PPPL Turbulence Lecture Notes

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This series of notes corresponds to a series of graduate plasma physics lectures given by Dr. Jason TenBarge at the Princeton Plasma Physics Laboratory. The current goal of this document is to grow into a large reference for the graduate students at the Bryn Mawr Plasma Laboratory.

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1. Introduction

The theory of magnetohydrodynamic (MHD) turbulence starts with the Kolmogorov 1941(a,b) theory, known as K41. This is marked as the first major breakthrough in the theory of incompressible turbulence predicting an isotropic power law energy spectrum $E(k) \sim k^{-5/3}$.

The K41 theory is based on non-magnetized turbulence and attempts to describe magnetic incompressible turbulence were made by Iroshnikov and Kraichnan (IK model). Their model predicts an energy spectrum with $k^{-3/2}$ scaling for both velocity and magnetic field. However, their model is isotropic even though the magnetic field forces the system to be anisotropic.

A similar theory to IK was proposed by Goldreich and Sridhar (hereinafter GS95) for incompressible MHD turbulence. It predicts a Kolmogorov-like energy spectra $E_{k_\perp} \sim k_\perp^{-5/3}$ in terms of wave-vector component $k_\perp$ which is perpendicular to the local direction of magnetic field. The parallel component of the wave-vector $k_{||} \sim k_\perp^{2/3}$.

This series of notes follows the incompressible development of MHD turbulent theory. K41 is reviewed and then, at some later time, IK and GS theories are introduced.

2. Introduction to Turbulence and K41

Turbulence is chaotic flow regime characterized by diffusivity, rotationality, and dissipation. To start we will look at the most basic of equations the Euler equations.

2.1. ***Equations of Hydrodynamic Turbulence*** *(Euler equations)*

(i) Incompressibility: $\rho = constant \implies \vec{\nabla} \cdot \vec{u}$
(ii) Navier-Stokes equation (NS):

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla}(\frac{P}{\rho}) + \nu \nabla^2 \vec{u} + \vec{f} \tag{2.1}$$

(iii) Energy:

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot [\vec{u}(E + P)] = 0; \;\; E = \rho u^2/2 + \rho l$$

One can assume **incompressibility** when the speed of sound in the medium is much faster than the phenomenon of interest. The Navier-Stokes equation can be thought as stating temporal change and convection of the flow are due to thermal, viscous, and external forces.

2.2. *Parameters of System Pertaining to Turbulence*

Essentially we will be defining characteristic scales of the turbulence system.
(i) characteristic velocity (outer scale); u_o
(ii) characteristic length (outer scale); L
(iii) viscosity, ν

The two "characteristic" scales are associated with external forces; while the viscosity is set by molecular properties of the fluid. Okay, let us look at the NS and compare the convection and the viscous term

$$\frac{\text{convection}}{\text{viscous}} \sim \frac{u_o^2/L}{\nu u_o/L^2} = \frac{u_o L}{\nu}$$

The result of the comparison is a characteristic parameter of fluids, the Reynolds number

$$R_e = \frac{u_o L}{\nu} \tag{2.2}$$

2.3. *Phenomenological Picture of Turbulence*

At every point in the fluid, the velocity is fluctuating around its mean value, $\vec{u}_o$,

$$\vec{u} = \vec{u}_o + \delta\vec{u}.$$

We can transform the mean field and at the outer-scale $\delta\vec{u}_o \sim \delta\vec{u}_L$. The important point here is that there is a large scale fluctuation. At this point we can redefine the Reynolds number with the large scale fluctuation

$$R_e = \frac{\delta\vec{u}_L L}{\nu} \tag{2.3}$$

What happens to the energy of the system?

$$E = \int d\vec{x}|u^2|$$

Recall the NS Eq:(2.1),

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}(\frac{P}{\rho}) + \nu\nabla^2\vec{u} + \vec{f},$$

If we were to take the integral of the inner product between the NS equation and the flow we would get the following,

$$\int d\vec{x}(NS)\cdot\vec{u} = \frac{dE}{dt},$$
$$\frac{dE}{dt} = \nu\int d\vec{x}(\vec{u}\cdot\nabla^2\vec{u}) + \int d\vec{x}(\vec{u}\cdot\vec{f})$$

One can think of the resulting equation as the change rate of the energy is equal to the viscous dissipation, the first term on the LHS, and the rate of energy injection†, the second term on the LHS.

† The external force includes the thermal forces

If our system is in a stationary state‡,

$$\frac{dE}{dt} = 0 = \nu \int d\vec{x} \langle \vec{u} \cdot \nabla^2 \vec{u} \rangle + \int d\vec{x} \langle \vec{u} \cdot \vec{f} \rangle,$$
$$V\epsilon \equiv \int d\vec{x} \langle \vec{u} \cdot \vec{f} \rangle,$$
$$V\epsilon = -\nu \int d\vec{x} \langle \vec{u} \cdot \nabla^2 \vec{u} \rangle.$$

In a steady state, the energy input rate, ϵ, must match the dissipation rate.
To move forward we are going to construct estimates of various quantities based on dimensional analysis.

(i) At the outer-scale: U_o, L

(ii) At smaller scales the rms velocity u_l at scale l is used: u_l, l

The eddy turnover time, also known as the nonlinear time, is the characteristic time for a structure of size l to undergo a significant distortion due to the relative motion of its components. The time scale is

$$t_l \sim \frac{l}{u_l},$$

while at the outer-scale,

$$t_L \sim \frac{L}{u_L}.$$

The energy injection rate is,

$$\epsilon \sim \frac{u_o^3}{L} \sim \frac{u_o^2}{t_L}.$$

Rearranging the terms we can cast outer-scale eddy turnover time as

$$t_L \sim \epsilon^{-1/3} L^{2/3}. \tag{2.4}$$

We can also setup a dissipation time scale:

$$t_l^{(\mathrm{diss})} \sim \frac{l_\nu^2}{\nu}. \tag{2.5}$$

At the viscous scale we assume that the eddy turnover time is equal to the dissipation time,

$$t_l = t_\nu.$$

Which we can say,

$$l_\nu \sim (\frac{\nu^3}{\epsilon})^{1/4} \sim L Re^{-3/4} \ll L,$$

the dissipation scale is much less than the outer-scale. This dissipation scale,

$$l_\nu \sim L Re^{-3/4}, \tag{2.6}$$

takes many names; viscous scale, inner-scale, and Kolmogorov scale. The picture that we are trying to build is the energy cascade. At the outer-scale energy is injected into the chaotic fluctuations of the fluid velocity and is transported to lower and lower scales traversing the inertial range. The energy cascade continues until ultimately dissipating at the dissipation scale. The inertial range is the a range of scales that much less than

‡ Formally we are considering the ensemble average $\langle ... \rangle$

the outer-scale while at the same time much greater than the dissipation scale,

$$L \gg l \gg l_\nu \sim LRe^{-3/4}.$$

2.4. *Observations to Constrain the Theory*

2/3 law The mean square velocity increments, $\langle \delta \vec{u}_l^2 \rangle$, between two points separated by l scales as

$$\langle \delta u_l^2 \rangle \sim l^{2/3}.$$

Finite energy dissipation The energy dissipation is always positive and finite. In a series of publications in 1941 Kolmolgorov proposed the following axioms

(i) **Universality** The turbulence, intertial range, is independent of the particular energy injection and dissipation.

(ii) **Locality of Eddy Interaction**

(iii) **Homogeneity** No special points†

(iv) **Isotropy** No special directions

(v) **Scale Invariance** No special scales‡

The 2/3 law is easy to see if we consider the energy injection rate at some scale l,

$$\epsilon \sim \frac{\delta u_l^3}{l}.$$

This would imply the following about the fluctuating flow

$$\delta u_l \sim (\epsilon l)^{1/3}, \tag{2.7}$$

or,

$$\delta u_l^2 \sim (\epsilon)^{2/3} l^{2/3}. \tag{2.8}$$

Equation (2.7) is known as the Kolmogorov-Obukhov law and equation (2.8) is known as the 2/3 law. The 2/3 law is a statement of energy in fluctuations at scale l.

† no intermittency
‡ constant cascade rate, ϵ