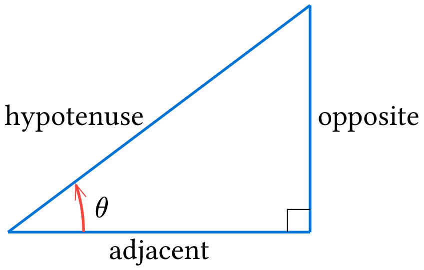
Thomas' Calculus

1. Preliminaries

This chapter is too easy, but still worth a figure drawn with `CeTZ`:



$$\textbf{sine:}\ \sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \textbf{cosecant:}\ \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\textbf{cosine:}\ \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \textbf{secant:}\ \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\textbf{tangent:}\ \sin\theta = \frac{\text{opp}}{\text{adj}} \qquad \textbf{cotangent:}\ \cot\theta = \frac{\text{adj}}{\text{opp}}$$

2. Limits and Continuity

Limits of Function Values.

Let $f(x)$ be defined on an open interval about x_0, *except possibly at x_0 itself.* If $f(x)$ gets arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L$$

Definition of Limit.

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x \to c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exist a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \varepsilon.$$

Definition of Limits involving Infinity.

1. We say that $f(x)$ has the **limit L as x approaches Infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x,

$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \varepsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x,

$$x < N \qquad \Rightarrow \qquad |f(x) - L| < \varepsilon.$$

***Limits Involving* $(\sin\theta)/\theta$.**

2.4. Limits Involving $(\sin\theta)/\theta$

$$\lim_{x\to 0}\frac{\sin\theta}{\theta}=1 \qquad (\theta \text{ in radians})$$

Proof Consider Figure 1. Notice that

$$\text{area} \triangle \text{OAP} < \text{area sector OAP} < \text{area} \triangle \text{OAT}.$$

Then express these areas in terms of θ, and thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

So

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$, the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=1.$$

Because $\frac{\sin\theta}{\theta}$ is an even function, so

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta}=1.$$

So $\lim_{x\to 0}\frac{\sin\theta}{\theta}=1$

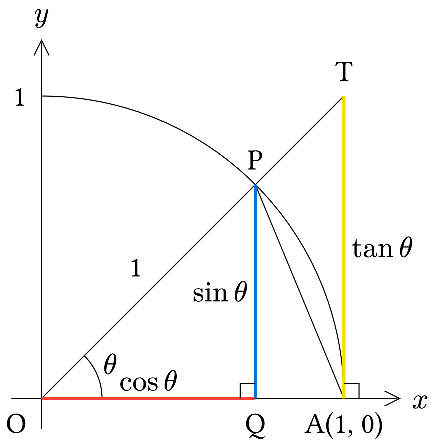


Figure 1: The proof of $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

Definition of Continuity.
Let c be a real number on the x-axis.
The function is **continuous at c** if

$$\lim_{x\to c} f(x) = f(c).$$

The function is **right-continuous at c (or continuous from the right)** if

$$\lim_{x\to c^+} f(x) = f(c).$$

The function is **left-continuous at c (or continuous from the left)** if

$$\lim_{x \to c^-} f(x) = f(c).$$

Continuity Test.
A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

Some kinds of discontinuities.

- removable discontinuity ($\frac{x^2}{x}$ at 0)
- jump discontinuity ($\lfloor x \rfloor$ at 1)
- infinite discontinuity ($\frac{1}{x^2}$ at 0)
- oscillating discontinuity ($\sin \frac{1}{x}$ at 0)

3. Derivatives

Definition of the derivative of functions.
The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or, alternatively

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

provided the limit exists.

The derivatives of some trigonometric functions.
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x,$

The Chain Rule.
If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

Implicit Differentiation.
1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.

2. Collect the terms with $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$

The Number e as a Limit.
The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$$

The Derivative Rule for Inverses.
If f has an interval I as domain and $f'(x)$ exists and is never zero on I, the f^{-1} is differentiable at every point in its domain. The value of $\left(f^{-1}\right)'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Derivatives of the inverse trigonometric functions.

$$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

$$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

$$\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$$

4. Applications of Derivatives

The Extreme Value Theorem.
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.