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Space-dependent heat source determination problem with nonlocal periodic boundary conditions

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ABSTRACT

The purpose of this paper is to identify the space-dependent heat source coefficient numerically, for the first time, in the third-order pseudo-parabolic equation with initial and nonlocal periodic boundary conditions from nonlocal integral observation. This problem emerges significantly in the modelling of numerous phenomena in physics, engineering, mechanics and science. Although, the inverse source problem considered in this article is ill-posed by being sensitive to noise but has a unique solution. For the numerical realization, we apply the finite difference method (FDM) for discretizing the forward problem and the Tikhonov regularization for finding a stable and accurate solution. The resulting nonlinear minimization problem is solved computationally using the MATLAB subroutine *lsqnonlin*. Numerical results presented for two benchmark test examples with linear and nonlinear source functions show the efficiency of the computational method and the accuracy and stability of the numerical solution even in the presence of noise in the input data. Furthermore, the von Neumann stability analysis is also discussed.

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1. Introduction

The identification of the unknown space- and/or time-dependent source term in the inverse problems for pseudo-parabolic equations was begun in the 1980s [1]. These equations arise in modelling of numerous phenomena such as heat conduction, diffusion, and wave processes [2,3], etc. The problems of heat transfer, fluid filtration, moisture transfer in soils lead to the third-order partial pseudo-parabolic equation [4–7]. The reconstruction of the unknown terms in the inverse problems of the third-order pseudo-parabolic equations have been investigated by only few researchers. For example, the authors of [8] investigated the inverse problems of identifying the time-dependent coefficients in the third-order pseudo-parabolic equation. The existence and uniqueness of a strong solution also proved in [9]. Mehraliyev and Shafiyeva [10] studied an inverse boundary-value problem for the aforesaid equation with integral condition and proved the existence and uniqueness of the classical solution.

Furthermore, Abylkairov and Khompysh [11] investigated an inverse problem of the pseudo-parabolic equation for reconstructing the space-dependent term while Lyubanova and Velisevich [12] investigated it for identifying the coefficient in the lower coefficient from nonlocal integral measurement. Ramazanova et al. [13] determined theoretically the time-dependent potential coefficient while Huntul et al. [14–17] determined it numerically in the fourth-order partial differential equations from various over-specification conditions. Lyubanova [18] investigated the inverse problems to reconstruct the time-dependent coefficient in the linear pseudo-parabolic equation based on the integral data over the entire boundary. Khonatbek [19] investigated the inverse problem of the multi-dimensional pseudo-parabolic equation

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for finding the velocity field, pressure and right-hand side while the authors of [20] studied it for identifying the leading coefficient with additional condition as an integral over-determination condition. Yaman and Gözükizil [21] considered an inverse problem for pseudo-parabolic equations to study the asymptotic behaviour of the solution of unknown source term with the integral condition. In [22], the authors studied an inverse problem of reconstructing the time-dependent source function for the population model with population density nonlocal boundary conditions and an integral over-determination measurement.

Recently, Huntul [23] identified the unknown time-dependent coefficient in the third-order equation from nonlocal integral observation. Khompysh and Shakir [24] determined theoretically the space-dependent heat source in a third-order pseudo-parabolic equation. The theorems of existence and uniqueness also proved. In this work, we identify numerically, for the first time, an unknown space-dependent heat source term in a third-order pseudo-parabolic equation with periodic boundary conditions from nonlocal integral observation. We use usual FDM to discretize the direct problem while the Tikhonov regularization is used to find a stable and accurate solution.

The paper is organized as follows. The mathematical formulation of the third-order pseudo-parabolic problem with initial and nonlocal periodic boundary conditions, is stated in Section 2 while the discretization of the direct problem is described in Section 3. The stability analysis is presented in Section 4. The numerical approach based on the minimization of the nonlinear least-squares objective function is introduced in Section 5. Numerical results are presented and discussed in Section 6. Finally, conclusions are highlighted in Section 7.

2. Mathematical formulation of the inverse problem

We consider the inverse problem of identifying the space-dependent heat source coefficient $f(x)$ in the third-order pseudo-parabolic equation

$$u_t(x,t) - u_{xx}(x,t) - u_{xxt}(x,t) = f(x), \quad (x,t) \in Q_T, \tag{1}$$

in the solution domain $Q_T = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$, subject to the initial condition

$$u(x,0) = \varphi(x), \quad x \in [0,1], \tag{2}$$

the nonlocal periodic boundary conditions

$$u(1,t) = 0, \quad u_x(0,t) = u_x(1,t) \quad t \in [0,T], \tag{3}$$

and the nonlocal integral observation

$$\int_0^T u(x,t)dt = \psi(x), \quad x \in [0,1], \tag{4}$$

where $\varphi(x)$, $\psi(x)$ are given functions, whilst the pair of functions $f(x)$ and $u(x,t)$ are unknown. Further, assume that the conditions (2)–(4) are compatible, i.e.

$$\varphi(1) = 0, \quad \varphi'(1) = \varphi'(0), \quad \psi(1) = 0, \quad \psi'(0) = \psi'(1). \tag{5}$$

The uniqueness of solution of the inverse problem (1)–(4) has been established in [24] and reads as follows.

Theorem 1 (*Uniqueness of the Solution*)**.** *Assume that the conditions of Eq. (5) are satisfied, and that $\varphi(x), \psi(x) \in C^3[0,1]$, $\varphi''(1) = \varphi''(0)$, $\psi''(0) = \psi''(1)$. Then, a solution $(f(x), u(x,t)) \in C[0,1] \times C^{2,1}_{x,t}(Q_T)$, to the problem (1)–(4) is unique.*

3. Numerical solution of the forward problem

In this section, we consider the forward (direct) initial boundary value problem given by Eqs. (1)–(3) when $f(x)$ and $\varphi(x)$ are given and $u(x,t)$ is to be determined along with the nonlocal integral measured data $\psi(x)$. The discrete form of the direct problem is as follows. We denote $u(x_i,t_j) = u_{i,j}$, and $f(x_i) = f_i$ where $x_i = i\Delta x$, $t_j = j\Delta t$, $\Delta x = \frac{1}{M}$ and $\Delta t = \frac{T}{N}$ for $i = 0,1,\dots,M$ and $j = 0,1,\dots,N$. Based on the FDM [23,25], Eq. (1) can be approximated as:

$$\begin{aligned}&\frac{u_{i,j+1} - u_{i,j}}{\Delta t} - \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} - \left(\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{(\Delta x)^2\,\Delta t}\right.\\ &\left. - \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2\,\Delta t}\right) = f_i, \quad i = 1,2,\dots,M-1, \quad j = 0,1,\dots,N-1.\end{aligned} \tag{6}$$

Simplifying Eq. (6), we get

$$-Au_{i-1,j+1} + (1+2A)u_{i,j+1} - Au_{i+1,j+1} = -Bu_{i-1,j} + (1+2B)u_{i,j} - Bu_{i+1,j} + \Delta t f_i, \tag{7}$$

where

$$A = \frac{1}{(\Delta x)^2}, \ B = \frac{1}{(\Delta x)^2} - \frac{\Delta t}{(\Delta x)^2}.$$

The discretization of the conditions (2) and (3) give

$$u(x, 0) = u_{i,0} = \varphi(x_i), \quad u(1, t) = u_{M,j} = 0, \quad i = 0, 1, \dots, M, \quad j = 0, 1, \dots, N. \tag{8}$$

Finally the discretization of the condition $u_x(0, t) = u_x(1, t)$ gives

$$u_{-1,j} = u_{1,j} + u_{M-1,j}, \quad j = 0, 1, \dots, N. \tag{9}$$

For $i = 0$, using Eq. (9) in Eq. (7), we have

$$(1 + 2A)u_{0,j+1} - 2Au_{1,j+1} - Au_{M-1,j+1} \quad = (1 + 2B)u_{0,j} - 2Bu_{1,j} - Bu_{M-1,j} + \Delta tf_0, \quad j = 0, 1, \dots, N - 1. \tag{10}$$

For $i = M$, from boundary condition (8), we have

$$u_{M,j} = 0, \quad j = 0, 1, \dots, N. \tag{11}$$

At time t_{j+1}, for $j = 0, 1, \dots, N - 1$, Eqs. (7), (10), and (11) can be reformulated as a $(M + 1) \times (M + 1)$ tridiagonal system as:

$$\begin{pmatrix} 1+2A & -2A & 0 & 0 & \cdots & -A & 0 \\ -A & 1+2A & -A & 0 & \cdots & 0 & 0 \\ 0 & -A & 1+2A & -A & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & -A & 1+2A & -A & 0 \\ 0 & \cdots & 0 & 0 & -A & 1+2A & -A \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{0,j+1} \\ u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{M-2,j+1} \\ u_{M-1,j+1} \\ u_{M,j+1} \end{pmatrix} = \begin{pmatrix} R_{0,j} \\ R_{1,j} \\ R_{2,j} \\ \vdots \\ R_{M-2,j} \\ R_{M-1,j} \\ R_{M,j} \end{pmatrix},$$

where

$$\begin{aligned} &R_{0,j} = (1 + 2B)u_{0,j} - 2Bu_{1,j} - Bu_{M-1,j} + \Delta tf_0, \\ &R_{i,j} = -Bu_{i-1,j} + (1 + 2B)u_{i,j} - Bu_{i+1,j} + \Delta tf_i, \ i = 1, 2, \dots, M - 1, \\ &R_{M,j} = 0. \end{aligned}$$

Finally, the integral condition (4) is discretized using trapezoidal rule as

$$\int_0^T u(x_i, t)dt = \frac{1}{2M}\left(u_{i,0} + 2\sum_{j=1}^{N-1} u_{i,j} + u_{i,N}\right), \quad i = 0, 1, \dots, M. \tag{12}$$

4. Stability analysis

This section presents the von Neumann stability analysis [26–28] for pseudo-parabolic equation (1). For stability, taking $f_i = 0$ in Eq. (7), we get

$$-Au_{i-1,j+1} + (1 + 2A)u_{i,j+1} - Au_{i+1,j+1} = -Bu_{i-1,j} + (2B + 1)u_{i,j} - Bu_{i+1,j}. \tag{13}$$

Now we consider Fourier mode $u_{i,j} = \delta^j e^{ki\phi}$ as trial solutions at x_i, where $\phi = \theta h$ and $k = \sqrt{-1}$. By using the Fourier mode into the difference equation (13), a value for the complex amplification factor δ is examined. If $|\delta| \leq 1$, then the discretized system for Eq. (1) will be stable. Now, substituting the trial solution in above equation, we get

$$\begin{aligned} &-A\delta^{j+1}e^{(i-1)k\phi} + (1 + 2A)\delta^{j+1}e^{ik\phi} - A\delta^{j+1}e^{(i+1)k\phi} \\ &\quad = -B\delta^j e^{(i-1)k\phi} + (1 + 2B)\delta^j e^{ik\phi} - B\delta^j e^{(i+1)k\phi}. \end{aligned} \tag{14}$$

Simplifying above equation, we get

$$((1 + 2A) - 2A\cos(\phi))\,\delta = (1 + 2B) - 2B\cos(\phi) \tag{15}$$

which can be written as

$$\delta = \frac{4B\sin^2(\phi) + 1}{4A\sin^2(\phi) + 1}. \tag{16}$$

Since $A > 0$, $B > 0$ and $B \leq A$. Hence, $|\delta| \leq 1$, the proposed numerical scheme is unconditionally stable.

5. Numerical solution of the inverse problem

Our aim is to obtain accurate and stable determinations of $f(x)$ and $u(x,t)$ satisfying Eqs. (1)–(4). The inverse problem can be formulated as minimizing the regularized objective function

$$J(f) = \left\| \int_0^T u(x,t)dt - \psi(x) \right\|^2 + \lambda \|f(x)\|^2, \tag{17}$$

where u solves the forward problem (1)–(3) for given $f(x)$, and $\lambda \geq 0$ is penalty parameter to be selected and the norm is usually the L^2-norm. In discrete form, Eq. (17) becomes

$$J(\underline{f}) = \sum_{i=1}^{M} \left[\int_0^T u(x_i,t)dt - \psi(x_i) \right]^2 + \lambda \sum_{i=1}^{M} f_i^2. \tag{18}$$

For minimizing the objective function (18), the MATLAB subroutine *lsqnonlin* [29] is utilized.

6. Numerical results and discussion

In this section, we present a couple of benchmark examples with linear as well as nonlinear continuous space-dependent heat source term to illustrate the accuracy and stability of the FDM combined with the Tikhonov regularization technique presented in Section 5. In order to test the accuracy of the approximations, let us introduce the root mean square error (RMSE) defined as

$$\mathrm{RMSE}(f) = \left[\frac{1}{M} \sum_{i=1}^{M} \left(f^{numerical}(x_i) - f^{exact}(x_i) \right)^2 \right]^{1/2}. \tag{19}$$

We take $T = 1$, for simplicity. Simple bounds (lower and upper) on the variable f are specified as $-10^2 \leq f \leq 10^2$.

In order to investigate the stability of the numerical solution, we add noise to the nonlocal integral overspecified in (4) as:

$$\psi^{\epsilon}(x_i) = \psi(x_i) + \epsilon_i, \quad i = 1, 2, \ldots, M, \tag{20}$$

where ϵ_i are random variables with mean zero and standard deviation σ, computed as:

$$\sigma = p \times \max_{x\in[0,1]} |\psi(x)|, \tag{21}$$

where p denotes the percentage of noise. For noisy data (20), we replace $\psi(x_i)$ by $\psi^{\epsilon}(x_i)$ in (18).

6.1. Example 1

We consider the inverse problem (1)–(4) with linear unknown space-dependent heat source coefficient $f(x)$, with the input data:

$$\varphi(x) = u(x,0) = \frac{1}{3}(1-x)(3+2x^2-2x^3), \;\; u(1,t) = 0, \;\; u_x(0,t) = u_x(1,t) = -1-t, \tag{22}$$

$$\psi(x) = \int_0^1 u(x,t)dt = \frac{1}{6}(9-9x+4x^2-8x^3+4x^4). \tag{23}$$

The conditions of Theorem 1 are fulfilled, and therefore, the uniqueness of the solution is guaranteed. In fact, it can easily be checked by direct substitution that the analytical solutions for the temperature $u(x,t)$ and heat source $f(x)$ are given by

$$u(x,t) = \frac{1}{3}(1-x)(3+3t+2x^2-2x^3), \quad (x,t) \in Q_T, \tag{24}$$

$$f(x) = -\frac{1}{3} + 7x - 8x^2, \quad x \in [0,1]. \tag{25}$$

First, we investigate the performance of the FDM described in Section 3 to solve the direct well-posed problem (1)–(3), when $f(x)$ is known and given by (25), with $M = N \in \{5, 10, 20, 40\}$ for the function $u(x,t)$. Table 1 gives the analytical and numerical solutions for the nonlocal integral condition (4). From this table it can be seen that the numerical results are convergent, as the mesh size decreases, and they are in very good agreement with the exact solution (23). The numerical and exact solutions for $u(x,t)$ at interior points are shown in Fig. 1 and one can observe that an excellent agreement is obtained.

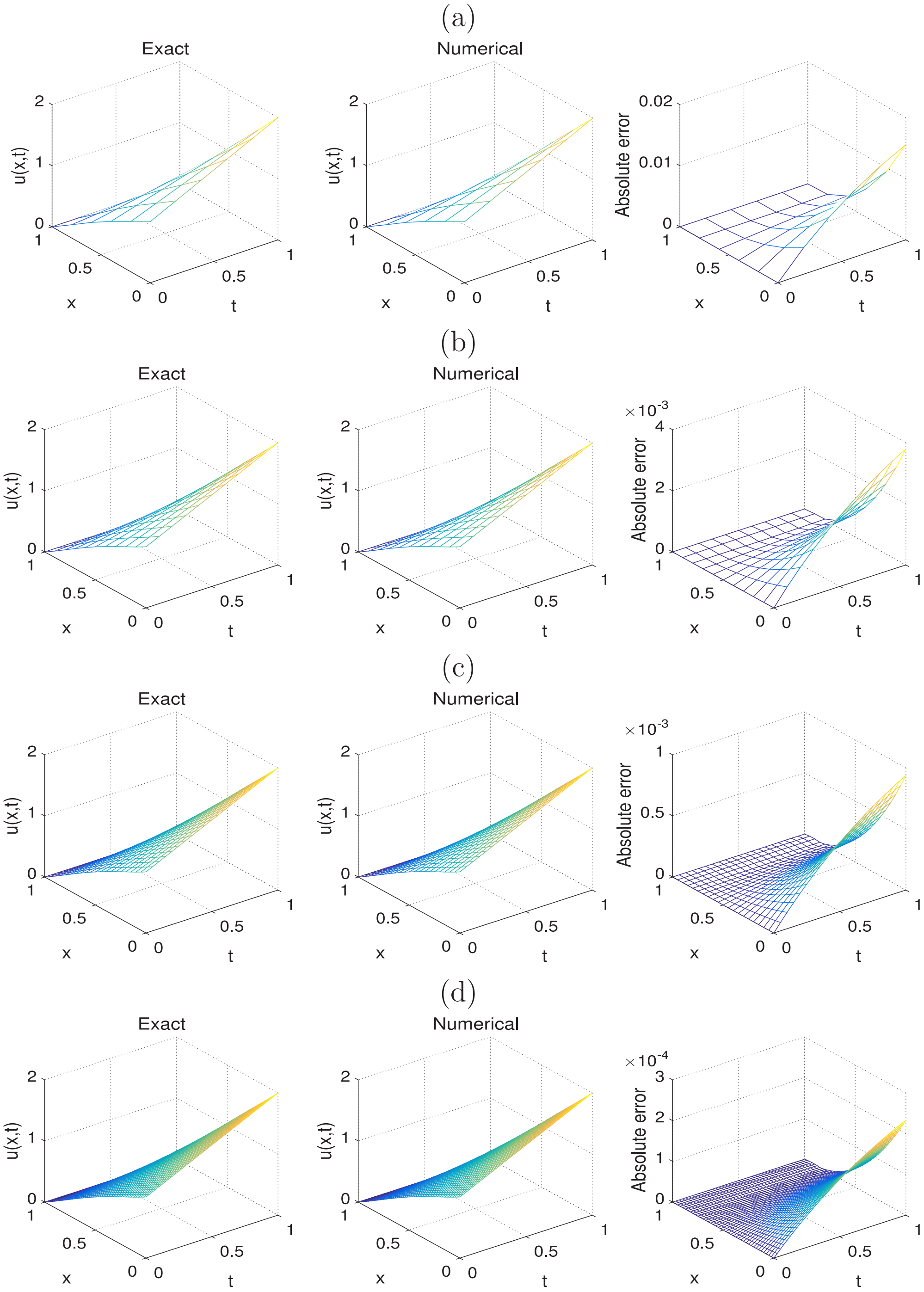

Fig. 1. Exact and numerical solutions for $u(x, t)$, and the absolute error between them for the direct problem, obtained with: (a) $M = N = 5$, (b) $M = N = 10$, (c) $M = N = 20$, and (d) $M = N = 40$, for Example 1.

The inverse problem given by Eqs. (1)–(4) with (22) and (23) is considered next. We take the initial guess for the vector $\underline{f}$ as:

$$f^0(x_i) = f(0) = -\frac{1}{3}, \quad i = 1, 2, \ldots, M. \tag{26}$$

Let us fix $\Delta x = \Delta t = 0.025$ and start the investigation for determining the space-dependent force term $f(x)$, with the case of exact input data, i.e. there is no noise included in (21). Fig. 2(a) represents the objective functional (19), as a function

Table 1
Exact and numerical solutions for $\psi(x)$ and the RMSE values, for direct problem.

x	0.2	0.4	0.6	...	0.8	$M=N$	RMSE(ψ)
	1.2118	0.9358	0.6375	...	0.3171	5	2.6E−3
	1.2158	0.9378	0.6382	...	0.3171	10	7.7E−4
$\psi(x)$	1.2168	0.9382	0.6383	...	0.3171	20	2.0E−4
	1.2170	0.9384	0.6384	...	0.3171	40	5.3E−5
	1.2171	0.9384	0.6384	...	0.3171	exact	0

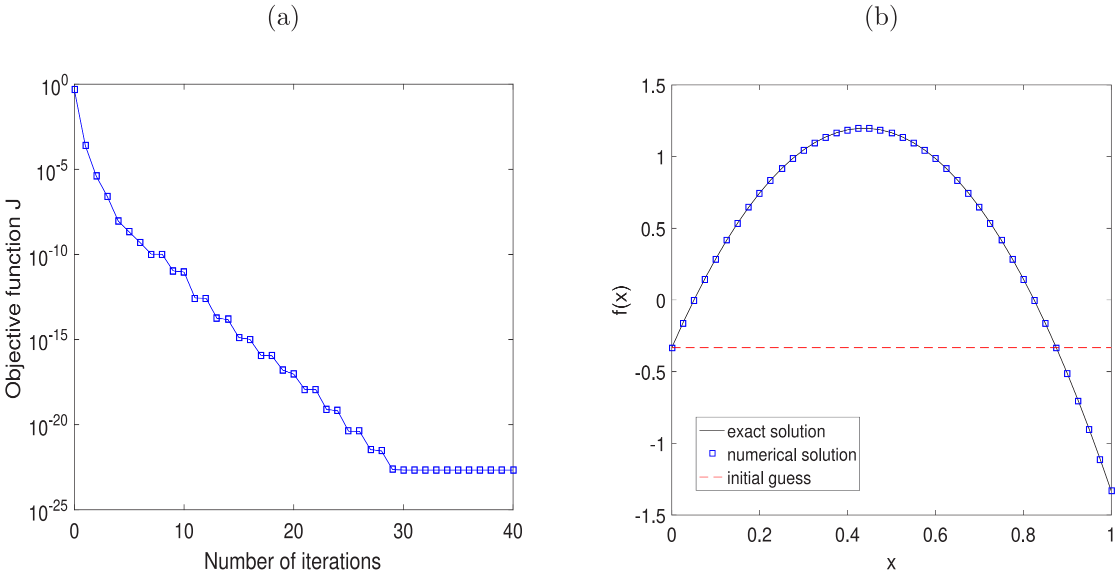

Fig. 2. (a) The objective function J (18), as a function of the number of iterations, and (b) the exact (25) and numerical solutions for the heat source $f(x)$, with no noise and no regularization, for Example 1.

of the number of iterations. From this figure it can be seen that the decreasing convergence is fast and is achieved in 40 iterations to reach a stationary value of $O(10^{-23})$. The numerical results for the space-dependent heat source coefficient $f(x)$ are depicted in Fig. 2(b). From this figure it can be seen that the agreement between the numerical (final iteration 40) and exact (25) solutions for $f(x)$ is excellent with RMSE(f) $= 8.1$E-4.

Next, we investigate the stability of the numerical solution with respect to noise in the data (4), defined by Eq. (20). We add $p \in \{0.01\%, 0.1\%\}$ noise to the nonlocal integral observation $\psi(x)$, as in Eq. (21) via (20). We have also experimented with higher amounts of noise p in Eq. (21), but the results obtained were less accurate and therefore they are not presented. Figs. 3 and 4 show the identification of the estimated $f(x)$. The heat source $f(x)$ is shown in Figs. 3(a) and 4(a), where the unstable and inaccurate results are obtained, if no regularization, i.e. $\lambda = 0$, is imposed with RMSE(f) $= 2.1549$ for $p = 0.01\%$ and RMSE(f) $= 21.5180$ for $p = 0.1\%$, because the problem is ill-posed. Therefore, regularization $\lambda > 0$ is needed for recovering the loss of stability. Among all chosen λ, we deduce that $\lambda \in \{10^{-7}, 10^{-6}\}$ for $p = 0.01\%$ noise, see Fig. 3(b), and $\lambda \in \{10^{-6}, 10^{-5}\}$ for $p = 0.1\%$ noise, see Fig. 4(b), give a stable and reasonable approximate solution for $f(x)$, achieving RMSE(f) $\in \{0.1948, 0.1790\}$, and RMSE(f) $\in \{0.4591, 0.2482\}$, respectively. Furthermore, from Figs. 2(b), 3, 4 and Table 2, it can be seen that as the percentage of noise p decreases from 0.1% to 0.01% and then to zero the numerically obtained results becomes more stable and accurate.

6.2. Example 2

In the previous example we have inverted linear coefficient given by Eq. (25). In this example, we consider the recovery of a nonlinear function $f(x)$ with the following input data:

$$\varphi(x) = \sin^2(2\pi x), \quad u(1,t) = 0, \quad u_x(0,t) = u_x(1,t) = -t, \tag{27}$$

$$\psi(x) = \frac{1}{2}(1-x) + \sin^2(2\pi x). \tag{28}$$

The analytical solutions for $f(x)$ and $u(x,t)$ are given by

$$f(x) = 1 - x - 8\pi^2\cos(4\pi x), \quad x \in [0,1], \tag{29}$$

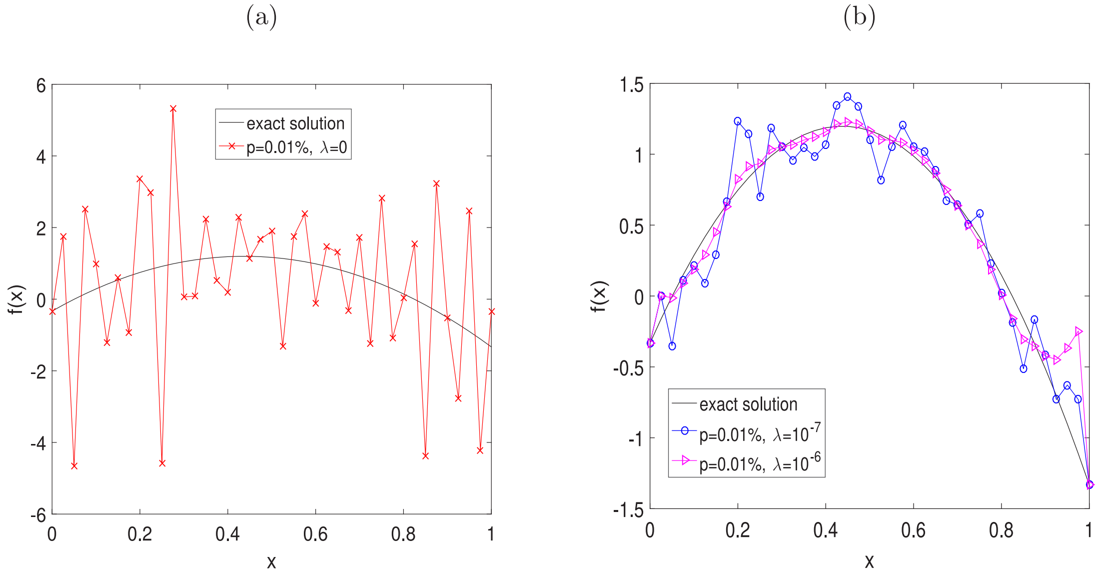

Fig. 3. The exact (25) and numerical solutions for the heat source $f(x)$, for $p = 0.01\%$ noise and without and with regularization, for Example 1.

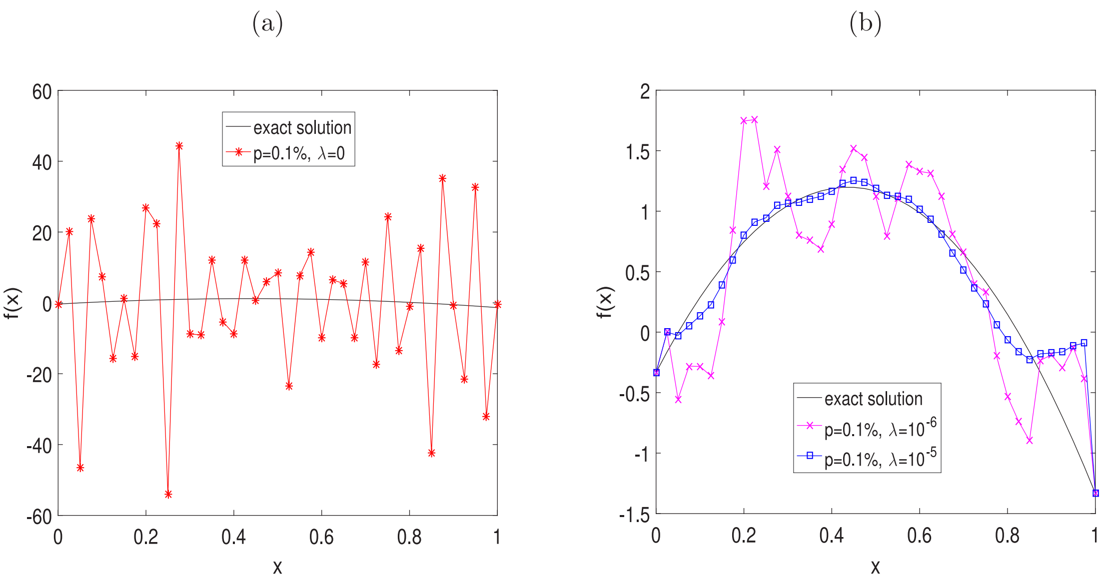

Fig. 4. The exact (25) and numerical solutions for the heat source $f(x)$, for $p = 0.1\%$ noise and without and with regularization, for Example 1.

$$u(x,t) = (1-x)t + \sin^2(2\pi x), \quad (x,t) \in Q_T. \tag{30}$$

Also, one can notice that with this input data the conditions of Theorem 1 are fulfilled and so, the solution is unique. The initial guess for f has been taken as

$$f^0(x_i) = f(0) = 1 - 8\pi^2, \quad i = 1, 2, \ldots, M. \tag{31}$$

Let us fix $M = N = 40$ as we did in Example 1, and start first with the case of exact data, i.e. $p = 0$. The convergence of the unregularized objective function J(18) achieved within 50 iterations using the *lsqnonlin* routine is illustrated in Fig. 5(a). From this figure it can be seen that a monotonic decreasing convergence is reached a low prescribed tolerance of $O(10^{-24})$. Fig. 5(b) demonstrates the outstanding agreement in the exact (29) and numerical solutions for $f(x)$, achieving $\text{RMSE}(f) = 9.3\text{E}{-3}$.

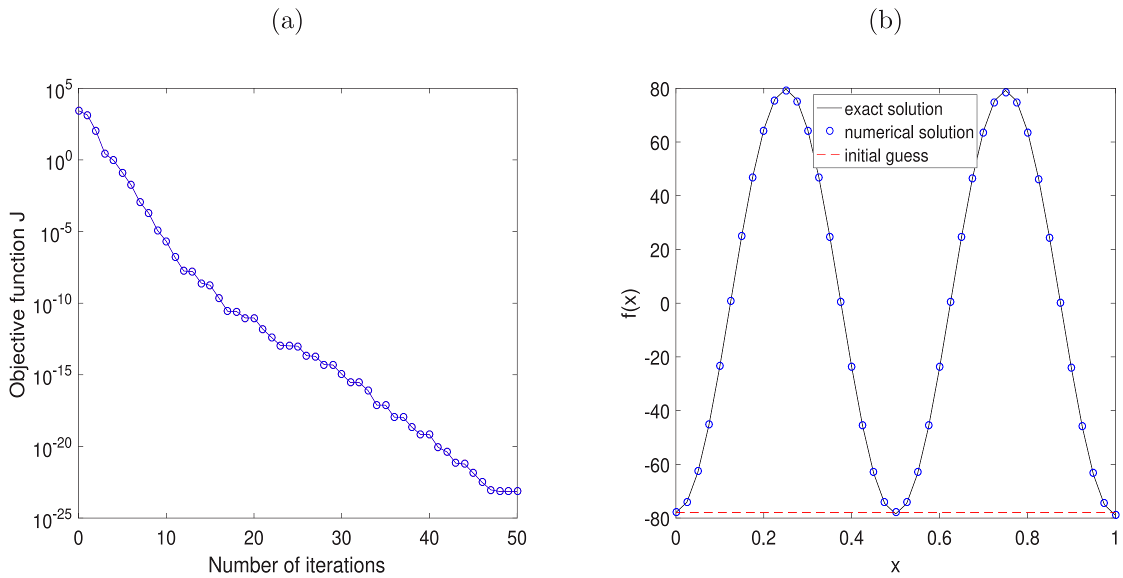

Fig. 5. (a) The objective function J (18), as a function of the number of iterations, and (b) the exact (29) and numerical solutions for the heat source $f(x)$, with no noise and no regularization, for Example 2.

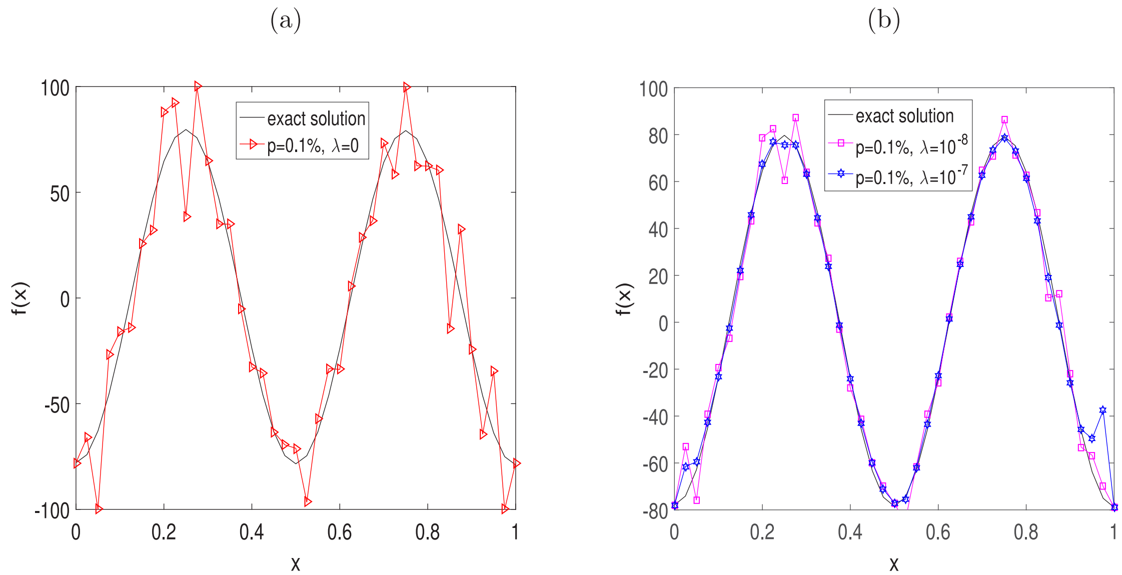

Fig. 6. The exact (29) and numerical solutions for the heat source $f(x)$, for $p = 0.1\%$ noise and without and with regularization, for Example 2.

Next, we investigate the stability of the solution with respect to noise. We include $p \in \{0.1\%, 1\%\}$ noise to the measured data $\psi(x)$, as in (21). The corresponding exact and numerical solutions for the heat source term $f(x)$ are shown in Figs. 6(a) and 7(a), where the unstable (highly oscillation) results are obtained, if no regularization, i.e. $\lambda = 0$, is imposed with RMSE(f) $= 17.5856$ for $p = 0.1\%$, and RMSE(f) $= 54.7615$ for $p = 1\%$ noise. However, the inclusion of some regularizations with $\lambda > 0$ in J (18) recovers the stability of the approximate solutions. One can observe that the choice $\lambda \in \{10^{-8}, 10^{-7}\}$, see Fig. 6(b) for $p = 0.1\%$, and $\lambda \in \{10^{-7}, 10^{-6}\}$, see Fig. 7(b) for $p = 1\%$, give a stable and accurate numerical solutions for the space-dependent term $f(x)$, obtaining RMSE(f) $\in \{2.6403, 2.2477\}$ and RMSE(f) $\in \{3.5217, 3.7120\}$, respectively. The exact (30) and numerical solutions for $u(x, t)$ together with absolute errors, without and with regularization, are demonstrated in Fig. 8, where the effect of $\lambda > 0$ in decreasing the unstable behaviour of the recovered temperature can be observed. For more details about the RMSE(f) values, the minimum value of J at final iteration without and with regularization, for both Example 1 and 2, we refer to Table 2. In general, the similar conclusions can be drawn about the stable identification for the unknown heat source coefficient $f(x)$.

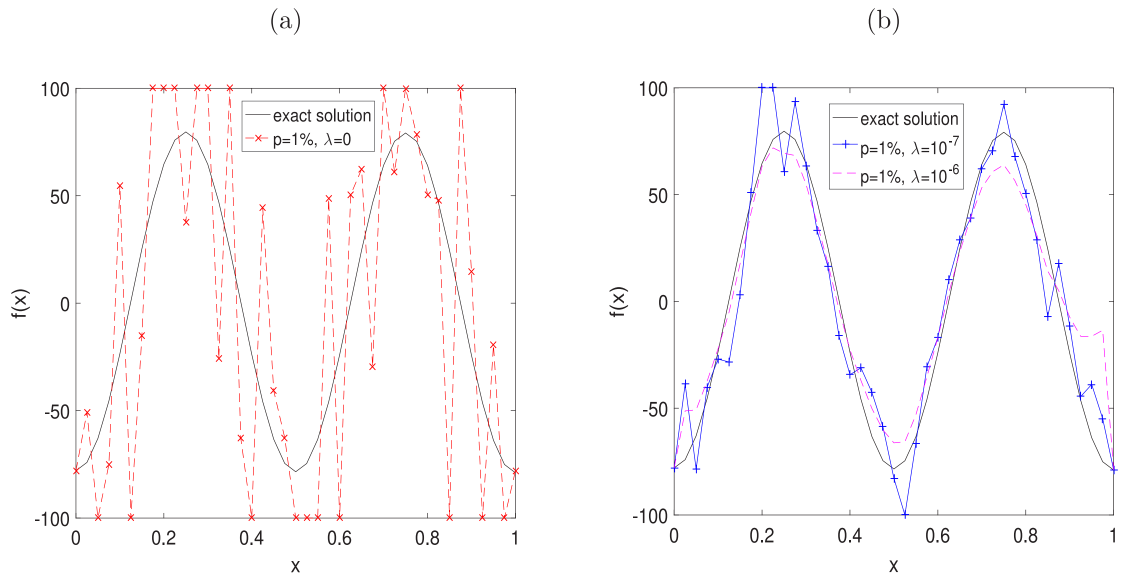

Fig. 7. The exact (29) and numerical solutions for the heat source $f(x)$, for $p = 1\%$ noise and without and with regularization, for Example 2.

Table 2
The RMSE values and the minimum value of (18) at final iteration for $p \in \{0, 0.01\%, 0.1\%, 1\%\}$ noise, without and with regularization, for Examples 1 and 2.

	p	λ	RMSE(f)	Minimum value of (18)	Iter
Example 1	0	0	8.1E−4	2.1E−23	40
	0.01%	0	2.1549	2.3E−10	50
		10^{-7}	0.1948	3.7E−6	15
		10^{-6}	0.1790	2.6E−5	15
		10^{-5}	0.2527	2.2E−4	15
	0.1%	0	21.5180	2.3E−8	100
		10^{-6}	0.4591	1.6E−4	20
		10^{-5}	0.2482	3.9E−4	20
		10^{-4}	0.3816	1.9E−3	20
Example 2	0	0	9.3E−3	7.1E−24	50
	0.1%	0	17.5856	2.3E−8	100
		10^{-8}	2.6403	1.1E−3	30
		10^{-7}	2.2477	9.5E−3	30
		10^{-6}	4.5755	0.0753	30
	1%	0	54.7615	5.4E−3	400
		10^{-7}	3.5217	0.0215	40
		10^{-6}	3.7120	0.0815	40
		10^{-5}	7.3201	0.1204	40

7. Conclusions

In this paper, the inverse problem involving the determination of a space-dependent coefficient $f(x)$ and the temperature $u(x,t)$ in the third-order pseudo-parabolic equation with initial and nonlocal periodic boundary conditions from nonlocal integral observation has been numerically solved. The forward solver based on the FDM was employed. The stability analysis of the solution has been discussed using the von Neumann method. The resulting nonlinear minimization objective function problem was solved computationally using the MATLAB subroutine *lsqnonlin*. Since the inverse source problem is ill-posed, therefore, the Tikhonov regularization was applied in order to overcome the stability. The RMSE values for various noise levels p with and without regularization were compared. The numerical results for the inverse source problem shows that stable and accurate approximate results have been obtained. Finally, the generalization of the proposed method to the real applications for identifying the space-dependent term in the third-order pseudo-parabolic equation [24] is an interesting topic for future research.

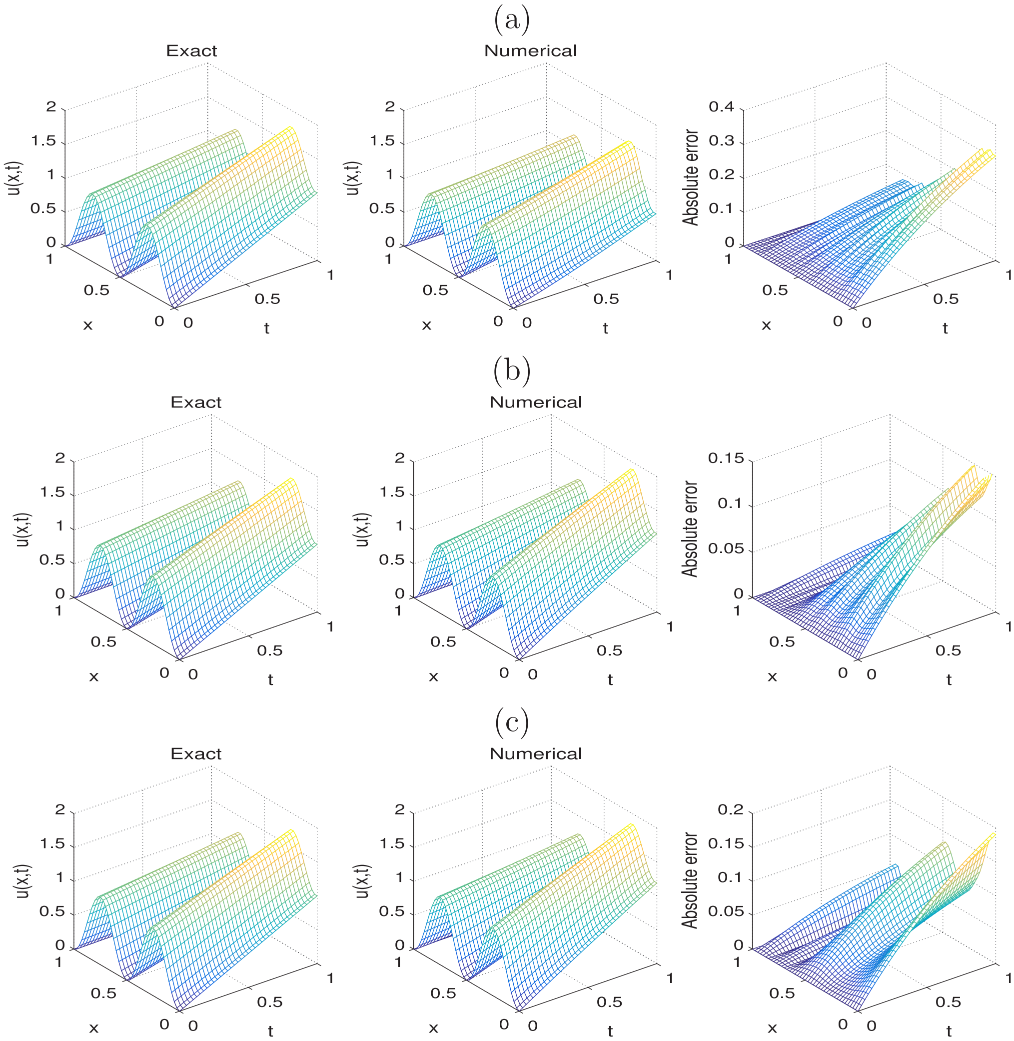

Fig. 8. The exact (30) and numerical temperature $u(x, t)$, for Example 2 with $p = 1\%$ noise and (a) $\lambda = 0$, (b) $\lambda = 10^{-7}$, and (c) $\lambda = 10^{-6}$. The absolute error between them is also included.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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