# 323
# ARTIFICIAL INTELLIGENCE & EXPERT SYSTEMS

## Constraint Satisfaction Problems (CSPs)

Chapter 5

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# Outline

- Constraint Satisfaction Problems (CSP)
- Backtracking search for CSPs
- Local search for CSPs

# Constraint satisfaction problems (CSPs)

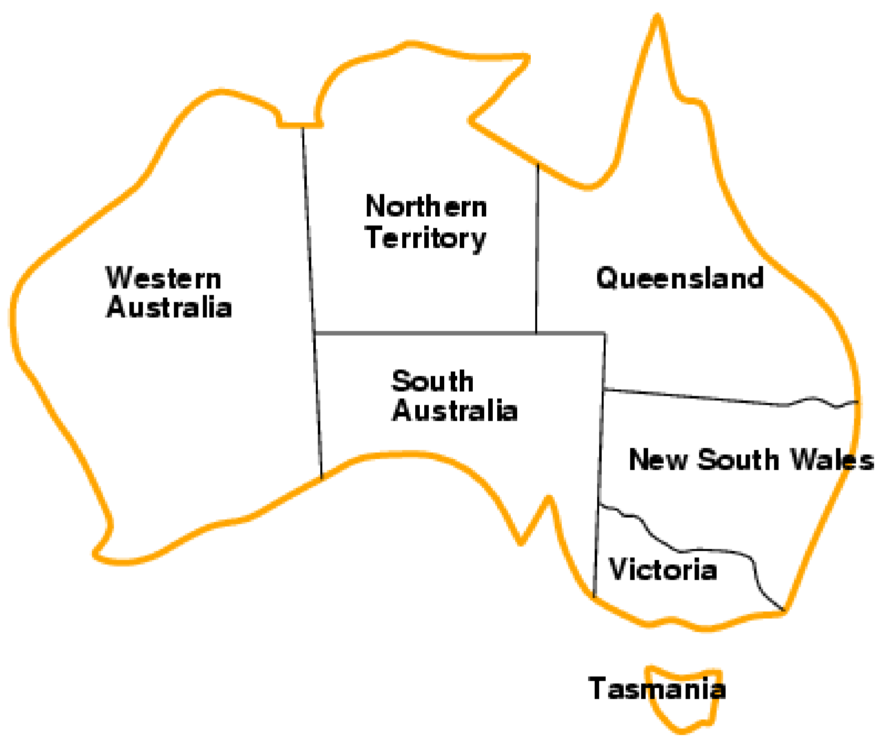
- Problems can be solved by searching in a space of states.
- Standard search problem:
  - state is a "black box" – any data structure that supports successor function, heuristic function, and goal test.
- CSP:
  - state is defined by variables $X_i$ with values from domain $D_i$
  - goal test is a set of constraints $C_m$ specifying allowable combinations of values for subsets of variables
- Allows useful *general-purpose* algorithms with more power than standard search algorithms – *problem specific*.

# Constraint satisfaction problems (CSPs)

- A state of the problem is defined by an assignment of values to some or all of the variables. $\{X_i=v_i, X_j=v_j, \ldots\}$
- An assignment that does not violate any constraints is called a consistent (legal) assignment.
- In a complete assignment every variable is mentioned.
- A solution to a CSP is a complete assignment that satisfies all the constraints.
- Some CSPs also require a solution that maximizes an objective function.

# Example: Map-Coloring

A map of Australia showing each of its states and territories:

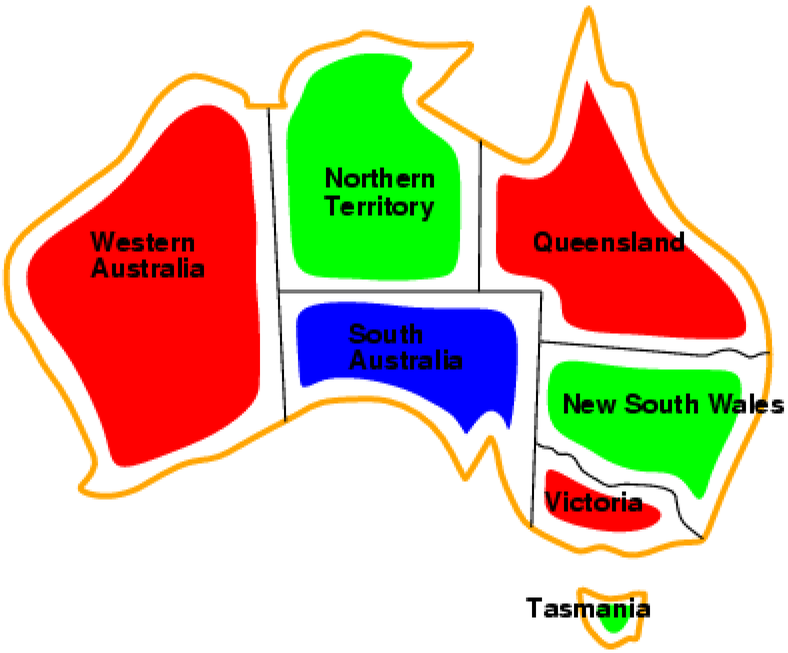



- Suppose that we are given the task of coloring each region either red,green or blue in such a way that no neighboring regions have the same color.
- To formulate this problem as a CSP, we first define
  - the variables,
  - the domain of each variable and
  - the constraints.

# Example: Map-Coloring


- Variables *WA, NT, Q, NSW, V, SA, T*
- Domains $D_i$ = {red,green,blue}
- Constraints: adjacent regions must have different colors
  - e.g., $WA \neq NT$ (if the language allows this), or
  - $(WA,NT) \in$ {(red,green),(red,blue),(green,red),(green,blue),(blue,red),(blue,green)}

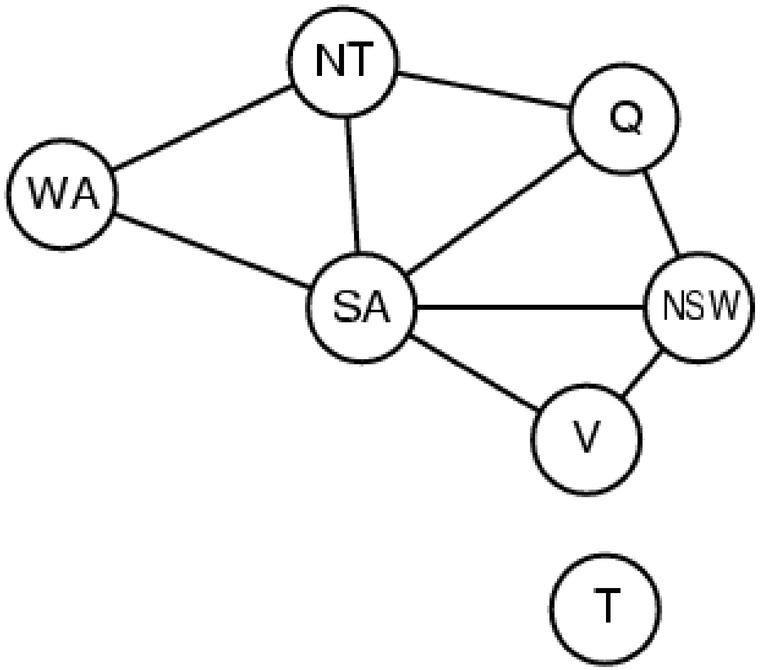
# Example: Map-Coloring




- Solutions are assignments satisfying all constraints,
- e.g., WA = red, NT = green, Q = red,
  NSW = green, V = red, SA = blue,
  T = green

# Constraint graph

- Binary CSP: each constraint relates two variables.
- Constraint graph: nodes are variables, arcs are constraints.
- General-purpose CSP algorithms use the graph structure to speed up search – an exp. reduction in complexity.
  - e.g., Tasmania is an independent subproblem!



# Varieties of CSPs

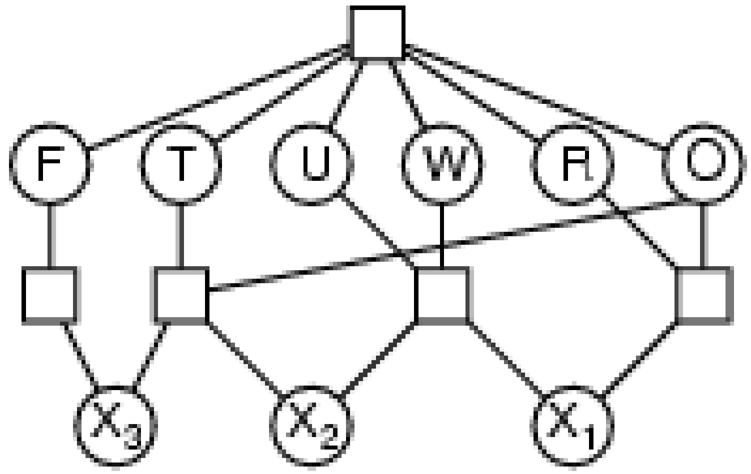
- Discrete variables
  - finite domains:
    - *n* variables, domain size *d* → $O(d^n)$ complete assignments
    - exponential in the number of variables.
    - e.g., map coloring, 8-queens problem, Boolean CSPs.
  - infinite domains:
    - set of integers, set of strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \le StartJob_3$
- Continuous domains– common in the real world
  - e.g., start/end times for Hubble Space Telescope observations.
  - **linear programming** problems can be solved in time polynomial in the number of variables.

# Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
  - it can be represented as a constraint graph.
- Higher-order constraints involve 3 or more variables,
  - e.g., cryptarithmetic puzzles.
  - each letter represents a different digit.
  - The aim is to find a substitution of digits for letters such that the resulting sum is arithmetically correct.

# Example: Cryptarithmetic

```
    T W O
  + T W O
  -------
  F O U R
```




- Variables: *F T U W R O* $X_1$ $X_2$ $X_3$
- Constraints: *Alldiff (F,T,U,W,R,O)*
  - $O + O = R + 10 \cdot X_1$
  - $X_1 + W + W = U + 10 \cdot X_2$
  - $X_2 + T + T = O + 10 \cdot X_3$
  - $X_3 = F$, $T \neq 0$, $F \neq 0$

# Varieties of constraints

- Preferences (soft constraints),
  - e.g., red is better than green.
  - often representable by a cost for each variable assignment.
  - CSPs with preferences can be solved using optimization search methods.
    - e.g. in a university timetabling problem :
    - Prof. X might prefer teaching in the morning whereas Prof. Y prefers teaching in the afternoon.
    - A timetable that has Prof. X teaching at 2 p.m. would still be a solution but would not be an optimal one.
      - assigning an afternoon slot for Prof. X costs 2 points, where as a morning slot costs 1.

# Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

- Notice that many real-world problems involve real-valued variables

# Standard search formulation (incremental)

- Initial state: the empty assignment { }, all variables are unassigned.
- Successor function: assign a value to an unassigned variable that does not conflict with current assignment.
  → fail if no legal assignments.
- Goal test: the current assignment is complete.

1. This is the same for all CSPs
2. Every solution appears at depth *n* with *n* variables
   → use depth-first search
3. Path is irrelevant, so can also use complete-state formulation – every state is a complete assignment that might or might not satisfy the constraints.
4. Local search methods work well with this formulation.

# Standard search formulation (incremental)

- We gave a formulation of CSPs as search problems.
- Using this formulation, any of the search algorithms can solve CSPs.
  - Suppose we apply breath-first search.
  - The branching factor at the top level is $nd$ – any of $d$ values can be assigned to any of $n$ variables.
  - At the next level, the branching factor is $(n-1)d$, and so on for $n$ levels.
  - We generate a tree with $n!d^n$ leaves.
  - However, there are only $d^n$ possible complete assignments!!!

# Backtracking search

- Variable assignments are commutative.
  - A problem is commutative if the order of application of any given set of actions has no effect on the outcome.

i.e., [ WA = red then NT = green ] same as [ NT = green then WA = red ]

- Only need to consider assignments to a single variable at each node.
  - e.g. we might have a choice between $SA=red$, $SA=green$ and $SA=blue$.
  - but, we would never choose between $SA=red$ and $WA=blue$

→ b = d and there are $d^n$ leaves.

# Backtracking search

- Depth-first search for CSPs with single-variable assignments is called backtracking search.

- Backtracking search is the basic uninformed algorithm for CSPs.

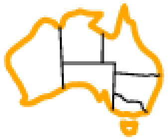
- Can solve $n$-queens for $n \approx 25$.

# Backtracking search **

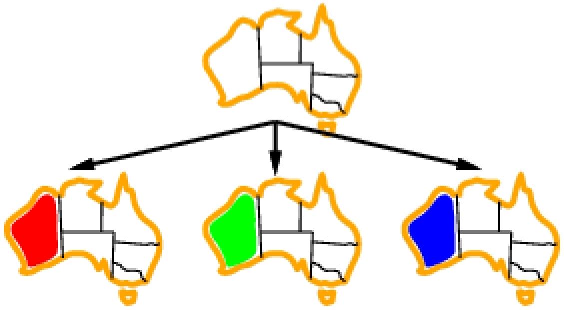
```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
   return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING( assignment, csp) returns a solution, or
failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment according to Constraints[csp] then
         add { var = value } to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failue then return result
         remove { var = value } from assignment
   return failure
```

The algorithm is modeled on the recursive depth-first search.

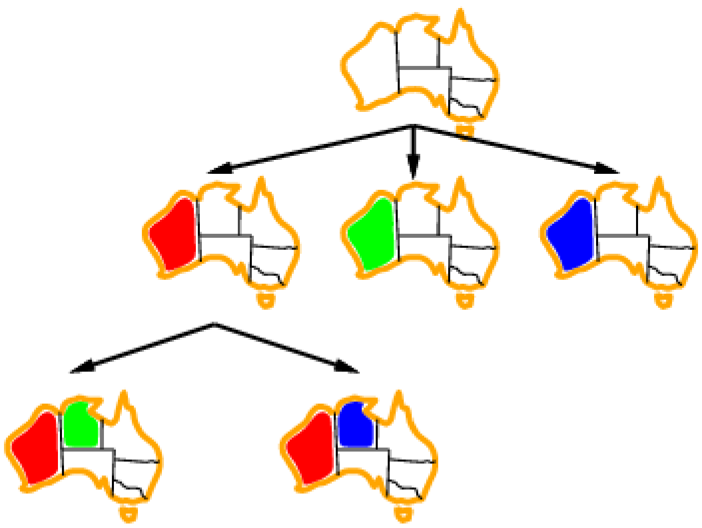
# Backtracking example



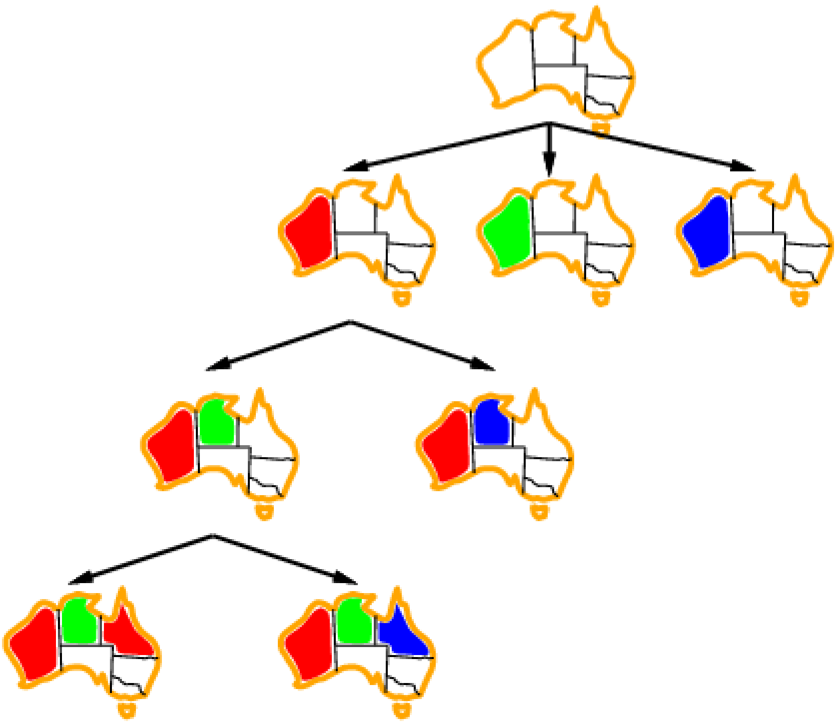
# Backtracking example



# Backtracking example



# Backtracking example



# Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?
  - When a path fails – it means, a state is reached in which a variable has no legal values – can the search avoid repeating this failure in the subsequent paths?

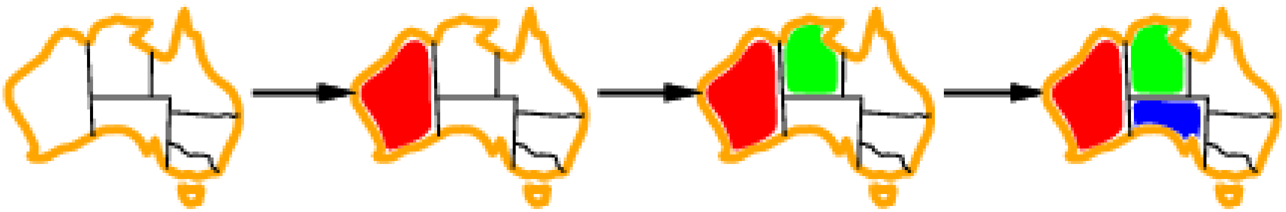
# Most constrained variable

$var \leftarrow \text{SELECT-UNASSIGNED-VARIABLE}(Variables[csp], assignment, csp)$

- Selects the next unassigned variable in the order given by the list *Variables[csp]*.
- This static variable ordering seldom results in the most efficient search.
  - e.g. after the assignments for $WA=red$ and $NT=green$, there is only one possible value for $SA$.
  - So it makes sense to assign $SA=blue$ next rather than assigning $Q$.
  - After $SA$ is assigned, the choices for $Q$, $NSW$, and $V$ are all forced.

# Most constrained variable

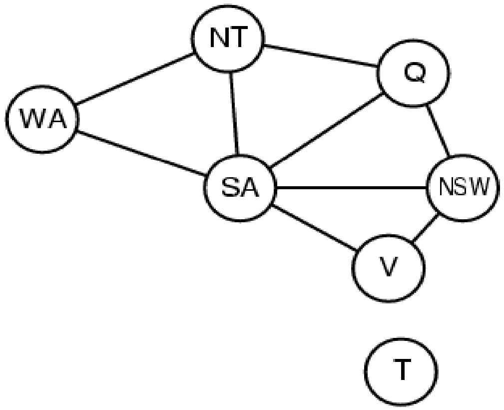
- Most constrained variable:
  choose the variable with the fewest legal values.



- minimum remaining values (MRV) heuristic.
  - It picks a variable that is most likely to cause a failure soon.
  - If there is a variable X with zero legal values remaining, the MRV will select X and failure will be detected immediately.

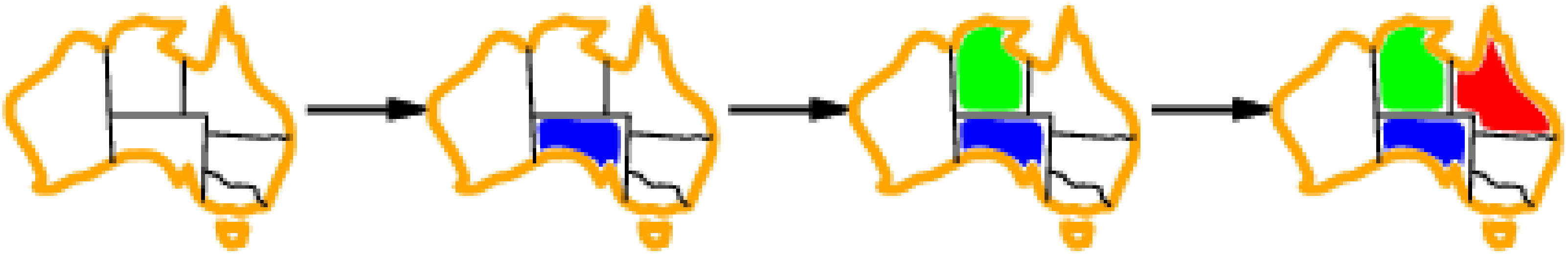
# MRV heuristic

- The MRV heuristic doesn't help in choosing the first region to color in Australia because every region has 3 legal colors – degree heuristic.
- Degree heuristic tries to reduce the branching factor by selecting the variable that has the largest number of constraints on.
  - $SA$ has the highest degree, 5; other variables have degree 2 or 3 and $T$ has 0.



# Most constraining variable

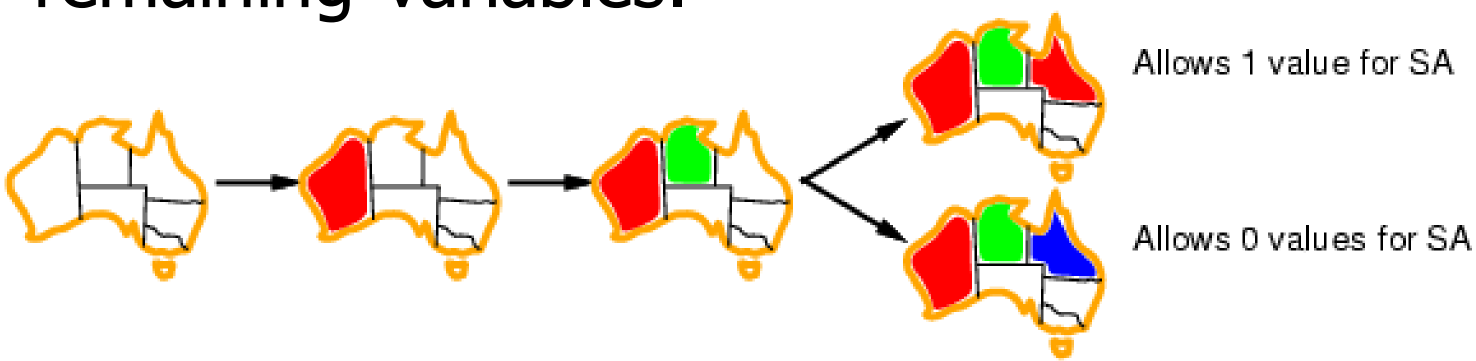
- Most constraining variable:
  - choose the variable with the most constraints on remaining variables



- The MRV heuristic is a more powerful guide, but the degree heuristic can be useful as tie-breaker.

# Least constraining value (LCV)

- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables.



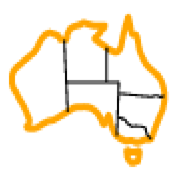

- In general, the heuristic is trying to leave the maximum flexibility for subsequent variable assignments.

# Forward checking

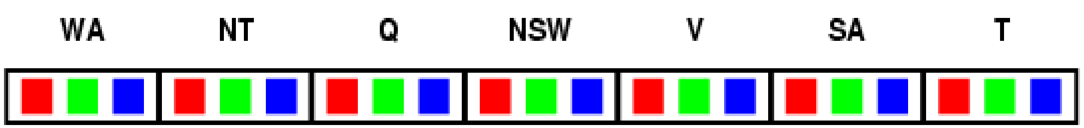
- So far our search algorithm considers the constraints on a variable only at the time that the variable is chosen by *SELECT-UNASSIGNED-VARIABLE*.
- However, by looking at some of the constraints earlier in the search or before the search has started, we can reduce the search space.
- One way to make better use of constraints during search is called forward checking.

# Forward checking

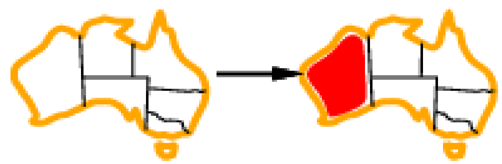
- Idea:
    - Keep track of remaining legal values for unassigned variables.
    - Terminate search when any variable has no legal values.

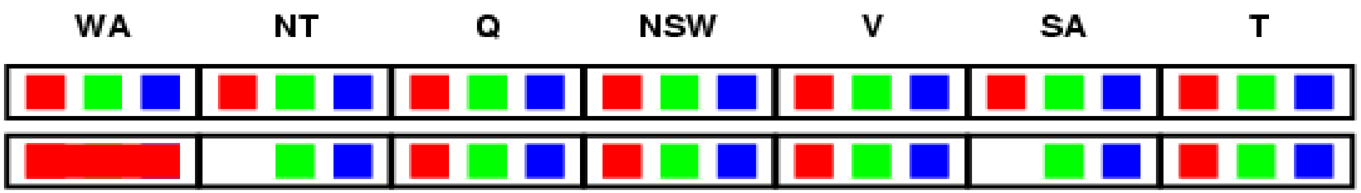


Initial domains



# Forward checking



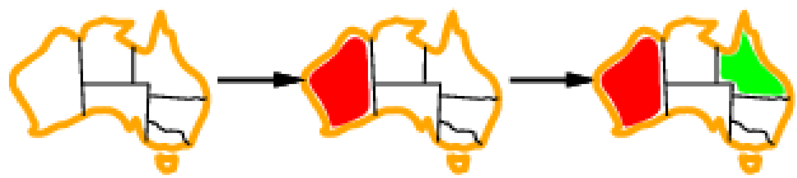


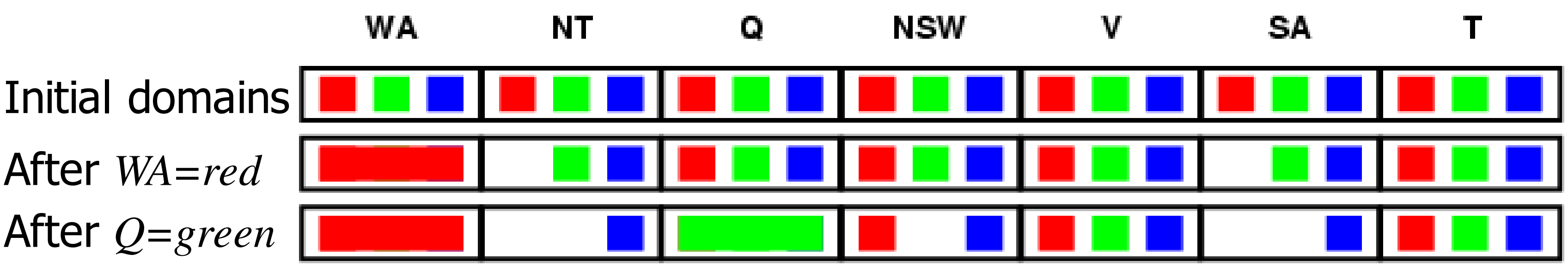

Initial domains

After $WA = red$

$WA = red$ is assigned first; then forward checking deletes red from the domains of the neighboring variables $NT$ and $SA$.

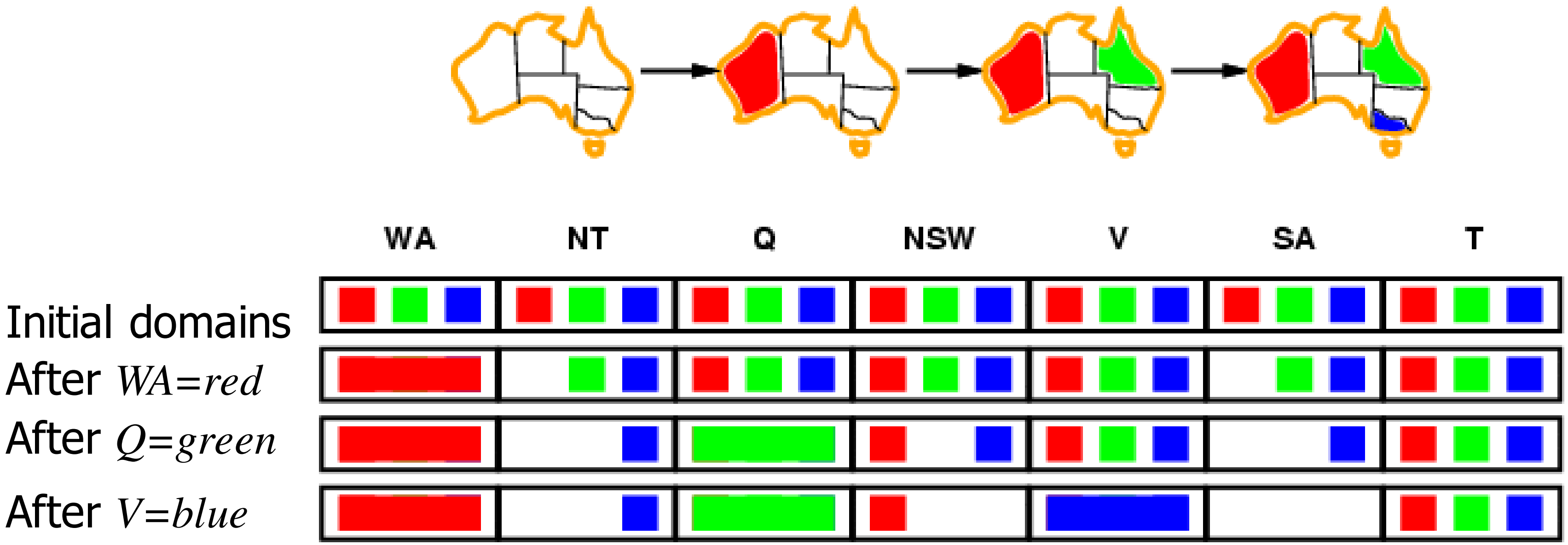
# Forward checking






After $Q=green$, green is deleted from the domains of $NT$, $SA$ and $NSW$.

# Forward checking



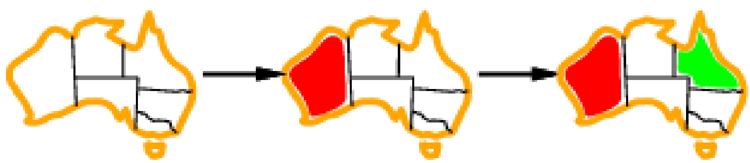

After *V=blue*, blue is deleted from the domains of *NSW* and *SA*, leaving *SA* with no legal values.

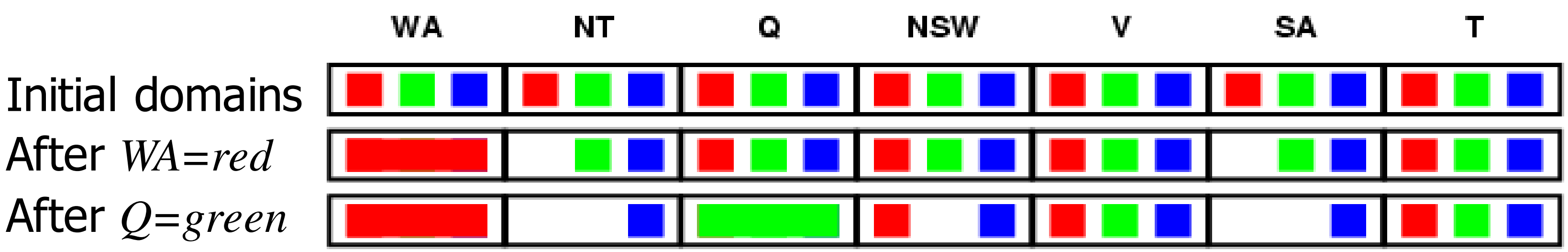
# Forward checking

- Forward checking has detected that the partial assignment *{WA=red, Q=green, V=blue}* is inconsistent with the constraints of the problem.
- The algorithm will therefore backtrack immediately.
- Although forward checking detects many inconsistencies, it does NOT detect all of them.
  - e.g. when *WA=red* and *Q=green*, both *NT* and *SA* are forced to be blue.
  - but they are adjacent and they can not have the same value.
  - forward checking does not detect this as an inconsistency.

# Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



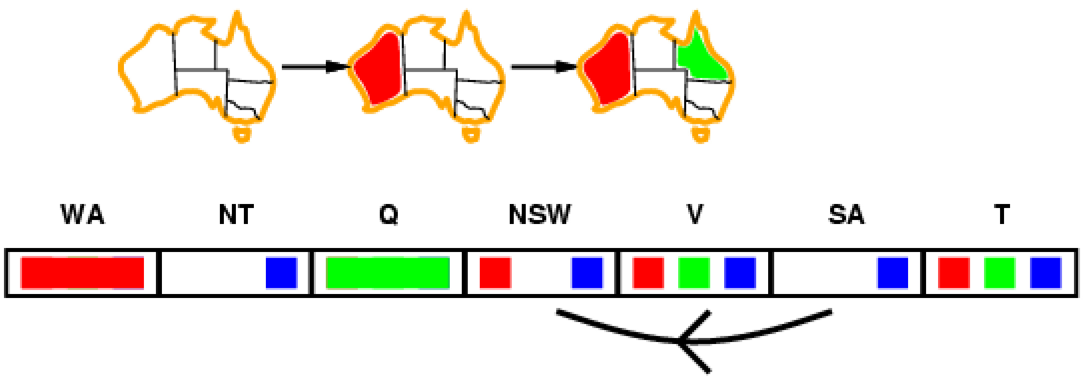



- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally.

# Arc consistency

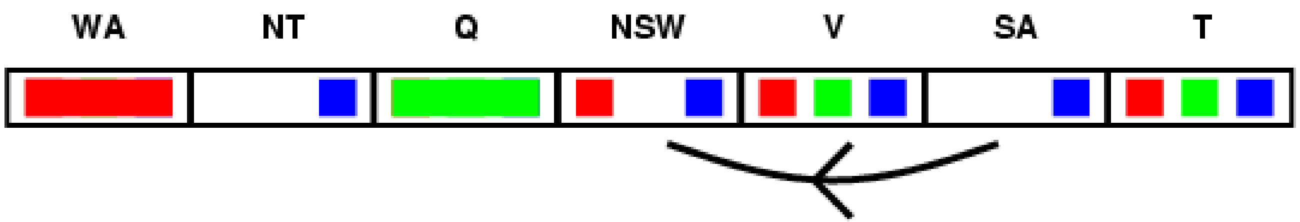
- Simplest form of propagation makes each arc consistent.
- $X \rightarrow Y$ is consistent iff

  for every value $x$ of $X$ there is some allowed $y$.



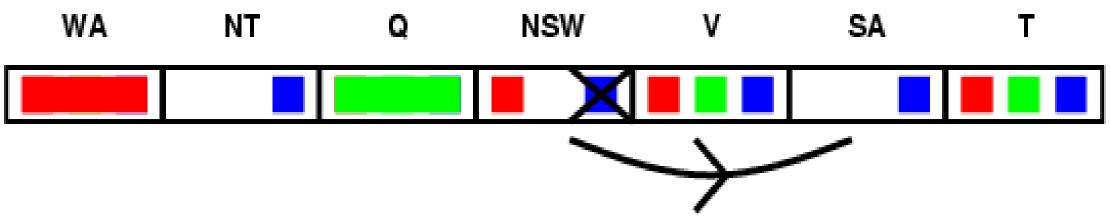

- e.g. the arc from $SA$ to $NSW$.
  - the arc is consistent if for every value $x$ of $SA$, there is some value $y$ of $NSW$ that is consistent with $x$.

# Arc consistency




- The current domains: $SA=\{blue\}$ and $NSW=\{red,blue\}$.
- For $SA=\{blue\}$, there is a consistent assignment for NSW. It is $NSW=\{red\}$.
- Therefore, the arc from $SA$ to $NSW$ is consistent.

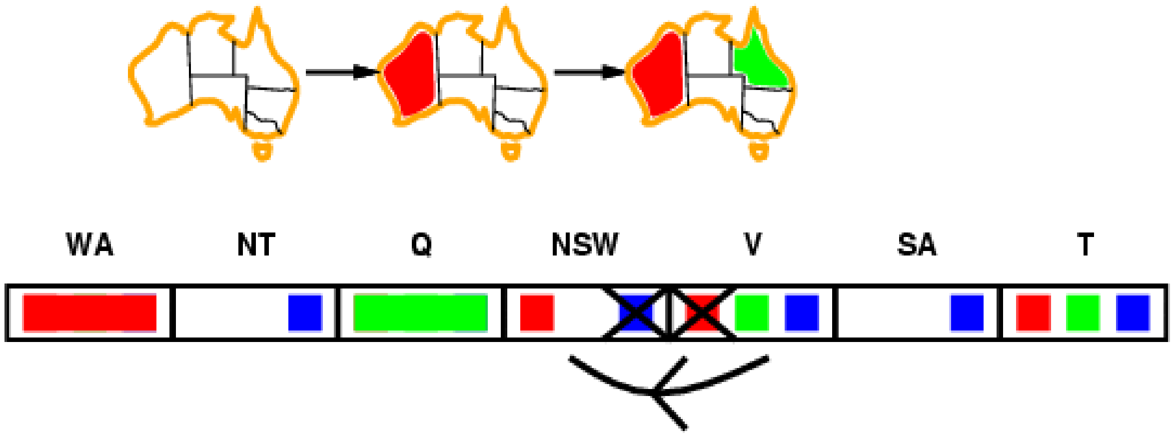
# Arc consistency



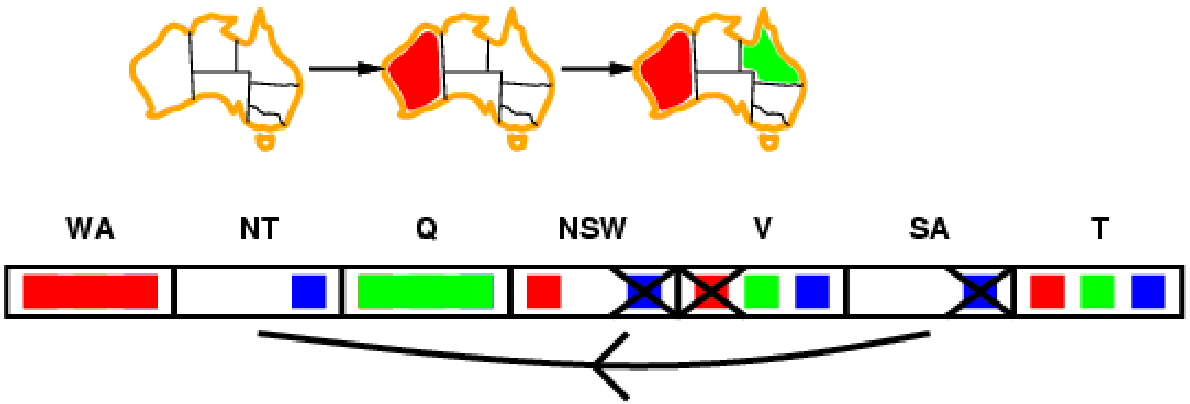

- The reverse arc from $NSW$ to $SA$ is NOT consistent.
- For the assignment $NSW{=}blue$, there is no consistent assignment for $SA$.
- The arc can be made consistent by deleting the value blue from the domain of $NSW$.

# Arc consistency

- If *X* loses a value, neighbors of *X* need to be rechecked.



# Arc consistency




- Apply arc consistency to the arc from $SA$ to $NT$.
- Both variables have the domain $\{blue\}$.
- The result is that blue must be deleted from the domain of $SA$, leaving the domain empty.
- Arc consistency detects failure earlier than forward checking.
- Can be run as a preprocessor or after each assignment.

# Arc consistency algorithm AC-3

**function** AC-3( *csp*) **returns** the CSP, possibly with reduced domains
  **inputs**: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  **local variables**: *queue*, a queue of arcs, initially all the arcs in *csp*

  **while** *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    **if** RM-INCONSISTENT-VALUES($X_i$, $X_j$) **then**
      **for each** $X_k$ **in** NEIGHBORS[$X_i$] **do**
        add $(X_k, X_i)$ to *queue*

---

**function** RM-INCONSISTENT-VALUES( $X_i$, $X_j$) **returns** true iff remove a value
  *removed* ← *false*
  **for each** $x$ **in** DOMAIN[$X_i$] **do**
    **if** no value $y$ in DOMAIN[$X_j$] allows $(x, y)$ to satisfy constraint($X_i$, $X_j$)
      **then** delete $x$ from DOMAIN[$X_i$]; *removed* ← *true*
  **return** *removed*

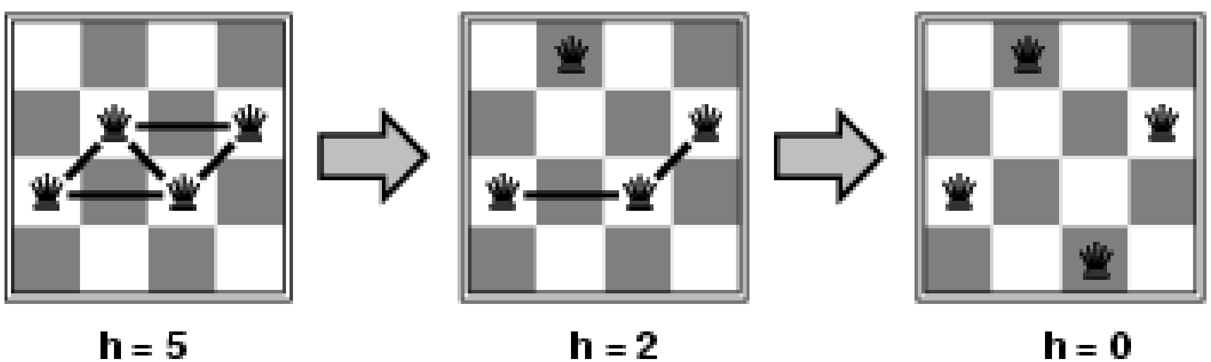
- AC-3 uses a queue to keep track of the arcs that need to be checked for inconsistency.
- After applying AC-3, either every arc is arc-consistent or some variable has an empty domain (thus the CSP cannot be solved).

# Local search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned.
- To apply to CSPs:
  - allow states with unsatisfied constraints.
  - operators reassign variable values.
- Variable selection: randomly select any conflicted variable.
- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints.
  - i.e., hill-climb with $h(n)$ = total number of violated constraints.

# Example: 4-Queens

- States: 4 queens in 4 columns ($4^4$ = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: *h(n)* = number of attacks




- Given random initial state, can solve *n*-queens in almost constant time for arbitrary *n* with high probability (e.g., *n* = 10,000,000)

# Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables.
  - goal test defined by constraints on variable values.
  - CSP can be represented by a constraint graph.
- Backtracking = depth-first search with one variable assigned per node.
- Variable ordering and value selection heuristics help significantly.
- Forward checking prevents assignments that guarantee later failure.
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies.
- Local search using the min-conflicts heuristic is usually effective in practice.