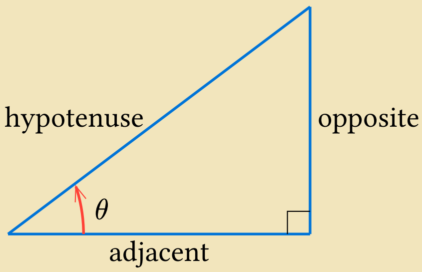
Thomas' Calculus

1. Preliminaries

This chapter is too easy, but still worth a figure drawn with `CeTZ`:



sine: $\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ **cosecant**: $\csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

cosine: $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ **secant**: $\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

tangent: $\sin\theta = \dfrac{\text{opp}}{\text{adj}}$ **cotangent**: $\cot\theta = \dfrac{\text{adj}}{\text{opp}}$

2. Limits and Continuity

2.1. Limits of Function Values

Let $f(x)$ be defined on an open interval about x_0, *except possibly at x_0 itself.* If $f(x)$ gets arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L$$

2.2. Definition of Limit

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x \to c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exist a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

2.3. Definition of Limits involving Infinity

1. We say that $f(x)$ has the **limit L as x approaches Infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x,

$$x > M \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x,

$$x < N \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

2.4. Limits Involving $(\sin\theta)/\theta$

$$\lim_{x\to 0}\frac{\sin\theta}{\theta}=1 \qquad (\theta \text{ in radians})$$

Proof Consider Figure 1. Notice that

$$\text{area}\,\triangle\,\mathrm{OAP} < \text{area sector OAP} < \text{area}\,\triangle\,\mathrm{OAT}.$$

Then express these areas in terms of θ, and thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

So

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$, the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=1.$$

Because $\frac{\sin\theta}{\theta}$ is an even function, so

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta}=1.$$

So $\lim_{x\to 0}\frac{\sin\theta}{\theta}=1$

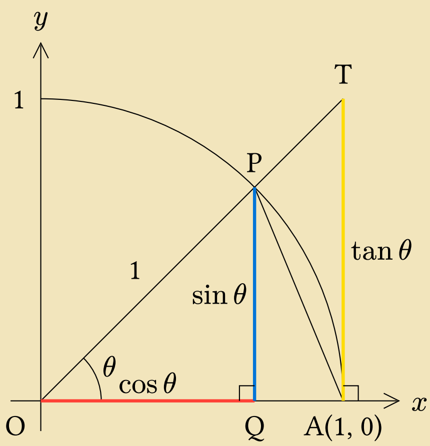


Figure 1: The proof of $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

2.5. Definition of Continuity

Let c be a real number on the x-axis.
The function is **continuous at c** if

$$\lim_{x\to c} f(x)=f(c).$$

The function is **right-continuous at c (or continuous from the right)** if

$$\lim_{x\to c^+} f(x)=f(c).$$

The function is **left-continuous at c (or continuous from the left)** if

$$\lim_{x \to c^-} f(x) = f(c).$$

2.6. Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

2.7. Some kinds of discontinuities

- removable discontinuity ($\frac{x^2}{x}$ at 0)
- jump discontinuity ($\lfloor x \rfloor$ at 1)
- infinite discontinuity ($\frac{1}{x^2}$ at 0)
- oscillating discontinuity ($\sin \frac{1}{x}$ at 0)

3. Derivatives

3.1. Definition of the derivative of functions

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or, alternatively

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

provided the limit exists.

3.2. The derivatives of some trigonometric functions

- $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$
- $\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\csc^2 x$
- $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$
- $\frac{\mathrm{d}}{\mathrm{d}x}(\csc x) = -\csc x \cot x,$

3.3. The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x},$$

where $\frac{\mathrm{d}y}{\mathrm{d}u}$ is evaluated at $u = g(x)$.

3.4. Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with $\frac{\mathrm{d}y}{\mathrm{d}x}$ on one side of the equation and solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$

3.5. The Number e as a Limit

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}$$

3.6. The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, the f^{-1} is differentiable at every point in its domain. The value of $\left(f^{-1}\right)'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\left(f^{-1}(b)\right)}$$

or

$$\left.\frac{\mathrm{d}f^{-1}}{\mathrm{d}x}\right|_{x=b} = \frac{1}{\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=f^{-1}(b)}}$$

3.7. Derivatives of the inverse trigonometric functions

$$\frac{\mathrm{d}\left(\sin^{-1} u\right)}{\mathrm{d}x} = \frac{1}{\sqrt{1-u^2}} \frac{\mathrm{d}u}{\mathrm{d}x}, \quad |u| < 1$$

$$\frac{\mathrm{d}\left(\cos^{-1} u\right)}{\mathrm{d}x} = -\frac{1}{\sqrt{1-u^2}} \frac{\mathrm{d}u}{\mathrm{d}x}, \quad |u| < 1$$

$$\frac{\mathrm{d}\left(\sec^{-1} u\right)}{\mathrm{d}x} = \frac{1}{|u|\sqrt{u^2-1}} \frac{\mathrm{d}u}{\mathrm{d}x}, \quad |u| > 1$$

$$\frac{\mathrm{d}\left(\csc^{-1} u\right)}{\mathrm{d}x} = -\frac{1}{|u|\sqrt{u^2-1}} \frac{\mathrm{d}u}{\mathrm{d}x}, \quad |u| > 1$$

$$\frac{\mathrm{d}\left(\tan^{-1} u\right)}{\mathrm{d}x} = \frac{1}{1+u^2} \frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}\left(\cot^{-1} u\right)}{\mathrm{d}x} = -\frac{1}{1+u^2} \frac{\mathrm{d}u}{\mathrm{d}x}$$

4. Applications of Derivatives

4.1. Definition of Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

4.2. The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

4.3. Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b),$$

then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

Proof Being continuous, f assumes absolute maximum and minimum values on $[a, b]$. These can occur only

1. at interior points where f' is zero,
2. at interior points where f' doesn't exist,
3. at the endpoints of the function's domain, in this case a and b.

The rest proof is easy.

4.4. The Mean Value Theorem

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b)at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

It's easy to proof, simply let $g(x) = f(x) - f(a) - \frac{x-a}{b-a}(f(b) - f(a))$, then it comes to Rolle's Theorem since $g(a) = g(b) = 0$.

4.5. Definition of Concavity and The Second Derivative Test for Concavity

The graph of a differentiable function $y = f(x)$ is

a) concave up on an open interval I if f' is increasing on I ($f'' > 0$ on I)

b) concave down on an open interval I if f' is decreasing on I ($f'' < 0$ on I)

4.6. Definition of Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**. It mostly exists where $y'' = 0$, but sometime not, and may occur where y'' doesn't exist.

4.7. L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

The proof requires Cauchy's Theorem, a Mean Value Theorem involving two functions instead of one.

Proof Suppose $x > a$. Then $g'(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from a to x, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

As x approaches a, c approaches a. Therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(x)}{g'(c)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

It similar when $x < a$,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

The L'Hôpital's Rule applies to the indeterminate form $\frac{\infty}{\infty}$ as well as to $\frac{0}{0}$. Not proved here.

4.8. Cauchy's Mean Value Theorem

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exist a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof According to the Mean Value Theorem, $g(a) \neq g(b)$, otherwise

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0.$$

Then, we apply the Mean Value Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(b)}[g(x) - g(a)].$$

of which $F(b) = F(a) = 0$, so there exists c

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}[g'(c)] = 0,$$

or

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

4.9. Newton's Method

pass.

4.10. Definition of Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Here are some basic and important antiderivatives.

Function	**General antiderivative**
x^n	$\frac{x^{n+1}}{n+1} + C, n \neq -1$
$\sin kx$	$-\frac{1}{k}\cos kx + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$
$\sec^2 kx$	$\frac{1}{k}\tan kx + C$
$\csc^2 kx$	$-\frac{1}{k}\cot kx + C$
$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
$\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$
e^{kx}	$\frac{1}{k}e^{kx} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C, x \neq 0$
$\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
$\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
$\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C, kx > 1$
a^{kx}	$\left(\frac{1}{k\ln a}\right)a^{kx} + C, a > 0, a \neq 1$

4.11. Definition of Indefinite Integral and Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,\mathrm{d}x.$$

The symbol $\int$ is an **integral sign**.The function f is the **integrand** of the integral, and x is the **variable of integration**.

5. Integration

5.1. Formulas for the sums of the squares and cubes

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

5.2. Riemann Sums

Consider an arbitrary function f defined on a closed interval $[a,b]$. We subdivide the interval $[a,b]$ into subintervals, not necessarily of equal widths. To do so, we choose $n-1$ points $\{x_1, x_2, x_3, \dots, x_{n-1}\}$ between a and b and satisfying

$$a < x_1 < x_2 < x_3 < \dots < x_{n-1} < b.$$

To make the notation consistent, we denote a by x_0 and b by x_n, so that

$$a = x_0 < x_1 < x_2 < x_3 < \dots < x_{n-1} < x_n = b.$$

The set

$$P = \{x_0, x_1, x_2, x_3, \dots, x_{n-1}, x_n\}$$

is called a **partition** of $[a,b]$.

In each subinterval $[x_{k-1}, x_k]$ we choose a point c_k, and we have

$$S_p = \sum_{k=1}^{n} f(c_k)\Delta x_k.$$

The S_p is called a **Riemann sum for f on the interval $[a,b]$**. And we define the **norm** of a partition P, written $\|P\|$, to be the largest of all the subinterval widths, that is, the maximum of Δx_k.

5.3. The Definite Integral as a Limit of Riemann Sum

Let $f(x)$ be a function defined on a closed interval $[a,b]$. We say that a number I is the **definite integral of f over $[a,b]$** and that I is the limit of the Riemann sums $\Sigma_{k=1}^{n} f(c_k)\Delta x_k$ is the following condition is satisfied:

Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ of $[a,b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n} f(c_k)\Delta x_k - I\right| < \varepsilon.$$

5.4. Notation of the Definite Integral

The symbol for the number I in the definition of the definite integral is

$$\int_a^b f(x)\,\mathrm{d}x$$

which is read as "the integral from a to b of f of x dee x".

5.5. The Existence of Definite Integrals

A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, the its definite integral over $[a, b]$ exists.

5.6. The Average or Mean Value of a Function

If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean value**, is

$$\mathrm{av}(f) = \frac{a}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

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5.7. The Mean Value Theorem for Definite Integral

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

Proof If we divide both sides of the Max-in Inequality

$$\min f \cdot (b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq \max f \cdot (b-a)$$

by $(b-a)$, we obtain

$$\min f \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \max f.$$

Since f is continuous, f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$ at some point c in $[a, b]$.

5.8. The Fundamental Theorem of Calculus

If $f(t)$ is an integrable function over a finite interval I, then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Then, at every value of x,

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x).$$

To calculate $\int_a^b f(x)\,\mathrm{d}x$, we let

$$G(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Thus

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= [G(b)+C] - [G(a)+C] \\ &= \int_a^b f(x)\,\mathrm{d}x - \int_a^a f(x)\,\mathrm{d}x \\ &= \int_a^b f(x)\,\mathrm{d}x.\end{aligned}$$

The usual notation for $F(b) - F(a)$ is

$$F(x)\Big]_a^b \quad \text{or} \quad \Big[F(x)\Big]_a^b,$$

depending on whether F has one or more terms.

5.9. The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int (f(g(x))g'(x))\,\mathrm{d}x = \int f(u)\,\mathrm{d}u.$$

5.10. Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)) \cdot g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

6. Applications of Definite Integrals