Identifying important prior hyperparameters in Bayesian inverse problems with efficient variance-based global sensitivity analysis

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Inverse problem

Consider the problem

$$\arg\min_{\boldsymbol{\theta}} J(\boldsymbol{\theta}), \quad J(\boldsymbol{\theta}) := \|\mathbf{B}\boldsymbol{y}(\boldsymbol{\theta}) - \mathbf{d}\|^2 \tag{1}$$

where $\boldsymbol{\theta}$ are unknown parameters, $\boldsymbol{y}$ is the solution to

$$\begin{cases} \boldsymbol{y}' = f(\boldsymbol{y}; \boldsymbol{\theta}) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases}, \quad \boldsymbol{y} \in \mathbb{R}^d \tag{2}$$

Observation operator $\mathbf{B}$ at each time selects only some of the responses from $\boldsymbol{y}(t_i)$ corresponding to data available in array $\mathbf{d}$

Bayesian context

Assume data measurement procedure involves i.i.d. random noise

$$y_i(\boldsymbol{\theta}) - d_i \sim \pi_{\text{noise}}$$

Noise model gives likelihood $\pi_{\text{like}}(\boldsymbol{\theta}) = \pi(\mathbf{d}|\boldsymbol{\theta})$

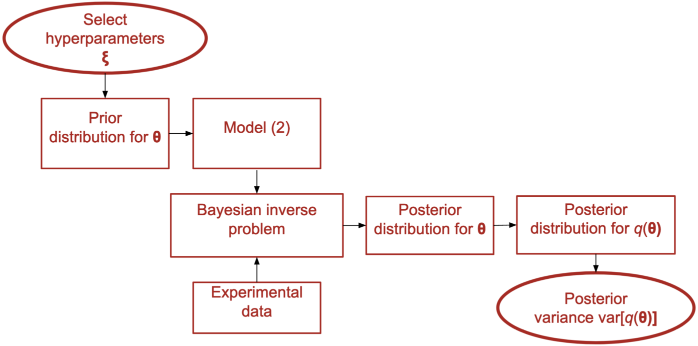
Incorporate prior beliefs/assumptions about $\boldsymbol{\theta}$ in prior distribution $\pi_{\text{prior}}(\boldsymbol{\theta})$

$$\textbf{Bayes' rule: } \pi_{\text{post}}(\boldsymbol{\theta}) = \frac{\pi_{\text{like}}(\boldsymbol{\theta})\pi_{\text{prior}}(\boldsymbol{\theta})}{\int_{\mathbb{R}^n} \pi_{\text{like}}(\boldsymbol{\theta})\pi_{\text{prior}}(\boldsymbol{\theta})d\boldsymbol{\theta}}$$

Posterior distribution in general constructed using Markov Chain Monte Carlo

GSA for Bayesian inverse problem

For Bayesian inference: Select a parameterized prior $\pi_{\text{prior}}(\boldsymbol{\theta}; \boldsymbol{\xi})$ with hyperparameters $\boldsymbol{\xi}$
Consider $F(\boldsymbol{\xi})$, which maps hyperparameters to a statistic of the posterior for QoI $q(\boldsymbol{\theta})$



Goal: Study sensitivity of $F(\boldsymbol{\xi})$ to prior hyperparameters

Variance-based global sensitivity analysis

- Consider a model $y = f(\boldsymbol{x})$ where $y \in \mathbb{R}$, and $\boldsymbol{x} \sim \pi(\boldsymbol{x})$ has independently distributed entries
- Sobol' indices are invaluable tools for GSA which measure the contribution of each input to variance in model output:

$$S_k := \frac{\mathrm{var}[f_k(x_k)]}{\mathrm{var}[f(\boldsymbol{x})]}, \quad S_k^T := 1 - \frac{\mathrm{var}[\mathbb{E}(f(\boldsymbol{x})|x_j,\ j \neq k)]}{\mathrm{var}[f(\boldsymbol{x})]}$$

- $f_k(x_k) := \int f(\boldsymbol{x}) d\boldsymbol{x}_{-k} - \mathbb{E}(f\boldsymbol{x})$, where $d\boldsymbol{x}_{-k}$ denotes integrating over all inputs **except** x_k
- First order Sobol' index S_k measures influence of x_k **outside of** interactions
- Total Sobol' index S_k^T measures influence of x_k **including** interactions with other inputs

Related Work

- Robust Bayesian analysis - tools to determine if posterior is robust to different priors in inference problems
- Hyper-differential sensitivity analysis (HDSA) has been used for Bayesian inverse problems to study measures of posterior uncertainty[1]
- Derivative-based global sensitivity measures (DGSM) has been used to study the sensitivity of information gain to uncertain model parameters [2]
- Variance-based GSA of function similar to $F(\xi)$. Emulated by Gaussian process, training data computed by MCMC [3]

[1] I. Sunseri, A. Alexanderian, J. Hart, B. van Bloemen Waanders. Hyper-differential sensitivity analysis for nonlinear Bayesian inverse problems. 2022.

[2] A. Chowdhary, A. Alexanderian. Sensitivity Analysis of the Information Gain in Infinite-Dimensional Bayesian Linear Inverse Problems. 2023.

[3] I. Vernon, J. P. Gosling. A Bayesian Computer Model Analysis of Robust Bayesian Analyses. 2022.

Method

- Most statistics or measures of uncertainty require estimating an integral
- Evaluating $F(\xi)$ using MCMC for different ξ is expensive
- **Note:** π_{like} does not depend on ξ
- **Question:** Can we re-use likelihood evaluations for different ξ's?
- Yes, but we have to be careful about what distribution we integrate over!

Importance Sampling

Consider integrating the following by Monte Carlo integration

$$F_{\text{mean}}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} q(\boldsymbol{\theta})\pi_{\text{post}}(\boldsymbol{\theta};\boldsymbol{\xi})d\boldsymbol{\theta}$$

Sampling uniformly over $\mathbb{R}^n$ will not work

Importance Sampling: Choose an auxiliary distribution π_{IS} to sample from

$$\int_{\mathbb{R}^n} q(\boldsymbol{\theta})\pi_{\text{post}}(\boldsymbol{\theta};\boldsymbol{\xi})d\boldsymbol{\theta} = \int_{\mathbb{R}^n} q(\boldsymbol{\theta})\frac{\pi_{\text{post}}(\boldsymbol{\theta};\boldsymbol{\xi})}{\pi_{\text{IS}}(\boldsymbol{\theta})}\pi_{\text{IS}}(d\boldsymbol{\theta}),$$

π_{IS} should be "close" to π_{post} for this to work

How to choose the auxiliary distribution

- For this scheme to work, π_{IS} should be "close" to $\pi_{\text{post}}(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$
- Take $\pi_{\text{IS}} \propto \pi_{\text{like}}\pi_{\text{prIS}}$ where π_{prIS} is same class distribution as priors
- Building IS sample set requires one MCMC run
- Find hyperparameters for π_{prIS} by minimizing the total KL-divergence[4]:

$$\boldsymbol{\xi}^* = \arg\min_{\boldsymbol{\xi}} \sum_{i=1}^{M} \int_{\mathbb{R}^n} \log\left(\frac{\pi_{\text{IS}}(\boldsymbol{\theta};\boldsymbol{\xi})}{\pi_{\text{post}}(\boldsymbol{\theta};\boldsymbol{\xi}_i)}\right)\pi_{\text{IS}}(\boldsymbol{\theta};\boldsymbol{\xi})d\boldsymbol{\theta}$$

[4] J. Zhang, M.D. Shields. On the quantification and efficient propagation of imprecise probabilities resulting from small datasets. 2017.

Importance sampling-Monte Carlo estimator

Estimate $F_{\text{mean}}(\boldsymbol{\xi})$ by Monte Carlo integration:

$$\int_{\mathbb{R}^n} q(\boldsymbol{\theta}) \frac{\pi_{\text{post}}(\boldsymbol{\theta};\boldsymbol{\xi})}{\pi_{\text{IS}}(\boldsymbol{\theta})} \pi_{\text{IS}}(d\boldsymbol{\theta}) = \frac{1}{\int_{\mathbb{R}^n} \frac{\pi_{\text{pr}}(\boldsymbol{\theta};\boldsymbol{\xi})}{\pi_{\text{prIS}}(\boldsymbol{\theta})} \pi_{\text{prIS}}(d\boldsymbol{\theta})} \int_{\mathbb{R}^n} q(\boldsymbol{\theta}) \frac{\pi_{\text{pr}}(\boldsymbol{\theta};\boldsymbol{\xi})}{\pi_{\text{prIS}}(\boldsymbol{\theta})} \pi_{\text{prIS}}(d\boldsymbol{\theta})$$

$$\approx \frac{1}{\sum_{j=1}^N \frac{\pi_{\text{pr}}(\boldsymbol{\theta}_j;\boldsymbol{\xi})}{\pi_{\text{prIS}}(\boldsymbol{\theta}_j)}} \sum_{j=1}^N q(\boldsymbol{\theta}_j) \frac{\pi_{\text{pr}}(\boldsymbol{\theta}_j;\boldsymbol{\xi})}{\pi_{\text{prIS}}(\boldsymbol{\theta}_j)}, \quad \boldsymbol{\theta}_j \sim \pi_{\text{IS}}$$

When we change $\boldsymbol{\xi}$, now we only need to re-evaluate the prior distribution

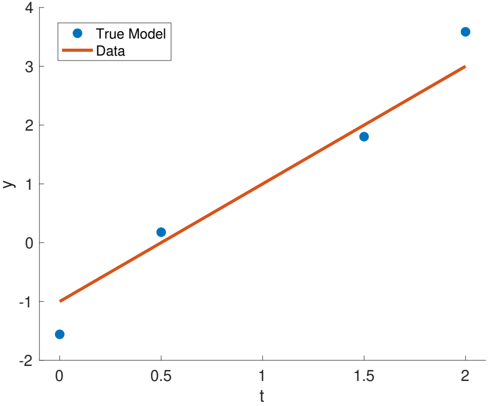
Proposed Method

1. Build ISMC sample set $\{\boldsymbol{\theta}_j\}_{j=1}^N$ by sampling from π_{IS} using MCMC
2. Evaluate the QoI $q(\boldsymbol{\theta})$ at the sample points
3. For hyperparameter samples $\{\xi_i\}_{i=1}^M$, evaluate π_{pr} at samples $\{\boldsymbol{\theta}_j\}_{j=1}^N$
4. Estimate by Monte Carlo integration $\{F(\boldsymbol{\theta}_j)\}_{j=1}^N$
5. Use $\{F(\boldsymbol{\theta}_j)\}_{j=1}^N$ to estimate Sobol' indices (surrogate-assisted or sampling)

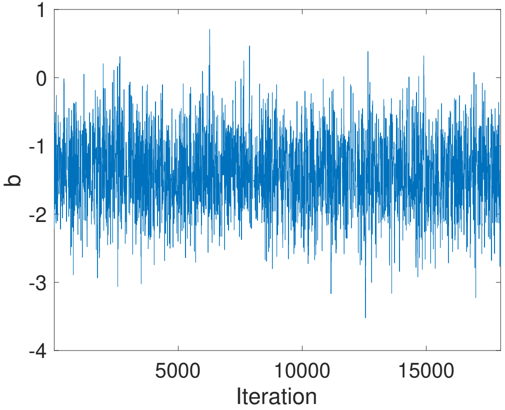
Assuming the likelihood is expensive to evaluate, the costliest step is the first one

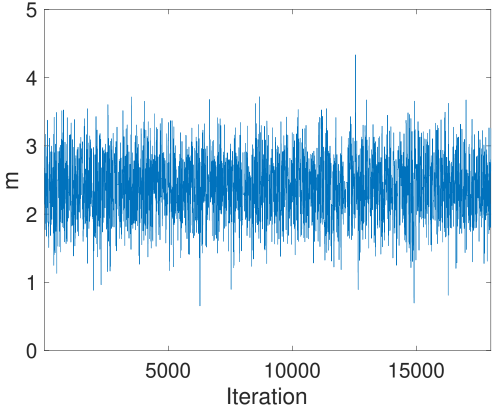
Example: Fitting Noisy Data to a Line

- Fit data to $y = mx + b$
- Estimate $\boldsymbol{\theta} = (m, b)$
- $b = -1,\ m = 2$
- Noise is i.i.d. normally distributed with $\sigma^2 = 1$
- Likelihood and prior are Gaussian $\implies$ posterior is Gaussian and can be analytically computed



Parameter estimation with MCMC





Problem Setup

- Prior is Gaussian with $\boldsymbol{\xi} = (\mu, \Gamma)$
- Nominal $\mu = \begin{bmatrix} 2.4 \\ -1.4 \end{bmatrix}$
- Nominal $\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- We let the hyperparameters vary by $\pm 50\%$ of the respective nominal value

- Consider two different QoIs
- Linear: $q(\boldsymbol{\theta}) = \boldsymbol{\theta}^\top \begin{bmatrix} 5 \\ 1 \end{bmatrix}$
- Nonlinear: $q(\boldsymbol{\theta}) = \boldsymbol{\theta}^\top \boldsymbol{\theta}$
- We are interested in both the posterior means and variances
- These can be analytically computed for both QoIs

ISMC total convergence for linear QoI

Do estimates of $F(\xi)$ converge on average for different choices of ξ?

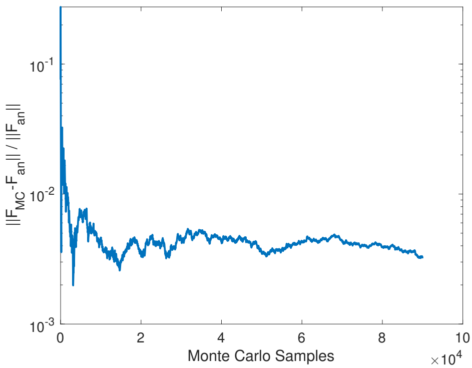


Figure: $F_{\mathrm{mean}}(\xi)$ for the linear QoI

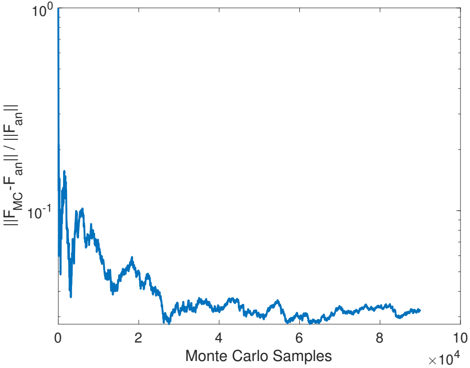


Figure: $F_{\mathrm{var}}(\xi)$ for the linear QoI

ISMC GSA for linear QoI

We use a polynomial chaos surrogate model and compare to true indices

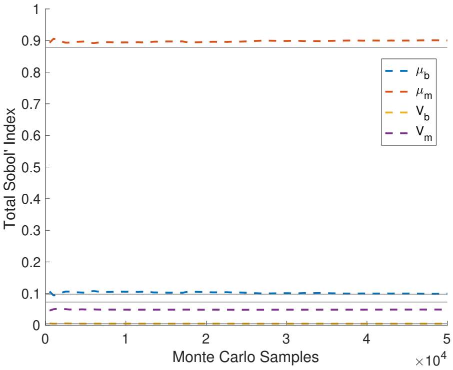


Figure: Total indices of $F_{\mathrm{mean}}(\xi)$ for the linear QoI

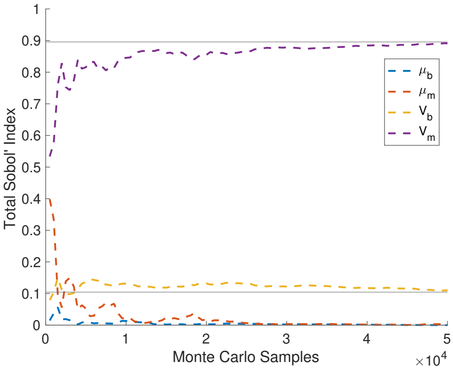


Figure: Total indices of $F_{\mathrm{var}}(\xi)$ for the linear QoI

ISMC total convergence for nonlinear QoI

Do estimates of $F(\xi)$ converge on average for different choices of ξ?

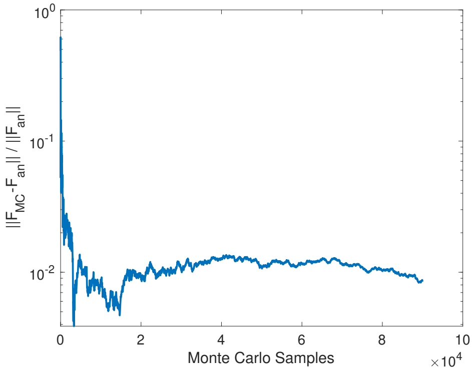
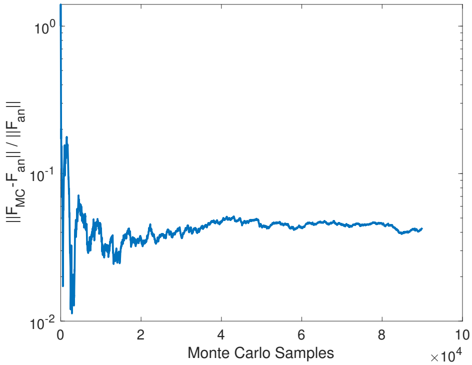


Figure: $F_{\mathrm{mean}}(\xi)$ for the nonlinear QoI



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ISMC GSA for nonlinear QoI

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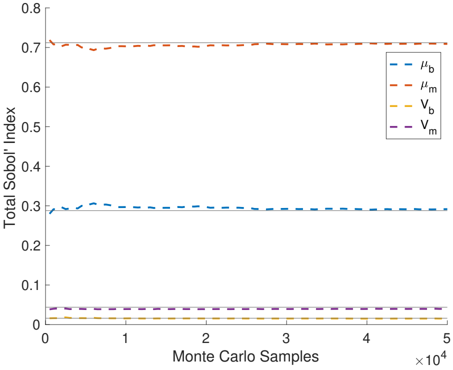

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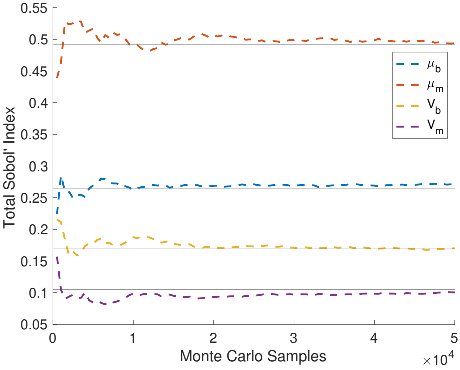

Figure: Total indices of $F_{\mathrm{var}}(\xi)$ for the nonlinear QoI

Nonlinear Bayesian inverse problems

Applying the method to nonlinear problems introduces more challenges:

- Model is a “black-box”
- Nothing to compare our results against
- Efficiently sampling from π_{IS} by MCMC could be difficult
- Importance sampling could fail for some priors

References

1. I. Sunseri, A. Alexanderian, J. Hart, B. van Bloemen Waanders. Hyper-differential sensitivity analysis for nonlinear Bayesian inverse problems. 2022.
2. A. Chowdhary, A. Alexanderian. Sensitivity Analysis of the Information Gain in Infinite-Dimensional Bayesian Linear Inverse Problems. 2023.
3. I. Vernon, J. P. Gosling. A Bayesian Computer Model Analysis of Robust Bayesian Analyses. 2022.
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5. S. Tokdar, R. Kass, Importance sampling: A review. 2010.
6. J. O. Berger, D. R. Insua, F. Ruggeri. Bayesian Robustness. 2000.