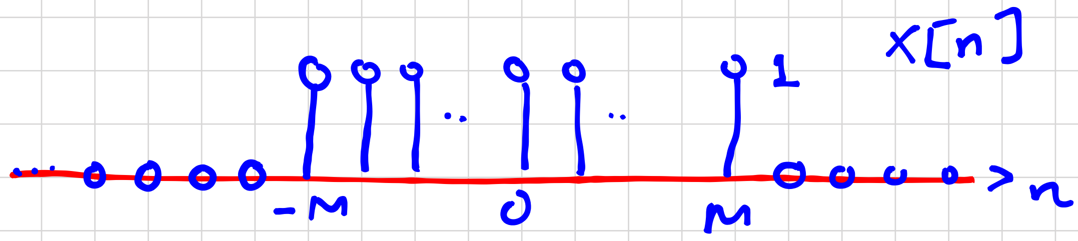
DTFT (Devam edelim)

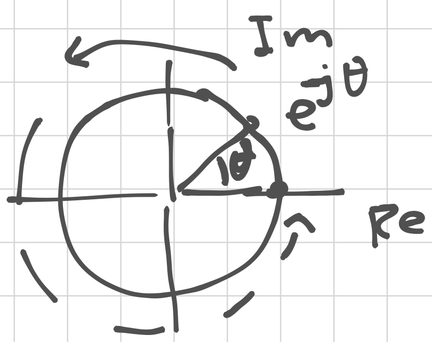
Summary :

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \cdot e^{j\Omega n} \, d\Omega$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] \cdot e^{-j\Omega n}$$

Ex DTFT of a rectangular pulse





$$x[n] = \begin{cases} 1 , & |n| \leq M \\ 0 , & |n| > M \end{cases}$$

find the DTFT of $x[n]$

$$X(\Omega) = \sum_{n=-M}^{M} 1 \cdot e^{-j\Omega n}$$

① when Ω is a multiple of 2π
(: $\Omega = 0, \mp 2\pi, \mp 4\pi \ldots$) then

$$X(\Omega) = \sum_{n=-M}^{M} 1 \cdot 1 = 2M + 1$$

② when Ω is not a multiple of 2π
(: $\Omega \neq 0, \mp 2\pi, \mp 4\pi \ldots$)

$$X(\Omega) = \sum_{n=-M}^{M} e^{-j\Omega n}$$

$$/* \quad \sum_{n=k}^{l} \beta^n = \frac{\beta^k - \beta^{l+1}}{1 - \beta} , \quad \text{if } \beta \neq 1 \quad */$$

$$X(\Omega) = \frac{e^{j\Omega M} - e^{-j\Omega(M+1)}}{1 - e^{-j\Omega}}$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$= \frac{e^{j\Omega M} - e^{-j\Omega(M+1)}}{e^{-j\Omega/2}\left(e^{j\Omega/2} - e^{-j\Omega/2}\right)}$$

$$= \frac{e^{j\Omega(M+\frac{1}{2})} - e^{-j\Omega(M+1-\frac{1}{2})}}{e^{j\Omega/2} - e^{-j\Omega/2}}$$

$$= \frac{e^{j\Omega\left(\frac{2M+1}{2}\right)} - e^{-j\Omega\left(\frac{2M+1}{2}\right)}}{e^{j\Omega/2} - e^{-j\Omega/2}} \cdot \frac{2j}{2j}$$

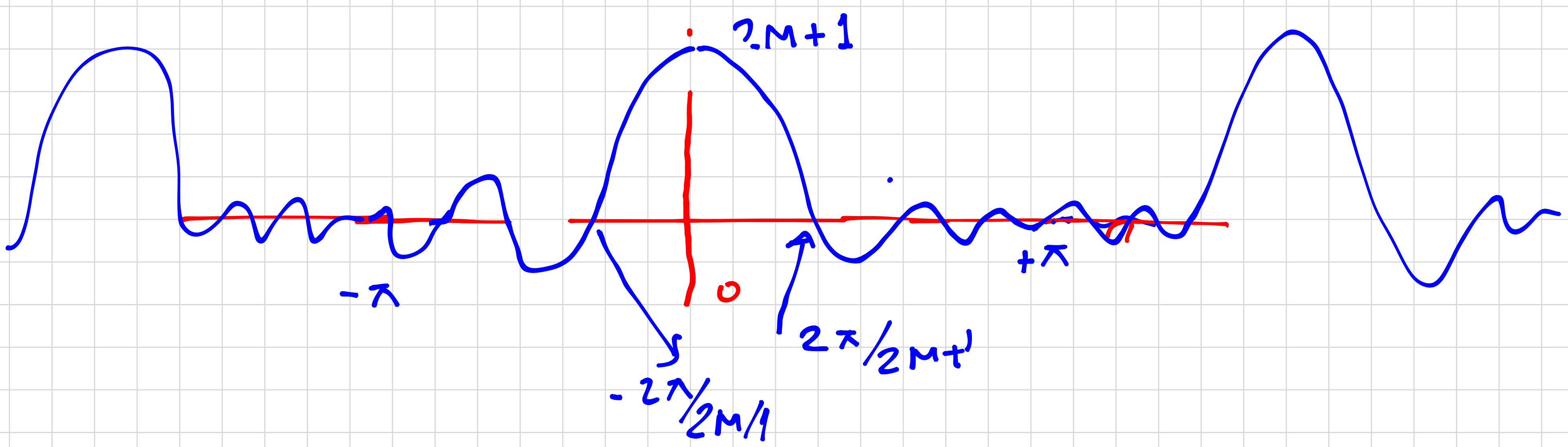
$$X(\Omega) = \frac{\sin\left(\Omega \cdot \frac{2M+1}{2}\right)}{\sin(\Omega/2)}$$

From L'Hôpital's rule

$$\lim_{\Omega \to 0, \mp 2\pi \ldots} \frac{\sin\left(\Omega \cdot \frac{2M+1}{2}\right)}{\sin(\Omega/2)} = 2M+1$$

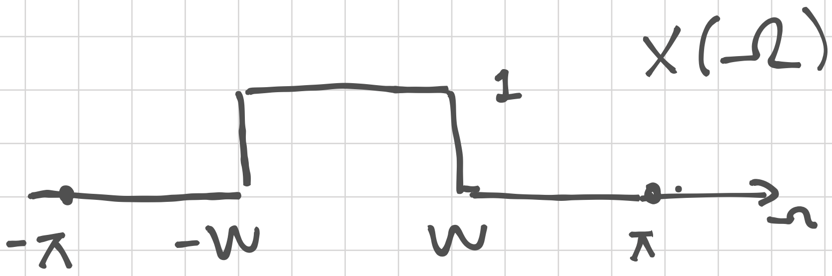
So, we can simply write

$$\boxed{X(\Omega) = \frac{\sin(\Omega(2M+1)/2)}{\sin(\Omega/2)}}$$



Ex Inverse DTFT of a Rectangular Spectrum

$$X(\Omega) = \begin{cases} 1, & |\Omega| < W \\ 0, & W < |\Omega| < \pi \end{cases}$$



$$x[n] = \frac{1}{2\pi} \int_{-W}^{W} e^{j\Omega n} \, d\Omega$$

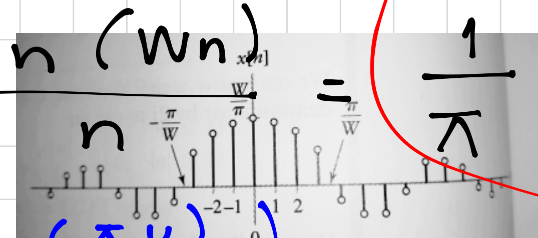
$$= \frac{1}{2\pi j n} \left[e^{j\Omega n} \right]_{-W}^{W}$$

$$= \frac{1}{\pi n \cdot 2j} \left(e^{jWn} - e^{-jWn} \right) \quad , n \neq 0$$

$$= \frac{1}{\pi n} \sin(Wn) \quad , \quad n \neq 0$$

$$\lim_{n \to 0} \frac{\sin(Wn)}{\pi n} = \frac{W}{\pi}$$

$$x[n] = \frac{1}{\pi} \cdot \frac{\sin(Wn)}{n} = \frac{1}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$$



$$: \operatorname{sinc}(u) \triangleq \frac{\sin(\pi u)}{\pi u}$$

Ex. DTFT of the unit impulse

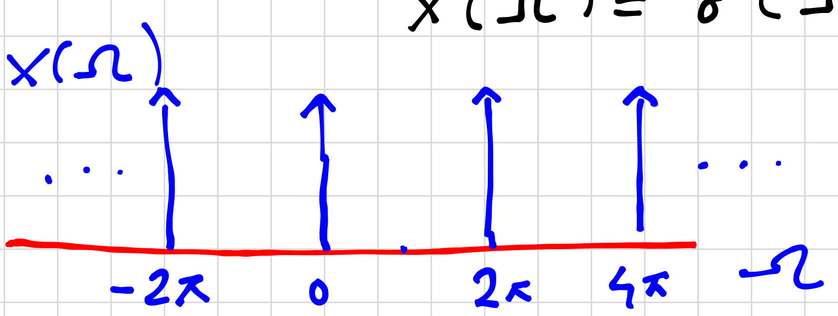
$$x[n] = \delta[n] \qquad X(\Omega) = ?$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} \delta[n] \cdot e^{-j\Omega n} = 1$$

$$\delta[n] \xleftrightarrow{DTFT} 1$$

Ex

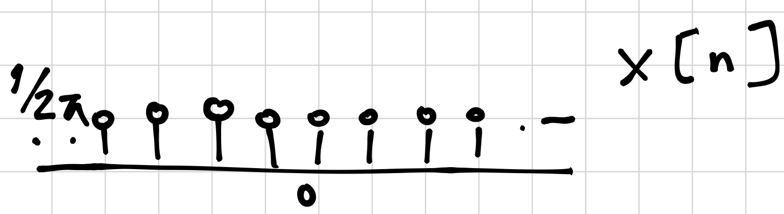
$$X(\Omega) = \delta(\Omega), \quad -\pi < \Omega < \pi$$



$$/* \int_{-\infty}^{+\infty} \delta(t)\, x(t)\, dt = x(0) */$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$x[n] = \frac{1}{2\pi}$$



The DTFT of $x[n] = \frac{1}{2\pi}$ does not converge, because it is not a square summable signal, however $X[n]$ is a valid DTFT !!!

→ In this case although the DTFT does not converge, we use this transform pair as a $\boxed{\text{problem solving tool.}}$

Pr. 3.12

a) $$x(n) = \begin{cases} 2^n, & 0 \le n \le 9 \\ 0, & \text{otherwise} \end{cases}$$

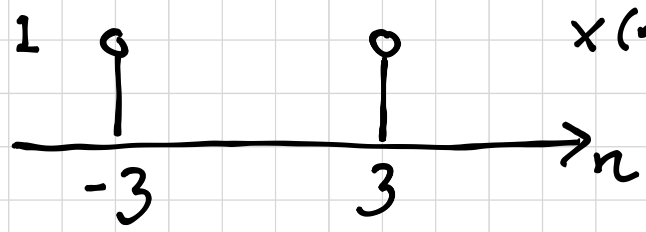
$$X(\Omega) = \sum_{n=0}^{9} 2^n \cdot e^{-j\Omega n}$$

$$= \sum_{n=0}^{9} \left(2 e^{-j\Omega}\right)^n$$

using the geometric series formula.

$$= \frac{1 - 2^{10} \cdot e^{-j \cdot 10 \cdot \Omega}}{1 - 2 \cdot e^{-j\Omega}}$$

b) $$x(n) = \delta(6 - 2n) + \delta(6 + 2n)$$



$\cos\theta = \frac{1}{2}\left(e^{-j\theta} + e^{j\theta}\right)$

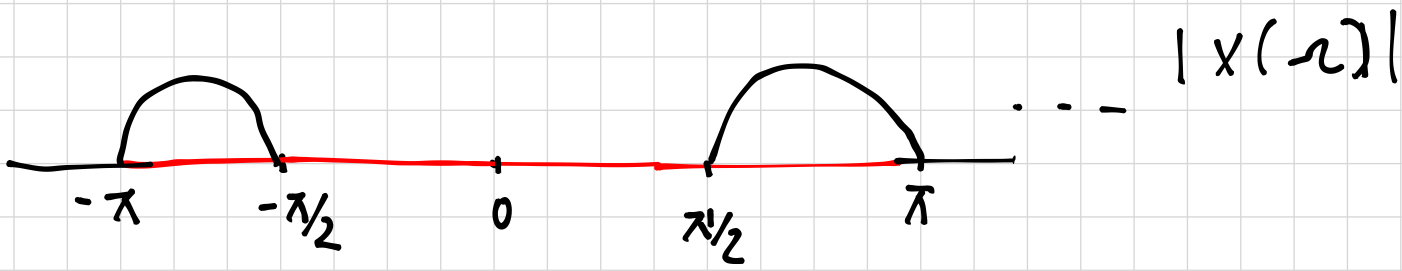
$$x(\Omega) = e^{-j\Omega 3} + e^{j\Omega 3}$$

$$= 2 \cdot \cos(3\Omega)$$

Pr. 3.13

$$X(\Omega) = e^{-j4\Omega}, \quad \pi/2 < |\Omega| < \pi$$

(on $-\pi < \Omega < \pi$)



$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \cdot e^{j\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{-\pi/2} e^{-j4\Omega} \cdot e^{j\Omega n} \, d\Omega + \int_{\pi/2}^{\pi} e^{-j \cdot 4\Omega} \cdot e^{j\Omega n} \, d\Omega \right]$$

$$= \frac{1}{2\pi} \cdot \frac{1}{j(n-4)} \left\{ \left[e^{j\Omega(n-4)} \right]_{-\pi}^{-\pi/2} + \left[e^{j\Omega(n-4)} \right]_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{2\pi} \cdot \frac{1}{j(n-4)} \cdot \Big[\underline{e^{-j\frac{\pi}{2}(n-4)}} - e^{-j\pi(n-4)} + e^{j\pi(n-4)} - \underline{e^{-j\frac{\pi}{2}(n-4)}} \Big]$$

$$= \frac{1}{\pi(n-4)} \Big[\sin(\pi(n-4)) - \sin\left(\frac{n-4}{2} \cdot \pi\right) \Big]$$

$$= \delta[n-4] - \frac{\sin\left(\pi \frac{(n-4)}{2}\right)}{\pi\,(n-4)}$$

CT Periodic Signals – The Fourier Series

If $x(t)$ is CT periodic signal, then it can be represented by a Fourier Series

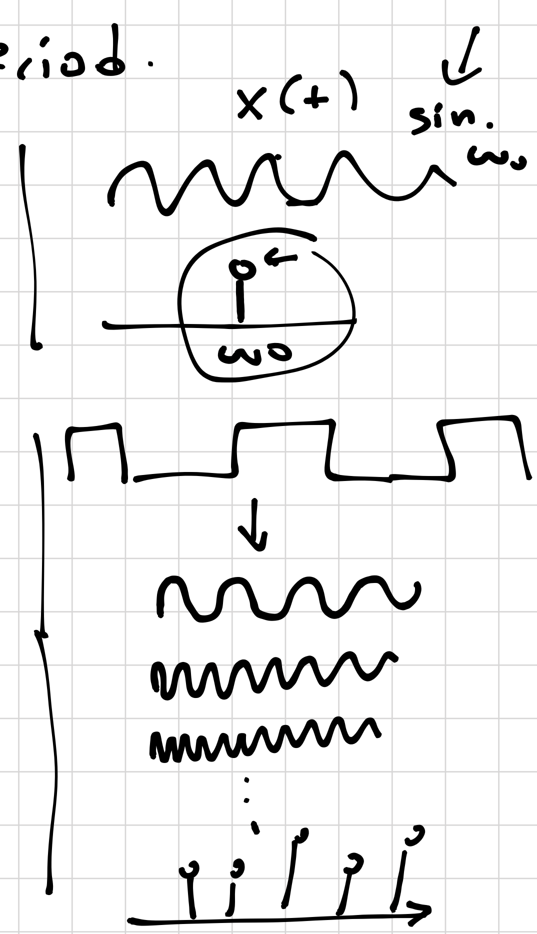
$$x(t) = \sum_{k=-\infty}^{+\infty} X[k] \cdot e^{jk\omega t}$$

where ω is the fundamental frequency, and $T = \frac{2\pi}{\omega}$, is the period.

The FS coefficients can be found via

$$X[k] = \frac{1}{T} \int_0^T x(t) \cdot e^{-jk\omega t} \, dt$$

$$x(t) \xleftrightarrow{FS;\ \omega} X[k]$$

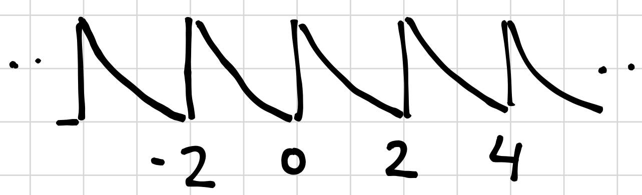


The FS coefficients are termed as the $\boxed{\text{frequency representation}}$ of the time domain signal because each FS coefficient is associated with a complex sinusoid of a different frequency.

The infinite series that represents $x(t)$ may not always converge. The requirement for convergence is that $x(t)$ must be squarely integrable

$$: \quad \frac{1}{T}\int_0^T |x(t)|^2\, dt < \infty$$

Example Direct calculation of FS coefficients



$T = 2 \qquad w = \frac{2\pi}{2} = \pi$

in the range $0 < t < 2$

$x(t) = e^{-2t}$

$$X[k] = \frac{1}{2}\int_0^2 e^{-2t} \cdot e^{jk\pi t} \cdot dt$$

$$= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t} \cdot dt$$

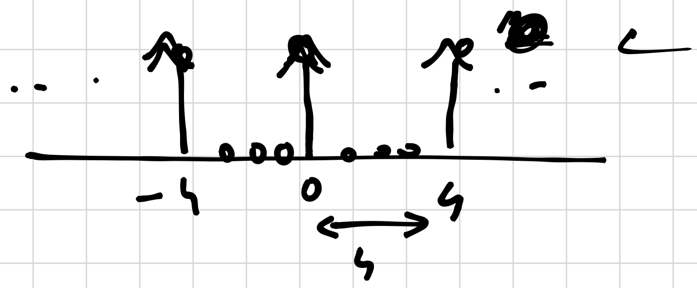
$$= \frac{1}{2} \cdot \frac{-1}{2+jk\pi} \cdot \left[e^{-(2+jk\pi)t} \right]_0^2$$

$$= \frac{1}{4+2jk\pi} \cdot \left[1 - e^{-4} \cdot \underbrace{e^{-jk2\pi}}_{1} \right]$$

$$X[k] = \frac{1 - e^{-4}}{4 + 2jk\pi}$$

Ex FS coefficients for impulse train

$$x(t) = \sum_{\ell=-\infty}^{+\infty} \delta(t - 4\ell)$$



$$X[k] = \frac{1}{4}\int \ldots = \frac{1}{4}$$

$x(t)$ is not squarely integrable.
However it can be used as problem solving tool!

Ex Calculating the FS coefficient by inspection

$$x(t) = 3 \cdot \cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$$

$$\omega = \frac{\pi}{2} \qquad T = 4$$

$$x(t) = \sum_{k=-\infty}^{+\infty} X[k] \cdot e^{+jk\frac{\pi t}{2}}$$

$$x(t) = \frac{3}{2} \cdot \left[e^{j(\pi t/2 + \pi/4)} + e^{-j(\pi t/2 + \pi/4)} \right]$$

$$= \underbrace{\frac{3}{2} \cdot e^{j\frac{\pi}{4}}}_{X[1]} \cdot \underbrace{e^{j\cdot\frac{\pi}{2}\cdot t}}_{k=1} + \underbrace{\cdots}_{X[-1]} \quad \underbrace{\qquad}_{k=-1}$$

$$X[k] = \begin{cases} \frac{3}{2} \cdot e^{-j\pi/4}, & k=-1 \\ \frac{3}{2} e^{j\pi/4}, & k=1 \\ 0, & \text{otherwise} \end{cases}$$

Ex Inverse FS

$$X[k] = \left(\frac{1}{2}\right)^{|k|} \cdot e^{jk\pi/20}$$

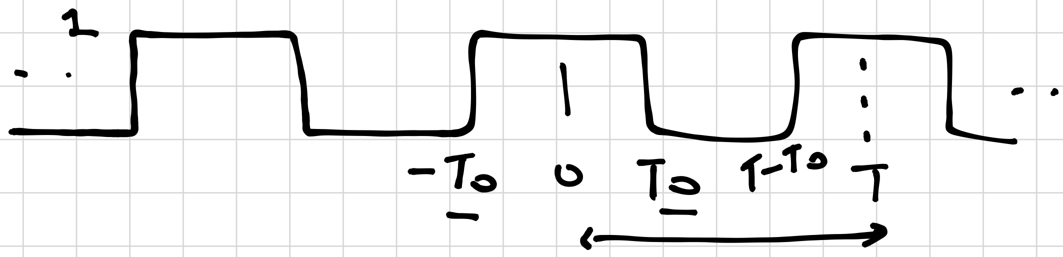
give $T = 2$, find the time domain signal.

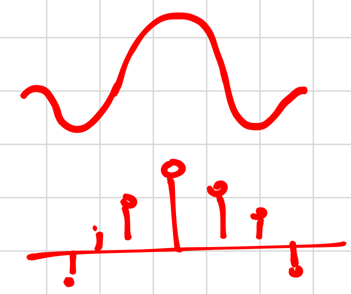
$$x(t) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} \cdot e^{+jk\pi/20} \cdot e^{jk\pi t}$$

$$+ \sum_{k=-\infty}^{-1} \left(\frac{1}{2}\right)^{-k} \quad '' \qquad)$$

$$= \cdots = \frac{3}{5 - 4\cos(\pi t + \pi/20)}$$

E+FS series for a square wave





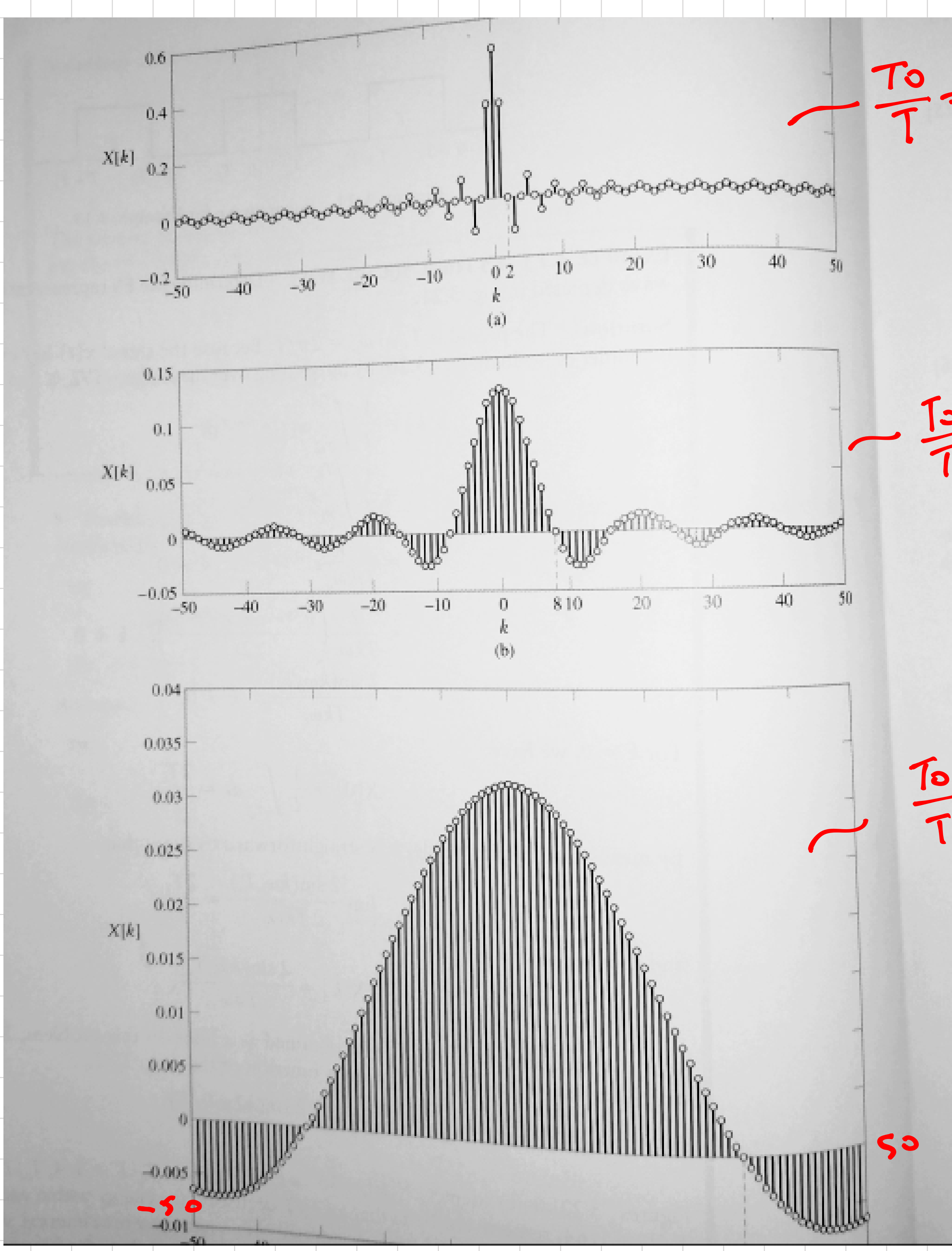
$T = T \quad , \quad \omega = 2\pi / T$

$$X[k] = \frac{1}{T} \int_{-T_0/2}^{T/2} x(t) \cdot e^{-jk\omega t} \, dt$$

$$= \ldots$$

$$X[k] = \frac{2}{T} \cdot \frac{\sin(k \omega T_0)}{T k \omega} \qquad \omega = 2\pi / T$$

$$= 2 \cdot \frac{\sin(k 2\pi T_0 / T)}{2\pi \cdot k}$$



$\sim \frac{T_0}{T} = \frac{1}{4}$

$\sim \frac{T_0}{T} = 1/16$

$\sim \frac{T_0}{T} = \frac{1}{6}$