CS 383: Machine Learning

Prof Adam Poliak

Fall 2024

10/31/2024

Lecture 19

Hyperplane

Divides space into positive (+1) and negative (-1)

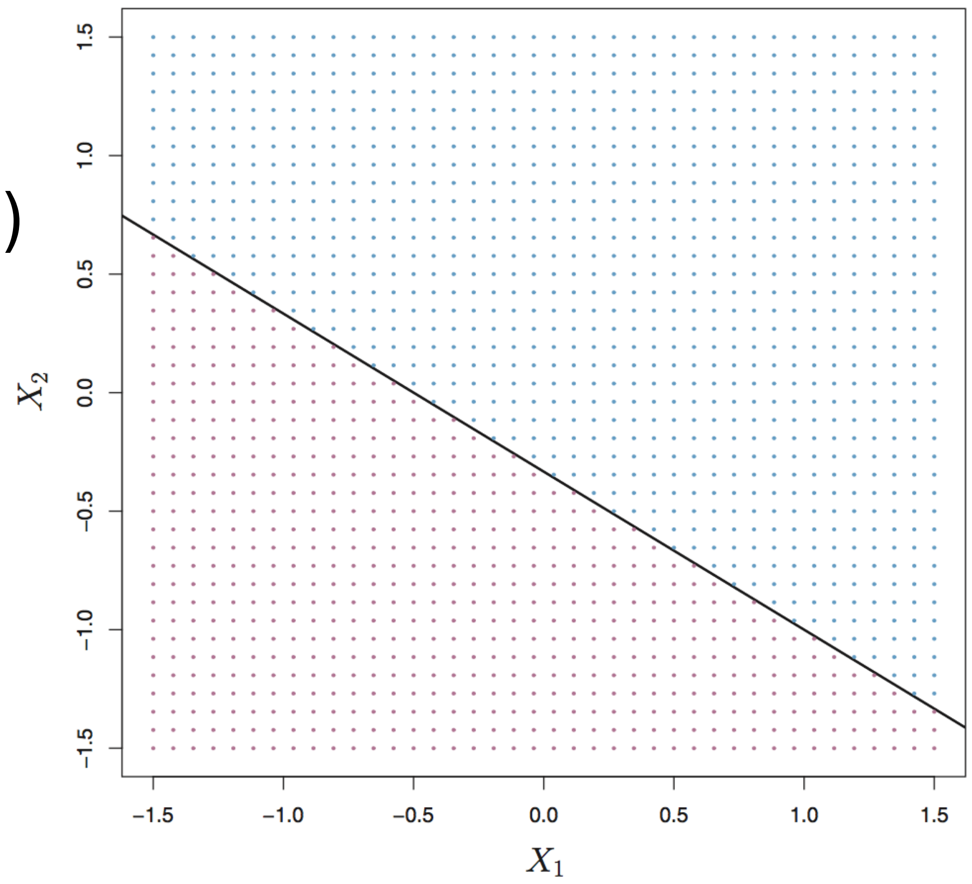


FIGURE 9.1. *The hyperplane* $1 + 2X_1 + 3X_2 = 0$ *is shown. The blue region is the set of points for which* $1 + 2X_1 + 3X_2 > 0$*, and the purple region is the set of points for which* $1 + 2X_1 + 3X_2 < 0$.

Goal: use training data to create a hyperplane

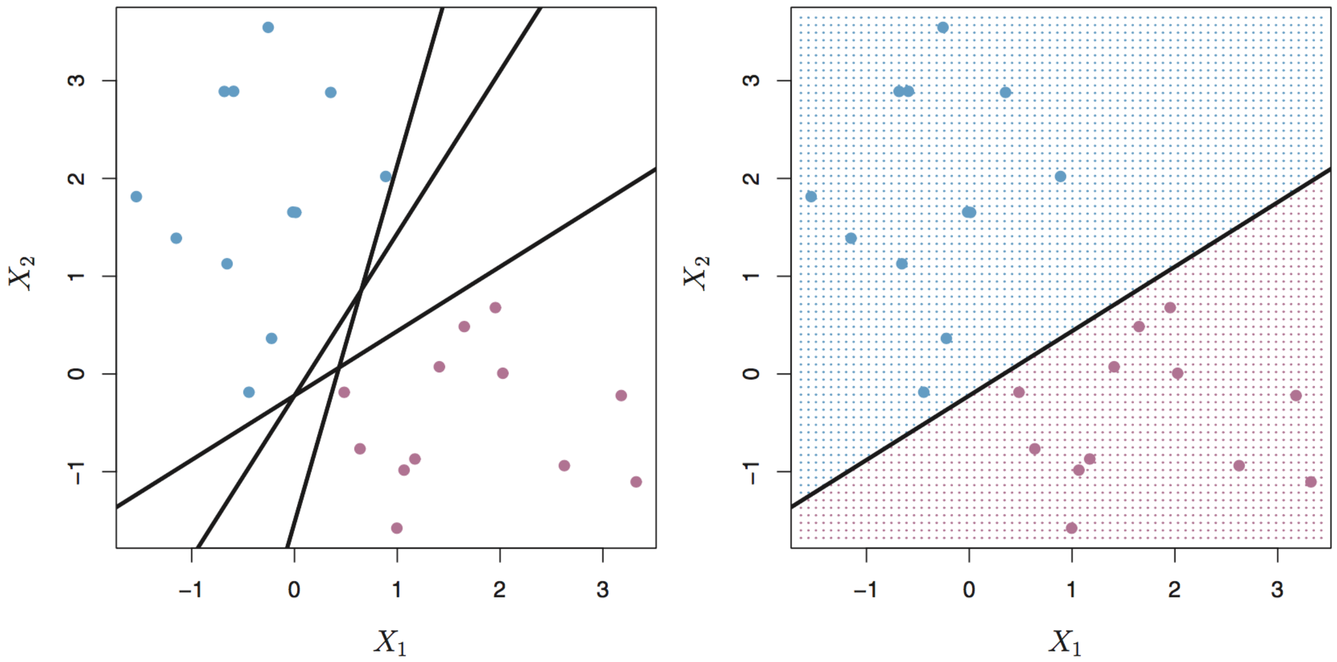


FIGURE 9.2. Left: *There are two classes of observations, shown in blue and in purple, each of which has measurements on two variables. Three separating hyperplanes, out of many possible, are shown in black.* Right: *A separating hyperplane is shown in black. The blue and purple grid indicates the decision rule made by a classifier based on this separating hyperplane: a test observation that falls in the blue portion of the grid will be assigned to the blue class, and a test observation that falls into the purple portion of the grid will be assigned to the purple class.*

ISL: Figure 9.2

Different Hyperplanes

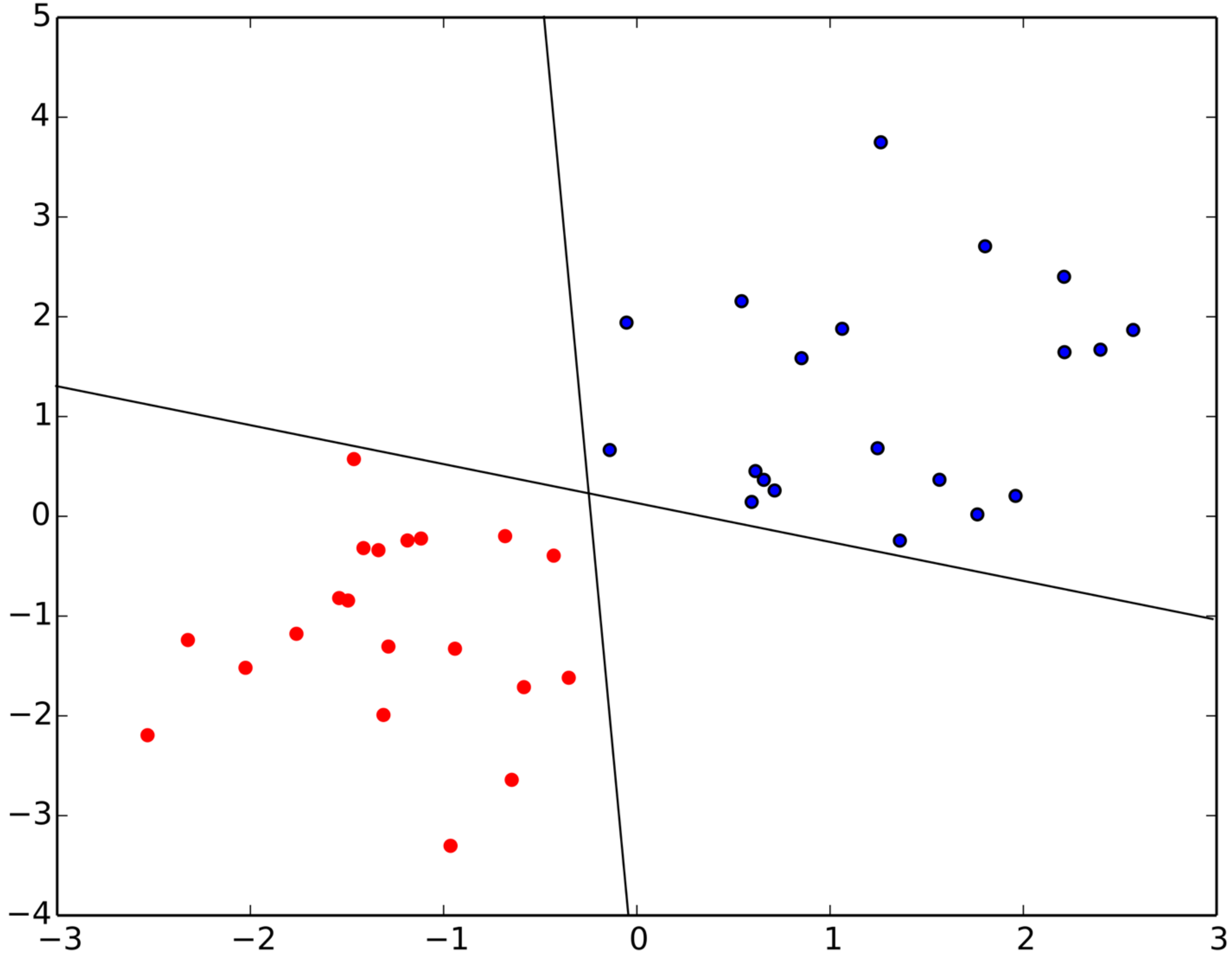


Image: Qwertyus, Wikipedia

Linear Algebra Review

- Matrix rank
- Linearly Independent rows/columns
- Null space

Perceptron Algorithm: Making a prediction

$y \in \{-1, +1\}$

$h(\vec{x}) = \text{sign}(\vec{w} \ * \text{x})$

If $\vec{w} \ * \text{x} > 0, \Rightarrow \ \ \hat{y} = +1$
If $\vec{w} \ * \text{x} < 0, \Rightarrow \ \ \hat{y} = -1$

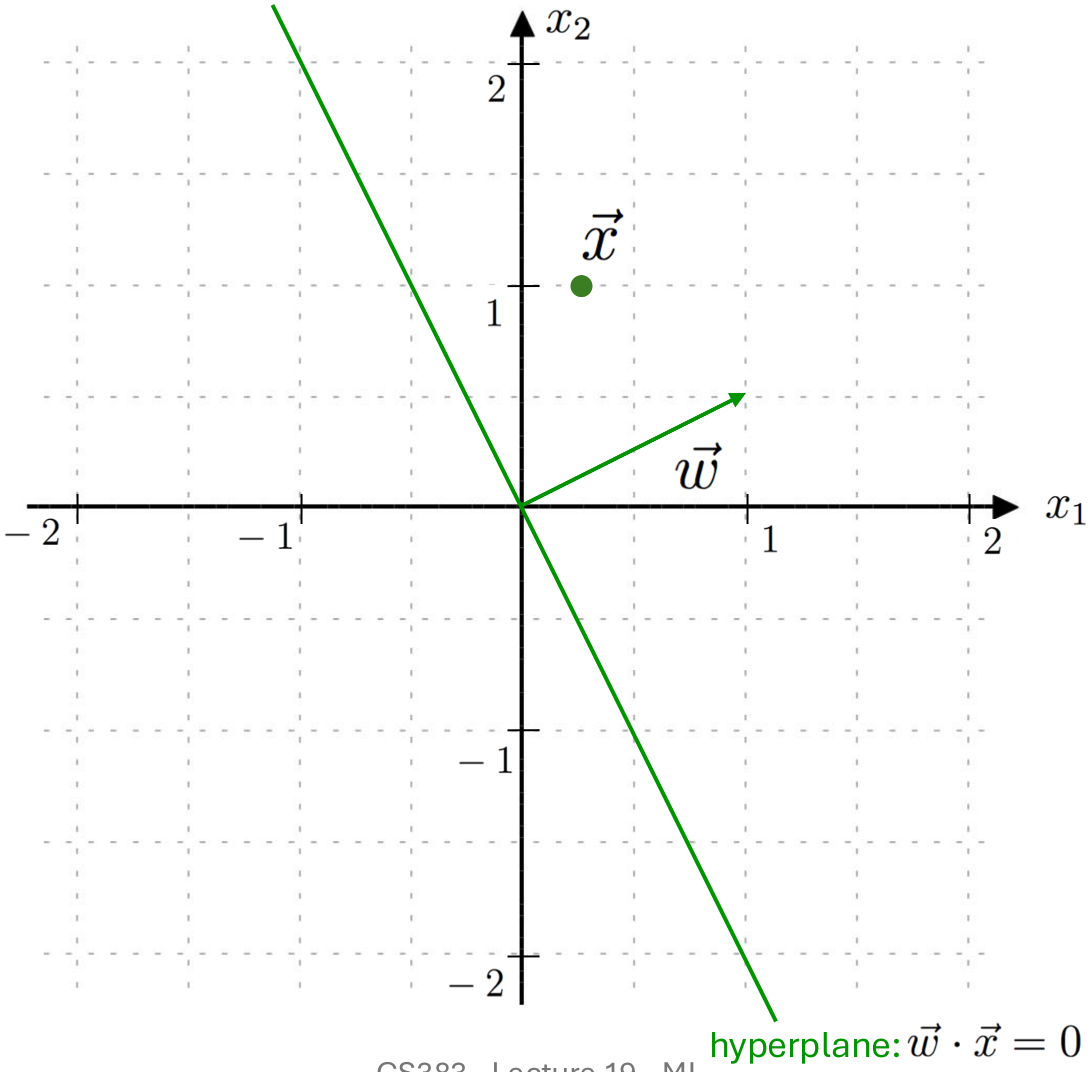
Perceptron Algorithm: updating weights

Set $\vec{w}$ to 0-vector

Repeat until training set is perfectly classified:

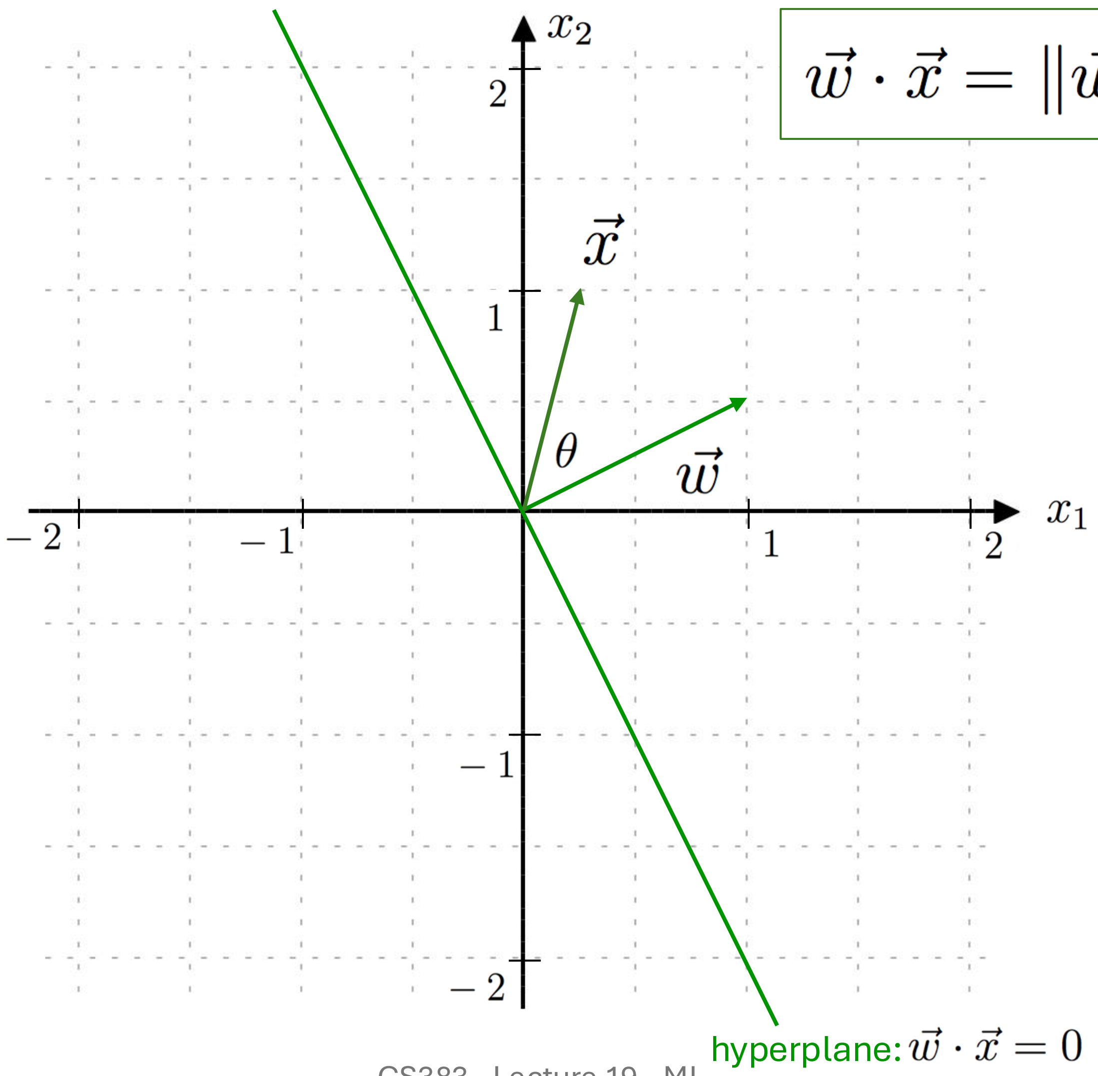
1. Randomly choose (x_i, y_i)
2. Predict $\widehat{y_i}$
3. If $\widehat{y_i} = y_i$:
    1. do nothing
4. Else:
    1. $\vec{w} \leftarrow \vec{w} + y_i \, \vec{x_i}$

Intuition behind the dot product

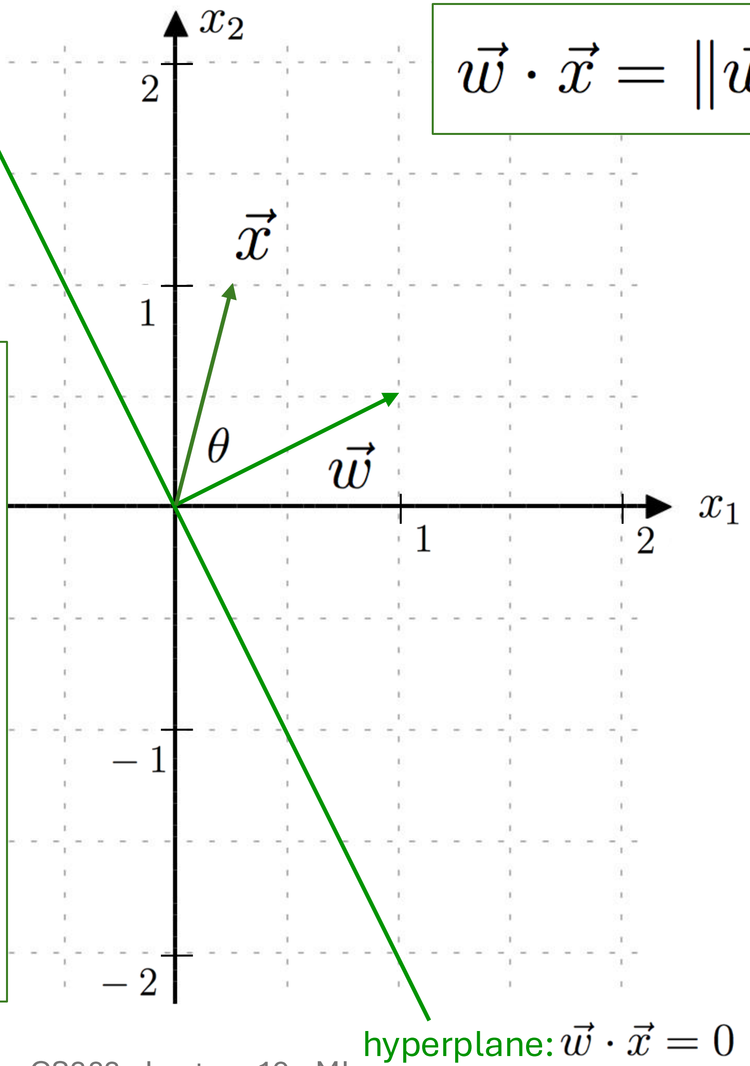


Intuition behind the dot product

$$\vec{w} \cdot \vec{x} = \|\vec{w}\| \|\vec{x}\| \cos\theta$$



Intuition behind the dot product



$$\vec{w} \cdot \vec{x} = \|\vec{w}\| \|\vec{x}\| \cos\theta$$

Takeaway: we only care about the sign of the angle between ***x*** and ***w***

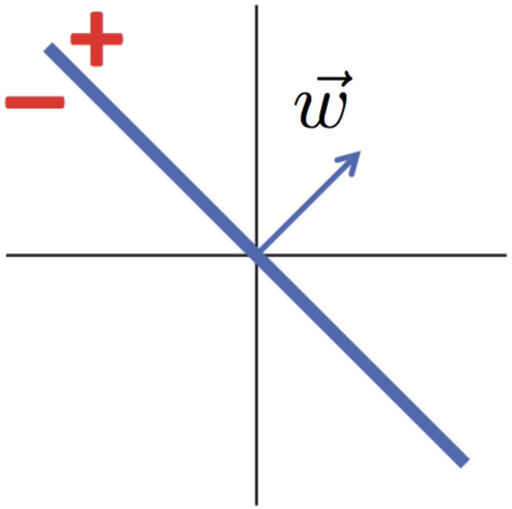
- If cos θ > 0, ***x*** is on the same side of the hyperplane as ***w***, so we classify it as positive
- If cos θ < 0, ***x*** is on the opposite side from ***w***, so we classify it as negative

The ***bias*** (*b*) and the *y*-intercept are different, but they both capture a "shift" away from the origin.

Example modified from Jessica Wu

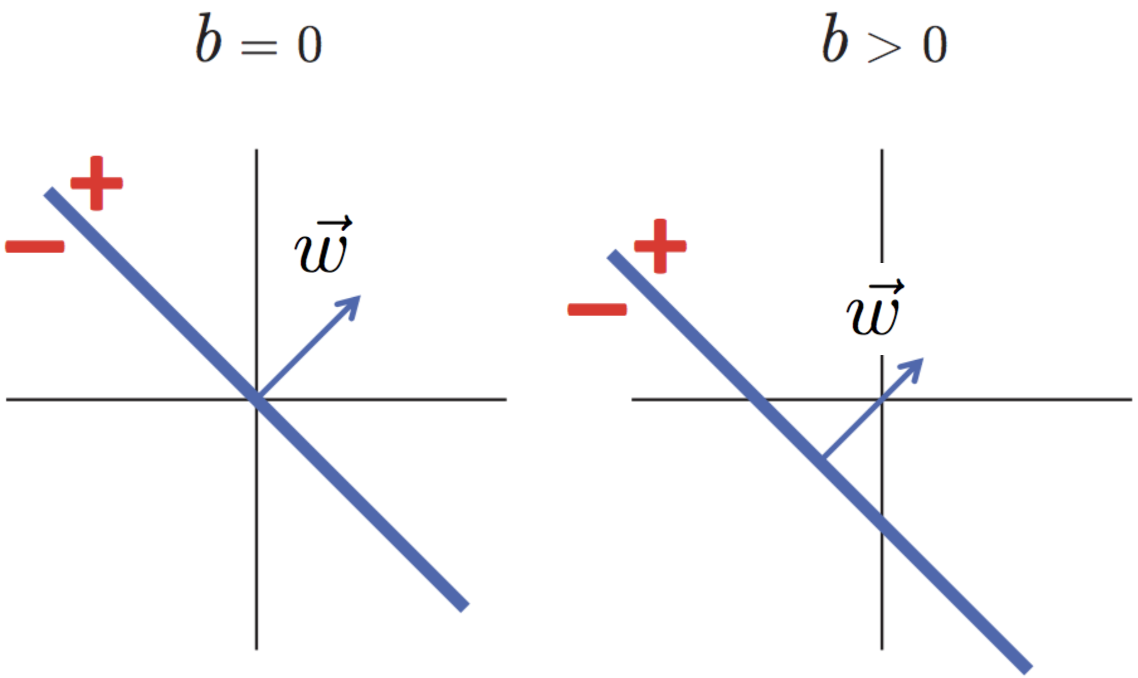
The ***bias*** (*b*) and the *y*-intercept are different, but they both capture a "shift" away from the origin.

$b = 0$



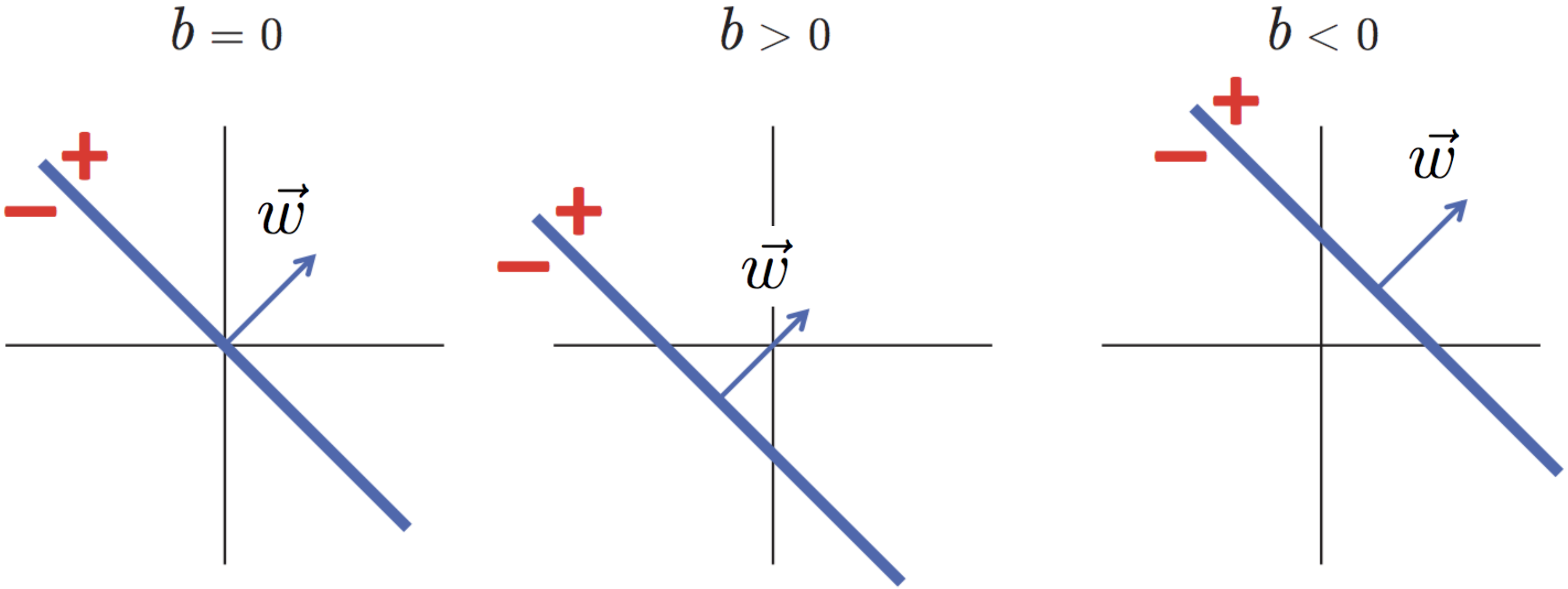
Example modified from Jessica Wu

The ***bias*** (*b*) and the *y*-intercept are different, but they both capture a "shift" away from the origin.



Example modified from Jessica Wu

The ***bias*** (*b*) and the *y*-intercept are different, but they both capture a “shift” away from the origin.



With p=2, if w_2 is positive, then the above example holds

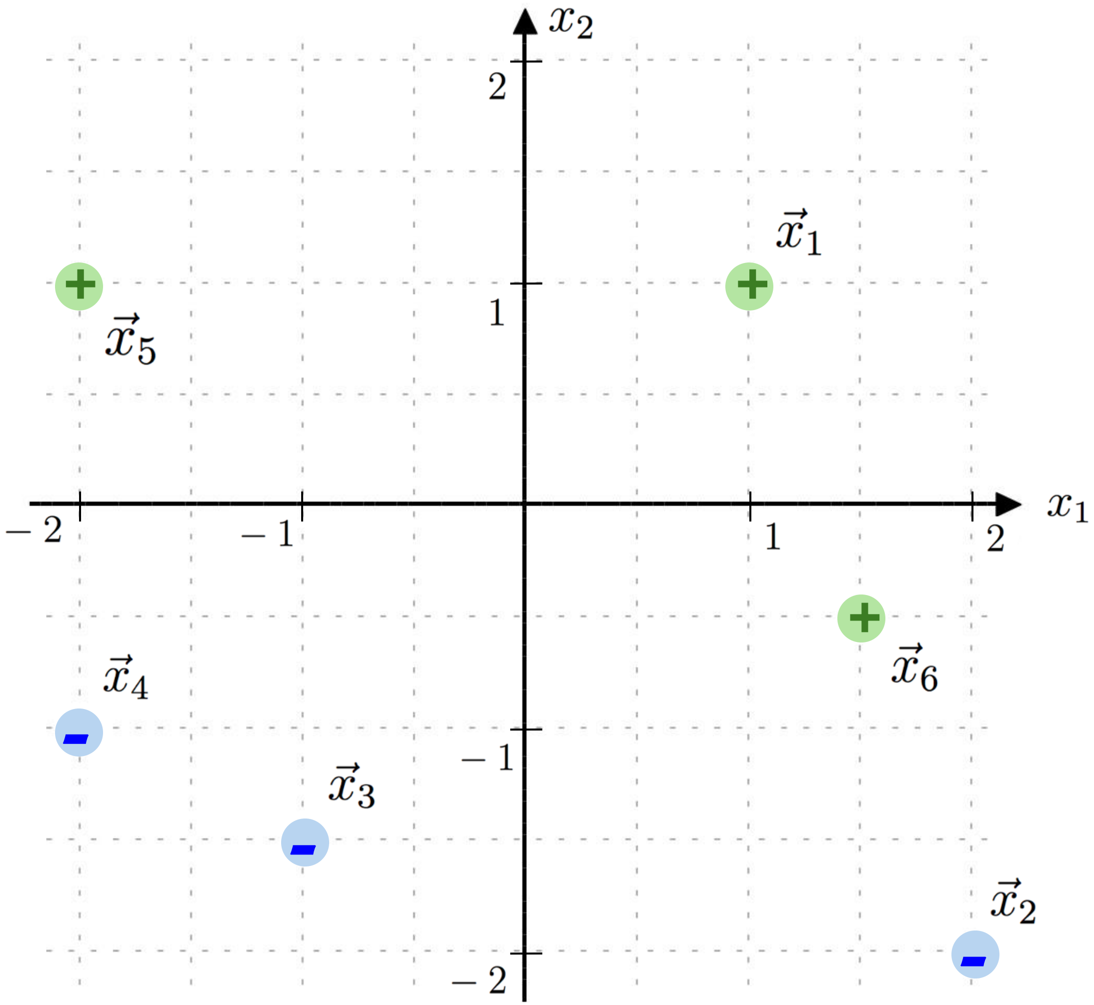
Example modified from Jessica Wu

Handout 19 example

Initial values:

$$\alpha = 0.2$$

$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$$



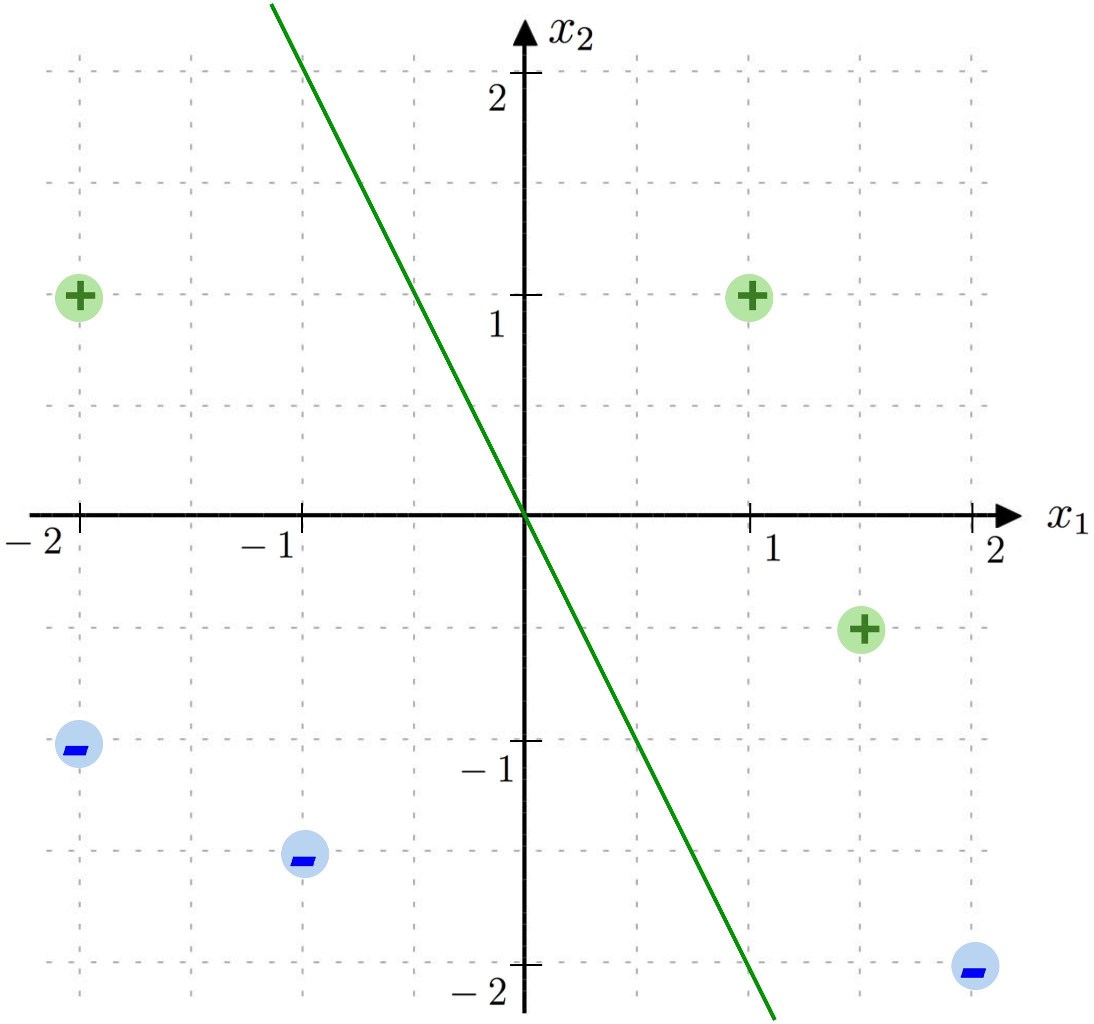
Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 19 example

Initial values:

$$\alpha = 0.2$$

$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$$



Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 19 example

$$\alpha = 0.2$$

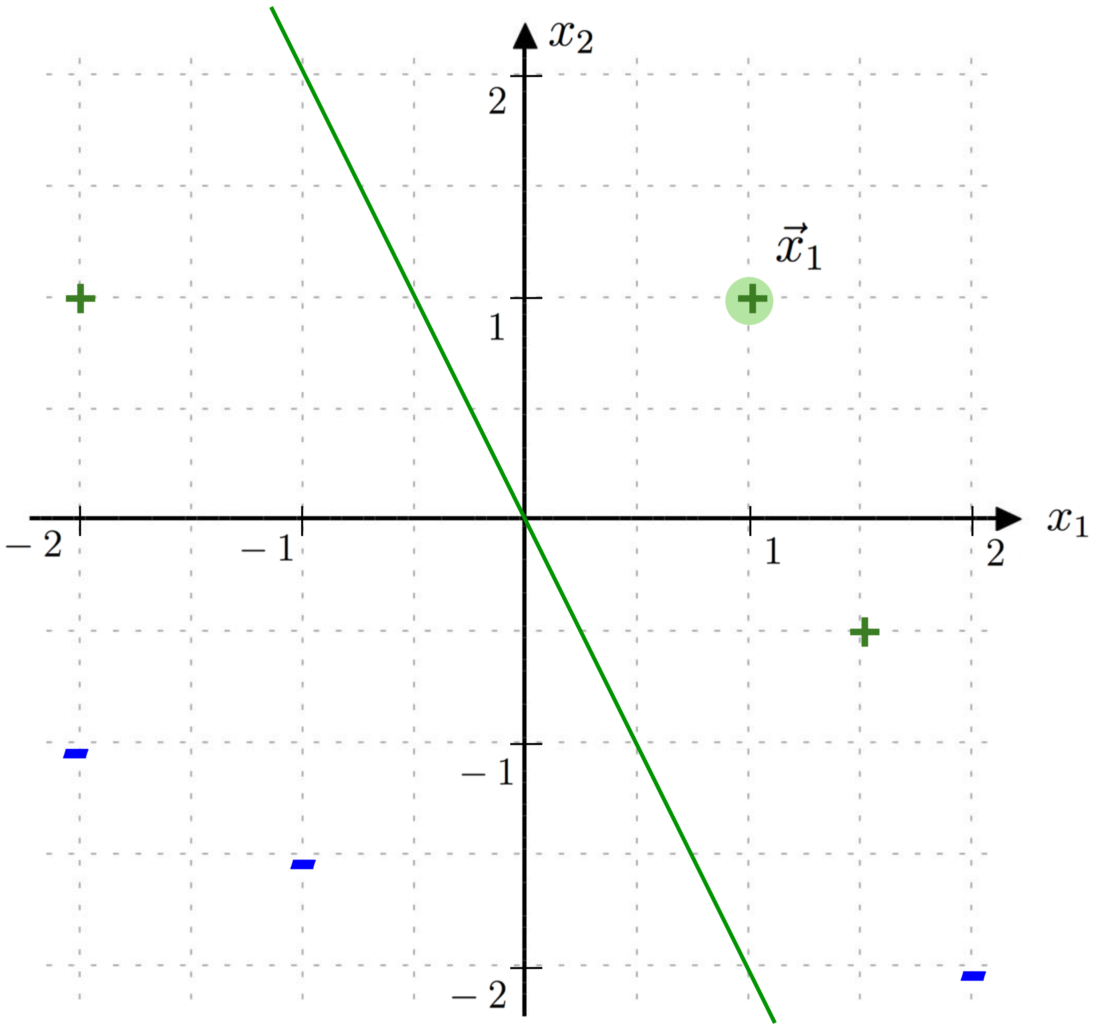
$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$$

Round 1:

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{w} \cdot \vec{x}_1 > 0$$

Correct classification, no action



Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 19 example

$$\alpha = 0.2$$

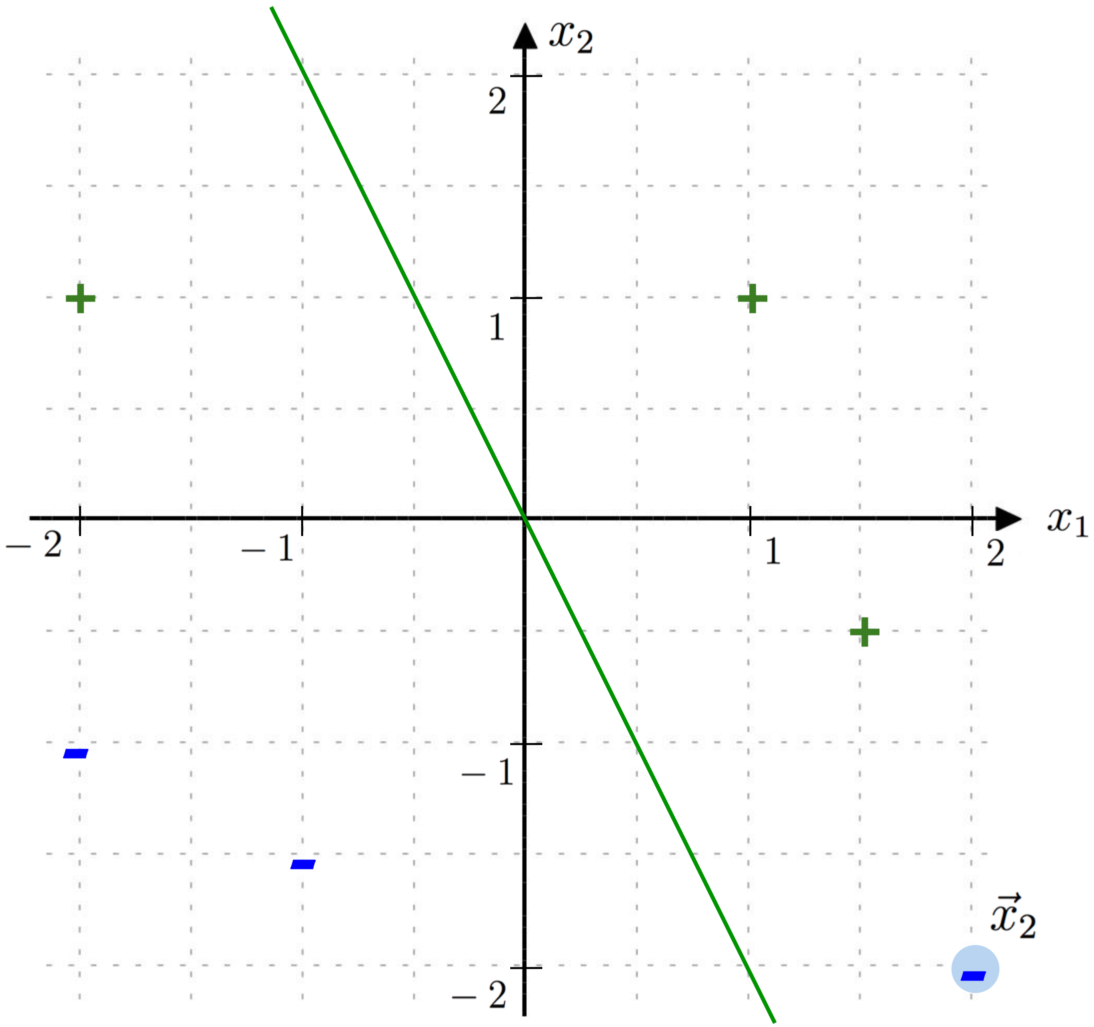
$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$$

Round 2:

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w} \cdot \vec{x}_2 > 0$$

Incorrect classification



Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 19 example

$$\alpha = 0.2$$

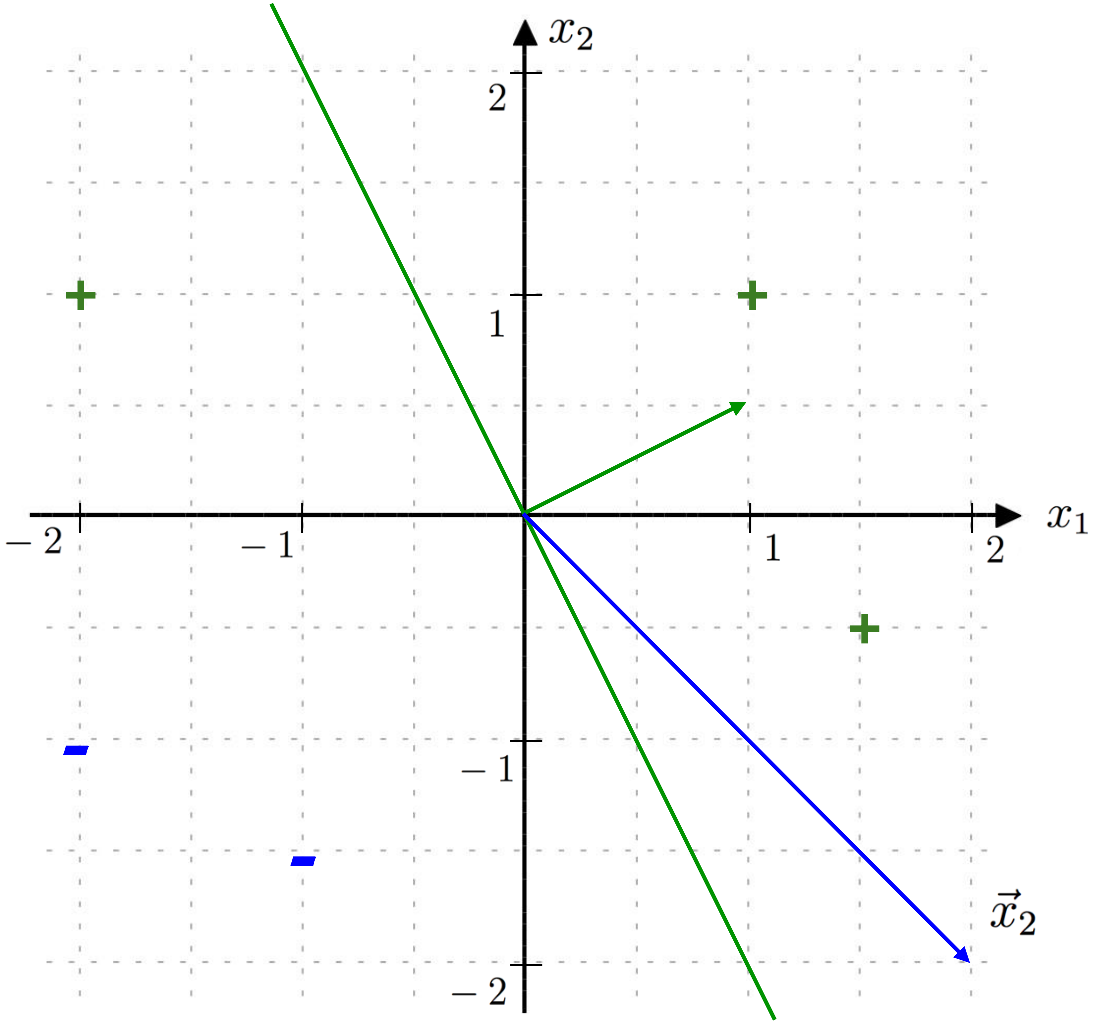
$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$$

Round 2:

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w} \cdot \vec{x}_2 > 0$$

Incorrect classification



Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 19 example

$$\alpha = 0.2$$

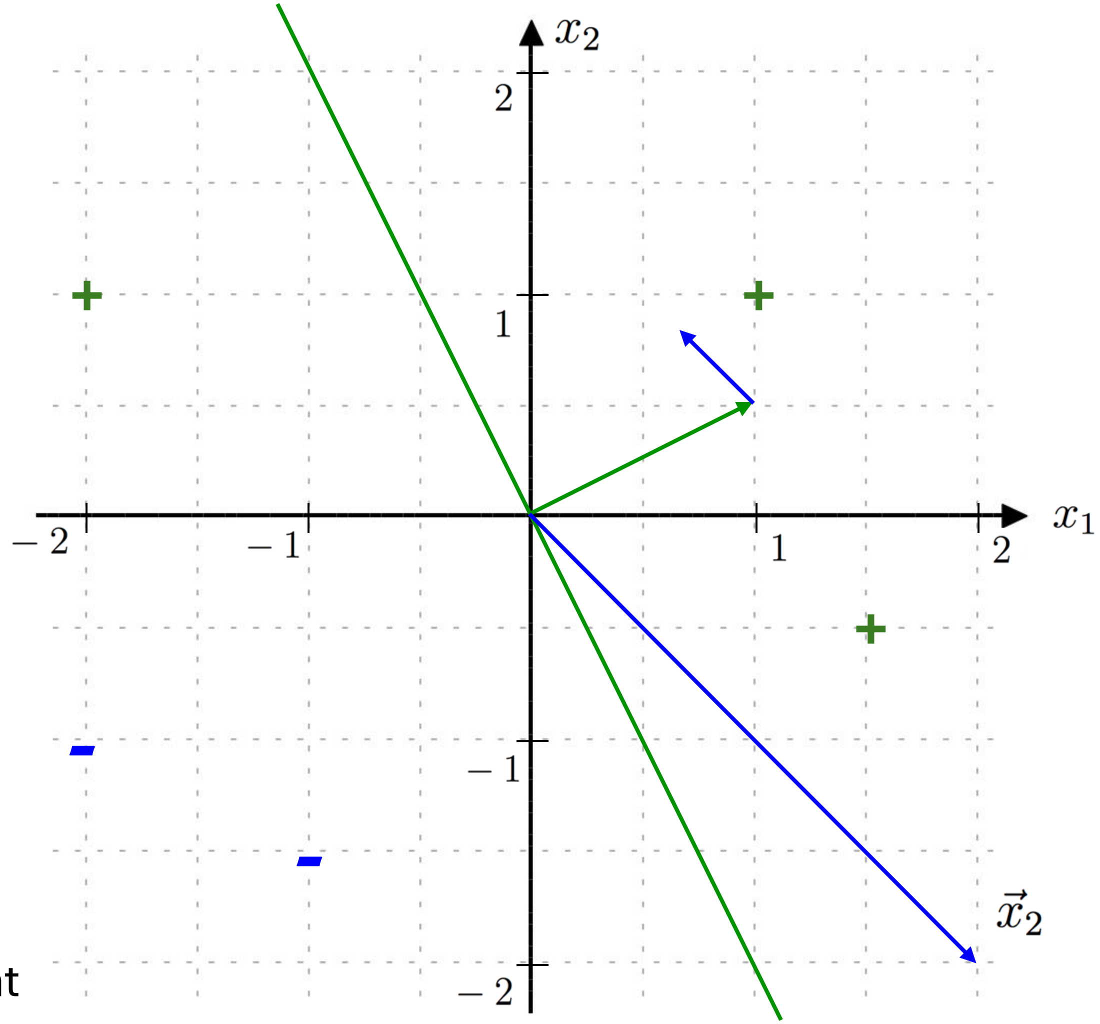
$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$$

Round 2:

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w} \cdot \vec{x}_2 > 0$$

Incorrect classification
"Push" ***w*** away from negative point



Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 19 example

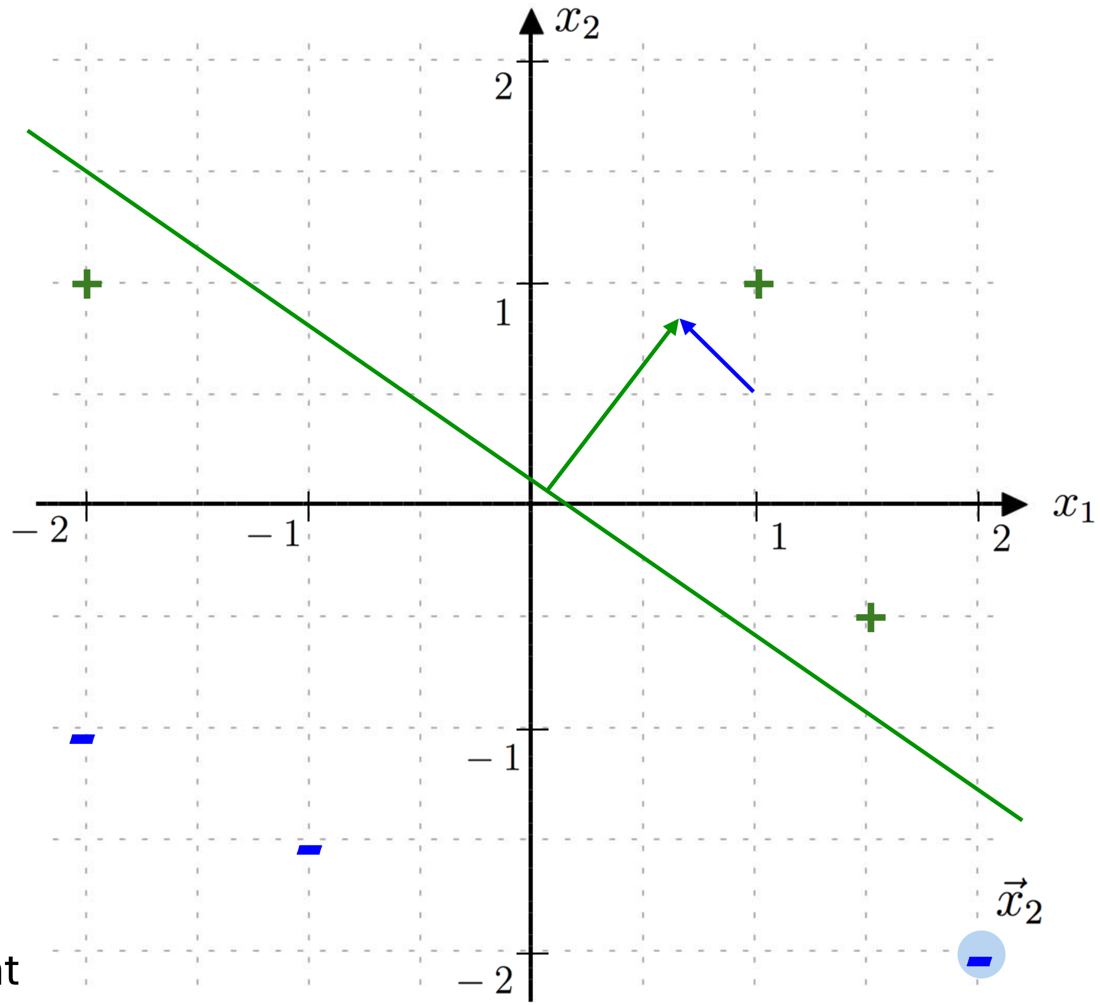
$$\alpha = 0.2$$

$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$$

Round 2:

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w} \cdot \vec{x}_2 > 0$$



Incorrect classification
"Push" ***w*** away from negative point

Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 19 example

$$\alpha = 0.2$$

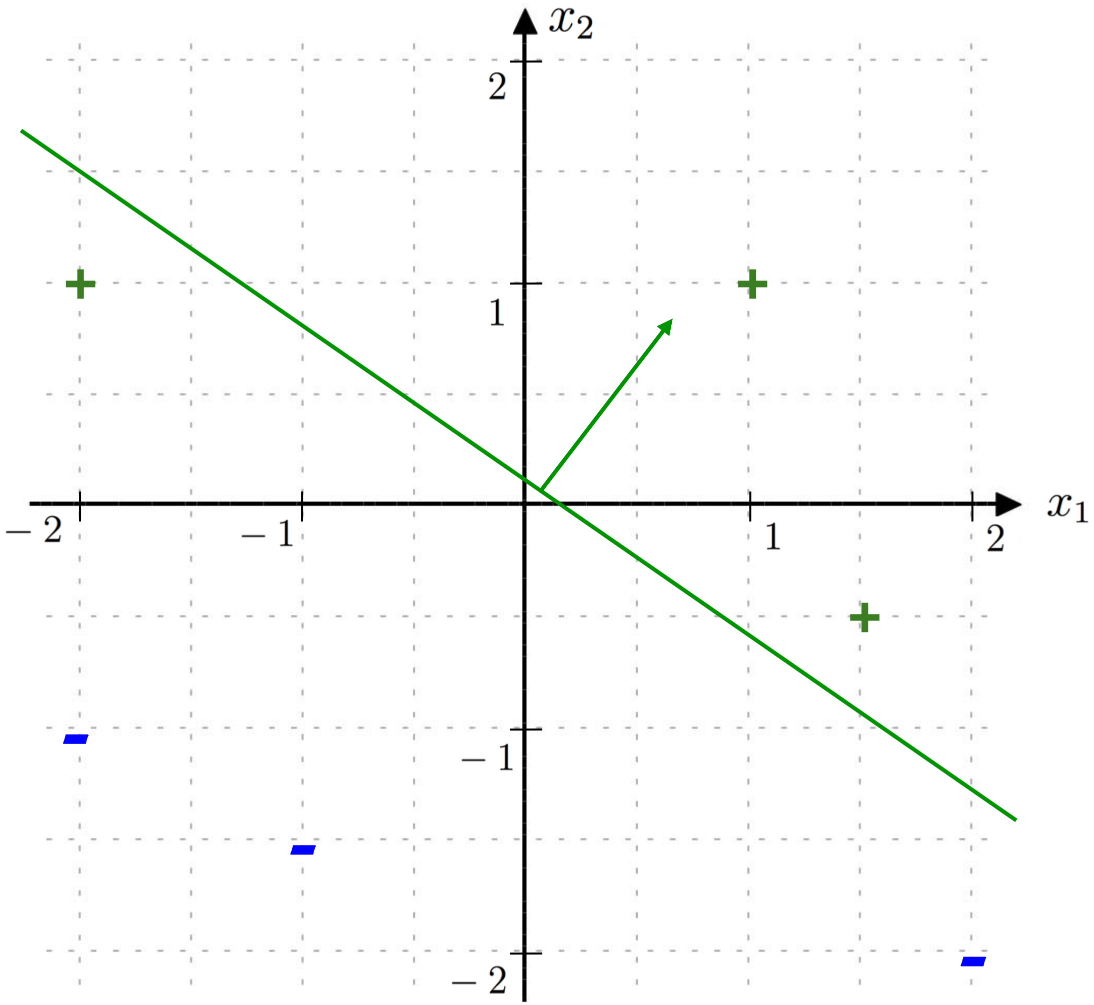
$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$$

Round 2:

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w} \cdot \vec{x}_2 > 0$$

What is the new weight vector?



Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 19 example

$$\alpha = 0.2$$

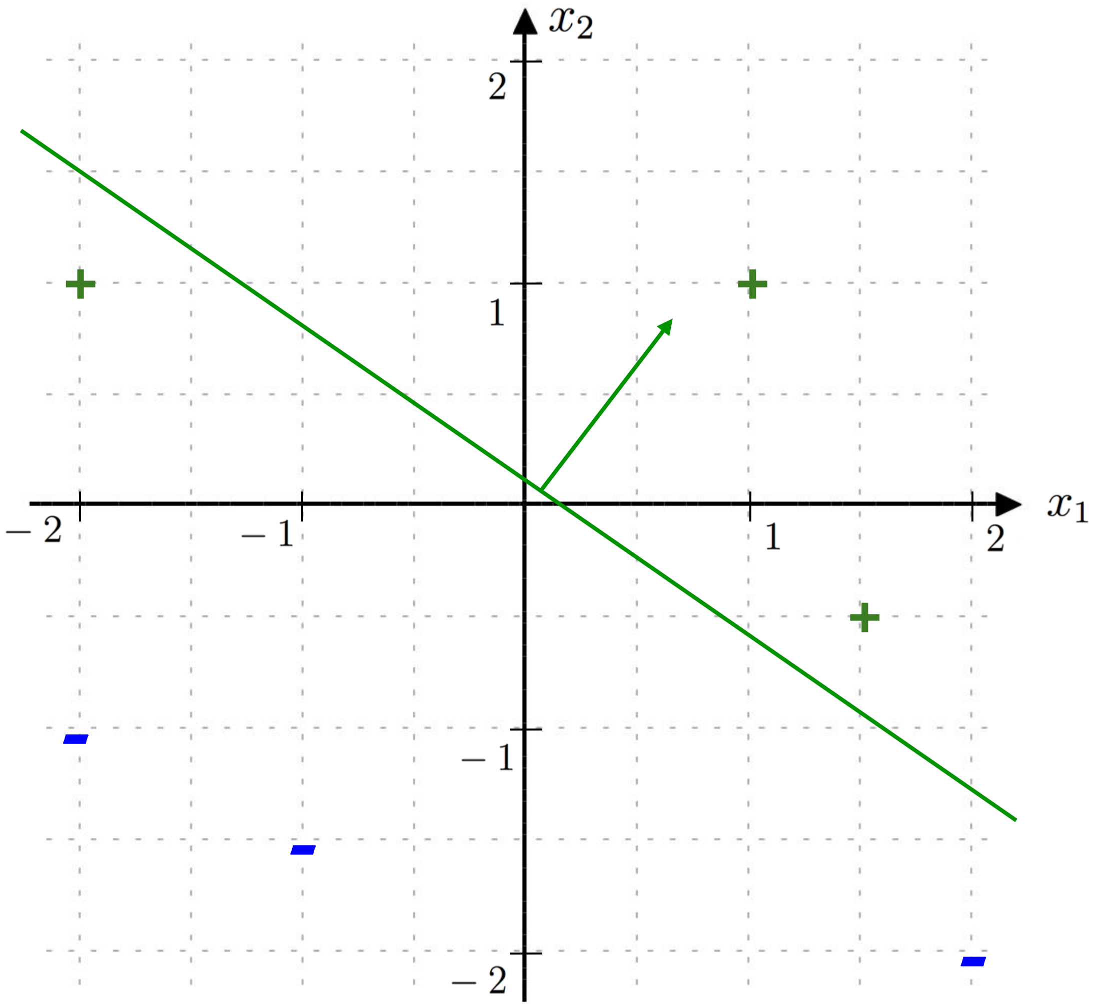
$$\vec{w} = \begin{bmatrix} -0.2 \\ 0.6 \\ 0.9 \end{bmatrix}$$

Round 5:

$$\overrightarrow{x_5} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{w} * \overrightarrow{x_5} < 0$$

What is the new weight vector?



Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 19 example

Final solution (so you can check your work):

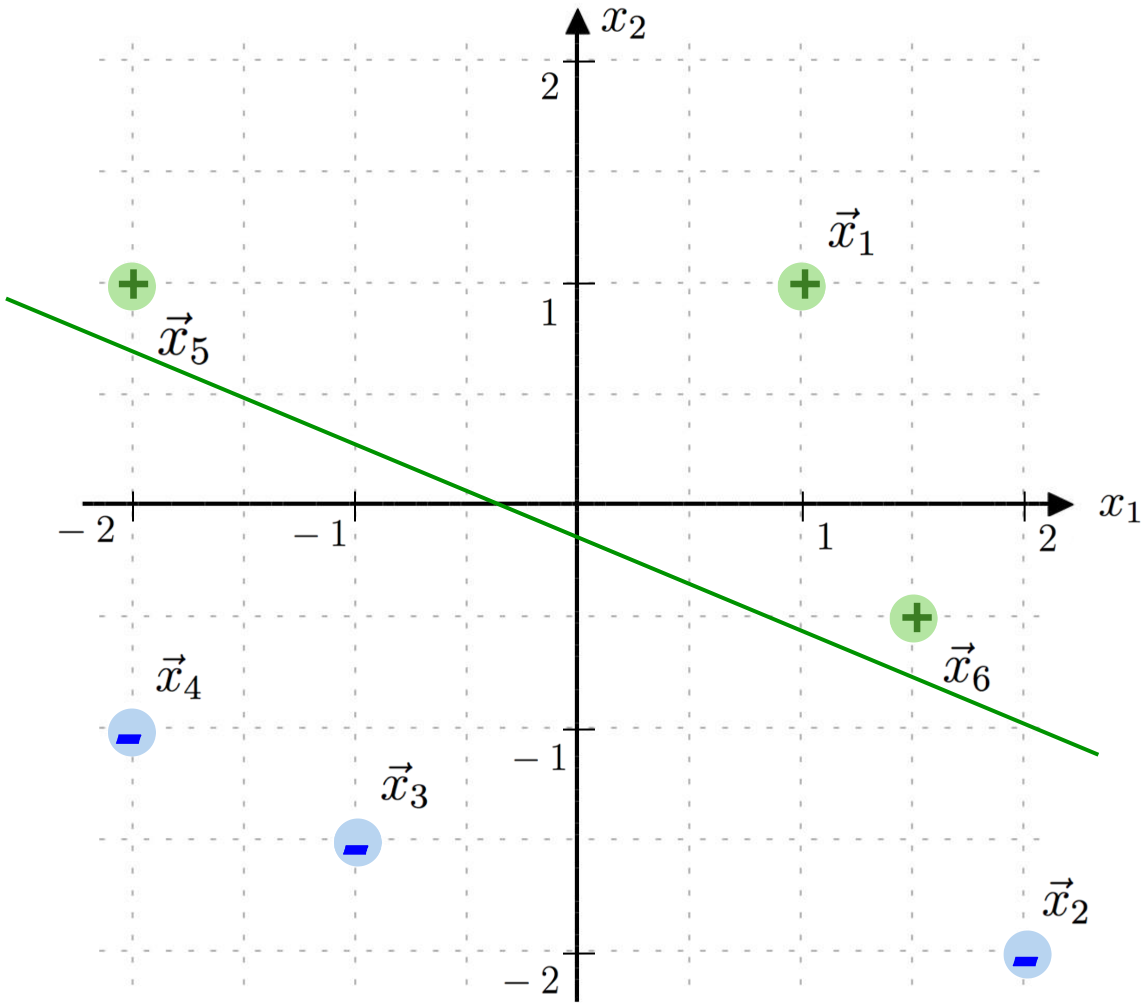
$$\vec{w}^* = \begin{bmatrix} 0.2 \\ 0.5 \\ 1 \end{bmatrix}$$

Final hyperplane:

$$0.2 + 0.5x_1 + x_2 = 0$$

$$\Rightarrow$$

$$x_2 = -0.2 - 0.5x_1$$



Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Linear Algebra Review

- Matrix rank
- Linearly Independent rows/columns
- Null space

Convergence Guarantee

- Perceptron is guaranteed to converge to a solution if a separating hyperplane exists
- Not guaranteed to converge to a "good" solution
- No guarantees about behavior if a separating hyperplane does not exist!