Quaternions

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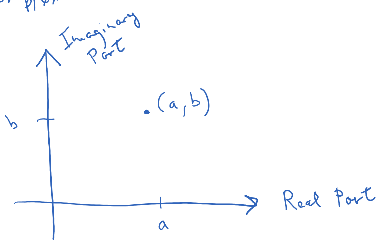
Review: Complex Numbers
→ quaternions are an extension of the complex numbers

Defn: A complex number, $a + bi$, consists of real part $a \in \mathbb{R}$ and a imaginary part $b \in \mathbb{R}$. The imaginary part has this property

$$i^2 = -1$$

Complex Numbers can be visualized in 2D



Analogous to 2D coordinate (a,b)

$$(a,b)^T = a\hat{i} + b\hat{j}$$

where $\hat{i} = (1,0)^T$
$\hat{j} = (0,1)^T$

Add complex numbers components

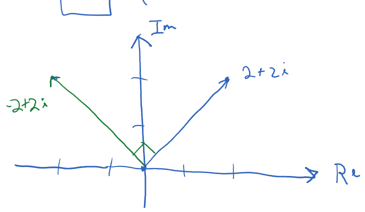
$\boxed{EX}$ $(3+5i) + (7+i) = 10 + 6i$

Multiply using distributive rule:

$\boxed{EX}$ $(3 + 5i) * (-3 + 3i) = 3(-3) + 3(3i) + (5i)(-3) + (5i)(3i)$
$= -9 + 9i - 15i + 15 i^2 \to -1$
$= -24 - 6i$

Complex Numbers are related to rotations:
→ multiplying by i rotates by $90°$

$\boxed{EX}$ $(2+2i) \times (0 + i) = 2i + 2i^2 = -2 + 2i$



$(2,2) \cdot (-2,2) = -4 + 4 = 0$
↑ dot product $= 0$
$\Rightarrow 90°$ angle

Quaternions: extend complex numbers to 4 dimensions

$$q = w + a\hat{i} + b\hat{j} + c\hat{k}, \text{ where } \hat{i}^2 = \hat{j}^2 = \hat{k}^2 = -1$$

We represent quaternions a 4-tuple vector: $(a, b, c, w)^T$

Unit quaternions correspond directly to angle/axis rotations

$$q = (\bar{v}, w) = \left(\sin\left(\frac{\theta}{2}\right)\hat{u}, \cos\left(\frac{\theta}{2}\right)\right)^T$$

↑ (a,b,c)

$\boxed{ex}$ $\theta = 45$, and $\hat{u} = (1,0,0)^T$. Then the corresponding quaternion

$$q = \left[\underbrace{\sin\left(\frac{45}{2}\right)\cdot 1}_{v_x}, \underbrace{\sin\left(\frac{45}{2}\right)\cdot 0}_{v_y}, \underbrace{\sin\left(\frac{45}{2}\right)\cdot 0}_{v_z}, \underbrace{\cos\left(\frac{45}{2}\right)}_{w}\right]^T$$

Quaternions share properties w/ vector types

$\|q\| = \sqrt{q_x^2 + q_y^2 + q_z^2 + q_w^2}$

$v_x \quad , \quad v_y \quad , \quad z \quad ,$

Quaternions share properties w/ vector types

Length of a quaternion q: $\|q\| = \sqrt{q_x^2 + q_y^2 + q_z^2 + q_w^2}$

Angle between quaternions: $\cos\theta = \dfrac{q_1 \cdot q_2}{\|q_1\| \|q_2\|}$

Dot product: $q_1 \cdot q_2 = q_{1x} q_{2x} + q_{1y} q_{2y} + q_{1z} q_{2z} + q_{1w} q_{2w}$

Unit quaternion: $\dfrac{q}{\|q\|}$

Multiplication: Let $q_1 = [x_1, y_1, z_1, w_1]^T$, $q_2 = [x_2, y_2, z_2, w_2]^T$

$q_1 q_2 = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + w_1)(x_2\hat{i} + y_2\hat{j} + z_2\hat{k} + w_2)$ ← distribute & apply rules

$= \left[\underbrace{w_1\bar{v}_2 + w_2\bar{v}_1 + \bar{v}_1 \times \bar{v}_2}_{\text{vector part}}, \underbrace{w_1 w_2 - v_1 \cdot v_2}_{\text{scalar}} \right]$

where $\bar{v}_1 = (x_1\ y_1\ z_1)^T$ & $\bar{v}_2 = (x_2\ y_2\ z_2)^T$

Conjugate: Denoted $q^* = [-\bar{v}, w]$

$q^* q = q q^* = q \cdot q = \|q\|^2$

Inverse: $q^{-1} = \dfrac{q^*}{\|q\|^2}$ ← when $\|q\| = 1$, this simplifies to $q^{-1} = q^*$

$q^{-1} q = q q^{-1} = q\left(\dfrac{q^*}{\|q\|^2}\right) = \dfrac{\|q\|^2}{\|q\|^2} = 1$

Quaternions: Advantages

→ 4-tuple to store (compact)

→ no gimbal lock problems

→ supports smooth, stable blending between orientations

→ efficient multiplication

How to rotate w/ a quaternion:

$p' = q\, p\, q^{-1}$

$\boxed{\text{EX}}$ Suppose we have $p = (1, 0, 0)^T$ & we want to rotate $90°$ around the z axis.

① Compute q & q^{-1}

$q = [\bar{v}, w] = \left[\sin\left(\frac{90}{2}\right)(0,0,1), \cos\left(\frac{90}{2}\right)\right]^T = [0, 0, \underbrace{\sin(45)}_{\sqrt{2}/2}, \underbrace{\cos(45)}_{\sqrt{2}/2}]^T$

$q^{-1} = [-\bar{v}, w] = [0, 0, -\sin(45), \cos(45)]^T = \left[0, 0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$

② Compute $p' = q\,p\,q^{-1}$ ← Let $q = [v, w]$ and $p = [P, 0]$

note: to rotate a point p using a rotation matrix R 3×3

② Compute $p' = q\,p\,q^{-1}$. Let $q = [v, w]$ and $p = [p, 0]$

note: to a point p rotation Rp

$$= \underbrace{[v, w][p, 0]}[-v, w]$$

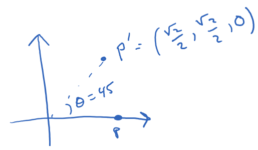
$$= [wp + \cancel{0 v} + \underbrace{v \times p}, \; w \cancel{0} - v \cdot p][-v, w]$$

$v = (0, 0, \tfrac{\sqrt{2}}{2})$, $p = (1, 0, 0)$

$$= [wp + v \times p, \; -\overset{0}{\cancel{v \cdot p}}][-v, w]$$

$$= \left[\frac{\sqrt{2}}{2} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}, 0\right][-v, w]$$

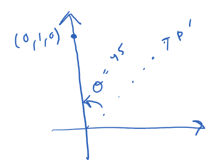
$$= \left[\begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}, 0\right]\underline{[-v, w]}$$



$$= \left[\begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}, 0\right]\left[\begin{pmatrix} 0 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix}, \frac{\sqrt{2}}{2}\right]$$

$$= \left[\frac{\sqrt{2}}{2}\begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix} + \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, 0\right]$$

$$= \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, 0\right] \checkmark$$



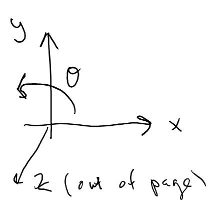
Notes:

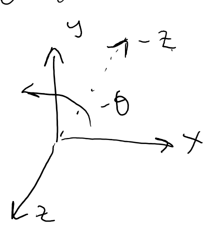
① q and $-q$ represent the same rotation

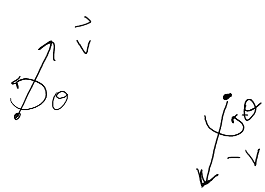
Intuition: $q = [\sin(\frac{\theta}{2})\hat{u}, \cos(\frac{\theta}{2})]$ ← rotation θ around axis $\hat{u}$

$-q = [-\sin(\frac{\theta}{2})\hat{u}, -\cos(\frac{\theta}{2})]$ ← rotation $-\theta$ around axis $-\hat{u}$

example: Think about rotation θ around $+z$







② $[\underbrace{0, 0, 0}_{\text{vector part}}, \underset{w}{1}]$ corresponds to no rotation (identity)

③ Unit quaternions are related to spheres

$\boxed{\text{EX}}$ The sphere S^1 is a circle: $x^2 + y^2 = 1$

The sphere S^2 is a ball: $x^2 + y^2 + z^2 = 1$

The sphere S^3 is a 4D ball: $x^2 + y^2 + z^2 + w^2 = 1$

Converting from a quaternion to a matrix:

Recall: $p' = q p q^{-1}$

$$= [\bar{v}, w][p, \emptyset][-\bar{v}, w]$$

$$= (w^2 - v \cdot v)p + 2w(v \times p) + 2(v \cdot p)v \qquad \text{// put in matrix form}$$

$$= \begin{pmatrix} 1-2(v_y^2+v_z^2) & 2(v_xv_y - wv_z) & 2(v_xv_z + wv_y) \\ 2(v_xv_y + wv_z) & 1-2(v_x^2+v_z^2) & 2(v_yv_z - wv_x) \\ 2(v_xv_z - wv_y) & 2(v_yv_z + wv_x) & 1-2(v_x^2+v_y^2) \end{pmatrix}$$

where does the above come from?

$$= (w^2 - v_x^2 - v_y^2 - v_z^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\bar{p} + 2w\begin{pmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{pmatrix}\bar{p} + 2\begin{pmatrix} v_x^2 & v_xv_y & v_xv_z \\ v_xv_y & v_y^2 & v_yv_z \\ v_xv_z & v_yv_z & v_z^2 \end{pmatrix}\bar{p}$$

Converting from a matrix to a quaternion:

Suppose $R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$. How to get v_x, v_y, v_z, w components of a quaternion back out?

Let's look at the sum of the diagonal term

$$r_{11} + r_{22} + r_{33} = 1 - 2(v_y^2 + v_z^2) + 1 - 2(v_x^2 + v_z^2) + 1 - 2(v_x^2 + v_y)^2$$

$$= 3 - 4(v_x^2 + v_y^2 + v_z^2)$$

Because q is a unit quaternion, we know $v_x^2 + v_y^2 + v_z^2 + w^2 = 1$

It follows that $v_x^2 + v_y^2 + v_z^2 = 1 - w^2$,

$$\underline{r_{11} + r_{22} + r_{33}} = 3 - 4(1 - \underline{w}^2)$$

$$\tfrac{1}{4}(r_{11} + r_{22} + r_{33} + 1) = w^2$$

$$\tfrac{1}{4}(1 + r_{11} - r_{22} - r_{33}) = v_x^2$$

$$\tfrac{1}{4}(1 - r_{11} + r_{22} - r_{33}) = v_y^2$$

$$\tfrac{1}{4}(1 - r_{11} - r_{22} + r_{33}) = v_z^2$$

Verify: $1 + r_{11} - r_{22} - r_{33} = 4v_x^2$

$\rightarrow 1 + 1 - 2(v_y^2 + v_z^2) - (1 - 2(v_x^2 + v_z^2)) - (1 - 2(v_x^2 + v_y^2))$

$\rightarrow \cancel{1} + \cancel{1} - \cancel{2v_y^2} - \cancel{2v_z^2} - \cancel{1} + \underline{2v_x^2} + \cancel{2v_z^2} - \cancel{1} + \underline{2(v_x^2)} + \cancel{2v_y^2}$

$\rightarrow 4v_x^2$ ✓

$\frac{1}{4}(1 - r_{11} - r_{22} + r_{33}) = v_z^2$ $\rightarrow 4v_x^2$ ✓

<u>NOTE</u>: We can't use the above eqns alone because we need the signs of each component

→ approach: use biggest of w^2, v_x^2, v_y^2, v_z^2 as a starting term & then use off diagonals to get remaining components.

<u>Using the off diagonal terms</u>:

Notice: $r_{21} - r_{12} = \cancel{2v_x v_y} + 2wv_z - \cancel{2v_x v_y} + 2wv_z$

$= 4wv_z$ ← ex. Suppose w^2 was the largest diagonal term we could find. We would then use this to solve for v_z.

$$v_z = \frac{1}{4w}(r_{21} - r_{12})$$

$$wx = \frac{1}{4}(r_{32} - r_{23})$$

$$wy = \frac{1}{4}(r_{13} - r_{31})$$

$$xy = \frac{1}{4}(r_{21} - r_{12})$$

$$xz = \frac{1}{4}(r_{13} + r_{31})$$

$$yz = \frac{1}{4}(r_{23} + r_{32})$$

sqr term always positive, but these can be negative (recall: $-q$ & q are the same rotation)

<u>The Alg</u>:
1. Solve for v_x^2, v_y^2, v_z^2, w^2 using diagonal terms
2. Choose largest term
3. Based on largest term, solve for other components using off diagonal terms.

<u>Matrix / Quaternion Summary</u>

* Subset of quats w/ unit length represent rotations
  → they work analogously to rotation matrices

1. Multiplying 2 rotation matrices produces a rotation matrix

   $\boxed{EX}$ $R_z(90)\, R_y(45) = R_{zy}$ ← combines both rots into a single matrix

   Multiplying 2 unit quats produces a unit quaternion

2. Adding / Subtracting rotation matrices & unit quaternions does not ...

guarantee a rotation.

→ aside: you can always renormalize a quat to make it a valid rotation

(3) Composing rotations corresponds to multiplying

$\boxed{EX}$ Matrix: Rotate an object by 45° around Y followed by 10° rotation around X

$$\Leftrightarrow R_x(10)\, R_y(45)\, \bar{p}$$ ← "points on our object"

$$\Leftrightarrow R_{xy}\, \bar{p}$$

Quaternions: Rotations correspond post/pre multiplication.

$$\Leftrightarrow q_x(10)\, q_y(45)\, \bar{p}\, q_y^{-1}(45)\, q_x^{-1}(10)$$

$$\Leftrightarrow q_{xy}\, \bar{p}\, q_{xy}^{-1}$$

(4) Multiplication not communitive

$$R_z(90)\, R_y(45) \neq R_z(45)\, R_y(90)$$

$$q_z(90)\, q_y(45) \neq q_z(45)\, q_y(90)$$

(5) Identity corresponds to no rotation

(6) Inverse corresponds to the reverse rotation

$\boxed{EX}$ $R_z(90)^{-1} = R_z(-90) = R_z(90)^T$

$q = [w, v]$ and $q^{-1} = [w, -v]$ ← angle stays the same, axis changes