

Computational Methods I)

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Goal: Modern Computational Techniques for PP.

PP uses Vlasov - Maxwell Eqs

$$\frac{\partial f_s}{\partial t} + \nabla_{\vec{x}} \cdot (\vec{v} f_s) + \nabla_{\vec{v}} \cdot (F_s f_s) = \left(\frac{\partial f_c}{\partial t} \right)_c \rightarrow \text{collison terms.}$$

Describe self-consistent evolution of distribution function $f(\vec{x}, \vec{v}, t)$ and electromagnetic fields.

\vec{F}_s = Lorentz force.

This equations are highly nonlinear \Rightarrow Highly difficult to solve Vlasov - Maxwell eqs

Because of difficulty, we have many approximations to simplify greatly.

Approximations

- Discovery of gyrokinetic equations \rightarrow asymptotic approx for plasmas in strong magnetic fields. Fast frequencies can be neglected, increasing simplicity.

- Fluid approximations \Rightarrow MHD (but lose kinetic effects).

- Computational physics can be an application tool, or as an area of legitimate research area.

(In situ \Rightarrow In space)

The EM fields are determined by Maxwell's equations.

Conservation properties are in Vlasov - Maxwell Eqs:

- Conserves particle number
- Conserves Total (particle + field) momentum
- Conserves Total (particle + field) energy.

(Computational Physics II)

Smyze Particle Motion in an EM Field

First, SHO)

$$\frac{d^2 z}{dt^2} = -\omega^2 z \Rightarrow z = a \cos(\omega t) + b \sin(\omega t)$$

Solve numerically first $\Rightarrow \frac{dz}{dt} = v, \frac{dv}{dt} = -\omega^2 z$, note (z, v) label the phase space of HO.

$$\frac{d}{dt} \left(\frac{1}{2} v^2 + \frac{1}{2} \omega^2 z^2 \right) = 0 \Rightarrow \text{Energy is conserved}$$

Numerically, can replace derivatives by finite differences.
(Forward Euler Scheme)

$$\frac{z^{n+1} - z^n}{\Delta t} = v^n, \frac{v^{n+1} - v^n}{\Delta t} = -\omega^2 z^n; \text{ thus, if extrapolated}$$

out, we see that energy is not conserved; energy increases.

Numerical methods of solving the SHO eqn, even if the energy is conserved (midpt. scheme), we still had phase errors in those graphs, getting worse in time.

Stability is good if $\det(\text{Jacobiam})$ of amplification factor.

Runge-Kutta Scheme \rightarrow constructed from forward Euler Scheme.

- even though Forward Euler scheme is unconditionally unstable, we can use multiple "steps" or "scheme" to get a more accurate, agreeable

Motion in EM Field) Even if E, B given, we can't use the midpoint scheme, b/c we get averages of E, B , which we're not given.

- Use a staggered scheme!

Boris Algorithm)

(Computational Methods II)

Boris Algorithm

- Boris algorithm exactly conserves energy for simple motion in EM field.
- It is also volume conserving scheme ($\det(\text{Jacobian}) = 1$)
+ Deals w/ conservation of phase space volume. \nearrow

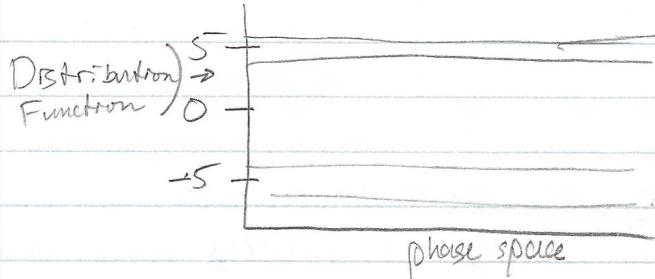
Current best \Rightarrow [Itiguer & Cary] for relativistic single particle motion in EM field. Preserves conservation of phase space that Boris got for nonrelativistic effect, and correctly accounts for $E \times B$ velocity in relativistic regime.

Vlasov Equation removes all single particle effects, so no radiative effects are accounted for in Vlasov; you'd have to account for that separately.

- Debye shielding does not exist in gravitational systems, like galactic dynamics, globular clusters, self-gravitating system.
 - kinetic theory is used in plasmas, now being used in gravitational astrophysics and galaxy theory.

Looking to solve Vlasov-Maxwell equations.

Simulation \rightarrow classic 2 stream instability



\Rightarrow Forms "B6K" modes w/ small perturbation.

Classical Helmholtz - Kelvin instabilities, super super cool simulation, has all 3 ranges, w/m the simulations.

(Computational Physics III)

Recall scheme conserves volume phase space $\Rightarrow \det(\text{Jacobian}) = 1$
 - MHD part scheme conserves energy.

Particle in motion in conservative force for given $\phi(x)$

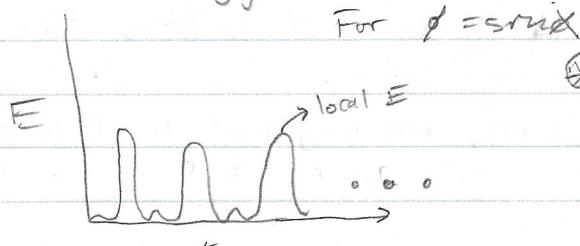
$$W = \frac{1}{2}v^2 + \phi = \text{constant}, \quad \text{for } \frac{dx}{dt} = v, \quad \frac{dv}{dt} = -\frac{\partial \phi}{\partial x}$$

Implicit methods won't work, need iterative process.

Look @ Boris (staggered timestep), for conservative force:

- Is energy conserved?

In the staggered method, energy oscillates about some mean method



For $\phi = \sin x$
 • Define some quantity that will actually be conserved, even though E oscillates.

Aste) How do you know what quantity to define that yields a conservative result?

Solving Maxwell Eqs) There are global conservation laws

- How can we actually solve M.E. efficiently and maintain (some) conservation + geometric properties?

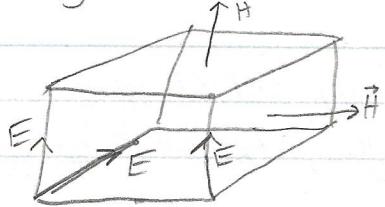
• The M.E. geometry doesn't readily come out from compact form.

\vec{E} is a vector, while \vec{B} is a bivector

↳ They are different, so solving M.E. must treat E, B differently.

(Computational Physics III)

Solving M.E. \Rightarrow The Yee-cell.



Electric field should be on edges, while
Magnetic field is a bivector, should be on faces

- Thus can also be flipped \Rightarrow Duality of $E + B$ fields.
evidence up

- In spacetime, EM Fields $\Rightarrow F_{\mu\nu}$, bivector.

Now calculate 2 curls

$$\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0, \quad \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{B} = 0$$

stagger
frame
by 1/2
step

Define two different grad operators, ∇_E , ∇_F , one on edges, other on faces.

The curl operator ∇_E takes stuff on edges, moves to faces,
 ∇_F " " " " faces, moves to edges!

What if we reintroduce currents? \rightarrow Geometry must be conserved;
so current must be a vector, needs to be on the edges!,

• Staggering in time reflects that in 4D, EM field is a bivector in space-time.

\Rightarrow staggering also applies to Dirac equations, can solve them.

Divergence in Yee-cell.

$$\begin{aligned} \nabla_F \cdot \vec{B}^{n+1/2} &= 0 \\ \nabla_E \cdot \vec{E}^n &= 0 \end{aligned} \quad \left. \right\} \text{Exactly maintained w/ the Yee-mesh.}$$

For plasma, we have current conservation: $\frac{\partial \rho_e}{\partial t} + \nabla \cdot \vec{J} = 0$,
on Yee-cell: $\frac{\rho_e^{n+1} - \rho_e^n}{\Delta t} + \nabla_E \cdot \vec{J}^{n+1/2} = 0$

Computational Physics III)

Maxwells eqns are hyperbolic

- Small disturbance in system described by set of eqns. If the speed @ which disturbance travels is finite, eqns are hyperbolic
- ↳ speed of light for EM.

$$\text{Consider } \frac{\partial Q}{\partial t} + \frac{\partial F}{\partial t} = 0, \quad \vec{F} = \vec{F}(Q)$$

Compute flux-Jacobian $A = \frac{\partial \vec{F}}{\partial Q}$. Calculate eigenvalues/vectors.
If real → and complete ↴

If \vec{A} is diagonalizable, they are hyperbolic equations. ↴

$$\text{Consider } \frac{\partial}{\partial t} \begin{bmatrix} E_y \\ B_z \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} c^2 B_z \\ E_y \end{bmatrix} = 0,$$

$$\text{Flux-Jacobian } A = \begin{bmatrix} 0 & c^2 \\ 1 & 0 \end{bmatrix}$$

Values: $r^2 = \begin{bmatrix} 1 \\ -1/c \end{bmatrix} \rightarrow$ real, and y holds max speed @ which disturbance propagates

Consider ideal fluid eqns, (Euler Eqns) \Rightarrow They're hyperbolic

3 eigenvalues: velocity of fluid, $u \pm c_s$, where $c_s = \text{sound speed}$
 \Rightarrow Hyperbolic, but nonlinear hyperbolic.

Other examples \Rightarrow ideal MHD eqns, relativistic fluid eqns, Einstein field eqns,

↓ If flux-Jacobian, and eigenvalues are found, values give all speeds w/in system \Rightarrow Alfvén speed, slow MD speed, other characteristics we should be familiar with.

Computational Physics III)

Hyperbolic problems require very special methods to solve!

- major features of H.E.'s exhibit shocks, rarefactions, contact discontinuities, etc.)

- Looked into Shock capturing methods.

- developed to simulate aerospace engineering (supersonic flight, transonic flight, re-entry into atmosphere)

NY → Paris in 90 mins \Rightarrow Mach 5, hypersonic flight.

Fusion physics, shock capturing methods aren't widely used, but Astrophys plasmas widely use shock capturing methods.

- apparently fusion machine plasma evolution is too slow to justify shock capturing methods.

- Are Vlasov eqns hyperbolic? \Rightarrow yes, but only finite dimensionally,

IV Galerkin scheme, DG
Discontinuous \Rightarrow solution represented using piecewise discontinuous functions. \Rightarrow solving hyperbolic PDE

- need Galerkin minimization,

\rightarrow state of the art currently,

• First introduced in 1973, as conference paper to solve steady state neutron transport in reactors. \rightarrow linear system.

- 25 yrs. later first application of DG in nonlinear systems.

Example) L_2 minimization of errors on finite dimensional set
Legendre Polynomials. Interval $[-1, 1]$

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$$

Select inner product, use to define norm.

Consider time-dependent problem on $x \in [-1, 1]$

$$f(x, t) \approx f_h(x, t) = \sum_{k=1}^N f_k(t) P_k(x) \text{ yields } \sum f'_k P_k(x) = G[f_h]$$



$$\|L\|_2 = (f, f)$$

(Computational Physics III)

Now we want to find the f_k 's. Error function $\Rightarrow L_2$ norm

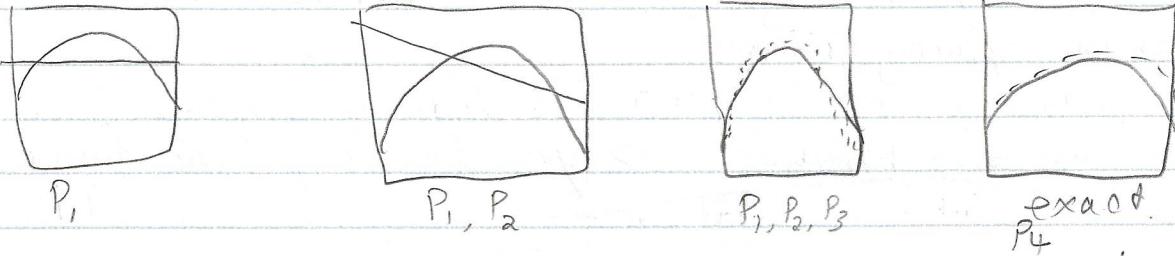
$$E_N = \left\| \sum f'_k P_k(x) - G[f_h] \right\|_2 = \int_{-1}^1 \left[\sum f'_k P_k(x) - G[f_h] \right]^2 dx$$

Find minimum: $\frac{\partial E_N}{\partial f_m} = 0$ for all $k = 1, 2, \dots, N$.

Gives a set of equations that minimize error in our problem.

Consider finding best fit on finite-dimensional space for

$$f(x) = 3 + (x-0.5)^4 + 2x^3 - 5x^2 \text{ on } x \in [-1, 1]$$



Highest poly. in $f(x)$ is 4, so gives exact fit when using P_4 .
 These are the piecewise functions \uparrow

DG, split intervals into cells and use Galerkin scheme in each cell.
 - Note the discontinuous lines are representations of our solution, keep in mind.

So we can relate representation w/ solution. Use weak-equality

Definition. Weak Equality.

2 funcs, f, g are weakly equal if

$$(\psi_k, f-g) = 0 \text{ for all } k = 1, \dots, N$$

Then $f \doteq g$

Computational Physics IV

We relate LHS, RHS via weak equality $f'(x, t) \doteq G[f]$,
Thrs implies

$$(\psi_k, f'(x, t) - G[f]) = 0$$

Consider a linear represent. of particle distribution fn:

$f_h(x) = f_0 + x f_1$ in a cell. Reconstruct an function to describe exponential
 This can be negative, not good. Exponentials make all > 0 , so no negatives, good.

"Example of recovery: Exponential recovery in a cell".

DG scheme for linear advection.

Consider 1D passive advective eqn on $I \in [L, R]$

$$\frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial x} = 0, \quad \lambda = \text{const. advective speed.}$$

Exact solution: $f(x, t) = f_0(x - \lambda t)$, for $f_0(x) \equiv \text{IC}$.

i) Discretize into elements

$$I_j \in [x_{j-1/2}, x_{j+1/2}]$$

2) Pick finite dimensional basis to work from (Legendre, etc).

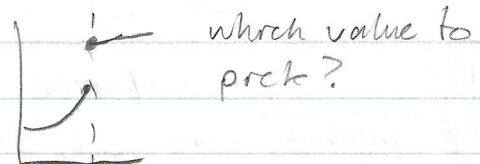
③ minimization

Find f_h in fn space such that for basis ϕ fn ϕ is

$$\int_I \phi \left(\frac{\partial f_h}{\partial t} + \lambda \frac{\partial f_h}{\partial x} \right) dx = 0$$

IBP to move derivative onto basis functions,

at discontinuous break, we have 2 values



- Look and see how information is flowing, and will inform which value above to pick.

Computational Physics IV

Also, no choice is wrong; we can average or just pick, etc.

But some choices yield better fits for the solution representation.

- Averages typically look better, but can go negative which isn't good.

- Choose upwinding value, and use higher order basis terms, and our fit looks quite good.

Does our scheme satisfy certain characteristics (like for continuous passive equation), do we get particle conservation from our representation?)

{ - The average value conserves particles, but is a bad representation

{ - The upwinding soln doesn't conserve particles but is good representation.

→ Need to look carefully at these things.

Now apply it to a problem \Rightarrow Putting everything together, Vlasov-Maxwell eqn using DG.

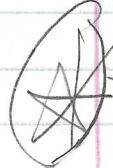
Treat Vlasov as PDE in (6D). How do we solve this and retain invariance. (particles, total momentum, total energy)

- Total energy is conserved, but momentum is not

Need to make sure entropy increases in your scheme.

Advice \Rightarrow Every day, read the plasma physics archive
arxiv physics.comp-ph and numerical analysis math.NA
and plasma archive

Learn C++)



Learn) expert@ (i) the physics, (ii) mathematics of the numerical methods, and (iii) program and software techniques

\hookrightarrow Apparently these skills are in insanely high demand!