Operads as double functors

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Virtual Double Categories Workshop
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Motivations

Aim

A more natural approach to colored operads
(symmetric multicategories).

Achievement

The non-skeletal approach to operads
seems in fact more natural in many respects.

Technical tools

It this approach, double categories play a pivotal role.

Byproduct

This is how I learned to love double categories.

Motivations

The operad of sets

- Objects are sets.
- Arrows $f : X_1; \cdots ; X_n \to Y$
  are maps which take a list of elements
  $x_1; \cdots ; x_n$ (with $x_i \in X_i$)
  and give an element $y \in Y$.

Motivations

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  are maps which take a list of elements
  $x_1; \cdots ; x_n$ (with $x_i \in X_i$)
  and give an element $y \in Y$.

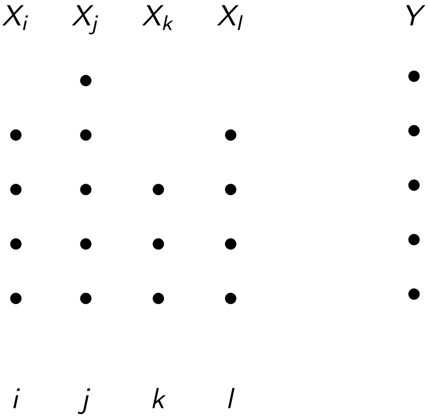
But order doesn't really matter...

Motivations

More naturally

- Objects are sets.
- Arrows $f : (X_i)_{i \in A} \to Y$
  are maps which take a family of elements
  $(x_i)_{i \in A}$ (with $x_i \in X_i$)
  and give an element $y \in Y$.

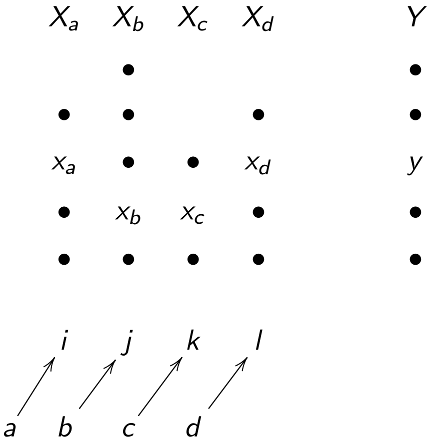
Motivations



Motivations

X_i	X_j	X_k	X_l	Y
	$\bullet$			$\bullet$
$\bullet$	$\bullet$		$\bullet$	$\bullet$
x_i	$\bullet$	$\bullet$	x_l	y
$\bullet$	x_j	x_k	$\bullet$	$\bullet$
$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
i	j	k	l	

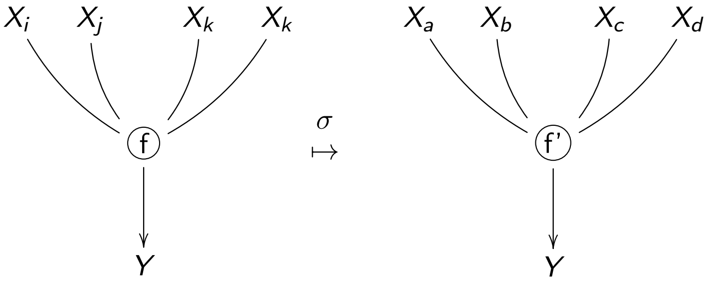
Motivations



Motivations

Reindexing of arrows

A bijection $\sigma : B \to A$ gives a reindexing,
taking any arrow whose domain is indexed by A
to an arrow whose domain is indexed by B.

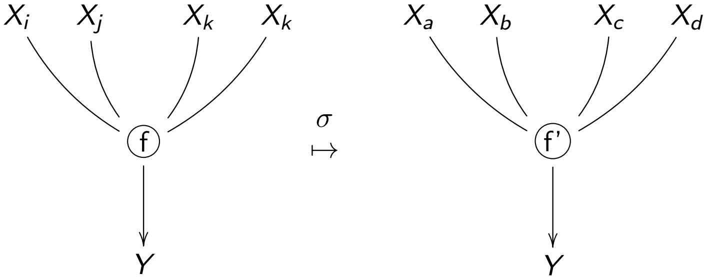


Motivations

Idea

The maps f and f' are the same, up to the indexing of domains.

But indexing is necessary in order to composing arrows.



Motivations

If, instead of single arrows, we consider families of arrows, we get a category with an underlying functor to Set_f.

$$\begin{tikzcd} (X_i)_{i\in A} \arrow[r, "(f_j)_{j\in B}"] \arrow[rr, bend right, "(h_k)_{k\in C}"'] & (Y_j)_{j\in B} \arrow[r, "(g_k)_{k\in C}"] & (Z_k)_{k\in C} \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\overline{f}"] \arrow[rr, bend right, "\overline{h}"'] & B \arrow[r, "\overline{g}"] & C \end{tikzcd}$$

Motivations

Reindexing of families of arrows
are the cells double category.

$$\begin{tikzcd}
(X'_j)_{j\in B} \arrow[d, "f'"'] & (X_i)_{i\in A} \arrow[l, "\sigma^*"'] \arrow[d, "f"] \\
(Y'_s)_{s\in D} & (Y_t)_{t\in C} \arrow[l, "\rho^*"']
\end{tikzcd}$$

Motivations

Main idea

To properly understand operads,
we need a framework allowing to express symmetry of arrows
and yet retaining the possibility of composing them.

Double categories provide this framework.

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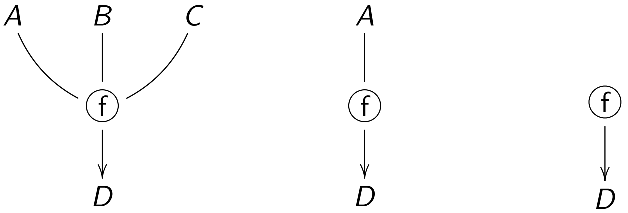
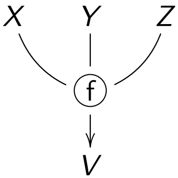
Operads (arity)

Arrows of any arity

A ternary arrow, a unary arrow and a nullary arrow.



Operads (composition)



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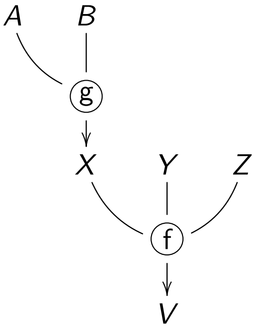

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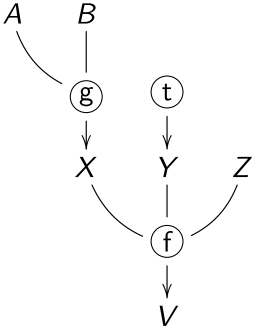

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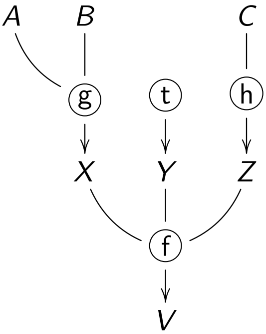

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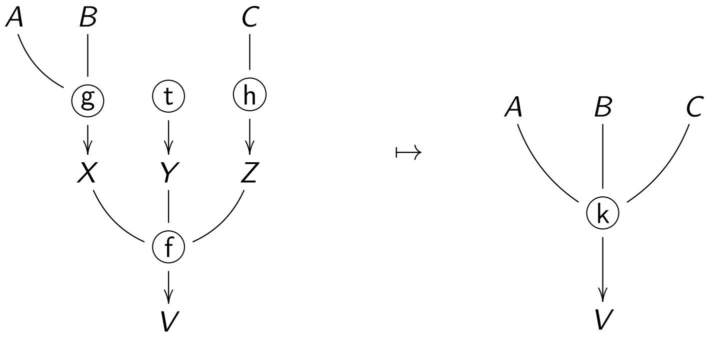
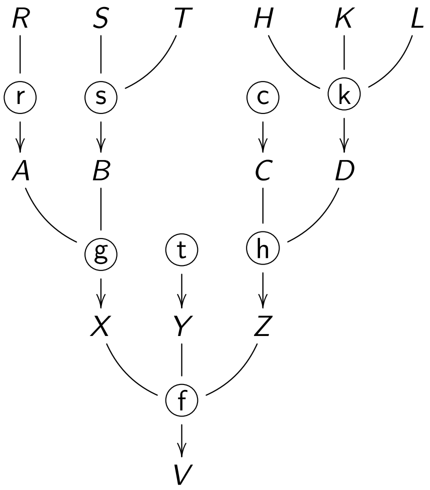
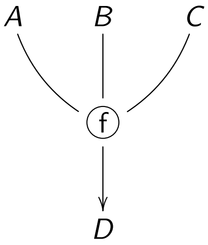


Operads (associativity)



Operads (symmetry)



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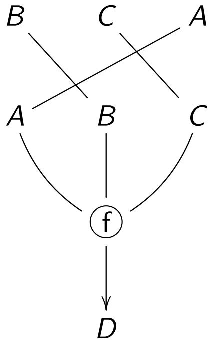

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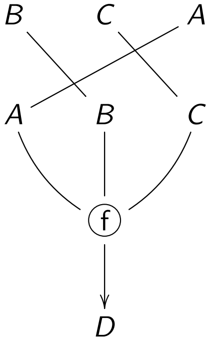
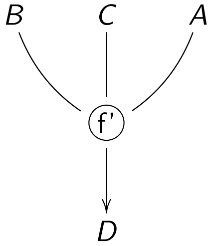
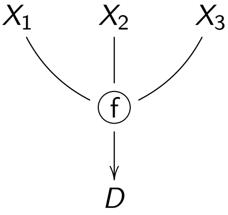


$\mapsto$



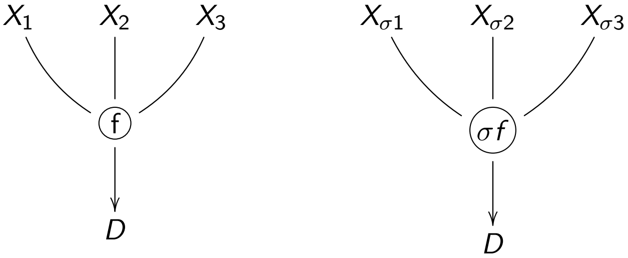
Operads: Classical approach

The domain of an arrow is a list $X : \mathrm{n} \to \mathcal{O}_0$
of objects in $\mathcal{O}$



Operads: Classical approach

Arrows can be transported along permutations σ
of the indexing set $\mathrm{n} = \{1, \cdots, \mathrm{n}\}$



Operads: Classical approach

Axioms

- Composition and associativity.
- Permutations act on arrows.
- The action is compatible with composition.

Operads: Classical approach

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- Composition and associativity.
- Permutations act on arrows.
- The action is compatible with composition.

When made explicit,
these conditions assume a rather unwieldy form
(involving for instance block permutations)
showing drawbacks of the skeletal choice for indexing.

Operads: Examples

Monoidal operads

Any symmetric monoidal category gives an operad $\mathcal{O}$,
whose arrows $f : X_1; \cdots ; X_n \to Y$
are arrows $f : X_1 \otimes \cdots \otimes X_n \to Y$.

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In particular, one can consider a cartesian monoidal category.

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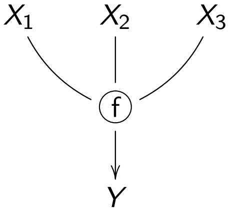
In particular, one can consider a cartesian monoidal category.

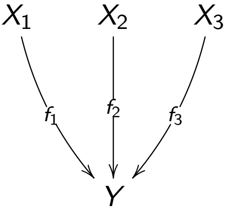
Starting with $(\mathrm{Set}, \times, 1)$ we get the operad of sets.

Operads: Examples

Starting with a cocartesian monoidal category $(\mathcal{C}, +, 0)$
we get the sequential operad $\mathcal{C}_{\blacktriangleright}$ whose maps
are sequences of concurrent arrows in $\mathcal{C}$ (discrete cocones).

One can consider $\mathcal{C}_{\blacktriangleright}$ for any category $\mathcal{C}$.





Operads: Aim

The examples also suggest a more natural notion of operad:

Non-skeletal operads

The domain of an arrow is a family of objects
indexed by an arbitrary finite set
(rather than by a set in a skeleton $\mathbb{N}$ of Set_f)
and reindexing of objects can be extended to arrows.

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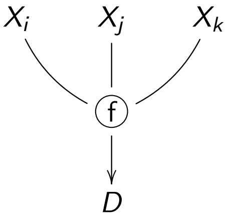
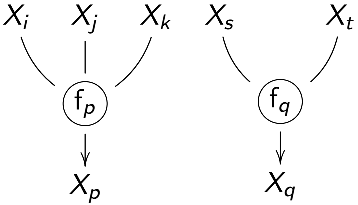
Operads: non-skeletal approach

The domain of an arrow is an arbitrary family
$X : A \to \mathcal{O}_0$ of objects



Operads: non-skeletal approach

Taking in account composition,
we need to consider families of arrows.



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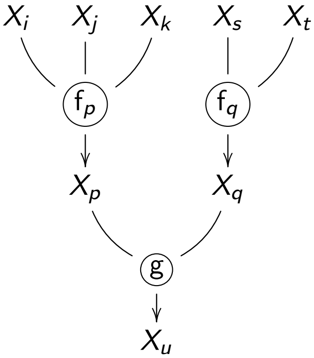
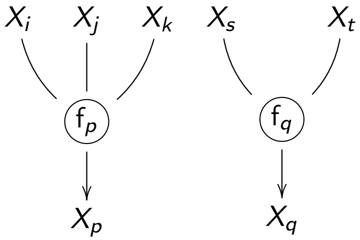


Operads: non-skeletal approach

Any family of arrows has an underlying mapping



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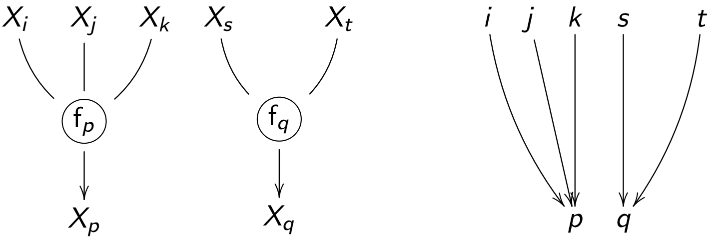

Operads: non-skeletal approach

Question

So, what do we get by considering families of arrows in a non-skeletal operad $\mathcal{O}$?

Answer

They form a category $\mathcal{D}_{\mathcal{O}}$ over finite sets: the functor $d : \mathcal{D}_{\mathcal{O}} \to \mathsf{Set}_f$ keeps track of the indexing of objects and maps.

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The category $\mathcal{D}_{\mathcal{O}}$, in its skeletal form, appears in the literature under several names, such as "operator" or "envelope" category of $\mathcal{O}$, or the free PROP generated by $\mathcal{O}$.

Operads: non-skeletal approach

Question

What further structure is inherited by $\mathcal{D}_{\mathcal{O}}$
from the operad structure $\mathcal{O}$?

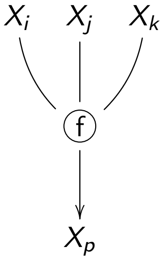
Answer

Its maps (families of arrows in $\mathcal{O}$)
can be reindexed along pullbacks in Set_f.

Reindexing along pullbacks

For instance, we can reindex a single arrow or a family of arrows along pullbacks whose horizontal sides are isomorphisms.

We so obtain the same arrow (up to indexing).



$$\begin{tikzcd}
i \arrow[dddr, bend right] & & \\
& j \arrow[dd] & \\
& & k \arrow[dl, bend left] \\
& p &
\end{tikzcd}$$

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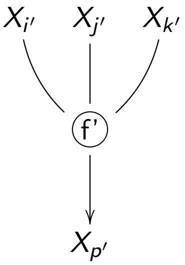
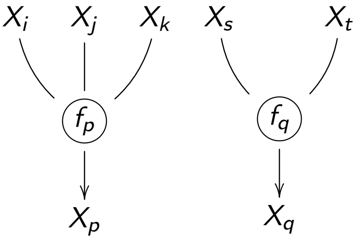


$$\begin{tikzcd}
i' \arrow[rr] \arrow[dddr, bend right] & & i \arrow[dddr, bend right] & & \\
& j' \arrow[rr] \arrow[dd] & & j \arrow[dd] & \\
& & k' \arrow[rr] \arrow[dl, bend left] & & k \arrow[dl, bend left] \\
& p' \arrow[rr] & & p &
\end{tikzcd}$$

Reindexing along pullbacks

Or we can reindex along injective mappings
to pick up just some arrows of the family.



$$
\begin{tikzcd}
i \arrow[dddr, bend right] & & & s \arrow[ddd] & \\
& j \arrow[dd] & & & t \arrow[ddl, bend left] \\
& & k \arrow[dl, bend left] & & \\
& p & & q &
\end{tikzcd}
$$

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$$
\begin{tikzcd}
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 & & j \arrow[dd] & & & t \arrow[dddl, bend left] \\
 & & & k \arrow[dl, bend left] & & \\
p' \arrow[rr] & & p & & q &
\end{tikzcd}
$$

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$$
\begin{tikzcd}
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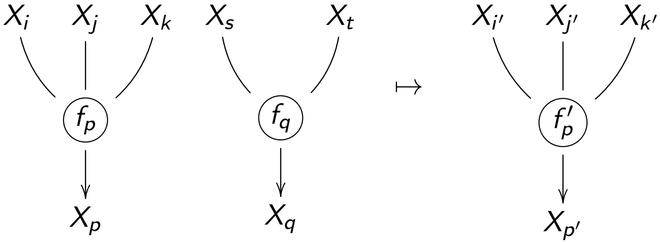
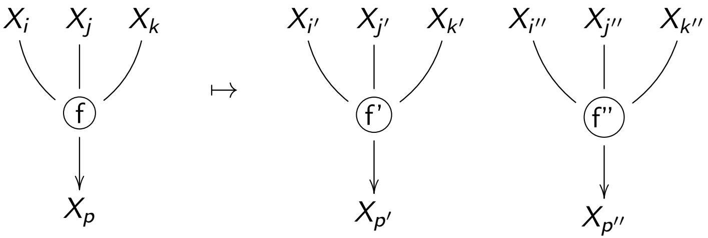
& j' \arrow[rr] \arrow[dd] & & j \arrow[dd] & & & t \arrow[ddl, bend left] \\
& & k' \arrow[rr] \arrow[dl, bend left] & & k \arrow[dl, bend left] & & \\
& p' \arrow[rr] & & p & & q &
\end{tikzcd}
$$

Reindexing along pullbacks



Reindexing along pullbacks

Or we can reindex along more general mappings
to obtain copies of some of the arrows in a family.



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So, what do we get by considering families of arrows in a non-skeletal operad $\mathcal{O}$?

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- They form a category over finite sets $d : \mathcal{D}_{\mathcal{O}} \to \mathsf{Set}_f$.
- For any pullback in Set_f there is a reindexing of objects and of maps over it.

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So, what do we get by considering families of arrows in a non-skeletal operad $\mathcal{O}$?

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- They form a category over finite sets $d : \mathcal{D}_{\mathcal{O}} \to \mathsf{Set}_f$.
- For any pullback in Set_f there is a reindexing of objects and of maps over it.
- The reindexing is compatible with composition.

Here they come double categories

For any pullback in Set_f there is a reindexing over it.

$$
\begin{tikzcd}
X \arrow[d, "f"] & & A \arrow[d, "df"] \\
Y & & B
\end{tikzcd}
$$

Here they come double categories

For any pullback in Set_f there is a reindexing over it.

$$
\begin{tikzcd}
X \arrow[d, "f"] \\
Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A' \arrow[r, "s"] \arrow[d, "k"'] \arrow[dr, phantom, "\mathrm{pb}"] & A \arrow[d, "df"] \\
B' \arrow[r, "t"] & B
\end{tikzcd}
$$

Here they come double categories

For any pullback in Set_f there is a reindexing over it.

$$\begin{tikzcd}
s^*X \arrow[r, dashed] & X \arrow[d, "f"] & & A' \arrow[r, "s"] \arrow[d] & A \arrow[d, "df"] \\
t^*Y \arrow[r, dashed] & Y & & B' \arrow[r, "t"] & B
\end{tikzcd}$$

the dashed arrows indicate that s^*X and t^*Y are the reindexing of the families X and Y along s and t.

Here they come double categories

For any pullback in Set_f there is a reindexing over it.

$$\begin{tikzcd}
s^*X \arrow[r, dashed] \arrow[d, dotted, "f'"'] & X \arrow[d, "f"] \\
t^*Y \arrow[r, dashed] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A' \arrow[r, "s"] \arrow[d, "df'=k"'] & A \arrow[d, "df"] \\
B' \arrow[r, "t"] & B
\end{tikzcd}$$

the vertical dotted arrow is uniquely determined.

The double category of an operad

The reindexing is compatible with composition

Reindexing squares can be composed vertically
(as well as horizontally).

The double category of an operad

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The double category $\mathbb{D}_{\mathcal{O}}$

- Objects are finite families $A \to \mathcal{O}_0$ of objects in $\mathcal{O}$.

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- Objects are finite families $A \to \mathcal{O}_0$ of objects in $\mathcal{O}$.
- Vertical arrows (proarrows) are the maps of $\mathcal{D}$, that is families of arrows in $\mathcal{O}$.
- Horizontal arrows are the arrows of the discrete fibration $\mathcal{O}_0^A$, $A \in \mathrm{Set}_f$, the family fibration on the set $\mathcal{O}_0$.

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The double category $\mathbb{D}_{\mathcal{O}}$

- Objects are finite families $A \to \mathcal{O}_0$ of objects in $\mathcal{O}$.
- Vertical arrows (proarrows) are the maps of $\mathcal{D}$, that is families of arrows in $\mathcal{O}$.
- Horizontal arrows are the arrows of the discrete fibration $\mathcal{O}_0^A$, $A \in \mathrm{Set}_f$, the family fibration on the set $\mathcal{O}_0$.
- Cells are the reindexing of families of arrows.

Operads as discrete fibrations

Furthermore, there is a discrete double fibration

$$d : \mathbb{D}_{\mathcal{O}} \to \mathbb{P}\mathrm{b}\,(\mathrm{Set}_{\mathrm{f}})$$

giving the reindexing of objects and of maps

discrete double fibration (Lambert, 2021)

That is, both the components
$d_0 : \mathbb{D}_0 \to \mathrm{Set}_f$ and $d_1 : \mathbb{D}_1 \to \mathrm{PbSet}_f$
are discrete fibrations.

Operads as discrete fibrations

Lastly, $d : \mathbb{D} \to \mathbb{P}\mathrm{b}\,(\mathrm{Set}_{\mathrm{f}})$
should satisfy the glueing conditions:

Operads as discrete fibrations

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should satisfy the glueing conditions:

Glueing condition for objects

If X and Y are objects in $\mathbb{D}$ over A and B respectively, there is a unique object Z over a sum $C = A + B$ in Set_f which restricts to X and Y along injections.

Operads as discrete fibrations

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Glueing condition for objects

If X and Y are objects in $\mathbb{D}$ over A and B respectively,
there is a unique object Z over a sum $C = A + B$ in Set_f
which restricts to X and Y along injections.

Glueing condition for maps

If f and g are maps over s and t respectively,
there is a unique map h over a sum $r = s + t$ in Set_f^2
which restricts to f and g along injections
(which are pullbacks in Set_f).

Operads as discrete fibrations

Objects are families of objects...

The glueing condition for objects
assures that the horizontal part d^h of $d : \mathbb{D} \to \mathbb{P}\mathrm{b}\,(\mathrm{Set}_{\mathrm{f}})$
is indeed the family fibration on $\mathcal{O}_0$
(where $\mathcal{O}_0$ is the fiber over a terminal set).

...and maps are families of arrows

The glueing condition for maps
assures that a proarrow in $\mathbb{D}$ (that is, an object in $\mathbb{D}_1$)
is indeed a family of "single arrows", that is of proarrows
with the codomain indexed by a terminal set.

Operads as discrete fibrations

We so arrive to our definition of operad:

Non-skeletal notion of operad

An operad is a double discrete fibration $d : \mathbb{D} \to \mathbb{P}\mathrm{b}\,(\mathrm{Set}_{\mathrm{f}})$ satisfying the glueing condition.

Operads as discrete fibrations

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Note that $\mathbb{D}$ is a strict double category,
and that $d : \mathbb{D} \to \mathbb{P}\mathrm{b}\,(\mathrm{Set_f})$ is a strict double functor.

The category of operads

This notion of non-skeletal operad
is essentially equivalent to the classical one.

Morphisms $\mathcal{O} \to \mathcal{O}'$ of non-skeletal operads
are double functors $\mathbb{D}_{\mathcal{O}} \to \mathbb{D}_{\mathcal{O}'}$ over Set_f.
The category of non-skeletal operads is equivalent
to the category of classical operads.

Operads as discrete fibrations (advantages)

Compatibility of permutation actions with composition

Figure from Leinster's book.

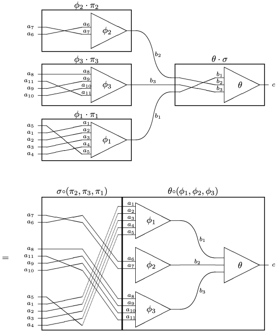


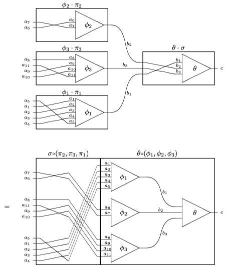
Figure 2-H: Symmetric multicategory axiom

$$
\begin{aligned}
& (\theta \cdot \sigma)\circ(\phi_{\sigma(1)} \cdot \pi_{\sigma(1)}, \ldots, \phi_{\sigma(n)} \cdot \pi_{\sigma(n)}) \\
= \quad & (\theta\circ(\phi_1, \ldots, \phi_n)) \cdot (\sigma\circ(\pi_{\sigma(1)}, \ldots, \pi_{\sigma(n)}))
\end{aligned}
$$

Operads as discrete fibrations (advantages)

Confronting two ways of expressing compatibility

In our context, compatibility is given by
vertical composition of cells.



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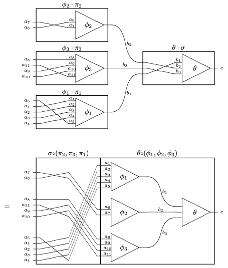

Figure 2-H: Symmetric multicategory axiom

$$\begin{tikzcd}
s^*X \arrow[r, dashed] \arrow[d, "f'"'] \arrow[dd, bend right=60, "(gf)'"'] & X \arrow[d, "f"] \\
t^*Y \arrow[r, dashed] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
Z \arrow[r, equal] & Z
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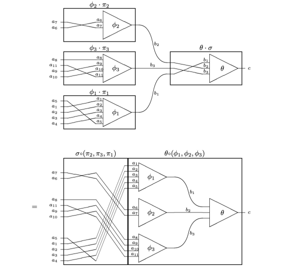

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\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "s"] \arrow[d] & A \arrow[d] \\
B' \arrow[r, "t"] \arrow[d] & B \arrow[d] \\
1 \arrow[r, equal] & 1
\end{tikzcd}$$

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Operads as double functors

Double Grothendieck correspondence (Lambert 2021, Paré 2011)

Double discrete fibrations $d : \mathbb{D} \to \mathbb{A}$ correspond to lax functors $F : \mathbb{A}^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$ to the (non-strict) double category of mappings and spans.

Operads as double functors

Double Grothendieck correspondence (Lambert 2021, Paré 2011)

Double discrete fibrations $d : \mathbb{D} \to \mathbb{A}$ correspond to
lax functors $F : \mathbb{A}^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$
to the (non-strict) double category of mappings and spans.

Universal property of the monoid construction (Cruttwell & Shulman 2010)

Since the monoid construction on $\mathbb{S}\mathrm{pan}$ gives $\mathbb{C}\mathrm{at}$,
the double category of functors and profunctors,
lax functors $F : \mathbb{A}^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$ correspond to
normal lax functors $F' : \mathbb{A}^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$.

Operads as double functors

Thus, given an non-skeletal operad

$$d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}}$$

there are corresponding lax functors

$$F_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$$

$$F'_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$$

Operads as double functors

Furthermore it is easy to see that the glueing condition on $d_{\mathcal{O}}$ corresponds to the fact that $F_{\mathcal{O}}$ and $F'_{\mathcal{O}}$ preserve products.

Products in $(\mathbb{P}\mathrm{b}\,\mathrm{Set_f})^{\mathrm{op}}$ are sums in $\mathbb{P}\mathrm{b}\,\mathrm{Set_f}$,
that is pair of commuting squares whose horizontal sides are sums in Set_f (since Set_f is extensive).

$$\begin{tikzcd}
A_1 \arrow[r, "i"] \arrow[d, "s"'] & A_1 + A_2 \arrow[d, "s+t"] & A_2 \arrow[l, "j"'] \arrow[d, "t"] \\
B_1 \arrow[r, "i"'] & B_1 + B_2 & B_2 \arrow[l, "j"]
\end{tikzcd}$$

What is an operad?

Summarizing

A (non-skeletal) operad $\mathcal{O}$
can be defined in three equivalent ways:

1. A double discrete fibration with glueing
   $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}}$.
2. A product-preserving lax functor
   $F_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$.
3. A product-preserving normal lax functor
   $F'_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$.

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3. A product-preserving normal lax functor
   $F'_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$.

Each definition gives a different point of view
best suited to treat some aspects of operads.

Operads as double functors (explicitly)

The functor $F_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$
takes a set $A \in \mathsf{Set}_f$ to the set $\mathcal{O}_0^A$,
and a mapping $t : A \to B$ to the span
whose vertex is formed by all families of arrows over t
and whose legs are given by domain and codomain.

Operads as double functors (explicitly)

The functor $F_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$
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and whose legs are given by domain and codomain.

The functor $F'_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$
takes a set $A \in \mathsf{Set}_f$ to the category $\mathcal{O}_1^A$,
(where $\mathcal{O}_1$ is the category of unary arrows in $\mathcal{O}$)
and a mapping $t : A \to B$ to the profunctor $\overline{t}$ such that
$\overline{t}(X, Y)$ is formed by all families of arrows $f : X \to Y$ over t.

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Special operads

Given an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}}$,
the horizontal part $d^h_{\mathcal{O}} : \mathbb{D}_0 \to \mathsf{Set}_f$ is forced to be
the discrete family fibration on the set $\mathcal{O}_0$
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(by the glueing or product preserving condition).

Thus, the character of $\mathcal{O}$ is in a sense
determined by the vertical part $d^v_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}_f$.

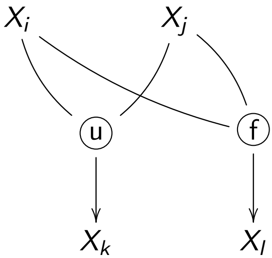
Symmetric monoidal categories

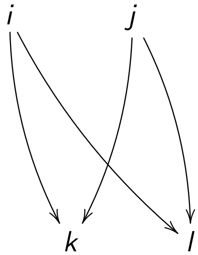
The vertical part $d^v_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}_f$ is an opfibration
if and only if $\mathcal{O}$ has tensor products.
That is, it is a symmetric monoidal category
in its universal form
(the representable multicategories of Hermida and Leinster).

Symmetric monoidal categories

Universal arrows

The opcartesian arrows for $d^{v}_{\mathcal{O}}$ are
the universal arrows defining tensor products.





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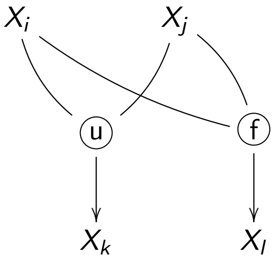

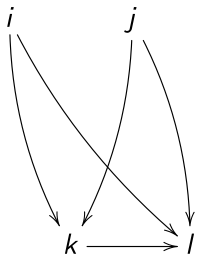

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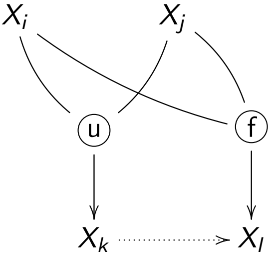

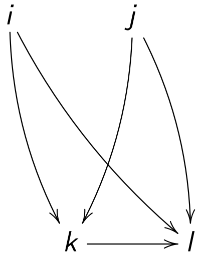

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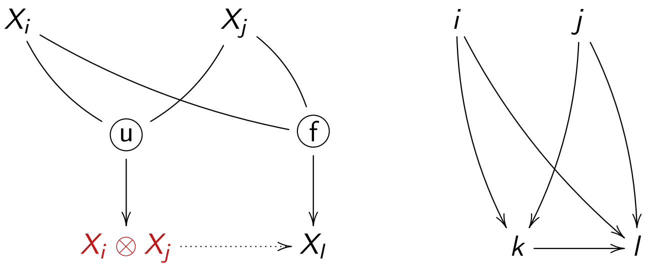


Symmetric monoidal categories as lax double functors

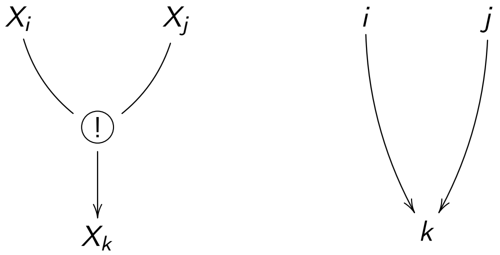
An operad $F_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$,
is a symmetric monoidal category
if and only if its vertical part $F_{\mathcal{O}}^{v} : \mathsf{Set}_{f} \to \mathsf{Prof}$,
(in general, a lax functor of bicategories)
lands in representable profunctors.

Commutative monoids

The vertical part $d^v_{\mathcal{O}} : \mathcal{D} \to \mathrm{Set}_f$ is a discrete opfibration if and only if $\mathcal{O}$ is a commutative monoid.
That is, it is a discrete symmetric monoidal category.

Commutative monoids

There is exactly one arrow out of any family of objects
(over a given mapping in Set_f)
whose codomain is the product of the family.



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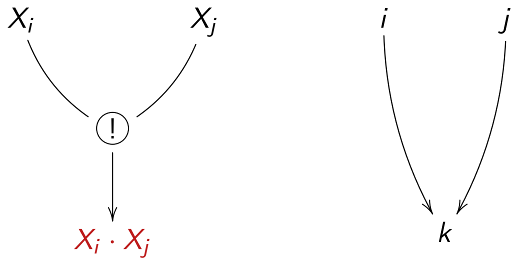

Commutative monoids

In elementary terms, a commutative monoid consists of
a discrete family fibration and a discrete opfibration
over finite sets, with the same objects which are compatible:

$$
\begin{tikzcd}
 & X & \\
 & & \\
C \arrow[r] \arrow[dr] & A \arrow[dr] & \\
 & D \arrow[r] & B
\end{tikzcd}
$$

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$$\begin{tikzcd}
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W \arrow[r] & Y
\end{tikzcd}$$

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& W'
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$$
\begin{tikzcd}
Z \arrow[r] \arrow[dr] & X \arrow[dr] & \\
& W' = W \arrow[r] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[r] \arrow[dr] & A \arrow[dr] & \\
& D \arrow[r] & B
\end{tikzcd}
$$

Commutative monoids as double functors

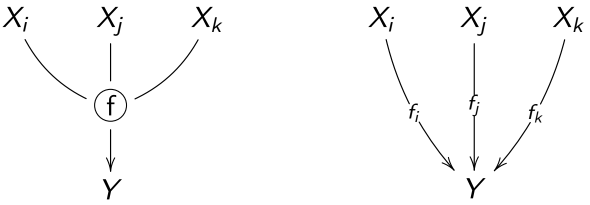
Corollary

A commutative monoid consists of a product-preserving strict double functor

$$(\mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}})^{\mathrm{op}} \to \mathbb{S}\mathrm{q}\,\mathsf{Set}$$

Sequential operads

The vertical part $d^v_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}_f$ is a fibration
if and only if $\mathcal{O}$ is a sequential operad.



Sequential operads

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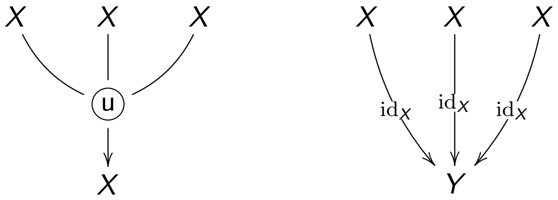

The cartesian arrows in $\mathcal{D}$ are those made up
of identities (or isomorphims) in $\mathcal{C}$.

They form a "central monoid" in the operad,
which is in fact a way to characterize sequential operads (P. 2014).

Cocartesian monoidal categories

Corollary

The vertical part $d^v_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}_f$ is a bifibration
if and only if $\mathcal{O}$ is both monoidal and sequential.
That is, $\mathcal{O}$ is a cocartesian monoidal category
(since universal arrows are colimiting cones).

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Copying and deleting

The well-known characterization of cartesian monoidal categories is a manifestation of (the dual of) the above fact:
the "copying-deleting" arrows are the cartesian maps of $d^v_{\mathcal{O}}$.

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Caution

The term "cartesian" is overworked:
cartesian arrow (of a fibration), cartesian monoidal category,
cartesian operad (to be considered later on)...

Exponentiable operads

An operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}_{\mathrm{f}}$, is exponentiable
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These include fibrations and opfibration, so that
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Exponentiable operads coincide with promonoidal symmetric multicategories.

Exponentiable operads as double functors

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(in general, a lax functor of bicategories)
is a pseudofunctor.

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Changing the base

Till now, we have presented a non-skeletal approach to operads.
The main advantages are:

- It avoids the introduction of spurious orders, rendering neater the notion.
- We can exploit the language of double categories, to capture in a smooth way various classes of operads and to highlight their connections.

Changing the base

Now, we briefly review a possible generalization, obtained by replacing the base category Set_f with another category $\mathcal{S}$.

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Infintary operads

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2. A product-preserving lax functor $F_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set})^{\mathrm{op}} \to \mathbb{S}\mathrm{et}$.
3. A product-preserving normal lax functor $F'_{\mathcal{O}} : (\mathbb{P}\mathrm{b}\,\mathsf{Set})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$.

Changing the base: example

Consider a category $\mathcal{C}$ and the family fibration $d : \mathrm{Fam}\,\mathcal{C} \to \mathsf{Set}$ given by $\mathcal{C}^A$; $A \in \mathsf{Set}$ is the vertical part of an operad
$d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}$
(the infinitary sequential operad on $\mathcal{C}$).

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(the infinitary sequential operad on $\mathcal{C}$).

If $\mathcal{C}$ has small sums the family fibration
$d : \mathrm{Fam}\,\mathcal{C} \to \mathsf{Set}$ is a bifibration.

We thus have a notion of infinitary monoidal category, namely, an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathsf{Set}$ such that the vertical part $d^v_{\mathcal{O}} : \mathcal{D} \to \mathsf{Set}$ is an opfibration.

Changing the base: example

Of course, we also have a notion of infinitary commutative monoid, namely, an operad $d_{\mathcal{O}}$ on Set such that the vertical part d^{v} is a discrete opfibration.

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And taking isomorphism classes of an infinitary monoidal category one gets an infinitary commutative monoid.

This is a way to make it precise the idea that universal sums or products can be "decategorified" to give algebraic structures, not only in the finite case.

Changing the base: example

More generally, we have a notion of monoidal category on $\mathcal{S}$, namely, an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathcal{S}$ such that the vertical part is an opfibration, and such that opcartesian arrows are stable with respect to reindexing.

This sort of Beck condition is necessary to assure that, also in this general case, by taking isomorphism classes one gets a commutative monoid on $\mathcal{S}$.

We now show how also the notion of cartesian operad
can be developed relatively to any category $\mathcal{S}$ is with pullbacks.

Cartesian operads

Idea 1

The notion of cartesian operad (or cartesian multicategory) is aimed to fill the missing term in the equality
operads :: symmetric monoidal = ?? :: cartesian monoidal

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The notion of cartesian operad (or cartesian multicategory)
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operads :: symmetric monoidal = ?? :: cartesian monoidal

Thus, one minimum requirement is:
representable cartesian operads = cartesian monoidal categories.
That is, if a cartesian operad $\mathcal{O}$ has tensor products,
these are cartesian (that is, universal) products.

Cartesian operads

Idea 2

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with an adjunctive structure which makes it possible
weakening and contraction of variables.

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with an adjunctive structure which makes it possible
weakening and contraction of variables.

Cartesian operads are a notion of algebraic theory
alternative to (and more flexible than) Lawvere theories.

Cartesian operads

weakening and contraction

For instance, in the operad of sets,
a map $f : X, Y, X \to T$ gives another map $f' : Y, Z, X \to T$
by the rule $f'(y, z, x) = f(x, y, x)$
which introduces the extra variable z (weakening)
and identifies the repeated variables x (contraction).

Cartesian operads

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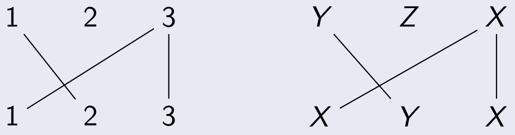
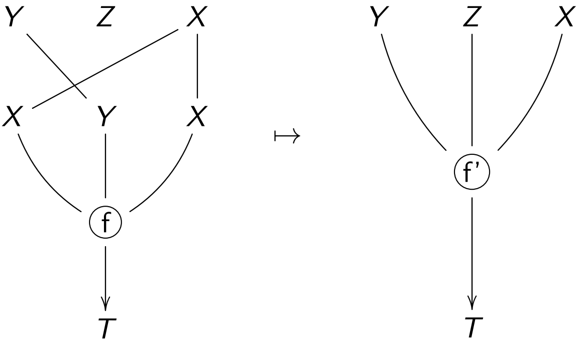
weakening and contraction

The map f' is then obtained by f
covariantly along the reindexing of the domain



Cartesian operads : “contraction” and “weakening”

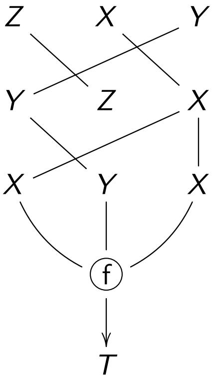
Reindexing arrows act covariantly on maps.



Cartesian operads

Reindexing arrows act on maps.

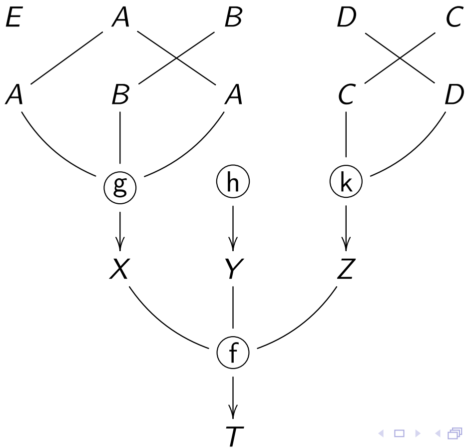
This is unambiguous:



Cartesian operads

The action is compatible with composition from below.

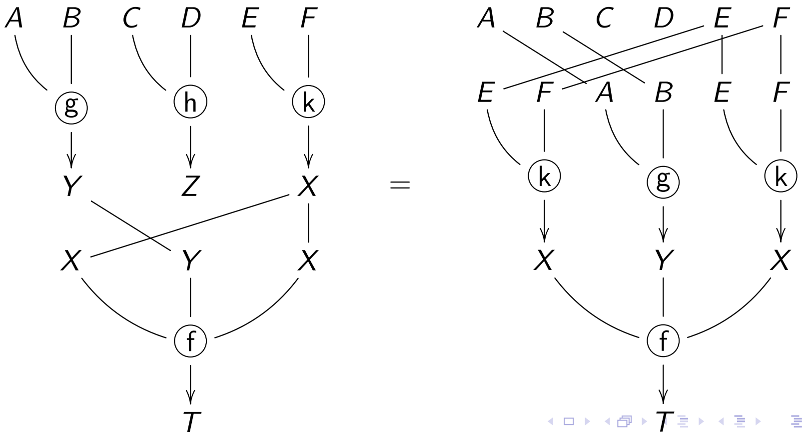
This is unambiguous:



Cartesian operads

Combing

The action is compatible with composition from above.



Cartesian operads

Cartesian operads

Let $\mathcal{S}$ be a category with pullbacks. A cartesian operad on $\mathcal{S}$ is an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathcal{S}$,
such that $\mathbb{D}$ has, in addition to ordinary cells, also
triangular cells, formed by two proarrows and an arrow.

Triangular cells (giving covariant reindexing)

$$\begin{tikzcd}
X \arrow[rr, dashed, "t"] \arrow[dr, "f"'] & \arrow[d, phantom, "\tau"] & Y \arrow[dl, "f'"] \\
& Z &
\end{tikzcd}$$

Cartesian operads

Cartesian operads

A cartesian operad on $\mathcal{C}$ is an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathcal{C}$,
such that $\mathbb{D}$ has, in addition to ordinary cells,
also triangular cells satisfying the conditions

- Maps in $\mathcal{D}$ (proarrows) can be covariantly reindexed along commutative triangles in $\mathcal{C}$.
- Triangular cells compose horizontally and with proarrows out of them.
- Triangular cells can be pasted with square cells.
- Triangular cells are stable with respect to reindexing.

Cartesian operads

Covariant reindexing of maps

Given a proarrow $f : t^*Y \to Z$ in $\mathcal{D}$,
and a commutative triangle in $\mathcal{S}$ completing df and t,
there is a unique extension to a triangular cell over it:

$$\begin{tikzcd} t^*Y \arrow[r, dashed] \arrow[dr, "f"'] & Y \\ & Z \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "t"] \arrow[dr, "df"'] & B \\ & C \end{tikzcd}$$

Cartesian operads

Covariant reindexing of maps

Given a proarrow $f : t^*Y \to Z$ in $\mathcal{D}$,
and a commutative triangle in $\mathcal{S}$ completing df and t,
there is a unique extension to a triangular cell over it:

$$
\begin{tikzcd}
t^*Y \arrow[rr, dashed] \arrow[dr, "f"'] & & Y \\
& Z &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[rr, "t"] \arrow[dr, "df"'] & & B \arrow[dl, "s"] \\
& C &
\end{tikzcd}
$$

Cartesian operads

Covariant reindexing of maps

Given a proarrow $f : t^*Y \to Z$ in $\mathcal{D}$,
and a commutative triangle in $\mathcal{S}$ completing df and t,
there is a unique extension to a triangular cell over it:

$$\begin{tikzcd}
t^*Y \arrow[rr, dashed] \arrow[dr, "f"'] & \arrow[d, phantom, "\mathcal{T}"] & Y \arrow[dl, dotted, "f'"] \\
& Z &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[rr, "t"] \arrow[dr, "df"'] & & B \arrow[dl, "s=df'"] \\
& C &
\end{tikzcd}$$

Cartesian operads

A triangular cell can be pasted with a square cell,
giving a triangular cell.

Pasting

$$
\begin{tikzcd}
t^*X \arrow[rr, dashed] \arrow[d, "h"'] \arrow[drr, phantom, "\alpha"] & & X \arrow[d, "f"] \\
s^*Y \arrow[rr, dashed] \arrow[dr, bend right, "k"'] & \arrow[d, phantom, "\tau"] & Y \arrow[dl, bend left, "g"] \\
& Z &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
t^*X \arrow[rr, dashed] \arrow[dr, bend right, "kh"'] & \arrow[d, phantom, "\tau'"] & Y' \arrow[dl, bend left, "gf"] \\
& Z &
\end{tikzcd}
$$

Cartesian operads

Triangular cells are stable with respect to reindexing.

Stability

$$
\begin{tikzcd}
 & & Y' \arrow[rrr, dashed] \arrow[dl] & & & Y \arrow[dl] \\
X' \arrow[urr, dashed] \arrow[rrr, dashed] \arrow[dr] & & & X \arrow[urr, dashed] \arrow[dr] & & \\
 & Z' \arrow[rrr, dashed] & & & Z &
\end{tikzcd}
$$

Algebraic products

Given a cartesian operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathcal{S}$,
an object $X \in \mathbb{D}$ and a map $t : dX \to J$ in $\mathcal{S}$,
an algebraic product for X along t
is an object $P \in \mathbb{D}$ over J along with a vertical map
$\pi : t^*P \to X$ and a map $u : X \to P$ over t...

$$\begin{tikzcd} t^*P \arrow[d, "\pi"'] \arrow[dr, dashed] & \\ X \arrow[r, "u"'] & P \end{tikzcd} \qquad \begin{tikzcd} dX \arrow[d, "\mathrm{id}"'] \arrow[dr, "t"] & \\ dX \arrow[r, "t"'] & J \end{tikzcd}$$

Algebraic products

Given a cartesian operad $d_{\mathcal{O}} : \mathbb{D} \to \mathbb{P}\mathrm{b}\,\mathcal{S}$,
an object $X \in \mathbb{D}$ and a map $t : dX \to J$ in $\mathcal{S}$,
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is an object $P \in \mathbb{D}$ over J along with a vertical map
$\pi : t^* P \to X$ and a map $u : X \to P$ over t...

...such that the following are both triangular cells:

$$\begin{tikzcd} t^* P \arrow[r, dashed] \arrow[d, "\pi"'] & P \arrow[ddl, bend left, "\mathrm{id}"] \\ X \arrow[dr, bend right, "u"'] & \\ & P \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dashed, "\Delta"] \arrow[ddr, bend right, "\mathrm{id}"'] & h^* X \arrow[d, "t^* u"] \\ & t^* P \arrow[dl, bend left, "\pi"] \\ X & \end{tikzcd}$$

Main result

Main result for cartesian operads

For a cartesian operad $\mathcal{O}$ on $\mathcal{S}$, the following are equivalent:

1. $\mathcal{O}$ has algebraic products.
2. $\mathcal{O}$ has universal products.
3. $\mathcal{O}$ is monoidal (representable).

This result indicates that we have indeed captured a proper notion of cartesian operad.

Cartesian + Sequential = Semiadditive

Further evidence

One can also generalize results such as the following:

Cartesian + Sequential = Semiadditive (Pisani 2014)

Cartesian structures on sequential operads
correspond to enrichments of the underlying category
in the category of commutative monoids.

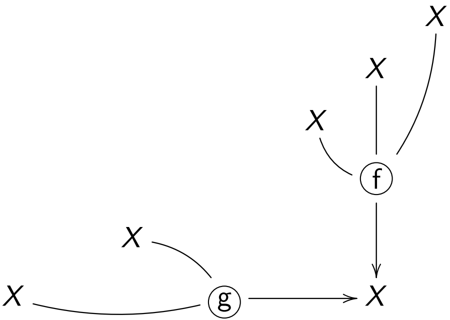
In the present context, objects are to be intended as sections $x : \mathcal{C} \to \mathbb{D}^h$ of $d^h_{\mathcal{O}}$, and the commutative monoid $\mathcal{O}(x; y)$ is a commutative monoid on $\mathcal{S}$ in the generalized sense.

Commuting internal operations

One important notion that can be considered in operads
is that of commuting internal operations
(that is, arrows involving just one object).

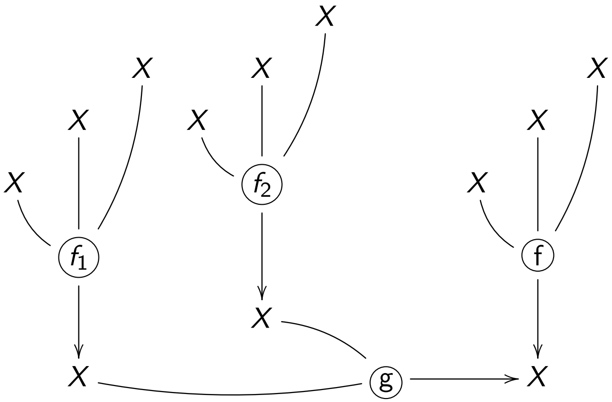
Commuting internal operations

Two internal operations with the same codomain.



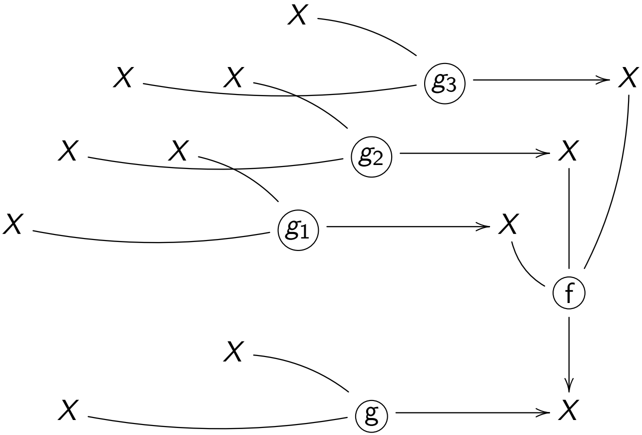
Commuting internal operations

Reindexing f along dg.



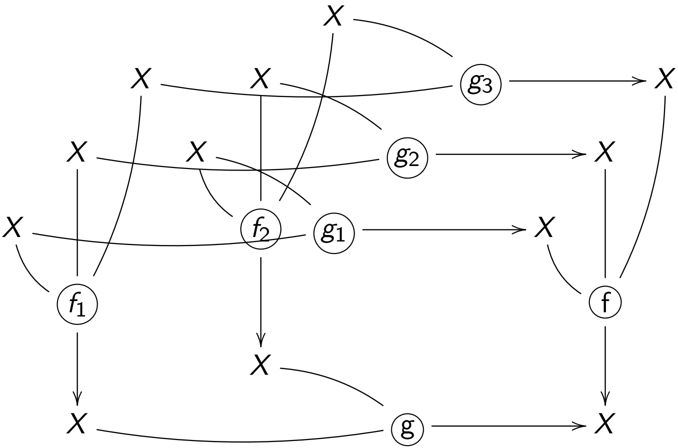
Commuting internal operations

Reindexing g along df.



Commuting internal operations

The two reindexing can be composed and may give the same result.



Commuting internal operation

Also this notion has a natural definition in the general setting of an operad $d_{\mathcal{O}}$ on a category $\mathcal{S}$ with pullbacks, and one can prove therein sort of Hilton-Eckman arguments.

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Internal operations

A map $f : X \to Y$ is an internal operation if $X = (df)^* Y$, that is, it is parallel to the horizontal lifting from Y of its image.

$$X \xrightarrow{f} Y \qquad dX \xrightarrow{df} dY$$

Commuting internal operation

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Internal operations

A map $f : X \to Y$ is an internal operation if $X = (df)^* Y$, that is, it is parallel to the horizontal lifting from Y of its image.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} dX \arrow[r, "df"] & dY \end{tikzcd}$$

Commuting internal operation

Commuting internal operations

Two internal operations f and g, with the same codomain, commute if the square below commutes in $\mathcal{D}$: $fg' = gf'$.

$$
\begin{tikzcd}
W \arrow[r, dotted, "g'"] \arrow[r, dashed, shift right] \arrow[d, dotted, "f'"'] \arrow[d, dashed, shift left] & X \arrow[d, "f"'] \arrow[d, dashed, shift left] \\
Z \arrow[r, "g"] \arrow[r, dashed, shift right] & Y
\end{tikzcd}
$$

f' and g' are obtained by reindexing
once “horizontally” and once “vertically”.
(The notion does not depend on the chosen pullback.)

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Fibrations as discrete double fibrations

Decoupled fibrations

The present approach to operads also suggests
the more general idea of decoupled fibration.

Indeed, an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathcal{S}$ can be seen
as a (split) fibration where the (chosen) cartesian arrows
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Fibrations as discrete double fibrations

Decoupled fibrations

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Indeed, an operad $d_{\mathcal{O}} : \mathbb{D} \to \mathcal{S}$ can be seen
as a (split) fibration where the (chosen) cartesian arrows
are separated from the other arrows (proarrows).

In fact, we have the following result:

Split fibrations $d : \mathcal{D} \to \mathcal{S}$
are discrete double fibrations $d : \mathbb{D} \to \mathbb{S}\mathrm{q}\,\mathcal{S}$
such that $\mathbb{D}$ has companions preserved by d.

Fibrations as lax double functors

From the point of view of lax functors, we have:

A lax functor $F : \mathbb{S}\mathrm{q}\,\mathcal{S} \to \mathbb{C}\mathrm{at}$ is a fibration
if and only if it preserves companions.

Which amounts to saying that the vertical part F^v
is determined by the horizontal part:
$F^v(f)$ is the profunctor represented by $F^h(f)$.

To explore

Another promising development is considering
operads on double categories which are more “relations-like”,
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Another promising development is considering operads on double categories which are more "relations-like", for instance cospans in Set_f.

In this case, it seems appropriate to consider the double category of summand squares rather than that of pullback squares.
There is no difference in extensive categories.