* Random Expeiment -
  Throwing a dice
  Flipping a coin

* Events
* Sample space
* Axioms of Probability

For event A

① $0 \leqslant P(A) \leqslant 1$

② $P(S) = 1$

③ For any countable collection, $A_1, A_2 \ldots$ of mutually exclusive events

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

- $P(\phi) = 0$
- $P(A^c) = 1 - P(A)$
- For any event A, B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

/* $P(A \cap B) = P(AB)$ */

Sets

* Mutually Exclusive Sets
* Exhaustive sets
* Partition
* Universal set.

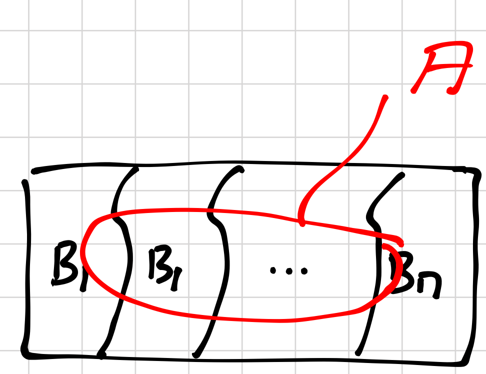
Conditional Probability

$P(A|B)$: Probability of A occurs under the condition that B occurs.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Total probability

$B_1, B_2, \ldots, B_n$ is a partition



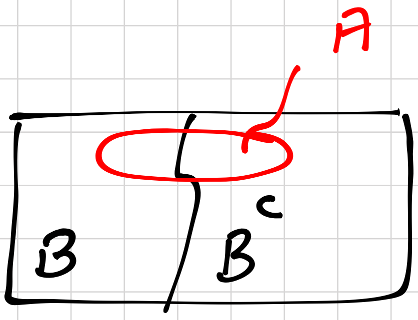
$$A = \underbrace{(A \cap B_1)}_{C_1} \cup \underbrace{(A \cap B_2)}_{C_2} \cup \cdots \cup \underbrace{(A \cap B_n)}_{C_n}$$

$C_1, C_2, \ldots, C_n$ are mutually exclusive

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \dots + P(A \cap B_n)$$

$$P(A) = P(A|B_1) \cdot P(B_1) + P(A|B_2)\,P(B_2) + \dots + P(A|B_n)\,P(B_n)$$

Total Probability



B and B^c makes a partition

$$P(A) = P(A|B)\,P(B) + P(A|B^c)\,P(B^c)$$

Bayes' Theorem

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Independent events

$$P(A \cap B) = P(A)\,P(B) \iff$$ A and B are independent.

$$P(A|B) = P(A) \qquad P(B|A) = P(B)$$

Textbook : "Roy D. Yates, David J Goodman
"Probability and Stochastic Processes : A Friendly Introduction for Electrical and Computer Engineers",
3rd Ed, Wiley, 2014

$\rightarrow$ Kitabın web sitesi $\rightarrow$ Student Solutions.

Discrete Random Variables

Random Variable – A $\underline{\underline{\text{R.V.}}}$ consists of an experiment with a probability measure $P(\cdot)$ defined on a sample space S and a function that assigns a real number to each outcome in the sample space of the experiment.

- X

For example

A : the number of students asleep during this lecture

C : the number of SMS messages you receive in the next hour

(Continuous R.V. – M : the number of minutes you wait until the next text arrives)

M can have any non-negative real value

$\therefore$ it is not a Discrete R.V.

Discrete R.V.s

X is discrete R.V. if the range of X is a countable set $S_X = \{x_1, x_2, x_3, \dots\}$

Probability Mass Function

$P(X=x)$: probability that X will take the value x

$$f(x) = P_X(x) = P(X=x)$$

the notation can be either

When the basketball player Wilt Chamberlain shot two free throws, each shot was equally likely either to be good (g) or bad (b). Each shot that was good was worth 1 point. What is the PMF of X, the number of points that he scored?

g : good
b : bad

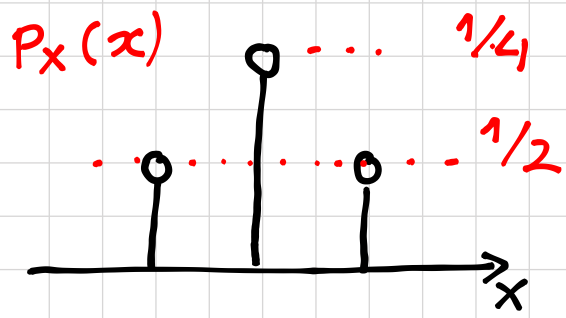
	bb	bg	gb	gg
	1/4	1/4	1/4	1/4
X	0	1	1	2

$\{X=0\} = \{bb\}$ $\qquad P(X=0) = 1/4$

$\{X=1\} = \{bg, gb\}$ $\qquad P(X=1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\{X=2\} = \{gg\}$ $\qquad P(X=2) = 1/4$

$$f(x) = P_X(x) = \begin{cases} 1/4, & x = 0 \\ 1/2, & x = 1 \\ 1/4, & x = 2 \\ 0, & \text{otherwise} \end{cases}$$



Theorem

For a Discrete R.V. with a PMF $P_X(x)$ and range S_X

a) For any x, $P_X(x) \geq 0$

b) $\sum_{x \in S_X} P_X(x) = 1$

c) For any event $B \subset S_X$, the probability that X is in the set B is

$$P[B] = \sum_{x \in B} P_X(x)$$

Families of Discrete R.V. (Probability Distributions)

Bernoulli Random Variable –

PMF of a Bernoulli R.V., X

$$P_X(x) = \begin{cases} 1-p, & x=0 \\ p, & x=1 \\ 0, & \text{otherwise} \end{cases}$$

→ (Bernoulli trial is experiment with only two possible outcomes)

Ex Test one circuit and X: the number of rejected circuits, the probability of the circuit being rejected is p

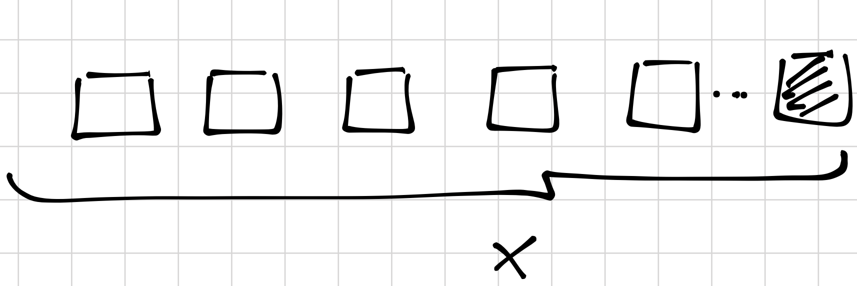
$P(X=0) = 1-p \qquad P(X=1) = p$

$$P_X(x) = \begin{cases} p, & x=1 \\ 1-p, & x=0 \\ 0, & \text{otherwise} \end{cases}$$

Geometric R.V.

The number of Bernoulli trials that takes place until the first observation of one of the two outcomes is a geometric R.V.

Ex In a sequence of independent tests of integrated circuits, each circuit is rejected with probability p. Let Y equal the number of tests up to and including the first test that results in a reject. PMF of Y?



The first $x-1$ circuits must be operating. The xth one must be a defective one.

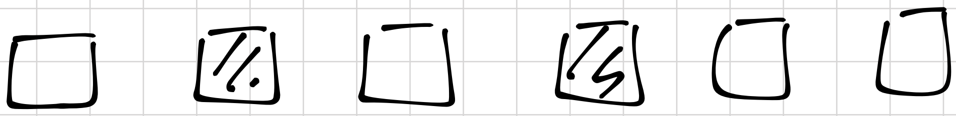
$$P_Y(y) = P(Y=y) = p(1-p)^{y-1}, \quad y \geq 1$$

X is a geometric R.V if its PMF is

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & , x = 1, 2, \ldots \\ 0 & , \text{otherwise} \end{cases}$$

Binomial R.V.

Example: In a sequence of independent tests of integrated circuits, each circuit is rejected with probability p. Let K equal the number of rejects in the n tests. PMF ?



For $K = k$

~~...~~ k successes (rejected circuits) , $n-k$ failures.

(Succes is the outcome happening)

$$P_K(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad , \quad k = 0, 1, \ldots n$$

Definition

X is Binomial R.V if

$$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & , x = 0, 1, \ldots, n \\ 0 & , \text{otherwise.} \end{cases}$$

Pascal (Negative Binomial) R.V.

Example : .. L, the number of tests performed until there are k rejects.

$$P(L = \ell) = P\{ k-1 \text{ rejects in } \ell-1 \text{ attempt}, \text{ reject on attemp } \ell \}$$

$$= \binom{\ell-1}{k-1} \cdot p^{k-1} \cdot (1-p)^{\ell-1-(k-1)} \cdot p$$

$$= \binom{\ell-1}{k-1} \cdot p^k \cdot (1-p)^{\ell-k}.$$

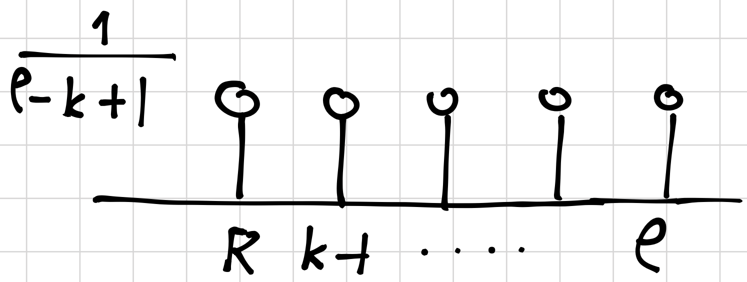
Definition

X is a Pascal (k,p) R.V. if

$$P_X(x) = \binom{x-1}{k-1} \cdot p^k \cdot (1-p)^{x-k} \qquad x = k, k+1 \ldots$$

Discrete Uniform R.V.

X is a discret uniform (k,l) R.V if PMF of X



$$P_X(x) = \begin{cases} \frac{1}{l-k+1}, & x = k, k+1, \ldots, l \\ 0, & \text{otherwise} \end{cases}$$

Poisson R.V.

X is a Poisson (α) R.V if

$$P_X(x) = \begin{cases} e^{-\alpha} \cdot \frac{\alpha^x}{x!}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

- The arrival of information requests at WWW server
- the initiation of telephone calls.

To describe a Poisson R.V. we will call the occurence of the phenomenon of interest an "arrival". A Poisson model often specifies an average rate, λ arrivals per second and a time interval T seconds. In this time interval the number of arrivals X has a Poisson PMF of $\alpha = \lambda T$.

The number of hits at a website in any time interval is a Poisson random variable. A particular site has on average $\lambda = 2$ hits per second. What is the probability that there are no hits in an interval of 0.25 seconds? b) What is the probability that there are no more than two hits in an interval of one second?

a) H: the number of hits

$\alpha = 2 \times 0.25 = 0.5$ hits.

$$P_H(h) = \begin{cases} 0.5^h \cdot e^{-0.5}/h!, & h = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$P(H=0) = P_H(0) = e^{-0.5} \times \frac{0.5^0}{0!} = 0.607$$

b) J, Poisson, $\alpha = 2 \times 1 = 2$

$$P(J \leq 2) = P(J=0) + P(J=1) + P(J=2)$$

$$= e^{-2} \cdot \left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} \right) = 0.677$$