Lab 1

1. Proofs with Sets

Let's try to show this statement about any three sets A, B, and C:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

1. Draw a Venn diagram of the three sets and shade the region $A \cap (B \cup C)$. Now shade the region $(A \cap B) \cup (A \cap C)$. What do you see?

This isn't a formal argument. What if it has something to do with the way we drew the Venn diagram? Or, more generally, what if we were trying to prove something that was too complicated to draw and look at?

In these cases we need to be more careful. Here we're going to learn a general recipe to help us think through problems where we want to show two sets F and G are really the same set $F = G$.

Equality and Subsets

The way to show equality is usually to break equality down into two set inclusion relationships:

$$F = G \text{ means that } F \subseteq G \text{ and } G \subseteq F$$

which we usually write

$$F = G \iff F \subseteq G \text{ and } G \subseteq F$$

This is actually the definition of set equality. If every element of F is also an element of G and every element of G is also an element of F, then the two sets are the same.

So if we want to show equality, what we should do is show that both inclusions hold. Specifically:

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$
$$A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$$

Inclusion and Arbitrary Elements

So we've reduced equality to two set inclusions. But how do we show that $F \subseteq G$?

The general technique is to pick some *arbitrary element* $\omega \in F$. If, for this element, $\omega \in G$ no matter how I picked it, then every element of F is also an element of G and thus $F \subseteq G$ by definition.

By "arbitrary" what I mean is that it doesn't matter what ω is or how you decide to pick it. For example, if F happened to be the set $\{1, 2, 3\}$ then one choice for ω could be the number 1. But since it's arbitrary then you could pick $\omega = 1$ and I could pick $\omega = 3$; it shouldn't matter which one we go with. When I write ω I mean one of these values- it's fixed from here on out, but it doesn't matter *how* we fixed it.

Once we pick $\omega \in F$ the way to show $\omega \in G$ varies from problem to problem, but the general idea is to exploit the general definitions of sets and elements. For example, continuing with our running problem, let's try to show

$$\underbrace{A \cap (B \cup C)}_{F} \subseteq \underbrace{(A \cap B) \cup (A \cap C)}_{G}$$

Pick any element $\omega \in F$. Since $\omega \in A \cap (B \cup C)$, this element has to be in *both* the sets A and $(B \cup C)$, otherwise it wouldn't be in their intersection. Therefore ω must be in A, but it could be in only one of B or C and not necessarily both. Therefore we have the three possible cases:

- $\omega \in A$, $\omega \in B$, $\omega \notin C$
- $\omega \in A$, $\omega \notin B$, $\omega \in C$
- $\omega \in A$, $\omega \in B$, $\omega \in C$

Now we can use the definition of G to prove that $\omega \in G$ in all three cases! To wit: if our element is in both A and B, then it must be in the intersection of the two by definition. Thus $\omega \in A \cap B$. Since G is the union of that set with another, G contains all elements that are in $A \cap B$ therefore G contains ω as desired.

2. Make similar arguments to show $\omega \in G$ for the other two cases.

The conclusion is that $\omega \in G$. Since this works no matter how we chose our element in F, it holds for any possible $\omega \in F$. Therefore all elements of F are elements of G which by definition means $F \subseteq G$.

3. Use a similar strategy to prove the inclusion holds the other way too: $A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$. Conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

DIY

4. Use this strategy to prove the following
   - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   - $(A \cup B)^\complement = A^\complement \cap B^\complement$

2. Power Set

The *power set* of a set A is the set of all subsets (with equality) of A. For reasons that may become apparent shortly, we sometimes write the powerset using the funny notation 2^A. So, for example, if $A = \{1, 2, 3\}$, then the power set would be given by

$$
\begin{aligned}
2^A = \{ & \\
& \{\}, \\
& \{1\}, \{2\}, \{3\}, \\
& \{1,2\}, \{2,3\}, \{3,1\}, \\
& \{1,2,3\} \\
\} &
\end{aligned}
$$

1. For a set A that has n elements, how many elements does its power set have? In other words, how many different subsets does A have?

2. What is the relationship between $2^{\{1...n\}}$ and $2^{\{1...n,n+1\}}$?

3. Let $\mathcal{C} = \{C_1, \ldots C_n\}$ be a set of sets. Let the notation $\bigcup_{\mathcal{C}} C_i$ denote the union of all of the sets inside of $\mathcal{C}$. What is $\bigcup_{C \in 2^A} C_i$?

3. Function Factory in R

Let's say we'd like to computationally represent the y position of a clock arm of length 5 inches based on the angle t (e.g. the hour of day). y is measured in inches from center of the clock. Using some trigonometry we can figure out that the relationship between y and time t in hours is $y = 5\sin(t \times 2\pi/12)$.

1. write a function in R that implements this (i.e. takes a number t as input and returns y). Plot this function for $t \in [0, 12]$ and verify that it matches your intuition.

This is nice, but it only works for clocks where the hours arm is 5 inches long. It also only works for the hour arm, not the minute or second arm.

2. If l is the length of the clock arm and r is the number of units of time per a 12 hour period (e.g $r = 12$ for the hour arm), figure out a general way express y as a function of t, l, r.

3. Code this function in R (make it have 3 arguments) and make plots to test it (eg for second and minute arms).

4. It's annoying to always have to specify l and r. One way to deal with this is to build default arguments into your function. Figure out how to do this with R so that your function defaults to `l=5` and `r=12`.

This is great if you like the defaults, but maybe you don't! One solution is to build a *function factory*. This is just a function that returns a function.

5. Make a function in R that takes the clock arm specifications `l` and `r` as arguments and which returns a function with `l` and `r` baked in as defaults. The returned function should take `t` as an argument and return `y`.

6. Use your function factory to recreate your original function.

4. Cardinality

What do we mean when we count the number of elements a set has? For example, how do we know that $\{\text{apple}, \text{orange}, \text{banana}\}$ has three elements? This might seem like a stupid question to ask since it's so trivial- everyone can count to three! But it actually gets at the meaning of "counting". Formally, this set has three elements because there is a 1-to-1 function between it and the numbers 1,2,3:

$$f(x) = \begin{cases} 1 & x = \text{apple} \\ 2 & x = \text{orange} \\ 3 & x = \text{banana} \end{cases} \qquad f^{-1}(y) = \begin{cases} \text{apple} & y = 1 \\ \text{orange} & y = 2 \\ \text{banana} & y = 3 \end{cases}$$

or we could write

$$\text{apple} \overset{f,f^{-1}}{\longleftrightarrow} 1$$
$$\text{orange} \overset{f,f^{-1}}{\longleftrightarrow} 2$$
$$\text{banana} \overset{f,f^{-1}}{\longleftrightarrow} 3$$

to convey the same thing. The point is that counting the number of things means assigning a natural number, in order, to each element of the set. Once we're out of elements, whatever the last number we got to is the number of elements in the set.

It doesn't matter *how* we assign numbers to the elements of the set as long as the assignment is invertible- we can't assign the same number twice (undercounting) or assign the same set element to two numbers (overcounting).

By extension, we can show that two sets have the same number of elements in them if there any 1-to-1 function between them. In other words, we can show two sets have the same number of elements without actually counting them explicitly. You can imagine that each set is in a different pot: you take one element out of each and pair them up and set them aside. If both pots end up empty at the same time, there were the same number of things in each, even if you didn't count how many there were.

We call this notion of "size" of a set A its *cardinality* and we usually denote it like $|A|$. The notation is evocative of the absolute value of a number, which is also like its "size" or absolute magnitude.

1. Pick a really big number n. Show formally that the sets $\{1, 2, 3, \ldots n\}$ and $\{2, 4, 6, \ldots 2n\}$ have the same number of elements by finding a 1-to-1 function between the two sets.

2. Show that for finite sets, $A \subset B$ implies that $|A| < |B|$.

This might seem silly to you but it actually gets really interesting for *infinite* sets. For example, consider the set of natural (counting) numbers $\mathbb{N} = \{1, 2, \ldots\}$ and the set of *integers* $\mathbb{Z} = \{\cdots - 2, -1, 0, 1, 2, \ldots\}$ (the symbol most commonly used for these is $\mathbb{Z}$ because German mathematicians came up with it and in German "Zahlen" means "numbers").

Clearly, $\mathbb{N} \subset \mathbb{Z}$. Every natural number is an integer, but there are integers that aren't natural numbers: namely, zero and the negative integers. So in that sense, the set of integers is "larger" than the set of natural numbers.

But what about in terms of the number of elements? Obviously both of these are infinite sets so we can't count the number of elements they have in the normal sense. However, using our notion of cardinality, we can still *compare* the cardinalities of the two sets. If the sets were finite, the result you proved above would imply that $|\mathbb{N}| < |\mathbb{Z}|$ but we'll see that's not the case.

3. Let $f(n) = \left(\frac{n+1}{2}\right)(-1)^{n+1}$. Calculate the value of f for the inputs $\{1, 2, 3, 4, 5\}$. Is this function 1-to-1? If so, what is the inverse? Use chatGPT or google or wolfram alpha to help you figure it out if you want.

The conclusion here is that there are exactly as many natural numbers as there are integers, which flies in the face of our understanding that the natural numbers are a strict subset of the integers. This should be somewhat unsettling to you. But really there isn't a contradiction: our intuition is wrong because we're used to the world of finite things, where a notion of size by "subsetting" and a notion of size by "counting" correspond to each other. But when you move to infinite sets that's no longer the case. This is definitely trippy and unintuitive. If you have a hard time wrapping your head around it, you're in good company, the chatGPT AI is confused too:

whats a bijective function between the integers and natural numbers? be terse

A bijective function between the integers and natural numbers is f(x) = x+1. It assigns to each integer x a unique natural number f(x) = x+1, and vice versa.

wrong. that's not bijective because x+1 is not a natural number for any negative integer

You are correct that the function f(x) = x+1 is not bijective between the integers and natural numbers, as it maps negative integers to non-natural numbers. A possible bijective function between the integers and natural numbers is f(x) = x+n where n is the smallest natural number such that f(x) is a natural number for all x in the integers.

there is no such smallest n, that doesn't work

You are correct. My previous statement was not correct. There is no bijective function between the integers and natural numbers, as the integers include negative numbers while the natural numbers do not.

You might think, oh, ok, then the cardinality of all infinite sets is the same, then? Right? *Wrong!* It turns out that some infinite sets cannot be put into 1-to-1 correspondence with others! The canonical example of this is the relationship between the real numbers $\mathbb{R}$ (i.e. the entire number line, all decimals, including infinite decimals like π or $\sqrt{2}$) and the natural numbers $\mathbb{N}$. Both of these are infinite sets but they don't have the same cardinality:

4. Watch this youtube video explaining Cantor's diagonalization argument. Summarize the gist of the argument in your own words.

The implication is that there are *different "sizes" of infinities!* Mathematicians call cardinality of the natural numbers *aleph naught:* $\aleph_0 = |\mathbb{N}|$, which is somehow "smaller" than the cardinaltiy of the real numbers. In fact, there is a 1-to-1 correspondence between the size of the *power set of natural numbers* and the size of the reals: $|2^{\mathbb{N}}| =$

$|\mathbb{R}|$. So there are as many real numbers as there are subsets of natural numbers. That's a shitton. And, in fact, this holds for any number line: any interval $[a, b]$, however small, has the same cardinality as the whole number line. As a matter of speaking, when a set has the cardinality of the counting numbers, we say it is "countably infinite", whereas when it has the cardinality of the number line we say it is "uncountably" or "continuously" infinite.

5. Over the domain $[-\pi/2, \pi/2]$, the function $\tan(x)$ is 1-to-1 with its range, $\mathbb{R}$. Use R to plot this function over this domain to visualize this fact. How can you use this to prove that any interval has the cardinality as the reals?

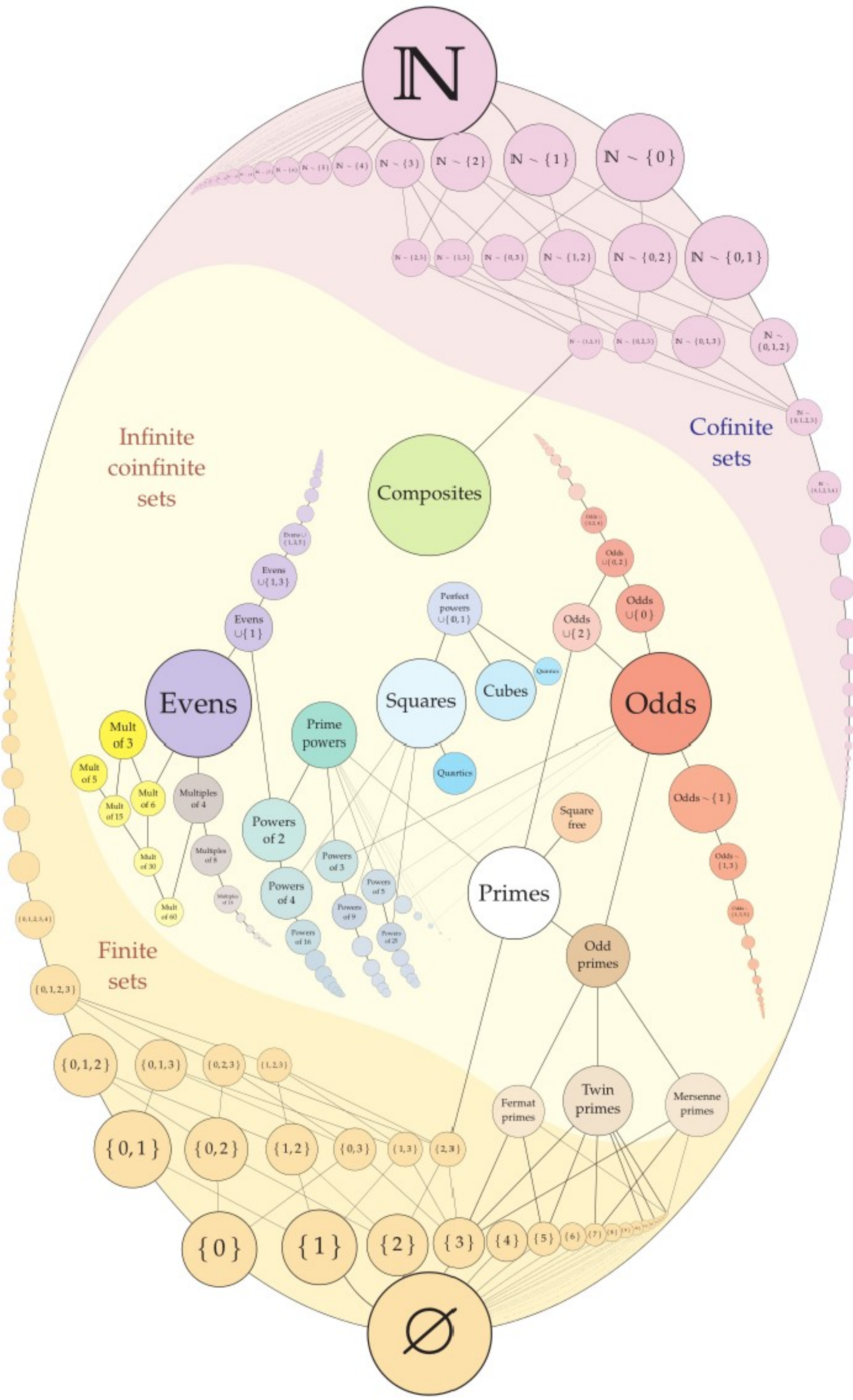

Figure 3.1

The power set lattice $P(\mathbb{N})$ of all sets of natural numbers, not to scale, some sets omitted…

The empty set $\varnothing$ is the global least element at the bottom of the lattice, of course, and the whole set $\mathbb{N}$ is the greatest element, at the top. The singletons $\{n\}$ are atoms, one step up from $\varnothing$, while their complements $\mathbb{N}\setminus\{n\}$ are coatoms, one step down from $\mathbb{N}$. One generally conceives of the

finite sets as clustering near the Earthly bottom of the lattice, finitely close to ∅, whereas the cofinite sets soar above, finitely close to the heavenly top. In the vast central regions between, in contrast, we find the infinite-coinfinite sets, including many deeply interesting sets such as the prime numbers, the squares, the square-free numbers, the composites, and uncountably many more.
- from "The lattice of sets of natural numbers is rich" by Joel Hamkins.

This is insane and really deep. There is something fundamentally different about the infinities of the "continuous" number line and that of the "countable" numbers. This is a really important distinction in higher math that actually matters for probability: there are lots of theorems that work for infinite sets when they are countable, but which totally fall apart when they are uncountable!

5. Look up the "continuum hypothesis". What does it say?