PPPL Grad Summer School 8/10/2020

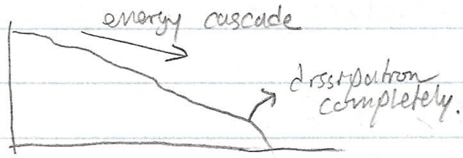
Introduction)

Mohammad Barani → MMS spacecraft related to paper we read.

Turbulence Introduction)

- For fusion, turbulence is bad, allows for transport of energy, momentum, mass, and disrupt confinement.
- Turbulence may play fundamental role in heating the corona → coronal heating problem.
  - Accelerates the solar wind in the corona.
- Energy injected @ large scale, forms one eddy; dissipates into smaller and smaller eddies (cascades) until dissipated into heat.
  - Would take ~month for creamer to diffuse totally if creamer didn't have a velocity.

(*) • Energy Spectrum →



Equations of Hydrodynamic Turbulence (Euler Eqns)

1) Incompressibility, $\rho \equiv$ constant, $M_a = u/c_s << 1$

$$\nabla \cdot \vec{u} = 0$$

2) Navier-Stokes (NS)

$$\underbrace{\frac{d\vec{u}}{dt}} = \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\vec{u} + \vec{f}$$

non linear forces. (convective) — thermal forces — viscosity — external forces.

3) Energy: $\frac{\partial E}{\partial t} + \nabla\cdot\left[\vec{u}(E+p)\right] = 0$

⇒

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$\delta \to$ Fluctuating

Turbulence I)

Characteristic Parameters from Eqns)

- characteristic velocity: δu_L
- characteristic length: L

} Fixed by the (External) forcing

- viscosity, ν → set by molecular scale

The length parameter → Just outside where turbulence begins!

L ⇒ outer scale, inertial scale, auto-correlation, energy containing scale.

$$\frac{\text{Convection}}{\text{viscous}} \approx \frac{\delta u_L^2 / L}{\nu \cdot \delta u_L / L^2} = \frac{\delta u_L \cdot L}{\nu} = Re \equiv \text{Reynolds } \#$$

Re small (≤ 100) ⇒ laminar (paint flow, slow faucett)

Re large (> 100) ⇒ Turbulent flow. (fast faucett)

Phenomenological Picture of Turbulence

Velocity

$$\vec{u} = \langle \vec{u} \rangle + \delta \vec{u}$$

↑ mean flow ⇒ dont care about it (remove mean flow from river, left w/ turbulence)

Energy

$$E = \frac{1}{2} \int d\vec{x} \, |\vec{u}|^2$$

, and $\vec{u} \cdot$ (Navier-Stokes)

Convective Derivative:

$$\frac{dE}{dt} = \underbrace{\nu \int d\vec{x} \, \vec{u} \cdot \nabla^2 \vec{u}}_{\text{Viscous dissipation}} + \underbrace{\int d\vec{x} \, \vec{u} \cdot \vec{f}}_{\text{Rate of energy injection}}$$

$$\int d\vec{x} \, \vec{u} \cdot \left(\frac{p}{\rho}\right) = \int d\vec{x} \, \vec{\nabla} \cdot \left(\vec{u} \cdot \frac{p}{\rho}\right) = \int d\hat{A} \cdot \vec{u} \frac{p}{\rho} = 0$$

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Turbulence I)

Consider a stationary state (formally → consider ensemble average)

$$\frac{d\langle E\rangle}{dt} = 0 = \nu\int d\vec{x}\,\langle \vec{u}\cdot\nabla^2\vec{u}\rangle + \underbrace{\int dx\,\langle \vec{u}\cdot\vec{f}\rangle}_{\text{4-Volume}} = \underset{\uparrow \text{ energy input rate}}{\varepsilon}$$

$$\Rightarrow \varepsilon V = -\nu\int d\vec{x}\,\langle \vec{u}\cdot\nabla^2\vec{u}\rangle$$

Parameters now → $\ell,\ \delta u_\ell,\ L,\ \delta u_L,\ \nu$

Eddy-turnover time : $t_\ell \approx \frac{\ell}{\delta u_\ell}$, then @ outer scale $\Rightarrow t_L \sim \frac{L}{\delta u_L}$

Energy injection rate : $\varepsilon \sim \frac{\delta u_L^2}{t_L} \sim \frac{\delta u_L^3}{L}$, then $\boxed{t_L \sim \varepsilon^{-1/3} L^{2/3}}$ ①

Dissipation time scale : $\boxed{t_\ell^{diss} \sim \frac{\ell}{\nu}}$ ②

Equating ① and ② : $\nu = \frac{\delta u_L \cdot L}{Re}$ $\Rightarrow$ Need viscous scale ;

$$\boxed{\ell_\nu \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \sim L\,Re^{-3/4} \ll L}$$

↳ $\ell_\nu \equiv$ dissipation scale, viscous scale, inner scale, Kolmogorov Scale.
↳ scale @ which viscosity becomes important.

Basic Picture

energy injection → Energy Transport → Energy dissipation

$$L \gg \underbrace{\ \ell\ }_{\text{Inertial Range}} \gg \ell_\nu \sim L\,Re^{-3/4}$$

✱Cascade ⟶

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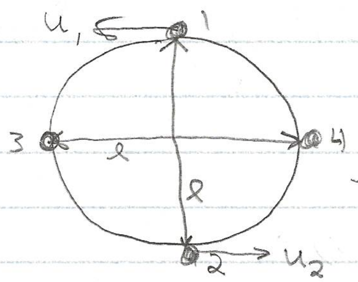
Turbulence I)

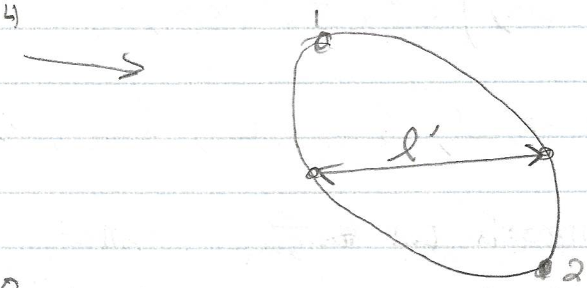
Look @ Eddy turnover time:

$t_\ell \sim \ell / \delta u_\ell$



$t_\ell \sim \frac{\ell}{|u_1 - u_2|} \sim \frac{\ell}{\delta u_\ell}$

↑ velocity increment



We assume $\nabla \cdot u = 0$,

so the area above is conserved!

- Energy is transported via deformation of the eddies!
  - ⇒ They don't actually break down like the cartoon suggests.

Eddies break down fractally; $\ell \to \frac{\ell}{2} \to \frac{\ell}{4} \to$

Turbulence II)

Kolmogorov (1941) Turbulence Theory

Observational facts

1) 2/3 Law $\langle \delta u_\ell^2 \rangle \sim \ell^{2/3}$
2) Finite energy dissipation.

Step back to Richardson (1922)

⇒ Energy transfer is local ($\ell \to \ell/2$, NOT $\ell \to \ell/16$)

⟹

Turbulence II)

1941) Kolmogorov Proposal)

0) Universality: Turbulence is independent of forcing and dissipation

1) Locality of interactions (in scaling, not like in space locality)

2) Homogeneity: No special points.

3) Isotropy: No special directions.

*4) Scale Invariance: Constant cascade rate, ϵ

$$\epsilon = \text{const} \simeq \frac{\delta u_L^3}{L} \simeq \frac{\delta u_\ell^3}{\ell}$$

$\therefore \ \delta u_\ell \cong (\epsilon \ell)^{1/3}$, or $\delta u_\ell^2 \cong \epsilon^{2/3} \ell^{2/3}$, so we have the $2/3$ law! $\delta u_\ell \sim (\epsilon \ell)^{2/3}$ Kolmogorov – Obukhov Law

But energy spectra have $5/3$ scaling $\Rightarrow$?

Energy Spectra

$$E = \int d\vec{k}\, E^{(3)}(\vec{k}), \quad E^{(3)}(\vec{k}) \Rightarrow \text{3D energy spectrum}$$

$\Rightarrow$ spherical coords: $\iiint dk\, d\theta\, d\phi\, k^2 \sin\theta\, E^{(3)}(\vec{k}) \sim \int dk\, k^2 E^{(3)}(\vec{k})$

$=: \int dk\, E^{(1)}(k)$, $E^{(1)}(k) \Rightarrow$ 1D E spectrum, where

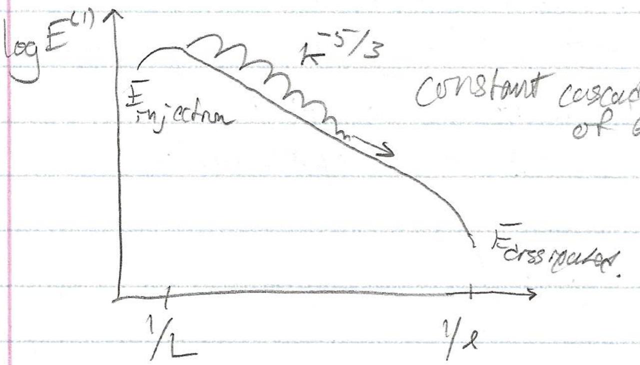
$$E = \delta u_\ell^2 \sim \epsilon^{2/3} \ell^{2/3} \sim \epsilon^{2/3} k^{-2/3} \sim \int_0^k dk'\, k'^2 E^{(3)}(k')$$

$$= \int_0^k dk'\, E^{(1)}(k') \sim k E^{(1)}(k), \quad \text{and} \quad \boxed{E^{(1)}(k) \simeq \epsilon^{2/3} k^{-5/3}}, \text{ and}$$

$5/3$ energy spectra!

$\Longrightarrow$

Turbulence II)



What about the limits outside injection/dissipation?

What about

1) $\ell < \ell_\nu \Rightarrow \epsilon \sim \nu \vec{u} \cdot \nabla^2 \vec{u} \sim \nu \dfrac{\delta u_\ell^2}{\ell^2}$,

$\delta u_\ell \sim \left(\dfrac{\epsilon}{\nu}\right)^{1/2} \ell \Rightarrow \boxed{E^{(1)} \sim k^{-3}}$

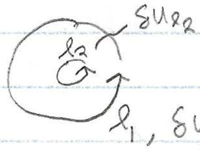
2) $\ell > L$; $E = \delta u_L^2 = \text{const.} \Rightarrow \boxed{E^{(1)} \simeq k^{-1}}$

We know that most energy is @ the largest length scales.

- $E \sim \ell^{2/3} \Rightarrow$ Dominated by large scales.
- Gradients $\sim \nabla \delta u \sim \dfrac{\delta u_\ell}{\ell} \sim \ell^{-2/3}$, dominated by small scales.
- Cascade time $t_\ell \sim \dfrac{\ell}{\delta u_\ell} \sim \ell^{2/3}$, decreases with scale

"Verify" that cascade is local

1) Can large scale motion shear apart small eddies?



Shearing time $\sim t_s \sim \dfrac{\ell}{\delta u_{\ell_1}}$

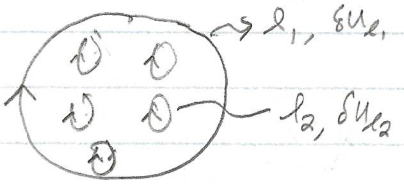
Cascade time for ℓ_2, $t_{\ell_2} \sim \dfrac{\ell_2}{\delta u_{\ell_2}}$

$t_s / t_{\ell_2} \sim \left(\dfrac{\ell_1}{\ell_2}\right)^{2/3} >> 1$, shearing @ large scales not important.

$\Longrightarrow$

Turbulence II)

2) Can small scale eddies diffuse large eddies?



Diffusion due to l_2 eddies

$$D = \frac{l_2^2}{t_{l_2}} \sim l_2\, \delta u_{l_2}$$

Time to diffusion, $t_D = \frac{l_1^2}{t_D} \sim \frac{l_1^2}{l_2\, \delta u_{l_2}}$; ratio

$t_D / t_{l_1} \sim \left(\frac{l_1}{l_2}\right)^{4/3} >> 1 \Rightarrow$ Diffusion is unimportant,

Therefore the energy cascade is local in scale!

(*) (*) • Jason TenBarge has semester worth of notes on hydroturbulence, ask if we want!

Introduction to Plasma Turbulence → start w/ new assumptions

Keep 2) I) Scale Invariance

II) Locality of Energy Transfer, $\epsilon \sim$ constant $\sim \frac{\delta u_l^3}{l}$

We had, for hydro turbulence, a time scale $\Rightarrow \tau_l \sim l/\delta u_l$ only time-scale.
But in plasma there are more (Alfvén modes, fast modes, etc).

Incompressible MHD Equations)

$$\nabla \cdot \vec{u} = 0$$

$$① \quad \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla\left(\frac{P}{\rho}\right) + \frac{\vec{B} \cdot \nabla \vec{B}}{4\pi\rho} + \nu \nabla^2 \vec{u}$$

$$② \quad \partial_t \vec{B} + \vec{u} \cdot \nabla \vec{B} = \vec{B} \cdot \nabla \vec{u} + \eta \nabla^2 \vec{B}$$

↑ resistivity

$$P = p + \frac{B^2}{8\pi}$$

Assume mean magnetic field $\vec{B_0} = B_0 \hat{z}$
Breaks isotropy assumption, so we have

III Anisotropy relative to B_0 w/ Alfvén waves III

Turbulence II)

Elsasser (1950) form from ①+② Not going to write in.

Important Notes about eqn ③)

1) For $\vec{z}^- = 0 \Rightarrow \vec{z}^+ = f(x, y, z + V_A t)$ ← Alfvèn speed, likewise

for $\vec{z}^+ = 0 \Rightarrow \vec{z}^- = f(x, y, z - V_A t)$

2) System supports two linear wave modes. Both have $\omega^2 = k_{\parallel}^2 V_A^2$, $k_{\parallel} = \vec{k} \cdot \hat{b}_0$

a) Alfvèn wave is polarized in the $\hat{z} \times \hat{k}$ direction.

b) Pseudo Alfvèn waves (Incompressible limit of slow modes), polarized in $\hat{k} \times (\hat{z} \times \hat{k})$ direction.

- Fast modes ordered out due to incompressible assumption.

3) Elsasser eqn ③ is a closed system;

$$\nabla \cdot ③ \rightarrow \nabla^2 (P/\rho) = -\nabla \cdot (z^{\mp} \cdot \nabla z^{\pm})$$

4) The nonlinear term, $\vec{z}^- \cdot \nabla \vec{z}^+$, requires oppositely propagating Alfvèn waves, and need to be relatively polarized
↳ in $\hat{x}$, the other can't be polarized in $\hat{x}$

Enstropy → Causes the energy flow to be from large scale down the energy cascade.

Turbulence III) Toroidal magnetized plasma turbulence + transport

- Turbulence is unpredictable, yet deterministic.
  - Not a property of fluid/plasma ⇒ feature of the flow.
- Turbulence spans wide range of spatial / temporal scales.
  - For fusion plasmas (hot low-collisional) wide range of scales in 6D phase space.

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Turbulence III)

Turbulent transport ad vective.

- Transport the result of 2nd order correlations btwn perturbed drift velocity, and perturbed fluid moments.

Ions (10^7), electrons (10^{10}) => gyro-radii in fusion plasmas.

Toroidicity leads to vertical drifts from ∇B and curvature.

→ Opposite effects for ions/e^-, then we get an $\vec{E}\times\vec{B}$ drift that effects everything in confinement.

• Solved by an applied field thats twists (introduce helicity into system)
↳ guide

Turbulence Characteristics in Tokamaks

• Spectroscopic imaging provides 2D picture of turbulence; cm spatial scales, μs time scales, <1% amplitude.

↳ 100 kHz frequency scales.

Transport is essential to understand for fusion reactors → hot center to cool edge.

- Transport is of order of Gyrobohm diffusivity.
  - Bohm uncovered transport scaling that goes $\approx \frac{1}{16}\frac{T_e}{B}$ = Bohm Diffusivity.
    → Low transport, high energy for large magnetic field ↑ diffusivity small!

Onset of turbulence has a critical threshold! GyroBohm scaling is important, but this linear threshold and scaling also matter!

- Want to stay below the threshold, can tweak knobs to maximize confinement time.
  + What ~~sets~~ sets the threshold.
    ⇒ Gyrokinetics in 5D, w/ gyro-rings.

We also need to solve gyrokinetic Maxwell equations self-consistently.

→

Turbulence III)

What are Drift Waves?

- Finite-frequency drifting waves driven by temp gradients, and density gradients.

Easiest way to characterize ⇒ treat as fluid, make assumptions.

+ consider uniform $\vec{B}$ field, only care about density, electrostatic, cool ions, no toroidicity. Very simple.

We can get the electron diamagnetic drift velocity and frequency

ω_{*e} and v_{*e}.

After everything, we get that the electrons follow a Boltzmann response to perturbation.

Ion continuity + Quasi neutrality + Boltzmann electron = electron drift wave

- Toroidicity leads to inhomogeneity in $|\vec{B}|$, gives $\nabla\vec{B}$ and κ drifts

Background temp gradient reinforce perturbation, lead to instability.

- Rayleigh-Taylor type instabilities arise in tokamaks (similar to two mixing fluids on top of each other, different density) → in presence of gravity.

Large scale flow shear (background) dramatically limits turbulence @ boundaries

Turbulence IV

Alfvèn waves need to be relatively polarized.

$$\vec{z}^{\pm} = z^{\pm} e^{i(\vec{k}^{\pm}\cdot\vec{x}-\omega t)}(\hat{z}\times\hat{k}^{\pm})$$

$$i k^{+} z^{-} z^{+}\left[(\hat{z}\times\hat{k}^{-})\cdot\hat{k}^{+}\right](\hat{z}\times\hat{k}^{+}) = i k^{+} z^{-} z^{+}(\hat{z}\times k^{+})\left[\hat{z}\cdot\underbrace{(\hat{k}^{-}\times\hat{k}^{+})}\right]$$

Waves must have nonzero relative polarization or this cross product ⇒ 0

Turbulence III)

Alfvènic fluctuations, and $\delta u \approx \delta B$ scale-by-scale IV

So now, can we get a Kolmogorov spectra, like w/ fluid dynamics.

$$\epsilon \sim \frac{\delta u_\ell^2}{\tau_\ell} \sim \text{constant}$$ is still ok, but we have 2 time scales:

- Eddy turnover time $\tau_{eddy} \sim \frac{\ell_\perp}{\delta u_\ell}$ and
- Alfvèn time $\tau_A \sim \frac{\ell_{\parallel}}{V_A}$

Iroshnikov (1964) - Kraichnan (1965) Theory.

If further assumed weak turbulence V

$$\frac{z^\mp \cdot \nabla_\perp z^\pm}{V_A \cdot \nabla_\parallel z^\pm} \sim \frac{z^\mp k_\perp}{V_A k_\parallel} \sim \frac{\delta u_\perp k_\perp}{V_A k_\parallel} \sim \frac{t_A}{t_{eddy}} =: \chi$$

Nonlinearity parameter (χ)

$\chi \ll 1$ weak turbulence, linear term dominates.

$\chi \gg 1$ Strong turbulence.

Assume $\chi \ll 1$

Crossing / interaction time: $\Delta t \approx \frac{\ell_\parallel}{V_A} \sim \tau_A$ → lots of alfvèn times to have order 1 change in amplitude.

Change in amplitude: δu

$$\frac{\partial \delta u}{\partial t} \sim \delta u \cdot \nabla \delta u$$

$$\Delta \delta u \sim \frac{\delta u^2 \Delta t}{\ell_\perp} \sim \delta u \frac{\delta u\, \ell_\parallel}{\ell_\perp V_A} \sim \delta u\, \chi$$

Consider N "kicks" (interactions)

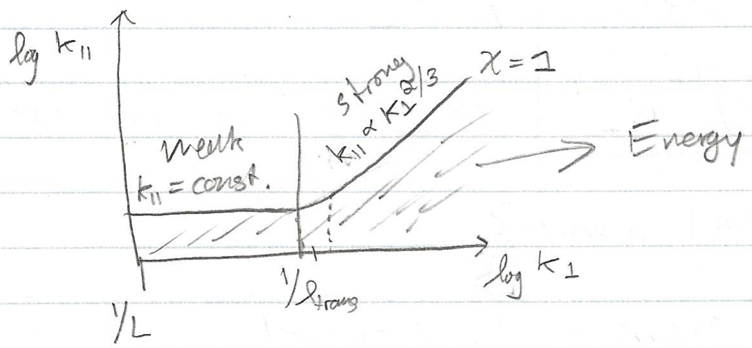
$$\sum_t \Delta \delta u \sim \delta u\, \chi \sqrt{N}, \quad N = \frac{t}{\tau_A} \implies \text{looks like random walk.}$$

Turbulence IV)

Now the parallel:

$$k_{\parallel} V_A \sim k_{\perp} \delta u_{\perp} \sim \epsilon^{1/3} k_{\perp}^{2/3} \Rightarrow \boxed{k_{\parallel}^2 \sim \frac{\epsilon^{1/3}}{V_A} k_{\perp}^{2/3}}$$ → Reduced parallel cascade

GS ⇒ Wave vector anisotropy grows w/ $k_{\perp}$



$$\epsilon \sim \frac{\delta u_{\ell}^2}{\tau_{\ell}} \sim \delta u_{\ell}^2 V_A k_{\parallel} \Rightarrow \delta u_{\ell}^2 \sim \left(\frac{\epsilon}{V_A}\right) k_{\parallel}^{-1}$$

$$\Rightarrow \boxed{E^{(1)}(k_{\parallel}) = \frac{\epsilon}{V_A} k_{\parallel}^{-2}}$$

Parallel spectra and perpendicular spectra.

For $Pr = \frac{\nu}{\eta} \gg 1$, viscous scale $\ell_{\perp\nu} \approx \frac{\nu^{3/4}}{\epsilon^{1/4}} \sim Re^{-3/4} L \left(\frac{V_A}{\delta u_{\ell}}\right)$ new

Solar wind spectra @ 1AU, Alfvenic inertial range transitions to other stuff, need to consider Landau damping, current sheets, all sorts of other types of effects.

Turbulence IV)

Since $k_{\parallel 1} = k_{\parallel 3}$, we have **NO** parallel cascade!

+ weak turbulence is mediated by $k_{\parallel} = 0$ modes.

VI' $\Rightarrow$ $l_{\parallel} \sim L$, large scale, then

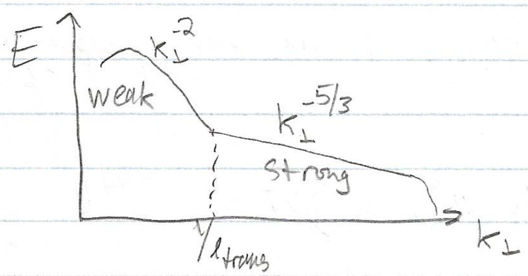
$$\delta u \sim (\epsilon v_A)^{1/4} l_{\perp}^{1/2} l_{\parallel}^{-1/4} \sim \left(\frac{\epsilon v_A}{L}\right)^{1/4} l_{\perp}^{1/2}$$

$$\Rightarrow \boxed{E^{(1)} \sim \left(\frac{\epsilon v_A}{L}\right)^{1/2} k_{\perp}^{-2}}$$ $\Rightarrow$ Weak Turbulence Spectrum.

Self consistent? Is $\chi \ll 1$ always?

$$\chi \sim \frac{\delta u_L}{v_A}\left(\frac{L}{l_{\perp}}\right)^{1/2}$$ $\Rightarrow$ grows as $l_{\perp}$ decreases!

$\chi \sim 1$ when $l_{\perp} \sim l_{trans} \sim L\left(\frac{\delta u_L}{v_A}\right)^2$



Goldreich-Sridhar (1995)

$\chi \sim 1$ (VI'') Critical balance $\tau_{eddy} \sim \tau_A$

$$|\vec{z}^{\mp} \cdot \nabla \vec{z}^{\pm}| \sim |v_A \nabla_{\parallel} \vec{z}^{\pm}|$$ $\Rightarrow$ Information propagates down a mag field line @ finite speed $\Rightarrow$ Alfvén speed.

For $\chi = 1$, $k_{\perp} \delta u_l \sim k_{\parallel} v_A$; $\tau_l \sim \frac{l_{\perp}}{\delta u_l}$

$$\epsilon \sim \frac{\delta u_l^2}{\tau_l} \sim \frac{\delta u_l^3}{l_{\perp}} \Rightarrow \delta u_l \sim (\epsilon l_{\perp})^{1/3}$$

GS spectrum

$$\Rightarrow \boxed{E^{(1)} \sim \epsilon^{2/3} k_{\perp}^{-5/3}}$$

Turbulence III)

Now

$\tau_\ell = \tau_{eddy} \Rightarrow$ time req'd for $\mathcal{O}(1)$ change in $\Delta \delta u$

$$\sum_{t}^{\tau_\ell} \Delta \delta u_\ell \sim \delta u_\ell \Rightarrow \chi \sqrt{N} \sim 1 \Rightarrow \tau_\ell \sim \frac{\tau_{eddy}^2}{\tau_A}$$

Cascade rate $\epsilon = \frac{\delta u_\ell^2}{\tau_\ell} \sim \delta u_\ell^2 \frac{\delta u_\ell^2 \lambda_{\parallel}}{V_A \lambda_\perp^2} \approx$ constant.

$$\delta u_\ell \sim (\epsilon V_A)^{1/4} \lambda_\perp^{1/2} \lambda_{\parallel}^{-1/4}$$

IK also assumed isotropy (VI) $\lambda_{\parallel} = \lambda_\perp$

so from

(I - VI) $\Rightarrow \delta u_\ell \sim \ell^{1/4} \Rightarrow \boxed{E^{(1)} \sim (\epsilon V_A)^{1/2} k^{-3/2}}$

IK spectrum

$E_B \sim E_V$

Is IK self consistent? yes.

Velocity spectrum 1.4, mag field spectra 1.6,

Total energy spectrum $\Rightarrow$ 3/2

Plots suggest that energy dissipates @ different rates in $\perp, \parallel$ directions.

IK doesn't account for this.

(Weak Turbulence)

Can we fix IK?

Energy: $\omega(\vec{k}_1) + \omega(\vec{k}_2) = \omega(\vec{k}_3)$

Momentum: $k_1 + k_2 = k_3$ assume $\omega = k_{\parallel} V_A > 0$

$$\left.\begin{array}{ll} \text{Ene)} & k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3} \\ \text{Mom)} & k_{\parallel 1} - k_{\parallel 2} = k_{\parallel 3} \end{array}\right\} \Rightarrow \text{Either } k_{\parallel 1}, k_{\parallel 2} = 0$$

Take $k_{\parallel 2} = 0$, $\therefore k_{\parallel 1} = k_{\parallel 3}$