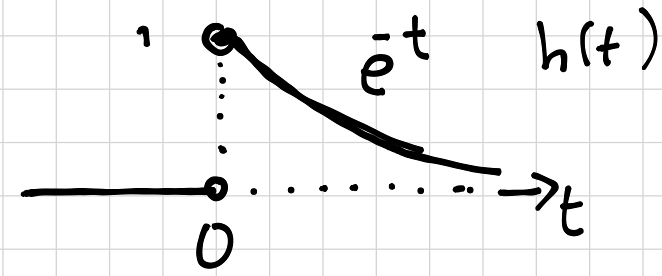
Ex

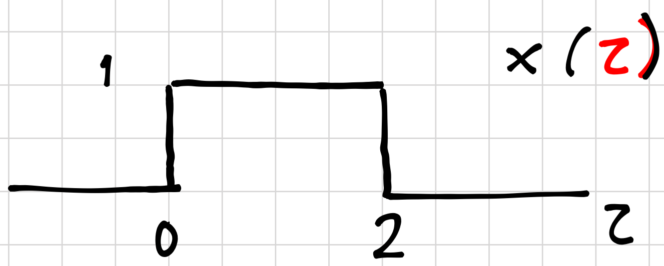
$$\left.\begin{aligned} x(t) &= u(t) - u(t-2) \\ h(t) &= e^{-t}\, u(t) \end{aligned}\right\}$$ Determine the output when $x(t)$ is the input.

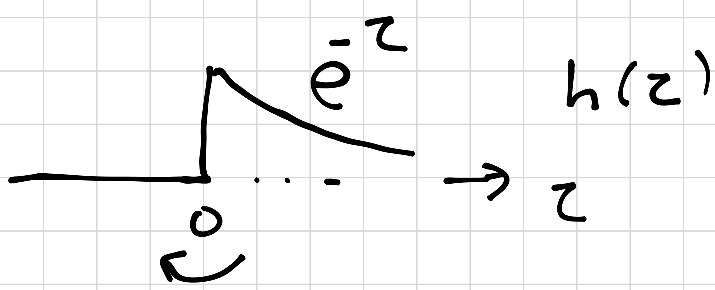
Sol

$$y(t) = x(t) * h(t)$$

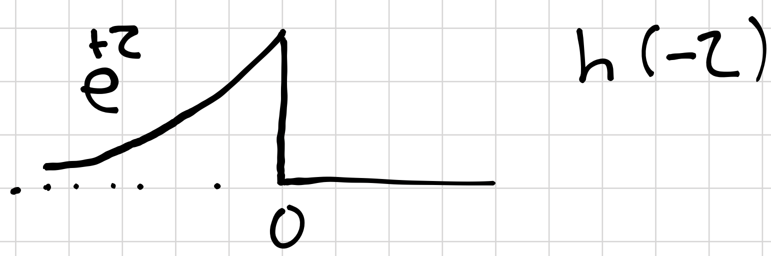
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

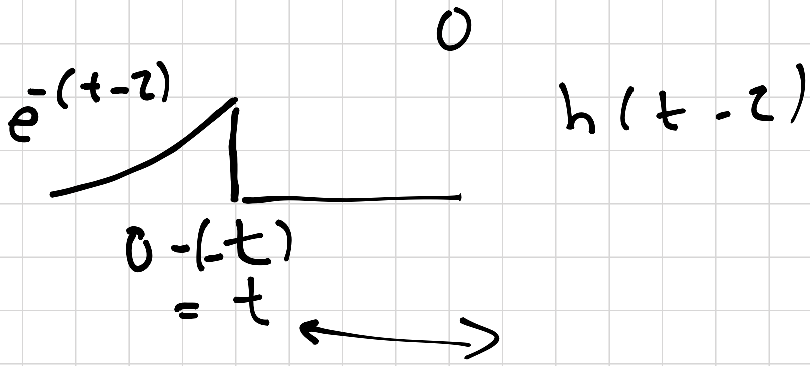




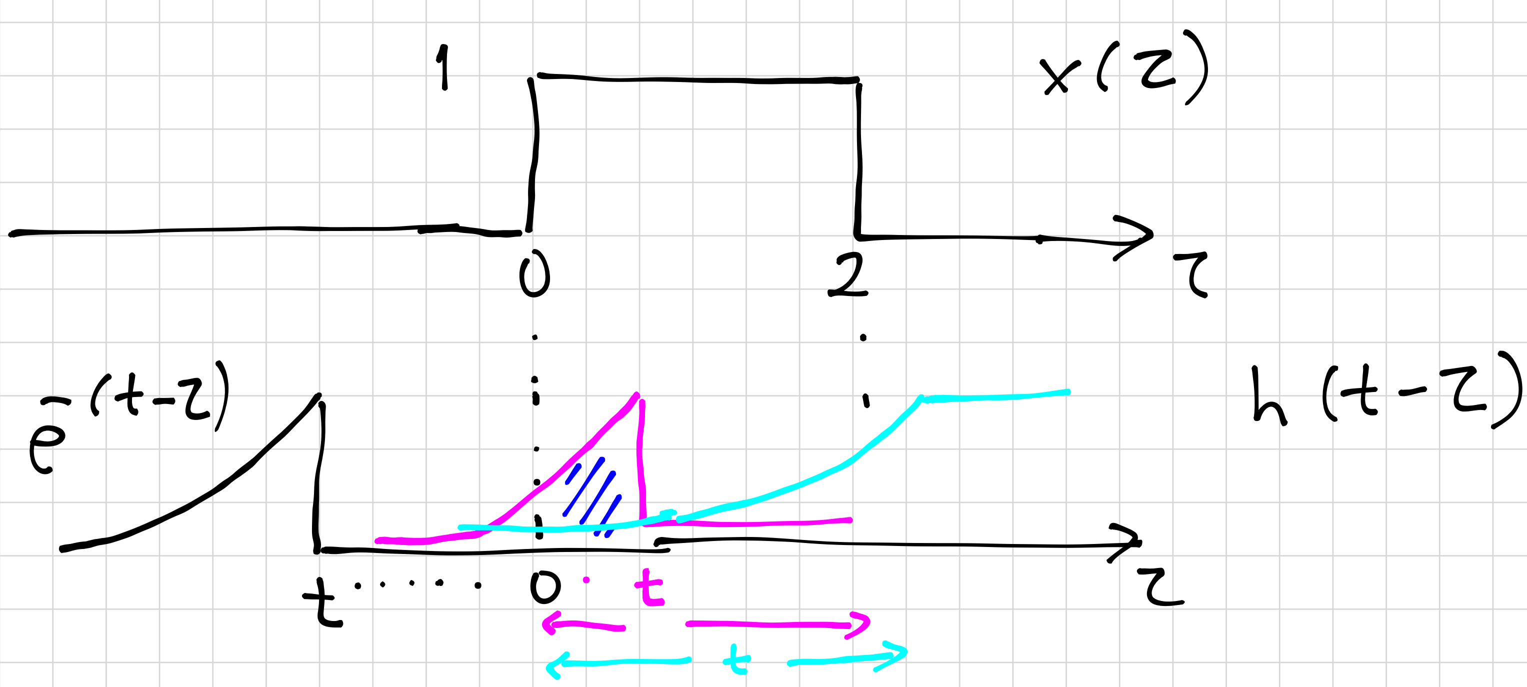


Reflect $x(\tau)$ around 0





Shift right by $-t$



① $t < 0$ $\quad w_t(\tau) = 0$

② $0 < t < 2$ $\quad w_t(\tau) = \begin{cases} e^{-(t-\tau)}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}$

③ $t > 2$ $\quad w_t(\tau) = \begin{cases} e^{\tau - t}, & 0 < \tau < 2 \\ 0, & \text{otherwise} \end{cases}$

$0 < t < 2$

$$y(t) = \int_0^t w_t(\tau)\, d\tau$$

$$= \int_0^t e^{\tau - t}\, d\tau$$

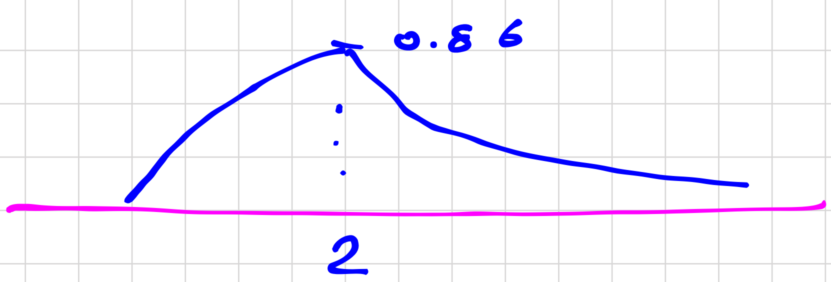
$$= e^{-t}\, e^{\tau}\Big|_0^t = 1 - e^{-t}$$

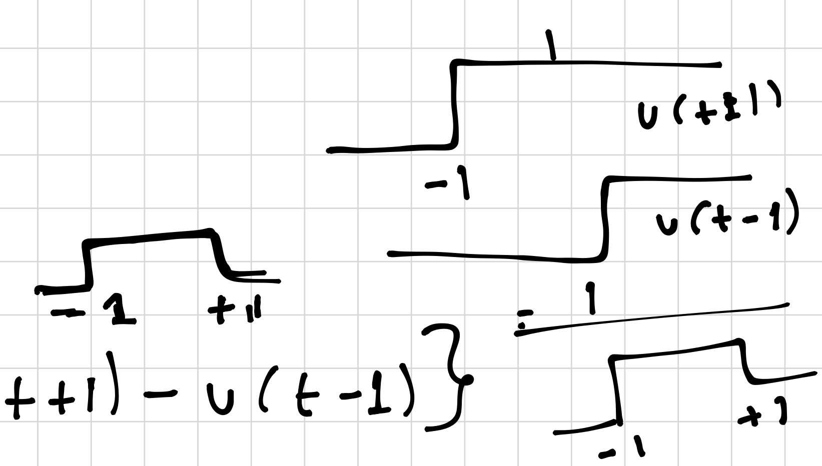
$t > 2$

$$y(t) = \int_0^2 w_t(\tau)\, d\tau$$

$$= e^{-t}\int_0^2 e^{\tau}\, d\tau = e^{-t}(e^2 - 1)$$

$$y(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t}, & 0 \leq t < 2 \\ e^{-t}(e^2-1), & t \geq 2 \end{cases}$$

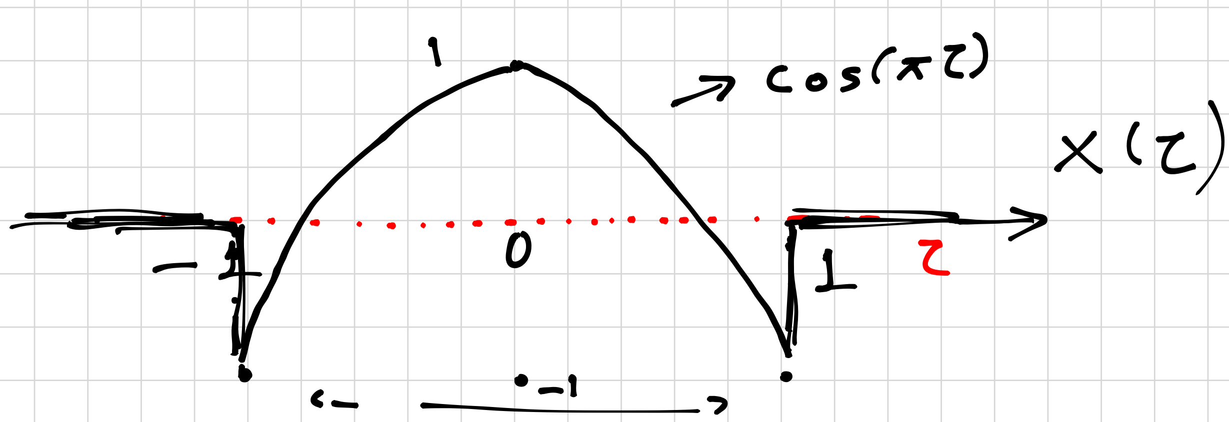


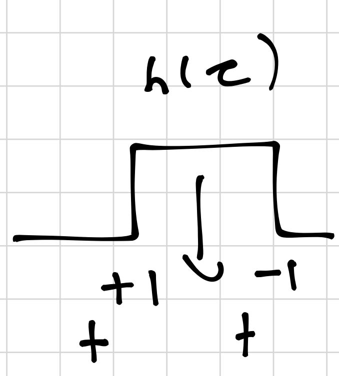


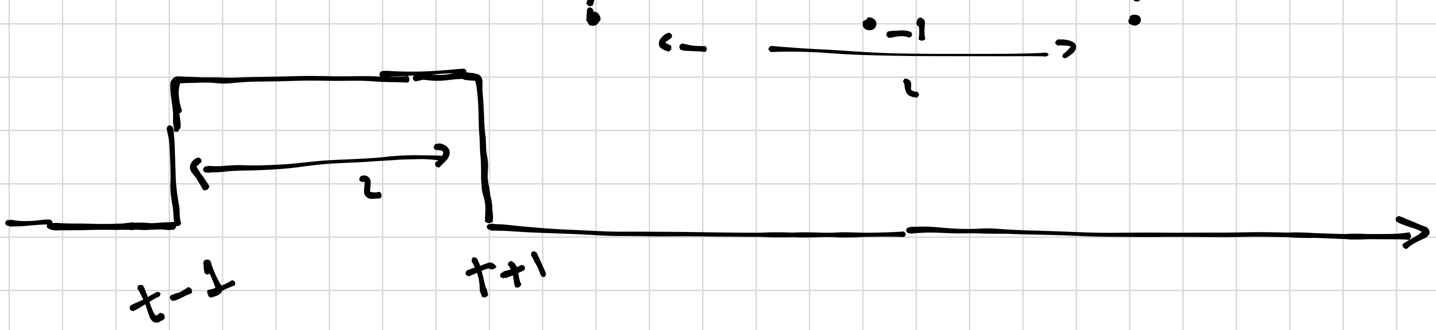
$\boxed{\text{EX}}$

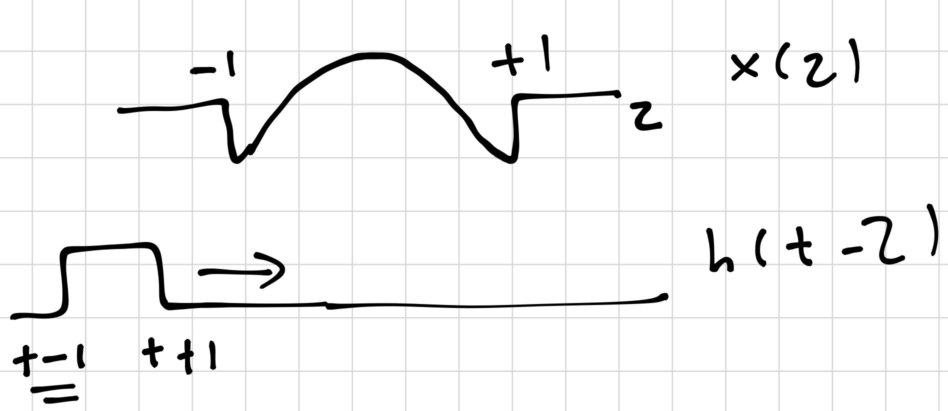
$$x(t) = \cos(\pi t)\,\{u(t+1) - u(t-1)\}$$

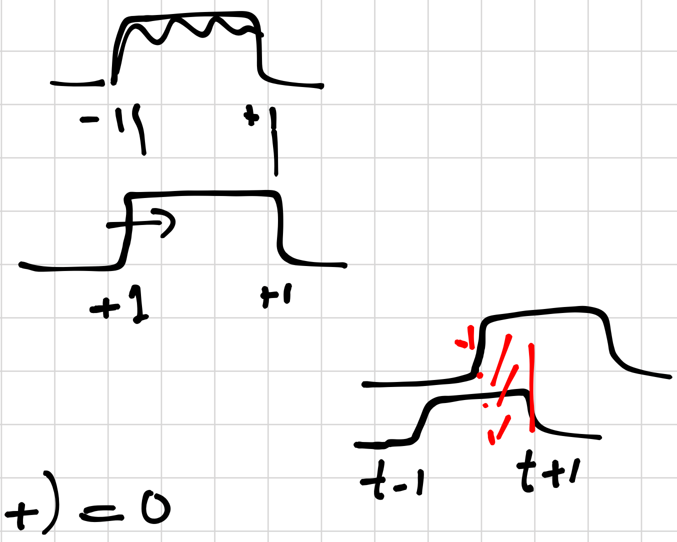
$$h(t) = u(t+1) - u(t-1)$$











① $t+1 \leq -1 \Rightarrow t \leq -2 \Rightarrow y(t) = 0$

② $\left.\begin{matrix} t+1 \geq -1 \\ t-1 \leq -1 \end{matrix}\right\} -2 \leq t < 0$

$$y(t) = \int_{-1}^{t+1} \cos(\pi\tau)\, d\tau$$

$$= \frac{1}{\pi} \sin\left[\pi(t+1)\right]$$

③ $\left.\begin{matrix} t+1 > 1 \\ t-1 < 1 \end{matrix}\right\} 0 < t < 2$

$$y(t) = \int_{t-1}^{1} \cos(\pi\tau)\, d\tau = -\frac{1}{\pi} \sin\left[\pi(t-1)\right]$$

④ $t-1 > 1$

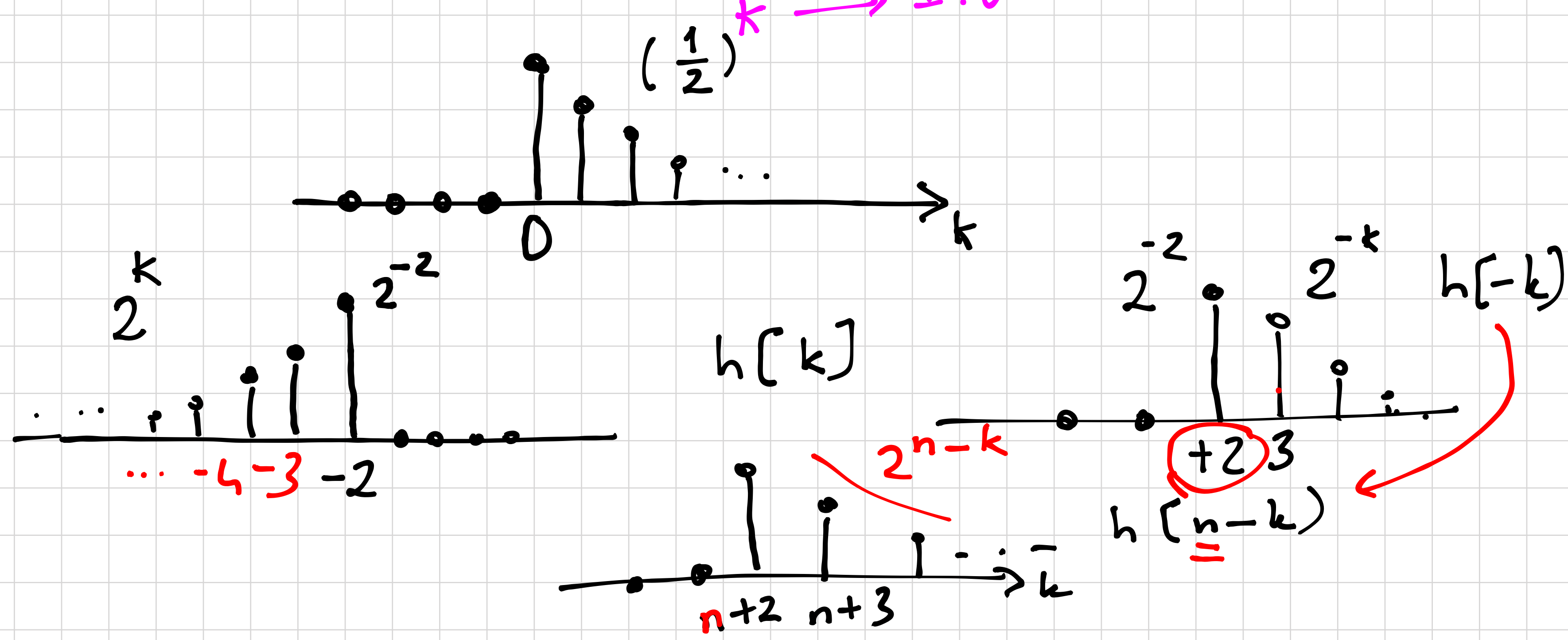
$t > 2 \quad y(t) = 0$

$$y(t) = \frac{1}{\pi} \begin{cases} \sin\left[\pi(t+1)\right], & -2 < t < 0 \\ -\sin\left[\pi(t-1)\right], & 0 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

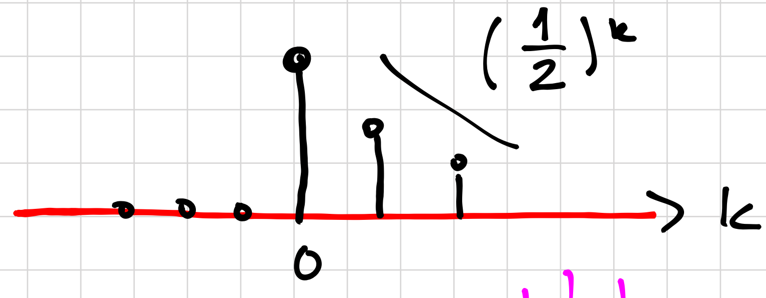
Ex

$$\left.\begin{matrix} x[n] = \left(\frac{1}{2}\right)^n \cdot u[n] \\ h[n] = 2^n \cdot u[-n-2] \end{matrix}\right\} y[n] = ?$$

I. Variable is k !



$x[k]$



2^{n-k} !!! $\leftarrow h(n-k)$

I $n+2 < 0$, $n < -2$

$$w_n[k] = \begin{cases} 2^{n-k} \cdot \left(\frac{1}{2}\right)^k, & 0 \le k < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$y[n] = \sum_{k=0}^{\infty} 2^{-k} \cdot 2^{+n-k}$$

$$= 2^n \cdot \sum_{k=0}^{\infty} 4^{-k}$$

$$= 2^n \cdot \frac{1}{1 - 1/4} = \frac{4}{3} \cdot 2^n$$

II $n \geq -2$

$$y[n] = \sum_{k=n+2}^{\infty} 2^{-k} \cdot 2^{n-k} = \ldots$$

$$= \frac{1}{12} \cdot 2^{-n}$$

$$y[n] = \begin{cases} 4/3 \cdot 2^n, & n < -2 \\ 1/12 \cdot 2^{-n}, & n \geq -2 \end{cases}$$

Properties of LTI Systems & Convolution

1) The Commutative Property (Değişme öz.)

$$x[n] * h[n] = h[n] * x[n]$$

$$x(t) * h(t) = h(t) * x(t)$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

2) The Distributive Property (Dağılma öz)

$$y(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

$$= [x_1(t) + x_2(t)] * h(t)$$

$$= h(t) * [x_1(t) + x_2(t)]$$

or

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

$$= x(t) * [h_1(t) + h_2(t)]$$

$$= [h_1(t) + h_2(t)] * x(t)$$

(Same for DT) !

3) The Associative Property (Birleşme)

$$x[n] \rightarrow \boxed{h_1[n]} \xrightarrow[w[n]]{} \boxed{h_2[n]} \rightarrow y[n]$$

$$w[n] = x[n] * h_1[n]$$

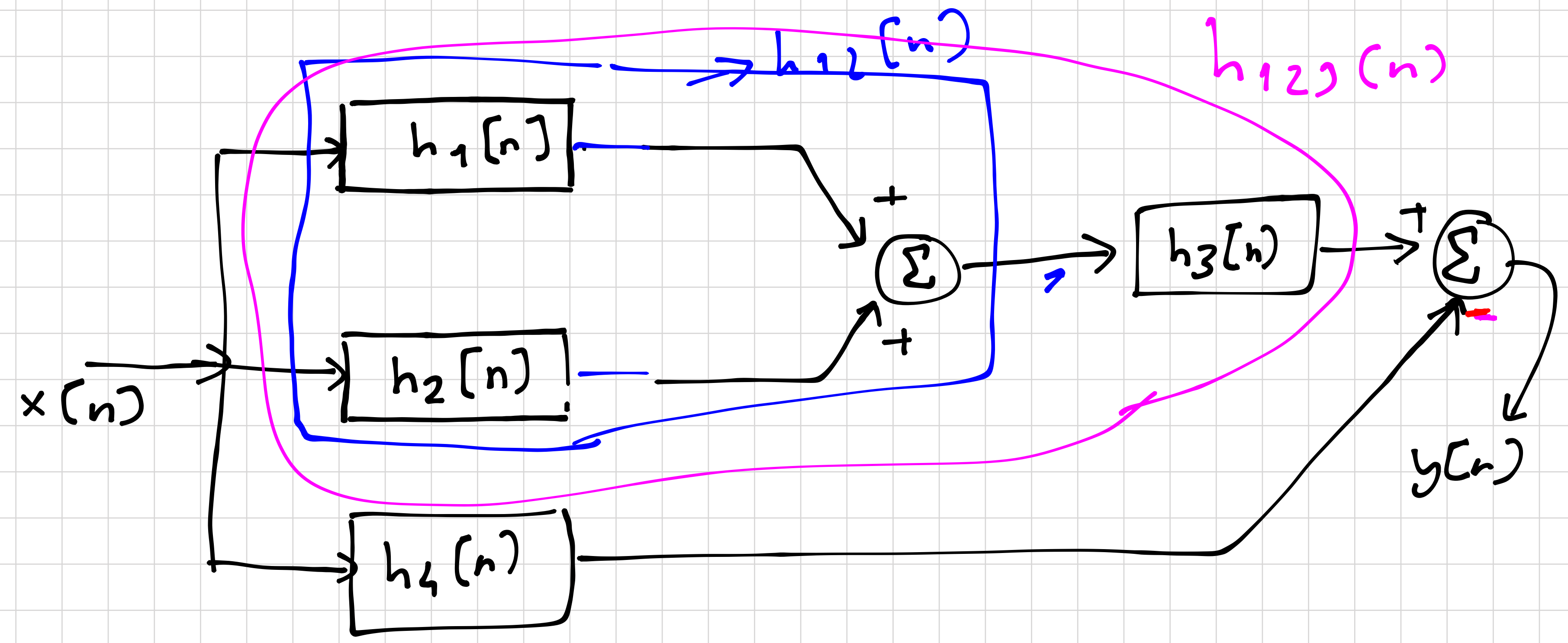
$$y[n] = w[n] * h_2[n]$$

$$= \{x[n] * h_1[n]\} * h_2[n]$$

$$= x[n] * \underbrace{\{h_1[n] * h_2[n]\}}_{h[n]}$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

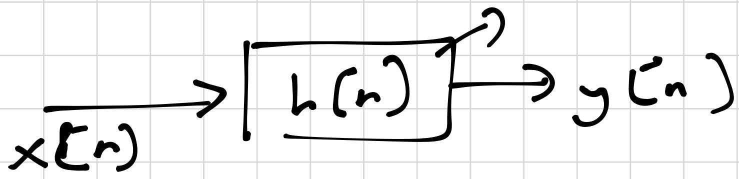
Ex



$h_1[n] = u(n)$ ✓ $h_2[n] = u(n+2) - u(n)$ ✓

$h_3[n] = \delta(n-2)$ ✓ $h_4[n] = \alpha^n u(n)$ ✓

Find the impulse response of the overall system.



$h_{12}[n] = h_1[n] + h_2(n)$ $h_{123}(n) = h_{12}(n) * h_3(n)$

$h_{123}(n) = (h_1(n) + h_2[n]) * h_3(n)$

$h[n] = h_{123}(n) - h_4(n)$

$h(n) = (\underbrace{h_1(n) + h_2[n]}_{h_{12}(n)}) * h_3(n) - h_4(n)$

$h_{12}(n) = u[\cancel{n}] + u(n+2) - \cancel{u(n)}$

$h_{12}[n] = u(n+2)$

$h_{123}(n) = u(n+2) * \delta[n-2]$

$= u(n+2-2) = u(n)$

$x[n] * \delta[n-n_0] = x[n-n_0]$

$h(n) = u(n) - \alpha^n u[n] = \boxed{(1-\alpha^n)\, u[n]}$

Relationship betwee LTI System Properties and the Impulse Response

1 memoryless LTI Systems

If $h[n]$ is the impulse response of an LTI system, given the $x(n)$, the output is

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{+\infty} h[k] \cdot x[n-k]$$

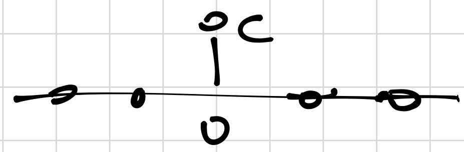
$$= \dots + \underline{h[-2]\, x[n+2]} + \underline{h[-1]\, x[n+1]}$$
$$+ h[0]x[n] + h(1)\, x(n-1) + \dots$$

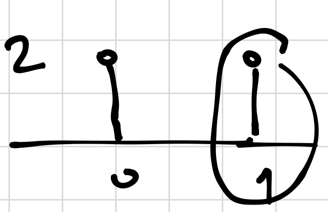
For this system to be memoryless $y[n]$ must depend on only $x[n]$, and it cannot depend on $x(n-k)$ for $k \neq 0$.

So,
$h[k]$ must be zero when $k \neq 0$

For a DT LTI system to be memoryless

$$h[k] = c\, \delta(k)$$

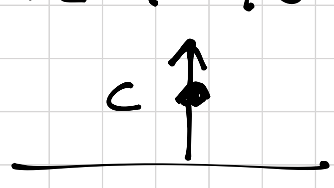


/* 

$h(n)$ $\Rightarrow$ Is it memoryless */
$\rightarrow$ No

Similarly for CT-LTI system to be memoryless

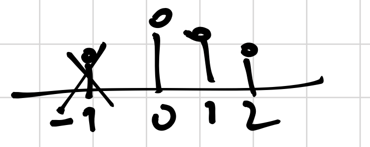
$$h(t) = c\, \delta(t)$$



Causality

For a causal LTI system

- $h[k] = 0$ for $k < 0$ (DT)
- $h(t) = 0$ for $t < 0$ (CT)



③ Stable LTI Systems

If a system is stable then:

- Assuming $|x[n]| \leq \underline{\underline{M_x}} < \infty$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{+\infty} |h[k]| \underbrace{|x[n-k]|}_{\leq M_x}$$

$$\leq M_x \underbrace{\sum_{k=-\infty}^{+\infty} |h[k]|}\ \overset{?}{<} \underline{\underline{M_y}}$$

$$\boxed{\begin{array}{l} a+b=c \\ |a|+|b| \leq |c| \end{array}}$$

If the system is stable then $\boxed{\sum_{k=-\infty}^{+\infty} |h[k]| < \infty}$

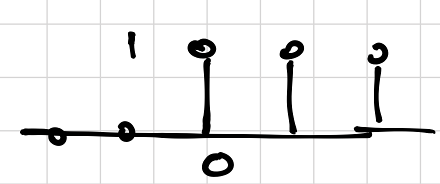
Similarly for CT

If is a stable system $\longrightarrow$ $\boxed{\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty}$

STEP RESPONSE

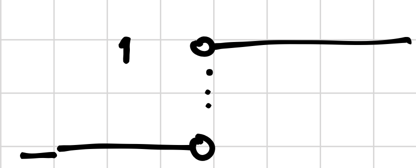
Step response of an LTI system shows how the system responds to sudden changes at the input.

DT $\quad u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$



u[n]

CT $\quad u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$



u(t)

$$u[n] \longrightarrow \boxed{H} \longrightarrow s[n]$$

$$s[n] \triangleq h[n] * u[n]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]\, u[\underset{\sim}{n-k}]$$

$u[n-k] \Rightarrow$

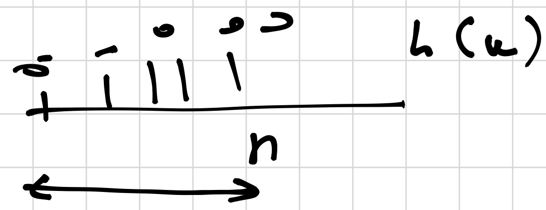
$k > n \rightarrow 0$

$k \leq n \rightarrow 1$

$u[n-k] = 0$ when $k > n$

$u[n-k] = 1$ when $k \leq n$

DT $\quad s[n] = \sum_{k=-\infty}^{n} h[k]$



CT $\quad s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$

Conversely ~~Inversely~~

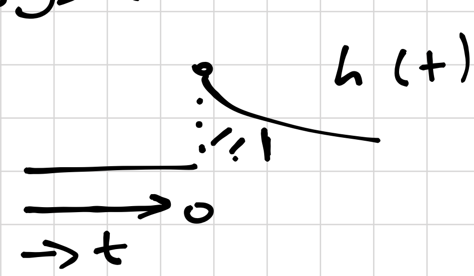
$$h(t) = \frac{d}{dt} s(t)$$

$$h[n] = s[n] - s[n-1]$$

Ex $\quad h(t) = \frac{1}{\alpha} \cdot e^{-t/\alpha} \cdot u(t) \Rightarrow$

what is the step response of this system?

$$S(t) = \int_{-\infty}^{t} h(\tau) d\tau$$



$t < 0 \quad s(t) = \int_{-\infty}^{t} 0 \cdot d\tau = 0$

$t \geq 0 \quad s(t) = \int_{-\infty}^{0} 0 \cdot d\tau + \int_{0}^{t} \frac{1}{\alpha} e^{-\tau/\alpha} d\tau$

$$= 1 - e^{-t/\alpha}$$

$$s(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t/\alpha}, & t \geq 0 \end{cases}$$