A} & \xrightarrow{\Phi} & \mathbf{A}' \end{array}$$

But what about oplax morphisms $(\Psi, \psi): \mathbb{T} \multimap \mathbb{T}'$?

Profunctors make a cameo appearance

$$\mathbf{EM}(\Psi, \psi): \mathbf{EM}(\mathbb{T}) \longrightarrow \mathbf{EM}(\mathbb{S})$$

$$\mathbf{EM}(\Psi, \psi): \mathbf{EM}(\mathbb{T})^{op} \times \mathbf{EM}(\mathbb{S}) \longrightarrow \mathbf{Set}$$

An element of $\mathbf{EM}(\Psi, \psi)((A, a), (B, b))$ is $x: \Psi A \longrightarrow B$

$$\begin{array}{ccccc} \Psi TA & \xrightarrow{\psi A} & S\Psi A & \xrightarrow{Sx} & SB \\ \Psi a \downarrow & & & & \downarrow b \\ \Psi A & \xrightarrow{x} & & & B \end{array}$$

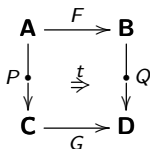
EM extends to cells in \mathbf{Monad}

$$\begin{array}{ccc}
 \mathbb{T} & \xrightarrow{(\Phi, \phi)} & \mathbb{T}' \\
 (\Psi, \psi) \downarrow & \alpha & \downarrow (\Psi', \psi') \\
 \mathbb{S} & \xrightarrow{(\Theta, \theta)} & \mathbb{S}'
 \end{array}$$

$$\begin{array}{ccc}
 \mathbf{EM}(\mathbb{T}) & \xrightarrow{\mathbf{EM}(\Phi, \phi)} & \mathbf{EM}(\mathbb{T}') \\
 (\Psi, \psi) \downarrow & \mathbf{EM}(\alpha) \Rightarrow & \downarrow (\Psi', \psi') \\
 \mathbf{EM}(\mathbb{S}) & \xrightarrow{\mathbf{EM}(\Theta, \theta)} & \mathbf{EM}(\mathbb{S}')
 \end{array}$$

$$\mathbf{EM}(\alpha): (\Psi A \xrightarrow{x} B) \mapsto (\Psi' \Phi A \xrightarrow{\alpha A} \Theta \Psi A \xrightarrow{\Theta x} \Theta B)$$

$$\mathbf{EM}: \mathbf{Mnd} \longrightarrow \mathbf{Cat}$$



- **A, B, C, D** categories
- F, G functors
- P, Q profunctors

$$P: \mathbf{A}^{op} \times \mathbf{C} \longrightarrow \mathbf{Set}, Q: \mathbf{B}^{op} \times \mathbf{D} \longrightarrow \mathbf{Set}$$

- t natural transformation

$$t: P(-, =) \longrightarrow Q(F-, G=)$$

- Composition of profunctors uses coends, and is not associative on the nose

Cat is a *weak double category*

EM: $\mathbf{Mnd} \longrightarrow \mathbf{Cat}$ is a *lax double functor*

Double categories

(Ehresmann) A *double category* is a category object in **Cat**

$$\mathbb{A}: \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{\bullet} \\ \xrightarrow{p_1} \end{array} \mathbf{A}_1 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{\text{id}} \\ \xleftarrow{d_1} \end{array} \mathbf{A}_0$$

$$\begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{p_2} & \mathbf{A}_1 \\ \downarrow p_1 & \boxed{\text{Pb}} & \downarrow d_0 \\ \mathbf{A}_1 & \xrightarrow{d_1} & \mathbf{A}_0 \end{array} \qquad \begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{\mathbf{A}_1 \times_{\mathbf{A}_0} \bullet} & \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \\ \downarrow \bullet \times_{\mathbf{A}_0} \mathbf{A}_1 & \boxed{\text{Assoc}} & \downarrow \bullet \\ \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{\bullet} & \mathbf{A}_\bullet \end{array}$$

Double functor

$$\begin{array}{ccccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \mathbf{A}_1 & \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{array} & \mathbf{A}_0 \\ \downarrow F_1 \times_{F_0} F_1 & & \downarrow F_1 & & \downarrow F_0 \\ \mathbf{B}_1 \times_{\mathbf{B}_0} \mathbf{B}_1 & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \mathbf{B}_1 & \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{array} & \mathbf{B}_0 \end{array}$$

Weak double categories

A *(weak) double category* is a weak category object in $\mathcal{C}at$

$$\mathbb{A}: \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{\bullet} \\ \xrightarrow{p_1} \end{array} \mathbf{A}_1 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{id} \\ \xleftarrow{d_1} \end{array} \mathbf{A}_0$$

$$\begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{p_2} & \mathbf{A}_1 \\ \downarrow p_1 & \boxed{\text{Pb}} & \downarrow d_0 \\ \mathbf{A}_1 & \xrightarrow{d_1} & \mathbf{A}_0 \end{array} \qquad \begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{\mathbf{A}_1 \times_{\mathbf{A}_0} \bullet} & \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \\ \downarrow \bullet \times_{\mathbf{A}_0} \mathbf{A}_1 & \alpha \swarrow \cong & \downarrow \bullet \\ \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{\bullet} & \mathbf{A} \end{array}$$

Double functor

$$\begin{array}{ccccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \mathbf{A}_1 & \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{array} & \mathbf{A}_0 \\ \downarrow F_1 \times_{F_0} F_1 & & \downarrow F_1 & & \downarrow F_0 \\ \mathbf{B}_1 \times_{\mathbf{B}_0} \mathbf{B}_1 & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \mathbf{B}_1 & \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{array} & \mathbf{B}_0 \end{array}$$

Lax double functors of weak double categories

A *(weak) double category* is a weak category object in \mathcal{Cat}

$$\mathbb{A}: \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{\bullet} \\ \xrightarrow{p_1} \end{array} \mathbf{A}_1 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{\text{id}} \\ \xleftarrow{d_1} \end{array} \mathbf{A}_0$$

$$\begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{p_2} & \mathbf{A}_1 \\ \downarrow p_1 & \boxed{\text{Pb}} & \downarrow d_0 \\ \mathbf{A}_1 & \xrightarrow{d_1} & \mathbf{A}_0 \end{array} \qquad \begin{array}{ccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{\mathbf{A}_1 \times_{\mathbf{A}_0} \bullet} & \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \\ \downarrow \bullet \times_{\mathbf{A}_0} \mathbf{A}_1 & \alpha \swarrow \cong & \downarrow \bullet \\ \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{\bullet} & \mathbf{A} \end{array}$$

Lax double functor

$$\begin{array}{ccccc} \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \xrightarrow{\bullet} & \mathbf{A}_1 & \xleftarrow{\text{id}} & \mathbf{A}_0 \\ \downarrow F_1 \times_{F_0} F_1 & \nearrow \phi & \downarrow F_1 & \nwarrow \phi_0 & \downarrow F_0 \\ \mathbf{B}_1 \times_{\mathbf{B}_0} \mathbf{B}_1 & \xrightarrow{\bullet} & \mathbf{B}_1 & \xleftarrow{\text{id}} & \mathbf{B}_0 \end{array}$$

And, this is where the story begins...

Thank you!