Ideal Magnetohydrodynamics and the Dimensionless Form

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1. The Basic Equations

Ideal MHD describes the motion of a perfectly conducting fluid interacting with a magnetic field. Hence, we need to combine Maxwell's equations with the equations of fluid dynamics and provide interaction equations.

1.1. *Maxwell's Equations*

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \cdot \vec{E} = \frac{\tau}{\epsilon_0} \tag{1.1}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \qquad \vec{\nabla} \cdot \vec{B} = 0 \tag{1.2}$$

1.2. *Fluid Equations*

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{v} = \frac{\partial \rho}{\partial t} + \vec{v}\cdot\vec{\nabla}(\rho) + \rho\vec{\nabla}\cdot\vec{v} = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{v} \tag{1.3}$$

$$\frac{DP}{Dt} + \gamma P\vec{\nabla}\cdot\vec{v} = \frac{\partial P}{\partial t} + \vec{v}\cdot\vec{\nabla}(P) + \gamma P\vec{\nabla}\cdot\vec{v} = 0 \tag{1.4}$$

Note that we have introduced a distinct time-derivative operator,

$$\frac{D()}{Dt} \equiv \frac{\partial()}{\partial t} + \vec{v}\cdot\vec{\nabla}().$$

The operator takes many names, but it can be seen as the total time derivative of fields dependent on $\vec{r}$ and t. The Lagrangian time-derivative, the material derivative, and the convective derivative are some names for the operator, which correspond to the interpretation of the operator.‡ The interaction between the two sets of equations comes into the equation of motion for a fluid element,

$$\vec{F} = \rho\frac{D\vec{v}}{Dt} = -\vec{\nabla}(P) + \rho\vec{g} + \vec{j}\times\vec{B} + \tau\vec{E}. \tag{1.5}$$

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‡ **Lagrangian time-derivative**: The name references the Lagrangian representation of the fluid. The Lagrangian representation follows the dynamics of a single fluid element. In contrast, the other fluid representation is called the Eulerian representation which looks at the dynamics of the fluid-field.
Convective time-derivative: The name references the second-term of the operator, the flow convects the quantity.

Another equation is needed, the equation for an electric field in a perfectly conducting nonrelativistic moving medium,

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B} = 0. \tag{1.6}$$

Equation (1.6) expresses that the electric field in a co-moving frame vanishes, which is expected for a perfect conductor.

1.3. *Relative Strengths*

In the nonrelativistic regime there are terms that are irrelevant. This can be seen through dimensional analysis‡. In dimensional analysis various operators are represented as characteristic scales, for example; $|\vec{\nabla}| \sim 1/l_0$, $\partial/\partial t \sim 1/t_0$, where l_0 and t_0 represent a characteristic length and a characteristic time. We can use equation (1.1a) as an example

$$\begin{aligned} |\vec{\nabla} \times \vec{E}| &= \left|\frac{\partial \vec{B}}{\partial t}\right| \\ |\vec{\nabla} \times \vec{E}| \sim \frac{E}{l_0} \quad &, \quad \left|\frac{\partial \vec{B}}{\partial t}\right| \sim \frac{B}{t_0}, \\ E &\sim \frac{l_0}{t_0} B = v_0 B, \ \textit{where } v_0 \equiv \frac{l_0}{t_0}; \end{aligned} \tag{1.7}$$

here we look at the scales on both sides of the equation and resolve the scale of a quantity, in our case we resolve the scale of the electric field. We can continue through the equations. In Ampere's equation we find that the displacement current is negligible,

$$\frac{1}{c^2}\left|\frac{\partial \vec{E}}{\partial t}\right| \sim \frac{v_0^2}{c^2}\frac{B}{l_0} \ll |\vec{\nabla} \times \vec{B}| \sim \frac{B}{l_0},$$

again we are in the nonrelativistic regime meaning the speeds associated with the system are much-much less that than the speed of light. This means that in a nonrelativistic plasma Ampere's law holds,

$$\vec{j} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B}. \tag{1.8}$$

Scaling Coulomb's law results in

$$\tau \sim \frac{\epsilon_0}{l_0} E. \tag{1.9}$$

So far we have not investigated scales of any other sets of equations except Maxwell's set. Let us now look at the equation of motion Eq.(1.5), in particular the Lorentz force. The electric force has a scale of

$$\tau |\vec{E}| \sim \frac{v_0^2}{c^2} \frac{\mu_0 B^2}{l_0}, \tag{1.10}$$

while the magnetic force has a scale of

$$|\vec{j} \times \vec{B}| \sim \frac{\mu_0 B^2}{l_0}. \tag{1.11}$$

The important point here is the relative strengths of the two forces; the electric force is much-much weaker than the magnetic force. This means that in nonrelativistic plasmas the electric force is ignorable.

‡ dimensional analysis is also known as order-of-magnitude analysis.

In summery, through dimensional analysis it is seen that the electric force is irrelevant to the dynamics of plasmas and electric induction effects are negligible.

1.4. *Quasi-neutrality*

Plasmas by definition have a property called quasi-neutrality. The property states that the difference between the number densities of negatively-charge and positively-charged constituents is negligible. A direct consequence of this property is $\tau \sim 0$, i.e. the charge density is about zero. This implies that in plasmas the electric field is entirely induced by fluctuating magnetic fields.

1.5. *Basic Set of Equations*

Combining the previous sections we find a basic set of equations describing ideal MHD. The set of eight nonlinear partial differential equations for the eight variables $\rho(\vec{r},t),\ \vec{v}(\vec{r},t),\ P(\vec{r},t),\ \&\ \vec{B}(\vec{r},t)$,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{1.12}$$

$$\rho(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla}(\vec{v})) + \vec{\nabla}(P) - \rho \vec{g} - \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) \times \vec{B} = 0 \tag{1.13}$$

$$\frac{\partial P}{\partial t} + \vec{v} \cdot \vec{\nabla}(P) + \gamma P \vec{\nabla} \cdot \vec{v} = 0 \tag{1.14}$$

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{B}) = 0 \quad , \quad \vec{\nabla} \cdot \vec{B} = 0. \tag{1.15}$$