Continuous-Time Non-Periodic Signals :
The Fourier Transform

Time Domain		Frequency Domain
Continuous non-periodic	$\longrightarrow$	(continuous); non-periodic.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(w) \cdot e^{jwt} \, dw$$

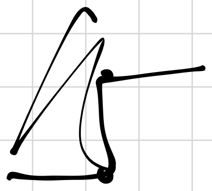
$$X(w) = \int_{-\infty}^{+\infty} x(t) \, e^{-jwt} \, dt$$

p. 242 'den başlayacak

$$\underset{\text{time domain}}{x(t)} \xleftrightarrow{FT} \underset{\text{frequency domain}}{X(w)}$$

(*) These integrals may not converge for all functions of $x(t)$ and $X(w)$. If the following condition is satisfied, the time domain signal will converge.

$$\left\{ \int_{-\infty}^{+\infty} |x(t)|^2 \, dt < \infty \right\}$$



Dirichlet Conditions :

Pointwise convergence is guaranteed excep on the discontinuities if the following conditions are met.

① $x(t)$ is absolutely integrable

$$\int_{-\infty}^{+\infty} |x(t)| \, dt < \infty$$

② $x(t)$ has finite number of maxima, minima and discontinuities in any finite interval.

③ The size of any discontinuity is finite.

In some cases, we use FT as a problem solving tool even if Dirichlet conditions are not met.

Ex Find the FT of $x(t) = e^{-\alpha t} u(t)$.

. This FT does not converge when $\alpha \leq 0$

Check 

$$\alpha \leq 0 \quad \int_0^{\infty} e^{-\alpha t}\, dt = \frac{-1}{\alpha} e^{-\alpha t}\Big]_0^{\infty} = \infty$$

$\alpha \leq 0 \quad -\alpha = \alpha$

. when $\alpha > 0$

$$X(\omega) = \int_{-\infty}^{+\infty} e^{-\alpha t} \cdot \underline{u(t)} \cdot e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-(\alpha + j\omega)t}\, dt$$

$$= -\frac{1}{(\alpha + j\omega)} \cdot \left[e^{-(\alpha + j\omega)t} \right]_0^{\infty}$$

$$\boxed{X(\omega) = \frac{1}{\alpha + j\omega}}$$

$$= \frac{\alpha - j\omega}{\alpha^2 - (j\omega)^2} = \frac{\alpha - j\omega}{\alpha^2 - (\underline{j})^2 \omega^2}$$

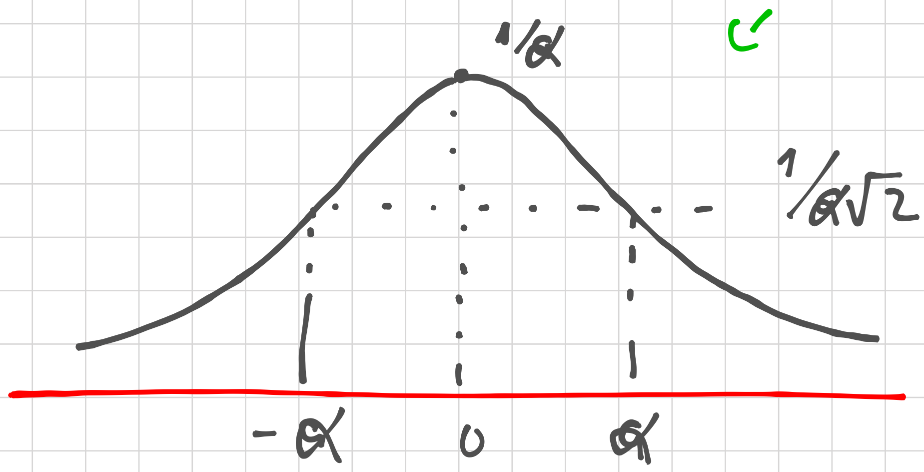
$$= \frac{\alpha - j\omega}{\alpha^2 + \omega^2}$$

$$|X(\omega)| = \sqrt{\left(\frac{\alpha}{\alpha^2 + \omega^2}\right)^2 + \left(\frac{\omega^2}{\alpha^2 + \omega^2}\right)^2}$$

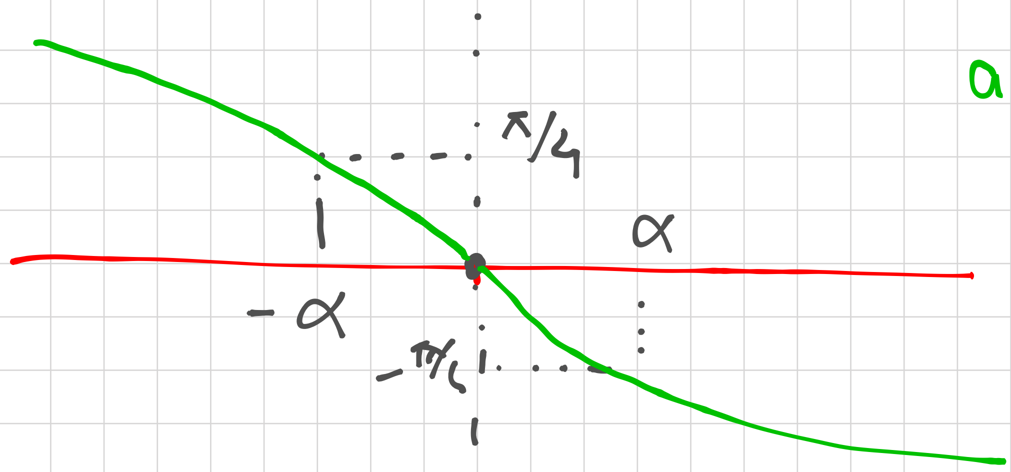
$$= \left(\frac{\alpha^2 + \omega^2}{(\alpha^2 + \omega^2)^2}\right)^{1/2}$$

$$= \left(\frac{1}{\alpha^2 + \omega^2}\right)^{1/2}$$

$$\arg\{X(\omega)\} = \arctan\left[\frac{-\omega / \cancel{\alpha^2 + \omega^2}}{\alpha / \cancel{\alpha^2 + \omega^2}}\right] = -\arctan(\omega/\alpha)$$

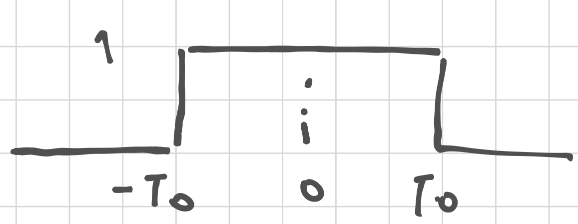


$|X(\omega)|$



$\arg\{X(\omega)\}$

Ex. FT of a rectangular signal



$$x(t) = \begin{cases} 1, & t < |T_0| \\ 0, & t \geq T_0 \end{cases}$$

$x(t)$ is absolutely integrable (assuming T_0 is finite) ∴ A FT pair exists

$$X(\omega) = \int_{-T_0}^{T_0} 1 \cdot e^{-j\omega t}\, dt$$

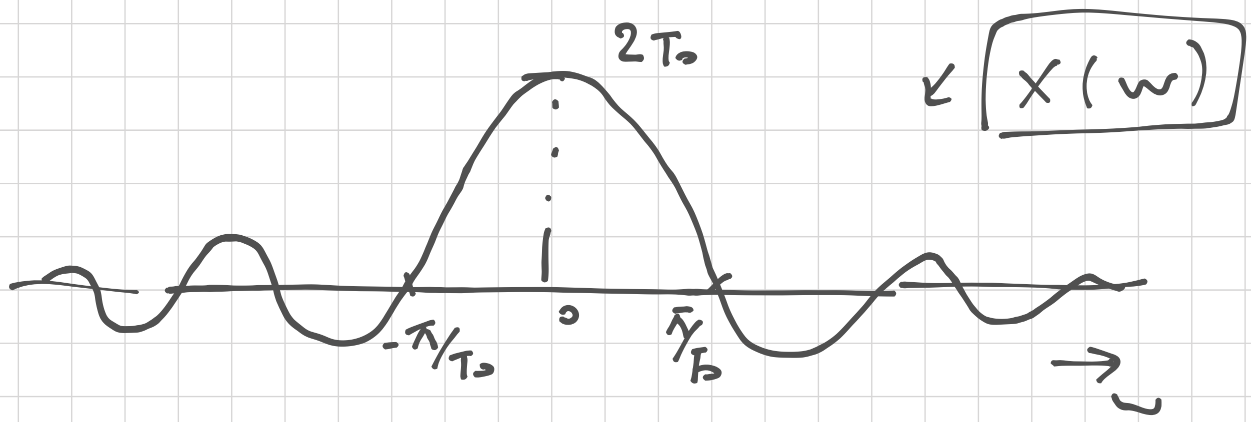
$$= -\frac{1}{j\omega}\left[e^{-j\omega t} \right]_{-T_0}^{T_0}$$

$$= \frac{2}{\omega} \sin(\omega T_0), \quad \omega \neq 0$$

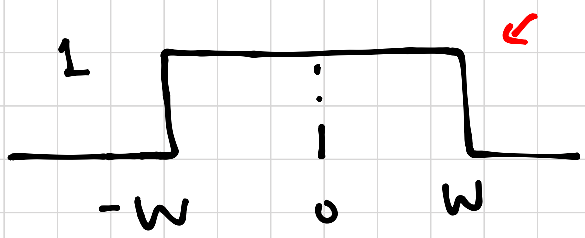
For $\omega = 0$, using L'Hôpital's rule

$$\lim_{\omega \to 0} \frac{2}{\omega} \cdot (\sin \omega T_0) = 2T_0$$

$$X(\omega) = \frac{2}{\omega} \sin(\omega T_0) = 2T_0 \cdot \mathrm{sinc}\left(\frac{\omega T_0}{\pi}\right)$$



Ex. Inverse FT of a Rectangular Spectrum.



$$X(\omega) = \begin{cases} 1, & \omega < |W| \\ 0, & \omega > |W| \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} \, d\omega$$

$$x(t) = \cdots = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$

Ex FT of a unit impulse

$$\int_{-\infty}^{+\infty} \delta(t)\, x(t) \, dt = x(0)$$

$$x(t) = \delta(t)$$

$$X(\omega) = \int_{-\infty}^{+\infty} \delta(t) \, e^{-j\omega t} \, dt = 1$$

Ex Inverse FT of an impulse spectrum

$$X(\omega) = 2\pi \, \delta(\omega)$$

$$x(t) = \frac{2\pi}{2\pi} \int_{-\infty}^{+\infty} \delta(\omega) \, e^{+j\omega t} \, d\omega = 1$$

$$\int_{-\infty}^{+\infty} 1 \, . \, dt = \infty$$

This is an example of those signals ~~whi~~ that do not satisfy the Dirichlet Conditions. But we use it as a problem solving tool, anyway.

Properties of Fourier Representations

TABLE 3.2 *The Four Fourier Representations.*

Time Domain		*Periodic* (t, n)	*Non periodic* (t, n)	Freq Domain	
Continuous	(t)	Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t}$ $X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt$ $x(t)$ has period T $\omega_o = \frac{2\pi}{T}$	Fourier Transform $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$ $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$	*Nonperiodic*	(k, ω)
Discrete	(n)	Discrete-Time Fourier Series $x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_o n}$ $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_o n}$ $x[n]$ and $X[k]$ have period N $\Omega_o = \frac{2\pi}{N}$	Discrete-Time Fourier Transform $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\,d\Omega$ $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ $X(\Omega)$ has period 2π	*Periodic*	(k, Ω)
		Discrete (k)	*Continuous* (ω, Ω)	*Frequency Domain*	

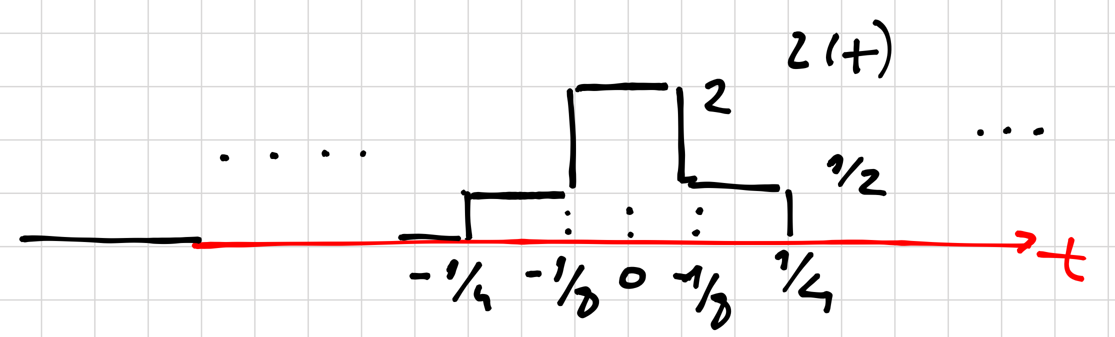
Linearity & Symmetry Properties

All four representations satisfy the linearity property, that is:

$$z(t) = a \cdot x(t) + b \cdot y(t) \xrightarrow{F} Z(\omega) = a \cdot X(\omega) + b \cdot Y(\omega)$$

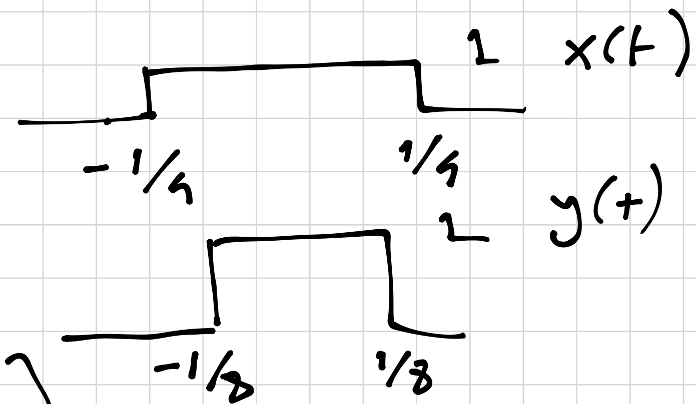
(This property applies all four classes)

Ex



Assume z(t) is periodic $T_0 = 1$

we can write



$$z(t) = \frac{3}{2} x(t) + \frac{1}{2} y(t)$$

Then, we find $X[k]$ and $Y[k]$ → use linearity property.

$$\frac{3}{2}\left(x(t) \xrightarrow{FS,\,2\pi} \frac{\sin(k\pi/4)}{k\pi} \right)$$

$$\frac{1}{2}\left(y(t) \xrightarrow{FS;\ 2\pi} \frac{\sin(k\pi/2)}{k\pi} \right)$$

$+$

$$Z[k] = \frac{3}{2k\pi}\cdot\sin(k\pi/4) + \frac{1}{2k\pi}\sin(k\pi/2)$$

Symmetry Properties: Real & Imaginary Signals

We will use FT for this section. Same applies to other three classes.

$$X^*(w) = \left[\int_{-\infty}^{+\infty} x(t)\cdot e^{-jwt}\cdot dt\right]^*$$

$$= \int_{-\infty}^{+\infty} x^*(t)\cdot e^{jwt}\cdot dt$$

- If $x(t)$ is real then $x^*(t) = x(t)$

which implies

$$X^*(w) = \int_{-\infty}^{+\infty} x(t)\cdot e^{-j(-w)t}\, dt$$

$$X^*(w) = X(-w)$$

$\therefore$ $X(w)$ is a complex-conjugate symmetric.

$$Re\{X(w)\} = Re\{X(-w)\}$$

$$Im\{X(w)\} = -Im\{X(-w)\}$$

$\therefore$ magnitude is even.
$\therefore$ Phase spectrum is odd.

Homework

Work the same issue when $x(t)$ is purely imaginary.

Later 3.9.2

3.10 } Convolution Property.

Convolution & Non-periodic Signals

Assume $x(t)$ and $h(t)$ are non-periodic.

$$y(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{+\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

Let's express $x(t-\tau)$ in terms of its FT.

$$x(t-\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) \cdot e^{j\omega(t-\tau)} d\omega$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) \cdot e^{j\omega(t-\tau)} d\omega \right] d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Bigg(\underbrace{\int_{-\infty}^{+\infty} h(\tau) \cdot e^{-j\omega\tau} d\tau}_{H(\omega)} \Bigg) X(\omega) \cdot e^{j\omega t} d\omega$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{H(\omega) X(\omega)}_{Y(\omega)} e^{j\omega t} d\omega$$

$$y(t) = h(t) * x(t) \overset{FT}{\longleftrightarrow} Y(\omega) = H(\omega) \cdot X(\omega)$$