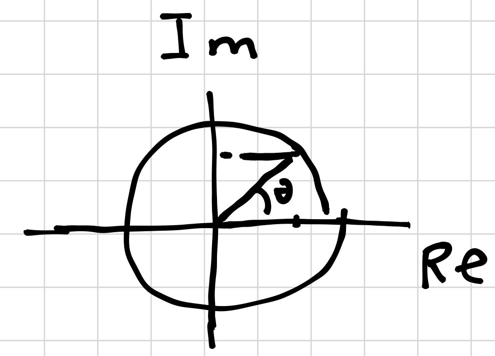
# Revisit

## CT Complex Exponential Signal

### Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$




Generally exponential and sinusoidal signals have the following form

$$x(t) = B \cdot e^{at}$$

In general, B and a may be complex numbers.

(*) If B and a are real numbers then $x(t)$ is called a "real exponential signal"

(*) Periodic Complex Exponential and Sinusoidal Signals

A second class is obtained when "a" is purely imaginary.

$$x(t) = e^{j\omega_0 t} \qquad (/* \exp\{j\omega_0 t\} */)$$

This signal is, in fact, peridic.

If $x(t)$ is periodic with a period $T$, then

$$x(t) = x(t+T)$$

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)}$$

$$e^{j\omega_0 t} = e^{j\omega_0 t} \cdot \underline{e^{j\omega_0 T}} \qquad \frac{\qquad 1}{\qquad}$$

thus $\boxed{e^{j\omega_0 T} = 1}$

- If $\boxed{\omega_0 = 0}$ then $x(t) = 1$, which means $x(t)$ is periodic with any values of T.

- If $w_0 \neq 0$ the "fundamental period", which is the smallest positive value of $T$ for which $e^{jw_0T}$ holds is:

$$T_0 = \frac{2\pi}{|w_0|}$$

$\boxed{j = \sqrt{-1}}$

: $e^{jw_0t}$ and $e^{-jw_0t}$ has the same period.

$$e^{jw_0t} = \cos(w_0 t) + j \sin(w_0 t)$$

A sinusoidal signal can be written in terms of periodic complex exponentials

$$A \cdot \cos(w_0 t + \phi) = \frac{A}{2} \cdot e^{j(w_0 t + \phi)} - \frac{A}{2} e^{-j(w_0 t + \phi)}$$

Also,

$$A \cdot \cos(w_0 t + \phi) = A \cdot \underbrace{Re}_{\text{real part}}\left\{ e^{j(w_0 t + \phi)} \right\}$$

$$A \cdot \sin(w_0 t + \phi) = A \cdot Im\left\{ e^{j(w_0 t + \phi)} \right\}$$

* Complex periodic exponential signals

Energy for a single periodic

$$E_{period} = \int_0^{T_0} |\underbrace{e^{jw_0t}}_{1}|^2 \, dt = \int_0^{T_0} 1^2 \cdot dt = \underline{\underline{T_0}}$$

$$P_{avg} = \lim_{T \to \infty} \frac{1}{T} \int_0^{T} |e^{jw_0t}|^2 \, dt = \underline{\underline{1}}$$

## General Complex Exponential Signals

$$x(t) = B \cdot e^{at}$$

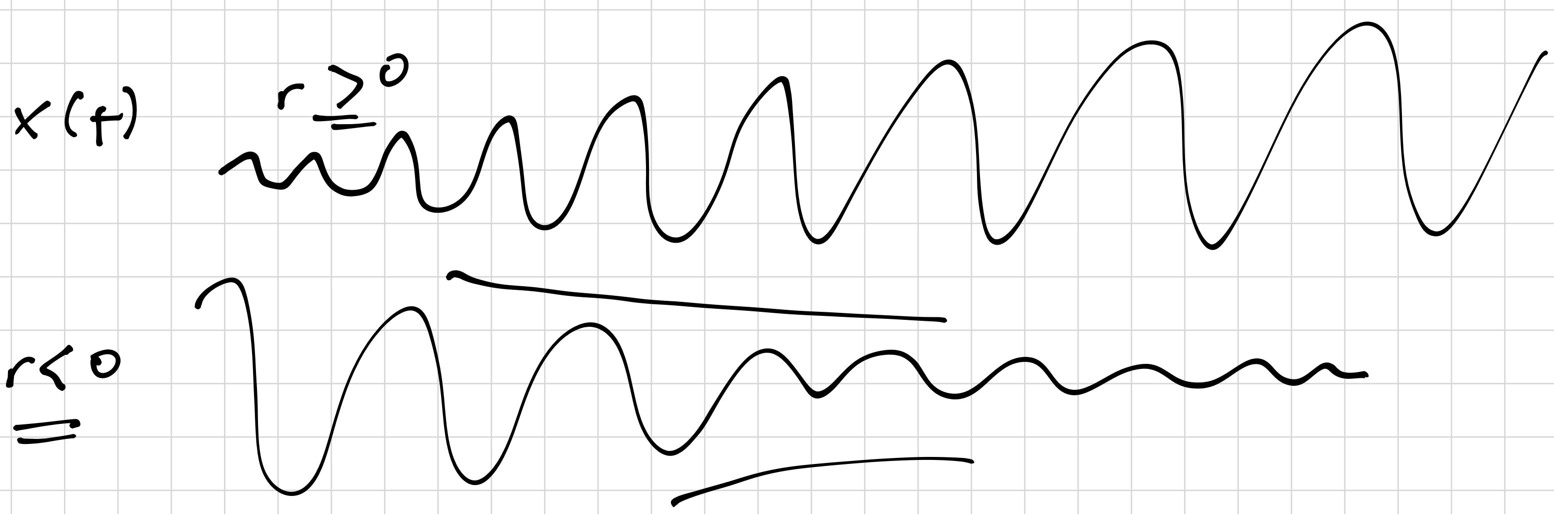
$$B = |B| \cdot e^{j\theta}$$

$$a = r + j w_0$$

$$B \cdot e^{at} = |B| e^{j\theta} \cdot e^{(r + j w_0) t}$$

$$= |B| \cdot e^{rt} \cdot e^{j(w_0 t + \theta)}$$

$$= |B| \cdot e^{rt} \cdot \cos(w_0 t + \theta)$$
$$+ j \cdot |B| \cdot e^{rt} \cdot \sin(w_0 t + \theta)$$




## Discrete Time Complex Exponential Sinusoidal Signals

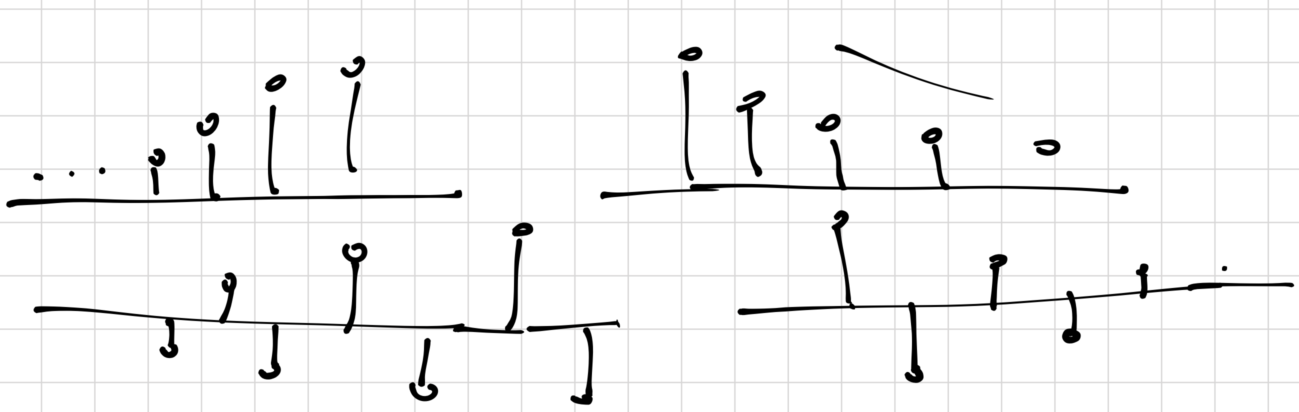
$$x[n] = B \cdot r^n$$

B and r are, in general, complex numbers.

$$x[n] = B \cdot e^{an} \qquad (r \triangleq e^{a})$$

## Real Exponential Signals

→ We already covered this.



# Sinusoidal Signals

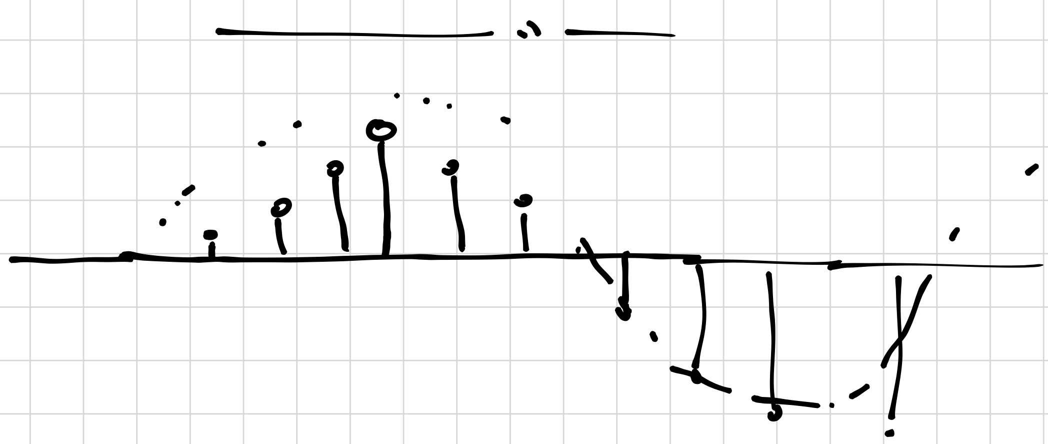
$$x[n] = e^{j\Omega n}$$

$$e^{j\Omega n} = \cos(\Omega n) + j\sin(\Omega n)$$

$$\underbrace{A\cdot\cos(\Omega n+\phi)}_{\text{real sinusoidal signal}} = \frac{A}{2}\cdot e^{j(\Omega n+\phi)} + \frac{A}{2}e^{-j(\Omega n+\phi)}$$

$\Omega \rightarrow$ frequency

$x[n]$ may not be a periodic signal!



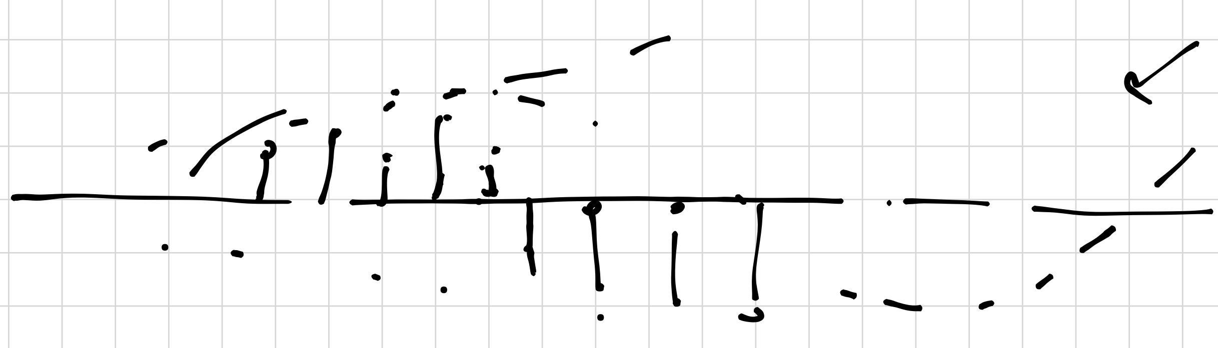
# General Complex Exponential Signals

$$x[n] = B\cdot e^{an} \qquad r = e^{a}$$

$$B = |B|\cdot e^{j\theta}$$

$$a = |a|\, e^{j\Omega n}$$

$$B r^{n} = |B|\cdot|r|^{n}\cdot\cos(\Omega n+\theta) + j\,|B|\,|r^{n}|\sin(\Omega n+\theta)$$



Look at the book.

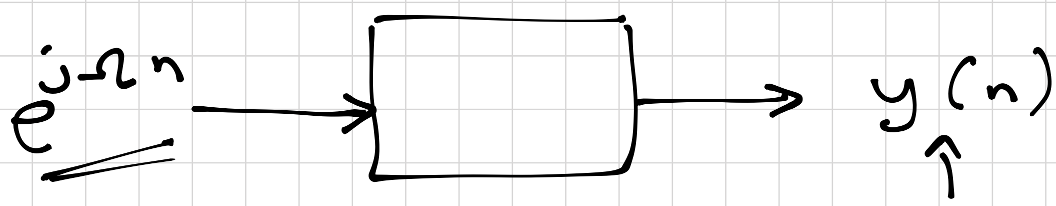
# Chapter 3
# Fourier Representation of Signals and LTI systems

- we will represent a signal as a weighted superposition of complex sinusoids.

Time domain → Frequency domain


## $\boxed{3.2}$ Complex Sinusoids and frequency Respons of LTI systems

Consider a system when the input is $x[n] = e^{j\Omega n}$




Freq. Respons

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k] \cdot e^{j\Omega(n-k)}$$

$$= e^{j\Omega n} \cdot \sum_{k=-\infty}^{+\infty} h[k] \cdot e^{-j\Omega k}$$

Let's say

$$\underline{H}\{e^{j\Omega}\} = \sum_{k=-\infty}^{+\infty} h(k) \cdot e^{-j\Omega k}$$

$$y(n) = \boxed{H\{e^{j\Omega}\}} \cdot e^{j\Omega n}$$

↳ Frequency response.

Similarly, for CT, $x(t) = e^{j\omega t}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) \cdot e^{+j\omega(t-\tau)} \, d\tau$$

$$= e^{+j\omega t} \cdot \underbrace{\int_{-\infty}^{+\infty} h(\tau) \cdot e^{-j\omega\tau} \, d\tau}_{\text{Frq. respons}}$$

$$\mathcal{H}\{j\omega\}$$

$$y(t) = e^{j\omega t} \cdot \mathcal{H}\{j\omega\}$$

## 3.3 Fourier Representation..

| Time → / ↓ Property | Periodic | Non-Periodic |
|---|---|---|
| CT (t) | Fourier Series (FS) | Fourier-Transform (FT) |
| DT (n) | Discrete Time Fourier Series (DTFS) | Discrete Time Fourier Transform (DTFT) |

### Periodic Signals - Fourier Representations

$x[n]$ is a DT signal with a fundamental period $N$. We wish to represent $x[n]$ ~~by~~ in frequency domain (DTFS)

$$\hat{x}[n] = \sum_{k} A[k] \cdot e^{j\underline{k}\Omega_0 n}$$

$\Omega_0 = \frac{2\pi}{N}$    $k$th sinusoid has a frequenc $\underline{k\Omega_0}$

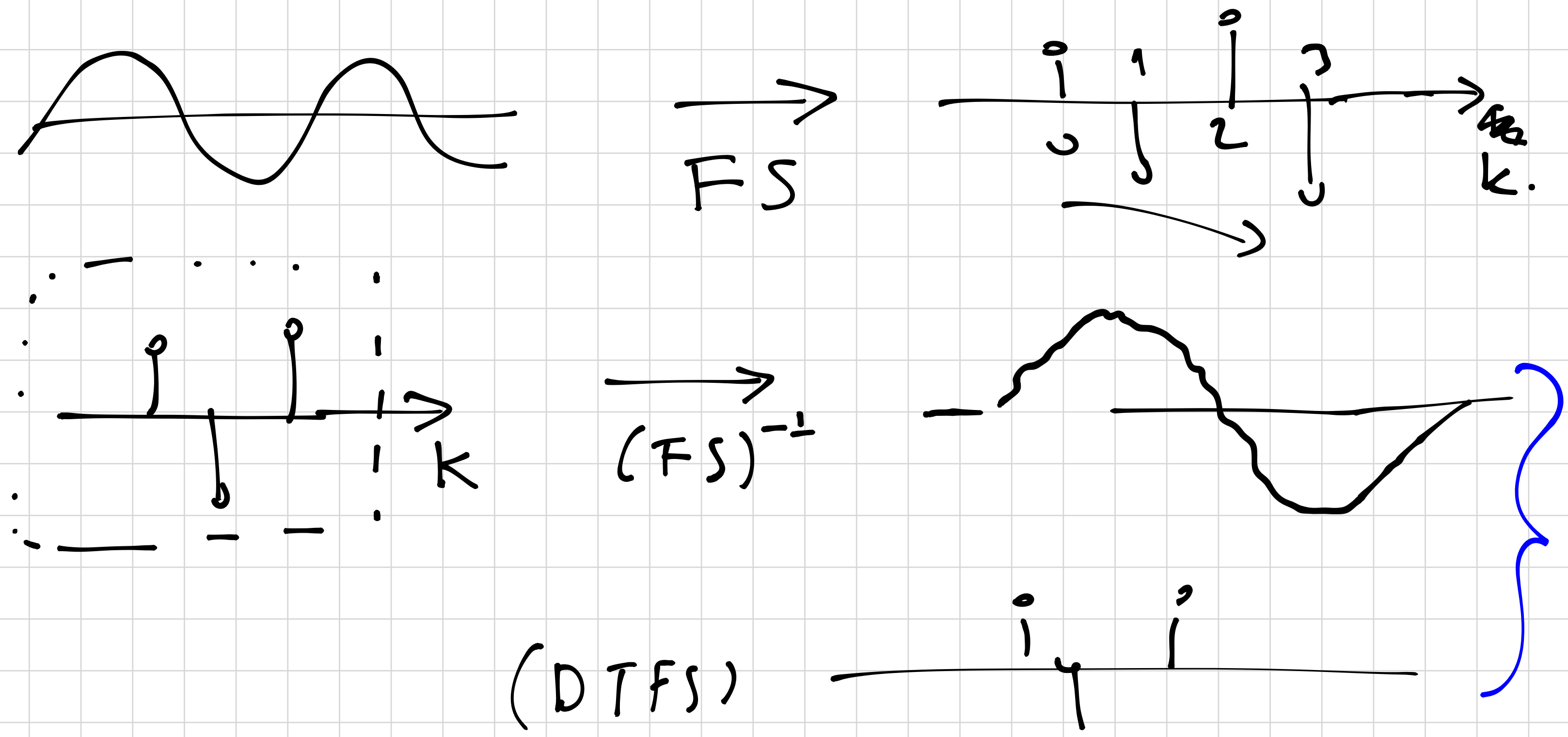
Fourier Series (FS)

CT. fundamenta period $T$

$$\hat{x}(t) = \sum_{k} A(k) \cdot e^{jk\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}$$

$k$th sinusoid has a frq. $k.\omega_0$




## Fourier-Transform Representation of Nonperiodic signal

- FT (Continuous)

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) \cdot e^{jwt} \, dw$$

- DTFT (Discrete)

$$x[n] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(e^{j\Omega}) \, e^{j\Omega n} \, d\Omega$$

dw

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## 3.4 Discrete-Time Fourier Series (DTFT) Periodic, Discrete

Let's say $x[n]$ is a DT signal with a fundamental period $N$, Fundamental frequency $\Omega_0 = 2\pi/N$

DTFS of $x[n]$

$$x[n] = \sum_{k=0}^{N-1} X[k] \cdot e^{jk\Omega_0 n} \leftarrow \textcircled{\cdot}$$

Inverse DTFS

$$X[k] = \frac{1}{N} \cdot \sum_{n=0}^{N-1} x[n] \cdot e^{-jk\Omega_0 n}$$

$X[k]$ → DTFS coefficients

• $X[k]$ is also called the "frequency domain" representation of $x[n]$. Each DTFS coefficient is associated with a different frequency.

• The limits on the sums can be chosen to be different from "0" to "N-1", as long as the summation is over N samples. (ex. $\sum_{k=-2}^{N-3}$ )

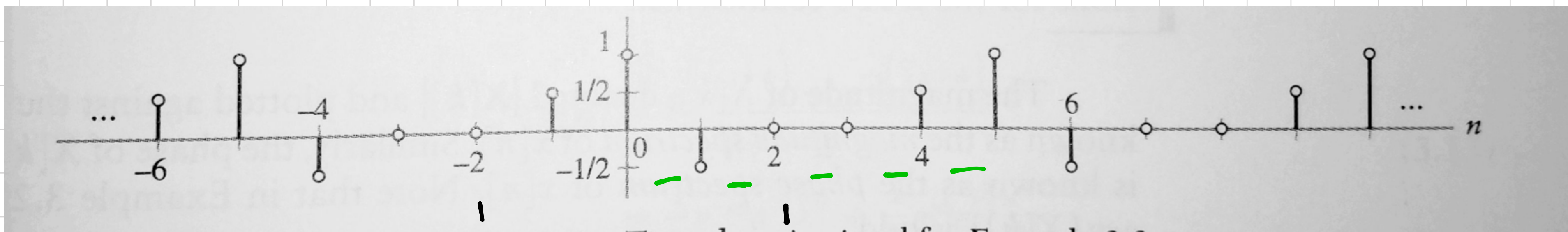



**Figure 3.5** Time-domain signal for Example 3.2.

$N = 5 \qquad \Omega_0 = 2\pi/5$

$$X[k] = \frac{1}{5}\sum_{n=-2}^{2} x[n]\cdot \exp\left\{-j\cdot k\cdot \frac{2\pi n}{5}\right\}$$

$$= \frac{1}{5}\Big\{ \underbrace{x[-2]}_{0} \exp\left\{j\cdot k\cdot \frac{4\pi}{5}\right\}$$

$$+ \underbrace{x[-1]}_{1/2} \exp\left\{j\cdot k\cdot \frac{2\pi}{5}\right\}$$

$$+ \underbrace{x[0]}_{1} \underbrace{\exp\{j\cdot k\cdot 0\}}_{1}$$

$$+ \underbrace{x[1]}_{-1/2} \exp\left\{-j\cdot k\cdot \frac{2\pi}{5}\right\}$$

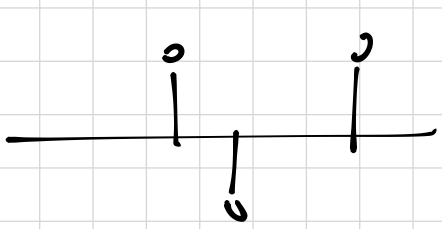
$$+ \underbrace{x[2]}_{0} \exp\left\{-j\cdot k\cdot \frac{4\pi}{5}\right\}$$

$$\Big\}$$

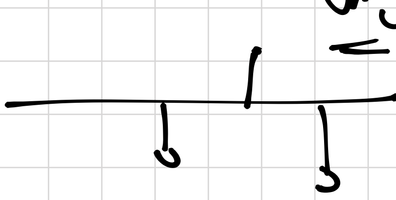
$$\boxed{e^{j\theta} - e^{-j\theta} = j2\sin(\theta)}$$

$$X[k] = \frac{1}{5}\Big\{1 + \underbrace{\frac{1}{2}\exp\left\{jk\frac{2\pi}{5}\right\} - \frac{1}{2}e^{-jk\frac{2\pi}{5}}}_{j\cdot\sin(2k\pi/5)}\Big\}$$

$$= \frac{1}{5}\left[1 + j\cdot\sin\left(\frac{2\pi k}{5}\right)\right]$$



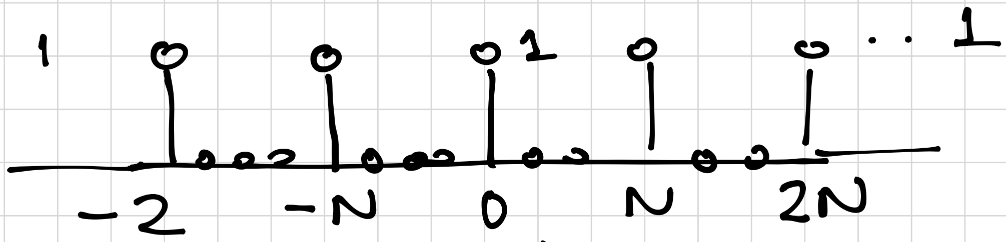
$|X[k]|$



$\arg(X[k])$

$$\boxed{x[n] \xleftrightarrow{\text{DTFS};\ N} X[k]}$$

## Ex. DTFS rerpesentation of an Impulse train




$$x[n] = \sum_{l=-\infty}^{+\infty} \delta[n - l \cdot N]$$

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{\delta[n] \cdot \exp\left\{-j \cdot k \cdot n \cdot \frac{2\pi}{N}\right\}}_{1}$$

$$X[k] = \frac{1}{N}$$