CAPITAL STRUCTURE MODELS

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1 Introduction

A firm's capital structure refers to how the firm finances its operations and growth. The sources of financing can come either from equity or debt. Capital structure *models* attempt to establish the relationship between a firm's value and its capital structure.
This report will compare some of the fundamental capital structure models in order to gain a better understanding of the underlying assumptions. We will seek to relax or remove some assumptions to see what effect they have on the model. Finally contingent capital will be introduced and applied to both the Merton model and the Black-Cox model. First we will look at some definitions that will be used throughout the paper.

Leverage will refer to the amount of debt a company has relative to their equity. A higher leverage ratio will imply the company is more risky.

$$Leverage = \frac{Debt}{Equity} = \frac{D}{E}$$

All values used in the report will be annualized unless stated otherwise.

2 The Merton Model

2.1 Introduction

In a 1974 paper by Robert C. Merton he proposed a method of determining a company's probability of defaulting on its debt, among other things. He makes use of the balance sheet equation, namely: $V_t = E_t + D_t$. We will first begin with some notation.

2.2 Notation & Assumptions

V_t = The value of the company's assets at time t.
σ_V = The volatility of the company's assets (assumed constant).
E_t = The value of the company's equity at time t.
σ_E = The instantaneous volatility of the company's equity.
D_t = The value of the company's debt at time t.
L = The promised debt payment at time T.
$t \in [0, T]$

Before we begin looking at results from the model it is important to look at some of the underlying assumptions listed below.

- Debt is formed by a single zero-coupon bond that matures at time T
- Default can only occur at time T
- Volatility of assets is constant
- Risk free rate of interest is constant
- Value of assets through time can by described by geometric Brownian motion as: $dV_t = rV_t dt + \sigma V_t dW_t$
- The company is liquidated at time T and there are no transaction or bankruptcy costs

The consequences of these assumptions will be highlighted in the next few sections.

2.3 Model Visualization

Now that the notation and assumptions have been stated we can consider what the model may look like visually.
In Figure 1, on the x-axis we have time starting at 0 and ending at T. The y-axis represents the value of the company, with the initial value V_0 being marked. The promised repayment of $\$L$ at time T is also drawn as a horizontal line. Taking a look at the green path we see that the firm value ends above the promised repayment, so debt holders will be repaid in full and equity holders will receive the residual $(V_T - L)$. Looking at the yellow path we notice firm value falls below L for a time but it finishes above L. Since default can only occur at time T, the debt holders will still be repaid in full and the equity holders again receive the residual $(V_T - L)$. Finally looking at the red path we see that it ends below L. This means the firm has defaulted and the debt holders will receive $\$V_T$ which is less than the promised payment of $\$L$. Equity holders will receive nothing.

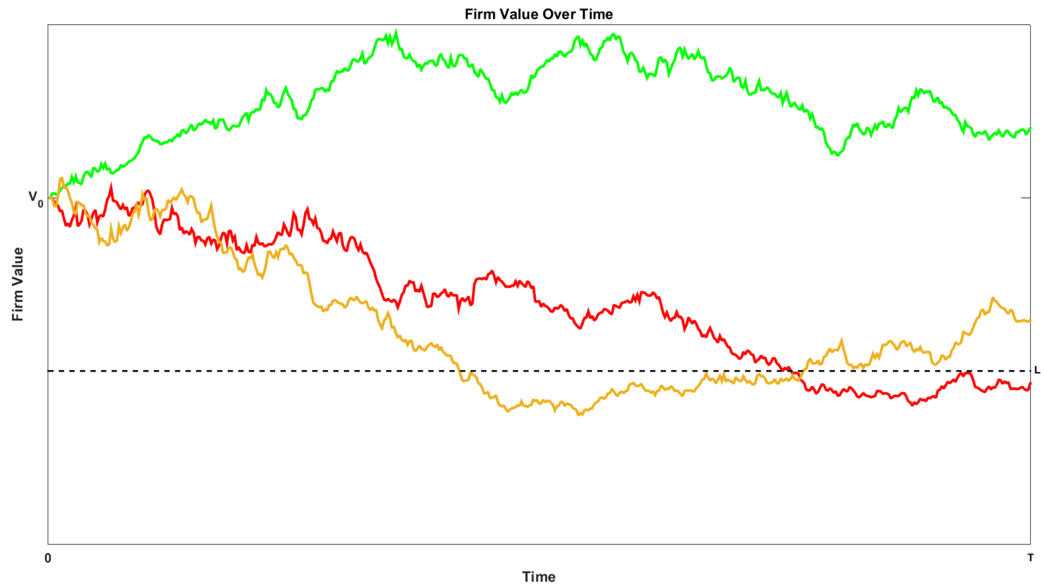

Figure 1: Three firm value sample paths.
$V_0 = 20,\ \sigma_V = 0.2,\ L = 10,\ r = 0.05,\ t = 0,\ T = 1.$

2.4 Accounting Identity Rewritten

Now that we have visualized the model we can begin to express the value of the risky debt. The first thing to notice is that since the company can only default on its debt at time T, we can express the debt payoff as:

$$D_T = \begin{cases} V_T & V_T \leq L \\ L & V_T > L \end{cases},$$

$$\Leftrightarrow D_T = \min(V_T, L),$$

$$\Leftrightarrow D_T = L - \max(L - V_T, 0).$$

This allows us to express the value of a company's debt as the difference between the amount owed and a European put on the value of the assets with a strike price equal to the debt owed. In other words it can be separated into the risk-free portion L and a short put on the company's assets.

Alternatively we can express the value of company's equity as a function of the debt repayment and the value of the company's assets.

$$E_T = \begin{cases} 0 & V_T \leq L \\ V_T - L & V_T > L \end{cases},$$

$$\Leftrightarrow E_T = \max(V_T - L, 0).$$

Then the value of a company's equity can be expressed as a European call on the value of the assets with a strike price equal to the debt owed. In the next section we see how Merton's model makes use of this equity expressed as a call relationship to help solve for the value of the company's assets.

2.5 Solving For Firm Value

A company's equity price at any point in time is observable (since its share price and shares outstanding are), and the volatility of those equity prices can then also be estimated. Then in the previous equity expressed as a call relationship, the value of assets and volatility are unknown. We can use the Black-Scholes formula for a European call to obtain one equation:

$$E_0 = V_0 N(d_1) - Le^{-rT} N(d_2),$$

where:

$d_1 = \frac{\ln(V_0/L)+(r+\sigma_V^2/2)T}{\sigma_V \sqrt{T}}$, and
$d_2 = d_1 - \sigma_V \sqrt{T}$.

This formula has its two unknowns as V_0 and σ_V. In order to solve for the unknowns we need another equation. From a result in stochastic calculus known as Itô's Lemma we are able to obtain the second equation:

$$\sigma_E E_0 = \frac{\partial E}{\partial V} \sigma_V V_0,$$

where we can write $\frac{\partial E}{\partial V} = N(d_1)$. Using these two equations we are now able to solve for the value of a company's assets and asset volatility using observable equity prices. As an example consider the values below.

For consistency we will use the following parameters for multiple examples:

$$E_0 = 3,\ \sigma_E = 0.8,\ L = 10,\ r = 0.05,\ T = 1.$$

Solving the two previous equations yields:

$$V_0 = 12.39, \ \sigma_V = 0.21.$$

This allows us to solve $D_0 = 9.39$. Comparing that to risk-free debt value $10e^{-0.05} = 9.51$ we see that the risky debt is less expensive. This makes sense as investors demand compensation for taking on risk, in this case by purchasing the risky bond at a discount to the risk-free debt.

2.6 Parameter Relationships

In this section we will be exploring the relationships between parameters by allowing one to vary and fixing the rest. We begin by allowing the initial equity value to vary. For each value of E_0 we recompute V_0 and σ_V. We then compute the leverage ratio and compare it to the volatility of the company's assets over the interval.

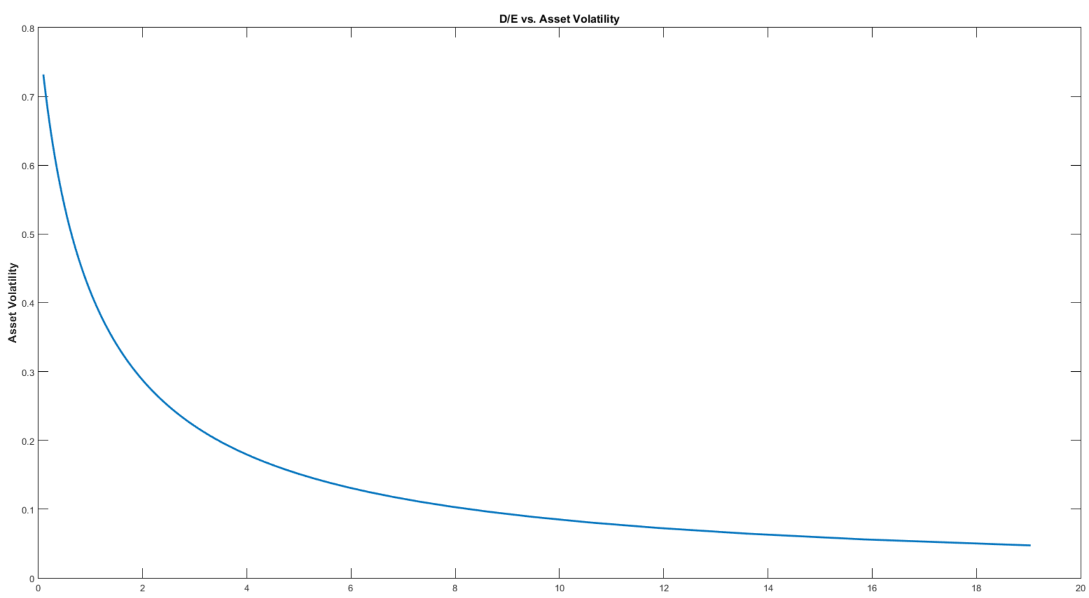

Figure 2: Asset Volatility vs. Leverage.
$\boldsymbol{E_0 \in (0, 100)}, \ \sigma_E = 0.8, \ L = 10, \ r = 0.05, \ t = 0, \ T = 1.$

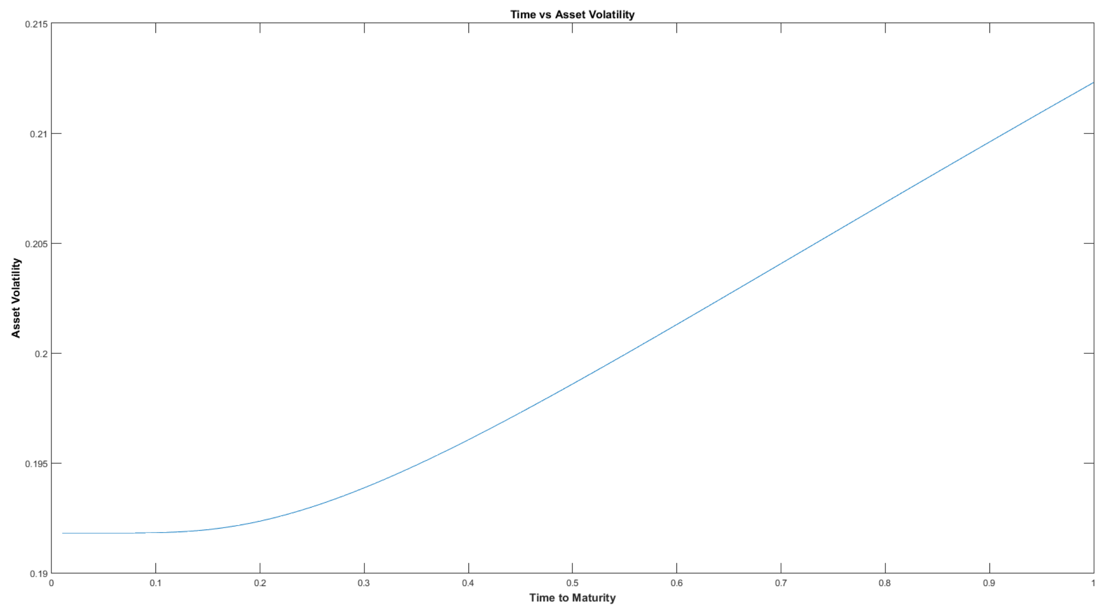

Figure 3: Asset Volatility vs. Time to Maturity
$E_t = 3,\ \sigma_E = 0.8,\ L = 10,\ r = 0.05,\ \boldsymbol{t \in [0, T)},\ T = 1.$

The first thing to notice about Figure 2 is that asset volatility is a decreasing function of the leverage ratio. As the company becomes more financed by debt relative to their equity financing, the volatility of the assets decreases. This makes sense since the value of the debt owed does not fluctuate as much relative to the value of equity does. The second thing to notice is that asset volatility approaches equity volatility as the leverage ratio decreases. This again makes sense as a company financed entirely by equity should have its asset volatility exactly equal to its equity volatility.

Next we allow time to flow from the beginning of the period to the end and note its effect on asset volatility.

It's important to notice that the x-axis is time-to-maturity rather than just time. With that in mind, we can see that the volatility of assets is an increasing function of time-to-maturity (or a decreasing function of time). This makes sense because as the company approaches the time at which the debt must be repaid there is less time for the equity prices to fluctuate, thereby reducing the asset's volatility. If we were to reduce L to say 1.5, the asset volatility would increase much faster as time to maturity does. Equity in this case would be playing a more significant role in the company's financing and therefore its asset volatility would be much more sensitive to time.

2.7 Default Statistics

From the system of equations used to solve for V_0 and σ_V come two measures we can use to compare and rank different companies.

Probability of Default is the probability that a company will not have enough money to pay off its debt at time T. From our model it is expressed as $N(-d_2)$.

Distance to Default is the distance from the expected firm value at time T and the promised debt repayment, measured in standard deviations. From our model it is expressed as d_2.

Using our usual parameters we have DD = 1.14 standard deviations and PD = 12.70%. Let's examine this graphically:

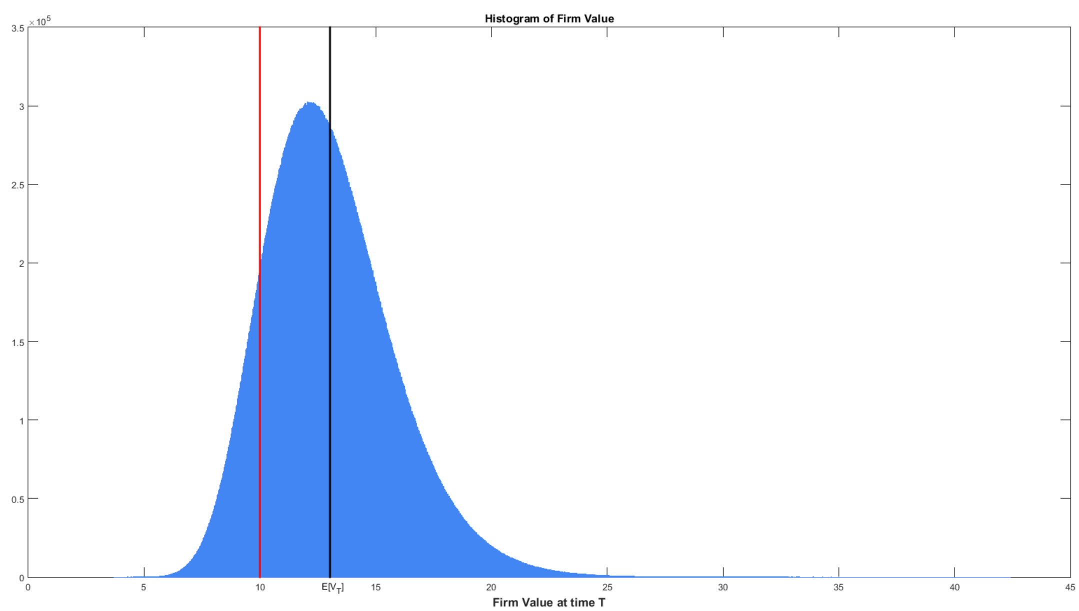

Figure 4: Histogram of Company's Terminal Asset Value.
$V_0 = 12.39,\ \sigma_V = 0.21,\ L = 10,\ r = 0.05,\ t = 0,\ T = 1$

On the x-axis we have the firm value at time T, which is log-normally distributed. The black line represents the expected firm value, which is about 13 using the usual parameters. The red line represents the promised payment of $L = 10$ at time T. The distance between the black line and the red line is the distance to default. The larger the distance is, the more of a negative "shock" the company would have to face in order to default. Probability of default is represented graphically as the blue area left of the red line. Of course keeping all else constant and moving the red line left will decrease our probability of default as it should since less money would be owed.

In Figure 5 we again allow the current time to vary and hold the other parameters constant. This time however we will examine the behaviour of the probability of default and distance to default.

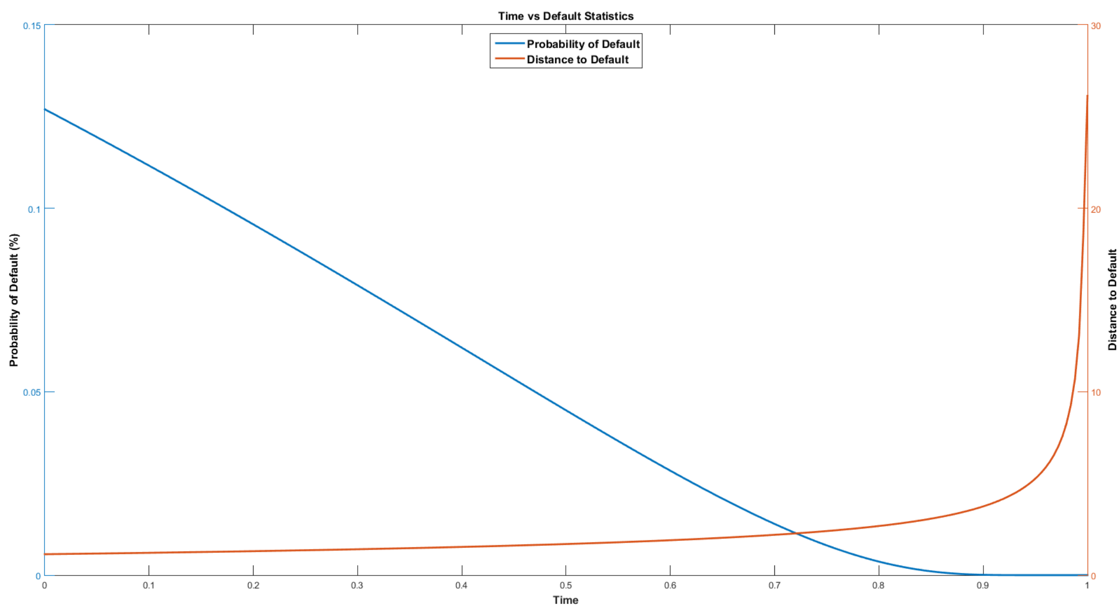

Figure 5: Default Statistics vs. Time.
$E_t = 3,\ \sigma_E = 0.8,\ L = 10,\ r = 0.05,\ \boldsymbol{t \in [0, T)},\ T = 1.$

Looking first at probability of default (blue line) we see it starts at 12.70% just like it did in our original system. We see that in this case the probability of default is a decreasing function of time. This makes sense since the expected value of this company's assets at time T is above L. As time approaches maturity, there is less time for equity to move adversely and cause default and therefore the probability of default decreases. We see this story being played out in the distance to default (orange line) as well. It starts at 1.14 as it did originally and increases as time moves towards maturity. What this means is that since the firm is expected to grow over time and be able to pay off its debt, it will take an increasingly large "shock" in order to default. As time-to-maturity decreases to 0 the distance to default increases to infinity, in this case.

2.8 Risk Structure of Debt

In Section 2.4 we showed that the value of a company's debt can be expressed by a risk free position (L) and a risky position (short put). We now define the risk premium (or credit spread) as the compensation you receive for taking on the risky debt. We let $\tau = T - t$. Define $R(\tau)$ as the yield to maturity on the risky debt provided that the company does not default. Then $R(\tau) - r$ is the risk premium. Merton provides the following formula for computing the risk premium:

$$R(\tau) - r = \frac{-1}{\tau}\log\{N\left[h_2(d, \sigma_V^2\tau)\right] + \frac{1}{d}N\left[h_1(d, \sigma_V^2\tau)\right]\}$$

where

$$h_1(d, \sigma_V^2\tau) = -\frac{\left[\frac{1}{2}\sigma_V^2\tau - \log(d)\right]}{\sigma_V\sqrt{\tau}}$$

$$h_2(d, \sigma_V^2\tau) = -\frac{\left[\frac{1}{2}\sigma_V^2\tau + \log(d)\right]}{\sigma_V\sqrt{\tau}}$$

$$d = \frac{Le^{-r\tau}}{V_t}$$

So from the term premium formula we notice it is a function of time to maturity, the volatility of assets and the discounted value of the promised payment relative to the firm's current assets.

For the following graph we say the firm is highly leveraged if $d > 1$. In other words the present value of the promised payment is more than the current value of the firm. We say a firm has low leverage otherwise.

In Figure 6 the term premium for both a highly leveraged (HL) company and a lowly leveraged (LL) company have been plotted as a function of time

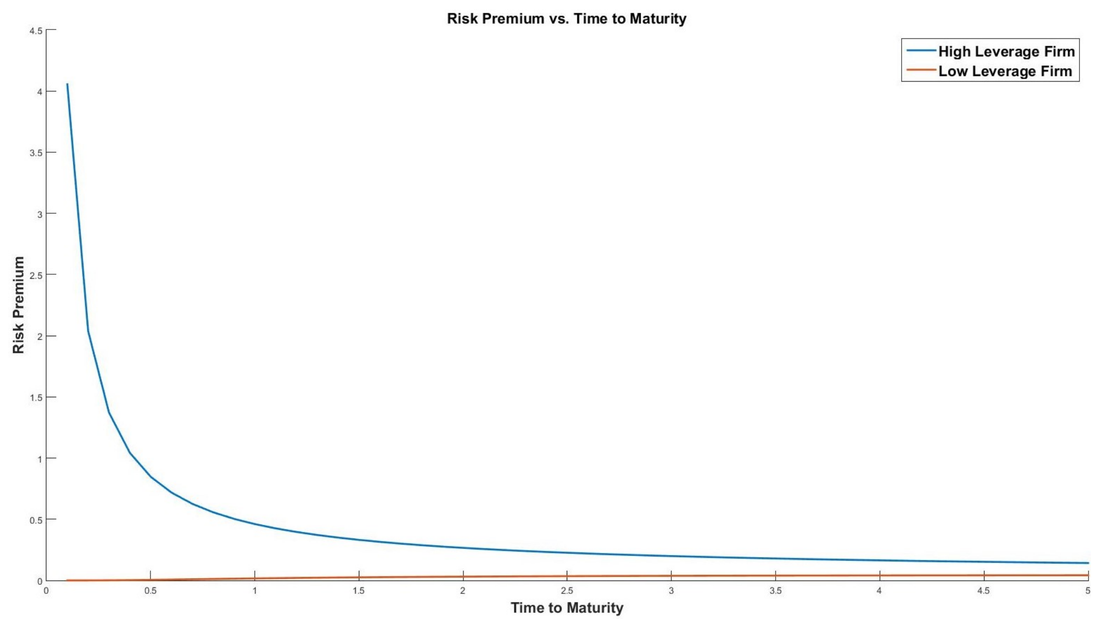

Figure 6: Risk Premium vs. Time to Maturity

to maturity. We notice that the risk premium for the HL company is always above that of the LL company. This makes sense as more leverage is more risky so the debt holders would require higher compensation. From this Figure we can also see that the risk premium for the HL company is a decreasing function of the time to maturity. As time to maturity approaches zero, the premium the HL company would have to pay on its debt will increase as it is much more likely it will not be able to make its promised payment.

In order to see the structure of the LL company's risk premium we have plotted it on its own axis in Figure 7. Here we are able to see that the risk premium for the LL company is an increasing function of the time to maturity, although the scale is significantly smaller. This is because the LL company has enough money to pay its debt-holders so as time to maturity approaches zero the debt becomes risk free and the premium goes to zero. As time to maturity increases it allows more time for equity to fluctuate and potentially cause the company to default at time T, resulting in the small risk premium.

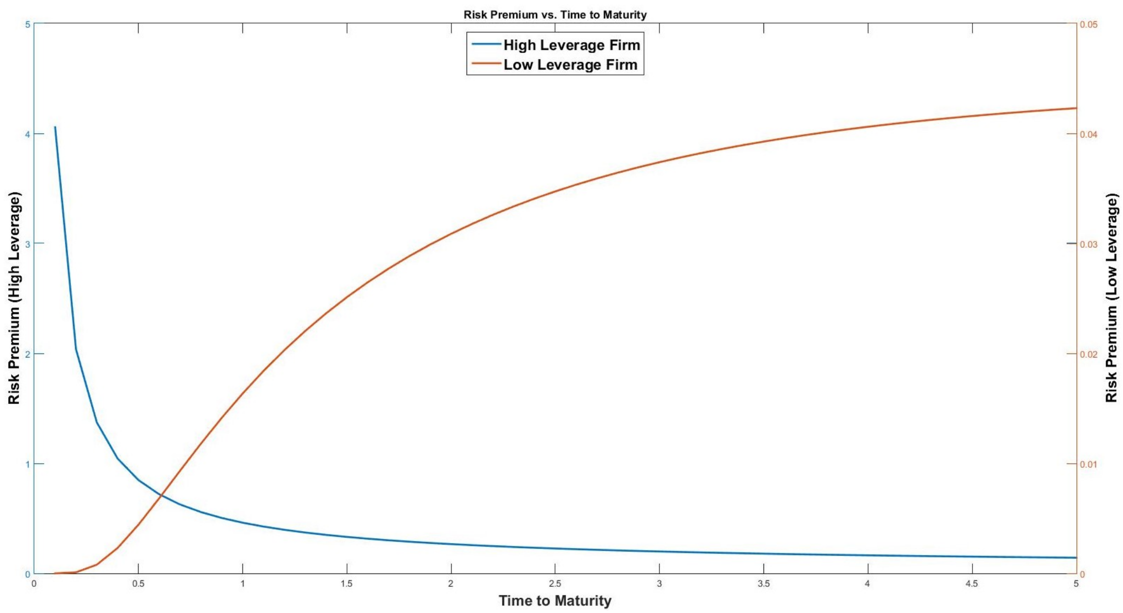

Figure 7: Risk Premium vs. Time to Maturity

2.9 Subordinated Debt

Subordinated or junior debt is debt that has a lower payment priority than a senior or secured debt. In other words, if bankruptcy occurs the junior debt is not repaid until the senior debt has been repaid. One of the initial model assumptions was that debt was formed by a *single* zero-coupon bond. We now will relax this assumption by allowing for two zero-coupon bonds with maturity T. One will be senior debt with face value L^S, and the other will be junior debt with face value L^J.

Figure 8 shows the same three paths we observed under the single debt Merton model but has now split the face value of the debt into L^S and L^J. This graph allows us to distinctly see the three terminal states and their effect on equity/bond payoff. The green path is well above the sum of the face values therefore the junior and senior bondholders are repaid in full and the equity retains the residual. The yellow path has enough money to pay off the senior debt holders in full but defaults on its promised payment to the junior debt holders, wiping out equity holders as well. Finally the red path defaults on its payment to the senior debt holders and so both the junior

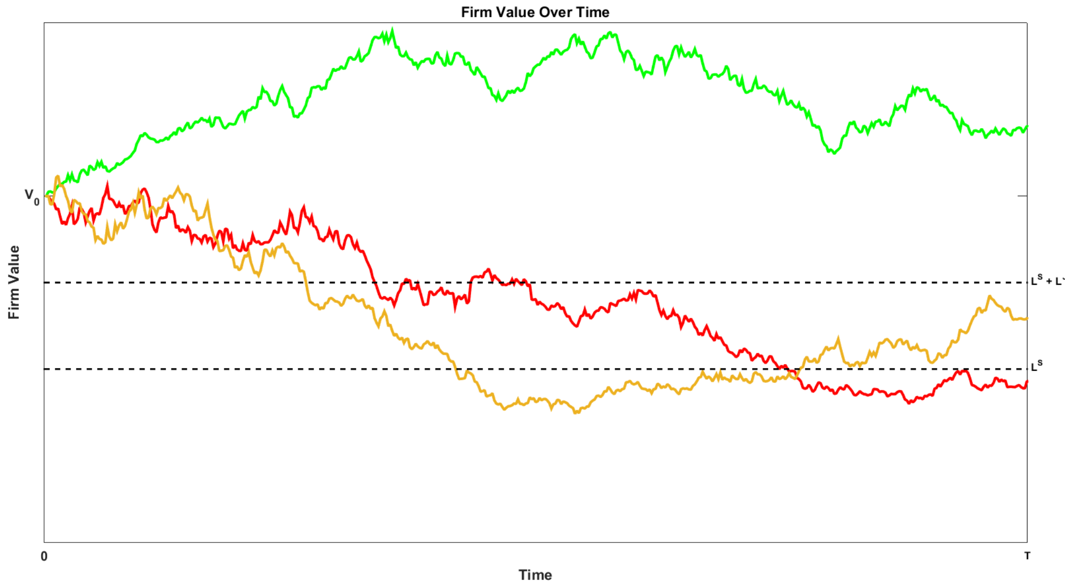

Figure 8: Three firm value sample paths.
$V_0 = 20,\ \sigma_V = 0.2,\ L^S = 5,\ L^J = 5,\ r = 0.05,\ t = 0,\ T = 1.$

debt holders and the equity holders receive nothing.

Figure 9 is introduced to provide an alternative way to visualize the time T payoff for the three different assets. If we fix a terminal state (red, yellow and green for simplicity) we can clearly see the effect it has on all asset holders. Additionally this Figure provides us with a nice way to determine the option payoff or strategy that can be used analogously to discuss the asset position. In other words the equity payoff looks identical to that of a call payoff, the junior debt looks like a bull call spread and the senior debt resembles a covered call position.
Similar to section 2.4 we now express the equity and debt payoffs in a familiar way. The senior debt is shown to be:

$$D_T{}^S = \begin{cases} V_T & V_T \leq L^S \\ L^S & V_T > L^S \end{cases},$$

$$\Leftrightarrow D_T{}^S = V_T - (V_T - L^S)^+,$$
$$\Leftrightarrow D_T{}^S = L^S - (L^S - V_T)^+.$$

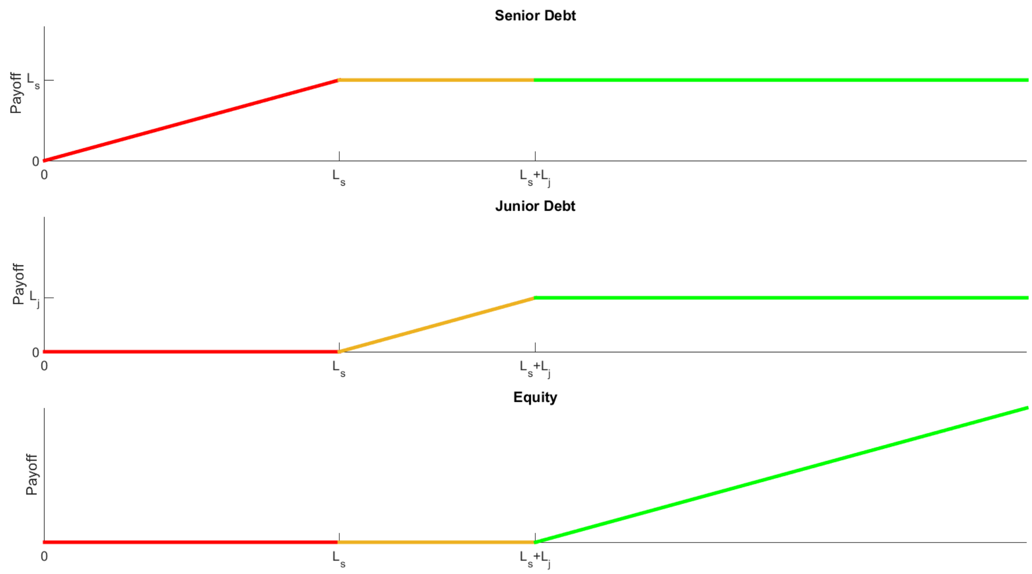

Figure 9: Payoff structure for three types of assets.

This gives us two ways to think about the senior debt position. Either as taking a long position in the firm with a short position in a call with strike L^S (i.e a covered call), or as holding L^S dollars with a short put position with strike L^S. We notice this senior debt position has the same payoff structure as the single debt did in section 2.4, the only difference being the strike value. Next we easily express the equity in the same way as section 2.4:

$$E_T = \begin{cases} 0 & V_T \leq L^S + L^J \\ V_T - (L^S + L^J) & V_T > L^S + L^J \end{cases},$$

$$\Leftrightarrow E_T = (V_T - (L^S + L^J))^+.$$

This is the same call on firm value expression that we originally began with, this time being written on the sum of the promised payments.
Finally the payoff of the junior debt is obtained by solving $V_T = E_T + D_T^S + D_T^J$ in each of the three terminal states. It can be represented as:

$$D_T{}^J = \begin{cases} 0 & V_T \leq L^S \\ V_T - L^S & L^S \leq V_T \leq L^S + L^J \\ L^J & V_T > L^S + L^J \end{cases},$$

$$\Leftrightarrow D_T{}^J = (V_T - L^S)^+ - (V_T - (L^S + L^J))^+.$$

Then the junior debt can be thought of as taking a long position in a call on firm value with strike L^S and a short position in a call on firm value with strike $L^S + L^J$. This strategy is also known as a bull call spread.

We can generalize these payoffs structures to support N types of subordinated debts where L^1 is the most senior debt and L^N is the least senior (or most junior) debt. Let $\mathbf{L} = \sum_{i=1}^{N} L^i$ be the sum of all the face values. The same three debt structures can then be expressed by modifying the strike values as below:

$$E_T = \begin{cases} 0 & V_T \leq \mathbf{L} \\ V_T - \mathbf{L} & V_T > \mathbf{L} \end{cases},$$

$$D_T{}^1 = \begin{cases} V_T & V_T \leq L^1 \\ L^1 & V_T > L^1 \end{cases},$$

and for $k = 2...N$

$$D_T{}^k = \begin{cases} 0 & V_T \leq \sum_{i=1}^{k-1} L^i \\ V_T - \sum_{i=1}^{k-1} L^i & \sum_{i=1}^{k-1} L^i \leq V_T \leq \sum_{i=1}^{k} L^i \\ L^k & V_T > \sum_{i=1}^{k} L^i \end{cases},$$

$$\Leftrightarrow D_T{}^k = (V_T - \sum_{i=1}^{k-1} L^i)^+ - (V_T - \sum_{i=1}^{k} L^i)^+.$$

We can easily recognize the same familiar payoffs as above, the only difference is how we choose our strike values. This allows us to support as many subordinated debt types as we wish, but that does not do much to improve the other limitations imposed by our initial assumptions.

2.10 Limitations

Merton's model provides a good starting point to understand how a company's assets relates to its risky debt and equity. We are able to compute the (risk-neutral) probability of default for a group of companies and then rank them. This would allow us to create our own credit rating system to determine the risk premium each company should offer on their junior and senior debt.

One major limitation is that the debt is assumed to be comprised of zero coupon bonds. This is far from realistic and in order to implement this to real companies we would have to transform their debt obligations into a zero coupon bond. Some weighted average or mapping of debt maturities might be appropriate. Another limitation is that there are assumed to be no liquidation or bankruptcy costs and no tax benefit for issuing debt. This has the effect of overstating the amount equity holders will receive if liquidation occurs and/or overstating the amount bondholders may receive if bankruptcy occurs. The final limitation to point out is that in this model default can only occur at maturity. This is unrealistic as a company can certainly default due to some unforeseen event or mismanagement long before a debt payment may be owed.

First passage time models allow for these default events to be incorporated and will be examined in the next section.

3 The Black-Cox Model

3.1 Introduction

In a 1976 paper by Fischer Black and John C. Cox they proposed an extension to the Merton model in which an event is triggered once a certain barrier is reached. If the barrier is hit from above, that event may be liquidation or a restructuring of the company's finances. If the barrier is hit from below, the event could be triggering a call provision on a bond. In either case it changes how the debt and therefore the company should be valued. All notation from the Merton model is maintained in this section with the addition of B, the value of the barrier.

3.2 Model Visualization

We begin the Black-Cox model discussion with a familiar picture from section 2.3, but this time allow liquidation to occur as soon as the barrier is reached. In Figure 10 debt is composed of a single zero coupon bond with face value L. We set the default barrier B to be equal to L, meaning that the bond holders will always be protected. Barrier selection and its consequences are discussed in the following subsection. In order to highlight the immediate differences between the Merton and Black-Cox models, the same three sample path have been plotted. The solid lines show how the firm evolves under the Black-Cox model while the dashed lines show how the firm would have evolved under the Merton model. The green path clearly has no difference in paths as the barrier is never breached. However there is a large difference for the yellow path. Under the Black-Cox model the firm defaults at some time before maturity, while under the Merton model that same path recovered and default would not have occurred at all. Finally we note that the red path defaulting early ended up protecting both the bond holders and the equity holders because of the choice of barrier. Bond holders would receive the present value of their promised payment (which we say is the "fair" value) and the equity holders would get the residual, however small. That same red path under the Merton model would have equity holders receive nothing and bond holders receive less than the time T amount they are owed.

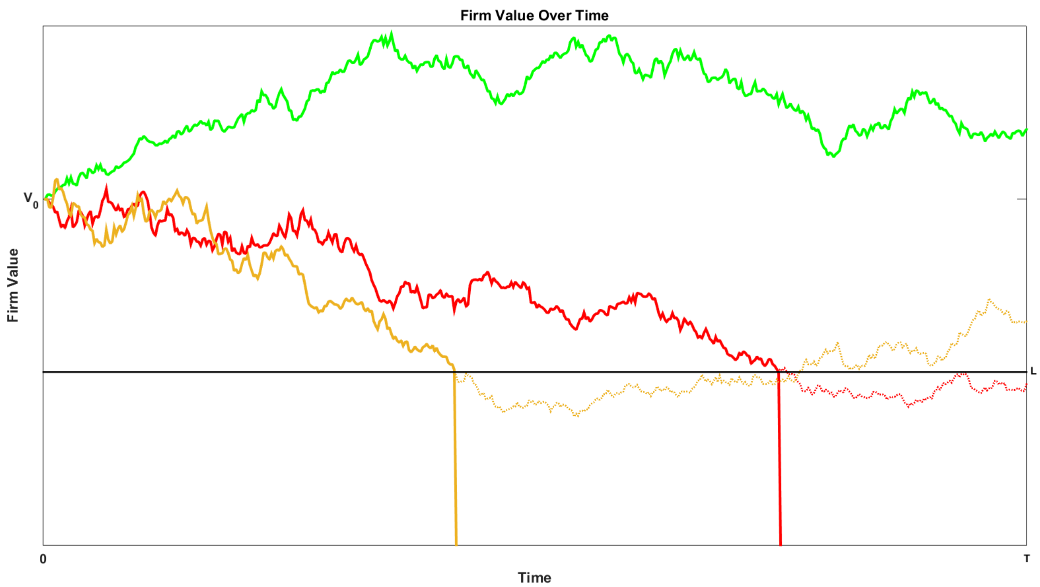

Figure 10: Three firm value sample paths.
Solid lines denote paths under the Black-Cox framework while dashed lines represent the same paths under the Merton framework.
$V_0 = 20,\ \sigma_V = 0.2,\ L = 10,\ B = 10,\ r = 0.05,\ t = 0,\ T = 1.$

3.3 Barrier Selection

All discussion about barrier selection is based on the assumption that there are no costs associated with defaulting and that in the event of default the debt holders will receive the minimum of the firm value and the discounted face value of their debt. With this in mind we present different choices of barrier levels and how these choices may or may not present perverse incentives for equity and/or debt holders.

A question we seek to answer is: *given* a firm evolves under the Black-Cox framework, what is the best choice of barrier? What does it mean for a barrier to be "best"? For our purposes the "best" choice of barrier will be the one that protects bondholders (at least partially) without introducing scenarios in which an equity or bond holder is incentivized to have the firm default early.

Initially we can see that the choice of barrier B should be in the interval $[0, V_0)$. Any choice of default barrier outside of these bounds is trivial, with either every path defaulting or no path defaulting. In Figure 10 the barrier was chosen to be equal to the face value of the debt. We saw that this choice of barrier created some scenarios (i.e the red path) where the equity holders could lock in value if the firm defaulted on its payments. Similarly, for any choice $B \in [L, V_0]$ the bond holders would be protected in full and equity holders would receive the residual (which is only zero in the case that default occurs at time T and $B = L$). Heuristically we could also note that a choice of barrier in this interval limits the investment choices the firm can make. They would need to make very low risk decisions in order to avoid triggering the default barrier with short term firm volatility. With this in mind we revise the bounds on B to $[0, L)$.

At the beginning of this discussion it was stated that in the event of liquidation the bond holders would receive $Le^{-r(T-t)}$. Since the payout to bond holders is time dependent it is natural to consider a time dependent default barrier as well. If we consider $B = Le^{-r(T-t)}$ then equity holders will receive \$0 if default occurs and bond holders will receive the fair value of their promised payment. This removes the perverse incentive that equity holders faced under the constant barrier choice as now they are only compensated if default does not occur. Bond holders should also be indifferent between default and no default as in either case they receive the fair time t value of their money (this is just from a bank account perspective, not considering any rights or privileges they received as bond holders while the company was operating).

More generally we could choose $B = Ke^{-\alpha(T-t)}$ where $K > 0$ and $\alpha \in [0, 1]$. This allows us much greater flexibility on how much bond holders should be protected and under what scenarios default is triggered. For example if $K = L$ and $\alpha >> r$ then this barrier allows for much greater volatility early on and gives the company more time to recover from short term decreases in firm value. Another choice could be $K = \frac{L}{2}$ and $\alpha = r$. This guarantees that the bondholders would receive 50% of the fair amount in the event of default while also allowing the firm value to fluctuate more throughout its business cycle.

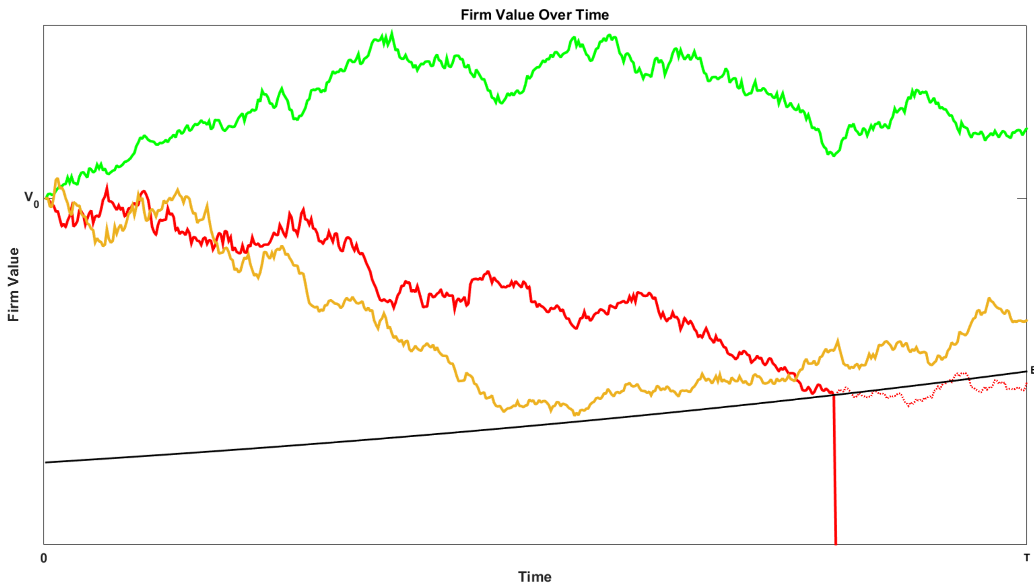

Figure 11: Choice of barrier to allow more short term volatility.
$V_0 = 20,\ \sigma_V = 0.2,\ L = 15,\ B = Le^{-0.75(T-t)},\ r = 0.05,\ t = 0,\ T = 1$

Taking a look at Figure 11 we see an example of the general time dependent barrier with $K = L$ and $\alpha = 0.75$. This barrier allows paths that have fallen the chance to recover while still paying equity holders \$0 in the event of default. We notice the yellow path is able to recover and does not end up defaulting at all, while the red paths still defaults.

We can see that there are many interesting choices for the barrier and that they each have a different effect on how the bond and equity will be valued. Too high of a barrier choice results in a firm that can not take on risky decisions for fear of defaulting, and too low of a barrier results in a firm that has no protection for its bond holders. This last point allows us to clearly see that if $B = 0$ then the Black-Cox model behaves exactly like the Merton model. The Black-Cox model is therefore a more general version of the Merton model.

To conclude this section we compare the value of equity and debt under both models when $B = Le^{-r(T-t)}$. If all other parameters are equal then:

$$E^{\text{Black-Cox}} \leq E^{\text{Merton}},$$
$$D^{\text{Black-Cox}} \geq D^{\text{Merton}}.$$

This should be evident since the Black-Cox model protects bond holders at the expense of equity holders. By introducing a barrier that allows early default it thereby removes some outcomes where the firm value would have recovered by time T and had positive payoff for the equity holders. As $B \searrow 0$, $E^{\text{Black-Cox}} \to E^{\text{Merton}}$ and $D^{\text{Black-Cox}} \to D^{\text{Merton}}$.

3.4 Effect of Barrier on Distribution

The introduction of a barrier into the model adds another distribution to work with. Looking at Figure 12 we see that in addition to the distribution of the terminal firm value, we also have the distribution of hitting times to look at. The choice for barrier was made to be the discounted value of L and the remaining parameters are the same as the previous three figures.

On the x-axis of Figure 12 we notice that there are few paths that default early on and that it appears more paths are defaulting as time goes on. The reason for this is that the barrier is sufficiently far away from V_0 at time 0 so very few paths are able to hit the default barrier early on. From this sample approximately 12.51% of the paths defaulted, whereas the same set of paths under the Merton model would have had a default rate of 6.84%. One of the reasons for the large difference in default rates is that many paths that default early under Black-Cox end up recovering if given enough time.

If we turn our attention to Figure 13 we see the distribution of the hitting times for all the paths that defaulted early. It confirms that very few paths hit the barrier early on but as time to maturity decreases, the barrier increases and paths are able to move down far enough to hit the barrier with more frequency. If we were to either increase firm volatility or decrease the distant between the barrier and the initial firm value we would see significantly more paths defaulting early and many less defaulting later on.

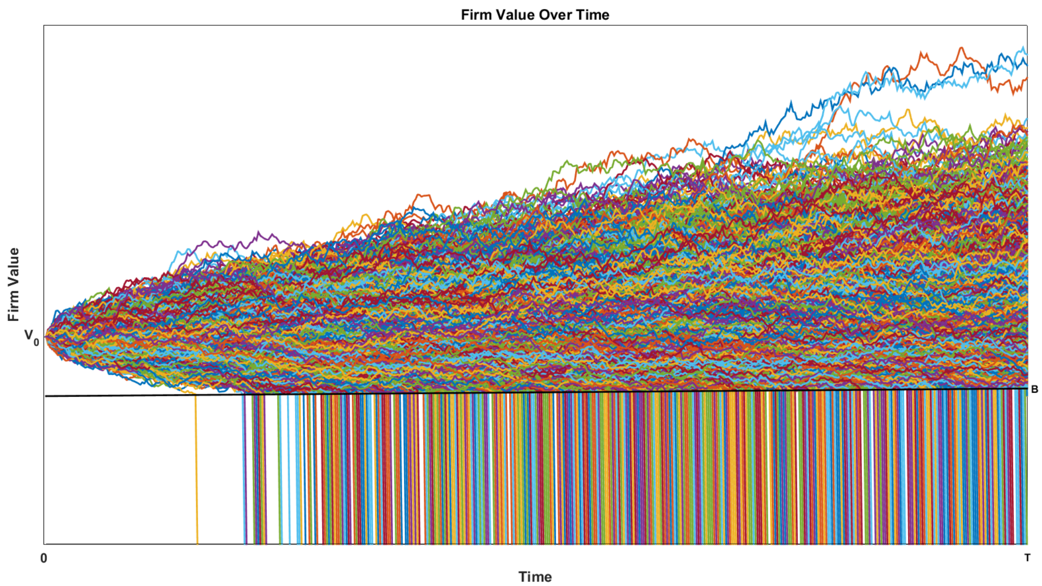

Figure 12: Ten thousand firm value sample paths.
$V_0 = 20,\ \sigma_V = 0.2,\ L = 15,\ B = Le^{-r(T-t)},\ r = 0.05,\ t = 0,\ T = 1$

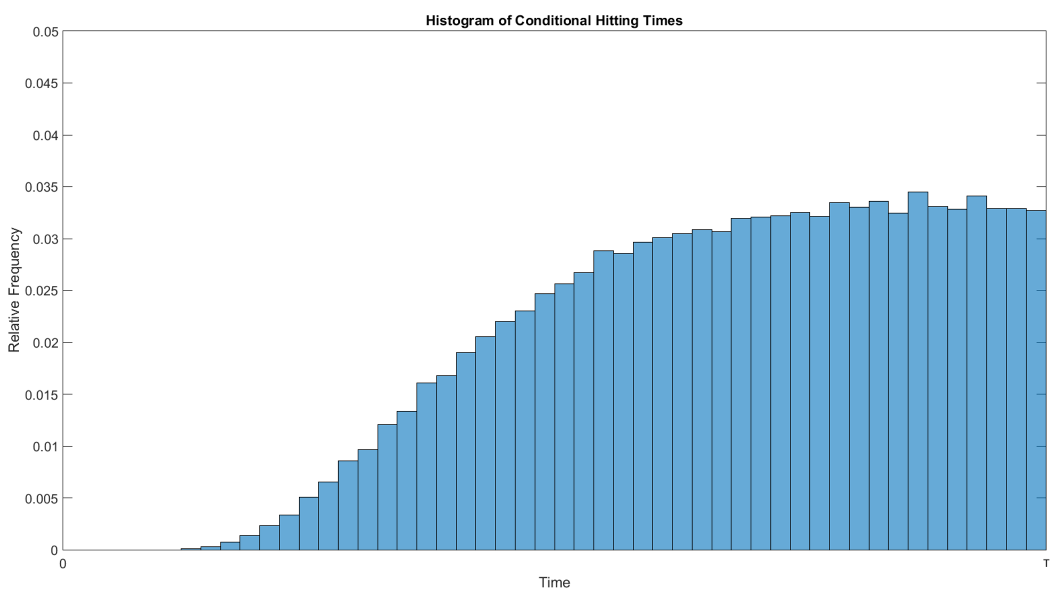

Figure 13: Histogram of conditional hitting times for ten thousand sample paths. $V_0 = 20,\ \sigma_V = 0.2,\ L = 15,\ B = Le^{-r(T-t)},\ r = 0.05,\ t = 0,\ T = 1$

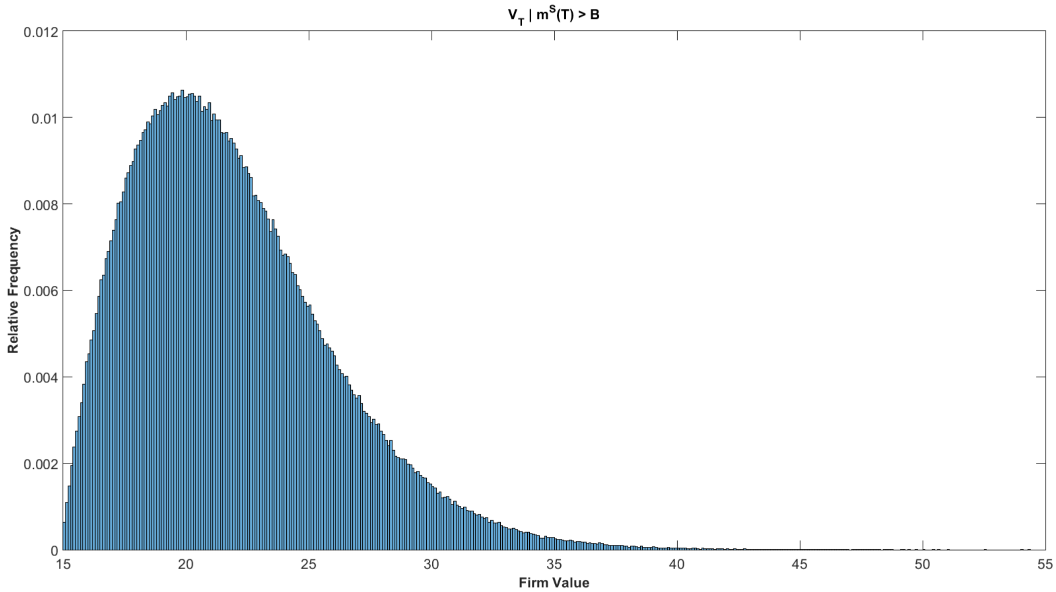

Figure 14: Histogram of conditional terminal firm value for ten thousand sample paths.
$V_0 = 20,\ \sigma_V = 0.2,\ L = 15,\ B = Le^{-r(T-t)},\ r = 0.05,\ t = 0,\ T = 1$

Figure 14 is a histogram of the terminal firm value for all the paths that did not default. It is the same log-normal shape we noticed under the Merton model. This makes sense since we have conditioned only on the paths that did not default, so the terminal distribution should closely resemble that of the Merton model.

In Figure 15 we have grouped all the paths that defaulted early and assigned them a terminal firm value of zero. Doing this allows us to clearly see the approximate 12.51% of paths that did not default. Due to our choice of barrier this figure accurately splits the group of equity holders into those who receive \$0 and those who receive some positive dollar amount.

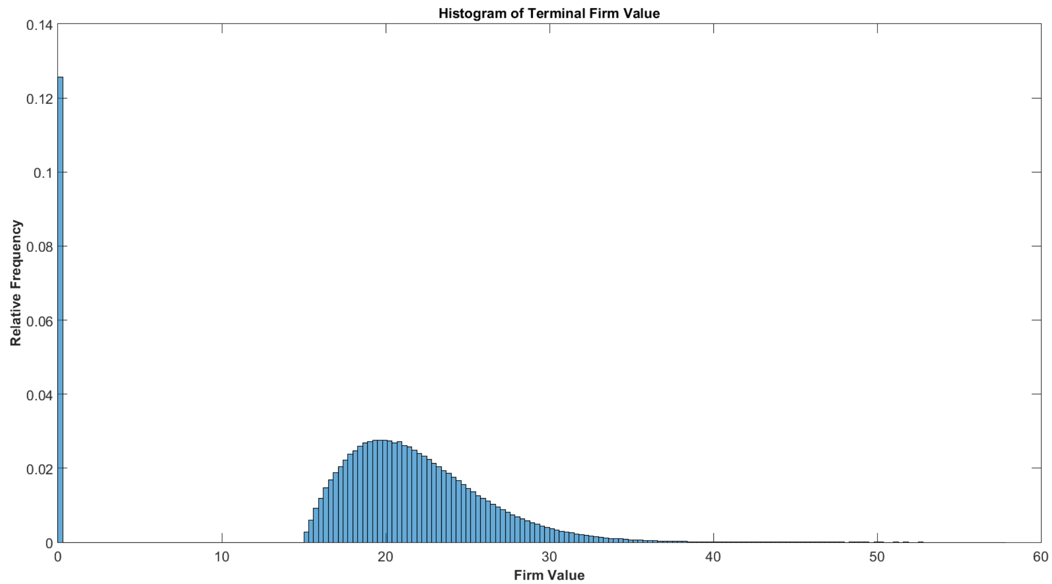

Figure 15: Histogram of terminal firm value for ten thousand sample paths.
$V_0 = 20,\ \sigma_V = 0.2,\ L = 15,\ B = Le^{-r(T-t)},\ r = 0.05,\ t = 0,\ T = 1$

3.5 Closed Form Solutions

For this subsection we introduce the notation that $\widetilde{\mathrm{P}}$ represents the risk-neutral probability measure. With this in mind we compute the risk-neutral probability that the firm value is below the barrier at some time $t < T$ as follows:

$$\begin{aligned}
\widetilde{\mathrm{P}}\left(V(t) < B\right) &= \widetilde{\mathrm{P}}\left(V_0 e^{(r-\frac{\sigma^2}{2})t+\sigma\widetilde{W}(t)} < Ke^{-\alpha(T-t)}\right) \\
&= \widetilde{\mathrm{P}}\left((r-\frac{\sigma^2}{2})t+\sigma\widetilde{W}(t) < \ln(\frac{K}{V_0}) - \alpha(T-t)\right) \\
&= \widetilde{\mathrm{P}}\left(\widetilde{Z} < \frac{\ln(\frac{K}{V_0}) - (r-\frac{\sigma^2}{2})t - \alpha(T-t)}{\sigma\sqrt{t}}\right) \\
&= N\left(-\frac{\ln(\frac{V_0}{K}) + (r-\frac{\sigma^2}{2})t + \alpha(T-t)}{\sigma\sqrt{t}}\right).
\end{aligned}$$

Note that since we are starting our firm process at time zero but starting the barrier value at time T and then discounting it adds the additional αT term in all the equations. If we wanted it all in terms of time t we could begin the barrier process at K_0 and accumulate it forward instead.
The time T payoff to the equity holders can be written as:

$$E_T = (V_T - L)^+ \mathbb{1}_{\left\{\inf\limits_{\forall t \leq T} V(t) > B\right\}}$$

Taking the expected discounted value under the risk neutral measure yields the pricing function for the equity value under the Black-Cox model. Much of the algebra is similar to that of above after obtaining the hitting time density. It can be shown to be a down-and-out call on firm value with strike L and barrier B. After obtaining the value for equity we can easily rearrange the original accounting equation to solve for the value of the debt. Alternatively, equation eight in the paper[1] by Black and Cox provides a closed form solution for the price of the bond. Using this we could solve for the equity value in the same fashion.

3.6 Subordinated Debt

We have now established the effects that introducing a barrier has on the model. The next step is to make the same subordinated debt extension that we made under the Merton model. For this subsection we use the notation $L = L^J + L^S$ is the sum of the promised payments at time T.

First let us briefly consider where the barrier could be chosen. If $B = Le^{-r(T-t)}$ then regardless of whether default occurs or not, both the junior and senior bondholders will receive the 'fair' value of the amount they are owed. This is not a very interesting choice for barrier as both the classes of debt holders are entirely protected, so the subordination is not affecting the payoff in any state. For the remainder of this section we use $B = L^S e^{-r(T-t)}$, which adds the risk of significant loss for junior debt holders. The discussion in Section 3.3 for the more general $B = Ke^{-\alpha(T-t)}$ still holds for the subordinated debt and different choices for K and α may significantly change the value of both the junior and senior debt.

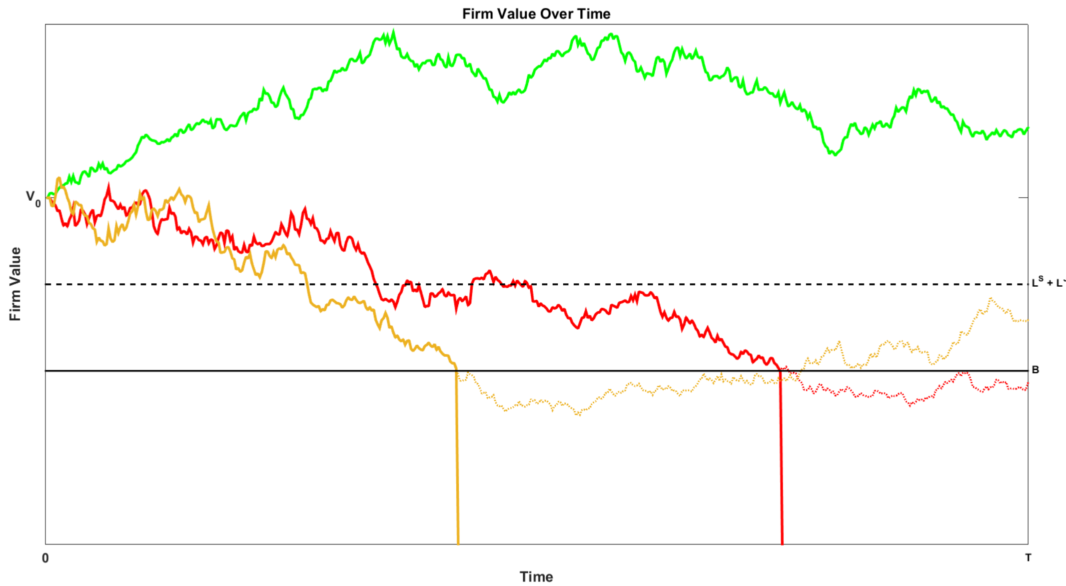

Figure 16: Three paths under Black-Cox model with subordinated debt. $V_0 = 20,\ \sigma_V = 0.2,\ L^S = 10,\ L^J = 5,\ B = L^S e^{-r(T-t)},\ r = 0.05,\ T = 1$

Figure 16 show our usual paths but now with subordinated debt added. To avoid clutter the value L^S was not plotted as it is extremely close to B (by our definition). For both the yellow and red paths the barrier is breached and so the equity holders are wiped out. The senior debt holders will receive $L^S e^{-r(T-t)}$ which leaves nothing left for the subordinated debt holders. The green path has payoffs as it did under the subordinated debt Merton model as the barrier is not breached.

The pricing functions for D_t^J and D_t^S may be found in equation nine in the paper[1] by Black and Cox. Using these prices we easily compute equity value as $E_t = V_t - D_t^J - D_t^S$.

4 Contingent Capital

4.1 Introduction

Section 1 presented the Merton model which gave us a very simple way to look at a firm's capital structure. The Black-Cox model was then introduced which allowed a company to default when a predetermined barrier was breached. One thing that both these models still lack is the ability for a company to protect itself from defaulting on its obligations. Contingent capital is debt that converts to equity when a predefined conversion barrier is breached. It acts as a buffer during emergency situations by converting debt to equity and thereby reducing the leverage ratio. This section aims to add contingent capital to both the Merton model and the Black-Cox model in order to further increase the flexibility of the company's capital structure.

4.2 Definitions & Assumptions

There are two items related to contingent capital that are important to note. The first is the idea of the conversion *trigger*. The trigger is the event at which the conversion process begins. The second is the conversion *rate*. The rate describes how the debt should be exchanged for equity. In other words it could state how many shares of equity will be received in exchange for the debt in the event of conversion. It is also interesting to note that in the event of conversion, new shares may be issued which will dilute the voting rights of the preexisting equity holders. This is something that a company must keep in mind when determining the conversion rate on their contingent capital. Too high of a rate may depress the value of the equity as the possibility of dilution would be factored into the market price.

For reasons that will become apparent in the following subsections we define a new process

$$v_t = \frac{V_t e^{r(T-t)}}{L}$$

as the leverage ratio. This process is an increasing function of the equity value which makes it desirable when working with hitting a barrier from above.

We will choose the conversion trigger to be when the firm's leverage ratio falls below some barrier C. The conversion rate will be $1:1$ debt for equity upon conversion for the sake of simplicity (although it would not be hard to choose a different rate).

An assumption that is important to make is that the firm will evolve in the same fashion before and after conversion if it were to occur. That is to say that all the parameters in the original stochastic process governing this firm stay constant, none of them will depend on any intermediate change in the capital structure.

Finally it should be clear that a poorly chosen conversion trigger *or* rate may result in a capital structure that is 'worse' than if there was no contingent capital at all. For example in the extreme case that the conversion rate for equity to debt tends to infinity we would expect the current equity value to go to zero since in the event of conversion the preexisting equity holders would retain zero percent of the firm.

4.3 Merton Model Revisited

For both of the following figures the firm starts with $L = 15$, where 5 of those dollars are contingent capital. In the event of conversion equity increases by 5 and the new debt payable at time T becomes 10.

Looking at Figure 17 we have the leverage ratio of the firm over two sample paths graphed. The conversion barrier C is also labeled as a dashed line. The first time the process hits the barrier (if at all) causes the firm to retire its contingent debt and issue new shares to compensate those debt holders. This has the effect of increasing the leverage ratio (from how we defined it) and then the firm continues to evolve as it would have otherwise. Also notice that the red path hits the conversion barrier a second time and nothing happens. That is because the contingent capital trigger was defined to entirely convert the first time the barrier was breached.

Figure 18 shows the same two paths that Figure 17 does, but this time it shows the value of the firm instead of its leverage ratio. It has an equivalent barrier C^* marked with a dotted line. Notice that nothing happens to the value of the firm when it crosses the conversion barrier since the rate at

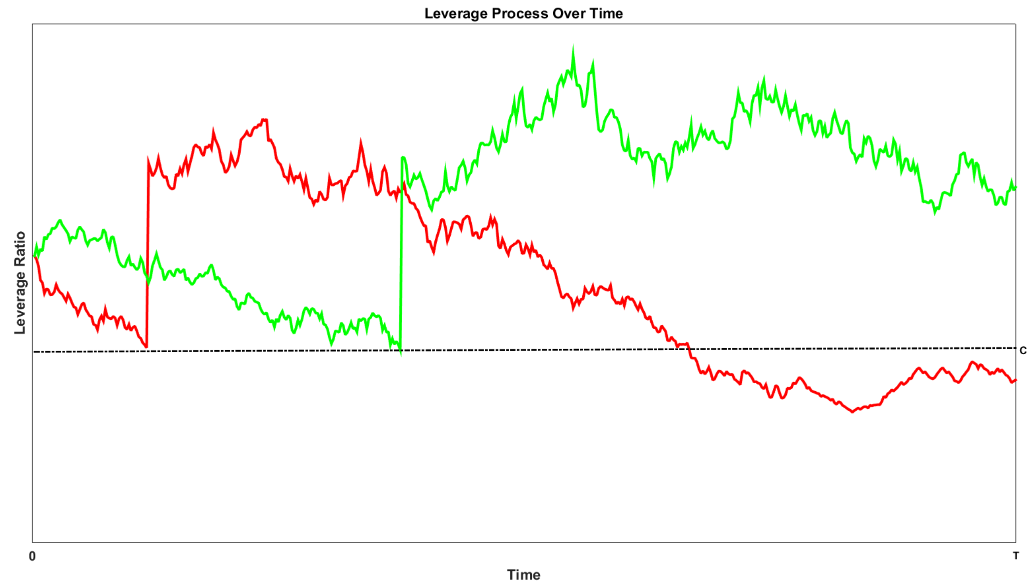

Figure 17: Leverage ratio under Merton model with contingent capital.
$V_0 = 20,\ \sigma_V = 0.2,\ L = 15,\ C = 1.5e^{-r(T-t)},\ r = 0.05$

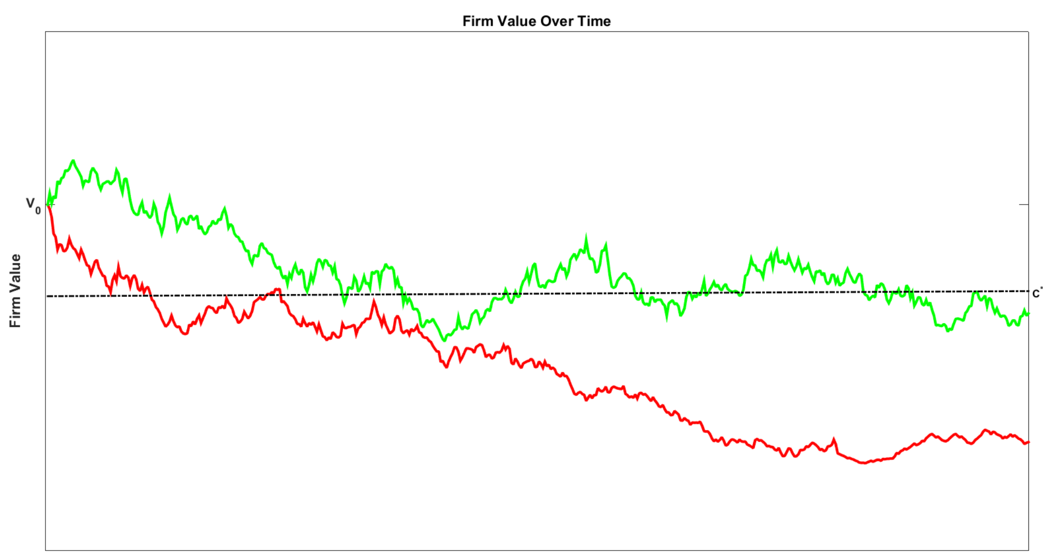

Figure 18: Firm value under Merton model with contingent capital.
$V_0 = 20,\ \sigma_V = 0.2,\ L = 15,\ C^* = 15e^{-r(T-t)},\ r = 0.05$

which we are swapping debt for equity is one-to-one. Then we can say that introducing contingent capital into the Merton model with our choice of conversion trigger and rate has the effect of protecting a firm from defaulting at the expense of equity holders voting rights.

4.4 Black-Cox Model Revisited

Under the Merton model with contingent capital the conversion barrier had the effect of protecting the bond holders by reducing the total amount of debt owed at time T. The Black-Cox model with contingent capital goes one step further by also having the default barrier as a second layer of protection.

For both of the following figures the firm starts with $L = 15$, where 5 of those dollars are contingent capital. In the event of conversion equity increases by 5 and the new debt payable at time T becomes 10. We also define the barrier $B^* = 10e^{-r(T-t)}$ as the discounted value of the debt (excluding contingent capital).

Looking at Figure 19 we have the leverage ratio of the firm over two sample paths graphed. C is again the conversion barrier and now additionally we have the default barrier B. Looking at the green path we see the conversion barrier 'successfully' pushes the leverage ratio far enough away from B in order to avoid default over its lifetime. However with the red path we can see that even with the conversion the path eventually drifts down and hits the default barrier. At that point the process stops and the firm is liquidated. Equity holders (which now includes the contingent capital holders) are entirely wiped out and the bond holders receive the 'fair' value of their promised payment. The dashed red line after the killed process is provided as reference to show what the path would have done had it been under the Merton model. We can see in this scenario it would have rebounded leaving some residual claim for the equity holders.

Finally we turn our attention to Figure 20 where have the same paths as Figure 19 converted to their corresponding firm value. We again see the conversion barrier has no effect on the firm value, but we notice the default barrier does. This makes sense as the conversion barrier is swapping one-to-one while the default barrier is liquidating the company and therefore removing its value as a firm.

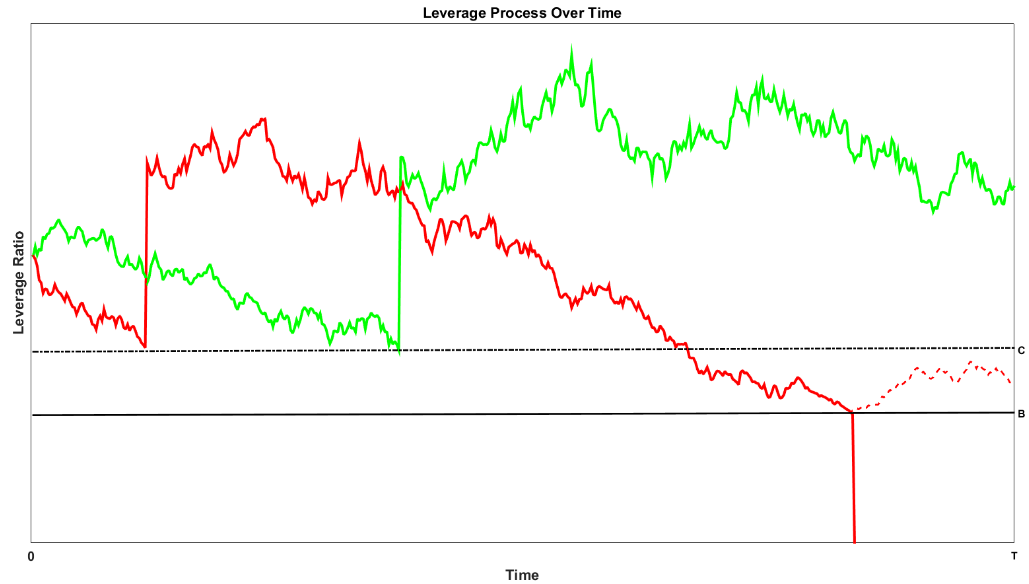

Figure 19: Leverage ratio under Black-Cox model with contingent capital.
$V_0 = 20,\ \sigma_V = 0.2,\ L = 15,\ C = 1.5e^{-r(T-t)},\ B = 1e^{-r(T-t)},\ r = 0.05$

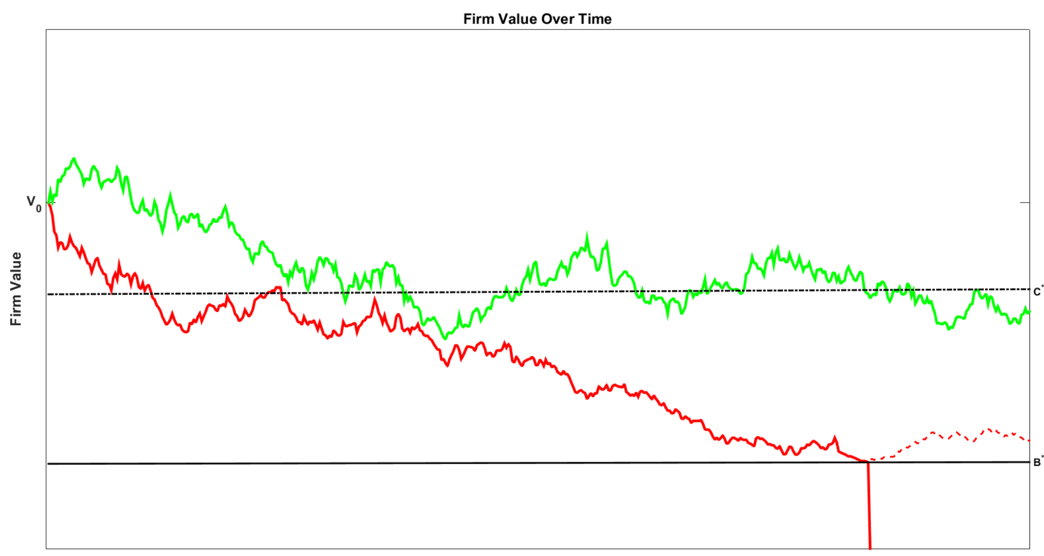

Figure 20: Firm value under Black-Cox model with contingent capital.
$V_0 = 20,\ \sigma_V = 0.2,\ L = 15,\ C^* = 15e^{-r(T-t)},\ B^* = 10e^{-r(T-t)},\ r = 0.05$

5 Conclusion

A firm's capital structure dictates how the firm will finance its operations and growth. The Merton model and Black-Cox model provided an introduction on how to determine this firm's capital structure. We determined that a firm's value could vary drastically between the two models and that the choice of default barrier is very much up to the discretion of the firm's management. Contingent capital was introduced as a tool to use in emergency situations by providing liquidity when it is most needed. Models are always being modified and extended and it is no different for these capital structure models. This report served as a good starting point to determine many of the critical assumptions that underlie these models and their shortcomings.

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